Krishna's Question Bank

Differential Calculus

(*For B.A. and B.Sc. I year students of All Colleges affiliated to Allahabad State University*)

As per Allahabad State University Syllabus

(*w.e.f.* 2017-2018)

By

A. R. Vasishtha
Retired Head, Dep't. of Mathematics
Meerut College, Meerut (U.P.)

Hemlata Vasishtha
M.Sc. (Gold Medalist), Ph.D.
C.C.S. University, Meerut (U.P.)

KRISHNA
KRISHNA Prakashan Media (P) Ltd.
KRISHNA HOUSE, 11, Shivaji Road, Meerut-250 001 (U.P.), India

Jai Shri Radhey Shyam

Dedicated to Lord Krishna

Authors & Publishers

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Krishna's

DIFFERENTIAL CALCULUS

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Problem 1: *If* $\lim_{x \to a} f(x)$ *and* $\lim_{x \to a} g(x)$ *do not exist, can* $\lim_{x \to a} [f(x) + g(x)]$ *or* $\lim_{x \to a} [f(x)\, g(x)]$ *exist ?*

(Kumaun 2008)

Solution: Yes. If we take $f(x) = \sin(1/x)$, $g(x) = -\sin(1/x)$ whenever $x \neq 0$, then $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ do not exist but $\lim_{x \to a} [f(x) + g(x)]$ exists.

Again if we take $f(x) = g(x) = 1$ for all rational x and $f(x) = g(x) = -1$ for all irrational x, then both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ do not exist for any real number a but $\lim_{x \to a} [f(x)\, g(x)]$ exists for every real number a.

Problem 2: *If* $\lim_{x \to a} f(x)$ *and* $\lim_{x \to a} [f(x) + g(x)]$ *both exist, must* $\lim_{x \to a} g(x)$ *exist ?*

Solution: Yes.

Let $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} [f(x) + g(x)] = m.$

Then $\lim_{x \to a} [\{f(x) + g(x)\} - f(x)] = \lim_{x \to a} [f(x) + g(x)] - \lim_{x \to a} f(x)$

$= m - l.$

$\therefore$ $\lim_{x \to a} g(x) = m - l$ and thus $\lim_{x \to a} g(x)$ must exist.

Problem 3: *If* $\lim_{x \to a} f(x)$ *and* $\lim_{x \to a} [f(x)\, g(x)]$ *both exist, must* $\lim_{x \to a} g(x)$ *exist ?*

Solution: No. If we take $f(x) = x \;\; \forall\, x \in \mathbf{R}$ and $g(x) = \sin(1/x)$, if $x \neq 0$, $g(0) = 0$, then $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} [f(x)\, g(x)]$ both exist but $\lim_{x \to 0} g(x)$ does not exist.

Problem 4: *Show that* $\lim_{x \to 0} f(x) = \lim_{x \to a} f(x - a)$.

Solution: Let $\lim_{x \to a} f(x - a) = l$. Then to show that $\lim_{x \to 0} f(x) = l$.

Take any given $\varepsilon > 0$.

Since $\lim_{x \to a} f(x - a) = l$, therefore there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x - a) - l| < \varepsilon$$

i.e., $0 < |y| < \delta \Rightarrow |f(y) - l| < \varepsilon$, putting $x - a = y$

i.e., $0 < |y - 0| < \delta \Rightarrow |f(y) - l| < \varepsilon$.

$\therefore$ By the definition of limit, $\lim_{y \to 0} f(y) = l$ *i.e.*, $\lim_{x \to 0} f(x) = l$.

Problem 5: *Using definition of limit, show that* $\lim_{x \to 0} f(x) = 1$ *when*

$$f(x) = \begin{cases} 1 + x^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Solution: It is required to prove that $\lim_{x \to 0} f(x) = 1$.

Take any given $\varepsilon > 0$.
If $x \neq 0$, then $|f(x) - 1| = |(1 + x^2) - 1| = |x^2| = |x|^2 < \varepsilon$, provided $|x| < \sqrt{\varepsilon}$.
So if we take $\delta > 0$ such that $\delta = \sqrt{\varepsilon}$, then $|f(x) - 1| < \varepsilon$, whenever $0 < |x - 0| < \delta$.
$\therefore$ By definition of limit, $\lim_{x \to 0} f(x) = 1$.

Problem 6: *If f is defined on* $\mathbf{R}$ *as* $f(x) = \begin{cases} 2, & \text{if } x \text{ is irrational} \\ 1, & \text{if } x \text{ is rational,} \end{cases}$

prove that $\lim_{x \to a} f(x)$ *does not exist for any* $a \in \mathbf{R}$.

Solution: We shall show that there is no real number l such that $\lim_{x \to a} f(x) = l$.

We know that $\lim_{x \to a} f(x) = l$ if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - l| < \varepsilon, \quad \text{whenever } 0 < |x - a| < \delta.$$

We have $|f(x) - l| = |2 - l|$, if x is irrational

and $|f(x) - l| = |1 - l|$, if x is rational.

Now between any two distinct real numbers there lie infinite rational and infinite irrational numbers. So whatever $\delta > 0$ we take, there are infinite rational and infinite irrational numbers x such that $0 < |x - a| < \delta$.

So whatever $\delta > 0$ we take there are infinite rational numbers x such that

$$0 < |x - a| < \delta \quad \text{and} \quad |f(x) - l| = |1 - l|$$

and there are infinite irrational numbers x such that

$$0 < |x - a| < \delta \text{ and } |f(x) - l| = |2 - l|.$$

Case 1: Let $l \neq 1$. Then $1 - l \neq 0 \rightarrow |1 - l| > 0$.

So if we take $\varepsilon = \frac{1}{2}|1 - l| > 0$, then whatever $\delta > 0$ we may take, there exist real numbers x such that $0 < |x - a| < \delta$ and $|f(x) - l| = |1 - l| > \frac{1}{2}|1 - l| = \varepsilon$

i.e., for $\varepsilon = \frac{1}{2}|1 - l| > 0$, there exists no $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

$\therefore$ We cannot have $\lim_{x \to a} f(x) = l$, where $l \neq 1$.

Case 2: Let $l = 1$. Then $l \neq 2 \Rightarrow |2 - l| > 0$. So if we take $\varepsilon = \frac{1}{2}|2 - l| > 0$, then whatever $\delta > 0$ we may take, there exist real numbers x such that

$$0 < |x - a| < \delta \text{ and } |f(x) - l| = |2 - l| > \frac{1}{2}|2 - l| = \varepsilon$$

i.e., for $\varepsilon = \frac{1}{2}|2 - l| > 0$, there exists no $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

$\therefore$ We cannot have $\lim_{x \to a} f(x) = l$, where $l = 1$.

Thus no real number l can be equal to $\lim_{x \to a} f(x)$ and so $\lim_{x \to a} f(x)$ does not exist.

Problem 7: *If f is defined on* **R** *as* $f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1, & \text{if } x \text{ is rational,} \end{cases}$

prove that $\lim_{x \to a} f(x)$ *does not exist for any* $a \in \mathbf{R}$.

Solution: Proceed exactly in the same way as in problem 6.

Here $\quad |f(x) - l| = |0 - l| = |l|$, if x is irrational

and $\quad |f(x) - l| = |1 - l|$, if x is rational.

While discussing case 2, when $l = 1$, take $\varepsilon = \frac{1}{2}|0 - l| = \frac{1}{2}|l| > 0.$

Then for $\varepsilon = \frac{1}{2}|l| > 0$, there exists no $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

Problem 8: *If $x \rightarrow 0$, then does the limit of the following function f exist or not ?*

$$f(x) = x, \text{ when } x < 0;\ f(x) = 1, \text{ when } x = 0;\ f(x) = x^2, \text{ when } x > 0.$$

Solution: We have $f(0+0)=\lim_{x\to 0+} f(x)=\lim_{h\to 0} f(0+h), h>0$ and sufficiently small

$$=\lim_{h\to 0} f(h)=\lim_{h\to 0} h^2=0.$$

Again $f(0-0)=\lim_{x\to 0-} f(x)=\lim_{h\to 0} f(0-h), h>0$ and sufficiently small

$$=\lim_{h\to 0} f(-h)=\lim_{h\to 0} (-h)=0.$$

Since $f(0+0)=f(0-0)=0$, therefore $\lim_{x\to 0} f(x)$ exists and is equal to 0 .

Problem 9: *Use the formula* $\lim_{x\to 0}\frac{a^x-1}{x}=\log a$ *to find* $\lim_{x\to 0}\frac{2^x-1}{(1+x)^{1/2}-1}$.

Solution: We have $\lim_{x\to 0}\frac{2^x-1}{(1+x)^{1/2}-1}$

$$=\lim_{x\to 0}\frac{2^x-1}{\left[1+\frac{1}{2}x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1.2}x^2+\ldots\right]-1},$$

expand $(1+x)^{1/2}$ by binomial theorem

$$=\lim_{x\to 0}\frac{2^x-1}{x\left[\frac{1}{2}-\frac{1}{8}x+\ldots\right]}=\lim_{x\to 0}\left[\frac{2^x-1}{x}\cdot\frac{1}{\frac{1}{2}-\frac{1}{8}x+\ldots}\right]$$

$$=(\log 2)\cdot\frac{1}{\frac{1}{2}}=2\log 2. \qquad \left[\because \lim_{x\to 0}\frac{a^x-1}{x}=\log a\right]$$

Problem 10: *If* $f(x)=e^{-1/x}$, *show that at* $x=0$, *the right hand limit is zero while the left hand limit is* $+\infty$, *and thus there is no limit of the function at* $x=0$.

Solution: We have $f(0+0)=\lim_{x\to 0+} f(x)$

$$=\lim_{h\to 0} f(0+h), h>0$$

$$=\lim_{h\to 0} f(h)=\lim_{h\to 0} e^{-1/h}=\lim_{h\to 0}\frac{1}{e^{1/h}}=\frac{1}{e^{\infty}}=\frac{1}{\infty}=0.$$

Again $f(0-0)=\lim_{x\to 0-} f(x)=\lim_{h\to 0} f(0-h), h>0$

$$=\lim_{h\to 0} f(-h)=\lim_{h\to 0} e^{-1/(-h)}=\lim_{h\to 0} e^{1/h}=e^{\infty}=\infty.$$

Since $f(0-0)\neq f(0+0)$, therefore $\lim_{x\to 0} f(x)$ does not exist.

Problem 11: *Give an example to show that* $\lim_{x\to a} f(x)$ *may exist even when the function is not defined for* $x=a$.

Solution: Consider the function $f(x)=\dfrac{x^2-a^2}{x-a}$.

We have $f(a)=\dfrac{0}{0}$ which is meaningless and so $f(x)$ is not defined for $x=a$.

But $$\lim_{x\to a} f(x)=\lim_{x\to a}\frac{x^2-a^2}{x-a}=\lim_{x\to a}\frac{(x-a)(x+a)}{x-a}$$

$$=\lim_{x\to a}(x+a)=2a \quad i.e., \quad \lim_{x\to a} f(x) \text{ exists.}$$

Thus though $f(x)$ is not defined for $x=a$, yet $\lim_{x\to a} f(x)$ exists.

Problem 12: *Let* $f(x)=\begin{cases} x, & 0\le x<1 \\ 3-x, & 1\le x\le 2. \end{cases}$

Show that $\lim_{x\to 1+} f(x)=2$. *Does the limit of* $f(x)$ *at* $x=1$ *exist ? Give reasons for your answer.*

Solution: We have

$$f(1+0)=\lim_{x\to 1+} f(x)=\lim_{h\to 0} f(1+h), h>0$$

and sufficiently small

$$=\lim_{h\to 0}\{3-(1+h)\}=\lim_{h\to 0}(2-h)=2.$$

Again $f(1-0)=\lim_{x\to 1-} f(x)=\lim_{h\to 0} f(1-h), h>0$ and sufficiently small

$$=\lim_{h\to 0}(1-h) \qquad [\because 1-h<1 \text{ and when } x<1,\ f(x)=x]$$

$$=1.$$

Since $f(1-0)\neq f(1+0)$, therefore $\lim_{x\to 1} f(x)$ does not exist.

Problem 13: *Evaluate :* $\lim_{x \to 0} \frac{x - |x|}{x}$. **(Meerut 2001)**

Solution: Let $f(x) = \frac{x - |x|}{x}$.

Then $f(0+0) = \lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h)$, $h > 0$ and sufficiently small

$= \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{h - |h|}{h} = \lim_{h \to 0} \frac{h - h}{h}$ $\quad [\because h > 0 \Rightarrow |h| = h]$

$= \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0.$

Again $f(0-0) = \lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h)$, $h > 0$ and sufficiently small

$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{-h - |-h|}{-h} = \lim_{h \to 0} \frac{-h - h}{-h}$

$[\because -h < 0 \Rightarrow |-h| = -(-h) = h]$

$= \lim_{h \to 0} \frac{-2h}{-h} = \lim_{h \to 0} 2 = 2.$

Since $f(0-0) \neq f(0+0)$, therefore $\lim_{x \to 0} f(x)$ does not exist.

Problem 14: *Evaluate :* $\lim_{x \to 0} \frac{|\sin x|}{x}$.

Solution: Let $f(x) = \frac{|\sin x|}{x}$.

Then $f(0+0) = \lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h), h > 0$

$= \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{|\sin h|}{h} = \lim_{h \to 0} \frac{\sin h}{h} = 1.$

$[\because$ When $h > 0$ and is sufficiently small, we have $\sin h > 0]$

Again $f(0-0) = \lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h), h > 0$

$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{|\sin(-h)|}{-h} = \lim_{h \to 0} \frac{|-\sin h|}{-h}$

$= \lim_{h \to 0} \frac{|\sin h|}{-h} = -\lim_{h \to 0} \frac{|\sin h|}{h} = -\lim_{h \to 0} \frac{\sin h}{h} = -1.$

Since $f(0-0) \neq f(0+0)$, therefore $\lim_{x \to 0} f(x)$ does not exist.

Problem 15: *Evaluate :* $\lim_{x \to 0} \dfrac{e^{1/x}}{e^{1/x}+1}$.

(Rohilkhand 2005, 08; Meerut 06; Avadh 10)

Solution: Let $f(x) = \dfrac{e^{1/x}}{e^{1/x}+1}$.

Then $f(0+0) = \lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h), h > 0$

$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{e^{1/h}}{e^{1/h}+1} = \lim_{h \to 0} \frac{1}{1+(1/e^{1/h})}$$

$$= \frac{1}{1+0} = 1. \quad \left[\because \lim_{h \to 0} \frac{1}{e^{1/h}} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0\right]$$

Again $f(0-0) = \lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h), h > 0$

$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{e^{-1/h}}{e^{-1/h}+1} = \frac{0}{0+1} = 0.$$

$$\left[\because \lim_{h \to 0} e^{-1/h} = \lim_{h \to 0} \frac{1}{e^{1/h}} = \frac{1}{\infty} = 0\right]$$

Since $f(0-0) \neq f(0+0)$, therefore $\lim_{x \to 0} f(x)$ does not exist .

Remark: Remember that $\lim_{x \to 0+} \dfrac{1}{x} = \infty$ and $\lim_{x \to 0-} \dfrac{1}{x} = -\infty$.

Problem 16: *If* $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n$ *then prove that*

$$\lim_{x \to c} f(x) = f(c).$$

(Garhwal 2011)

Solution: First we shall prove that $\lim_{x \to c} x = c$.

Take any given $\varepsilon > 0$.

Let $g(x) = x$. Then $|g(x) - c| = |x - c| < \varepsilon$, provided $|x - c| < \varepsilon$.

So if we take $\delta > 0$ such that $\delta = \varepsilon$ or $\delta < \varepsilon$, then $|g(x) - c| = |x - c| < \varepsilon$, whenever $0 < |x - c| < \delta$.

Thus for any given $\varepsilon > 0$, there exists $\delta (= \varepsilon) > 0$ such that

$$|x - c| < \varepsilon \text{ whenever } 0 < |x - c| < \delta.$$

$$\therefore \quad \lim_{x \to c} x = c.$$

Now we know that if $\lim_{x \to c} \phi(x) = l$, $\lim_{x \to c} \psi(x) = m$ and $k \in \mathbf{R}$, then

$$\lim_{x \to c} [\phi(x) + \psi(x)] = \lim_{x \to c} \phi(x) + \lim_{x \to c} \psi(x) = l + m,$$

$$\lim_{x \to c} \{k\, \phi(x)\} = k. \lim_{x \to c} \phi(x) = kl$$

and $$\lim_{x \to c} \{\phi(x) . \psi(x)\} = \lim_{x \to c} \phi(x) . \lim_{x \to c} \psi(x) = lm.$$

$\therefore$ If $n \in \mathbf{N}$, then

$$\lim_{x \to c} x^n = \left(\lim_{x \to c} x\right)\left(\lim_{x \to c} x\right) \ldots n \text{ times} = (c)(c) \ldots n \text{ times} = c^n.$$

$$\therefore \quad \lim_{x \to c} f(x) = \lim_{x \to c} [a_0 x^n a_1 x^{n-1} + \ldots + a_{n-1} x + a_n]$$

$$= \lim_{x \to c} (a_0 x^n) + \lim_{x \to c} (a_1 x^{n-1}) + \ldots + \lim_{x \to c} (a_{n-1} x) + \lim_{x \to c} a_n$$

$$= a_0 \lim_{x \to c} x^n + a_1 \lim_{x \to c} x^{n-1} + \ldots + a_{n-1} \lim_{x \to c} x + a_n$$

$$= a_0 c^n + a_1 c^{n-1} + \ldots + a_{n-1} c + a_n = f(c).$$

Comprehensive Problems 2

Problem 1: *Discuss the continuity and discontinuity of the following functions :*

(*i*) $f(x) = x^3 - 3x$.

(*ii*) $f(x) = x + x^{-1}$.

(*iii*) $f(x) = e^{-1/x}$.

(*iv*) $f(x) = \sin x$. **(Kanpur 2015)**

(*v*) $f(x) = \cos\left(\frac{1}{x}\right)$ when $x \neq 0$; $f(0) = 0$. **(Lucknow 2005)**

(*vi*) $f(x) = \sin(1/x)$ *when* $x \neq 0$, $f(0) = 0$. **(Lucknow 2011)**

(*vii*) $f(x) = \dfrac{\sin x}{x}$ *when* $x \neq 0$ *and* $f(0) = 1$. **(Kanpur 2007; Avadh 08)**

(*viii*) $f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ *when* $x \neq 0$ *and* $f(0) = 1$. **(Meerut 2004B; Kumaun 10)**

(*ix*) $f(x) = \dfrac{e^{1/x}}{1 + e^{1/x}}$ *when* $x \neq 0$, $f(0) = 0$. **(Bundelkhand 2011; Lucknow 11)**

(*x*) $f(x) = \dfrac{xe^{1/x}}{1 + e^{1/x}} + \sin\left(\dfrac{1}{x}\right)$ *when* $x \neq 0$, $f(0) = 0$.

(*xi*) $f(x) = \sin x \cos(1/x)$ *when* $x \neq 0$, $f(0) = 0$.

Solution: **(i)** Here $f(x) = x^3 - 3x$. The domain f is the whole **R**. Let $c \in \mathbf{R}$.

Then $$\lim_{x \to c} f(x) = \lim_{x \to c} (x^3 - 3x) = \lim_{x \to c} x^3 - 3 \lim_{x \to c} x$$
$$= c^3 - 3c = f(c).$$

$\therefore$ $f(x)$ is continuous at every point c of **R** and thus $f(x)$ is continuous on the whole real line.

(ii) Here $f(x) = x + x^{-1} = x + (1/x)$.

The function $f(x)$ is not defined at $x = 0$ and is defined at every other real number.

Thus domain $f = \mathbf{R} - \{0\}$ *i.e.*, $\mathbf{R}_0$.

Let $c \in \mathbf{R}$ and $c \neq 0$.

We have $$\lim_{x \to c} f(x) = \lim_{x \to c} \left(x + \frac{1}{x}\right) = c + \frac{1}{c} = f(c).$$

$\therefore$ $f(x)$ is continuous at every point c of **R** if $c \neq 0$.

Now $$f(0+0) = \lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h), h > 0$$
$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} \left(h + \frac{1}{h}\right) = 0 + \infty = \infty$$

and $$f(0-0) = \lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h), h > 0$$
$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} \left(-h - \frac{1}{h}\right) = -0 - \infty = -\infty.$$

Thus $f(x)$ has a discontinuity of the second kind at $x = 0$. In fact it is an infinite discontinuity.

(iii) Here $f(x) = e^{-1/x}$.

The function $f(x)$ is not defined at $x = 0$ and is defined at every other real number.

Thus domain $f = \mathbf{R} - \{0\}$ *i.e.*, $\mathbf{R}_0$.

Let $c \in \mathbf{R}$ and $c \neq 0$.

We have $$\lim_{x \to c} f(x) = \lim_{x \to c} e^{-1/x} = e^{-1/c} = f(c).$$

$\therefore$ $f(x)$ is continuous at every point c of **R** if $c \neq 0$.

Now $$f(0+0) = \lim_{x \to 0+} f(x) = \lim_{h \to 0} f(0+h), h > 0$$
$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} e^{-1/h} = \lim_{h \to 0} \frac{1}{e^{1/h}} = \frac{1}{\infty} = 0$$

and $$f(0-0) = \lim_{x \to 0-} f(x) = \lim_{h \to 0} f(0-h), h > 0$$

$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} e^{-1/(-h)} = \lim_{h \to 0} e^{1/h} = \infty.$$

Thus $f(x)$ has an infinite discontinuity at $x = 0$. The discontinuity is not ordinary but is of the second kind since $\lim_{x \to 0-} f(x)$ does not exist .

(iv) $f(x) = sin\ x$

We have $f(x) = \sin x \ \forall\ x \in \mathbf{R}$ and and domain $f = \mathbf{R}$

Let $c \in \mathbf{R}$.

Then $\lim_{x \to c} f(x) = \lim_{x \to c} \sin x = \sin c = f$

Hence, $f(x) = \sin x$ is continuous at $x = c$

But c is an arbitrary point of R. Hence $f(x)$ is continuous at all points of R i.e. the function $f(x) = c$ is continuous on the whole real line.

(v) Here $f(x) = \cos(1/x)$ when $x \neq 0$, $f(0)$.

Let $c \in \mathbf{R}$ and $c \neq 0$.

We have $\lim_{x \to c} f(x) = \lim_{x \to c} \cos\left(\frac{1}{x}\right) = \cos\left(\frac{1}{c}\right) = f(c).$

$\therefore$ $f(x)$ is continuous at every point c of $\mathbf{R}$ if $c \neq 0$.

Now to check the continuity of $f(x)$ at $x = 0$.

We have $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h),\ h > 0$

$$= \lim_{h \to 0} \cos \frac{1}{h},\ h > 0 \text{ which does not exist.}$$

As $h \to 0$, the value of $\cos(1/h)$ oscillates between $+1$ and -1, passing through zero and intermediate values an infinite number of times. Hence there is no definite real number l to which $\cos(1/h)$ tends as h tends to zero. Therefore the right hand limit $f(0+0)$ does not exist.

Again $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h),\ h > 0$

$$= \lim_{h \to 0} \cos\left(\frac{1}{-h}\right) = -\lim_{h \to 0} \cos\left(\frac{1}{h}\right), \text{which does not exist.}$$

Thus both $f(0-0)$ and $f(0+0)$ do not exist and so $f(x)$ has a discontinuity of the second kind at $x = 0$.

(vi) Here $f(x) = \sin(1/x)$ when $x \neq 0$, $f(0) = 0$.

Let $c \in \mathbf{R}$ and $c \neq 0$.

We have $\lim_{x \to c} f(x) = \lim_{x \to c} \sin \frac{1}{x} = \sin \frac{1}{c} = f(c).$

$\therefore$ $f(x)$ is continuous at every point c of $\mathbf{R}$ if $c \neq 0$.

Now to check the continuity of $f(x)$ at $x = 0$.

We have $$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h),\ h > 0$$

$$= \lim_{h \to 0} \sin\frac{1}{h}, \text{ which does not exist.}$$

Again $$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h),\ h > 0$$

$$= \lim_{h \to 0} \sin\left(\frac{1}{-h}\right) = -\lim_{h \to 0} \sin\frac{1}{h}, \text{ which does not exist.}$$

Thus both $f(0-0)$ and $f(0+0)$ do not exist and so $f(x)$ has a discontinuity of the second kind at $x = 0$.

(vii) Here $f(x) = \dfrac{\sin x}{x}$ when $x \neq 0$ and $f(0) = 1$.

Let $c \in \mathbf{R}$ and $c \neq 0$.

We have $$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{\sin x}{x} = \frac{\sin c}{c} = f(c).$$

$\therefore$ $f(x)$ is continuous at every point c of $\mathbf{R}$ if $c \neq 0$.

Now to check the continuity of $f(x)$ at $x = 0$.

We have $$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h),\ h > 0 = \lim_{h \to 0} \frac{\sin h}{h} = 1.$$

Again $$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h),\ h > 0$$

$$= \lim_{h \to 0} \frac{\sin(-h)}{(-h)} = \lim_{h \to 0} \frac{\sin h}{h} = 1.$$

Also $f(0) = 1$.

Since $f(0-0) = f(0) = f(0+0)$, therefore $f(x)$ is continuous at $x = 0$.

Hence $f(x)$ is continuous on the whole real line.

(viii) Here $f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ when $x \neq 0$ and $f(0) = 1$.

Let $c \in \mathbf{R}$ and $c \neq 0$.

We have $$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{e^{1/x} - 1}{e^{1/x} + 1} = \frac{e^{1/c} - 1}{e^{1/c} + 1} = f(c).$$

$\therefore$ $f(x)$ is continuous at every point c of $\mathbf{R}$ if $c \neq 0$.

Now to check the continuity of $f(x)$ at $x = 0$.

We have $$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h),\ h > 0$$

$$= \lim_{h \to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1} = \lim_{h \to 0} \frac{1 - (1/e^{1/h})}{1 + (1/e^{1/h})} = 1.$$

$$\left[\because \lim_{h \to 0} \frac{1}{e^{1/h}} = \frac{1}{\infty} = 0\right]$$

Again $\quad f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h), h > 0$

$$= \lim_{h \to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1} = \frac{0-1}{0+1} = -1.$$

$$\left[\because \lim_{h \to 0} e^{-1/h} = \lim_{h \to 0} \frac{1}{e^{1/h}} = 0\right]$$

Also $\quad f(0) = 1.$

Thus $\quad f(0-0) \neq f(0) = f(0+0).$

$\therefore$ At $x = 0$, $f(x)$ is discontinuous from the left and is continuous from the right.

Thus $f(x)$ is discontinuous at $x = 0$ and the discontinuity is of the first kind because both the limits $f(0-0)$ and $f(0+0)$ exist. The jump of the function at $x = 0$ is

$$f(0+0) - f(0-0) \text{ i.e., } 1 - (-1) \text{ i.e., } 2.$$

(ix) Here $f(x) = \dfrac{e^{1/x}}{1 + e^{1/x}}$ when $x \neq 0$, $f(0) = 0$.

Let $\quad c \in \mathbf{R}$ and $c \neq 0$.

We have $\quad \lim_{x \to c} f(x) = \lim_{x \to c} \dfrac{e^{1/x}}{1 + e^{1/x}} = \dfrac{e^{1/c}}{1 + e^{1/c}} = f(c).$

$\therefore$ $f(x)$ is continuous at every point c of $\mathbf{R}$ if $c \neq 0$.

Now to check the continuity of $f(x)$ at $x = 0$.

We have $f(0+0) = 1$ and $f(0-0) = 0$.

[See Problem 15, Comprehensive Problems 1]

Also $\quad f(0) = 0.$

Thus $\quad f(0-0) = f(0) \neq f(0+0).$

$\therefore$ At $x = 0$, $f(x)$ is continuous from the left and is discontinuous from the right.

Thus $f(x)$ has an ordinary discontinuity at $x = 0$. The jump of the function at $x = 0$ is $f(0+0) - f(0-0)$ i.e., $1 - 0$ i.e., 1.

(x) Here $\quad f(x) = \dfrac{xe^{1/x}}{1 + e^{1/x}} + \sin\left(\dfrac{1}{x}\right)$ when $x \neq 0$, $f(0) = 0$.

Let $\quad c \in \mathbf{R}$ and $c \neq 0$.

We have $\quad \lim_{x \to c} f(x) = \lim_{x \to c} \left[\dfrac{xe^{1/x}}{1 + e^{1/x}} + \sin \dfrac{1}{x}\right] = \dfrac{c\, e^{1/c}}{1 + e^{1/c}} + \sin \dfrac{1}{c} = f(c).$

$\therefore$ $f(x)$ is continuous at every point c of $\mathbf{R}$ if $c \neq 0$.

Now to check the continuity of $f(x)$ at $x = 0$.

We have $f(0) = 0$.

Also $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h), h > 0$

$$= \lim_{h \to 0} \left[\frac{he^{1/h}}{1+e^{1/h}} + \sin\frac{1}{h}\right] = \lim_{h \to 0} \left[\frac{h}{e^{-1/h}+1} + \sin\frac{1}{h}\right]$$

$$= \frac{0}{0+1} + \lim_{h \to 0} \sin\frac{1}{h} = 0 + \lim_{h \to 0} \sin\frac{1}{h},$$

which does not exist because $\lim_{h \to 0} \sin(1/h)$ does not exist.

Again $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h), h > 0$

$$= \lim_{h \to 0} \left[\frac{(-h)\,e^{-1/h}}{1+e^{-1/h}} + \sin\left(\frac{1}{-h}\right)\right]$$

$$= \frac{0}{1+0} - \lim_{h \to 0} \sin\frac{1}{h} = 0 - \lim_{h \to 0} \sin\frac{1}{h},$$ which does not exist.

Thus both $f(0-0)$ and $f(0+0)$ do not exist. It follows that $f(x)$ is discontinuous at $x = 0$ and the discontinuity is of the second kind.

(xi) Here $f(x) = \sin x \cos(1/x)$ when $x \neq 0$, $f(0) = 0$.

Let $c \in \mathbf{R}$ and $c \neq 0$.

We have $\lim_{x \to c} f(x) = \lim_{x \to c} \sin x \cos(1/x) = \sin c \cos(1/c) = f(c)$.

$\therefore$ $f(x)$ is continuous at every point c of $\mathbf{R}$ if $c \neq 0$.

Now to check the continuity of $f(x)$ at $x = 0$.

We have $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h), h > 0$

$$= \lim_{h \to 0} \sin h \cos(1/h) = 0.$$ [See theorem 10, article 3]

$\left[\because \lim_{h \to 0} \sin h = 0 \text{ and } |\cos(1/h)| \leq 1 \text{ when } h \neq 0 \text{ i.e.,}\right.$

$\left.\cos(1/h) \text{ is bounded in some deleted neighbourhood of zero}\right]$

Again $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h), h > 0$

$$= \lim_{h \to 0} \sin(-h)\cos(-1/h) = -\lim_{h \to 0} \sin h \cos(1/h) = -0 = 0.$$

Also $f(0) = 0.$

Thus $f(0-0) = f(0) = f(0+0).$

Hence the function $f(x)$ is continuous at $x = 0.$

Problem 2 (i) : *Examine at $x = 0$, the continuity of* $f(x) = \begin{cases} \dfrac{e^{1/x^2}}{1 - e^{1/x^2}}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0. \end{cases}$

(Meerut 2008)

Solution: We have $f(0) = 1.$

Also $f(0+0) = \lim_{h \to 0} f(0+h)$, where h is + ive and sufficiently small

$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{e^{1/h^2}}{1 - e^{1/h^2}} = \lim_{h \to 0} \frac{1}{e^{-1/h^2} - 1},$$

dividing the Nr. and Dr. by e^{1/h^2}

$$= \frac{1}{0-1} = -1. \qquad \left[\because \lim_{h \to 0} e^{-1/h^2} = \lim_{h \to 0} \frac{1}{e^{1/h^2}} = \frac{1}{\infty} = 0\right]$$

Again $f(0-0) = \lim_{h \to 0} f(0-h)$, where h is + ive and sufficiently small

$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{e^{1/h^2}}{1 - e^{1/h^2}} = -1.$$

Thus $f(0-0) = -1,\ f(0) = 1,\ f(0+0) = -1.$

Since $f(0-0) = f(0+0) \neq f(0)$, therefore $f(x)$ is not continuous at $x = 0$. In fact $f(x)$ has a removable discontinuity at $x = 0$.

Problem 2(ii): *If $f(x) = \dfrac{1}{x-a} \sin \dfrac{1}{x-a}$, find $f(a+0)$ and $f(a-0)$. Is the function continuous at $x = a$?*

Solution: We have $f(a+0) = \lim_{h \to 0} f(a+h),\ h > 0$

$$= \lim_{h \to 0} \frac{1}{(a+h)-a} \cdot \sin \frac{1}{(a+h)-a}$$

$$= \lim_{h \to 0} \frac{1}{h} \sin \frac{1}{h}, \text{ which does not exist.}$$

Again $f(a-0) = \lim_{h \to 0} f(a-h),\ h > 0$

$$= \lim_{h \to 0} \frac{1}{(a-h)-a} \sin \frac{1}{(a-h)-a} = \lim_{h \to 0} \left\{-\frac{1}{h} \sin\left(-\frac{1}{h}\right)\right\}$$

$$= \lim_{h \to 0} \frac{1}{h} \sin \frac{1}{h}, \quad \text{which does not exist.}$$

Thus both $f(a+0)$ and $f(a-0)$ do not exist and so $f(x)$ has a discontinuity of the second kind at $x = a$.

Problem 3: *Find out the points of discontinuity of the following functions :*

(*i*) $f(x) = (2 + e^{1/x})^{-1} + \cos e^{1/x}$ *for* $x \neq 0$, $f(0) = 0$.

(*ii*) $f(x) = 1/2^n$ *for* $1/2^{n+1} < x \leq 1/2^n$, $n = 0, 1, 2, \ldots$ *and* $f(0) = 0$.

Solution: **(i)** Let $c \in \mathbf{R}$ and $c \neq 0$. We have

$$\lim_{x \to c} f(x) = \lim_{x \to c} \left[\frac{1}{2 + e^{1/x}} + \cos e^{1/x}\right]$$

$$= \frac{1}{2 + e^{1/c}} + \cos e^{1/c} = f(c).$$

$\therefore$ $f(x)$ is continuous at every point c of $\mathbf{R}$ if $c \neq 0$.

Now to check the continuity of $f(x)$ at $x = 0$.

We have $f(0) = 0$.

Now $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h), h > 0$

$$= \lim_{h \to 0} \left[\frac{1}{2 + e^{1/h}} + \cos e^{1/h}\right] = \frac{1}{2+\infty} + \lim_{h \to 0} \cos e^{1/h}$$

$$= 0 + \lim_{h \to 0} \cos e^{1/h} = \lim_{h \to 0} \cos e^{1/h}.$$

As $h \to 0$, $\cos e^{1/h}$ oscillates between -1 and 1 and so $\lim_{h \to 0} \cos e^{1/h}$ does not exist.

Hence $f(0+0)$ does not exist.

Again $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h), h > 0$

$$= \lim_{h \to 0} \left[\frac{1}{2 + e^{-1/h}} + \cos e^{-1/h}\right]$$

$$= \frac{1}{2 + e^{-\infty}} + \cos e^{-\infty} = \frac{1}{2+0} + \cos 0 = \frac{1}{2} + 1 = \frac{3}{2}.$$

Since $f(0+0)$ does not exist, $f(x)$ is discontinuous at $x = 0$ and discontinuity is of second kind. It is a kind of **mixed discontinuity** since the limit on the right does not exist whereas the limit on the left exists.

(ii) We have $f(x) = 1$ for $\frac{1}{2} < x \le 1,$

$$f(x) = \frac{1}{2} \text{ for } \frac{1}{2^2} < x \le \frac{1}{2}$$

$$f(x) = \frac{1}{2^2} \text{ for } \frac{1}{2^3} < x \le \frac{1}{2^2}$$

....

....

$$f(x) = \frac{1}{2^{n-1}} \text{ for } \frac{1}{2^n} < x \le \frac{1}{2^{n-1}}$$

$$f(x) = \frac{1}{2^n} \text{ for } \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$$

and so on. Also $f(0) = 0$. The domain $f(x) = [0, 1]$.

Obviously $f(x)$ is continuous at $x = 1$ and at all points x where $\frac{1}{2^{n+1}} < x < \frac{1}{2^n}$, $n = 0, 1, 2, ...$ because in each such interval $f(x)$ is constant.

Now it remains to check the continuity of $f(x)$ at $x = \frac{1}{2^n}$, $n = 1, 2, 3, \ldots$. We consider $x = 1/2^n$.

We have $$f\left(\frac{1}{2^n}\right) = \frac{1}{2^n},\ f\left(\frac{1}{2^n} - 0\right) = \frac{1}{2^n} \text{ and } f\left(\frac{1}{2^n} + 0\right) = \frac{1}{2^{n-1}}.$$

Since $f\left(\frac{1}{2^n} - 0\right) \ne f\left(\frac{1}{2^n} + 0\right)$, the function f is discontinuous at $x = 1/2^n$, $n = 1, 2, 3, \ldots$ At each such point $f(x)$ is continuous from the left and is discontinuous from the right and the discontinuity is of the first kind.

A little consideration shows that $\lim\limits_{x \to 0+} f(x) = 0.$

Also $f(0) = 0$. Since $f(0 + 0) = f(0)$, therefore $f(x)$ is continuous at $x = 0$. The question of finding $f(0 - 0)$ does not arise because $f(x)$ is not defined for $x < 0$.

Hence $f(x)$ is discontinuous at $x = \frac{1}{2^n}$, $n = 1, 2, 3, \ldots$

Problem 4: *If $f(x) = \frac{1}{x} \sin \frac{1}{x}$ for $x \ne 0$ and $f(0) = 0$, show that $f(x)$ is finite for every value of x in the interval $[-1, 1]$ but is not bounded. Determine the points of discontinuity of the function if any.*

Solution: Let $c \in [-1, 1]$ and $c \ne 0$.

Then $f(c) = \frac{1}{c} \sin \frac{1}{c}$ which is finite because both $1/c$ and $\sin(1/c)$ are definite real numbers.

Also $f(0) = 0$ which is also finite.

Thus $f(x)$ is finite for every value of x in the interval $[-1, 1]$.

However $f(x)$ is not bounded in $[-1, 1]$. Take any positive real number k, however large. Then there exists a real number x lying in $]0, 1[$ such that

$$\sin\frac{1}{x} = 1 \text{ and } \frac{1}{x} > k \text{ so that } \frac{1}{x}\sin\frac{1}{x} > k.$$

$\therefore$ $f(x)$ is not bounded in $[-1, 1]$.

Obviously $f(x)$ is continuous at every real number c if $c \neq 0$.

Also $f(0-0)$ and $f(0+0)$ both do not exist and so $f(x)$ has a discontinuity of the second kind at $x = 0$.

Problem 5: *A function f defined on* $[0, 1]$ *is given by* $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational.} \end{cases}$

Show that f takes every value between 0 *and* 1 (*both inclusive*), *but it is continuous only at the point* $x = \frac{1}{2}\cdot$

(Rohilkhand 2012B)

Solution: Let $c \in [0, 1]$.

If c is rational, then $f(c) = c$.

If c is irrational, then $1 - c$ is also irrational

and $0 < 1 - c < 1$ *i.e.*, $1 - c \in [0, 1]$.

We have $f(1-c) = 1 - (1-c) = c$.

Thus f takes every value c in $[0, 1]$.

Now to show that f is continuous only at the point $x = \frac{1}{2}\cdot$

Let x_0 be any point of $[0, 1]$. For each positive integer n we select a rational number a_n and an irrational number b_n, both in $[0, 1]$, such that

$$|a_n - x_0| < 1/n, |b_n - x_0| < 1/n.$$

$$\therefore \quad \lim_{n \to \infty} a_n = x_0 = \lim_{n \to \infty} b_n.$$

If f is to be continuous at x_0, then we must have

$$\lim_{n \to \infty} f(a_n) = f(x_0) = \lim_{n \to \infty} f(b_n).$$

Now $f(a_n) = a_n$ for all n and $f(b_n) = 1 - b_n$ for all n.

$$\therefore \quad \lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} a_n = x_0$$

and

$$\lim_{n \to \infty} f(b_n) = \lim_{n \to \infty} (1 - b_n) = 1 - x_0.$$

So for f to be continuous at x_0, we must have

$$x_0 = f(x_0) = 1 - x_0 \quad i.e., \ x_0 = \frac{1}{2}\cdot$$

Thus $x = \frac{1}{2}$ is the only possible point of [0, 1] where f can be continuous.

Now we shall show that f is actually continuous at the point $x = 1/2$.

We have $f(1/2) = \frac{1}{2}\cdot$

Let $\varepsilon > 0$ be given.

Take a positive real number $\delta = \frac{1}{2}\varepsilon$. Then if x is rational, we have

$$\left|x - \frac{1}{2}\right| < \delta \Rightarrow \left| f(x) - f\left(\frac{1}{2}\right)\right| = \left|x - \frac{1}{2}\right| < \delta = \frac{1}{2}\varepsilon < \varepsilon$$

and if x is irrational, we have

$$\left| x - \frac{1}{2} \right| < \delta \Rightarrow \left| f(x) - f\left(\frac{1}{2}\right)\right| = \left|(1 - x) - \frac{1}{2}\right| = \left|x - \frac{1}{2}\right| < \delta = \frac{1}{2}\varepsilon < \varepsilon.$$

Thus, we have

$$\left|x - \frac{1}{2}\right| < \delta \Rightarrow \left| f(x) - f\left(\frac{1}{2}\right)\right| < \varepsilon,$$

so that f is continuous at $x = \frac{1}{2}\cdot$

Hence f is continuous only at the point $x = 1/2$.

Problem 6: *Prove that the function f defined by* $f(x) = \begin{cases} \frac{1}{2}, & \text{if } x \text{ is rational} \\ \frac{1}{3}, & \text{if } x \text{ is irrational} \end{cases}$

is discontinuous everywhere.

Solution: We shall show that $f(x)$ is discontinuous at every point a of **R**.

Take any $\delta > 0$. Then $a - \delta$ and $a + \delta$ are two distinct real numbers and between two distinct real numbers there lie infinite rational and infinite irrational numbers. Thus for every $\delta > 0$, there exist infinite rational and infinite irrational numbers in the open interval $]\,a - \delta, a + \delta\,[$.

$\therefore$ For every $\delta > 0$, there exists a point x in $]\,a - \delta, a + \delta\,[$ at which

$$|f(x) - f(a)| = \left|\frac{1}{2} - \frac{1}{3}\right| = \frac{1}{6}, \text{ taking } x \text{ as rational if } a \text{ is irrational}$$

or at which $$|f(x) - f(a)| = \left|\frac{1}{3} - \frac{1}{2}\right| = \frac{1}{6}, \text{ taking } x \text{ as irrational if } a \text{ is rational.}$$

Thus whatever the point a in **R** may be, for every $\delta > 0$ there exists a point x in $]\,a - \delta, a + \delta\,[$ at which $|f(x) - f(a)| = \frac{1}{6}\cdot$

Thus for $\varepsilon = \frac{1}{6} > 0$, there exists no $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon, \text{ whenever } |x - a| < \delta \quad i.e., \text{ whenever } x \in]a, -\delta, a + \delta[.$$

$\therefore \quad f(x)$ is not continuous at $x = a$.

Hence $f(x)$ is discontinuous at every point a of $\mathbf{R}$.

Problem 7(i): *Show that the function f defined by* $f(x) = \dfrac{x\, e^{1/x}}{1 + e^{1/x}}, x \neq 0, f(0) = 1$ *is not continuous at* $x = 0$ *and also show how the discontinuity can be removed.*

(Rohilkhand 2006; Lucknow 08; Meerut 11)

Solution: We have $f(0 + 0) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h), h > 0$

$$= \lim_{h \to 0} \left(h \cdot \frac{e^{1/h}}{1 + e^{1/h}} \right) = \lim_{h \to 0} \left(h \cdot \frac{1}{e^{-1/h} + 1} \right)$$

$$= 0 \cdot \frac{1}{0 + 1} = 0 \cdot 1 = 0$$

and $\quad f(0 - 0) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h), h > 0$

$$= \lim_{h \to 0} \left\{ (-h) \cdot \frac{e^{-1/h}}{1 + e^{-1/h}} \right\} = 0 \cdot \frac{0}{1 + 0} = 0 \cdot 0 = 0.$$

Since $\quad f(0 - 0) = 0 = f(0 + 0)$, therefore $\lim_{x \to 0} f(x) = 0$.

But $f(0) = 1 \neq \lim_{x \to 0} f(x)$. Hence $f(x)$ is discontinuous at $x = 0$ and the discontinuity is removable and can be removed by defining $f(x)$ as follows :

$$f(x) = \frac{x\, e^{1/x}}{1 + e^{1/x}}, x \neq 0, f(0) = 0.$$

Problem 7(ii): *Show that the function* $f(x) = 3x^2 + 2x - 1$ *is continuous for* $x = 2$.

Solution: We have

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (3x^2 + 2x - 1)$$

$$= 3 \lim_{x \to 2} x^2 + 2 \lim_{x \to 2} x - \lim_{x \to 2} 1$$

$$= 3 \cdot 2^2 + 2 \cdot 2 - 1 \qquad \left[\because \lim_{x \to c} x = c \text{ and } \lim_{x \to c} x^2 = c^2 \right]$$

$$= f(2).$$

Since $\lim_{x \to 2} f(x) = f(2)$, therefore $f(x)$ is continuous at $x = 2$.

Problem 7(iii): *Show that the function* $f(x)=(1+2x)^{1/x}$, $x\neq 0$ *and* $f(x)=e^2$, $x=0$ *is continuous at* $x=0$.

Solution: We know that $\lim_{h\to 0}(1+ax)^{1/x}=e^a$.

$\therefore \quad \lim_{x\to 0} f(x)=\lim_{x\to 0}(1+2x)^{1/x}=e=f(0)$ and so $f(x)$ is continuous at $x=0$

Problem 8: *Examine the continuity of the function* $f(x)=\begin{cases} -x^2 & \text{if } x\le 0\\ 5x-4 & \text{if } 0<x\le 1\\ 4x^2-3x & \text{if } 1<x<2\\ 3x+4 & \text{if } x\ge 2\end{cases}$

at $x=0,1$ *and* 2. **(Meerut 2004, 06B, 07B; Purvanchal 06; 10; Avadh 06; Lucknow 06; Gorakhpur 15)**

Solution: **(i) Continuity at $x=0$.** We have $f(0)=-0^2=0$;

$$f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h),\ h>0 \text{ and is sufficiently small}$$

$$=\lim_{h\to 0}\{-(-h)^2\}=\lim_{h\to 0}(-h^2)=0;$$

and $$f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h),\ h>0 \text{ and is sufficiently small}$$

$$=\lim_{h\to 0}(5h-4) \qquad [\because 0<h<1]$$

$$=-4.$$

Since $f(0-0)\neq f(0+0)$, the function $f(x)$ is discontinuous at $x=0$. It is continuous at $x=0$ from the left but is discontinuous from the right and the discontinuity is ordinary.

(ii) Continuity at $x=1$. We have $f(1)=5\cdot 1-4=1$;

$$f(1-0)=\lim_{h\to 0} f(1-h),\ h>0=\lim_{h\to 0}\{5(1-h)-4\} \qquad [\because 0<1-h<1]$$

$$=\lim_{h\to 0}(1-5h)=1;$$

and $$f(1+0)=\lim_{h\to 0} f(1+h),\ h>0$$

$$=\lim_{h\to 0}[4(1+h)^2-3(1+h)] \qquad [\because 1<1+h<2]$$

$$=\lim_{h\to 0}(4h^2+5h+1)=1.$$

Thus $f(1-0)=f(1)=f(1+0)$ and so $f(x)$ is continuous at $x=1$.

(iii) Continuity at $x = 2$. We have $f(2) = 3 \cdot 2 + 4 = 10$;

$$f(2-0) = \lim_{h \to 0} f(2-h), h > 0$$

$$= \lim_{h \to 0} [4(2-h)^2 - 3(2-h)] \qquad [\because 1 < 2 - h < 2]$$

$$= \lim_{h \to 0} (4h^2 - 13h + 10) = 10;$$

and $$f(2+0) = \lim_{h \to 0} f(2+h), h > 0 = \lim_{h \to 0} \{3(2+h)+4\} \qquad [\because 2 + h > 2]$$

$$= \lim_{h \to 0} (3h + 10) = 10.$$

Since $f(2-0) = f(2) = f(2+0)$, $f(x)$ is continuous at $x = 2$.

Problem 9(i): *Show that the function* $f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$, $x \neq 0$ *and* $f(0) = 0$ *is discontinuous at* $x = 0$.

Solution: (i) Proceeding as in part (viii) of problem 1, we have $f(0+0) = 1$ and $f(0-0) = -1$.

Also $f(0) = 0$. Thus $f(0-0) \neq f(0)$ and $f(0+0) \neq f(0)$.

$\therefore$ $f(x)$ is discontinuous at $x = 0$ both from the left and from the right.

Hence $f(x)$ is discontinuous at $x = 0$ and the discontinuity is ordinary because both the limits $f(0-0)$ and $f(0+0)$ exist. Here $\lim_{x \to 0} f(x)$ does not exist because $f(0-0) \neq f(0+0)$.

Problem 9(ii): *Show that the following function is continuous at* $x = 0$.

$$f(x) = \frac{\sin^{-1} x}{x}, x \neq 0, f(0) = 1.$$

(Agra 2003)

Solution: We have $$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin^{-1} x}{x}$$

$$= \lim_{\theta \to 0} \frac{\theta}{\sin \theta},$$

putting $\sin^{-1} x = \theta$ so that $x = \sin \theta$ and $\theta \to 0$ as $x \to 0$

$$= 1 = f(0).$$

$\therefore$ $f(x)$ is continuous at $x = 0$.

Problem 10: *Discuss the continuity of the function* $f(x) = \dfrac{1}{1 - e^{1/x}}$ *when* $x \neq 0$ *and* $f(0) = 0$ *for all values of* x. **(Meerut 2004; Rohilkhand 10B; Lucknow 10)**

Solution: Let $c \in \mathbf{R}$ and $c \neq 0$.

We have $\lim_{x \to c} f(x) = \lim_{x \to c} \dfrac{1}{1 - e^{1/x}} = \dfrac{1}{1 - e^{1/c}} = f(c)$.

$\therefore$ $f(x)$ is continuous at every point c of $\mathbf{R}$ if $c \neq 0$.

Now to check the continuity of $f(x)$ at $x = 0$.

We have $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h), h > 0$

$$= \lim_{h \to 0} \frac{1}{1 - e^{1/h}} = \frac{1}{1 - \infty} = -\frac{1}{\infty} = -0 = 0.$$

Again $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h), h > 0$

$$= \lim_{h \to 0} \frac{1}{1 - e^{-1/h}} = \frac{1}{1 - 0} = 1.$$

Also $f(0) = 0$. Thus $f(0-0) \neq f(0) = f(0+0)$.

$\therefore$ $f(x)$ is discontinuous at $x = 0$ and the discontinuity is ordinary. The jump of the function at $x = 0$ is $f(0-0) - f(0+0)$ *i.e.*, $1 - 0$ *i.e.*, 1. The function $f(x)$ is continuous at $x = 0$ from the right but is discontinuous from the left.

Problem 11: *Prove that the function* $f(x) = \dfrac{|x|}{x}$ *for* $x \neq 0$, $f(0) = 0$ *is continuous at all points except* $x = 0$. **(Kanpur 2008; Meerut 09; Gorakhpur 11)**

Solution: If $x > 0$, then $f(x) = \dfrac{x}{x} = 1$. $[\because x > 0 \Rightarrow |x| = x]$

If $x < 0$, then $f(x) = \dfrac{-x}{x} = -1$. $[\because x < 0 \Rightarrow |x| = -x]$

Thus
$$f(x) = \begin{cases} -1, \text{ if } x < 0 \\ 0, \text{ at } x = 0 \\ 1, \text{ if } x > 0. \end{cases}$$

If $x < 0$, $f(x) = -1$ *i.e.*, $f(x)$ is a constant function and a constant function is continuous at each point of its domain.

$\therefore$ $f(x)$ is continuous at each point x where $x < 0$. Similarly $f(x)$ is continuous at each point x where $x > 0$.

Now to check the continuity of $f(x)$ at $x = 0$.

We have $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h), \ h > 0$

$$= \lim_{h \to 0} -1 = -1$$

and similarly $f(0+0) = 1$.

Also $f(0) = 0$.

Since $f(0-0) \neq f(0+0)$, therefore $\lim_{x \to 0} f(x)$ does not exist and so $f(x)$ is discontinuous at $x = 0$.

Since both $f(0-0)$ and $f(0+0)$ exist, therefore the discontinuity is ordinary and the jump of the function at $x = 0$ is $f(0+0) - f(0-0)$ *i.e.*, $1 - (-1)$ *i.e.*, 2.

Problem 12: *Test the continuity of the function $f(x)$ at $x = 0$ if*

$$f(x) = \frac{e^{1/x} \sin(1/x)}{1 + e^{1/x}}, \; x \neq 0 \text{ and } f(x) = 0, \; x = 0.$$

(Meerut 2005)

Solution: We have

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h), \; h > 0$$

$$= \lim_{h \to 0} \left\{ \frac{e^{-1/h}}{1 + e^{-1/h}} \cdot \sin\left(-\frac{1}{h}\right) \right\} = 0.$$

$$\left[\because \lim_{h \to 0} \frac{e^{-1/h}}{1 + e^{-1/h}} = \frac{0}{1+0} = 0 \text{ and } |\sin(-1/h)| \leq 1 \text{ when } h \neq 0 \right.$$

$$\left. \text{i.e., } \sin(-1/h) \text{ is bounded in some deleted neighbourhood of zero} \right]$$

Again $$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h), \; h > 0$$

$$= \lim_{h \to 0} \left\{ \frac{e^{1/h}}{1 + e^{1/h}} \cdot \sin(1/h) \right\} = \lim_{h \to 0} \left\{ \frac{1}{e^{-1/h} + 1} \cdot \sin(1/h) \right\}$$

which does not exist because $\lim_{h \to 0} \frac{1}{e^{-1/h} + 1} = \frac{1}{0+1} = 1$ but $\lim_{h \to 0} \sin(1/h)$ does not exist.

Also $f(0) = 0.$

Thus $f(0-0) = f(0)$ and $f(0+0)$ does not exist.

$\therefore \lim_{x \to 0} f(x)$ does not exist and $f(x)$ has discontinuity of the second kind at $x = 0$ from the right.

We observe that $f(x)$ is continuous at $x = 0$ from the left.

Problem 13: *Examine the following function for continuity at $x = 0$ and $x = 1$:*

$$f(x) = \begin{cases} x^2 & \text{for } x \leq 0 \\ 1 & \text{for } 0 < x \leq 1 \\ \dfrac{1}{x} & \text{for } x > 1 \end{cases}$$

(Meerut 2001, 03, 04B, 05)

Solution: **(i)** Continuity at $x = 0$

we have $f(0) = (0)^2 = 0$

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} (-h)^2 = 0$$

and $$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} 1 = 1$$

Since $f(0-0) \neq f(0+0)$, the function $f(x)$ is discontinuous at $x = 0$.

(ii) Continuity at $x = 0$

We have $f(1) = 1 = 1$

$$f(1-0) = \lim_{h \to 0} f(1-h), h > 0 = \lim_{h \to 0} 1 = 1$$

and $$f(1+0) = \lim_{h \to 0} f(1+h), h > 0 = \lim_{h \to 0} \frac{1}{1+h} = 1$$

Thus $f(1-0) = f(1) = f(1+0)$ and so $f(x)$ is continuous at $x = 1$.

Problem 14: *Discuss the continuity of the following function at $x = 0$:*

$$f(x) = \begin{cases} \cos x, & x \geq 0 \\ -\cos x, & x < 0. \end{cases}$$

Solution: We have $f(0) = \cos 0 = 1$

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \cos h = \cos 0 = 1$$

and $$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} -\cos(-h)$$

$$= \lim_{h \to 0} -\cos h = -\cos 0 = -1$$

Since $f(0-0) \neq f(0+0)$, the function $f(x)$ is discontinuous at $x = 0$.

Problem 15: *Test the continuity of the following functions at $x = 0$:*

(*i*) $f(x) = x \cos(1/x)$, *when* $x \neq 0$, $f(0) = 0$. **(Meerut 2007)**

(*ii*) $f(x) = x \log x$, *for* $x > 0$, $f(0) = 0$.

Solution: **(i)** We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h), h > 0$$

$$= \lim_{h \to 0} h \cos(1/h) = 0.$$

[$\because \lim_{h \to 0} h = 0$ and $|\cos(1/h)| \leq 1$ when $h \neq 0$ *i.e.*, $\cos(1/h)$ is bounded in some deleted neighbourhood of zero]

Again $$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h), h > 0$$

$$= \lim_{h \to 0} (-h)\cos(-1/h) = -\lim_{h \to 0} h\cos(1/h) = -0 = 0.$$

Also $f(0) = 0.$

Thus $f(0-0) = f(0) = f(0+0)$ and so $f(x)$ is continuous at $x = 0$.

(ii) We have $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h), h > 0$

$$= \lim_{h \to 0} h \log h \qquad [\text{From } 0 \times \infty]$$

$$= \lim_{h \to 0} \frac{\log h}{1/h} \qquad \left[\text{From } \frac{\infty}{\infty}\right]$$

$$= \lim_{h \to 0} \frac{1/h}{-1/h^2} = -\lim_{h \to 0} h = -0 = 0.$$

Also $f(0) = 0.$

Since $f(x)$ is not defined for $x < 0$, therefore the question of finding $\lim_{x \to 0-} f(x)$ does not arise.

Since $f(0) = f(0+0)$ and $f(x)$ is not defined for $x < 0$, *i.e.*, domain $f(x)$ is $[\,0, \infty\,[$, therefore $f(x)$ is continuous at $x = 0$.

Problem 16: *Discuss the nature of discontinuity at $x = 0$ of the function $f(x) = [\,x\,] - [-\,x]$ where $[\,x\,]$ denotes the integral part of x.*

Solution: We have $f(0) = [0] - [-0] = [0] - [0] = 0 - 0 = 0.$

Also $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h), h > 0$ and is sufficiently small

$$= \lim_{h \to 0} ([h] - [-h]) = \lim_{h \to 0} \{0 - (-1)\} = \lim_{h \to 0} 1 = 1.$$

$$[\because\ 0 < h < 1 \Rightarrow [h] = 0 \text{ and } -1 < -h < 0 \Rightarrow [-h] = -1]$$

Again $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h), h > 0$ and is sufficiently small

$$= \lim_{h \to 0} ([-h] - [-(-h)]) = \lim_{h \to 0} ([-h] - [h])$$

$$= \lim_{h \to 0} (-1 - 0) = -1.$$

Since $f(0-0) \neq f(0+0)$, therefore $\lim_{x \to 0} f(x)$ does not exist and so $f(x)$ is discontinuous at $x = 0$. Also $f(0) \neq f(0-0)$ and $f(0) \neq f(0+0)$ implies that $f(x)$ is discontinuous at $x = 0$ both from the left and from the right. Since $f(0-0)$ and $f(0+0)$ both exist, therefore the discontinuity is of the first kind and the jump of the function at $x = 0$ is $f(0+0) - f(0-0)$ *i.e.*, $1 - (-1)$ *i.e.*, 2.

Problem 17: *Discuss the continuity of $f(x) = (1/x)\cos(1/x)$.*

Solution: Let $c \in \mathbf{R}$ and $c \neq 0$.

We have $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} \frac{1}{x}\cos\frac{1}{x} = \frac{1}{c}\cos\frac{1}{c} = f(c)$.

$\therefore$ $f(x)$ is continuous at every point c of $\mathbf{R}$ if $c \neq 0$.

Now to check the continuity of $f(x)$ at $x = 0$.

We have $f(0+0) = \lim\limits_{h \to 0} f(0+h) = \lim\limits_{h \to 0} f(h), h > 0$

$= \lim\limits_{h \to 0} \frac{1}{h}\cos\frac{1}{h}$ which does not exist.

Again $f(0-0) = \lim\limits_{h \to 0} f(0-h) = \lim\limits_{h \to 0} f(-h), h > 0$

$= \lim\limits_{h \to 0} \left\{-\frac{1}{h}\cos\left(-\frac{1}{h}\right)\right\} = \lim\limits_{h \to 0} \left\{-\frac{1}{h}\cos\left(\frac{1}{h}\right)\right\}$

which does not exist.

Thus both $f(0-0)$ and $f(0+0)$ do not exist and so $f(x)$ has a discontinuity of the second kind at $x = 0$.

Problem 18: *Give an example of each of the following types of functions :*

(*i*) *The function which possesses a limit at $x = 1$ but is not defined at $x = 1$.*

(*ii*) *The function which is neither defined at $x = 1$ nor has a limit at $x = 1$.*

(*iii*) *The function which is defined at two points but is nevertheless discontinuous at both the points.*

Solution: **(i)** $f(x) = x^2$ for $x > 1$, $f(x) = x^3$ for $x < 1$.

(ii) $f(x) = -x^2$ for $x < 1$, $f(x) = x^2$ for $x > 1$.

(iii) $f(x) = 0$ for $x \leq 0$, $f(x) = \frac{3}{2} - x$ for $0 < x \leq \frac{1}{2}$, $f(x) = \frac{3}{2} + x$ for $x > \frac{1}{2}$.

Problem 19: *In the closed interval $[-1, 1]$ let f be defined by*

$$f(x) = x^2 \sin(1/x^2) \text{ for } x \neq 0 \text{ and } f(0) = 0.$$

In the given interval (i) Is the function bounded ? (ii) Is it continuous ?

Solution: **(i)** If $x \in [-1, 1]$ and $x \neq 0$, we have

$$|f(x)| = |x^2 \sin(1/x^2)| = |x^2| \cdot |\sin(1/x^2)|$$
$$= |x|^2 \cdot |\sin(1/x^2)| \leq 1 \cdot 1 = 1.$$

$[\because |\sin(1/x^2)| \leq 1 \text{ and } -1 \leq x \leq 1 \Rightarrow |x| \leq 1]$

Also $f(0) = 0 \Rightarrow |f(0)| = 0 < 1$.

Thus $|f(x)| \leq 1, \forall x \in [-1, 1]$ and so f is bounded in $[-1, 1]$.

(ii) Let $c \in [-1, 1]$ and $c \neq 0$.

We have $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} x^2 \sin \frac{1}{x^2} = c^2 \sin \frac{1}{c^2} = f(c)$.

$\therefore$ $f(x)$ is continuous at every point c of $[-1, 1]$ if $c \neq 0$.

Now to check the continuity of $f(x)$ at $x = 0$.

We have $f(0-0) = \lim\limits_{h \to 0} f(0-h) = \lim\limits_{h \to 0} f(-h), h > 0$

$$= \lim_{h \to 0} (-h)^2 \sin\left\{\frac{1}{(-h)^2}\right\} = \lim_{h \to 0} h^2 \sin \frac{1}{h^2} = 0.$$

$$\left[\because \lim_{h \to 0} h^2 = 0 \text{ and } \left|\sin \frac{1}{h^2}\right| \leq 1 \text{ if } h \neq 0\right]$$

Again $f(0+0) = \lim\limits_{h \to 0} f(0+h) = \lim\limits_{h \to 0} f(h)$

$$= \lim_{h \to 0} h^2 \sin \frac{1}{h^2} = 0.$$

Also $f(0) = 0$.

Since $f(0-0) = f(0) = f(0+0)$, therefore $f(x)$ is continuous at $x = 0$.

Thus $f(x)$ is continuous at each point of $[-1, 1]$ and so it is continuous in $[-1, 1]$.

Hints to Objective Type Questions

Multiple Choice Questions

1. We have $\lim\limits_{x \to 0+} \frac{|x|}{x} = \lim\limits_{h \to 0} \frac{|h|}{h}, h > 0 = \lim\limits_{h \to 0} \frac{h}{h} = \lim\limits_{h \to 0} 1 = 1$

   Again, $\lim\limits_{x \to 0-} \frac{|x|}{x} = \lim\limits_{h \to 0} \frac{|-h|}{-h}, h > 0 = \lim\limits_{h \to 0} \frac{|h|}{-h} = \lim\limits_{h \to 0} \frac{h}{-h} = \lim\limits_{h \to 0} -1 = -1.$

   Since, $\lim\limits_{x \to 0+} \frac{|x|}{x} \neq \lim\limits_{x \to 0-} \frac{|x|}{x}$, therefore $\lim\limits_{x \to 0} \frac{|x|}{x}$ does not exist.

2. We have $\lim\limits_{x \to 0} \frac{e^x - 1}{x} = \lim\limits_{x \to 0} \frac{\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\right) - 1}{x}$

$$= \lim_{x \to 0} \frac{x\left(\frac{1}{1!} + \frac{x}{2!} + \ldots\right)}{x} = \lim_{x \to 0} \left(1 + \frac{x}{2!} + \ldots\right) = 1.$$

3. We have $\lim_{x \to 2+} \frac{|x-2|}{x-2} = \lim_{h \to 0} \frac{|2+h-2|}{(2+h)-2}, h > 0$

$$= \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1.$$

4. We have $\lim_{x \to 3-} \frac{|x-3|}{x-3} = \lim_{h \to 0} \frac{|(3-h)-3|}{(3-h)-3}, h > 0$

$$= \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{|h|}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} -1 = -1$$

5. See Example 6(ii).

6. We have $f(0) = K$;

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \frac{\sin 5h}{3h} = \frac{5h}{3h} \lim_{h \to 0} \left(\frac{\sin 5h}{5h}\right) = \frac{5}{3}$$

and $$f(0-0) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{\sin 5(-h)}{3(-h)} = \frac{5}{3}\cdot$$

If the given function is continuous then

$$f(0) = f(0+0) = f(0-0).$$

$\therefore \quad K = \frac{5}{3}\cdot$

7. See Example 5(e).

8. See Example 5(i).

9. See Example 5(b).

10. See Example 5(c).

11. Proceed as Example 5(f).

Fill in the Blanks

1. See article 7. An alternative definition of continuity.

2. We have $\lim_{x \to 1} \frac{x^2 - 1}{x-1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x \to 1} (x+1) = 2$

3. We have $\lim_{x \to 0} \frac{\sin(x/4)}{x} = \lim_{x \to 0} \frac{1}{4} \cdot \frac{\sin(x/4\}}{x/4} = \frac{1}{4} \lim_{x \to 0} \frac{\sin(x/4)}{x/4} = \frac{1}{4} \cdot 1 = \frac{1}{4}$

4. We have $\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} = \lim_{x \to 1} \frac{x^2+x+1}{x+1} = \frac{3}{2}$.

5. See Example 5, part (b).

6. See Example 5, part (f).

7. If $3 < x < 4$, then $[x] = 3$. So, if $3 < x < 4$, then $f(x) = x - 3$.

8. We have $\lim_{x \to 1-} f(x) = \lim_{x \to 1-} x = 1$ $[\because f(x) = x, \text{ when } x < 1]$

Also, See Problem 13, of Comprehensive Problems 1.

9. (i) We have $f\left(\frac{3}{2}\right) = 2 - \frac{3}{2} = \frac{1}{2}$ $\left[\because 1 < \frac{3}{2} < 2\right]$

(ii) We have $\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (2 - x)$ $[\because f(x) = 2 - x, \text{ when } 1 \le x < 2]$

$$= 2 - 1 = 1$$

(iii) We have $\lim_{x \to 2-} f(x) = \lim_{x \to 2-} (2 - x)$ $[\because f(x) = 2 - x, \text{ when } 1 \le x < 2]$

$$= 2 - 2 = 0$$

10. See Problem 14 of Comprehensive Problems 1.

11. See article 8, part (i).

12. The function $f(x) = \frac{\sin x}{x}$ is defined for all real numbers x except $x = 0$. So, domain $f = \mathbf{R} - \{0\}$.

13. The given function $f(x)$ is defined for all real numbers x. So, domain $f = \mathbf{R}$.

True or False

1. We have $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} x$ $[\because f(x) = x \text{ when } x < 0]$

Again, $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} x^2$ $[\because f(x) = x^2 \text{ when } x > 0]$

Since, $\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = 0$,

therefore $\lim_{x \to 0} f(x) = 0$.

2. We have $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1$

But, $f(0) = 2$

Since, $\lim_{x \to 0} f(x) \ne f(0)$, therefore $f(x)$ is discontinuous at $x = 0$.

3. We have $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (-\sin x)$ $[\because f(x) = -\sin x, \text{ when } x < 0]$

$$= 0$$

Again, $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \sin x$ $[\because f(x) = \sin x, \text{ when } x > 0]$

Since $\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = 0.$

therefore $\lim_{x \to 0} f(x) = 0.$

Also, $f(0) = \sin 0 = 0.$ [$\because f(x) = \sin x$, when $x = 0$]

Since $\lim_{x \to 0} f(x) = f(0)$, therefore $f(x)$ is continuous at $x = 0$.

4. The function $f(x) = \dfrac{x^2 - a^2}{x - a}$ is not defined at $x = a$, but

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} \frac{(x-a)(x+a)}{x - a}$$

$$= \lim_{x \to a} (x + a) = 2a, \text{ i.e., } \lim_{x \to a} f(x) \text{ exists.}$$

5. See Problem 15, part (i) of Comprehensive Problems 2.

6. See Theorem 5 after article 10.

7. We have $\lim_{x \to 0} f(x) = \lim_{x \to 0} \dfrac{\sin x}{x} = 1.$

Also, $f(0) = 1.$

Since, $\lim_{x \to 0} f(x) = f(0)$, therefore $f(x)$ is continuous at $x = 0$.

8. We have $\lim_{x \to 1-} f(x) = \lim_{x \to 1-} 1$ [$\because f(x) = 1$, when $x < 1$]

Again, $\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (2 - x)$ [$\because f(x) = 2 - x$, when $1 \le x < 2$]

$= 2 - 1 = 1.$

Also, $f(1) = 2 - 1 = 1.$

Since, $\lim_{x \to 1-} f(x) = f(1) \lim_{x \to 1+} f(x)$, therefore $f(x)$ is continuous at $x = 1$.

9. See Example 5, part (d).

10. We have $\lim_{x \to 0} \dfrac{\sin x}{x} = \lim_{x \to 0} 3 \cdot \dfrac{\sin 3x}{3x} = 3 \lim_{x \to 0} \dfrac{\sin 3x}{3x} = 3.1 = 3.$

11. We have $\lim_{x \to 0} \dfrac{\sin 2x}{x} = \lim_{x \to 0} 2 \cdot \dfrac{\sin 2x}{2x} = 2 \lim_{x \to 0} \dfrac{\sin 2x}{2x} = 2 \cdot 1 = 2.$

12. We have $\lim_{x \to 0} \dfrac{\sin x}{2x} = \dfrac{1}{2} \lim_{x \to 0} \dfrac{\sin x}{x} = \dfrac{1}{2} \cdot 1 = \dfrac{1}{2}.$

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Chapter-2

Differentiability

Comprehensive Problems 1

Problem 1: *Show that $f(x) = |x-1|, 0 \le x \le 2$ is not derivable at $x = 1$. Is it continuous in* $[0, 2]$?

Solution: We have $f(x) = \begin{cases} 1-x, & \text{when } 0 \le x \le 1 \\ x-1, & \text{when } 1 \le x \le 2. \end{cases}$

To test $f(x)$ for differentiability at $x = 1$.

We have

$$R\ f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h-1) - 0}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1;$$

and

$$L\ f'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{1-(1-h) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} -1 = -1.$$

Since $R\ f'(1) \ne L\ f'(1)$, the function $f(x)$ is not differentiable at $x = 1$.

To test $f(x)$ for continuity in [0, 2].

When $0 \le x < 1$, $f(x) = 1 - x$ which is a polynomial and when $1 < x \le 2$, $f(x) = x - 1$ which is also a polynomial.

Now a polynomial function is continuous at each point of its domain. Therefore $f(x)$ is continuous when $x \le 0 < 1$ and also when $1 < x \le 2$.

Now to check $f(x)$ for continuity at $x = 1$.

We have $f(1-0) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} \{1-(1-h)\} = \lim_{h \to 0} h = 0,$

$$f(1+0) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} \{(1+h) - 1\} = \lim_{h \to 0} h = 0$$

and $f(1) = 0.$

Since $f(1-0) = f(1) = f(1+0)$, $f(x)$ is continuous at $x = 1$.

Thus $f(x)$ is continuous at each point of $[0, 2]$ and so $f(x)$ is continuous in $[0, 2]$.

Problem 2 (i): *If* $f(x) = \dfrac{x}{1 + e^{1/x}}$, $x \neq 0$, $f(0) = 0$, *show that* f *is continuous at* $x = 0$, *but* $f'(0)$ *does not exist.* **(Lucknow 2005, 10; Gorakhpur 13; Purvanchal 14)**

Solution: We have

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h), h > 0$$

$$= \lim_{h \to 0} \frac{-h}{1 + e^{-1/h}} = \frac{0}{1+0} = 0,$$

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h), h > 0$$

$$= \lim_{h \to 0} \frac{h}{1 + e^{1/h}} = 0 \cdot \frac{1}{1 + \infty} = 0 \cdot 0 = 0$$

and $f(0) = 0$.

Since $f(0-0) = f(0) = f(0+0)$, therefore $f(x)$ is continuous at $x = 0$.

We now proceed to find the derivative of $f(x)$ at $x = 0$. We have

$$R\ f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}, h > 0$$

$$= \lim_{h \to 0} \frac{\dfrac{h}{1 + e^{1/h}} - 0}{h} = \lim_{h \to 0} \frac{1}{1 + e^{1/h}} = \frac{1}{1 + \infty} = 0$$

and

$$L\ f'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}, h > 0$$

$$= \lim_{h \to 0} \frac{\dfrac{-h}{1 + e^{-1/h}} - 0}{-h}$$

$$= \lim_{h \to 0} \frac{1}{1 + e^{-1/h}} = \frac{1}{1 + e^{-\infty}} = \frac{1}{1+0} = 1.$$

Since $R\ f'(0) \neq L\ f'(0)$, the derivative of $f(x)$ at $x = 0$ does not exist.

Problem 2 (ii): *If* $f(x) = \dfrac{x\, e^{1/x}}{1 + e^{1/x}}$ *for* $x \neq 0$ *and* $f(0) = 0$, *show that* $f(x)$ *is continuous at* $x = 0$, *but* $f'(0)$ *does not exist.* **(Lucknow 2006)**

Solution: We have $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h), h > 0$

$$= \lim_{h \to 0} \frac{-he^{-1/h}}{1 + e^{-1/h}} = \frac{0}{1+0} = 0,$$

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h), h > 0$$

$$= \lim_{h \to 0} \frac{he^{1/h}}{1 + e^{1/h}} = \lim_{h \to 0} \frac{h}{e^{-1/h} + 1} = \frac{0}{0 + 1} = 0$$

and $f(0) = 0$.

Since $f(0 - 0) = f(0) = f(0 + 0)$, $f(x)$ is continuous at $x = 0$.

We now proceed to find the derivative of $f(x)$ at $x = 0$.

We have
$$R\ f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{he^{1/h}}{1 + e^{1/h}} - 0}{h} = \lim_{h \to 0} \frac{e^{1/h}}{1 + e^{1/h}}$$

$$= \lim_{h \to 0} \frac{1}{e^{-1/h} + 1} = \frac{1}{0 + 1} = 1$$

and
$$L\ f'(0) = \lim_{h \to 0} \frac{f(0 - h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{\frac{-h\,e^{-1/h}}{1 + e^{-1/h}} - 0}{-h} = \lim_{h \to 0} \frac{e^{-1/h}}{1 + e^{-1/h}} = \frac{0}{1 + 0} = 0.$$

Since $R\ f'(0) \neq L\ f'(0)$, the derivative of $f(x)$ at $x = 0$ does not exist.

Problem 3: *A function ϕ is defined as follows : $\phi(x) = -x$ for $x \le 0$, $\phi(x) = x$ for $x \ge 0$. Test the character of the function at $x = 0$ as regards continuity and differentiability.*

Solution: We have
$$\phi(0 - 0) = \lim_{h \to 0} \phi(0 - h) = \lim_{h \to 0} \phi(-h),\ h > 0$$

$$= \lim_{h \to 0} -(-h) = \lim_{h \to 0} h = 0$$

$$\phi(0 + 0) = \lim_{h \to 0} \phi(0 + h) = \lim_{h \to 0} \phi(h) = \lim_{h \to 0} h = 0$$

and $\phi(0) = 0$.

Since $\phi(0 - 0) = \phi(0) = \phi(0 + 0)$, $\phi(x)$ is continuous at $x = 0$.

Again
$$R\ \phi'(0) = \lim_{h \to 0} \frac{\phi(0 + h) - \phi(0)}{h} = \lim_{h \to 0} \frac{\phi(h) - \phi(0)}{h}$$

$$= \lim_{h \to 0} \frac{h - 0}{h} = \lim_{h \to 0} 1 = 1$$

and
$$L\ \phi'(0) = \lim_{h \to 0} \frac{\phi(0 - h) - \phi(0)}{-h} = \lim_{h \to 0} \frac{\phi(-h) - \phi(0)}{-h},\ h > 0$$

$$= \lim_{h \to 0} \frac{-(-h) - 0}{-h} = \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} -1 = -1.$$

Since $R\phi'(0) \neq L\phi'(0)$, $\phi(x)$ is not differentiable at $x = 0$.

Problem 4: *Show that the function f defined on* **R** *by* $f(x) = |x-1| + 2|x-2| + 3|x-3|$ *is continuous but not differentiable at the points* 1, 2, *and* 3. **(Bundelkhand 2009)**

Solution: From the definition of the function f, we have

$$f(x) = 1 - x + 2(2-x) + 3(3-x) = 14 - 6x \quad \text{when } x \le 1$$
$$f(x) = x - 1 + 2(2-x) + 3(3-x) = 12 - 4x \quad \text{when } 1 \le x \le 2$$
$$f(x) = x - 1 + 2(x-2) + 3(3-x) = 4 \quad \text{when } 2 \le x \le 3$$

and
$$f(x) = x - 1 + 2(x-2) + 3(x-3) = 6x - 14 \quad \text{when } x \ge 3.$$

Continuity of $f(x)$ at $x = 1$

We have $f(1) = 14 - 6 = 8$,

$$f(1-0) = \lim_{x \to 1-} f(x) = \lim_{x \to 1} (14 - 6x) = 8$$

and
$$f(1+0) = \lim_{x \to 1+} f(x) = \lim_{x \to 1} (12 - 4x) = 8.$$

Since $f(1-0) = f(1) = f(1+0)$, $f(x)$ is continuous at $x = 1$.

Differentiability of $f(x)$ at $x = 1$.

We have
$$Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h \to 0} \frac{12 - 4(1+h) - 8}{h} = \lim_{h \to 0} \frac{-4h}{h} = \lim_{h \to 0} -4 = -4$$

and
$$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}, h > 0$$
$$= \lim_{h \to 0} \frac{14 - 6(1-h) - 8}{-h} = \lim_{h \to 0} \frac{6h}{-h} = \lim_{h \to 0} -6 = -6.$$

Since $R\,f'(1) \ne L\,f'(1)$, the function f is not differentiable at $x = 1$.
Similarly check $f(x)$ for continuity and differentiability at $x = 2$ and at $x = 3$.

Problem 5: *Show that the function*

$$f(x) = x, 0 < x \le 1 = x - 1, 1 < x \le 2$$

has no derivative at $x = 1$.

Solution: We have

$$R\,f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h \to 0} \frac{(1+h) - 1 - 1}{h} = \lim_{h \to 0} \frac{h-1}{h} = -\infty$$

and
$$L\,f'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}, h > 0$$
$$= \lim_{h \to 0} \frac{(1-h) - 1}{-h} = \lim_{h \to 0} 1 = 1.$$

Since $R\ f'(1) \neq L\ f'(1)$, the function $f(x)$ has no derivative at $x = 1$.

Remark: Here $f(1-0) = 1$, $f(1) = 1$, $f(1+0) = 0$ and so $f(x)$ is not continuous at $x = 1$ and consequently $f(x)$ is not differentiable at $x = 1$.

Problem 6: *Show that the function* $f(x) = x^2 - 1, x \geq 1 = 1 - x, x < 1$ *has no derivative at* $x = 1$.

Solution: We have $L\ f'(1) = \lim_{h \to 0} \dfrac{f(1-h) - f(1)}{-h}, h > 0$

$$= \lim_{h \to 0} \frac{1 - (1-h) - (1-1)}{-h} = \lim_{h \to 0} \frac{h}{-h}$$

$$= \lim_{h \to 0} -1 = -1$$

and $$R\ f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2 - 1 - (1-1)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 2h}{h} = \lim_{h \to 0} \frac{h(h+2)}{h} = \lim_{h \to 0} (h+2) = 2.$$

Since $L\ f'(1) \neq R\ f'(1)$, $f(x)$ is not differentiable at $x = 1$.

Problem 7: *The following limits are derivatives of certain functions at a certain point. Determine these functions and the points.*

(*i*) $\displaystyle \lim_{x \to 2} \frac{\log x - \log 2}{x - 2}$ (*ii*) $\displaystyle \lim_{h \to 0} \frac{\sqrt{(a+h)} - \sqrt{a}}{h}$.

Solution: We know that

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

(i) The function is $f(x) = \log x$ and the point is $x = 2$.

(ii) The function is $f(x) = \sqrt{x}$ and the point is $x = a$.

Problem 8: *Let* $f(x) = x^2 \sin(x^{-4/3})$ *except when* $x = 0$ *and* $f(0) = 0$. *Prove that* $f(x)$ *has zero as a derivative at* $x = 0$.

Solution: We have $R\ f'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}, h > 0$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin(h^{-4/3}) - 0}{h}$$

$$= \lim_{h \to 0} h \sin(h^{-4/3}) = 0.$$

$$\left[\because \lim_{h \to 0} h = 0 \text{ and } |\sin(h^{-4/3})| \leq 1 \text{ when } h \neq 0\right]$$

Again $L\ f'(0) = \lim_{h \to 0} \dfrac{f(0-h) - f(0)}{-h}$

$$= \lim_{h \to 0} \frac{f(-h) - f(0)}{-h} = \lim_{h \to 0} \frac{(-h)^2 \sin\{(-h)^{-4/3}\} - 0}{-h}$$

$$= \lim_{h \to 0} [(-h) . \sin\{(-h)^{-4/3}\}] = 0, \text{ as before.}$$

Since $R\ f'(0) = L\ f'(0) = 0$, the function f has zero as a derivative at $x = 0$.

Problem 9: *A function ϕ is defined as follows :*

$$\phi(x) = 1 + x \text{ if } x \le 2,\ \phi(x) = 5 - x \text{ if } x > 2.$$

Test the character of the function at $x = 2$ as regards its continuity and differentiability.

(Avadh 2007)

Solution: We have $\phi(2) = 1 + 2 = 3$,

$$\phi(2-0) = \lim_{x \to 2-} \phi(x) = \lim_{x \to 2} (1 + x) = 3$$

and $$\phi(2+0) = \lim_{x \to 2+} \phi(x) = \lim_{x \to 2} (5 - x) = 3.$$

Since $\phi(2-0) = \phi(2) = \phi(2+0)$, the function ϕ is continuous at $x = 2$.

Again $R\ \phi'(2) = \lim_{h \to 0} \dfrac{\phi(2+h) - \phi(2)}{h} = \lim_{h \to 0} \dfrac{5 - (2+h) - 3}{h}$

$$= \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} -1 = -1$$

and $L\ \phi'(2) = \lim_{h \to 0} \dfrac{\phi(2-h) - \phi(2)}{-h} = \lim_{h \to 0} \dfrac{1 + (2-h) - 3}{-h}$

$$= \lim_{h \to 0} \frac{-h}{-h} = \lim_{h \to 0} 1 = 1.$$

Since $R\ \phi'(2) \ne L\ \phi'(2)$, the function ϕ is not differentiable at $x = 2$.

Problem 10: *Examine the following curve for continuity and differentiability at $x = 0$ and $x = 1$:*

$$y = x^2 \quad \textit{for} \quad x \le 0$$
$$y = 1 \quad \textit{for} \quad 0 < x \le 1$$
$$y = 1/x \quad \textit{for} \quad x > 1.$$

Also draw the graph of the function. **(Meerut 2003, 04B, 09B)**

Solution: Let $y = f(x)$. We need to test $f(x)$ for continuity and differentiability at the points $x = 0$ and 1. It is obviously continuous and differentiable at all other points.

At $x = 0$. We have $f(0) = 0^2 = 0$;

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} 1 = 1,$$

and $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} (-h)^2$

$= \lim_{h \to 0} h^2 = 0.$

Thus $f(0-0) = f(0) \neq f(0+0)$ and so the function $f(x)$ is not continuous at $x = 0$. Consequently it is also not differentiable at $x = 0$.

Here $f(x)$ is continuous at $x = 0$ from the left.

At $x = 1$. We have $f(1-0) = \lim_{x \to 1-} f(x) = \lim_{x \to 1} 1 = 1, \; f(1) = 1$

and $f(1+0) = \lim_{x \to 1+} f(x) = \lim_{x \to 1} \frac{1}{x} = 1.$

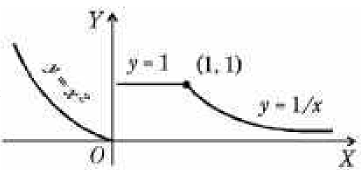
Since $f(1-0) = f(1) = f(1+0)$, $f(x)$ is continuous at $x = 1$.

Now $L\, f'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{1-1}{-h} = \lim_{h \to 0} 0 = 0$

and $R\, f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\frac{1}{1+h} - 1}{h}$

$= \lim_{h \to 0} \frac{1-1-h}{h(1+h)} = \lim_{h \to 0} \frac{-1}{1+h} = -1.$

Since $L\, f'(1) \neq R\, f'(1)$, $f(x)$ is not differentiable at $x = 1$.



The graph of the function consists of the following curves :

$y = x^2$ for $x \leq 0$, (parabola)

$y = 1$ for $0 < x \leq 1$, (straight line)

$y = 1/x$ for $x > 1$, (rectangular hyperbola).

Problem 11: *A function $f(x)$ is defined as follows :*

$f(x) = 1 + x$ *for* $x \leq 0$,

$f(x) = x$ *for* $0 < x < 1$,

$f(x) = 2 - x$ *for* $1 \leq x \leq 2$,

$f(x) = 3x - x^2$ *for* $x > 2$.

Discuss the continuity of $f(x)$ and the existence of $f'(x)$ at $x = 0, 1$ and 2.

Solution: **At $x = 0$.** We have $f(0) = 1 + 0 = 1$,

$f(0-0) = \lim_{x \to 0-} f(x) = \lim_{x \to 0} (1 + x) = 1$

$$f(0+0) = \lim_{x \to 0+} f(x) = \lim_{x \to 0} x = 0.$$

Since $f(0-0) \neq f(0+0)$, therefore $\lim_{x \to 0} f(x)$ does not exist and so $f(x)$ is not continuous at $x = 0$. Consequently $f(x)$ is also not differentiable at $x = 0$.

At $x = 1$. We have $f(1) = 2 - 1 = 1$,

$$f(1-0) = \lim_{x \to 1-} f(x) = \lim_{x \to 1} x = 1$$

and $$f(1+0) = \lim_{x \to 1+} f(x) = \lim_{x \to 1} (2-x) = 1.$$

Since $f(1-0) = f(1) = f(1+0)$, $f(x)$ is continuous at $x = 1$.

Now $$L\ f'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{(1-h)-1}{-h}$$

$$= \lim_{h \to 0} \frac{-h}{-h} = \lim_{h \to 0} 1 = 1$$

and $$R\ f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{2-(1+h)-1}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} -1 = -1.$$

Since $L\ f'(1) \neq R\ f'(1)$, $f(x)$ is not differentiable at $x = 1$ and so $f'(1)$ does not exist.

At $x = 2$. We have $f(2) = 2 - 2 = 0$,

$$f(2-0) = \lim_{x \to 2-} f(x) = \lim_{x \to 2} (2-x) = 0$$

and $$f(2+0) = \lim_{x \to 2+} f(x) = \lim_{x \to 2} (3x - x^2) = 2.$$

Since $f(2-0) \neq f(2+0)$, $f(x)$ is not continuous at $x = 2$ and consequently it is also not differentiable at $x = 2$.

Problem 12: *Discuss the continuity and differentiability of the following function :*

$$f(x) = x^2 \quad \textit{for} \quad x < -2$$
$$f(x) = 4 \quad \textit{for} \quad -2 \leq x \leq 2$$
$$f(x) = x^2 \quad \textit{for} \quad x > 2.$$

Also draw the graph. **(Meerut 2007, 10B)**

Solution: When $x < -2$, $f(x) = x^2$ and when $x > 2$, $f(x) = x^2$. Thus $f(x)$ is a polynomial when $x < -2$ or when $x > 2$ and a polynomial function is continuous as well as differentiable at each point of its domain. So $f(x)$ is continuous as well as differentiable at every point where $x < -2$ or $x > 2$.

Again when $-2 < x < 2$, $f(x) = 4$ *i.e.*, $f(x)$ is a constant function. So $f(x)$ is continuous as well as differentiable at every point where $-2 < x < 2$.

Now it remains to check the continuity and differentiability of $f(x)$ at $x = -2$ and 2.

At $x = -2$. We have $f(-2) = 4$,

$$f(-2-0) = \lim_{x \to -2-} f(x) = \lim_{x \to -2} x^2 = (-2)^2 = 4$$

and $$f(-2+0) = \lim_{x \to -2+} f(x) = \lim_{x \to -2} 4 = 4.$$

Since $f(-2-0) = f(-2) = f(-2+0)$, $f(x)$ is continuous at $x = -2$.

Now $$Lf'(-2) = \lim_{h \to 0} \frac{f(-2-h) - f(-2)}{-h}, h > 0$$

$$= \lim_{h \to 0} \frac{(-2-h)^2 - 4}{-h} = \lim_{h \to 0} \frac{4h + h^2}{-h} = \lim_{h \to 0} \frac{h(4+h)}{-h}$$

$$= \lim_{h \to 0} -(4+h) = -4$$

and $$Rf'(-2) = \lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h}, h > 0$$

$$= \lim_{h \to 0} \frac{4-4}{h} = \lim_{h \to 0} 0 = 0.$$

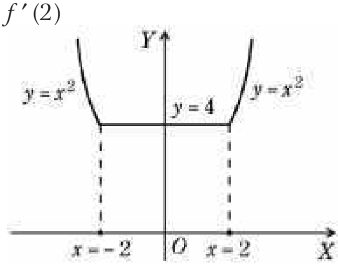
Since $Lf'(-2) \neq Rf'(-2)$, $f(x)$ is not differentiable at $x = -2$.

At $x = 2$. Here proceeding as above,

$$f(2-0) = 4 = f(2) = f(2+0) \text{ and so } f(x) \text{ is continuous at } x = 2.$$

Again $Lf'(2) = 0$, $Rf'(2) = 4$ and thus $Lf'(2) \neq Rf'(2)$ and so $f(x)$ is not differentiable at $x = 2$.

To draw the graph of the function, put $y = f(x)$. Then the graph of the function consists of the following curves :



$y = x^2$ when $x < -2$ (a parabola)

$y = 4$ when $-2 \leq x \leq 2$ (a straight line)

$y = x^2$ when $x > 2$ (a parabola).

Problem 13: *A function $f(x)$ is defined as follows :*

$$f(x) = x \quad \textit{for} \quad 0 \leq x \leq 1, \quad f(x) = 2 - x \quad \textit{for} \quad x \geq 1.$$

Test the character of the function at $x = 1$ as regards the continuity and differentiability.

(Meerut 2003)

Solution: Proceed yourself.

Here $f(1-0)=1=f(1)=f(1+0)$ and so $f(x)$ is continuous at $x=1$.

Again $L\ f'(1)=1$ and $R\ f'(1)=-1$ and thus $L\ f'(1)\neq R\ f'(1)$ and so $f(x)$ is not differentiable at $x=1$.

Problem 14: *Examine the function defined by* $f(x)=x^2\cos(e^{1/x}),\ x\neq 0,\ f(0)=0$ *with regard to (i) continuity (ii) differentiability in the interval* $]-1,1[$.

Solution: If $c\in\ ]-1,1[$ and $c\neq 0$, then

$$\lim_{x\to c} f(x)=\lim_{x\to c} x^2\cos(e^{1/x})=c^2\cos(e^{1/c})=f(c).$$

$\therefore$ $f(x)$ is continuous at every point $x=c$ if $c\neq 0$.

Again
$$f'(x)=2x\cos(e^{1/x})-x^2\{\sin(e^{1/x})\}.e^{1/x}.(-1/x^2)$$
$$=2x\cos(e^{1/x})+e^{1/x}\sin(e^{1/x})$$

which exists at every point $x=c$ if $c\neq 0$.

$\therefore$ $f(x)$ is differentiable at every point $x=c$ if $c\neq 0$.

Continuity of $f(x)$ at $x=0$. We have $f(0)=0$,

$$f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h),\ h>0$$
$$=\lim_{h\to 0} h^2\cos(e^{1/h})=0 \qquad \left[\because \lim_{h\to 0} h^2=0 \text{ and } |\cos(e^{1/h})|\leq 1 \text{ when } h\neq 0 \text{ i.e., } \cos(e^{1/h}) \text{ is bounded in some deleted neighbourhood of zero}\right]$$

and
$$f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h),\ h>0$$
$$=\lim_{h\to 0}(-h)^2\cos(e^{-1/h})=\lim_{h\to 0} h^2\cos(e^{-1/h})=0,$$

as discussed above.

Since $f(0-0)=f(0)=f(0+0)$, $f(x)$ is continuous at $x=0$.

Differentiability of $f(x)$ at $x=0$. We have

$$R\ f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h},\ h>0$$
$$=\lim_{h\to 0}\frac{h^2\cos(e^{1/h})-0}{h}=\lim_{h\to 0} h\cos(e^{1/h})=0$$

and
$$L\ f'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h},\ h>0$$
$$=\lim_{h\to 0}\frac{(-h)^2\cos(e^{-1/h})-0}{-h}=\lim_{h\to 0} -h\cos(e^{-1/h})=0.$$

Since $L\ f'(0) = R\ f'(0)$, $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$.
Thus $f(x)$ is continuous as well as differentiable throughout **R**.

Problem 15: *(i) Define continuity and differentiability of a function at a given point. If a function possesses a finite differential coefficient at a point, show that it is continuous at this point. Is the converse true ? Give example in support of your answer.*

Solution: A function $f(x)$ is said to be continuous at a point a of its domain if

$$\lim_{x \to a} f(x) = f(a).$$

For definition of differentiability of a function at a point see article 1.

If a function $f(x)$ possesses a finite differential coefficient $f'(x_0)$ at a point $x = x_0$, it is continuous at $x = x_0$. For proof refer theorem 4 of article 5. The converse is not true.

Consider the function $f(x) = |x|, x \in \mathbf{R}$.

Then $$f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0. \end{cases}$$

This function is continuous at $x = 0$ but is not differentiable at $x = 0$. For complete solution see solved example 1 after article 7.

Another example: Consider the function $f(x) = x \sin(1/x), x \neq 0$ and $f(0) = 0$.

This function is continuous at $x = 0$ but is not differentiable at $x = 0$. For complete solution see article 4.

Problem 15: *(ii) What do you understand by the derivative of a real valued function at the point $b \in \mathbf{R}$? Apply your definition to discuss the derivative of $f(x) = |x|, x \in \mathbf{R}$ at $x = 0$.*

Solution: Let I denote the open interval $]\,p, q\,[$ in **R** and let $b \in I$. Then a function $f : I \to \mathbf{R}$ is said to be differentiable or derivable at $x = b$ if

$$\lim_{h \to 0} \frac{f(b+h) - f(b)}{h} \quad \text{or equivalently} \quad \lim_{x \to b} \frac{f(x) - f(b)}{x - b}$$

exists finitely and this limit, if it exists finitely, is called the differential coefficient or derivative of f with respect to x at $x = b$ and is denoted by $f'(b)$.

To discuss the derivative of $f(x) = |x|, x \in \mathbf{R}$ at $x = 0$ see solved example 1 after article 7.

Problem 15 : *(iii) Prove that if a function $f(x)$ possesses a finite derivative in a closed interval $[a, b]$, then $f(x)$ is continuous in $[a, b]$.*

Solution: It is given that $f(x)$ possesses a finite derivative at each point of $[a, b]$ and to prove that $f(x)$ is continuous in $[a, b]$ *i.e.,* $f(x)$ is continuous at each point c of $[a, b]$.

By hypothesis, $f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists and is finite.

We can write $f(x) - f(c) = \dfrac{f(x) - f(c)}{x - c} \cdot (x - c)$, if $x \neq c$....(1)

Taking limit of both sides of (1) as $x \to c$, we get

$$\lim_{x \to c} \{ f(x) - f(c) \} = \lim_{x \to c} \left\{ \frac{f(x) - f(c)}{x - c} \cdot (x - c) \right\}$$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} (x - c) = f'(c) . 0 = 0.$$

$\therefore$ $\lim_{x \to c} \{ f(x) - f(c) \} = 0 \Rightarrow \lim_{x \to c} f(x) - \lim_{x \to c} f(c) = 0$

$\Rightarrow$ $\lim_{x \to c} f(x) - f(c) = 0 \Rightarrow \lim_{x \to c} f(x) = f(c)$

$\Rightarrow$ $f(x)$ is continuous at $x = c$.

Comprehensive Problems 2

Problem 1: *(i) State Rolle's theorem.*

(Kanpur 2005, 08; Lucknow 07)

(ii) Verify Rolle's theorem when $f(x) = e^{x} \sin x, a = 0, b = \pi.$ **(Gorakhpur 2012)**

Solution: (i) See article 8.

(ii) The function $f(x) = e^{x} \sin x$ is continuous as well as differentiable on the whole **R**. So $f(x)$ is continuous in $[0, \pi]$ and differentiable in $]0, \pi[$.

We have $f(0) = e^{0} \sin 0 = 0$ and $f(\pi) = e^{\pi} \sin \pi = 0.$

$\therefore$ $f(0) = f(\pi)$.

Thus $f(x)$ satisfies all the three conditions of Rolle's theorem in $[0, \pi]$. Therefore there must exist at least one number, say c, in the open interval $]0, \pi[$ for which $f'(c) = 0$.

Now $f'(x) = e^{x} \sin x + e^{x} \cos x = e^{x} (\sin x + \cos x)$

$$= e^{x} . \sqrt{2} . \left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x \right)$$

$$= \sqrt{2} . e^{x} \sin \left(x + \frac{1}{4} \pi \right) \cdot$$

From $f'(x) = 0$ we get $\sin \left(x + \frac{1}{4} \pi \right) = 0$ $[\because e^{x} \neq 0, \ \forall x \in \mathbf{R}]$

$\Rightarrow$ $x + \frac{1}{4} \pi = 0, \pm \pi, \pm 2\pi, \ldots.$

Out of these values $x = \pi - \frac{1}{4} \pi = \frac{3}{4} \pi$ lies in the open interval $]0, \pi[$. Thus the Rolle's theorem is verified.

Problem 2: *Verify Rolle's theorem for the following functions :*

(*i*) $f(x) = (x-4)^5 (x-3)^4$ *in the interval* $[3, 4]$.

(*ii*) $f(x) = x^3 - 6x^2 + 11x - 6$.

(*iii*) $f(x) = x^3 - 4x$ *in* $[-2, 2]$. **(Kanpur 2007)**

(*iv*) $f(x) = e^x (\sin x - \cos x)$ *in* $[\pi/4, 5\pi/4]$. **(Meerut 2013B)**

(*v*) $f(x) = 10x - x^2$ *in* $[0, 10]$. **(Kanpur 2006)**

Solution: **(i)** We have $f(x) = (x-4)^5 (x-3)^4$ which is a polynomial in x of degree 9 and a polynomial function is continuous as well as differentiable on the whole **R**. So $f(x)$ is continuous in $[3, 4]$ and differentiable in $]3, 4[$.

Also $f(3) = 0$ and $f(4) = 0$ so that $f(3) = f(4)$.

Thus all the three conditions of Rolle's theorem are satisfied so that there is at least one value of x in the open interval $]3, 4[$ where $f'(x) = 0$.

Now
$$f'(x) = 5(x-4)^4 (x-3)^4 + 4(x-3)^3 (x-4)^5$$
$$= (x-3)^3 (x-4)^4 [5(x-3) + 4(x-4)]$$
$$= (x-3)^3 (x-4)^4 (9x-31).$$

Solving the equation $f'(x) = 0$, we get $x = 3, 4, 31/9$.

Out of these values the value $31/9$ *i.e.*, $3\frac{4}{9}$ is a point which lies in the open interval $]3, 4[$ since $3 < 3\frac{4}{9} < 4$. Hence the Rolle's theorem is verified.

(ii) We have $f(x) = x^3 - 6x^2 + 11x - 6$ which is a polynomial in x of degree 3 and so it is continuous as well as differentiable for all real values of x.

Now $f(x) = 0$ gives $x^3 - 6x^2 + 11x - 6 = 0$

or $$x^2 (x-1) - 5x(x-1) + 6(x-1) = 0$$

or $$(x-1)(x^2 - 5x + 6) = 0$$

or $$(x-1)(x-2)(x-3) = 0 \quad i.e., \quad x = 1, 2, 3.$$

Thus $$f(1) = 0 = f(2) = f(3).$$

If we take the interval $[1, 3]$, then all the three conditions of Rolle's theorem are satisfied in this interval. Consequently there is at least one value of x in the open interval $]1, 3[$ for which $f'(x) = 0$.

Now $$f'(x) = 0 \Rightarrow 3x^2 - 12x + 11 = 0$$

$$\Rightarrow \quad x = \frac{12 \pm \sqrt{(144 - 12 . 11)}}{6} = \frac{6 \pm \sqrt{3}}{3} = 2 \pm \frac{1}{\sqrt{3}} \cdot$$

Since both the points $x = 2 + \frac{1}{\sqrt{3}}$ and $x = 2 - \frac{1}{\sqrt{3}}$

lie in the open interval $]1, 3[$, Rolle's theorem is verified.

If we take the interval [1, 2], then the point $x = 2 - (1/\sqrt{3})$ lies in the open interval]1, 2 [and $f'(x) = 0$ at this point. If we take the interval [2, 3], then the point $x = 2 + (1/\sqrt{3})$ lies in the open interval]2, 3 [and $f'(x) = 0$ at this point.

(iii) Here $f(x) = x^3 - 4x$ which is a polynomial in x of degree 3 and so it is continuous and differentiable for every real value of x. Also $f(-2) = 0 = f(2)$.

Thus $f(x)$ satisfies all the three conditions of Rolle's theorem in $[-2, 2]$.

$\therefore$ there must exist at least one number, say c, in the open interval $]-2, 2[$ for which $f'(c) = 0$.

Now $\quad f'(x) = 0$ gives $3x^2 - 4 = 0$

or $\quad x = \pm \dfrac{2}{\sqrt{3}} = \pm 1{\cdot}55$ (approx).

Both these values lie in the open interval $]-2, 2[$ and thus the Rolle's theorem is verified.

(iv) Here $\quad f(x) = e^x (\sin x - \cos x)$.

We have $\quad f\left(\dfrac{\pi}{4}\right) = e^{\pi/4}\left[\sin\dfrac{\pi}{4} - \cos\dfrac{\pi}{4}\right] = e^{\pi/4}\left[\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}\right] = 0$

and $\quad f\left(\dfrac{5\pi}{4}\right) = e^{5\pi/4}\left[\sin\dfrac{5\pi}{4} - \cos\dfrac{5\pi}{4}\right] = e^{5\pi/4}\left[-\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}\right] = 0.$

$\therefore \quad f(\pi/4) = f(5\pi/4)$.

The function $f(x)$ is continuous as well as differentiable for all real values of x and so $f(x)$ is continuous in $[\pi/4, 5\pi/4]$ and differentiable in $]\pi/4, 5\pi/4[$. Thus $f(x)$ satisfies all the three conditions of Rolle's theorem in $[\pi/4, 5\pi/4]$.

$\therefore$ There must exist at least one real number x in the open interval $]\pi/4, 5\pi/4[$ at which $\quad f'(x) = 0$.

Now $\quad f'(x) = e^x (\cos x + \sin x) + e^x (\sin x - \cos x) = 2e^x \sin x$.

From $f'(x) = 0$ we get $2e^x \sin x = 0$

or $\quad \sin x = 0 \qquad [\because e^x \neq 0 \; \forall \; x \in \mathbf{R}]$

or $\quad x = 0, \pm \pi, \pm 2\pi, \pm 3\pi, \ldots$

Out of these values $x = \pi$ lies in the open interval $]\pi/4, 5\pi/4[$. Thus the Rolle's theorem is verified.

(v) Here $f(x) = 10x - x^2$ which is a polynomial in x of degree 2 and so it is continuous and differentiable for every real value of x

Also $\quad f(0) = 0 = f(10)$

Thus $f(x)$ satisfies all, the three conditions of Rolle's theorem in [0, 10].

$\therefore$ There must exist at least one number say c in the open interval] 0, 10 [for which $f'(c) = 0$.

Now $f'(x) = 0$ gives $10 - 2x = 0$ or $x = \dfrac{10}{2} = 5$ this value lies in the open interval] 0, 10 [and thus the Rolle's theorem is verified.

Problem 3: *Discuss the applicability of Rolle's theorem to the function*

$$f(x) = x^2 + 1, \text{ when } 0 \le x \le 1$$
$$= 3 - x, \text{ when } 1 < x \le 2.$$

Solution: Here the function $f(x)$ is defined in the closed interval $[0, 2]$.

We have $f(0) = 0^2 + 1 = 1$, $f(2) = 3 - 2 = 1$ so that $f(0) = f(2)$.

When $0 \le x < 1$, $f(x) = x^2 + 1$ and when $1 < x \le 2$, $f(x) = 3 - x$ each of which is a polynomial and a polynomial function is continuous as well as differentiable at each point of its domain.

So $f(x)$ is continuous and differentiable at each point x when $0 \le x < 1$ or when $1 < x \le 2$.

Now to check the continuity and differentiability of $f(x)$ at $x = 1$.

We have $f(1) = 1^2 + 1 = 2$,

$$f(1-0) = \lim_{x \to 1-} f(x) = \lim_{x \to 1} (x^2 + 1) = 2$$

and $$f(1+0) = \lim_{x \to 1+} f(x) = \lim_{x \to 1} (3 - x) = 2.$$

Since $f(1-0) = f(1) = f(1+0)$, $f(x)$ is continuous at $x = 1$. Thus $f(x)$ is continuous in the closed interval $[0, 2]$.

Now $$R\ f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{\{3 - (1+h)\} - 2}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} -1 = -1$$

and $$L\ f'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{\{(1-h)^2 + 1\} - 2}{-h}$$

$$= \lim_{h \to 0} \frac{-2h + h^2}{-h} = \lim_{h \to 0} \frac{-h(2-h)}{-h} = \lim_{h \to 0} (2 - h) = 2.$$

Since $R\ f'(1) \ne L\ f'(1)$, $f(x)$ is not differentiable at $x = 1$ which is a point of the open interval $]0, 2[$.

Thus $f(x)$ is not differentiable in $]0, 2[$.

Hence Rolle's theorem is not applicable to $f(x)$.

Problem 4: *Show that between any two roots of $e^x \cos x = 1$ there exists at least one root of $e^x \sin x - 1 = 0$.*

Solution: If $x = a$ and $x = b$ are two distinct roots of $e^x \cos x = 1$, then

$$e^a \cos a = 1 \quad \text{and} \quad e^b \cos b = 1. \qquad \ldots(1)$$

Let f be the function defined as follows :

$$f(x) = e^{-x} - \cos x.$$

We observe that

(i) f is continuous in $[a, b]$ as both e^{-x} and $\cos x$ are continuous.

(ii) $f'(x) = -e^{-x} + \sin x$, which exists $\forall\ x \in \,]a, b[$;

so f is differentiable in $]a, b[$.

(iii) $f(a) = e^{-a} - \cos a = e^{-a} - e^{-a} = 0$ [By (1)]

$f(b) = e^{-b} - \cos b = e^{-b} - e^{-b} = 0$ [By (1)]

i.e., $f(a) = f(b) = 0$.

Thus f satisfies all the three conditions of Rolle's theorem in $[a, b]$. Hence there is at least one value of x in the open interval $]a, b[$, say c, such that $f'(c) = 0$.

Now $f'(c) = 0 \Rightarrow -e^{-c} + \sin c = 0 \Rightarrow e^{c} \sin c - 1 = 0$

$\Rightarrow$ c is a root of the equation $e^{x} \sin x - 1 = 0$.

Hence between any two roots of the equation $e^{x} \cos x = 1$ there is at least one root of the equation $e^{x} \sin x - 1 = 0$.

Problem 5: *State and prove Rolle's theorem. Interpret it geometrically. Verify Rolle's theorem for the function*

$$f(x) = x^2 \text{ in } [-1, 1].$$ **(Lucknow 2010)**

Solution: For the first part of this question refer article 8.

The function $f(x) = x^2$ is continuous as well as differentiable on the whole **R**. So $f(x)$ is continuous in $[-1, 1]$ and differentiable in $]-1, 1[$.

We have $f(1) = 1^2 = 1$ and $f(-1) = (-1)^2 = 1$ so that $f(-1) = f(1)$.

Thus $f(x)$ satisfies all the three conditions of Rolle's theorem in $[-1, 1]$. Therefore there must exist at least one number, say c, in $]-1, 1[$ for which $f'(c) = 0$.

Now $f'(x) = 2x$.

We have $f'(x) = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$.

We observe that the root $x = 0$ of the equation $f'(x) = 0$ lies in the open interval $]-1, 1[$. Hence the Rolle's theorem is verified.

Problem 6: *Verify the truth of Rolle's theorem for the function* $f(x) = x^2 - 3x + 2$ *on the interval* $[1, 2]$.

Solution: The function $f(x) = x^2 - 3x + 2$ is a polynomial in x of degree 2 and so it is continuous as well as differentiable on the whole **R**.

$\therefore$ $f(x)$ is continuous in the closed interval $[1, 2]$ and differentiable in the open interval $]1, 2[$.

We have $f(1) = 1^2 - 3.1 + 2 = 0$ and $f(2) = 2^2 - 3.2 + 2 = 0$ so that $f(1) = f(2)$.

Thus $f(x)$ satisfies all the three conditions of Rolle's theorem in $[1, 2]$. Therefore there must exist at least one point, say c, in $]1, 2[$ at which $f'(c) = 0$.

Now $f'(x) = 2x - 3$. We have $f'(x) = 0 \Rightarrow 2x - 3 = 0 \Rightarrow x = 3/2$. We observe that the root $x = 3/2$ of the equation $f'(x) = 0$ lies in the open interval $]1, 2[$ because $1 < 3/2 < 2$. Hence the Rolle's theorem is verified.

Problem 7: *Does the function $f(x) = |x - 2|$ satisfy the conditions of Rolle's theorem in the interval $[1, 3]$? Justify your answer with correct reasoning.*

Solution: In the interval $[1, 3]$, the function $f(x) = |x - 2|$ is defined as follows :

$$f(x) = \begin{cases} 2 - x, & 1 \le x \le 2 \\ x - 2, & 2 \le x \le 3. \end{cases}$$

We have $f(1) = 2 - 1 = 1$ and $f(3) = 3 - 2 = 1$ and so $f(1) = f(3)$.

Obviously the function $f(x)$ is continuous in $[1, 2[$ and in $]2, 3]$ because in each of these intervals it is represented by polynomials. Let us test the continuity of $f(x)$ at $x = 2$.

We have $f(2) = 0$,

$$f(2 - 0) = \lim_{h \to 0} f(2 - h), h > 0$$

$$= \lim_{h \to 0} \{2 - (2 - h)\} = \lim_{h \to 0} h = 0$$

and

$$f(2 + 0) = \lim_{h \to 0} f(2 + h), h > 0$$

$$= \lim_{h \to 0} \{(2 + h) - 2\} = \lim_{h \to 0} h = 0.$$

Thus $f(2 - 0) = f(2) = f(2 + 0)$ and so $f(x)$ is continuous at $x = 2$.

$\therefore$ $f(x)$ is continuous in the closed interval $[1, 3]$.

Now $f(x)$ is obviously differentiable in $[1, 2[$ and in $]2, 3]$. Let us test its differentiability at $x = 2$.

We have

$$Rf'(2) = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}, h > 0$$

$$= \lim_{h \to 0} \frac{\{(2 + h) - 2\} - 0}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

and

$$Lf'(2) = \lim_{h \to 0} \frac{f(2 - h) - f(2)}{-h}, h > 0$$

$$= \lim_{h \to 0} \frac{\{2 - (2 - h)\} - 0}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} -1 = -1.$$

Since $Rf'(2) \ne Lf'(2)$, therefore $f(x)$ is not differentiable at $x = 2$.

Thus $f(x)$ is not differentiable at $x=2$ which is a point in the open interval $]1,3[$. Hence out of the three conditions of Rolle's theorem, $f(x)$ satisfies the two conditions that $f(1)=f(3)$ and $f(x)$ is continuous in the closed interval $[1,3]$. But it does not satisfy the third condition that $f(x)$ must be differentiable in the open interval $]1,3[$.

Problem 8: *The function f is defined in* $[0,1]$ *as follows :*

$$f(x)=1 \quad \text{for } 0\le x<\frac{1}{2}$$
$$=2 \quad \text{for } \frac{1}{2}\le x\le 1.$$

Show that $f(x)$ *satisfies none of the conditions of Rolle's theorem, yet* $f'(x)=0$ *for many points in* $[0,1]$.

Solution: Here $f\left(\frac{1}{2}+0\right)=2,\ f\left(\frac{1}{2}-0\right)=1.$

Since $f\left(\frac{1}{2}+0\right)\ne f\left(\frac{1}{2}-0\right)$, f is discontinuous at $x=\frac{1}{2}$ and so it is not differentiable at $x=\frac{1}{2}\cdot$ Also $f(0)=1,\ f(1)=2$ so that $f(0)\ne f(1)$.

Thus all the three conditions of Rolle's theorem are not satisfied by f in $[0,1]$. But f is a constant function in $[0,\frac{1}{2}[$ and in $[\frac{1}{2},1]$.

Hence $f'(x)=0$ for many points in $[0,1]$.

Problem 9: *If* $a+b+c=0$, *then show that the quadratic equation* $3ax^2+2bx+c=0$ *has at least one root in* $]0,1[$.

Solution: Consider the function f, defined by $f(x)=ax^3+bx^2+cx+d$.

We have $f(0)=d$ and $f(1)=a+b+c+d=d$,

because it is given that $a+b+c=0$.

Since the function $f(x)$ is a polynomial, therefore it is continuous and differentiable for all real x. Consequently $f(x)$ is continuous in the closed interval $[0,1]$ and differentiable in the open interval $]0,1[$. Also $f(0)=d=f(1)$.

Thus $f(x)$ satisfies all the three conditions of Rolle's theorem in $[0,1]$. Hence there is at least one value of x in the open interval $]0,1[$ where $f'(x)=0$ *i.e.*, $3ax^2+2bx+c=0$.

Hence the equation $3ax^2+2bx+c=0$ has at least one root in $]0,1[$.

Problem 10: *Let* $\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+\ldots+\frac{a_{n-1}}{2}+a_n=0$. *Show that there exists at least one real x between* 0 *and* 1 *such that*

$$a_0x^n+a_1x^{n-1}+\ldots+a_n=0.$$

(Lucknow 2009)

Solution: Consider the function, *f*, defined by

$$f(x) = a_0 \frac{x^{n+1}}{n+1} + a_1 \frac{x^n}{n} + \ldots + a_{n-1} \frac{x^2}{2} + a_n x.$$

Since $f(x)$ is a polynomial, it is continuous and differentiable for all x. Consequently $f(x)$ is continuous in the closed interval $[0, 1]$ and differentiable in the open interval $]0, 1[$.

Also $f(0) = 0$ and

$$f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \ldots + \frac{a_{n-1}}{2} + a_n = 0 \text{ (given)}$$

i.e., $f(0) = f(1)$.

Thus all the three conditions of Rolle's theorem are satisfied. Hence there is at least one value of x in the open interval $]0, 1[$ where $f'(x) = 0$

i.e., $a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0.$

Problem 11: *If* $f(x) = \begin{vmatrix} \sin x & \sin\alpha & \sin\beta \\ \cos x & \cos\alpha & \cos\beta \\ \tan x & \tan\alpha & \tan\beta \end{vmatrix}$ *where* $0 < \alpha < \beta < \pi/2$, *show that* $f'(\xi) = 0$, *where* $\alpha < \xi < \beta$.

Solution: We have

$$f(x) = (\cos\alpha \tan\beta - \cos\beta \tan\alpha)\sin x$$
$$- (\sin\alpha \tan\beta - \sin\beta \tan\alpha)\cos x + (\sin\alpha \cos\beta - \sin\beta \cos\alpha)\tan x.$$

Obviously $f'(x)$ exists at each point of $]0, \pi/2[$. So $f(x)$ is continuous in $]0, \pi/2[$. Since $0 < \alpha < \beta < \pi/2$, therefore $f(x)$ is continuous in $[\alpha, \beta]$ and $f(x)$ is differentiable in $]\alpha, \beta[$.

Also $$f(\alpha) = \begin{vmatrix} \sin\alpha & \sin\alpha & \sin\beta \\ \cos\alpha & \cos\alpha & \cos\beta \\ \tan\alpha & \tan\alpha & \tan\beta \end{vmatrix} = 0$$

and $$f(\beta) = \begin{vmatrix} \sin\beta & \sin\alpha & \sin\beta \\ \cos\beta & \cos\alpha & \cos\beta \\ \tan\beta & \tan\alpha & \tan\beta \end{vmatrix} = 0.$$

$\therefore$ $f(\alpha) = f(\beta)$.

Thus $f(x)$ satisfies all the three conditions of Rolle's theorem in $[\alpha, \beta]$. Hence there exists at least one real number x in the open interval $]\alpha, \beta[$ at which $f'(x) = 0$.

$\therefore$ $f'(\xi) = 0$ where $\alpha < \xi < \beta$.

Problem 12: *Show that there is no real number k for which the equation* $x^3 - 3x + k = 0$, *has two distinct roots in* $]0, 1[$.

Solution: Suppose, if possible, there are two distinct roots *a, b* of the given equation in $]0, 1[$ such that $0 < a < b < 1$. Let

$$f(x) = x^3 - 3x + k.$$

Since $f(x)$ is a polynomial, so it is continuous and differentiable for all values of x *i.e,* $f(x)$ is continuous in $[a, b]$ and differentiable in $]a, b[$. Also, we have $f(a) = f(b) = 0$.

Thus f satisfies all the three conditions of Rolle's theorem in $[a, b]$. Hence there is a value c of x in $]a, b[$ such that $f'(c) = 0$.

Now $f'(x) = 0 \Rightarrow 3x^2 - 3 = 0 \Rightarrow x = \pm 1$, which contradicts the fact that $a < c < b$, as $0 < a < b < 1$.

Hence our assumption is wrong. So there cannot be two distinct roots of $f(x) = 0$ in $]0, 1[$ for any value of k.

Comprehensive Problems 3

Problem 1: *State Lagrange's mean value theorem.*

Test if Lagrange's mean value theorem holds for the function $f(x) = |x|$ *in the interval* $[-1, 1]$.

(Kanpur 2010; Rohilkhand 13B)

Solution: See article 9.

The function $f(x)$ is continuous throughout the closed interval $[-1, 1]$ but it is not differentiable at $x = 0$ which is a point of the open interval $]-1, 1[$. Thus $f(x)$ is not differentiable in $]-1, 1[$. Hence Lagrange's mean value theorem does not hold for the function $f(x) = |x|$ in the interval $[-1, 1]$.

Problem 2: *If* $f(x) = 1/x$ *in* $[-1, 1]$, *will the Lagrange's mean value theorem be applicable to* $f(x)$?

(Meerut 2012B)

Solution: The function $f(x) = 1/x$ is continuous and differentiable on $\mathbf{R} - \{0\}$. Thus it is not continuous and differentiable at $x = 0$.

Since $0 \in [-1, 1]$, therefore $f(x)$ is not continuous on $[-1, 1]$. Hence Lagrange's mean value theorem is not applicable to $f(x) = 1/x$ in $[-1, 1]$.

Problem 3: *Verify Lagrange's mean value theorem for the function*

$$f : [-1, 1] \to \mathbf{R} \text{ given by } f(x) = x^3.$$

Solution: The function $f(x) = x^3$ is a polynomial and so it is continuous and differentiable at all $x \in \mathbf{R}$. In particular it is continuous in the closed interval $[-1, 1]$ and differentiable in the open interval $]-1, 1[$ as is required for the application of Lagrange's mean value theorem.

By Lagrange's mean value theorem, there must exist at least one value 'c' of x lying in the open interval $]-1, 1[$ such that

$$\frac{f(1) - f(-1)}{1 - (-1)} = f'(c). \qquad ...(1)$$

Let us verify it.

We have $\quad f(1) = 1^3 = 1, \; f(-1) = (-1)^3 = -1.$

Also $\qquad f'(x) = 3x^2$ gives $f'(c) = 3c^2$.

Putting these values in (1), we have

$$\frac{1-(-1)}{1-(-1)} = 3c^2 \text{ or } c^2 = \frac{1}{3} \text{ or } c = \pm \frac{1}{\sqrt{3}}.$$

As both of these values of c lie in the open interval $]-1, 1[$, hence both of these are the required values of c and this verifies Lagrange's mean value theorem.

Problem 4: *Find 'c' of the mean value theorem, if* $f(x) = x(x-1)(x-2)$; $a = 0, b = \frac{1}{2}$.

(Kumaun 2012)

Solution: We have $f(a) = f(0) = 0$,

$$f(b) = f\left(\frac{1}{2}\right) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right) = \frac{3}{8}.$$

$$\therefore \qquad \frac{f(b)-f(a)}{b-a} = \frac{\frac{3}{8}-0}{\frac{1}{2}-0} = \frac{3}{4}.$$

Now $\qquad f(x) = x^3 - 3x^2 + 2x.$

$\therefore \qquad f'(x) = 3x^2 - 6x + 2$ gives $f'(c) = 3c^2 - 6c + 2$.

Putting these values in Lagrange's mean value theorem

$$\frac{f(b)-f(a)}{b-a} = f'(c), \ (a < c < b), \text{ we get}$$

$$\frac{3}{4} = 3c^2 - 6c + 2 \text{ or } 12c^2 - 24c + 5 = 0.$$

$$\therefore \qquad c = \frac{24 \pm \sqrt{(24 \times 24 - 4 \times 12 \times 5)}}{24} = \frac{24 \pm 4\sqrt{(36-15)}}{24} = 1 \pm \frac{\sqrt{21}}{6}.$$

Out of these two values of c only $1 - \frac{\sqrt{21}}{6}$ lies in the open interval $]0, \frac{1}{2}[$ which is therefore the required value of c.

Problem 5: *Find 'c' of Mean value theorem when*

(*i*) $f(x) = x^3 - 3x - 2$ *in* $[-2, 3]$

(*ii*) $f(x) = 2x^2 + 3x + 4$ *in* $[1, 2]$

(*iii*) $f(x) = x(x-1)$ *in* $[1, 2]$ **(Meerut 2013B)**

(*iv*) $f(x) = x^2 - 3x - 1$ *in* $\left[-\frac{11}{7}, \frac{13}{7}\right]$.

Solution: **(i)** We have $f(x) = x^3 - 3x - 2, a = -2, b = 3$.

$\therefore \qquad f(a) = f(-2) = (-2)^3 - 3.(-2) - 2 = -8 + 6 - 2 = -4$

and $\qquad f(b) = f(3) = 3^3 - 3.3 - 2 = 16.$

$\therefore$ $$\frac{f(b)-f(a)}{b-a}=\frac{16-(-4)}{3-(-2)}=\frac{20}{5}=4.$$

Also $f'(x)=3x^2-3$ gives $f'(c)=3c^2-3.$

Putting these values in Lagrange's mean value theorem

$$\frac{f(b)-f(a)}{b-a}=f'(c),(a<c<b), \text{ we get } \quad 4=4c^2-3$$

or $3c^2-7=0$ or $c=\pm\sqrt{(7/3)}.$

As both these values of c lie in the open interval $]-2,3[$, hence both of these are the required values of c.

(ii) We have $f(x)=2x^2+3x+4, a=1, b=2.$

$\therefore$ $f(a)=2\cdot 1^2+3\cdot 1+4=9$ and $f(b)=2\cdot 2^2+3\cdot 2+4=18.$

$\therefore$ $$\frac{f(b)-f(a)}{b-a}=\frac{18-9}{2-1}=9.$$

Also $f'(x)=4x+3$ gives $f'(c)=4c+3.$

Putting these values in Lagrange's mean value theorem

$$\frac{f(b)-f(a)}{b-a}=f'(c),(a<c<b), \text{ we get}$$

$9=4c+3$ or $c=3/2$ which lies in the open interval $]1,2[$.

(iii) We have $f(x)=x^2-x, a=1, b=2.$

$\therefore$ $f(a)=0$ and $f(b)=2.$

$\therefore$ $$\frac{f(b)-f(a)}{b-a}=\frac{2-0}{2-1}=2.$$

Also $f'(x)=2x-1$ gives $f'(c)=2c-1.$

Putting these values in Lagrange's mean value theorem

$$\frac{f(b)-f(a)}{b-a}=f'(c),(a<c<b), \text{ we get } 2=2c-1 \text{ or } c=3/2$$

which lies in the open interval $]1,2[$.

(iv) We have $f(x)=x^2-3x-1, a=-11/7, b=13/7.$

$\therefore$ $$f(a)=f\left(-\frac{11}{7}\right)=\frac{121}{49}+\frac{33}{7}-1=\frac{303}{49}$$

and $$f(b)=f\left(\frac{13}{7}\right)=\frac{169}{49}-\frac{39}{7}-1=-\frac{153}{49}.$$

$\therefore$ $$\frac{f(b)-f(a)}{b-a}=\frac{-456/49}{24/7}=-\frac{19}{7}.$$

Now $f'(x)=2x-3;$

$\therefore$ $f'(c)=2c-3.$

From Lagrange's mean value theorem, we have $2c-3=-19/7$ or $c=1/7.$

Problem 6: *Show that any chord of the parabola $y = Ax^2 + Bx + C$ is parallel to the tangent at the point whose abscissa is same as that of the middle point of the chord.*

Solution: Let a and b $(a < b)$ be the abscissae of the ends of the chord and let $f(x) = Ax^2 + Bx + C$. Since f is a polynomial function, f is continuous on $[a, b]$ and differentiable in $]a, b[$. Consequently by Lagrange's mean value theorem there exists $c \in\,]a, b[$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

i.e.,

$$Ab^2 + Bb + C - Aa^2 - Ba - C = (b - a)(2Ac + B).$$

$$[\because\ f'(c) = 2\,Ac + B]$$

On simplification it gives $c = \frac{1}{2}(a + b)$ *i.e.,* the abscissa of the point at which the tangent is parallel to the chord is the same as that of the middle point of the chord.

Problem 7: *State the conditions for the validity of the formula*

$$f(x + h) = f(x) + h\,f'(x + \theta h)$$

and investigate how far these conditions are satisfied and whether the result is true, when $f(x) = x \sin(1/x)$ (being defined to be zero at $x = 0$) and $x < 0 < x + h$.

Solution: The conditions for the validity of the given formula are :

(i) The function $f(x)$ must be continuous in the closed interval $[x, x + h]$.

(ii) The function $f(x)$ must be differentiable in the open interval $]x, x + h[$.

and (iii) θ is a real number such that $0 < \theta < 1$.

Now consider the function $f(x)$ defined as :

$$f(x) = x \sin(1/x) \text{ for } x \neq 0,\ f(0) = 0.$$

The first condition is satisfied because $f(x)$ is continuous in the closed interval $[x, x + h]$ for $x < 0 < x + h$. Obviously $f(x)$ is continuous at every point $x = c$ if $c \neq 0$ and it can be easily shown that $f(x)$ is continuous at $x = 0$.

But the second condition is not satisfied because $f(x)$ is not differentiable at $x = 0$ which is a point lying in the open interval $]x, x + h[$ for $x < 0 < x + h$. [Show here that $f(x)$ is not differentiable at $x = 0$].

Hence the result of the given formula is not true for this function $f(x)$.

Problem 8: *Show that $x^3 - 3x^2 + 3x + 2$ is monotonically increasing in every interval.*

Solution: Let $f(x) = x^3 - 3x^2 + 3x + 2.$

Then $$f'(x) = 3x^2 - 6x + 3 = 3(x - 1)^2.$$

We observe that $f'(x) > 0$ for every real value of x except 1 where its value is zero. Hence $f(x)$ is monotonically increasing in every interval.

Problem 9: *Show that $\log(1 + x) - \dfrac{2x}{2 + x}$ is increasing when $x > 0$.*

Solution: Let $f(x) = \log(1 + x) - \dfrac{2x}{2 + x}$.

We have $$f'(x) = \frac{1}{1+x} - \frac{2.(2+x) - 2x.1}{(2+x)^2}$$

$$= \frac{1}{1+x} - \frac{4}{(2+x)^2} = \frac{(2+x)^2 - 4(1+x)}{(1+x)(2+x)^2}$$

$$= \frac{4 + 4x + x^2 - 4 - 4x}{(1+x)(2+x)^2} = \frac{x^2}{(1+x)(2+x)^2}.$$

We observe that $f'(x) > 0$ for all $x > 0$.

∴ The function $f(x)$ is monotonically increasing in the interval $[0, \infty[$.

Hence the function $\log(1 + x) - \dfrac{2x}{2 + x}$ is increasing when $x > 0$.

Problem 10: *Determine the intervals in which the function*

$$(x^4 + 6x^3 + 17x^2 + 32x + 32)e^{-x}$$

is increasing or decreasing.

Solution: Let $f(x) = (x^4 + 6x^3 + 17x^2 + 32x + 32)e^{-x}$.

Then $$f'(x) = (x^4 + 6x^3 + 17x^2 + 32x + 32)(-e^{-x}) + (4x^3 + 18x^2 + 34x + 32)e^{-x}$$

$$= -e^{-x}(x^4 + 2x^3 - x^2 - 2x) = -xe^{-x}(x^3 + 2x^2 - x - 2)$$

$$= -x(x+2)(x-1)(x+1)e^{-x} = x(1-x)(1+x)(2+x)e^{-x}.$$

Now $f'(x)$ is positive in the intervals $]-2, -1[$ and $]0, 1[$ and negative in $]-\infty, -2[$, $]-1, 0[$ and $]1, \infty[$. Hence the function $f(x)$ is monotonically increasing in the intervals $[-2, -1]$ and $[0, 1]$ and monotonically decreasing in the intervals $]-\infty, -2]$, $[-1, 0]$ and $[1, \infty[$.

Problem 11: *Use the function* $f(x) = x^{1/x}$, $x > 0$ *to determine the bigger of the two numbers* e^{π} *and* π^{e}.

Solution: Let $f(x) = x^{1/x}$.

Then $$\log f(x) = \frac{1}{x}\log_e x.$$

Differentiating w.r.t. x, we get

$$\frac{1}{f(x)} f'(x) = \frac{1}{x}\cdot\frac{1}{x} - \frac{1}{x^2}\log_e x$$

or $$f'(x) = \frac{x^{1/x}}{x^2}\left[1 - \log_e x\right].$$

For $x > e$, $f'(x) < 0$. $[\because \log_e x > 1 \text{ for } x > e]$

∴ $f(x)$ is a decreasing function of x for $x > e$.

Hence $\pi > e \Rightarrow f(\pi) < f(e) \Rightarrow \pi^{1/\pi} < e^{1/e}$

$\Rightarrow$ $(\pi^{1/\pi})^{e\pi} < (e^{1/e})^{e\pi} \Rightarrow \pi^e < e^{\pi} \Rightarrow e^{\pi}$ is bigger than π^e.

Problem 12: *If $a = -1, b \geq 1$ and $f(x) = 1/|x|$, show that the conditions of Lagrange's mean value theorem are not satisfied in the interval $[a, b]$, but the conclusion of the theorem is true if and only if $b > 1 + \sqrt{2}$.*

Solution: This function is not defined at $x = 0$. So we take $f(0) = A$, where A is some definite real number.

Now $$R\, f'(0) = \lim_{h \to 0} \frac{\frac{1}{|h|} - A}{h} = \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{h} - A\right) = \infty \times \infty = \infty$$

and $$Lf'(0) = \lim_{h \to 0} \frac{\frac{1}{|-h|} - A}{-h} = \lim_{h \to 0} -\frac{1}{h}\left(\frac{1}{h} - A\right) = -\infty \times \infty = -\infty.$$

Since $R\, f'(0) \neq L\, f'(0)$ so f is not differentiable at $x = 0$. Thus the conditions of the mean value theorem are not satisfied in the interval $[a, b]$ which includes the origin.

Again, the conclusion of the mean value theorem is

$$\frac{f(b) - f(a)}{b - a} = f'(c) \text{ where } a < c < b.$$

If this result is true, we have

$$\frac{1}{|b|} - \frac{1}{|a|} = (b - a)\left\{\frac{d}{dx}\frac{1}{|x|}\right\}_{x = c} = (b - a)\left\{-\frac{1}{|c|^2}\right\}$$

or $$\frac{1}{b} - 1 = (b + 1)\left(-\frac{1}{|c|^2}\right) = -\frac{b + 1}{c^2}$$

or $$c^2 = \frac{b^2 + b}{b - 1} \quad \text{or} \quad \frac{b^2 + b}{b - 1} < b^2 \qquad [\because b^2 > c^2]$$

or $$b + 1 < b^2 - b \quad \text{or} \quad b^2 - 2b - 1 > 0$$

or $$(b - 1)^2 > 2 \quad \text{or} \quad (b - 1) > \sqrt{2} \quad \text{or} \quad b > 1 + \sqrt{2}.$$

Hence the conclusion of the mean value theorem is true iff $b > 1 + \sqrt{2}$.

Problem 13: *(a) State Cauchy's mean value theorem.* **(Kanpur 2007)**

(b) *Verify Cauchy's mean value theorem for $f(x) = \sin x, g(x) = \cos x$ in $[-\pi/2, 0]$.* **(Lucknow 2007)**

Solution: (a) **Cauchy's Mean Value Theorem :** *If two functions $f(x)$ and $g(x)$ are*

(*i*) *continuous in a closed interval $[a, b]$,*

(*ii*) *differentiable in the open interval $]a, b[$,* *(iii)* $g'(x) \neq 0$ *for any point of the open interval* $]a, b[$*, then there exists at least one value c of x in the open interval* $]a, b[$*, such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)},\ a<c<b.$$

Note that here c_1 is not necessarily equal to c_2.

(b) Here both the functions $f(x)=\sin x$ and $g(x)=\cos x$ are continuous in the closed interval $[-\pi/2, 0]$ and differentiable in the open interval $]-\pi/2, 0[$. Also $g'(x)=-\sin x \neq 0$ for any point in the open interval $]-\pi/2, 0[$. Hence by Cauchy's mean value theorem there exists at least one real number c in the open interval $]-\pi/2, 0[$ such that

$$\frac{f(0)-f(-\pi/2)}{g(0)-g(-\pi/2)}=\frac{f'(c)}{g'(c)}. \qquad ...(1)$$

Let us verify it.

We have $$\frac{f(0)-f(-\pi/2)}{g(0)-g(-\pi/2)}=\frac{\sin 0-\sin(-\pi/2)}{\cos 0-\cos(-\pi/2)}=\frac{1}{1}=1.$$

Also $f'(x)=\cos x,\ g'(x)=-\sin x.$

$\therefore$ $$\frac{f'(c)}{g'(c)}=\frac{\cos c}{-\sin c}=-\cot c.$$

Putting these values in (1), we get $-\cot c=1$ or $\cot c=-1$ whose solution $c=-\pi/4$ lies in the open interval $]-\pi/2, 0[$. Hence Cauchy's mean value theorem is verified.

Problem 14: *If* $f(x)=x^2$, $g(x)=\cos x$*, then find the point* $c \in\,]0, \pi/2[$ *which gives the result of Cauchy's mean value theorem in the interval* $[0, \pi/2]$ *for the functions* $f(x)$ *and* $g(x)$.

Solution: Both the functions $f(x)=x^2$ and $g(x)=\cos x$ are continuous in the closed interval $[0, \pi/2]$ and differentiable in the open interval $]0, \pi/2[$. Also $g'(x)=-\sin x \neq 0$ for any point in the open interval $]0, \pi/2[$. Hence by Cauchy's mean value theorem there exists at least one real number c in the open interval $]0, \pi/2[$, such that

$$\frac{f(\pi/2)-f(0)}{g(\pi/2)-g(0)}=\frac{f'(c)}{g'(c)}. \qquad ...(1)$$

We have $$\frac{f(\pi/2)-f(0)}{g(\pi/2)-g(0)}=\frac{\frac{1}{4}\pi^2-0}{\cos\frac{1}{2}\pi-\cos 0}=-\frac{\pi^2}{4}.$$

Also $f'(x)=2x,\ g'(x)=-\sin x.$

$\therefore$ $$\frac{f'(c)}{g'(c)}=\frac{2c}{-\sin c}.$$

Putting these values in (1), we get

$$-\frac{2c}{\sin c}=-\frac{\pi^2}{4} \quad \text{or} \quad \sin c-\frac{8c}{\pi^2}=0.$$

Let $F(c) = \sin c - \frac{8c}{\pi^2}\cdot$

We have $F\left(\frac{\pi}{6}\right) = \frac{1}{2} - \frac{8}{\pi^2}\cdot\frac{\pi}{6} = \frac{1}{2} - \frac{4}{3\pi} > 0$

and $F\left(\frac{\pi}{2}\right) = 1 - \frac{8}{\pi^2}\cdot\frac{\pi}{2} = 1 - \frac{4}{\pi} < 0.$

Since $F(\pi/6)$ and $F(\pi/2)$ are of opposite signs, therefore the equation $F(c) = 0$ has a root lying in the open interval $]\pi/6, \pi/2[$ and this root of the equation $\sin c - (8c/\pi^2) = 0$ is the required value of c.

Problem 15: *Show that* $\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta$, *where* $0 < \alpha < \theta < \beta < \frac{\pi}{2}\cdot$

Solution: Let $f(x) = \sin x$ and $g(x) = \cos x$, for $x \in [\alpha, \beta]$ where $0 < \alpha < \beta < \pi/2$.

$\therefore$ $f'(x) = \cos x$ and $g'(x) = -\sin x$.

Here both the functions $f(x)$ and $g(x)$ are continuous in the closed interval $[\alpha, \beta]$ and differentiable in the open interval $]\alpha, \beta[$. Also $g'(x) = -\sin x \neq 0$ for any point in the open interval $]\alpha, \beta[$. Hence by Cauchy's mean value theorem there exists at least one real number, say θ, in the open interval $]\alpha, \beta[$ such that

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(\theta)}{g'(\theta)} \Rightarrow \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta$$

$$\Rightarrow \frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta, \text{ where } 0 < \alpha < \theta < \beta < \frac{\pi}{2}\cdot$$

Example 16: *Use Cauchy's mean value theorem to evaluate*

$$\lim_{x\to 1}\left[\frac{\cos\frac{1}{2}\pi x}{\log(1/x)}\right].$$

Solution: Let $f(x) = \cos\left(\frac{1}{2}\pi x\right)$; $g(x) = \log x$, $a = x$, $b = 1$.

Putting these values in Cauchy's mean value theorem,

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}, a < c < b, \text{ we get}$$

$$\frac{\cos\frac{1}{2}\pi - \cos\frac{1}{2}\pi x}{\log 1 - \log x} = \frac{-\frac{1}{2}\pi\sin\left(\frac{1}{2}\pi c\right)}{1/c}, x < c < 1.$$

Taking limits as $x \to 1$ which implies that $c \to 1$, we get

$$\lim_{x\to 1}\left\{\frac{0 - \cos\left(\frac{1}{2}\pi x\right)}{\log(1/x)}\right\} = \lim_{c\to 1}\left\{\frac{-\frac{1}{2}\pi\sin\left(\frac{1}{2}\pi c\right)}{(1/c)}\right\}$$

or $$\lim_{x \to 1} \left\{ \frac{-\cos\left(\frac{1}{2}\pi x\right)}{\log(1/x)} \right\} = -\frac{1}{2}\pi \text{ as } \sin\left(\frac{1}{2}\pi c\right) \to 1 \text{ as } c \to 1$$

or $$\lim_{x \to 1} \left\{ \frac{\cos\left(\frac{1}{2}\pi x\right)}{\log(1/x)} \right\} = \frac{\pi}{2}.$$

Comprehensive Problems 4

Problem 1: *If* $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x+\theta h)$, ...(1)

find the value of θ *as* $x \to a$, $f(x)$ *being* $(x-a)^{5/2}$. **(Lucknow 2010)**

Solution: We have

$$f(x+h) = (x+h-a)^{5/2}$$

$$f'(x) = \frac{5}{2}(x-a)^{3/2},$$

$$f''(x+\theta h) = \frac{15}{4}(x+\theta h-a)^{1/2}.$$

Putting these values in (1), we get

$$(x+h-a)^{5/2} = (x-a)^{5/2} + \frac{5}{2}h(x-a)^{3/2} + \frac{15}{4}(x+\theta h-a)^{1/2} \cdot \frac{h^2}{2!} \quad ...(2)$$

Therefore as $x \to a$, we get from (2),

$$h^{5/2} = \frac{15}{4}(\theta h)^{1/2} \cdot \frac{h^2}{2!} \text{ or } \theta = \frac{64}{225}.$$

Problem 2: *Find* θ, *if* $f(x+h) = f(x) + hf'(x) = \frac{h^2}{2!} f''(x+\theta h)$, $0 < \theta < 1$, *and*

(i) $f(x) = ax^3 + bx^2 + cx + d$ *(ii)* $f(x) = x^3$.

Solution: (i) We have $f(x) = ax^3 + bx^2 + cx + d$

$\therefore$ $$f(x+h) = a(x+h)^3 + b(x+h)^2 + c(x+h) + d,$$

$$f'(x) = 3ax^2 + bx + c, \; f''(x) = 6ax + 2b,$$

so that $$f''(x+\theta h) = 6a(x+\theta h) + 2b$$

Substituting these values in the given relation

$$f(x+h) = f(x) + hf'(x) + (h^2/2!) f''(x+\theta h),$$

we get $a(x+h)^3 + b(x+h)^2 + c(x+h) + d$

$$= ax^3 + bx^2 + cx + d + h\,(3ax^2 + 2bx + c) + (h^2/2!)\,\{6a\,(x+\theta h) + 2b\} \quad \ldots(1)$$

The relation (1) being an identity in x, letting $x \to 0$ on both sides of (1), we get

$$ah^3 + bh^2 + ch + d = d + ch + (h^2/2)\,(6a\theta h + 2b)$$

or $$ah^3 + bh^2 + cd + d = d + ch + 3a\theta h^3 + bh^2$$

or $$ah^3 = 3a\theta h^3 \quad \text{or} \quad \theta = \frac{1}{3} \qquad [\because ah^3 \neq 0]$$

(ii) Proceed as in part (i) of this question. The required value of θ is $\frac{1}{3}$.

Problem 3: *Show that 'θ' (which occurs in the Lagrange's mean value theorem approaches the limit $\frac{1}{2}$ as 'h' approaches zero provided that $f''(a)$ is not zero. It is assumed that $f''(x)$ is continuous.*

Solution: Since $f''(x)$ is continuous at $x = a$, therefore, $f''(a)$ exists. Hence by Taylor's theorem, we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a + \theta' h) \quad \ldots(1)$$

Also by mean value theorem, we get

$$f(a+h) = f(a) + hf'(a + \theta h) \quad \ldots(2)$$

Subtracting (2) from (1), we have

$$0 = hf'(a) + \frac{h^2}{2!} f''(a+\theta' h) - hf'(a+\theta h)$$

or $$f'(a+\theta h) - f'(a) = \frac{h}{2} f''(a + \theta' h) \quad \ldots(3)$$

Further since f' is continuous and differentiable, we get by mean value theorem,

$$f'(a+\theta h) = f'(a) + \theta h\, f''(a + \theta''\theta h)$$

or $$f'(a+\theta h) - f'(a) = \theta h\, f''(a + \theta''\theta h) \quad \ldots(4)$$

Thus from (3) and (4), we get

$$\theta h\, f''(a + \theta''\theta h) = \frac{h}{2} f''(a + \theta' h)$$

or $$\theta = \frac{1}{2} \cdot \frac{f''(a+\theta' h)}{f''(a + \theta''\theta h}$$

Hence $$\lim_{h \to 0} \theta = \frac{1}{2} \frac{f''(a)}{f''(a)} = \frac{1}{2}, \text{ provided } f''(a) \neq 0.$$

Problem 4: *Show that the number θ which occurs in the Taylor's theorem with Lagrange's form of remainder after n terms approaches the limit $1/(n+1)$ as $h \to 0$ provided that $f^{(n+1)}(x)$ is continuous and different from zero at $x = a$.*

Solution: Applying Taylor's theorem with Lagrange's from of remainder after n terms and $(n+1)$ terms successively, we get for $\theta, \theta' \in\,]0,\ 1[$,

$$f(a+h) = f(a) + h f'(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a+\theta h)$$

and $$f(a+h) = f(a) + h f'(a) + \ldots + \frac{h^n}{n!} f^{(n)}(a) + \frac{h^n+1}{(n+1)!} f^{(n+1)}(a+\theta' h).$$

Subtracting these, we have

$$\frac{h^n f^{(n)}(a)}{n!} + \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(a+\theta' h) = \frac{h^n}{n!} f^{(n)}(a+\theta h)$$

or $$f^{(n)}(a+\theta h) - f^{(n)}(a) = \frac{h}{n+1} f^{(n+1)}(a+\theta' h) \qquad \ldots(1)$$

Applying Lagrange's mean value theorem to the function $f^{(n)}(x)$ in the interval $[a, a+\theta h]$, we get

$$f^{(n)}(a+\theta h) - f^{(n)}(a) = \theta h f^{(n+1)}(a+\theta\theta'' h),\ 0 < \theta'' < 1. \qquad \ldots(2)$$

From (1) and (2), we have

$$\theta h\, f^{(n+1)}(a+\theta\theta' h) = \frac{h}{n+1} f^{(n+1)}(a+\theta' h)$$

or $$\theta = \frac{1}{n+1} \frac{f^{(n+!)}(a+\theta' h)}{f^{(n+1)}(as+\theta\theta'' h)}.$$

$\therefore$ $$\lim_{h\to 0} \theta = \frac{1}{n+1} \frac{f^{(n+1)}(a)}{f^{(n+1)}(a)} = \frac{1}{(n+1)}, \text{ provided } f^{(n+1)}(a) \neq 0.$$

Hints to Objective Type Questions

Multiple Choice Questions

1. The function $f(x) = |x-1| = \begin{cases} x-1, & x \geq 1 \\ 1-x, & x < 1. \end{cases}$

   This function is not differentiable at $x = 1$.
2. The function $f(x) = |x+3|$ is not differentiable at $x = -3$.
3. A function $f(x)$ is differentiable at $x = a$ if $Rf'(a) = Lf'(a)$. See article 1.
4. See Problem 9 of Comprehensive Problems 1.
5. For the function $f(x) = \sin x$ in $[0, \pi]$, we have $f(0) = 0$ and $f(\pi) = 0$ so that $f(0) = f(\pi)$. Also $f(x)$ is continuous in $[0, \pi]$ and differentiable in $]0, \pi[$. Thus $f(x) = \sin x$ satisfies all the three conditions of Rolle' theorem in $[0, \pi]$.
6. The function $f(x) = \sin x$ is continuous on the whole **R** .

   We have $f'(x) = \cos x$. Out of the given four intervals $f'(x) = \cos x$ is positive only in $\left[0, \frac{\pi}{2}\right]$ and is positive as well as negative in each of the intervals $[0, \pi]$, $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ and $\left[\frac{\pi}{2}, \pi\right]$. So, out of the four given intervals the function $f(x) = \sin x$ is increasing only in $\left[0, \frac{\pi}{2}\right]$.

7. See Problem 5(iii) of Comprehensive Problems 3.

8. See Problem 1(ii) of Comprehensive Problems 2.

9. See Example 1.

10. See article 14(iv).

Fill in the Blank(s)

1. See article 1, definition of derivative of a function at a point.

2. See article 1, definition of right hand derivative of a function at a point.

3. See article 1, definition of left hand derivative of a function at a point.

4. See article 1, definition of differentiability of a function in an open interval.

5. See article 2.

6. See article 4.

7. See Example 1.

8. See article 8.

9. See article 9, Lagrange's mean value theorem.

10. See article 11, Cauchy's mean value theorem.

11. See article 10, theorem 4.

12. We have $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$. If $f(x) = \sin x$, then $f'(x) = \cos x$.

True or False

1. For example, the function $f(x) = |x|$ is continuous at $x = 0$ but it is not differentiable at $x = 0$.

2. If a function $f(x)$ possesses a finite derivative $f'(a)$ at a point $x = a$, we know that $f(x)$ must be continuous at $x = a$. See article 4.

3. The given statement is false. If a function $f(x)$ is differentiable at $x = a$, we know that it must be continuous at $x = a$.

4. The function $f(x) = |x|$ is not differentiable at $x = 0$.

5. For the function $f(x) = \sin x$ in $[0, 2\pi]$,
we have $f(0) = 0$, $f(2\pi) = 0$ so that $f(0) = f(2\pi)$.
Also $f(x)$ is continuous in $[0, 2\pi]$ and differentiable in $]0, 2\pi[$. Thus $f(x) = \sin x$ satisfies all the three conditions of Rolle's theorem in $[0, 2\pi]$. So Rolle's theorem is applicable for $f(x) = \sin x\,[0, 2\pi]$.

6. Rolle's theorem is not applicable for $f(x) = |x|$ in $[-1, 1]$ because $f(x) = |x|$ is not differentiable at $x = 0$ which is a point in $]-1, 1[$.

7. Lagrange's mean value theorem is not applicable for $f(x) = |x|$ in $[-1, 1]$ because $f(x) = |x|$ is not differentiable at $x = 0$ which is a point in $[-1, 1[$.

8. The function $f(x) = \sin x$ is continuous on the whole **R** and so also in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

We have $f'(x) = \cos x$ which is positive in $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. So the function $f(x) = \sin x$ is increasing in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

9. If $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has at least one root in $]0, 1[$. See Problem 9, of Comprehensive Problems 2.

10. If f is continuous on $[a, b]$ and $f'(x) \le 0$ in $]a, b[$, then f is decreasing in $[a, b]$. See article 10, Theorem 4.

11. If $f(x) = 2x^3 - 15x^2 + 36x + 1$, then $f'(x) = 6x^2 - 30x + 36 = 6(x-2)(x-3)$.

Now, $f(x)$ is continuous in (2, 3) and $f'(x) < 0$ in $]2, 3[$. so $f(x)$ is decreasing in [2, 3].

12. If $f(x) = |x| + |x-1|$, then $R\ f'(0) = 0$. See Example 2.

13. For the function $f(x) = x(x+2)\,e^{-x/2}$, we have $f(-2) = 0$ and $f(0) = 0$, so that $f(-2) = f(0)$. Also $f(x)$ is continuous in $[-2, 0]$ and differentiable in $[-2, 0[$. Thus $f(x)$ satisfies all the three conditions of Rolle's theorem in $[-2, 0]$ and so Rolle's theorem is applicable for $f(x)$ in $[-2, 0]$.

14. See Problem 5 part (ii) of Comprehensive Problems 3.

We note that the value of c is given by $c = \frac{3}{2}$.

15. If $f(x) = x^n$, then $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = f'(x) = nx^{n-1}$

16. We have $\lim\limits_{h \to 0} \dfrac{f(x) - f(a)}{x - a} = f'(a)$.

If $f(x) = \cos x$, then $f'(x) = -\sin x$ so that $f'(a) = -\sin a$.

So if $f(x) = \cos x$, then $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = -\sin a$

17. We have $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$

If $f(x) = e^x$, then $f'(x) = e^x$ so that $f'(x_0) = e^x 0$.

So if $f(x) = e^x$, then $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = e^{x_0}$ and not e^x.

OOO

Chapter-4

Successive Differentiation

Comprehensive Problems 1

Problem 1: *If $x = a(t - \sin t)$ and $y = a(1 + \cos t)$, prove that*

$$\frac{d^2 y}{dx^2} = \frac{1}{4a} \operatorname{cosec}^4 \left(\frac{t}{2}\right).$$

Solution: Proceed as in Example 3.

Problem 2: *(i) If $y = A \sin mx + B \cos mx$, prove that $y_2 + m^2 y = 0$.*

(ii) If $y = A e^{ax} + B e^{-ax}$, show that $y_2 - a^2 y = 0$.

Solution: (i) We have $y = A \sin mx + B \cos mx$. ...(1)

Differentiating both sides w.r.t. 'x', we get

$$y_1 = Am \cos mx - Bm \sin mx.$$

Again $$y_2 = -Am^2 \sin mx - Bm^2 \cos mx$$

$$= -m^2 (A \sin mx + B \cos mx) = -m^2 y, \text{ from (1).}$$

$\therefore$ $$y_2 + m^2 y = 0.$$

(ii) $$y = A e^{ax} + B e^{-ax} \qquad ...(1)$$

Differentiating both sides w.r.t. x, we get

$$y_1 = A e^{ax} \cdot a + B e^{-ax} \cdot (-a)$$

Again $$y_2 = a^2 A e^{ax} + a^2 B e^{-ax}$$

$$= a^2 (A e^{ax} + B e^{-ax}) = a^2 y$$

or $$y_2 - a^2 y = 0.$$

Problem 3: *If $y = e^{ax} \cos bx$, prove that $y_2 - 2ay_1 + (a^2 + b^2) y = 0$.*

Also prove that $y_{n+1} = 2ay_n - (a^2 + b^2) y_{n-1}$. **(Lucknow 2007; Kumaun 13)**

Solution: We have $y = e^{ax} \cos bx$ ...(1)

Differentiating both sides w.r.t. x, we get

$$y_1 = e^{ax} (-b \sin bx) + a e^{ax} \cos bx$$

or $\qquad y_1 = -b e^{ax} \sin bx + ay$ ...(2)

Again $\qquad y_2 = -b e^{ax} (b \cos bx) - b\,(a e^{ax}) \sin bx + ay_1$

$$y_2 = -b^2 e^{ax} \cos bx - a\,(b e^{ax} \sin bx) + ay_1$$

$$= -b^2 y - a\,(-y_1 + ay) + ay_1 \qquad \text{[By using (2)]}$$

$$= -(a^2 + b^2)\,y + 2ay_1$$

or $\qquad y_2 - 2ay_1 + (a^2 + b^2)\,y = 0$ (1st Proved) ...(3)

Differentiating both sides, w.r.to x, $(n-1)$ times, we get

$$y_{n+1} - 2ay_n + (a^2 + b^2)\,y_{n-1} = 0$$

or $\qquad y_{n+1} = 2a\,y_n - (a^2 + b^2)\,y_{n-1}.$ Proved.

Comprehensive Problems 2

Find the n^{th} differential coefficients of:

Problem 1: *(i)* $\log\,[(ax + b)(cx + d)]$. **(Bundelkhand 2008)**

(ii) $\cos 2x \cos 3x$. **(Bundelkhand 2001; Kashi 2011)**

(iii) $\cos x \cos 2x \cos 3x$.

(iv) $\cos^4 x$.

Solution : (i) Let $y = \log\,[(ax + b)(cx + d)] = \log\,(ax + b) + \log\,(cx + d)$.

We know that $D^n \log\,(ax + b) = (-1)^{n-1}\,(n-1)!\,a^n\,(ax + b)^{-n}$

[See result (v) of article 2]

$$\therefore \quad y_n = (-1)^{n-1}\,(n-1)!\,a^n\,(ax + b)^{-n} + (-1)^{n-1}\,(n-1)!\,c^n\,(cx + d)^{-n}$$

$$= (-1)^{n-1}\,(n-1)!\left[\frac{a^n}{(ax + b)^n} + \frac{c^n}{(cx + d)^n}\right].$$

(ii) Let $\qquad y = \cos 2x \cos 3x = \frac{1}{2}(2 \cos 2x \cos 3x) = \frac{1}{2}(\cos 5x + \cos x)$

$$\therefore \quad y_n = \frac{1}{2}\left\{5^n \cos\,(5x + \tfrac{1}{2} n\pi) + \cos\,(x + \tfrac{1}{2} n\pi)\right\}$$

(iii) $\qquad y = \cos x \cos 2x \cos 3x = \frac{1}{2} \cos x\,(2 \cos 2x \cos 3x)$

$$= \frac{1}{2} \cos x\,(\cos 5x + \cos x) = \frac{1}{4}\,(2 \cos x \cos 5x + 2 \cos^2 x)$$

$$= \frac{1}{4}\,(\cos 6x + \cos 4x + \cos 2x + 1).$$

$$\therefore \quad y_n = \frac{1}{4}\,\{6^n \cos\,(6x + \tfrac{1}{2} n\pi) + 4^n \cos\,(4x + \tfrac{1}{2} n\pi) + 2^n \cos\,(2x + \tfrac{1}{2} n\pi)\}.$$

(iv) Let $y = \cos^4 x = (\cos^2 x)^2 = [\frac{1}{2}(1 + \cos 2x)]^2$

$$= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$$

$$= \frac{1}{4}[1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)]$$

$$= \frac{1}{4}(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x)$$

$$= \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x.$$

Now $D^n \cos(ax + b) = a^n \cos(ax + b + \frac{1}{2}n\pi).$

$\therefore$ $y_n = 0 + \frac{1}{2}\cdot 2^n \cos(2x + \frac{1}{2}n\pi) + \frac{1}{8}\cdot 4^n \cos(4x + \frac{1}{2}n\pi)$

$$= 2^{n-1}\cos(2x + \frac{1}{2}n\pi) + 2^{2n-3}\cos(4x + \frac{1}{2}n\pi).$$

Problem 2: *(i)* $\cos^2 x \sin^3 x$. *(ii)* $e^{ax}\cos^3 bx$.

(iii) $e^{ax}\sin bx \cos cx$. *(iv)* $e^{2x}\sin^3 x$.

Solution: (i) Let $y = \cos^2 x \sin^3 x = \frac{1}{2}(1 + \cos 2x)\frac{1}{4}(3\sin x - \sin 3x)$

$$= \frac{1}{8}(3\sin x - \sin 3x + 3\sin x\cos 2x - \sin 3x\cos 2x)$$

$$= \frac{1}{8}[3\sin x - \sin 3x + \frac{3}{2}(2\sin x\cos 2x) - \frac{1}{2}(2\sin 3x\cos 2x)]$$

$$= \frac{1}{8}[3\sin x - \sin 3x + \frac{3}{2}(\sin 3x - \sin x) - \frac{1}{2}(\sin 5x + \sin x)]$$

$$= \frac{1}{16}(6\sin x - 2\sin 3x + 3\sin 3x - 3\sin x - \sin 5x - \sin x)$$

$$= \frac{1}{16}(2\sin x + \sin 3x - \sin 5x).$$

Now using the standard formula $D^n \sin(ax + b)$, we get

$$y_n = \frac{1}{16}[2\sin(x + \frac{1}{2}n\pi) + 3^n\sin(3x + \frac{1}{2}n\pi) - 5^n\sin(5x + \frac{1}{2}n\pi)].$$

(ii) Let $y = e^{ax}\cos^3 bx$

$$= e^{ax}\left\{\frac{\cos 3bx + 3\cos bx}{4}\right\} \qquad [\because \cos 3\theta = 4\cos^3\theta - 3\cos\theta]$$

$$= \frac{1}{4}e^{ax}.\cos 3bx + \frac{3}{4}e^{ax}\cos bx$$

Differentiating both sides, w.r.t. to x, n times, we get

$$y_n = \frac{1}{4}D^n[e^{ax}\cos 3bx] + \frac{3}{4}D^n[e^{ax}\cos bx]$$

$$= \frac{1}{4}(a^2 + 9b^2)^{n/2} e^{ax} \cos\left\{3bx + n\tan^{-1}\frac{3b}{a}\right\}$$
$$+ \frac{3}{4}(a^2 + b^2)^{n/2} e^{ax} \cos\left\{bx + n\tan^{-1}\frac{b}{a}\right\}$$

(iii) Let $y = e^{ax} \sin bx \cos cx = \frac{1}{2} e^{ax} (2 \sin bx \cos cx)$

$$= \frac{1}{2} e^{ax} [\sin(bx + cx) + \sin(bx - cx)]$$
$$= \frac{1}{2} [e^{ax} \sin(b + c)x + e^{ax} \sin(b - c)x].$$

Now $D^n \{e^{ax} \sin(bx + c)\} = (a^2 + b^2)^{n/2} e^{ax} \sin\{bx + c + n\tan^{-1}(b/a)\}$.

$\therefore$ $y_n = \frac{1}{2} [\{a^2 + (b + c)^2\}^{n/2} e^{ax} \sin\{(b + c)x + n\tan^{-1}(b + c)/a\}$

$+ \{a^2 + (b - c)^2\}^{n/2} e^{ax} \sin\{(b - c)x + n\tan^{-1}(b - c)/a\}].$

(iv) Let $y = e^{2x} \sin^3 x$.

We know that $\sin 3x = 3\sin x - 4\sin^3 x$.

$\therefore$ $4\sin^3 x = 3\sin x - \sin 3x$ or $\sin^3 x = (1/4)(3\sin x - \sin 3x)$.

$\therefore$ $y = \frac{1}{4} e^{2x} [3\sin x - \sin 3x] = \frac{3}{4} e^{2x} \sin x - \frac{1}{4} e^{2x} \sin 3x.$

$\therefore$ $y_n = \frac{3}{4} [(2^2 + 1^2)^{1/2}]^n e^{2x} \sin[x + n\tan^{-1}(1/2)]$

$- \frac{1}{4} [(2^2 + 3^2)^{1/2}]^n e^{2x} \sin[2x + n\tan^{-1}(3/2)].$

Problem 3: *(i)* $\frac{1}{1 - 5x + 6x^2}$. *(ii)* $\frac{1}{x^2 - a^2}$.

(iii) $\frac{x^2}{(x + 2)(2x + 3)}$. **(Kumaun 2014)** *(iv)* $\frac{x}{(x - a)(x - b)(x - c)}$.

Solution: (i) Let $y = \frac{1}{6x^2 - 5x + 1} = \frac{1}{(3x - 1)(2x - 1)} = \frac{2}{2x - 1} - \frac{3}{3x - 1}$

[On resolving into partial fractions]

$$= 2(2x - 1)^{-1} - 3(3x - 1)^{-1}.$$

Now $D^n (ax + b)^{-1} = (-1)^{-n} n!\, a^n (ax + b)^{-n-1}$.

$\therefore$ $y_n = 2(-1)^n n!\, 2^n (2x - 1)^{-n-1} - 3(-1)^n n!\, 3^n (3x - 1)^{-n-1}$

$$= (-1)^n n! [2^{n+1} (2x - 1)^{-n-1} - 3^{n+1} (3x - 1)^{-n-1}].$$

(ii) Let $y = \frac{1}{x^2 - a^2} = \frac{1}{(x - a)(x + a)} = \frac{1}{2a}\left[\frac{1}{x - a} - \frac{1}{x + a}\right]$

$$= \frac{1}{2a}[(x-a)^{-1}-(x+a)^{-1}].$$

Then $$y_n = \frac{1}{2a}(-1)^n\, n!\,\{(x-a)^{-n-1}-(x+a)^{-n-1}\}.$$

(iii) Let $y = x^2/[(x+2)(2x+3)]$

The given fraction is not a proper one. If we divide the numerator by the denominator, we observe orally that the quotient will be 1/2. So let

$$\frac{x^2}{(x+2)(2x+3)} \equiv \frac{1}{2}+\frac{A}{x+2}+\frac{B}{2x+3}.$$

Then $A = -4$ and $B = 9/2$.

Hence $$y = \frac{1}{2}-\frac{4}{x+2}+\frac{9}{2(2x+3)} = \frac{1}{2}-4(x+2)^{-1}+\frac{9}{2}(2x+3)^{-1}.$$

$$\therefore \quad y_n = -4(-1)^n\, n!\,(x+2)^{-n-1}+\frac{9}{2}\cdot(-1)^n\, n!\,.2^n\,(2x+3)^{-n-1}$$

$$= (-1)^n\, n!\left[\frac{9\cdot 2^{n-1}}{(2x+3)^{n+1}}-\frac{4}{(x+2)^{n+1}}\right].$$

(iv) Let $$y = \frac{x}{(x-a)(x-b)(x-c)}$$

$$= \frac{a}{(a-b)(a-c)(x-a)}+\frac{b}{(b-a)(b-c)(x-b)}+\frac{c}{(c-a)(c-b)(x-c)}$$

{on resolving into partial fractions}

$$= \frac{a}{(a-b)(a-c)}(x-a)^{-1}+\frac{b}{(b-a)(b-c)}(x-b)^{-1}+\frac{c}{(c-a)(c-b)}(x-c)^{-1}$$

Now differentiating both sides w. r.to x, n times, we get

$$y_n = \frac{a}{(a-b)(a-c)}D^n\left[\frac{1}{(x-a)}\right]+\frac{b}{(b-a)(b-c)}D^n\left[\frac{1}{(x-b)}\right]+\frac{c}{(c-a)(c-b)}D^n\left[\frac{1}{(x-c)}\right]$$

$$y_n = \frac{a}{(a-b)(a-c)}\frac{(-1)^n\, n!}{(x-a)^{n+1}}+\frac{b}{(b-a)(b-c)}\frac{(-1)^n\, n!}{(x-b)^{n+1}}+\frac{c}{(c-a)(c-b)}\frac{(-1)^n\, n!}{(x-c)^{n+1}}$$

$$y_n = (-1)^n\, n!\left[\frac{a}{(a-b)(a-c)(x-a)^{n+1}}+\frac{b}{(b-a)(b-c)(x-b)^{n+1}}+\frac{c}{(c-a)(c-b)(x-c)^{n+1}}\right]$$

Problem 4: $\dfrac{x^4}{(x-1)(x-2)}$, $n \geq 3$.

(Agra 2014)

Solution: Let $y = \dfrac{x^4}{(x-1)(x-2)} = \left[x^2 + 3x + 7 + \dfrac{15x - 14}{(x-1)(x-2)}\right]$,

dividing N^r by the D^r

$$= \left[x^2 + 3x + 7 + \frac{16}{x-2} - \frac{1}{x-1}\right], \text{ on resolving into partial fractions.}$$

Then $y_n = 16\,(-1)^n \cdot n\,!\,(x-2)^{-1-n} - (-1)^n\, n\,!\,(x-1)^{-1-n}$, if $n > 2$

$$= (-1)^n \cdot n\,!\,[16\,(x-2)^{-n-1} - (x-1)^{-n-1}].$$

Problem 5: *(i)* $\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$.

(Purvanchal 2011)

(ii) $\tan^{-1}\left\{\dfrac{2x}{(1-x^2)}\right\}$.

(Lucknow 2011)

Solution: (i) Let $y = \tan^{-1}\left(\dfrac{1+x}{1-x}\right) = \tan^{-1}\left\{\dfrac{1+x}{1-1\,.\,x}\right\} = \tan^{-1} 1 + \tan^{-1} x.$

Then $y_1 = 1/(1+x^2)$. Now proceeding as in problem 5(ii), we have

$$y_n = (-1)^{n-1}\,(n-1)\,!\,\sin^n \phi \sin n\,\phi, \text{ where } \phi = \tan^{-1}(1/x).$$

(ii) Let $y = \tan^{-1}\{2x/(1-x^2)\} = 2\tan^{-1} x.$

Then $y_1 = \dfrac{2}{1+x^2} = \dfrac{2}{(x-i)(x+i)} = \dfrac{1}{i}\left[\dfrac{1}{x-i} - \dfrac{1}{x+i}\right]$,

on resolving into partial fractions.

Now differentiating both sides $(n-1)$ times w.r.t. 'x', we have

$$y_n = \frac{(-1)^{n-1}\,(n-1)\,!}{i}\,[(x-i)^{-n} - (x+i)^{-n}].$$

Putting $x = r\cos\phi$ and $1 = r\sin\phi$, we have

$$y_n = \frac{(-1)^{n-1}\,(n-1)\,!}{i}\,[r^{-n}(\cos\phi - i\sin\phi)^{-n} - r^{-n}(\cos\phi + i\sin\phi)^{-n}]$$

$$= \frac{(-1)^{n-1}\,(n-1)\,!}{i}\,r^{-n}\,[(\cos n\phi + i\sin n\phi) - (\cos n\phi - i\sin n\phi)]$$

$$= 2\,(-1)^{n-1}\,(n-1)\,!\,r^{-n}\sin n\phi$$

$$= 2\,(-1)^{n-1}\,(n-1)\,!\,(1/\sin\phi)^{-n}\sin n\phi, \quad \text{since } r = 1/\sin\phi$$

$$= 2\,(-1)^{n-1}\,(n-1)\,!\,\sin^n\phi\,\sin n\phi, \quad \text{where } \phi = \tan^{-1}(1/x).$$

Problem 6: *If* $y = \tan^{-1}\left\{\dfrac{\sqrt{(1+x^2)}-1}{x}\right\}$, *show that*

$$y_n = \frac{1}{2}(-1)^{n-1}(n-1)!\,\sin^n\theta\,\sin n\theta,\ \text{where}\ \theta = \cot^{-1}x.$$

(Kanpur 2015)

Solution: Let $\quad y = \tan^{-1}\left\{\dfrac{\sqrt{(1+x^2)}-1}{x}\right\}.$

Put $x = \tan\phi$. Then

$$y = \tan^{-1}\left[\frac{\sqrt{(1+\tan^2\phi)}-1}{\tan\phi}\right] = \tan^{-1}\frac{\sec\phi-1}{\tan\phi}$$

$$= \tan^{-1}\frac{1-\cos\phi}{\sin\phi} = \tan^{-1}\frac{2\sin^2(\phi/2)}{2\sin(\phi/2)\cos(\phi/2)}$$

$$= \tan^{-1}\tan(\phi/2) = \phi/2 = \frac{1}{2}\tan^{-1}x.$$

$\therefore \qquad y_1 = 1/\{2(1+x^2)\}.$

Now proceeding as in problem 5(ii), we get

$$y_n = \frac{1}{2}(-1)^{n-1}(n-1)!\,\sin^n\theta\,\sin n\theta,\ \text{where}\ \theta = \cot^{-1}x.$$

Problem 7: *If* $y = x/(a^2+x^2)$, *prove that*

$$y_n = (-1)^n\,n!\,a^{-n-1}\sin^{n+1}\phi\,\cos(n+1)\phi,\ \text{where}\ \phi = \tan^{-1}(a/x).$$

(Kanpur 2009; Kumaun 13)

Solution: We have $y = \dfrac{x}{a^2+x^2} = \dfrac{x}{(x-ia)(x+ia)}$

$$= \frac{1}{2}\left[\frac{1}{(x-ia)} + \frac{1}{(x+ia)}\right],\ \text{on resolving into partial fractions.}$$

Differentiating both sides n times w.r.t. 'x', we get

$$y_n = \frac{1}{2}[(-1)^n\,n!\,(x-ia)^{-n-1} + (-1)^n\,n!\,(x+ia)^{-n-1}]$$

$$= \frac{1}{2}(-1)^n\,n!\,[(x-ia)^{-n-1} + (x+ia)^{-n-1}].$$

Putting $x = r\cos\phi$ and $a = r\sin\phi$, we get

$$y_n = \frac{1}{2}(-1)^n\,n!\,[r^{-n-1}(\cos\phi - i\sin\phi)^{-n-1} + r^{-n-1}(\cos\phi + i\sin\phi)^{-n-1}]$$

$$= \frac{1}{2}(-1)^n\,n!\,r^{-n-1}[\{\cos(n+1)\phi + i\sin(n+1)\phi\} + \{\cos(n+1)\phi - i\sin(n+1)\phi\}]$$

$$= \frac{1}{2} (-1)^n \, n! \, r^{-n-1} \, 2 \cos (n+1) \phi$$

$$= (-1)^n \, n! \, (a/\sin \phi)^{-n-1} \cos (n+1) \phi, \text{ since } r = a/\sin \phi$$

$$= (-1)^n . n! \, a^{-n-1} \sin^{n+1} \phi \cos (n+1) \phi, \text{ where } \phi = \tan^{-1} (a/x).$$

Problem 8: *If* $y = \sin mx + \cos mx$, *show that* $y_n = m^n \{1 + (-1)^n \sin 2mx\}^{1/2}$.

(Meerut 2000, 09B)

Solution: Here $y = \sin mx + \cos mx$.

$$\therefore \quad y_n = m^n \sin (mx + \tfrac{1}{2} n\pi) + m^n \cos (mx + \tfrac{1}{2} n\pi)$$

$$= m^n [\{\sin (mx + \tfrac{1}{2} n\pi) + \cos (mx + \tfrac{1}{2} n\pi)\}^2]^{1/2} \qquad \textbf{(Note)}$$

$$= m^n [\sin^2 (mx + \tfrac{1}{2} n\pi) + \cos^2 (mx + \tfrac{1}{2} n\pi)$$

$$+ 2 \sin (mx + \tfrac{1}{2} n\pi) \cos (mx + \tfrac{1}{2} n\pi)]^{1/2}$$

$$= m^n [1 + \sin 2 (mx + \tfrac{1}{2} n\pi)]^{1/2} = m^n [1 + \sin (2mx + n\pi)]^{1/2}$$

$$= m^n [1 + (-1)^n \sin 2mx], \text{ since } \sin (n\pi + \theta) = (-1)^n \sin \theta.$$

Problem 9: *Prove that the value of the nth differential coefficient of* $x^3/(x^2 - 1)$ *for* $x = 0$, *is zero if n is even, and is* $-n!$ *if n is odd and greater than 1.*

Solution: Let

$$y = \frac{x^3}{x^2 - 1} = x + \frac{x}{x^2 - 1} = x + \frac{x}{(x-1)(x+1)}$$

$$= x + \frac{1}{(1+1)(x-1)} + \frac{-1}{(-1-1)(x+1)}$$

$$= x + \frac{1}{2(x-1)} + \frac{1}{2(x+1)}.$$

$\therefore$ When $n > 1$, we have

$$y_n = \frac{(-1)^n \, n!}{2} \left[\frac{1}{(x-1)^{n+1}} + \frac{1}{(x+1)^{n+1}} \right].$$

Putting $x = 0$ in the expression for y_n, we get

$$(y_n)_0 = \frac{(-1)^n \, n!}{2} \left[\frac{1}{(-1)^{n+1}} + \frac{1}{1} \right] = \frac{(-1)^n \, n!}{2} \left[\frac{1}{(-1)^{n+1}} + 1 \right].$$

When n is even, $(y_n)_0 = \dfrac{n!}{2} \left[\dfrac{1}{(-1)} + 1 \right] = \dfrac{n!}{2} \cdot 0 = 0$.

When n is odd, $(y_n)_0 = -\dfrac{n!}{2} [1 + 1] = -n!$.

Problem 10: *If* $y = (\tan^{-1} x)^2$, *prove that* $(x^2+1)^2 y_2 + 2x(x^2+1) y_1 - 2 = 0$.

Solution: Given $y = (\tan^{-1} x)^2$ ...(1)

Differentiating (1) w.r.to x, we have

$$y_1 = 2\tan^{-1} x \cdot \frac{1}{1+x^2} \quad \text{or} \quad (1+x^2) y_1 = 2\tan^{-1} x$$

On squaring both sides, we get

$$(1+x^2)^2 y_1^2 = 4(\tan^{-1} x)^2 = 4y \qquad \text{...(2)}$$

Differentiating both sides w.r. to x

$$2y_1 y_2 (1+x^2)^2 + y_1^2\, 2(1+x^2) \cdot 2x = 4y_1$$

On dividing by $2y_1$, we get

$$(1+x^2)^2 y_2 + 2x(1+x^2) y_1 = 2$$

Comprehensive Problems 3

Problem 1: *State Leibnitz's theorem.*

(Meerut 2005B, 08, 11; Bundelkhand 08; Agra 08)

Solution: See article 6.

Problem 2: *Find the* 4^{th} *differential coefficients of (i)* $x^3 \log x$; *(ii)* $x^2 \sin 3x$; *(iii)* $e^{2x} \sin 2x$.

Solution: (i) Let $y = x^3 \log x$

Differentiating both sides w.r.to x, 4 times (by Leibnitz's theorem),

we get $y_4 = D^4 (x^3 \log x)$

$$= x^3 D^4(\log x) + {}^4C_1 (Dx^3)(D^3 \log x) + {}^4C_2 (D^2 x^3)(D^2 \log x) + {}^4C_3 (D^3 x^3) D(\log x)$$

$$= x^3 \frac{(-1)^3\, 3!}{x^4} + 4(3x^2)\frac{(-1)^2 2!}{x^3} + 6(6x)\frac{(-1)^1\, 1!}{x^2} + \frac{4(6)}{x}$$

$$\left\{\because D^n \log x = \frac{(-1)^{n-1}(n-1)!}{x^n}\right\}$$

$$= -\frac{6}{x} + \frac{24}{x} - \frac{36}{x} + \frac{24}{x} = \frac{6}{x}$$

(ii) Let $y = x^2 \sin 3x$

Differentiating w.r.to x, 4 times (by Leibnitz's theorem), we get

$$y_4 = D^4 (x^2 \sin 3x)$$

$$= x^2 D^4(\sin 3x) + {}^4C_1 (Dx^2)(D^3 \sin 3x) + {}^4C_2 (D^2 x^2)(D^2 \sin 3x)$$

$$= x^2 . 3^4 \sin\left(3x + \frac{4\pi}{2}\right) + 4(2x)\,3^3 \sin\left(3x + \frac{3\pi}{2}\right)$$
$$+ 6(2) . 3^2 \sin\left(3x + \frac{2\pi}{2}\right)$$
$$\left\{\because D^n \sin(ax+b) = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)\right\}$$
$$= 3^4 x^2 \sin 3x + 8x\, 3^3 (-\cos 3x) - 12.3^2 \sin 3x$$
$$= 3^3 (3x^2 - 4) \sin 3x - 6^3 x \cos 3x$$

(iii) Let $y = e^{2x} \sin 2x$

Differentiating w.r.to x, 4 times (by Leibnitz's theorem)

$$y_4 = D^4 (e^{2x} \sin 2x)$$
$$= e^{2x} D^4 (\sin 2x) + {}^4C_1 (D\, e^{2x})(D^3 \sin 2x)$$
$$+ {}^4C_2 (D^2 e^{2x})(D^2 \sin 2x) + {}^4C_3 (D^3 e^{2x})(D \sin 2x)$$
$$+ {}^4C_4 (D^4 e^{2x}) \sin 2x$$
$$= e^{2x} . 2^4 \sin\left(2x + \frac{4\pi}{2}\right) + 4 . 2\, e^{2x} 2^3 \sin\left(2x + \frac{3\pi}{2}\right)$$
$$+ 6 . 4\, e^{2x}\, 2^2 \sin\left(2x + \frac{2\pi}{2}\right) + 4.8\, e^{2x} . 2 \sin\left(2x + \frac{\pi}{2}\right)$$
$$+ 1.16\, e^{2x} \sin 2x$$
$$= 16\, e^{2x} \sin 2x - 64\, e^{2x} \cos 2x - 96\, e^{2x} \sin 2x$$
$$+ 64\, e^{2x} \cos 2x + 16\, e^{2x} \sin 2x$$
$$= e^{2x} \sin 2x (-64) = -64\, e^{2x} \sin 2x.$$

Problem 3: *Find the n^{th} differential coefficients of:*

(i) $x^2 e^{-x}$. *(ii)* $x^3 \log x$. *(iii)* $e^x \log x$. *(iv)* $x^2 \tan^{-1} x$.

Solution: (i) Let $y = x^2 e^{-x}$

Let $u = x^2, v = e^{-x}$

$$D^n (e^{-x}) = (-1)^n e^x$$

$\therefore$ $D^n (y) = D^n (x^2 e^{-x})$

$$y_n = x^2 D^n (e^{-x}) + n (Dx^2)(D^{n-1} e^{-x})$$
$$+ \frac{n(n-1)}{2!} (D^2 x^2)(D^{n-2} e^{-x})$$
$$y_n = x^2 (-1)^n e^x + n\, 2x\, (-1)^{n-1} e^x + \frac{n(n-1)}{2!} . 2(-1)^{n-1} e^x$$
$$= (-1)^n e^x [x^2 - 2nx + n(n-1)].$$

(ii) By Leibnitz's theorem, we have

$$D^n(uv) = (D^n u).v + {}^nC_1(D^{n-1}u)(Dv) + {}^nC_2(D^{n-2}u)(D^2 v) + \dots$$

Taking $u = \log x$ and $v = x^3$, we find that

$$D^n u = (-1)^{n-1}(n-1)!/x^n,$$

$$D^{n-1} u = (-1)^{n-2}(n-2)!/x^{n-1} \quad \text{and} \quad Dv = 3x^2,$$

$$D^{n-2} u = (-1)^{n-3}(n-3)!/x^{n-2} \quad \text{and} \quad D^2 v = 6x,$$

$$D^{n-3} u = (-1)^{n-4}(n-4)!/x^{n-3} \quad \text{and} \quad D^3 v = 6.$$

Therefore by Leibnitz's theorem, we have

$$D^n(x^3 \log x) = \frac{(-1)^{n-1}(n-1)!}{x^n} x^3 + {}^nC_1 \frac{(-1)^{n-2}(n-2)!}{x^{n-1}} 3x^2$$

$$+ {}^nC_2 \frac{(-1)^{n-3}(n-3)!}{x^{n-2}} 6x + {}^nC_3 \frac{(-1)^{n-4}(n-4)!}{x^{n-3}} 6,$$

since all other terms become zero.

$$= \frac{(-1)^{n-1}(n-4)!}{x^{n-3}}[(n-1)(n-2)(n-3) - 3n(n-2)(n-3)$$

$$+ 3n(n-1)(n-3) - n(n-1)(n-2)]$$

$$= (-1)^{n-1}(n-4)!\, x^{3-n}[(n-1)(n-2)(n-3-n)$$

$$+ 3n(n-3)(n-1-n+2)]$$

$$= 6(-1)^n (n-4)!\, x^{3-n}.$$

(iii) By Leibnitz's theorem, we have

$$D^n(e^x \log x) = (D^n e^x).\log x + {}^nC_1(D^{n-1}e^x).(D \log x)$$

$$+ {}^nC_2(D^{n-2}e^x)(D^2 \log x) + \dots + e^x D^n \log x$$

$$= e^x.\log x + {}^nC_1 e^x.\left(\frac{1}{x}\right) + {}^nC_2 e^x\left(-\frac{1}{x^2}\right) + {}^nC_3 e^x.\frac{2!}{x^3} + \dots$$

$$+ \dots + e^x(-1)^{n-1}.(n-1)!.\,x^{-n}$$

$$= e^x[\log x + {}^nC_1 x^{-1} - {}^nC_2 x^{-2} + {}^nC_3 2!\,x^{-3} - \dots$$

$$+ (-1)^{n-1}.(n-1)!\,x^{-n}].$$

(iv) Let $u = \tan^{-1} x$ and $v = x^2$. Then $y = uv$.

Differentiating n times by Leibnitz's theorem, we get

$$y_n = D^n(uv) = (D^n u).v + {}^nC_1(D^{n-1}u).Dv$$

$$+ {}^nC_2(D^{n-2}u).(D^2 v) + \dots$$

Now $\qquad D^n u = D^n(\tan^{-1} x) = (-1)^{n-1}(n-1)!\sin^n\theta \sin n\theta,$

where $\qquad \theta = \tan^{-1}(1/x).$

Also $Dv = Dx^2 = 2x$, $D^2 v = 2$ = constant and so the third and the higher derivatives of v all vanish .

Hence $y_n = [(-1)^{n-1}(n-1)!\sin^n\theta\sin n\theta].x^2$

$$+ {}^nC_1[(-1)^{n-2}(n-2)!\sin^{n-1}\theta\sin(n-1)\theta].2x$$

$$+ {}^nC_2[(-1)^{n-3}(n-3)!\sin^{n-2}\theta\sin(n-2)\theta].2$$

$$=(-1)^{n-1}(n-3)![(n-1)(n-2)x^2\sin^n\theta\sin n\theta$$

$$-2nx(n-2)\sin^{n-1}\theta\sin(n-1)\theta$$

$$+n(n-1)\sin^{n-2}\theta\sin(n-2)\theta], \text{ where } \theta=\tan^{-1}(1/x).$$

Problem 4: *If* $y = x^2 e^x$, *prove that*

$$y_n = \frac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \frac{1}{2}(n-1)(n-2)y.$$

(Bundelkhand 2008)

Solution: We have $y = x^2 e^x$. ...(1)

Differentiating (1) n times by Leibnitz's theorem , we get

$$y_n = e^x.x^2 + {}^nC_1 e^x.2x + {}^nC_2 e^x.2$$

$$= x^2e^x + 2nxe^x + n(n-1)e^x. \quad ...(2)$$

Also differentiating (1) only once w.r.t. x, we have

$$y_1 = e^x.x^2 + 2xe^x$$

or $y_1 = y + 2xe^x$. ...(3)

Now differentiating (3) w.r.t. x, we have

$$y = y_1 + 2xe^x + 2e^x = y_1 + (y_1 - y) + 2e^x,$$

[$\because$ from (3), $y_1 - y = 2xe^x$]

or $y_2 = 2y_1 - y + 2e^x$. ...(4)

Hence substituting for x^2e^x, $2xe^x$ and e^x respectively from (1), (3) and (4) in (2), we get

$$y_n = y + n(y_1 - y) + \frac{1}{2}n(n-1)(y_2 - 2y_1 + y)$$

i.e., $y_n = \frac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \frac{1}{2}(n-1)(n-2)y.$

Problem 5: *Prove that the* n^{th} *differential coefficient of* $x^n(1-x)^n$ *is equal to*

$$n!(1-x)^n\left\{1 - \frac{n^2}{1^2}\frac{x}{1-x} + \frac{n^2(n-1)^2}{1^2.2^2}\frac{x^2}{(1-x)^2} - \ldots\right\}.$$

(Rohilkhand 2007; Kanpur 08)

Solution: Let $y = x^n(1-x)^n$ ...(1)

Differentiating (1), n times by Leibnitz's theorem, we get

$$y_n = (D^n x^n)(1-x)^n + {}^nC_1 (D^{n-1} x^n) . D(1-x)^n + {}^nC_2 (D^{n-2} x^n) . D^2 (1-x)^n + ...$$

or $$= n!(1-x)^n + n \frac{n!}{1!} x . n (1-x)^{n-1}(-1) + \frac{n(n-1)}{2!} . \frac{n!}{2!} x^2 . n(n-1)(1-x)^{n-2}(-1)^2 + ...$$

or $$y_n = n!(1-x)^n \left\{1 - \frac{n^2}{1^2} . \frac{x}{(1-x)} + \frac{n^2 (n-1)^2}{1^2 . 2^2} \frac{x^2}{(1-x)^2} - ......\right\}.$$

Problem 6: *Prove that* $\dfrac{d^n}{dx^n}\left(\dfrac{\log x}{x}\right) = \dfrac{(-1)^n . (n)!}{x^{n+1}}\left[\log x - 1 - \dfrac{1}{2} - \dfrac{1}{3} - \ldots - \dfrac{1}{n}\right].$

Solution: We have $y = (1/x) . \log x = (x^{-1}) . \log x.$ ...(1)

Differentiating (1) n times by Leibnitz's theorem taking x^{-1} as first function, we have

$$y_n = \frac{(-1)^n n!}{x^{n+1}} \log x + {}^nC_1 \frac{(-1)^{n-1}(n-1)!}{x^n} \cdot \frac{1}{x} + {}^nC_2 \frac{(-1)^{n-2}(n-2)!}{x^{n-1}} \cdot (-1) \cdot \frac{1}{x^2} + {}^nC_3 \frac{(-1)^{n-3}(n-3)!}{x^{n-2}} \cdot (-1)(-2) \cdot \frac{1}{x^3} + \ldots + \frac{1}{x} \cdot \frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$= \frac{(-1)^n n!}{x^{n+1}} \log x + n \cdot \frac{(-1)^{n-1}(n-1)!}{x^{n+1}} + \frac{n(n-1)}{1 . 2} \cdot \frac{(-1)^{n-1}(n-2)!}{x^{n+1}} + \frac{n(n-1)(n-2)}{1 . 2 . 3} \cdot \frac{(-1)^{n-1}(n-3)!}{x^{n+1}} . 1 . 2 + \ldots + \frac{(-1)^{n-1}(n-1)!}{x^{n+1}}$$

$$= \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \ldots - \frac{1}{n}\right].$$

Note: Solve this question also by differentiating (1) n times by Leibnitz's theorem taking $\log x$ as first function.

Problem 7: *If* $y = x^n \log x$, *show that* $y_{n+1} = n!/x.$

(Meerut 2001; Bundelkhand 09; Rohilkhand 11B)

Solution: We have $y = x^n \log x.$

$\therefore$ $y_1 = x^n (1/x) + nx^{n-1} \log x$ or $xy_1 = x^n + nx^n \log x$

or $xy_1 = x^n + ny.$

Now differentiating both sides n times and using Leibnitz's theorem, we get

$$D^n (y_1 x) = D^n x^n + n D^n y$$

or $$(D^n y_1) . x + {}^nC_1 (D^{n-1} y_1) . (Dx) = n! + n y_n$$

or $\quad y_{n+1} \,.\, x + n y_n \,.\, 1 = n\,! + n y_n \quad$ or $\quad y_{n+1} \,.\, x = n\,!.$

Therefore $\quad y_{n+1} = n\,!/\,x.$

Problem 8: *By forming in two different ways the n^{th} derivative of x^{2n}, prove that*

$$1 + \frac{n^2}{1^2} + \frac{n^2\,(n-1)^2}{1^2 \,.\, 2^2} + \frac{n^2\,(n-1)^2\,(n-2)^2}{1^2 \,.\, 2^2 \,.\, 3^2} + \ldots = \frac{(2n)\,!}{(n\,!)^2} \cdot$$

Solution: Let $y = x^{2n}$, then

$$y_n = \frac{(2n)\,!}{(2n-n)\,!} \cdot x^{2n-n} = \frac{(2n)\,!}{n\,!} \cdot x^n. \qquad \ldots(1)$$

Again, $\quad y = x^n \,.\, x^n. \qquad \ldots(2)$

Differentiating (2), n times by Leibnitz's theorem, we get

$$y_n = n\,!\,x^n + {}^nC_1\,\frac{n\,!}{1\,!}\,x \,.\, n x^{n-1} + {}^nC_2\,\frac{n\,!}{2\,!}\,x^2 \,.\, n\,(n-1)\,x^{n-2}$$

$$+ \,{}^nC_3\,\frac{n\,!}{3\,!}\,x^3 \,.\, n\,(n-1)\,(n-2)\,x^{n-3} + \ldots + {}^nC_n\,x^n \,.\, n\,!$$

$$= n\,!\,x^n \left[1 + \frac{n^2}{1^2} + \frac{n^2\,(n-1)^2}{1^2 \,.\, 2^2} + \frac{n^2\,(n-1)^2\,(n-2)^2}{1^2 \,.\, 2^2 \,.\, 3^2} + \ldots \right] \qquad \ldots(3)$$

Equating the two values of y_n obtained in (1) and (3), we get

$$1 + \frac{n^2}{1^2} + \frac{n^2\,(n-1)^2}{1^2 \,.\, 2^2} + \frac{n^2\,(n-1)^2\,(n-2)^2}{1^2 \,.\, 2^2 \,.\, 3^2} + \ldots = \frac{(2n)\,!}{(n\,!)^2} \cdot$$

Problem 9: *Prove that*

$$D^n \left(\frac{\sin\,x}{x} \right) = \{P\,\sin\,(x + \tfrac{1}{2}\,n\pi) + Q\,\cos\,(x + \tfrac{1}{2}\,n\pi)\}\,/\,x^{n+1},$$

where $\quad P = x^n - n\,(n-1)\,x^{n-2} + n\,(n-1)\,(n-2)\,(n-3)\,x^{n-4} - \ldots,$

and $\quad Q = n x^{n-1} - n\,(n-1)\,(n-2)\,x^{n-3} + \ldots.$

Solution: Differentiating n times by Leibnitz's theorem taking $\sin x$ as first function and x^{-1} as second function, we get

$$D^n\,[(\sin x) . x^{-1}] = [\sin(x + \tfrac{1}{2}\,n\pi)] \,.\, (x^{-1})$$

$$+ \,{}^nC_1\,[\sin\,\{x + \tfrac{1}{2}\,(n-1)\,\pi\}]\,[(-1)\,x^{-2}]$$

$$+ \,{}^nC_2\,[\sin\,\{x + \tfrac{1}{2}\,(n-2)\,\pi\}] \,.\, [(-1)\,(-2)\,x^{-3}]$$

$$+ \,{}^nC_3\,[\sin\,\{x + \tfrac{1}{2}\,(n-3)\,\pi\}] \,.\, [(-1)\,(-2)\,(-3)\,x^{-4}]$$

$$+ \,{}^nC_4\,[\sin\,\{x + \tfrac{1}{2}\,(n-4)\,\pi\}] . [(-1)\,(-2)\,(-3)\,(-4)\,x^{-5}] + \ldots$$

$$= \frac{1}{x^n + 1} [x^n \sin (x + \tfrac{1}{2} n\pi) + n x^{n-1} \cos (x + \tfrac{1}{2} n\pi)$$

$$- n(n-1) x^{n-2} \sin (x + \tfrac{1}{2} n\pi)$$

$$- n(n-1)(n-2) x^{n-3} \cos (x + \tfrac{1}{2} n\pi)$$

$$+ n(n-1)(n-2)(n-3) x^{n-4} \sin (x + \tfrac{1}{2} n\pi) + \ldots]$$

$$= \frac{1}{x^{n+1}} [\{x^n - n(n-1) x^{n-2}$$

$$+ n(n-1)(n-2)(n-3) x^{n-4} - \ldots\} \sin (x + \tfrac{1}{2} n\pi)$$

$$+ \{n x^{n-1} - n(n-1)(n-2) x^{n-3} + \ldots\} \cos (x + \tfrac{1}{2} n\pi)]$$

$$= \frac{1}{x^{n+1}} [P \sin (x + \tfrac{1}{2} n\pi) + Q \cos (x + \tfrac{1}{2} n\pi)].$$

Problem 10: *If $y = e^{\tan^{-1} x}$, prove that*

$$(1 + x^2) y_{n+2} + [2(n+1)x - 1] y_{n+1} + n(n+1) y_n = 0.$$

(Avadh 2010; Kanpur 14)

Solution: We have $y = e^{\tan^{-1} x}$.

Therefore $\quad y_1 = e^{\tan^{-1} x} \cdot \dfrac{1}{(1 + x^2)} = \dfrac{y}{(1 + x^2)}$

or $\quad y_1 (1 + x^2) - y = 0.$ ...(1)

Differentiating (1) $(n + 1)$ times by Leibnitz's theorem, we get

$$y_{n+2} (1 + x^2) + {}^{n+1}C_1 \, y_{n+1} . 2x + {}^{n+1}C_2 \, y_n . 2 - y_{n+1} = 0$$

or $\quad (1 + x^2) y_{n+2} + [2(n+1)x - 1] y_{n+1} + (n+1) n y_n = 0.$

Problem 11: *If $y = \cos(\log x)$, prove that*

$$x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2 + 1) y_n = 0.$$

Solution: Proceed as in Example 13.

Problem 12: *If $y = (\sin^{-1} x)^2$, prove that $(1 - x^2) y_2 - x y_1 - 2 = 0$,*

and $(1 - x^2) y_{n+2} - x(2n+1) y_{n+1} - n^2 y_n = 0.$

(Meerut 2002; Agra 08; Rohilkhand 09B)

Solution: We have $y = (\sin^{-1} x)^2$.

Therefore $\quad y_1 = 2(\sin^{-1} x)/\sqrt{(1 - x^2)},$

or $$y_1^2 = \frac{4(\sin^{-1} x)^2}{1-x^2} = \frac{4y}{1-x^2}, \qquad [\because (\sin^{-1} x)^2 = y]$$

or $$y_1^2 (1-x^2) - 4y = 0.$$

Differentiating again, we get

$$2y_1 y_2 (1-x^2) - 2xy_1^2 - 4y_1 = 0$$

or $$2y_1 [y_2 (1-x^2) - xy_1 - 2] = 0.$$

Cancelling $2y_1$, since $2y_1 \neq 0$, we get $y_2 (1-x^2) - y_1 x - 2 = 0$, ...(1)

proving the first result.

Differentiating (1) n times by Leibnitz's theorem, we get

$$y_{n+2}(1-x^2) + {}^nC_1 y_{n+1}.(-2x) + {}^nC_2\, y_n.(-2) - y_{n+1}.x - {}^nC_1\, y_n.1 = 0$$

or $$(1-x^2)\, y_{n+2} - (2n+1)\, xy_{n+1} - n^2 y_n = 0.$$

Problem 13: *If* $y = (x^2-1)^n$, *prove that*

$$(x^2-1)\, y_{n+2} + 2xy_{n+1} - n(n+1)\, y_n = 0.$$

(Kumaun 2003; Rohilkhand 06,11B; Meerut 08; Kashi 2013)

Hence if $P_n = \frac{d^n}{dx^n}(x^2-1)^n$, *show that* $\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} + n(n+1)P_n = 0.$

Solution: We have $y = (x^2-1)^n$. Therefore $y_1 = n(x^2-1)^{n-1}.2x$

or $$(x^2-1)\, y_1 = n(x^2-1)^n.2x = 2nxy \qquad [\text{Replacing } (x^2-1)^n \text{ by } y]$$

or $$(x^2-1)y_1 - 2nxy = 0. \qquad ...(1)$$

Differentiating (1), $(n+1)$ times by Leibnitz's theorem, we have

$$D^{n+1}[y_1(x^2-1)] - 2n\, D^{n+1}(yx) = 0$$

or $$y_{n+2}(x^2-1) + (n+1)\, y_{n+1}.2x + \frac{(n+1)n}{2!}\cdot y_n.2 - 2ny_{n+1}.x - 2n(n+1)\, y_n.1 = 0$$

or $$(x^2-1)\, y_{n+2} + 2xy_{n+1} - n(n+1)\, y_n = 0, \qquad ...(2)$$

giving the first result. From (2), we get

$$(x^2-1)\, D^2 y_n + 2x\, Dy_n - n(n+1)\, y_n = 0. \qquad ...(3)$$

Putting $y_n = \frac{d^n}{dx^n}(x^2-1)^n = P_n$, (3) becomes

$$(x^2-1)\, D^2P_n + 2x\, D(P_n) - n(n+1)\, P_n = 0$$

or $\quad -(1-x^2)D^2(P_n)+2xD(P_n)-n(n+1)P_n=0$

or $\quad -\frac{d}{dx}\{(1-x^2)D(P_n)\}-n(n+1)P_n=0$

or $\quad \frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P_n\right\}+n(n+1)P_n=0.$

Problem 14: *If* $\cos^{-1}(y/b)=\log(x/n)^n$, *prove that*

$$x^2y_{n+2}+(2n+1)xy_{n+1}+2n^2y_n=0.$$

(Lucknow 2006, 07; Meerut 06B; Rohilkhand 13; Purvanchal 14)

Solution: Here $\cos^{-1}(y/b)=\log(x/n)^n=n\log(x/n)=n(\log x-\log n)$.

Differentiating both sides with respect to x, we have

$$-\frac{1}{\sqrt{\{1-(y^2/b^2)\}}}\frac{y_1}{b}=\frac{n}{x}$$

or $\quad -\frac{y_1}{\sqrt{(b^2-y^2)}}=\frac{n}{x}$

or $\quad {y_1}^2x^2=n^2(b^2-y^2).$

Differentiating again, we get

$$2y_1y_2x^2+2x{y_1}^2=-2n^2yy_1$$

or $\quad y_2x^2+y_1x+n^2y=0$, since $2y_1\neq 0$. ...(1)

Differentiating (1), n times by Leibnitz's theorem, we get

$$y_{n+2}.x^2+{}^nC_1\,y_{n+1}.(2x)+{}^nC_2\,y_n.(2)+y_{n+1}.x$$
$$+{}^nC_1\,y_n+n^2y_n=0$$

or $\quad x^2y_{n+2}+(2n+1)xy_{n+1}+2n^2y_n=0.$

Problem 15: *If* $y=[x+\sqrt{(1+x^2)}]^m$, *prove that* $(1+x^2)y_2+xy_1-m^2y=0$

and $\quad (1+x^2)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0.$

(Kanpur 2006; Avadh 09,11; Purvanchal 09; Bundelkhand 14)

Solution: Here $\quad y=[x+\sqrt{(1+x^2)}]^m.$ ...(1)

$\therefore \quad y_1=m[x+\sqrt{(1+x^2)}]^{m-1}.\left[1+\frac{1}{2}\cdot\frac{2x}{\sqrt{(1+x^2)}}\right]$

$$=\frac{m}{\sqrt{(1+x^2)}}[x+\sqrt{(1+x^2)}]^m=\frac{my}{\sqrt{(1+x^2)}} \quad ...(2)$$

or $\quad {y_1}^2(1+x^2)-m^2y^2=0.$

Differentiating again, we get

$$2y_1y_2(1+x^2)+2x{y_1}^2-2m^2yy_1=0$$

or $$y_2(1+x^2)+xy_1-m^2y=0, \qquad ...(3)$$

cancelling $2y_1$, since $2y_1\neq 0$.

Again differentiating (3) n times, we get

$$y_{n+2}(1+x^2)+{}^nC_1.2xy_{n+1}+{}^nC_2.2y_n+xy_{n+1}$$
$$+{}^nC_1.y_n-m^2y_n=0$$

or $$y_{n+2}(1+x^2)+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0. \qquad ...(4)$$

Problem 16: *If $y=[\log\{x+\sqrt{(1+x^2)}\}]^2$, prove that*

$$(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+n^2y_n=0.$$

(Agra 2005; Purvanchal 09)

Solution: Here $y=[\log\{x+\sqrt{(1+x^2)}\}]^2$. ...(1)

$\therefore$ $$y_1=[2\log\{x+\sqrt{(1+x^2)}\}]\cdot\frac{1}{x+\sqrt{(1+x^2)}}\cdot\left[1+\frac{1}{2}\cdot\frac{1}{\sqrt{(1+x^2)}}.2x\right]$$
$$=\frac{2}{\sqrt{(1+x^2)}}\log\{x+\sqrt{(1+x^2)}\}. \qquad ...(2)$$

Squaring both sides of (2), we get

$$(1+x^2)y_1^2=4[\log\{x+\sqrt{(1+x^2)}\}]^2=4y \qquad \text{[From (1)]}$$

or $$(1+x^2)y_1^2-4y=0.$$

Differentiating again, we get

$$2y_1y_2(1+x^2)+2xy_1^2-4y_1=0$$

or $$2y_1[y_2(1+x^2)+xy_1-2]=0$$

or $$y_2(1+x^2)+xy_1-2=0, \qquad ...(3)$$

since $y_1\neq 0$.

Differentiating (3) n times by Leibnitz's theorem, we get

$$y_{n+2}(1+x^2)+{}^nC_1.y_{n+1}.2x+{}^nC_2\,y_n.2$$
$$+xy_{n+1}+{}^nC_1\,y_n.1=0$$

or $$(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+n^2y_n=0. \qquad ...(4)$$

Problem 17: *If $y=(\sin^{-1}x)/\sqrt{(1-x^2)}$, prove that*

$$(1-x^2)y_{n+1}-(2n+1)xy_n-n^2y_{n-1}=0.$$

(Meerut 2007B; Kanpur 10)

Solution: We have $y\sqrt{(1-x^2)}=\sin^{-1}x$.

Differentiating both sides, we get

$$y_1 \sqrt{(1-x^2)} + y \cdot \frac{1}{2} \frac{-2x}{\sqrt{(1-x^2)}} = \frac{1}{\sqrt{(1-x^2)}}$$

or $$y_1 (1-x^2) - yx - 1 = 0. \quad \text{...(1)}$$

Differentiating (1) n times by Leibnitz's theorem, we get

$$y_{n+1} (1-x^2) + {}^nC_1\, y_n (-2x) + {}^nC_2\, y_{n-1} (-2) - y_n \,.\, x - {}^nC_1\, y_{n-1} \,.\, 1 = 0$$

or $$(1-x^2)\, y_{n+1} - (2n+1)\, xy_n - n^2 y_{n-1} = 0.$$

Comprehensive Problems 4

Problem 1: *If $y = \sin^{-1} x$, prove that*

$$(1-x^2) y_{n+2} - (2n+1)\, x y_{n+1} - n^2 y_n = 0,$$

and hence find the value of $(y_n)_0$. **(Agra 2005; Lucknow 05; Bundelkhand 11)**

Solution: We have $$y = \sin^{-1} x. \quad \text{...(1)}$$

$\therefore$ $$y_1 = \frac{1}{\sqrt{(1-x^2)}} \quad \text{...(2)}$$

or $$(1-x^2)\, y_1^2 - 1 = 0. \quad \text{...(3)}$$

Differentiating (3) w.r.t. x, we get

$$(1-x^2)\, 2 y_1 y_2 - 2x y_1^2 = 0 \quad \text{or} \quad 2 y_1 [(1-x^2)\, y_2 - x y_1] = 0.$$

Cancelling $2y_1$, since $2y_1$ is not identically equal to zero, we get

$$(1-x^2)\, y_2 - x y_1 = 0. \quad \text{...(4)}$$

Differentiating (4), n times by Leibnitz's theorem, we get

$$y_{n+2} (1-x^2) + {}^nC_1 y_{n+1} \,.\, (-2x) + {}^nC_2\, y_n \,.\, (-2) - y_{n+1} \,.\, x - {}^nC_1 y_n \,.\, 1 = 0$$

or $$y_{n+2} (1-x^2) - (2n+1)\, x y_{n+1} - n^2 y_n = 0. \quad \text{...(5)}$$

Putting $x = 0$ in (1), (2) and (4), we get

$$(y)_0 = 0,\ (y_1)_0 = 1 \quad \text{and} \quad (y_2)_0 - 0\, (y_1)_0 = 0 \ \textit{i.e.},\ (y_2)_0 = 0.$$

Also putting $x = 0$ in (5), we get

$$(y_{n+2})_0 = n^2\, (y_n)_0. \quad \text{...(6)}$$

(6) is a reduction formula which expresses $(y_{n+2})_0$ in terms of $(y_n)_0$.

Putting $n-2$ in place of n in (6), we get

$$(y_n)_0 = (n-2)^2\, (y_{n-2})_0 = (n-2)^2\, (n-4)^2\, (y_{n-4})_0.$$

[$\because$ From (6), $(y_{n-2})_0 = (n-4)^2\, (y_{n-4})_0$]

Now there arise two cases.

Case I: When *n* is odd. Putting $n = 1, 3, 5, \ldots$ in (6), we have

$$(y_3)_0 = 1^2 (y_1)_0 = 1^2 . 1, \qquad [\because (y_1)_0 = 1]$$

$$(y_5)_0 = 3^2 (y_3)_0 = 3^2 . 1^2 . 1,$$

$$(y_7)_0 = 5^2 (y_5)_0 = 5^2 . 3^2 . 1^2 . 1, \text{ and so on.}$$

Thus if n is odd, we have

$$(y_n)_0 = (n-2)^2 (n-4)^2 \ldots 5^2 . 3^2 . 1^2 . 1 .$$

Case II: When *n* is even. Putting $n = 2, 4, 6, \ldots$ in (6), we have

$$(y_4)_0 = 2^2 (y_2)_0 = 0, \qquad [\because (y_2)_0 = 0]$$

$$(y_6)_0 = 4^2 (y_4)_0 = 4^2 . 0 = 0, (y_8)_0 = 6^2 (y_6)_0 = 0, \text{ and so on.}$$

Thus if n is even, we have $(y_n)_0 = 0$.

Problem 2: *Find* $(y_n)_0$, *when* $y = \log [x + \sqrt{(1 + x^2)}]$. **(Agra 2005; Rohilkhand 06)**

Solution: Here $y = \log [x + \sqrt{(1 + x^2)}]$. ...(1)

$$\therefore \quad y_1 = \frac{1}{x + \sqrt{(1+x^2)}} \cdot \left[1 + \frac{1}{2} \cdot \frac{1}{\sqrt{(1+x^2)}} \cdot 2x \right] = \frac{1}{\sqrt{(1+x^2)}} \qquad \text{...(2)}$$

Squaring both sides of (2), we get $(1 + x^2) y_1^2 - 1 = 0$.

Differentiating again, we get

$$2 y_1 y_2 (1 + x^2) + 2xy_1^2 = 0 \quad \text{or} \quad 2 y_1 [y_2 (1 + x^2) + xy_1] = 0$$

or $\quad y_2 (1 + x^2) + x y_1 = 0,$ ...(3)

since $\quad y_1 \neq 0$.

Differentiating (3), n times by Leibnitz's theorem, we get

$$y_{n+2} (1 + x^2) + {}^nC_1 . y_{n+1} 2x + {}^nC_2 . y_n . 2 + y_{n+1} . x + {}^nC_1 . y_n . 1 = 0$$

or $\quad (1 + x^2) y_{n+2} + (2n+1) xy_{n+1} + n^2 y_n = 0.$ ...(4)

Putting $x = 0$ in (1), (2), (3) and (4), we have

$$(y)_0 = 0, (y_1)_0 = 1, (y_2)_0 = 0 \quad \text{and} \quad (y_{n+2})_0 + n^2 (y_n)_0 = 0$$

i.e., $\quad (y_{n+2})_0 = - n^2 (y_n)_0 .$ ...(5)

Putting $n - 2$ in place of n on both sides of (5), we get

$$(y_n)_0 = -(n-2)^2 (y_{n-2})_0 = [-(n-2)^2][-(n-4)^2](y_{n-4})_0,$$

since from (5), we have $(y_{n-2})_0 = -(n-4)^2 (y_{n-4})_0$.

Now there arise two cases :

Case I: When n is odd, we have

$$(y_n)_0 = [-(n-2)^2][-(n-4)^2][-(n-6)^2][-3^2][-1^2](y_1)_0$$
$$= (-1)^{(n-1)/2}(n-2)^2(n-4)^2 \ldots 3^2 . 1^2, \text{ since } (y_1)_0 = 1.$$

Case II: When n is even, we have

$$(y_n)_0 = [-(n-2)^2][-(n-4)^2]\ldots(y_2)_0 = 0, \text{ since } (y_2)_0 = 0.$$

Problem 3: *If* $y = [\log\{x + \sqrt{(1+x^2)}\}]^2$, *prove that* $(y_{n+2})_0 = -n^2(y_n)_0$, *hence find* $(y_n)_0$.

(Meerut 2005, 09B)

Solution: Here $y = [\log\{x + \sqrt{(1+x^2)}\}]^2$. ...(1)

$$\therefore \quad y_1 = [2\log\{x+\sqrt{(1+x^2)}\}] \cdot \frac{1}{x+\sqrt{(1+x^2)}} \cdot \left[1 + \frac{1}{2} \cdot \frac{1}{\sqrt{(1+x^2)}} . 2x\right]$$
$$= \frac{2}{\sqrt{(1+x^2)}} \log\{x + \sqrt{(1+x^2)}\}. \qquad ...(2)$$

Squaring both sides of (2), we get

$$(1+x^2)y_1^2 = 4[\log\{x+\sqrt{(1+x^2)}\}]^2 = 4y \qquad \text{[From (1)]}$$

or $$(1+x^2)y_1^2 - 4y = 0.$$

Differentiating again, we get

$$2y_1y_2(1+x^2) + 2xy_1^2 - 4y_1 = 0$$

or $$2y_1[y_2(1+x^2) + xy_1 - 2] = 0$$

or $$y_2(1+x^2) + xy_1 - 2 = 0, \qquad ...(3)$$

since $y_1 \neq 0$.

Differentiating (3) n times by Leibnitz's theorem, we get

$$y_{n+2}(1+x^2) + {}^nC_1 . y_{n+1} . 2x + {}^nC_2 \, y_n . 2$$
$$+ xy_{n+1} + {}^nC_1 \, y_n . 1 = 0$$

or $$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0. \qquad ...(4)$$

Putting $x = 0$ in (1), (2), (3) and (4), we get

$$(y)_0 = 0, (y_1)_0 = 0, (y_2)_0 = 2, \quad \text{and} \quad (y_{n+2})_0 + n^2(y_n)_0 = 0$$

i.e., $$(y_{n+2})_0 = -n^2(y_n)_0. \qquad ...(5)$$

Putting $n-2$ in place of n on both sides of (5), we get

$$(y_n)_0 = -(n-2)^2(y_{n-2})_0 = [-(n-2)^2][-(n-4)^2](y_{n-4})_0,$$

since from (5), we have $(y_{n-2})_0 = -(n-4)^2(y_{n-4})_0$.

Now there arise two cases :

Case I: When n is odd, we have

$$(y_n)_0 = [-(n-2)^2][-(n-4)^2]\ldots(y_1)_0 = 0, \text{ since } (y_1)_0 = 0.$$

Case II: When n is even, we have

$$(y_n)_0 = [-(n-2)^2][-(n-4)^2][-(n-6)^2]\ldots[-4^2][-2^2](y_2)_0$$
$$= (-1)^{(n-2)/2}(n-2)^2(n-4)^2(n-6)^2\ldots 4^2.2^2.2.$$

Problem 4: *If $y = (\sinh^{-1} x)^2$, prove that*

$$(1+x^2)\frac{d^{n+2}y}{dx^{n+2}} + (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} + n^2\frac{d^n y}{dx^n} = 0.$$

Hence find, at $x = 0$, the value of $(d^n y/dx^n)$.

Solution: Given $y = (\sinh^{-1}x)^2$ ...(1)

Differentiating both sides w.r.to x, we get

$$y_1 = 2\sinh^{-1}x \cdot \frac{1}{\sqrt{x^2+1}} \quad \text{...(2)}$$

or $\sqrt{1+x^2}\, y_1 = 2\sinh^{-1}x$

$(1+x^2)y_1^2 = 4(\sinh^{-1}x)^2$, squaring both sides

$(1+x^2)y_1^2 = 4y$

Again $(1+x^2)2yy_1 + y_1^2.2x = 4y_1$

Dividing by $2y_1$, we get

$$(1+x^2)y_2 + xy_1 = 2 \quad \text{...(3)}$$

Differentiating w.r.to x, n times (by Leibnitz's theorem)

$$(1+x^2)y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{2!}(2)y_n + xy_{n+1} + ny_n = 0$$

or $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$ ...(4)

or $(1+x^2)\dfrac{d^{n+2}y}{dx^{n+2}} + (2n+1)x\dfrac{d^{n+1}y}{dx^{n+1}} + n^2\dfrac{d^n y}{dx^n} = 0$

Putting $x = 0$ in (1), (2), (3) and (4), we get

$(y)_0 = 0, (y_1)_0 = 0, (y_2)_0 = 2,$

and $(y_{n+2})_0 = -n^2(y_n)_0$ ...(5)

Putting $n = 1, 2, 3$ .... in eqn. (5)

$(y_3)_0 = 0$

$(y_4)_0 = -2^2 \cdot (y_2)_0 = -2^2.2$

$(y_5)_0 = 0$

$$(y_6)_0 = -4^2 (y_4)_0 = -4^2 \{-2^2 \cdot 2\}$$

If *n* is odd: $(y_n)_0 = 0$

If *n* is even: $(y_n)_0 = (-1)^{\frac{n-2}{2}} (n-2)^2 (n-4)^2 .....4^2.2^2.2$

{By putting $n = n - 2$ in eqn. (5)}

Problem 5: *If* $y = \cos(m \sin^{-1} x)$, *find* $(y_n)_0$. **(Kumaun 2002)**

Solution: We have $y = \cos(m \sin^{-1} x)$. ...(1)

Differentiating, we get

$$y_1 = [-\sin(m \sin^{-1} x)] . [m/\sqrt{(1-x^2)}]. \quad ...(2)$$

Squaring both sides of (2) and multiplying by $(1-x^2)$, we get

$$(1-x^2) y_1^2 = m^2 \sin^2(m \sin^{-1} x) = m^2 [1 - \cos^2 (m \sin^{-1} x)]$$
$$= m^2 (1 - y^2).$$

$$\therefore \quad (1-x^2) y_1^2 + m^2 y^2 - m^2 = 0. \quad ...(3)$$

Differentiating (3), we get

$$(1-x^2) 2 y_1 y_2 - 2xy_1^2 + 2m^2 y y_1 = 0$$

or $$2 y_1 [(1-x^2) y_2 - xy_1 + m^2 y] = 0.$$

Cancelling $2y_1$, since $2y_1$ is not identically equal to zero, we get

$$(1-x^2) y_2 - xy_1 + m^2 y = 0. \quad ...(4)$$

Differentiating (4), *n* times by Leibnitz's theorem, we get

$$y_{n+2} (1-x^2) + {}^nC_1 y_{n+1} (-2x) + {}^nC_2 \; y_n (-2) - xy_{n+1}$$
$$- {}^nC_1 y_n + m^2 y_n = 0$$

or $$(1-x^2) y_{n+2} - (2n+1) xy_{n+1} - (n^2 - m^2) y_n = 0. \quad ...(5)$$

Putting $x = 0$ in (1), (2) and (4), we get

$$(y)_0 = \cos 0 = 1, (y_1)_0 = 0, (y_2)_0 + m^2 (y)_0 = 0$$

i.e., $$(y_2)_0 = -m^2 (y)_0 = -m^2.$$

Also putting $x = 0$ in (5), we get $(y_{n+2})_0 = (n^2 - m^2)(y_n)_0$. ...(6)

Putting $n - 2$ in place of n in the reduction formula (6), we get

$$(y_n)_0 = \{(n-2)^2 - m^2\} (y_{n-2})_0$$
$$= \{(n-2)^2 - m^2\} \{(n-4)^2 - m^2\} (y_{n-4})_0,$$ since from (6),

we have $$(y_{n-2})_0 = \{(n-4)^2 - m^2\} (y_{n-4})_0.$$

Now there arise two cases.

Case I: When n is odd, we have

$$(y_n)_0 = \{(n-2)^2 - m^2\}\{(n-4)^2 - m^2\}\dots(3^2 - m^2)(1^2 - m^2)(y_1)_0$$
$$= 0, \text{ since } (y_1)_0 = 0.$$

Case II: When n is even, we have

$$(y_n)_0 = \{(n-2)^2 - m^2\}\{(n-4)^2 - m^2\}\dots(4^2 - m^2)(2^2 - m^2)(y_2)_0$$
$$= -\{(n-2)^2 - m^2\}\{(n-4)^2 - m^2\}\dots(4^2 - m^2)(2^2 - m^2)m^2,$$

since $(y_2)_0 = -m^2$.

Problem 6: *If* $x = \sin\left(\frac{1}{a}\log y\right)$ *or if* $y = e^{a\sin^{-1}x}$, *show that*

$$(1-x^2)y_2 - xy_1 - a^2y = 0, \qquad \textbf{(Bundelkhand 2007)}$$
$$(1-x^2)y_{n+2} - x(2n+1)y_{n+1} - (n^2 + a^2)y_n = 0,$$

and hence find the value of $(y_n)_0$.

(Garhwal 2000, 01; Gorakhpur 05; Rohilkhand 05, 08; Agra 06, 08)

Solution: Proceeding as in Example 13, we have

$$y = e^{a\sin^{-1}x} \qquad \dots(1)$$
$$y_1 = e^{a\sin^{-1}x}\,.\,a/\sqrt{(1-x^2)}, \qquad \dots(2)$$
$$(1-x^2)y_2 - xy_1 - a^2y = 0, \qquad \dots(3)$$

and
$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + a^2)y_n = 0. \qquad \dots(4)$$

Putting $x = 0$ in (1), (2), (3) and (4), we have $(y)_0 = 1, (y_1)_0 = a, (y_2)_0 = a^2$,

and
$$(y_{n+2})_0 = (n^2 + a^2)(y_n)_0. \qquad \dots(5)$$

Now there arise two cases.

Case I: When n is odd.

Putting $n = 1, 3, 5, \dots$ in (5), we have

$$(y_3)_0 = (1^2 + a^2)(y_1)_0 = (1^2 + a^2)\,.\,a,$$
$$(y_5)_0 = (3^2 + a^2)(y_3)_0 = (3^2 + a^2)(1^2 + a^2)\,.\,a,$$
$$(y_7)_0 = (5^2 + a^2)(y_5)_0 = (5^2 + a^2)(3^2 + a^2)(1^2 + a^2)\,.\,a, \text{and so on.}$$

Thus if n is odd, we have

$$(y_n)_0 = [(n-2)^2 + a^2]\dots(3^2 + a^2)(1^2 + a^2)\,a.$$

Case II: When n is even.

Putting $n = 2, 4, 6, \dots\dots$ in (5), we have

$$(y_4)_0 = (2^2 + a^2)(y_2)_0 = (2^2 + a^2)\,.\,a^2,$$
$$(y_6)_0 = (4^2 + a^2)(y_4)_0 = (4^2 + a^2)(2^2 + a^2)\,.\,a^2,$$

$$(y_8)_0 = (6^2 + a^2)(y_6)_0 = (6^2 + a^2)(4^2 + a^2)(2^2 + a^2)\,a^2, \text{ and so on.}$$

Thus if n is even, we have $(y_n)_0 = [(n-2)^2 + a^2]\ldots(4^2 + a^2)(2^2 + a^2)\,a^2$.

Problem 7: *If $y = e^{a\cos^{-1}x}$, prove that*

$$(1 - x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (n^2 + a^2)\,y_n = 0.$$

Hence find the value of y_n for $x = 0$. **(Meerut 2001; Purvanchal 14)**

Solution: Given $y = e^{a\cos^{-1}x}$ ...(1)

Differentiating $y_1 = e^{a\cos^{-1}x}\left\{\dfrac{-a}{\sqrt{1-x^2}}\right\} = \dfrac{-ay}{\sqrt{1-x^2}}$ {From (1)}

or $y_1^2\,(1 - x^2) = a^2 y^2$, squaring both sides ...(2)

Again differentiating, we get

$$y_1^2\,(-2x) + 2\,y_1 y_2\,(1 - x)^2 = a^2 . 2\,y\,y_1$$

or $(1 - x^2)\,y_2 - xy_1 - a^2 y = 0$ ...(3)

Differentiating (3), n times by Leibnitz's theorem, we get

$$\left[(1 - x^2)\,y_{n+2} + n\,(-2x)\,y_{n+1} + \frac{n(n-1)}{2!}(-2)\,y_n\right] - [xy_{n+1} + n(1)\,y_n] - a^2 y_n = 0$$

or $(1 - x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (n^2 + a^2)\,y_n = 0$ ...(4)

Putting $x = 0$ in (1), (2), (3) and (4), we get

$$(y)_0 = e^{a\pi/2},\ (y_1)_0 = -\,a e^{a\pi/2},\ (y_2)_0 = a^2 (y)_0 = a^2 e^{a\pi/2}$$

$$(y_{n+2})_0 = (a^2 + n^2)(y_n)_0 \qquad \ldots(5)$$

If *n* is odd

Putting $n = 1, 3, 5, \ldots\ldots, (n-2)$ in (5), we get

$$n = 1,\ (y_3)_0 = (1^2 + a^2)(y_1)_0 = -(1^2 + a^2)\,a e^{a\pi/2}$$

$$n = 3,\ (y_5)_0 = (3^2 + a^2)(y_3)_0 = -(3^2 + a^2)(1^2 + a^2)\,a e^{a\pi/2}$$

$$\vdots \qquad \vdots$$

$$(y_n)_0 = -\{(n-2)^2 + a^2\}\{(n-4)^2 + a^2\}\ldots(3^2 + a^2)(1^2 + a^2\ e^{a\pi/2})$$

If *n* is even

Putting $n = 2, 4, 6, \ldots, (n-2)$ in (5), we get

$$n = 2,\ (y_4)_0 = (2^2 + a^2)(y_2)_0 = (2^2 + a^2)\,a^2 e^{a\pi/2}$$

$$n = 4,\ (y_6)_0 = (4^2 + a^2)(y_4)_0 = (4^2 + a^2)(2^2 + a^2)\,a^2 e^{a\pi/2}$$

$$\vdots \qquad \vdots$$

$$(y_n)_0 = \{(n-2)^2 + a^2\}\{(n-4)^2 + a^2\}\ldots(2^2 + a^2)\,a^2 e^{a\pi/2}$$

Problem 8: *If $y = \tan^{-1} x$, prove that $(1+x^2) y_2 + 2xy_1 = 0$, and hence find the value of all the derivatives of y with respect to x, when $x = 0$. Also show that $(y_n)_0$ is 0, $(n-1)!$ or $-(n-1)!$ according as n is of the form $2p, 4p+1$ or $4p+3$ respectively.*

Solution: We have $y = \tan^{-1} x$. ...(1)

$\therefore \quad y_1 = 1/(1+x^2),$ ...(2)

or $\quad (1+x^2) y_1 - 1 = 0.$ ...(3)

Differentiating (3), we get

$(1+x^2) y_2 + 2xy_1 = 0.$ ...(4)

Differentiating (4), n times by Leibnitz's theorem, we get

$$y_{n+2}(1+x^2) + n y_{n+1}(2x) + \frac{n(n-1)}{2!} y_n . 2 + 2xy_{n+1} + 2n y_n = 0$$

or $\quad (1+x^2) y_{n+2} + 2(n+1) xy_{n+1} + n(n+1) y_n = 0.$ ...(5)

Putting $x = 0$ in (1), (2) and (4), we get $(y)_0 = 0, (y_1)_0 = 1, (y_2)_0 = 0$.

Also putting $x = 0$ in (5), we get $(y_{n+2})_0 = -\{(n+1)n\}(y_n)_0$. ...(6)

Putting $n-2$ in place of n in the reduction formula (6), we get

$$(y_n)_0 = -\{(n-1)(n-2)\}(y_{n-2})_0$$
$$= [-\{(n-1)(n-2)\}][-\{(n-3)(n-4)\}](y_{n-4})_0,$$

since from (6), we have

$$(y_{n-2})_0 = -\{(n-3)(n-4)\}(y_{n-4})_0.$$

Now there arise two cases.

Case I: When *n* is even, we have

$$(y_n)_0 = [-\{(n-1)(n-2)\}][-\{(n-3)(n-4)\}] \ldots [-(3)(2)](y_2)_0 = 0, \text{ since } (y_2)_0 = 0.$$

Case II: When *n* is odd, we have

$$(y_n)_0 = [-\{(n-1)(n-2)\}][-\{(n-3)(n-4)\}] \ldots [-(4)(3)][-(2)(1)](y_1)_0$$
$$= (-1)^{(n-1)/2}(n-1)!, \text{ since } (y_1)_0 = 1.$$

Hints to Objective Type Questions

Multiple Choice Questions

1. We have, $y = \log x$.

$\therefore \; y_1 = \dfrac{1}{x} = x^{-1}, y_2 = (-1)x^{-2}, y_3 = (-1)(-2)x^{-3}, y_4 = (-1)(-2)(-3)x^{-4}, \ldots\ldots,$

$y_n = (-1)(-2)(-3)\ldots\ldots\{-(n-1)\}x^{-n} = (-1)^{n-1}(n-1)!\,x^{-n}.$

Hence, $D^n \log x = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$.

So, (b) is correct answer.

2. See Example 3.
3. Leibnitz's theorem is used to find the nth differential coefficient of the product of two functions. So, the correct answer is (c).
4. See Problem 10 of Comprehensive Problems 3.
5. See article 2, part (vii).
6. See article 2, part (vi).
7. See Problem 7 of Comprehensive Problems 3.
8. See article 12 of Chapter 3 (Differentiation).

Fill in the Blank(s)

1. See article 2, part (vii).
2. See article 2, part (iv).
3. Let $y = e^x \sin^2 x = \frac{1}{2}e^x(2\sin^2 x) = \frac{1}{2}e^x(1-\cos 2x) = \frac{1}{2}[e^x - e^x \cos 2x]$.

Now $D^n e^{ax} = a^n e^{ax}$.

Also $D^n e^{ax}\cos(bx+c) = r^n e^{ax}\cos(bx+c+n\phi)$, where

$r = (a^2+b^2)^{1/2}$ and $\phi = \tan^{-1}(b/a)$.

$\therefore \quad y_n = \frac{1}{2}[e^x - r^n e^x \cos(2x+n\phi)]$,

where $r = \sqrt{(1^2+2^2)} = \sqrt{5}$ and $\phi = \tan^{-1}\frac{2}{1} = \tan^{-1}2$.

4. See Example 12.
5. See Problem 17 of Comprehensive Problems 3.
6. Proceed as in Problem 1(ii) of Comprehensive Problems 3.

True or False

1. We have, $D^n y_r = y_{n+r} = D^{n+r} y$. So, the given statement is true.
2. The given statement is false because

$$D^n \{e^{ax}\sin(bx+c)\} = r^n e^{ax}\sin(bx+c+n\phi).$$

See article 2 , part (viii).

3. The given statement is true. In the application of Leibnitz's theorem, if we take thefunction, whose derivatives after a certain stage all become zero, as the second function, then all the remaining terms in the application of Leibnitz's theorem will vanish.

❍❍❍

Chapter-5

Expansions of Functions

Comprehensive Problems 1

Problem 1: *(i) State Maclaurin's theorem.*

(Meerut 2000; Bundelkhand 01, 06, 08, 11; Agra 07; Kashi 12,13)

(ii) State Taylor's theorem.

(Bundelkhand 2005, 06, 08, 11; Avadh 09,10,14; Kashi 11,13,14)

Solution: ***(i) Maclaurin's theorem :*** *Let $f(x)$ be a function of x which possesses continuous derivatives of all orders in the interval $[0, x]$. Assuming that $f(x)$ can be expanded as an infinite power series in x, we have*

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots$$

(ii) Taylor's theorem: *Let $f(x)$ be a function of x which possesses continuous derivatives of all orders in the interval $[a, a+h]$. Assuming that $f(a+h)$ can be expanded as an infinite power series in h, we have*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^n}{n!} f^{(n)}(a) + \ldots$$

Problem 2(i): *Expand a^x by Maclaurin's theorem.* **(Meerut 2012B)**

Solution: Let $f(x) = a^x$. Then $f(0) = a^0 = 1$,

$f'(x) = a^x \log a$ so that $f'(0) = \log a$,

$f''(x) = a^x (\log a)^2$ so that $f''(0) = (\log a)^2$,

$f'''(x) = a^x (\log a)^3$ so that $f'''(0) = (\log a)^3$, and so on.

In general,

$$f^n(x) = a^x (\log a)^n \quad \text{so that} \quad f^n(0) = (\log a)^n.$$

Now by Maclaurin's theorem, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots + \frac{x^n}{n!} f^n(0) + \ldots$$

$$\therefore \quad a^x = 1 + x \log a + \frac{x^2}{2!} (\log a)^2 + \frac{x^3}{3!} (\log a)^3 + \ldots + \frac{x^n}{n!} (\log a)^n + \ldots$$

Problem 2(ii): *Expand* $\tan x$ *by Maclaurin's theorem.* **(Kanpur 2014, 15)**

Solution: Let $y = \tan x$. Then $(y)_0 = \tan 0 = 0$,

$$y_1 = \sec^2 x = 1 + \tan^2 x = 1 + y^2 \text{ so that}$$

$$(y_1)_0 = 1 + (y)_0{}^2 = 1 + 0 = 1,$$

$$y_2 = 2yy_1 \text{ so that } (y_2)_0 = 2(y)_0 (y_1)_0 = 2 \times 0 \times 1 = 0,$$

$$y_3 = 2y_1y_1 + 2yy_2 = 2y_1{}^2 + 2yy_2 \text{ so that } (y_3)_0 = 2 \times 1^2 + 0 = 2,$$

$$y_4 = 4y_1y_2 + 2y_1y_2 + 2yy_3 = 6y_1y_2 + 2yy_3 \text{ so that}$$

$$(y_4)_0 = 6 \times 1 \times 0 + 2 \times 0 \times 2 = 0,$$

$$y_5 = 6y_2{}^2 + 6y_1y_3 + 2y_1y_3 + 2yy_4 = 6y_2{}^2 + 8y_1y_3 + 2yy_4$$

$$\text{so that } (y_5)_0 = 0 + 8 \times 1 \times 2 + 0 = 16, \text{ and so on.}$$

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \frac{x^5}{5!}(y_5)_0 + \ldots$$

$$\therefore \quad \tan x = 0 + x \,.\, 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \cdot 2 + \frac{x^4}{4!} \cdot 0 + \frac{x^5}{5!} \cdot 16 + \ldots$$

$$= x + \frac{x^3}{3} + \frac{2}{15} x^5 + \ldots .$$

Problem 2(iii): *Expand* $e^{x \cos x}$ *by Maclaurin's theorem.*

Solution: Let $y = e^{x \cos x}$. Then $(y)_0 = e^0 = 1$

$$y_1 = e^{x \cos x} (\cos x - x \sin x)$$

$$= y(\cos x - x \sin x) \text{ so that } (y_1)_0 = 1.(1-0) = 1,$$

$$y_2 = y_1 (\cos x - x \sin x) + y(-2 \sin x - x \cos x)$$

$$\text{giving } (y_2)_0 = 1.(1-0) + 1.0 = 1,$$

$$y_3 = y_2 (\cos x - x \sin x) + y_1 (-2 \sin x - x \cos x) + y_1 (-2 \sin x - x \cos x) + y(-3 \cos x + x \sin x)$$

or

$$y_3 = y_2 (\cos x - x \sin x) + 2y_1 (-2 \sin x - x \cos x) + y(-3 \cos x - x \sin x)$$

$$\text{so that } (y_3)_0 = 1.(1-0) + 0 + 1.(-3) = -2,$$

$$y_4 = y_3 (\cos x - x \sin x) + 3y_2 (-2 \sin x - x \cos x) + 3y_1 (-3 \cos x - x \sin x) + y(4 \sin x + x \cos x)$$

$$\text{giving } (y_4)_0 = -2 \,.\, (1-0) + 0 + 3 \,.\, 1 \,.\, (-3+0) + 0 = -11,$$

$$y_5 = y_4 (\cos x - x \sin x) + 4y_3 (-2 \sin x - x \cos x) + 6y_2(-3 \cos x + x \sin x) + 4y_1(4 \sin x + x \cos x) + y(5 \cos x - x \sin x)$$

so that

$$(y_5)_0 = -11 + 6.1.(-3) + 5 = -24, \text{ and so on.}$$

Substituting these values in Maclaurin's theorem, we get

$$e^{x\cos x} = 1 + x\,.\,1 + \frac{x^2}{2!}\,.\,1 + x^3 3!\,.\,(-2) + \frac{x^4}{4!}\,.\,(-11) + \frac{x^5}{5!}\,.\,(-24) + \ldots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11x^4}{24} - \frac{x^5}{5} + \ldots$$

Problem 2(iv): *Expand* $\tan^{-1} x$ *by Maclaurin's theorem. Write also the general term.*

(Bundelkhand 2001)

Solution: Let $y = \tan^{-1} x$. Proceeding as in Problem 8, Comprehensive problems 4 of the chapter 'Successive Differentiation', we get

$$(y)_0 = 0, (y_1)_0 = 1, (y_2)_0 = 0,$$

and $$(y_{n+2})_0 = -\{(n+1)\,n\}\,(y_n)_0. \qquad \ldots(1)$$

Putting $n = 1, 2, 3, \ldots$ in (1), we get

$$(y_3)_0 = -(2\,.\,1)\,(y_1)_0 = -2\,!, (y_4)_0 = -(3\,.\,2)\,(y_2)_0 = 0,$$

$$(y_5)_0 = -(4\,.\,3)\,(y_3)_0 = -(4\,.\,3)\,.\,(-2\,!) = 4\,!, \text{ etc.}$$

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x\,(y_1)_0 + \frac{x^2}{2\,!}\,(y_2)_0 + \frac{x^3}{3\,!}\,(y_3)_0 + \ldots$$

$$\therefore \qquad \tan^{-1} x = 0 + x\,.\,1 + \frac{x^2}{2\,!}\cdot 0 + \frac{x^3}{3\,!}\cdot(-2\,!) + \frac{x^4}{4\,!}\cdot 0 + \frac{x^5}{5\,!}\cdot(4\,!) + \ldots$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$$

The general term in this expansion is $(x^n / n\,!)\,(y_n)_0$.

So we need the value of $(y_n)_0$. Putting $(n-2)$ in place of n in (1), we get

$$(y_n)_0 = -\{(n-1)\,(n-2)\}(y_{n-2})_0$$

$$= [-\{(n-1)\,(n-2)\}]\,[-\{(n-3)\,(n-4)]\,(y_{n-4})_0.$$

Now there arise two cases :

Case I: When n is even, we have

$$(y_n)_0 = [-\{(n-1)\,(n-2)\}]\,[-\{(n-3)\,(n-4)\}]\ldots[-(3)\,(2)]\,(y_2)_0$$

$$= 0, \text{ since } (y_2)_0 = 0.$$

Case II: When n is odd, we have

$$(y_n)_0 = [-\{(n-1)\,(n-2)\}]\,[-\{(n-3)\,(n-4)\}]$$

$$\ldots[-(4)\,(3)]\,[-(2)\,(1)]\,(y_1)_0$$

$$= (-1)^{(n-1)/2}\,(n-1)\,!, \text{ since } (y_1)_0 = 1.$$

Thus in the expansion of $\tan^{-1} x$, the coefficient of x^n is 0 if n is even

and is $$\frac{(-1)^{(n-1)/2}\,(n-1)\,!}{n\,!} \text{ i.e., } \frac{(-1)^{(n-1)/2}}{n} \text{ if } n \text{ is odd.}$$

$$\therefore \qquad \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + \frac{(-1)^{[(2n-1)-1]/2}}{2n-1} x^{2n-1} + \ldots$$

$$= x - \frac{x^3}{3} + \ldots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + \ldots$$

Problem 2(v): *Expand sec x by Maclaurin's theorem .*

Solution: Let $y = \sec x$. Then $(y)_0 = \sec 0 = 1$,

$y_1 = \sec x \tan x$ so that $(y_1)_0 = 1 \times 0 = 0$,

$y_2 = \sec x \sec^2 x + \sec x \tan x \tan x = \sec^3 x + \sec x \tan^2 x$

$= \sec^3 x + \sec x (\sec^2 x - 1) = 2 \sec^3 x - \sec x = 2 y^3 - y$

so that $(y_2)_0 = 2 \times 1^3 - 1 = 1$,

$y_3 = 6 y^2 y_1 - y_1$ so that $(y_3)_0 = 0 - 0 = 0$,

$y_4 = 6 y^2 y_2 + 12 y y_1^2 - y_2$ so that $(y_4)_0 = 6 \times 1^2 \times 1 + 0 - 1 = 5$,

and so on.

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x (y_1)_0 + \frac{x^2}{2!} (y_2)_0 + \frac{x^3}{3!} (y_3)_0 + \frac{x^4}{4!} (y_4)_0 + \ldots$$

$$\therefore \qquad \sec x = 1 + x \, . \, 0 + \frac{x^2}{2!} \cdot 1 + \frac{x^3}{3!} \cdot 0 + \frac{x^4}{4!} \cdot 5 + \ldots$$

$$= 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \ldots$$

Problem 3(i): *Obtain by Maclaurin's theorem the first five terms in the expansion of* $e^{\sin x}$.

(Bundelkhand 2007)

Solution: Let $y = e^{\sin x}$. Then $(y)_0 = e^{\sin 0} = e^0 = 1$,

$y_1 = e^{\sin x} \cos x = y \cos x$ so that $(y_1)_0 = (y)_0 \cos 0 = 1 \times 1 = 1$,

$y_2 = y_1 \cos x - y \sin x$ so that $(y_2)_0 = 1 \times 1 - 1 \times 0 = 1$,

$y_3 = y_2 \cos x - y_1 \sin x - y_1 \sin x - y \cos x$

$= y_2 \cos x - 2 y_1 \sin x - y \cos x$ so that $(y_3)_0 = 1 - 0 - 1 = 0$,

$y_4 = y_3 \cos x - 3 y_2 \sin x - 3 y_1 \cos x + y \sin x$ so that

$(y_4)_0 = -3$, and so on.

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x (y_1)_0 + \frac{x^2}{2!} (y_2)_0 + \frac{x^3}{3!} (y_3)_0 + \frac{x^4}{4!} (y_4)_0 + \ldots$$

$\therefore \qquad e^{\sin x} = 1 + x.1 + \frac{x^2}{2!} \cdot 1 + \frac{x^3}{3!} \cdot 0 + \frac{x^4}{4!} \cdot (-3) + \ldots$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots$$

Problem 3(ii): *Expand by Maclaurin's theorem* $\frac{e^x}{1+e^x}$ *as far as the term* x^3.

(Meerut 2006B; Lucknow 06)

Solution: Let $\quad y = \frac{e^x}{1+e^x} = \frac{1+e^x-1}{1+e^x} = 1 - \frac{1}{1+e^x} \cdot$

Then $\quad (y)_0 = \frac{e^0}{1+e^0} = \frac{1}{2},$

$y_1 = 0 + \frac{e^x}{(1+e^x)^2} = \frac{e^x}{1+e^x} \cdot \frac{1}{1+e^x} = y(1-y) = y - y^2$ so that

$(y_1)_0 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4},$

$y_2 = y_1 - 2yy_1$ so that $(y_2)_0 = \frac{1}{4} - 2 \cdot \frac{1}{2} \cdot \frac{1}{4} = 0,$

$y_3 = y_2 - 2y_1^2 - 2yy_2$ so that

$(y_3)_0 = 0 - 2.(1/4)^2 - 0 = -1/8,$ and so on.

Substituting these values in Maclaurin's theorem, we get

$$\frac{e^x}{1+e^x} = \frac{1}{2} + x \cdot \frac{1}{4} + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \cdot \left(-\frac{1}{8}\right) + \ldots = \frac{1}{2} + \frac{x}{4} - \frac{1}{48} x^3 + \ldots$$

Problem 3(iii): *Obtain by Maclaurin's theorem the first five terms in the expansion of* $\log(1 + \sin x)$.

(Meerut 2007; Lucknow 07)

Solution: Let $\quad y = \log(1 + \sin x)$. Then $(y)_0 = 0,$

$y_1 = \frac{\cos x}{1 + \sin x}$ so that $(y_1)_0 = 1,$

$$y_2 = \frac{-\sin x(1+\sin x) - \cos^2 x}{(1+\sin x)^2} = \frac{-(1+\sin x)}{(1+\sin x)^2}$$

$= -\frac{1}{1+\sin x}$ so that $(y_2)_0 = -1,$

$y_3 = \frac{\cos x}{(1+\sin x)^2} = \frac{\cos x}{1+\sin x} \cdot \frac{1}{1+\sin x} = -y_1 y_2$

so that $(y_3)_0 = -1.(-1) = 1,$

$y_4 = -y_1 y_3 - y_2^2$ so that $(y_4)_0 = -1.1 - (-1)^2 = -1 - 1 = -2,$

$$y_5 = -y_1 y_4 - y_2 y_3 - 2y_2 y_3 = -y_1 y_4 - 3y_2 y_3 \text{ so that}$$
$$(y_5)_0 = -1.(-2) - 3.(-1).1 = 2 + 3 = 5, \text{and so on.}$$

Substituting these values in Maclaurin's theorem, we get

$$\log(1 + \sin x) = 0 + x.1 + (x^2/2!).(-1) + (x^3/3!).1 + (x^4/4!).(-2) + (x^5/5!).5 + \ldots$$
$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} + \ldots$$

Problem 3(iv): *Find the first four terms in the expansion, in powers of x, of $\log(1 + \tan x)$.*

(Rohilkhand 2011B)

Solution: Let $y = \log(1 + \tan x)$. Then $(y)_0 = \log(1 + \tan 0) = 0$.

Now $e^y = 1 + \tan x$. Differentiating both sides w.r.t. x, we get

$$e^y y_1 = \sec^2 x. \qquad \ldots(1)$$

Putting $x = 0$ on both sides of (1), we get

$$e^0 (y_1)_0 = 1 \quad \text{or} \quad (y_1)_0 = 1.$$

Differentiating (1), we get

$$e^y y_1^2 + e^y y_2 = 2\sec^2 x \tan x$$

or $$e^y (y_1^2 + y_2) = 2\sec^2 x \tan x. \qquad \ldots(2)$$

Putting $x = 0$ in (2), we get

$$1 + (y_2)_0 = 0 \quad \text{or} \quad (y_2)_0 = -1.$$

Differentiating (2), we get

$$e^y y_1 (y_1^2 + y_2) + e^y (2y_1 y_2 + y_3) = 2\sec^4 x + 4\sec^2 x \tan^2 x$$

or $$e^y (y_1^3 + 3y_1 y_2 + y_3) = 2\sec^4 x + 4\sec^2 x \tan^2 x. \qquad \ldots(3)$$

Putting $x = 0$ in (3), we get

$$1 + 3.1.(-1) + (y_3)_0 = 2 \quad \text{or} \quad (y_3)_0 = 4.$$

Differentiating (3), we get

$$e^y y_1 (y_1^3 + 3y_1 y_2 + y_3) + e^y (3y_1^2 y_2 + 3y_2^2 + 3y_1 y_3 + y_4)$$
$$= 8\sec^4 x \tan x + 8\sec^2 x \tan^3 x + 8\sec^4 x \tan x$$

or $$e^y (y_1^4 + 6y_1^2 y_2 + 4y_1 y_3 + 3y_2^2 + y_4)$$
$$= 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x. \qquad \ldots(4)$$

Putting $x = 0$ in (4), we get

$$1 + 6.1.(-1) + 4.1.4 + 3.(-1)^2 + (y_4)_0 = 0$$

or $$(y_4)_0 + 14 = 0$$

or $$(y_4)_0 = -14.$$

Now substituting these values in Maclaurin's theorem, we get

$$\log (1 + \tan x) = 0 + x \cdot 1 + \frac{x^2}{2!} \cdot (-1) + \frac{x^3}{3!} \cdot 4 + \frac{x^4}{4!} \cdot (-14) + \ldots$$
$$= x - \frac{1}{2} x^2 + \frac{2}{3} x^3 - \frac{7}{12} x^4 + \ldots$$

Problem 4(i): *Apply Maclaurin's theorem to prove that*
$$\log \sec x = \frac{1}{2} x^2 + \frac{1}{12} x^4 + \frac{1}{45} x^6 + \ldots.$$
(Garhwal 2002; Lucknow 10; Bundelkhand 11; Rohilkhand 13)

Solution: Let $y = \log \sec x$. Then $(y)_0 = \log \sec 0 = \log 1 = 0$,

$$y_1 = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x \text{ so that } (y_1)_0 = 0,$$
$$y_2 = \sec^2 x = 1 + \tan^2 x = 1 + y_1^2 \text{ so that } (y_2)_0 = 1 + (y_1)_0^2 = 1,$$
$$y_3 = 2 y_1 y_2 \text{ so that } (y_3)_0 = 2 (y_1)_0 (y_2)_0 = 0,$$
$$y_4 = 2 y_2^2 + 2 y_1 y_3 \text{ so that } (y_4)_0 = 2 \times 1^2 + 0 = 2,$$
$$y_5 = 4 y_2 y_3 + 2 y_2 y_3 + 2 y_1 y_4 = 6 y_2 y_3 + 2 y_1 y_4 \text{ so that}$$
$$(y_5)_0 = 6 \times 1 \times 0 + 2 \times 0 \times 2 = 0,$$
$$y_6 = 6 y_3^2 + 6 y_2 y_4 + 2 y_2 y_4 + 2 y_1 y_5 = 6 y_3^2 + 8 y_2 y_4 + 2 y_1 y_5$$
$$\text{so that } (y_6)_0 = 0 + 8 \times 1 \times 2 + 0 = 16, \text{ and so on.}$$

Now by Maclaurin's theorem, we have
$$y = (y)_0 + x (y_1)_0 + \frac{x^2}{2!} (y_2)_0 + \frac{x^3}{3!} (y_3)_0 + \frac{x^4}{4!} (y_4)_0 + \ldots$$

$\therefore$
$$\log \sec x = 0 + x \cdot 0 + \frac{x^2}{2!} \cdot 1 + \frac{x^3}{3!} \cdot 0 + \frac{x^4}{4!} \cdot 2 + \frac{x^5}{5!} \cdot 0 + \frac{x^5}{6!} \cdot 16 + \ldots$$
$$= \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \ldots$$

Problem 4(ii): *Use Maclaurin's formula to show that* $e^x \sec x = 1 + x + \frac{2x^2}{2!} + \frac{4x^3}{3!} + \ldots$
(Kumaun 2003; Meerut 04; Rohilkhand 08B)

Solution: Let $y = e^x \sec x$. Then $(y)_0 = 1$,

$$y_1 = e^x \sec x + e^x \sec x \tan x = y + y \tan x \text{ so that } (y_1)_0 = 1,$$
$$y_2 = y_1 + y_1 \tan x + y \sec^2 x \text{ so that } (y_2)_0 = 1 + 0 + 1 = 2,$$
$$y_3 = y_2 + y_2 \tan x + 2 y_1 \sec^2 x + 2 y \sec^2 x \tan x \text{ so that}$$
$$(y_3)_0 = 2 + 2 = 4, \text{ and so on.}$$

Substituting these values in Maclaurin's theorem, we get
$$e^x \sec x = 1 + x + \frac{2x^2}{2!} + \frac{4x^3}{3!} + \ldots$$

Problem 4(iii): *Expand sinh x cos x to fifth power of x.*

Solution: Let $y = \sinh x \cos x$, Then $(y)_0 = \sinh 0 \cos 0 = 0.1 = 0$

$y_1 = \cosh x \cos x - \sinh x \sin x,$

Then $(y_1)_0 = \cosh 0 \cos 0 - \sinh 0 \sin 0 = 1.1 - 0.0 = 1$

$y_2 = \sinh x \cos x - \cosh x \sin x - \cosh x \sin x - \sinh x \cos x$

$= -2 \cosh x \sin x.$

Then $(y_2)_0 = -2 \cosh 0 \sin 0 = 0$

$y_3 = -2 \sinh x \sin x - 2 \cosh x \cos x$

Then $(y_3)_0 = -2 \sinh 0 \sin 0 - 2 \cosh 0 \cos 0 = -2$

$y_4 = -2 \cosh x \sin x - 2 \sinh x \cos x + 2 \cosh x \sin x - 2 \sinh x \cos x$

$= -4 \sinh x \cos x = -4y$

Then $(y_4) = -4 \sinh 0 \cos 0 = -4.0 = 0$

$y_5 = -4y$ Then $(y_5)_0 = -4(y_1)_0 = -4.1 = -4$

Now substituting these values in Maclaurin's theorem, we get

$$\sinh x \cos x = 0 + x.1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!} \cdot (-2) + \frac{x^4}{4!} \cdot 0 + \frac{x^5}{5!} \cdot (-4) + ...$$

$$= x - \frac{2}{3!} x^3 - \frac{4}{5!} x^5 + ...$$

Problem 5: *Show that*

(i) $$e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2 x^4}{4!} - \frac{2^2 x^5}{5!} + \frac{2^3 x^7}{7!} + \dots + 2^{n/2} \cos \frac{n\pi}{4} \cdot \frac{x^n}{n!} + \dots$$

(Bundelkhand 2014; Agra 14)

(ii) $$e^x \sin x = x + x^2 - \frac{2}{3!} x^3 - \frac{2^2}{5!} x^5 - \dots + \sin\left(\frac{1}{4} n\pi\right) \frac{2^{n/2}}{n!} x^n + \dots$$

(Meerut 2003; Gorakhpur 06; Lucknow 09, 10; Bundelkhand 2014)

Solution: **(i)** Let $y = e^x \cos x$. Then $(y)_0 = e^0 \cos 0 = 1$,

$y_1 = e^x \cos x - e^x \sin x = e^x (\cos x - \sin x)$

so that $(y_1)_0 = 1(1 - 0) = 1$,

$y_2 = e^x (\cos x - \sin x) + e^x (-\sin x - \cos x) = -2e^x \sin x$

so that $(y_2)_0 = 0$,

$y_3 = -2e^x \sin x - 2e^x \cos x = -2e^x (\sin x + \cos x)$

so that $(y_3)_0 = -2$,

$y_4 = -2e^x (\sin x + \cos x) - 2e^x (\cos x - \sin x) = -4e^x \cos x = -2^2 y$

so that $(y_4)_0 = -2^2$,

$y_5 = -2^2 y_1$ so that $(y_5)_0 = -2^2$,

$y_6 = -2^2 y_2$ so that $(y_6)_0 = 0$,

$$y_7 = -2^2 y_3 \text{ so that } (y_7)_0 = 2^3, \text{ and so on.}$$

In general $\quad y_n = (1+1)^{n/2} \cos(x + n \tan^{-1} 1) = 2^{n/2} \cos(x + n\pi/4)$

so that $(y_n)_0 = 2^{n/2} \cos(\frac{1}{4} n\pi)$.

Now by Maclaurin's theorem, we have

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \dots + \frac{x^n}{n!}(y_n)_0 + \dots$$

$$= 1 + x.1 + \frac{x^2}{2!}\cdot 0 + \frac{x^3}{3!}\cdot(-2) + \frac{x^4}{4!}\cdot(-2^2) + \frac{x^5}{5!}\cdot(-2^2)$$

$$+ \frac{x^6}{6!}\cdot 0 + \frac{x^7}{7!}\cdot 2^3 + \dots + \frac{x^n}{n!} 2^{n/2} \cos(\frac{1}{4} n\pi) + \dots$$

$$= 1 + x - \frac{2x^3}{3!} - \frac{2^2 x^4}{4!} - \frac{2^2 x^5}{5!} + \frac{2^3 x^7}{7!} + \dots + 2^{n/2} \cos(\frac{1}{4} n\pi) \frac{x^n}{n!} + \dots$$

(ii) Proceed as in part (i).

Problem 6: *Apply Maclaurin's theorem to prove that*

(i) $$e^{ax} \sin bx = bx + abx^2 + \frac{3a^2 b - b^3}{3!} x^3 + \dots + \frac{(a^2 + b^2)^{n/2}}{n!} x^n \sin\{n \tan^{-1}(b/a)\} + \dots$$

(ii) $$e^{ax} \cos bx = 1 + ax + \frac{a^2 - b^2}{2} x^2 + \frac{a(a^2 - 3b^2)}{3!} x^3 + \dots$$

$$+ \frac{(a^2 + b^2)^{n/2}}{n!} x^n \cos\{n \tan^{-1}(b/a)\} + \dots$$

(Avadh 2011; Kumaun 12)

Solution: **(i)** Let $y = e^{ax} \sin bx$. Then $(y)_0 = e^0 \sin 0 = 0$,

$$y_1 = ae^{ax} \sin bx + be^{ax} \cos bx = ay + be^{ax} \cos bx$$

so that $(y_1)_0 = b$,

$$y_2 = ay_1 + abe^{ax} \cos bx - b^2 e^{ax} \sin bx = ay_1 - b^2 y + abe^{ax} \cos bx$$

so that $(y_2)_0 = ab - 0 + ab = 2ab$,

$$y_3 = ay_2 - b^2 y_1 + a^2 be^{ax} \cos bx - ab^2 e^{ax} \sin bx$$

$$= ay_2 - b^2 y_1 - ab^2 y + a^2 be^{ax} \cos bx$$

so that $\quad (y_3)_0 = 2a^2 b - b^3 + a^2 b = 3a^2 b - b^3$, and so on.

In general, $\quad y_n = (a^2 + b^2)^{n/2} \sin\{bx + n \tan^{-1}(b/a)\}$

so that $\quad (y_n)_0 = (a^2 + b^2)^{n/2} \sin\{n \tan^{-1}(b/a)\}$.

Now by Maclaurin's theorem, we have

$$y = (y)_0 + \frac{x}{1!} \cdot (y_1)_0 + \frac{x^2}{2!} (y_2)_0 + \frac{x^3}{3!} (y_3)_0 + \dots + \frac{x^n}{n!} (y_n)_0 + \dots$$

$$= 0 + \frac{x}{1!} \cdot b + \frac{x^2}{2!} \cdot (2ab) + \frac{x^3}{3!} (3a^2 b - b^3) + \dots + \frac{x^n}{n!} (a^2 + b^2)^{n/2} \sin \{n \tan^{-1} (b/a)\} + \dots$$

$$= bx + abx^2 + \frac{3a^2 b - b^3}{3!} x^3 + \dots + \frac{(a^2 + b^2)^{n/2}}{n!} x^n \sin \{n \tan^{-1} (b/a)\} + \dots$$

(ii) Let $y = e^{ax} \cos bx$. Then $(y)_0 = e^0 \cos 0 = 1$,

$y_1 = ae^{ax} \cos bx - be^{ax} \sin bx = ay - be^{ax} \sin bx$ so that $(y_1)_0 = a$,

$y_2 = ay_1 - abe^{ax} \sin bx - b^2 e^{ax} \cos bx = ay_1 - b^2 y - abe^{ax} \sin bx$

so that $(y_2)_0 = a^2 - b^2$,

$y_3 = ay_2 - b^2 y_1 - a^2 be^{ax} \sin bx - ab^2 e^{ax} \cos bx$

$= ay_2 - b^2 y_1 - ab^2 y - a^2 be^{ax} \sin bx$ so that

$(y_3)_0 = a(a^2 - b^2) - b^2 a - ab^2 = a(a^2 - 3b^2)$, and so on.

In general, $y_n = (a^2 + b^2)^{n/2} \cos \{bx + n \tan^{-1} (b/a)\}$ so that

$(y_n)_0 = (a^2 + b^2)^{n/2} \cos \{n \tan^{-1} (b/a)\}$.

Substituting these values in Maclaurin's theorem, we get

$$e^{ax} \cos bx = 1 + ax + \frac{a^2 - b^2}{2!} x^2 + \frac{a(a^2 - 3b^2)}{3!} x^3 + \dots + \frac{(a^2 + b^2)^{n/2}}{n!} x^n \cos \{n \tan^{-1} (b/a)\} + \dots$$

Problem 7: *Show that* $e^{x \cos \alpha} \cos (x \sin \alpha) = 1 + x \cos \alpha + \frac{x^2}{2!} \cos 2\alpha + \frac{x^3}{3!} \cos 3\alpha + \dots$

(Rohilkhand 2007; Avadh 11)

Solution: Putting $a = \cos \alpha$ and $b = \sin \alpha$, in the problem 6(ii), we get

$(y_n)_0 = (\cos^2 \alpha + \sin^2 \alpha)^{n/2} \cos (n \tan^{-1} \tan \alpha) = \cos n\alpha$ so that

$(y_1)_0 = \cos \alpha$, $(y_2)_0 = \cos 2\alpha$, $(y_3)_0 = \cos 3\alpha$, etc.

$\therefore$ $e^{x \cos \alpha} \cos (x \sin \alpha) = 1 + x \cos \alpha + (x^2 / 2!) \cos 2\alpha + (x^3 / 3!) \cos 3\alpha + \dots$

Problem 8: *(i) Expand* $\sin^{-1}(x+h)$ *in powers of x as far as the term in* x^3.

(Garhwal 2003)

(ii) Show that $\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - \ldots$

Solution: (i) First we observe that we are to expand $\sin^{-1}(x+h)$ in ascending powers of x.

So let $f(h) = \sin^{-1} h$. Then $f(h+x) = \sin^{-1}(h+x)$.

Thus we are to expand $f(h+x)$ in powers of x. So by Taylor's theorem, we have

$$f(h+x) = f(h) + xf'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \ldots \qquad \ldots(1)$$

Now $f(h) = \sin^{-1} h$. Therefore

$$f'(h) = \frac{1}{\sqrt{(1-h^2)}} = (1-h^2)^{-1/2}, \quad f''(h) = h(1-h^2)^{-3/2},$$

$$f'''(h) = (1-h^2)^{-3/2} + h(-3/2)(1-h^2)^{-5/2}(-2h)$$

$$= (1-h^2)^{-3/2} + 3h^2(1-h^2)^{-5/2}$$

$$= (1-h^2)^{-5/2}[(1-h^2) + 3h^2] = (1-h^2)^{-5/2}(1+2h^2), \text{ etc.}$$

Substituting these values in (1), we have

$$\sin^{-1}(h+x) = \sin^{-1} h + (1-h^2)^{-1/2} x + (x^2/2!)\, h(1-h^2)^{-3/2}$$
$$+ (x^3/3!)(1-h^2)^{-5/2}(1+2h^2) + \ldots$$

(ii) First we observe that we are to expand $\log(x+h)$ in ascending powers of x. So let $f(h) = \log h$. Then $\boldsymbol{f(h+x) = \log(h+x)}$.

Now proceed as in part (i).

Here $f(h) = \log h$, $f'(h) = 1/h$, $f''(h) = -1/h^2$, $f'''(h) = 2/h^3, \ldots$ etc.

Substituting these values in Taylor's expansion, we get

$$\log(h+x) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - \ldots$$

Problem 9(i): *Expand* $\tan^{-1} x$ *in powers of* $(x - \frac{1}{4}\pi)$.

(Agra 2001; Garhwal 06)

Solution: Let $f(x) = \tan^{-1} x$. Then, we have

$$\tan^{-1} x = f(x) = f[\tfrac{1}{4}\pi + (x - \tfrac{1}{4}\pi)]$$

[$\because$ We have to expand $f(x)$ in powers of $(x - \frac{1}{4}\pi)$]

$$= f(\pi/4) + (x - \tfrac{1}{4}\pi) f'(\pi/4) + \frac{1}{2!}(x - \tfrac{1}{4}\pi)^2 f''(\pi/4) + \ldots,$$

on expanding $f[\frac{1}{4}\pi + (x - \frac{1}{4}\pi)]$ by Taylor's theorem in powers of $(x - \frac{1}{4}\pi)$.

Now $\quad f(x) = \tan^{-1} x$. Therefore

$$f(\pi/4) = \tan^{-1}(\pi/4),\ f'(x) = \frac{1}{1+x^2}$$

so that $f'(\pi/4) = 1/(1+\pi^2/16)$,

$$f''(x) = -2x/(1+x^2)^2 \text{ so that } f''(\pi/4) = -\pi/\{2(1+\pi^2/16)^2\}$$

and so on.

Substituting these values in the above expansion, we get

$$\tan^{-1} x = \tan^{-1}(\pi/4) + (x - \frac{1}{4}\pi)/(1+\pi^2/16)$$
$$- \pi(x - \frac{1}{4}\pi)^2/\{4(1+\pi^2/16)^2\} + \ldots$$

Problem 9(ii): *Expand* $\sin\left(\frac{1}{4}\pi + \theta\right)$ *in powers of* θ.

(Lucknow 2009, 11)

Solution: We have $f(a+h) = f(a) + h\, f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) \ldots$

putting $a = \frac{\pi}{4}$ and $h = \theta$, we get

$$f\left(\frac{\pi}{4}+\theta\right) = f\left(\frac{\pi}{4}\right) + \theta f'\left(\frac{\pi}{4}\right) + \frac{\theta^2}{2!} f''\left(\frac{\pi}{4}\right) + \frac{\theta}{3!} f'''\left(\frac{\pi}{4}\right) + \ldots \quad \ldots(1)$$

Let $\quad f(\theta) = \sin\theta$

Differentiating w.r.to θ, we get

$$f'(\theta) = \cos\theta$$
$$f''(\theta) = -\sin\theta$$
$$f'''(\theta) = -\cos\theta$$

Putting $\theta = \frac{\pi}{4}$, we get

$$f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}},\ f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}},\ f''\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}},\ f'''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

Putting these value in eqn. (1)

$$f\left(\frac{\pi}{4}+\theta\right) = \sin\left(\frac{\pi}{4}+\theta\right) = \frac{1}{\sqrt{2}} + \theta\left(\frac{1}{\sqrt{2}}\right) + \frac{\theta^2}{2!}\left(-\frac{1}{\sqrt{2}}\right)$$
$$+ \frac{\theta^3}{3!}\left(-\frac{1}{\sqrt{2}}\right) + \ldots$$

$$\therefore \quad \sin\left(\frac{\pi}{4}+\theta\right) = \frac{1}{\sqrt{2}}\left\{1 + \theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \ldots\ldots\right\}.$$

Problem 9(iii): *Expand* $2x^3 + 7x^2 + x - 1$ *in powers of* $(x - 2)$.

(Meerut 2004B, 05B, 12B; Gorakhpur 06; Rohilkhand 09; Kanpur 11; Kashi 11; Purvanchal 11)

Solution: Let $f(x) = 2x^3 + 7x^2 + x - 1.$

We can write $f(x) = f[2 + (x - 2)]$.

Now expanding $f[2 + (x - 2)]$ by Taylor's theorem in powers of $x - 2$, we get

$$f(x) = f[2 + (x - 2)] = f(2) + (x - 2) f'(2) + (1/2!)(x - 2)^2 f''(2) + \dots \quad ...(1)$$

Now $f(x) = 2x^3 + 7x^2 + x - 1$ so that $f(2) = 2.2^3 + 7.2^2 + 2 - 1 = 45$,

$f'(x) = 6x^2 + 14x + 1$ so that $f'(2) = 53$,

$f''(x) = 12x + 14$ so that $f''(2) = 38$,

$f'''(x) = 12$ so that $f'''(2) = 12$,

$f^{iv}(x) = 0$ so that $f^{iv}(2) = 0$. Obviously $f^n(2) = 0$ when $n \geq 4$.

Now substituting these values in (1), we get

$$f(x) = 45 + (x - 2).53 + \frac{(x-2)^2}{2!} \cdot 38 + \frac{(x-2)^3}{3!} \cdot 12$$

$$= 45 + 53(x - 2) + 19(x - 2)^2 + 2(x - 2)^3.$$

Problem 9(iv): *Write the value of* α, *if by Taylor's theorem*

$$2x^3 + 7x^2 + x - 1 = \alpha + 53(x - 2) + 19(x - 2)^2 + 2(x - 2)^3.$$

(Meerut 2001)

Solution: Proceed as in part (iii). Ans. 45.

Problem 9(v): *Expand log sin x in powers of* $(x - a)$.

(Meerut 2001, 06, 13; Rohilkhand 07B; Kumaun 07; Avadh 10)

Solution: Let $f(x) = \log \sin x$. We can write $f(x) = f[a + (x - a)]$.

Expanding $f[a + (x - a)]$ by Taylor's theorem in powers of $(x - a)$, we get

$$f(x) = f(a) + (x - a) f'(a) + (1/2!)(x - a)^2 f''(a) + (1/3!)(x - a)^3 f'''(a) + \dots \quad ...(1)$$

Now $f(x) = \log \sin x$. Therefore $f(a) = \log \sin a$,

$f'(x) = (1/\sin x).\cos x = \cot x$, giving $f'(a) = \cot a$

$f''(x) = -\operatorname{cosec}^2 x$ so that $f''(a) = -\operatorname{cosec}^2 a$,

$f'''(x) = 2\operatorname{cosec}^2 x \cot x$ so that $f'''(a) = 2\operatorname{cosec}^2 a \cot a$,

and so on.

Substituting these values in (1), we get

$$\log \sin x = \log \sin a + (x-a) \cot a - \frac{(x-a)^2}{2\,!} \operatorname{cosec}^2 a + \frac{(x-a)^3}{3\,!} 2 \operatorname{cosec}^2 a \cot a + \ldots$$

Problem 10(i): *If* $y = e^{a \sin^{-1} x}$, *show that*

$$(1-x^2) y_{n+2} - (2n+1) x y_{n+1} - (n^2 + a^2) y_n = 0.$$

Hence by Maclaurin's theorem show that

$$e^{a \sin^{-1} x} = 1 + ax + \frac{a^2}{2\,!} x^2 + \frac{a(1^2 + a^2)}{3\,!} x^3 + \ldots .$$

(Kumaun 2008)

Also deduce that $e^{\theta} = 1 + \sin \theta + \frac{1}{2\,!} \sin^2 \theta + \frac{2}{3\,!} \sin^3 \theta + \ldots$.

(Agra 2000; Kumaun 03)

Solution: Let $y = e^{a \sin^{-1} x}$. Proceeding as in problem 7 of the Comprehensive Problems 4 of chapter 'Successive Differentiation', we get

$$y(0) = 1,\ y_1(0) = a,\ y_2(0) = a^2$$

and $$y_{n+2}(0) = (n^2 + a^2)\, y_n(0). \qquad \ldots(1)$$

Putting $n = 1, 2, 3, 4, \ldots$ in (1), we get

$$y_3(0) = (1^2 + a^2)\, y_1(0) = (1^2 + a^2)\, a,$$

$$y_4(0) = (2^2 + a^2)\, y_2(0) = (2^2 + a^2)\, a^2,$$

$$y_5(0) = (3^2 + a^2)\, y_3(0) = (3^2 + a^2)(1^2 + a^2)\, a,$$

$$y_6(0) = (4^2 + a^2)\, y_4(0) = (4^2 + a^2)(2^2 + a^2)\, a^2, \text{ etc.}$$

In general, $$y_n(0) = \begin{cases} a(1^2 + a^2)(3^2 + a^2)\ldots[(n-2)^2 + a^2] \text{ if } n \text{ is odd} \\ a^2 (2^2 + a^2)(4^2 + a^2)\ldots[(n-2)^2 + a^2] \text{ if } n \text{ is even} . \end{cases}$$

Substituting these values in Maclaurin's expansion

$$y = y(0) + x y_1(0) + \frac{x^2}{2\,!} y_2(0) + \ldots + \frac{x^n}{n\,!} y_n(0) + \ldots, \text{ we get}$$

$$e^{a \sin^{-1} x} = 1 + ax + \frac{a^2}{2\,!} x^2 + \frac{a(1^2 + a^2)}{3\,!} x^3 + \frac{a^2 (2^2 + a^2)}{4\,!} x^4 + \ldots \qquad \ldots(2)$$

The general term is $(x^n / n\,!)\, y_n(0)$, where $y_n(0)$ is as given above.

Now putting $x = \sin \theta$ and $a = 1$ in (2), we get

$$e^{\theta} = 1 + \sin \theta + \frac{1}{2\,!} \sin^2 \theta + \frac{2}{3\,!} \sin^3 \theta + \ldots$$

Problem 10(ii): *If* $y = \sin(m \sin^{-1} x)$, *then show that* $(1 - x^2)\frac{d^2 y}{dx^2} - x\frac{dy}{dx} + m^2 y = 0.$

Hence or otherwise expand $\sin m\theta$ *in powers of* $\sin \theta$. **(Garhwal 2003; Lucknow 08)**

Solution: Let $y = \sin(m \sin^{-1} x)$. Proceeding as in Example 16, after article 7 of the chapter 'Successive Differentiation', we get

$$y(0) = 0,\ y_1(0) = m,\ y_2(0) = 0,$$

and
$$y_{n+2}(0) = (n^2 - m^2)\, y_n(0). \quad \text{...(1)}$$

Putting $n = 1, 2, 3, \ldots$ in (1), we get

$$y_3(0) = (1^2 - m^2)\, y_1(0) = (1^2 - m^2)\, m,$$

$$y_4(0) = (2^2 - m^2)\, y_2(0) = 0,$$

$$y_5(0) = (3^2 - m^2)\, y_3(0) = (3^2 - m^2)(1^2 - m^2)\, m, \text{ etc.}$$

Substituting these values in Maclaurin's theorem, we get

$$\sin(m \sin^{-1} x) = y(0) + x\, y_1(0) + \frac{x^2}{2!}\, y_2(0) + \ldots$$

$$= mx + \frac{m(1^2 - m^2)}{3!} x^3 + \frac{m(1^2 - m^2)(3^2 - m^2)}{5!} x^5 + \ldots$$

Putting $x = \sin\theta$ on both sides, we get

$$\sin m\theta = m \sin\theta + \frac{m(1^2 - m^2)}{3!} \sin^3\theta + \frac{m(1^2 - m^2)(3^2 - m^2)}{5!} \sin^5\theta + \ldots$$

Problem 10(iii): *If* $y = \sin\log(x^2 + 2x + 1)$, *prove that*

$$(x+1)^2 y_{n+2} + (2n+1)(x+1)\, y_{n+1} + (n^2 + 4)\, y_n = 0.$$ **(Agra 2002)**

Hence or otherwise expand y in ascending powers of x as far as x^6.

Solution: Here $y = \sin\log(x^2 + 2x + 1) = \sin\log(x+1)^2$ ...(1)

$$\therefore \quad y_1 = [\cos\log(x+1)^2] \cdot \frac{1}{(x+1)^2} \cdot 2(x+1)$$

$$= [\cos\log(x+1)^2] \cdot \frac{2}{x+1}. \quad \text{...(2)}$$

Squaring both sides of (2), we get

$$(x+1)^2 y_1^2 = 4\cos^2\log(x+1)^2$$

$$= 4[1 - \sin^2\log(x+1)^2] = 4(1 - y^2)$$

or
$$(x+1)^2\, y_1^2 + 4y^2 - 4 = 0. \quad \text{...(3)}$$

Differentiating (3), we get

$$(x+1)^2\, 2y_1y_2 + 2(x+1)\,y_1^2 + 8yy_1 = 0$$

or $$2y_1\,[(x+1)^2\,y_2 + (x+1)\,y_1 + 4y] = 0$$

or $$(x+1)^2\,y_2 + (x+1)\,y_1 + 4y = 0, \qquad ...(4)$$

since $2y_1 \neq 0$.

Differentiating (4) n times by Leibnitz's theorem, we get

$$(x+1)^2\,y_{n+2} + {}^nC_1 \,.\, y_{n+1} \,.\, 2(x+1) + {}^nC_2 \,.\, y_n \,.\, 2$$
$$+ (x+1)\,y_{n+1} + {}^nC_1 \,.\, y_n \,.\, 1 + 4y_n = 0$$

or $$(x+1)^2\,y_{n+2} + (2n+1)(x+1)\,y_{n+1} + (n^2+4)\,y_n = 0. \qquad ...(5)$$

Putting $x = 0$ in (1), (2) and (4), we get

$$(y)_0 = 0, (y_1)_0 = 2, (y_2)_0 + (y_1)_0 + 4(y)_0 = 0 \text{ or } (y_2)_0 = -2.$$

Also putting $x = 0$ in (5), we get

$$(y_{n+2})_0 + (2n+1)(y_{n+1})_0 + (n^2+4)(y_n)_0 = 0$$

or $$(y_{n+2})_0 = -[(2n+1)(y_{n+1})_0 + (n^2+4)(y_n)_0]. \qquad ...(6)$$

Now putting $n = 1, 2, 3, 4$ in (6), we get

$$(y_3)_0 = -[3(y_2)_0 + 5(y_1)_0] = -[3\,.\,(-2) + 5\,.\,2] = -4,$$
$$(y_4)_0 = -[5(y_3)_0 + 8(y_2)_0] = -[5\,.\,(-4) + 8\,.\,(-2)] = 36,$$
$$(y_5)_0 = -[7(y_4)_0 + 13(y_3)_0] = -[7\,.\,(36) + 13\,.\,(-4)] = -200,$$

and $$(y_6)_0 = -[9(y_5)_0 + 20(y_4)_0] = -[9\,.\,(-200) + 20\,.\,(36)] = 1080.$$

Now by Maclaurin's theorem, we have

$$y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \frac{x^5}{5!}(y_5)_0 + \ldots$$

$$= 0 + \frac{x}{1}\cdot 2 + \frac{x^2}{2}\cdot(-2) + \frac{x^3}{1\,.\,2\,.\,3}\cdot(-4) + \frac{x^4}{1\,.\,2\,.\,3\,.\,4}\cdot 36 + \frac{x^5}{1\,.\,2\,.\,3\,.\,4\,.\,5}\cdot(-200) + \frac{x^6}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6}\cdot(1080) + \ldots$$

$$= 2x - x^2 - \frac{2}{3}x^3 + \frac{3}{2}x^4 - \frac{5}{3}x^5 + \frac{3}{2}x^6 + \ldots$$

Problem 11: *By Maclaurin's theorem or otherwise find the expansion of $y = \sin(e^x - 1)$ upto and including the term in x^4. Find also the first two non-vanishing terms in the expansion of x as a series of ascending powers of y.*

Solution: Let $y = \sin(e^x - 1)$. Then $(y)_0 = \sin 0 = 0$,

$$y_1 = [\cos(e^x - 1)].\, e^x \text{ so that } (y_1)_0 = (\cos 0).\, e^0 = 1,$$

$$y_2 = [\cos(e^x - 1)].\, e^x - [\sin(e^x - 1)].\, e^{2x} = y_1 - ye^{2x}$$

$$\text{so that } (y_2)_0 = (y_1)_0 - (y)_0\, e^0 = 1 - 0 = 1,$$

$$y_3 = y_2 - y_1\, e^{2x} - 2\, y\, e^{2x} \text{ so that } (y_3)_0 = 1 - 1 - 0 = 0,$$

$$y_4 = y_3 - y_2\, e^{2x} - 4\, y_1\, e^{2x} - 4\, y\, e^{2x} \text{ so that } (y_4)_0 = -5, \text{ etc.}$$

Hence by Maclaurin's theorem, we get

$$\sin(e^x - 1) = (y)_0 + x\,(y_1)_0 + \frac{x^2}{2\,!}(y_2)_0 + \frac{x^3}{3\,!}(y_3)_0 + \ldots$$

$$= 0 + x\,.\,1 + \frac{x^2}{2\,!}\cdot 1 + \frac{x^3}{3\,!}\cdot 0 + \frac{x^4}{4\,!}\cdot(-5) + \ldots$$

$$= x + \frac{1}{2}x^2 - \frac{5}{24}x^4 + \ldots$$

Second Part: We have

$$y = \sin(e^{\,x} - 1) \;\Rightarrow\; e^{\,x} - 1 = \sin^{-1} y \;\Rightarrow\; e^{\,x} = 1 + \sin^{-1} y. \qquad \ldots(1)$$

Differentiating (1) with respect to 'y', we get

$$e^{\,x}.\, x_1 = 1/\sqrt{(1 - y^2)}, \text{ where } x_1 = dx/dy. \qquad \ldots(2)$$

From (2), we get $(1 - y^2)x_1^{\,2} = e^{-2x}$.

Differentiating it with respect to 'y', we get

$$(1 - y^2)2x_1x_2 - 2\,y\,x_1^{\,2} = e^{-2x}(-2x_1)$$

or $$(1 - y^2)x_2 - y\,x_1 = -e^{-2x}, \text{ since } 2x_1 \neq 0. \qquad \ldots(3)$$

From (1), we have $x = \log(1 + \sin^{-1} y)$. ...(4)

Now putting $y = 0$ in (4), (2) and (3), we get

$$(x)_0 = \log(1 + 0) = 0,\, e^{\,0}.(x_1)_0 = 1/\sqrt{(1 - 0)} \text{ giving}$$

$$(x_1)_0 = 1,\; (x_2)_0 = -e^0 = -1. \qquad [\text{Note that } (x)_{y=0} = 0]$$

Hence by Maclaurin's theorem, we get

$$x = (x)_0 + y(x_1)_0 + (y^2/2!)(x_2)_0 + \ldots$$

$$= 0 + y.1 + (y^2/2!)(-1) + \ldots = y - \frac{1}{2}y^2 + \ldots$$

Problem 12: *Expand* $\log\{1 - \log(1 - x)\}$ *in powers of x by Maclaurin's theorem as far as the term* x^3. **(Avadh 2009)**

By substituting $x/(1 + x)$ *for x deduce the expansion of* $\log\{1 + \log(1 + x)\}$ *as far as the term in* x^3.

Solution: Let $y = \log\{1 - \log(1 - x)\}$. Then $(y)_0 = 0$.

Now $e^{y} = 1 - \log(1-x)$. Differentiating, we get

$$e^{y} y_1 = (1-x)^{-1}. \quad \text{...(1)}$$

Putting $x = 0$ in (1), we get $(y_1)_0 = 1$.

Differentiating (1), we get

$$e^{y} y_1^2 + e^{y} y_2 = (1-x)^{-2}$$

or $$e^{y}(y_1^2 + y_2) = (1-x)^{-2} \quad \text{...(2)}$$

Putting $x = 0$ in (2), we get

$$1 + (y_2)_0 = 1 \quad \text{or} \quad (y_2)_0 = 0.$$

Differentiating (2), we get

$$e^{y}(y_1^3 + 3y_1y_2 + y_3) = 2(1-x)^{-3} \quad \text{...(3)}$$

Putting $x = 0$ in (3), we get

$$1 + (y_3)_0 = 2 \quad \text{or} \quad (y_3)_0 = 1.$$

Substituting these values in Maclaurin's theorem, we get

$$\log\{1 - \log(1-x)\} = 0 + x\,.\,1 + \frac{x^2}{2\,!}\cdot 0 + \frac{x^3}{3\,!}\cdot 1 + \ldots$$

$$= x + \frac{x^3}{6} + \ldots \quad \text{...(4)}$$

Now substituting $x/(1+x)$ for x on both sides of (4), we get

$$\log\left\{1 - \log\left(1 - \frac{x}{1+x}\right)\right\} = \frac{x}{1+x} + \frac{1}{6}\left(\frac{x}{1+x}\right)^3 + \ldots$$

or $$\log\{1 + \log(1+x)\} = x(1+x)^{-1} + (1/6)\,x^3(1+x)^{-3} + \ldots$$

$$= x\left\{1 + (-1)\,x + \frac{(-1)(-2)}{1\,.\,2}\,x^2 + \ldots\right\} + \frac{1}{6}\,x^3\,\{1 + (-3)\,x + \ldots\},$$

on expanding by binomial theorem

$$= (x - x^2 + x^3 + \ldots) + (\tfrac{1}{6}x^3 + \ldots) = x - x^2 + \frac{7}{6}x^3 + \ldots$$

Problem 13: *If* $y = (\sin^{-1} x)/\sqrt{(1-x^2)}$, *where* $-1 < x < 1$ *and* $-\pi/2 < \sin^{-1} x < \pi/2$,

prove that $$(1-x^2)\frac{d^{n+1}y}{dx^{n+1}} - (2n+1)\,x\frac{d^n y}{dx^n} - n^2\frac{d^{n-1}y}{dx^{n-1}} = 0.$$ **(Meerut 2007B)**

Assuming that y *can be expanded in ascending powers of* x *in the form*

$$a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots,$$

prove that $(n+1)\,a_{n+1} = n\,a_{n-1}$ *and hence obtain the general term of the expansion.*

Solution: Here $y = (\sin^{-1} x)/\sqrt{(1-x^2)}$. ...(1)

$\therefore$ $$y^2(1-x^2) = (\sin^{-1} x)^2.$$

Differentiating w.r.t. x, we get

$$2yy_1(1-x^2)-2xy^2=2(\sin^{-1}x)/\sqrt{(1-x^2)}=2y.$$

Since $2y\neq 0$, therefore $y_1(1-x^2)-xy=1$

i.e. $\quad y_1(1-x^2)-xy-1=0.$ ...(2)

Differentiating (2) n times by Leibnitz's theorem, we get

$$y_{n+1}(1-x^2)+ny_n(-2x)+\frac{n(n-1)}{2!}y_{n-1}\cdot(-2)-xy_n-ny_{n-1}=0$$

or $\quad (1-x^2)\,y_{n+1}-(2n+1)\,xy_n-n^2y_{n-1}=0.$ ...(3)

Now putting $x=0$ in (1), (2) and (3), we get $(y)_0=0, (y_1)_0=1$

and $\quad (y_{n+1})_0=n^2(y_{n-1})_0.$ ...(4)

By Maclaurin's theorem, we have

$$y=(y)_0+x(y_1)_0+(x^2/2!)(y_2)_0+\ldots+(x^n/n!)(y_n)_0+\ldots$$

Also we are given that $y=a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots$

Comparing the coefficients of x^n in the two expansions for y, we get $a_n=(y_n)_0/n!$

$$\therefore \quad \frac{a_{n+1}}{a_{n-1}}=\frac{(y_{n+1})_0}{(n+1)!}\div\frac{(y_{n-1})_0}{(n-1)!}=\frac{(y_{n+1})_0}{(y_{n-1})_0}\cdot\frac{(n-1)!}{(n+1)!}$$

$$=n^2\cdot\frac{1}{n(n+1)}, \quad \text{[From (4)]}$$

$$=\frac{n}{n+1}.$$

$\therefore \quad (n+1)\,a_{n+1}=n\,a_{n-1}$ **Proved**

or $\quad a_{n+1}=\dfrac{n}{n+1}a_{n-1}.$ ...(5)

Now $\quad a_0=(y)_0=0, a_1=(y_1)_0=1.$

Putting $\quad n=1, 3, 5, \ldots$ in (5),

we get $\quad a_2=\dfrac{1}{2}a_0=0, a_4=\dfrac{3}{4}a_2=0, a_6=\dfrac{5}{6}a_4=0$, etc.

Thus $\quad a_n=0$ if n is even *i.e.*, $a_{2m}=0$.

Again putting $n=2, 4, 6, \ldots$ in (5), we get

$$a_3=\frac{2}{3}a_1=\frac{2}{3}, a_5=\frac{4}{5}\cdot a_3=\frac{4}{5}\cdot\frac{2}{3}, a_7=\frac{6}{7}a_5=\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}, \text{etc.}$$

In general, if n is odd, we have

$$a_n=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3}.$$

Thus $$a_{2m+1} = \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdots \frac{4}{5} \cdot \frac{2}{3}.$$

Problem 14: *If* $y = e^{m \tan^{-1} x} = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, *prove that*

$$(n+1)\, a_{n+1} + (n-1)\, a_{n-1} = m a_n.$$ **(Purvanchal 2007)**

Solution: Given $y = e^{m \tan^{-1} x}$ ...(1)

Differentiating $y_1 = e^{m \tan^{-1} x} \left[\frac{m}{1+x^2} \right]$ ...(2)

or $y_1 (1 + x^2) = my$

Differentiating again $(1 + x^2)\, y_2 + 2xy_1 = my_1$

or $(1 + x^2)\, y_2 + (2x - m)\, y_1 = 0$ ...(3)

Differentiating (3) n times by Leibnitz's theorem, we get

$$(1 + x^2)\, y_{n+2} + n \,.\, 2x\, y_{n+1} + \frac{n(n-1)}{2!} 2\, y_n + (2x - m)\, y_{n+1} + n\, y_n \,.\, 2 = 0 \quad \text{...(4)}$$

Putting $x = 0$ in all equations, we get

$$(y)_0 = 1, (y_1)_0 = m, (y_2)_0 = m^2$$

and $(y_{n+2})_0 = m\, (y_{n+1})_0 - n\,(n+1)\,(y_n)_0$ ... (5)

Given $y = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n + \ldots\ldots$ ...(6)

Also from Maclaurin's Theorem, we get

$$y = (y)_0 + x\,(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \ldots\ldots + \frac{x^n}{n!}(y_n)_0 + \ldots. \quad \text{...(7)}$$

Comparing the coefficients of x^n in (6) and (7), we get

$$a_n = \frac{(y_n)_0}{n!} \text{ or } (y_n)_0 = n!\, a_n$$

Similarly $(y_{n+1})_0 = a_{n+1}\,(n+1)!$ and $(y_{n-1})_0 = a_{n-1}\,(n-1)!$

Also replacing n by $n - 1$ in (5), we get

$$(y_{n+1})_0 = m\,(y_n)_0 - (n-1)\, n\,(y_{n-1})_0$$

or $$a_{n+1}\,(n+1)! = m\, n!\, a_n - n\,(n-1)\,(n-1)!\, a_{n-1}$$

$$a_{n+1}\,(n+1) = m a_n - (n-1)\, a_{n-1}.$$

Problem 15: *Prove that*

$$f(mx) = f(x) + (m-1)\, x f'(x) + (1/2!)\,(m-1)^2 x^2 f''(x) + (1/3!)(m-1)^3 x^3 f'''(x) + \ldots$$

(Meerut 2001; Agra 07; Rohilkhand 13)

Solution: First we observe that we are to expand $f(mx)$ in powers of $(m-1)x$.

We can write $\quad f(mx) = f\{x + (m-1)x\}$.

Expanding $f\{x+(m-1)x\}$ in powers of $(m-1)x$ by Taylor's theorem, we get

$$f\{x+(m-1)x\} = f(x) + (m-1)xf'(x) + (1/2!)(m-1)^2 x^2 f''(x) + \ldots$$

$$\therefore \quad f(mx) = f(x) + (m-1)xf'(x) + (1/2!)(m-1)^2 x^2 f''(x) + \ldots$$

Problem 16: *Prove that*

(i) $$f\left(\frac{x^2}{1+x}\right) = f(x) - \frac{x}{1+x} f'(x) + \frac{x^2}{(1+x)^2}\frac{1}{2!} f''(x) - \ldots$$ **(Rohilkhand 2008)**

(ii) $$f(x) = f(0) + xf'(x) - \frac{x^2}{2!} f''(x) + \frac{x^3}{3!} f'''(x) - \ldots$$

Solution: (i) First we observe that we are to expand $f\left(\frac{x^2}{1+x}\right)$ in powers of $\left(-\frac{x}{1+x}\right)$.

We can write $\quad f\left(\frac{x^2}{1+x}\right) = f\left[x + \left(-\frac{x}{1+x}\right)\right]$.

Now expanding $f\left[x+\left(-\frac{x}{1+x}\right)\right]$ by Taylor's theorem in powers of $-\frac{x}{1+x}$, we get

$$f\left(\frac{x^2}{1+x}\right) = f(x) + \left(-\frac{x}{1+x}\right) f'(x) + \frac{1}{2!}\left(-\frac{1}{1+x}\right)^2 f''(x) + \ldots$$

$$= f(x) - \frac{x}{1+x} f'(x) + \frac{x^2}{(1+x)^2}\frac{1}{2!} f''(x) - \ldots$$

(ii) We can write $f(0) = f[x + (-x)]$. Now expanding $f[x+(-x)]$ by Taylor's theorem in powers of $-x$, we get

$$f(0) = f[x+(-x)] = f(x) + (-x)f'(x) + \frac{(-x)^2}{2!} f''(x) + \frac{(-x)^3}{3!} f'''(x) + \ldots$$

or $$f(0) = f(x) - xf'(x) + \frac{x^2}{2!} f''(x) - \frac{x^3}{3!} f'''(x) + \ldots.$$

$$\therefore \quad f(x) = f(0) + xf'(x) - \frac{x^2}{2!} f''(x) + \frac{x^3}{3!} f'''(x) - \ldots, \text{ by transposition.}$$

Hints to Objective Type Questions

Multiple Choice Questions

1. See Problem 9(v) of Comprehensive Problems 1.

2. Let $y = \dfrac{e^x}{1+e^x} = \dfrac{(1+e^x)-1}{1+e^x} = 1 - \dfrac{1}{1+e^x} = 1-(1+e^x)^{-1}.$

Then, $y_1 = (1+e^x)^{-2}\, e^x = \dfrac{e^x}{(1+e^x)^2}$.

$\therefore \quad (y_1)_0 = \dfrac{1}{4}.$ $\qquad [\because\ e^0 = 1]$

Now, by Maclaurin's theorem, we have

$$y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \ldots$$

The second term in the expansion $= \dfrac{x}{1!}(y_1)_0 - \dfrac{x}{1!}\cdot\dfrac{1}{4} = \dfrac{x}{4}$.

3. Let $y = \log(1+\sin^2 x)$. Then $(y)_0 = 0$.

Now $\quad e^y = 1 + \sin^2 x.$

Differentiating, we get $\quad e^y\, y_1 = 2\sin x\cos x = \sin 2x.$ ...(1)

Putting $x = 0$ in (1), we get $\quad e^0\,(y_1)_0 = 0$ or $(y_1)_0 = 0.$

Differentiating (1), we get $\quad e^y\,(y_1^2 + y_2) = 2\cos 2x.$ ...(2)

Putting $x = 0$ in (2), we get $\quad (y_2)_0 = 2.$

Now substituting these values in Maclaurin's theorem, we get

$$\log(1+\sin^2 x) = 0 + x\,.\,0 + \frac{x^2}{2\,!}\cdot 2 + \ldots\ .$$

4. See Problem 9(v) of Comprehensive Problems 1.

5. See Problem 9(i) of Comprehensive Problems 1.

6. See Problem 4(ii) of Comprehensive Problems 1.

7. Like Problem 2(v) of Comprehensive Problems 1.

8. See Problem 2(iii) of Comprehensive Problems 1.

9. Like Problem 8(i) of Comprehensive Problems 1.

10. See Problem 9(iii) of Comprehensive Problems 1.

11. See article 2.

Fill in the Blanks

1. Like Problem 2(iv) of Comprehensive Problems 1.

2. By Maclaurin's theorem,

$$y = (y_0) + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \dots$$

But, it is given that $y = x + \frac{x^3}{6} + \frac{x^5}{24} + \dots$

So, comparing the coefficients of x^3 in both the expansions of y, we get

$$\frac{(y_3)_0}{3!} = \frac{1}{6} \Rightarrow (y_3)_0 = 1$$

3. We have $\frac{(y_3)_0}{3!} = \frac{2}{3!}$

$\therefore \quad (y_3)_0 = 2.$

4. See Problem 8(ii) of Comprehensive Problems 1.

True or False

1. If $y = f(x)$ is to be expanded by Maclaurin's theorem, y must be defined at $x = 0$. The function $\log x$ is not defined at $x = 0$ because $\log 0 = -\infty$.

2. In the Taylor's series $f(a + h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots$

if we put $a = 0$ and $h = x$, we get $f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots$

which is Maclaurin's series.

3. See article 3.

❍❍❍

Chapter-6

Indeterminate Forms

Comprehensive Problems 1

Problem 1: *State L' Hospital's rule.*

Solution: If $f(x)$ and $\phi(x)$ be two functions of x which can be expanded by Taylor's theorem in the neighbourhood of $x = a$ and if $f(a) = \phi(a) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)},$$

provided, the latter limit exists, finite or infinite.

This is generally known as *L' Hospital's Rule.*

Note: If $f'(a) = f''(a) = \ldots = f^{n-1}(a) = 0$

and $\phi'(a) = \phi''(a) = \ldots = \phi^{n-1}(a) = 0$

but $f^n(a)$ and $\phi^n(a)$ are not both zero, then by repeated application of Hospital's rule, we have

$$\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f^n(x)}{\phi^n(x)}.$$

Problem 2(i): *Evaluate* $\lim_{x \to 0} \frac{\sin x}{x}$.

Solution: Here $\lim_{x \to 0} \frac{\sin x}{x}$ [From 0/0]

$$= \lim_{x \to 0} \frac{\cos x}{1} = 1.$$

Problem 2(ii): *Evaluate* $\lim_{x \to 0} \frac{x - \sin x}{x^3}$.

Solution: Here $\lim_{x \to 0} \frac{x - \sin x}{x^3}$ [Form 0 /0 so we shall apply, L'Hospital's rule]

$= \lim_{x \to 0} \frac{1 - \cos x}{3x^2}$ [Form again 0/0]

$= \lim_{x \to 0} \frac{\sin x}{6x}$ [Form again 0/0]

$$= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}.$$

Aliter : $\lim_{x\to 0} \frac{x - \sin x}{x^3} = \lim_{x\to 0} \frac{x - [x - (x^3/3!) + (x^5/5!) - \dots]}{x^3}$

$$= \lim_{x\to 0} \frac{(x^3/6) - (x^5/120) + \dots}{x^3}$$

$$= \lim_{x\to 0} \left(\frac{1}{6} - \frac{x^2}{120} + \dots\right) = \frac{1}{6}.$$

Problem 2(iii): *Evaluate* $\lim_{x\to 0} \frac{e^x - 1}{x}$.

Solution: We have $\lim_{x\to 0} \frac{e^x - 1}{x}$ $\left[\text{Form } \frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{e^x - 0}{1} = 1.$$

Problem 2(iv): *Evaluate* $\lim_{x\to 0} \frac{1 - \cos x}{x^2}$.

(Meerut 2012B)

Solution: We have $\lim_{x\to 0} \frac{1 - \cos x}{x^2}$ $\left[\text{Form } \frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{\sin x}{2x} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

Problem 3(i): *Evaluate* $\lim_{x\to 1} \frac{x^5 - 2x^3 - 4x^2 + 9x - 4}{x^4 - 2x^3 + 2x - 1}$

Solution: We have

$$\lim_{x\to 1} \frac{x^5 - 2x^3 - 4x^2 + 9x - 4}{x^4 - 2x^3 + 2x - 1},$$

[Form 0 /0 so we shall apply L'Hospital's rule]

$$= \lim_{x\to 1} \frac{5x^4 - 6x^2 - 8x + 9}{4x^3 - 6x^2 + 2}, \qquad \text{[Form again 0 /0]}$$

$$= \lim_{x\to 1} \frac{20x^3 - 12x - 8}{12x^2 - 12x}, \quad \text{[Form 0 /0, } \therefore \text{ again apply L'Hospital's rule]}$$

$$= \lim_{x\to 1} \frac{60x^2 - 12}{24x - 12}$$

$$= \frac{60 - 12}{24 - 12} = \frac{48}{12} = 4.$$

Problem 3(ii): *Evaluate* $\lim_{x\to 0} \dfrac{a^x - b^x}{x}$.

(Agra 2003)

Solution: We have $\lim_{x\to 0} \dfrac{a^x - b^x}{x}$, [Form 0 /0]

$$= \lim_{x\to 0} \frac{a^x \log a - b^x \log b}{1}, \quad \text{by Hospital's rule}$$

$$= \log a - \log b = \log(a/b).$$

Problem 3(iii): *Evaluate* $\lim_{x\to 0} \dfrac{\log(1-x^2)}{\log \cos x}$.

Solution: We have $\lim_{x\to 0} \dfrac{\log(1-x^2)}{\log \cos x}$,

[Form 0 /0 so we shall apply L'Hospital's rule]

$$= \lim_{x\to 0} \frac{-2x/(1-x^2)}{-\sin x/\cos x} = \lim_{x\to 0} \left(\frac{2}{1-x^2} \cdot \frac{x}{\tan x} \right)$$

$$= \left(\lim_{x\to 0} \frac{2}{1-x^2} \right) \left(\lim_{x\to 0} \frac{x}{\tan x} \right) = 2 \times 1 = 2.$$

Problem 3(iv): *Evaluate* $\lim_{x\to 0} \dfrac{xe^x - \log(1+x)}{x^2}$,

Solution: We have $\lim_{x\to 0} \dfrac{xe^x - \log(1+x)}{x^2}$,

[Form 0 /0 so we shall apply, Hospital's rule]

$$= \lim_{x\to 0} \frac{xe^x + e^x - \{1/(1+x)\}}{2x}, \quad \text{[Form 0/0]}$$

$$= \lim_{x\to 0} \frac{xe^x + e^x + e^x + \{1/(1+x)^2\}}{2} = \frac{0+1+1+1}{2} = \frac{3}{2}.$$

Problem 4(i): *Evaluate* $\lim_{x\to 0} \dfrac{x - \sin x}{\tan^3 x}$.

Solution: We have $\lim_{x\to 0} \dfrac{x - \sin x}{\tan^3 x} = \lim_{x\to 0} \left\{ \dfrac{x - \sin x}{x^3} \cdot \left(\dfrac{x}{\tan x} \right)^3 \right\}$ **(Note)**

$$= \lim_{x\to 0} \frac{x - \sin x}{x^3} \cdot \lim_{x\to 0} \left(\frac{x}{\tan x} \right)^3 = \lim_{x\to 0} \frac{x - \sin x}{x^3}, \quad \left[\because \lim_{x\to 0} \frac{x}{\tan x} = 1 \right]$$

$$= \frac{1}{6}. \quad \text{[Proceeding as in problem 2(ii)]}$$

Problem 4(ii): *Evaluate* $\lim_{x\to 0} \dfrac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}$.

Solution: We have $\lim_{x\to 0} \dfrac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}$, [Form 0/0]

$$= \lim_{x\to 0} \frac{2\cos 2x + 4\sin x\cos x - 2\cos x}{-\sin x - 2\cos x(-\sin x)}$$

$$= \lim_{x\to 0} \frac{2\cos 2x + 2\sin 2x - 2\cos x}{-\sin x + \sin 2x}, \qquad \text{[Form 0/0]}$$

$$= \lim_{x\to 0} \frac{-4\sin 2x + 4\cos 2x + 2\sin x}{-\cos x + 2\cos 2x} = \frac{-0+4+0}{-1+2} = 4.$$

Problem 4(iii): *Evaluate* $\lim_{x\to 0} \dfrac{(1+x)^n - 1}{x}$.

Solution: We have $\lim_{x\to 0} \dfrac{(1+x)^n - 1}{x}$, [Form 0/0]

$$= \lim_{x\to 0} \frac{n(1+x)^{n-1}}{1}, \qquad \text{[By L'Hospital's rule]}$$

$$= n.$$

Aliter: We have $\lim_{x\to 0} \dfrac{(1+x)^n - 1}{x} = \lim_{x\to 0} \dfrac{1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots - 1}{x}$

$$= \lim_{x\to 0} \left\{ n + \frac{n(n-1)}{2!}x + \ldots \right\} = n.$$

Problem 4(iv): *Evaluate* $\lim_{x\to 1} \dfrac{\log x}{x-1}$.

(Garhwal 2001)

Solution: We have $\lim_{x\to 1} \dfrac{\log x}{x-1}$ [Form 0/0]

$$= \lim_{x\to 1} \frac{1/x}{1} = 1.$$

Problem 5(i): *Evaluate* $\lim_{x\to 0} \dfrac{e^x - e^{\sin x}}{x - \sin x}$.

Solution: Here $Nr. = e^x - e^{\sin x} = e^x - e^{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots} = e^x - e^x . e^{-\frac{x^3}{3!} + \frac{x^5}{5!} - \ldots}$

$$= e^x(1 - e^z), \quad \text{where } z = -\frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$= e^x\left[1 - \left(1 + z + \frac{z^2}{2!} + \ldots\right)\right] = -e^x\left[z + \frac{z^2}{2!} + \ldots\right]$$

$$= -e^x\left[\left(-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)+\frac{1}{2!}\left(-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)^2+\ldots\right]$$

$$= -e^x\left[-\frac{x^3}{6}+\frac{x^5}{120}-\ldots\right]= x^3e^x\left(\frac{1}{6}-\frac{x^2}{120}+\ldots\right).$$

Also Denom. $= x-\sin x = x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)$

$$= \frac{x^3}{6}-\frac{x^5}{120}+\ldots = x^3\left(\frac{1}{6}-\frac{x^2}{120}+\ldots\right).$$

$$\therefore \quad \lim_{x\to 0}\frac{e^x-e^{\sin x}}{x-\sin x}=\lim_{x\to 0}\frac{x^3 \,.\, e^x[(1/6)-(x^2/120)+\ldots]}{x^3[(1/6)-(x^2/120)+\ldots]}$$

$$= \lim_{x\to 0}\frac{e^x[(1/6)-(x^2/120)+\ldots]}{(1/6)-(x^2/120)+\ldots}=\frac{1/6}{1/6}=1.$$

Problem 5(ii): *Evaluate* $\lim_{x\to 0}\frac{1-\sqrt{(1-x^2)}}{x^2}$.

Solution: We have $\lim_{x\to 0}\frac{1-\sqrt{(1-x^2)}}{x^2}$,

[Form 0 / 0 so we shall apply, L'Hospital's rule]

$$= \lim_{x\to 0}\frac{0-\{\frac{1}{2}(1-x^2)^{-1/2}(-2x)\}}{2x}$$

$$= \lim_{x\to 0}\frac{x/\sqrt{(1-x^2)}}{2x}=\lim_{x\to 0}\frac{1}{2\sqrt{(1-x^2)}}=\frac{1}{2}.$$

Problem 5(iii): *Evaluate* $\lim_{x\to 0}\frac{\tan x-x}{x^2\tan x}$.

(Avadh 2014)

Solution: We have $\lim_{x\to 0}\frac{\tan x-x}{x^2\tan x}=\lim_{x\to 0}\left[\frac{\tan x-x}{x^3}\cdot\frac{x}{\tan x}\right]$, **(Note)**

$$= \lim_{x\to 0}\frac{\tan x-x}{x^3}, \qquad \text{[Form 0 /0]}$$

$$= \lim_{x\to 0}\frac{\sec^2 x-1}{3x^2}, \qquad \text{[Form 0 /0]}$$

$$= \lim_{x\to 0}\frac{2\sec x \,.\, \sec x\tan x}{6x}$$

$$= \lim_{x\to 0}\left(\frac{1}{3}\cdot\sec^2 x\cdot\frac{\tan x}{x}\right)=\frac{1}{3}\cdot 1\,.\,1=\frac{1}{3}.$$

Problem 5(iv): *Evaluate* $\lim_{x\to 0} \dfrac{\tan x - x}{x - \sin x}$.

Solution: We have $\lim_{x\to 0} \dfrac{\tan x - x}{x - \sin x}$ [Form 0/0]

$$= \lim_{x\to 0} \frac{\sec^2 x - 1}{1 - \cos x} \quad \text{[Form 0/0]}$$

$$= \lim_{x\to 0} \frac{2\sec x \,.\, \sec x \tan x}{\sin x} = \lim_{x\to 0} \frac{2\sec^2 x \,.\, \tan x}{\sin x} \quad \text{[form 0/0]}$$

$$= \lim_{x\to 0} \frac{4\sec^2 x \,.\, \tan^2 x + 2\sec^4 x}{\cos x}$$

$$= \frac{0 + 2}{1} = 2.$$

Problem 6(i): *Evaluate* $\lim_{x\to 0} \dfrac{\{\cosh x + \log(1 - x) - 1 + x\}}{x^2}$.

Solution: We have $\lim_{x\to 0} \dfrac{\cosh x + \log(1 - x) - 1 + x}{x^2}$ $\left[\text{Form } \dfrac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} \ldots\right) + \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\right) - 1 + x}{x^2}$$

$$= \lim_{x\to 0} \frac{-\frac{x^3}{3} + \frac{x^4}{4!} + (\text{terms containing } x \text{ and its higher powers})}{x^2}$$

$$= \lim_{x\to 0} \frac{x^3}{x^2}\left[-\frac{1}{3} + \frac{x}{4!} + \text{terms containing } x \text{ and its higher powers}\right]$$

$$= 0.$$

Problem 6(ii): *Evaluate* $\lim_{x\to 0} \dfrac{5\sin x - 7\sin 2x + 3\sin 3x}{\tan x - x}$.

(Meerut 2001)

Solution: We have $\lim_{x\to 0} \dfrac{5\sin x - 7\sin 2x + 3\sin 3x}{\tan x - x}$, [Form 0/0]

$$= \lim_{x\to 0} \frac{5\left(x - \frac{x^3}{3!} + \ldots\right) - 7\left\{(2x) - \frac{(2x)^3}{3!} + \ldots\right\} + 3\left\{(3x) - \frac{(3x)^3}{3!} + \ldots\right\}}{\left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots\right) - x}$$

$$= \lim_{x\to 0} \frac{x^3\left(-\frac{5}{6} + \frac{28}{3} - \frac{27}{2} + \text{higher powers of } x\right)}{\left(\frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots\right)}$$

$$= \lim_{x\to 0} \frac{-5 + \text{terms containing } x \text{ and its higher powers}}{(\frac{1}{3} + \frac{2}{15}x^2 + \ldots)}$$

$$= -\frac{5}{1/3} = -15.$$

Problem 6(iii): *Evaluate* $\lim_{x\to 0} \frac{[e^x + \log\{(1-x)/e\}]}{\tan x - x}$.

Solution: We have $\lim_{x\to 0} \frac{e^x + \log\left(\frac{1-x}{e}\right)}{\tan x - x} = \lim_{x\to 0} \frac{e^x + \log(1-x) - \log e}{\tan x - x}$ **(Note)**

$$= \lim_{x\to 0} \frac{e^x + \log(1-x) - 1}{\tan x - x}, \qquad \text{[Form 0/0]}$$

$$= \lim_{x\to 0} \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\right) - 1}{\left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\right) - x}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{6}x^3 (1 + \text{terms containing } x \text{ and its higher powers})}{\frac{1}{3}x^3 (1 + \text{terms containing } x \text{ and its higher powers})} = -\frac{1}{2}.$$

Problem 6(iv): *Evaluate* $\lim_{x\to 1} \frac{x\sqrt{3x - 2x^4} - x^{6/5}}{1 - x^{2/3}}$

Solution: We have $\lim_{x\to 1} \frac{x\sqrt{3x-2x^4} - x^{6/5}}{1 - x^{2/3}}$ $\left[\text{Form } \frac{0}{0}\right]$

$$= \lim_{x\to 1} \frac{x\frac{1}{2}(3x-2x^4)^{-1/2}(3-8x^3) + \sqrt{3x-2x^4} - \frac{6}{5}x^{\frac{1}{5}}}{-\frac{2}{3}x^{-1/3}}$$

$$= \frac{1\frac{1}{2}(3-2)^{-1/2}(3-8) + \sqrt{3-2} - \frac{6}{5}}{-2/3} = \frac{\frac{-5}{2} + 1 - \frac{6}{5}}{-2/3} = \frac{27}{10} \times \frac{3}{2} = \frac{81}{20}.$$

Problem 7(i): *Evaluate* $\lim_{x\to \pi/2} \frac{\cos x}{x - \frac{1}{2}\pi}$.

Solution: We have $\lim_{x\to \pi/2} \frac{\cos x}{x - \frac{\pi}{2}}$. $\left[\text{Form } \frac{0}{0}\right]$

$$= \lim_{x\to \pi/2} -\frac{\sin x}{1} = -\sin\frac{\pi}{2} = -1.$$

Problem 7(ii): *Evaluate* $\lim_{x \to a} \dfrac{a^x - x^a}{x^x - a^a}$

Solution: We have $\lim_{x \to a} \dfrac{a^x - x^a}{x^x - a^a}$ $\left[\text{Form } \dfrac{0}{0}\right]$

$$= \lim_{x \to a} \frac{a^x \log a - a\, x^{a-1}}{x^x (1 + \log x) - 0}$$

$$= \frac{a^a \log a - a\, a^{a-1}}{a^a (1 + \log a)} = \frac{(\log a - 1)}{(\log a + 1)}.$$

Problem 7(iii): *Evaluate* $\lim_{x \to 0} \dfrac{(1 + x)^{1/x} - e + \frac{1}{2} ex}{x^2}$.

(Agra 2002; Avadh 06; Purvanchal 14)

Solution: Here $\lim_{x \to 0} \dfrac{(1 + x)^{1/x} - e + \frac{1}{2} ex}{x^2}$, [Form 0 / 0]

$$= \lim_{x \to 0} \frac{e\left[1 - \frac{1}{2} x + \frac{11}{24} x^2 + \ldots\ldots\right] - e + \frac{1}{2} ex}{x^2}$$

(See Example 5 after article 3)

$$= \lim_{x \to 0} \frac{e\,[(11/24)\, x^2 + \text{terms containing higher powers of } x]}{x^2}$$

$$= \lim_{x \to 0} e\,[(11/24) + \text{terms containing } x \text{ and its higher powers}]$$

$$= (11/24)\, e.$$

Problem 7(iv): *Evaluate* $\lim_{x \to 0} \dfrac{\sin x . \sin^{-1} x - x^2}{x^6}$.

Solution: We have $\lim_{x \to 0} \dfrac{\sin x . \sin^{-1} x - x^2}{x^6}$ $\left[\text{Form} \dfrac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{1}{x^6} \left[\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \right) \left(x + \frac{1}{2} \frac{x^3}{3} + \frac{1.3}{2.4} \frac{x^5}{5} + \ldots \right) - x^2 \right]$$

$$= \lim_{x \to 0} \frac{1}{x^6} \left[x^2 + x^4 \left(\frac{1}{6} - \frac{1}{6} \right) + x^6 \left(\frac{3}{10} - \frac{1}{36} + \frac{1}{120} \right) + \ldots \right]$$

$$= \lim_{x \to 0} \frac{1}{x^6} \left[x^6 . \frac{1}{18} + \text{higher powers of } n \right]$$

$$= \frac{1}{18}.$$

Problem 8(i): *Evaluate* $\lim_{x\to 0} \frac{x^2 + 2\cos x - 2}{x \sin^3 x}$.

Solution: We have $\lim_{x\to 0} \frac{x^2 + 2\cos x - 2}{x \sin^3 x} = \lim_{x\to 0}\left[\frac{x^2 + 2\cos x - 2}{x^4} \cdot \frac{x^3}{\sin^3 x}\right]$

$$= \lim_{x\to 0}\left(\frac{x}{\sin x}\right)^3 \cdot \lim_{x\to 0} \frac{x^2 + 2\cos x - 2}{x^4}$$

$$= \lim_{x\to 0} \frac{x^2 + 2\cos x - 2}{x^4}, \qquad \left[\because \lim_{x\to 0} \frac{x}{\sin x} = 1\right]$$

$$= \lim_{x\to 0} \frac{2x - 2\sin x}{4x^3}, \qquad \text{[By L'Hospital's rule for the form } 0/0]$$

$$= \lim_{x\to 0} \frac{2 - 2\cos x}{12x^2} \qquad \text{[Form again } 0/0]$$

$$= \lim_{x\to 0} \frac{2\sin x}{24x} = \frac{1}{12} \lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{12} \cdot 1 = \frac{1}{12}.$$

Problem 8(ii): *Evaluate* $\lim_{x\to 0} \frac{e^x - e^{x\cos x}}{x - \sin x}$.

(Kumaun 2012)

Solution: We have $\lim_{x\to 0} \frac{e^x - e^{x\cos x}}{x - \sin x}$, [Form 0/0]

$$= \lim_{x\to 0} \frac{e^x - e^{x\cos x}(\cos x - x\sin x)}{1 - \cos x}, \text{ by Hospital's Rule}$$

[Form again 0/0]

$$= \lim_{x\to 0} \frac{e^x - e^{x\cos x}(\cos x - x\sin x)^2 - e^{x\cos x}(-2\sin x - x\cos x)}{\sin x},$$

by Hospital's rule

$$= \lim_{x\to 0} \frac{e^x - e^{x\cos x}(\cos x - x\sin x)^2 + e^{x\cos x}(2\sin x + x\cos x)}{\sin x},$$

[Form 0/0]

$$= \lim_{x\to 0} \frac{\begin{aligned} e^x - e^{x\cos x}&(\cos x - x\sin x)^3 \\ &- 2e^{x\cos x}(\cos x - x\sin x)(-2\sin x - x\cos x) \\ &+ e^{x\cos x}(\cos x - x\sin x).(2\sin x + x\cos x) \\ &+ e^{x\cos x}(3\cos x - x\sin x)\end{aligned}}{\cos x}$$

$$= \frac{1 - 1(1-0)^3 - 2\times 1\times 1\times 0 + 1\times 1\times 0 + 1\times(3-0)}{1} = \frac{1 - 1 + 3}{1} = 3.$$

Problem 8(iii): *Evaluate* $\lim\limits_{x \to 0} \dfrac{\cosh x - \cos x}{x \sin x}$.

(Meerut 2012)

Solution: We have $\lim\limits_{x \to 0} \dfrac{\cosh x - \cos x}{x \sin x}$, $\left[\text{Form } \dfrac{0}{0}\right]$

$$= \lim_{x \to 0} \left\{ \frac{\cosh x - \cos x}{x^2} \cdot \left(\frac{x}{\sin x} \right) \right\}$$ **(Note)**

$$= \lim_{x \to 0} \frac{\cosh x - \cos x}{x^2}, \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{\sinh x + \sin x}{2x}, \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{\cosh x + \cos x}{2} = \frac{1+1}{2} = 1.$$

Problem 8(iv): *Evaluate* $\lim\limits_{x \to 0} \dfrac{\sin 2x + a \sin x}{x^2}$

Solution: We have $\lim\limits_{x \to 0} \dfrac{\sin 2x + a \sin x}{x^2}$ $\left[\text{Form } \dfrac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{2 \cos 2x + a \cos x}{2x} = \infty, \text{ if } a \neq -2$$

$$= \lim_{x \to 0} \frac{-4 \sin 2x - a \sin x}{2} = 0, \text{ if } a = 2$$

Problem 9(i): *Evaluate* $\lim\limits_{x \to 0} \dfrac{\sin^2 x - x^2}{x^4}$

Solution: We have $\lim\limits_{x \to 0} \dfrac{\sin^2 x - x^2}{x^4}$ $\left[\text{Form } \dfrac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{\left(x - \frac{x^3}{3!} + \dots \right)^2 - x^2}{x^4} = \lim_{x \to 0} \frac{\left(x^2 + \left(\frac{x^3}{3!} \right)^2 - 2.x.\frac{x^3}{3!} \dots \right) - x^2}{x^4}$$

$$= \lim_{x \to 0} \frac{x^4 \left[-\frac{2}{6} + \text{ higher powers of } x \right]}{x^4} = -\frac{1}{3}$$

Problem 9(ii): *Evaluate* $\lim\limits_{x \to 0} \dfrac{\sin x \sin^{-1} x}{x^2}$.

Solution: We have $\lim\limits_{x \to 0} \dfrac{\sin x \sin^{-1} x}{x^2} = \lim\limits_{x \to 0} \left(\dfrac{\sin^{-1} x}{x} \cdot \dfrac{\sin x}{x} \right)$

$$= \lim_{x \to 0} \frac{\sin^{-1} x}{x}, \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{1/\sqrt{(1-x^2)}}{1} = 1.$$

Problem 10: *Find the values of a and b in order that*

$$\lim_{x\to 0} \frac{x(1 + a\cos x) - b\sin x}{x^3} \text{ may be equal to } 1.$$

(Meerut 2013B)

Solution: We have $\lim_{x\to 0} \dfrac{x(1 + a\cos x) - b\sin x}{x^3}$,

[Form 0/0 so we shall apply Hospital's rule]

$$= \lim_{x\to 0} \frac{1 + a\cos x - ax\sin x - b\cos x}{3x^2}. \quad \ldots(1)$$

Now the denominator of (1) $\to 0$ as $x \to 0$. Therefore if the numerator of (1) does not tend to 0 as $x \to 0$, then the given limit cannot be equal to 1. Hence for the given limit to be equal to 1 the numerator of (1) must also $\to 0$ as $x \to 0$.

$\therefore \quad 1 + a - b = 0 \quad$ or $\quad a - b = -1. \quad \ldots(2)$

Now if $1 + a - b = 0$, then (1) takes the form $0/0$. Hence by applying L' Hospital's rule to (1), the given limit is equal to

$$\lim_{x\to 0} \frac{-a\sin x - a\sin x - ax\cos x + b\sin x}{6x}$$

$$= \lim_{x\to 0} \frac{-ax\cos x + (b - 2a)\sin x}{6x}, \quad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{-a\cos x + ax\sin x + (b-2a)\cos x}{6}, \quad \text{[By L'Hospital's rule]}$$

$$= \frac{-a + b - 2a}{6} = \frac{b - 3a}{6} = 1, \quad \text{(as given)}$$

$\therefore \quad b - 3a = 6. \quad \ldots(3)$

Adding (2) and (3), we have $-2a = 5$ or $a = -5/2$.

$\therefore \quad b = a + 1 = (-5/2) + 1 = -3/2.$

Hence $\quad a = -5/2, \quad b = -3/2.$

Problem 11: *Find the values of a, b, c so that* $\lim_{x\to 0} \dfrac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2.$

(Kumaun 2007)

Solution: Here the given limit

$$= \lim_{x\to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x}. \quad \left[\text{Form } \frac{a - b + c}{0}\right]$$

$\therefore$ For the given limit to be equal to 2, we must have

$a - b + c = 0. \quad \ldots(1)$

Now applying L'Hospital's rule for the form $0/0$, we have the given limit

$$= \lim_{x\to 0} \frac{ae^x + b\sin x - ce^{-x}}{\sin x + x\cos x}. \quad \left[\text{form } \frac{a - c}{0}\right]$$

$\therefore$ For the given limit to be equal to 2, we must have

$$a - c = 0. \qquad \ldots(2)$$

Now again applying L' Hospital's rule for the form 0 /0, we have the given limit

$$= \lim_{x \to 0} \frac{ae^{x} + b \cos x + ce^{-x}}{\cos x + \cos x - x \sin x} = \frac{a + b + c}{2}.$$

$\therefore$ for the given limit to be equal to 2, we must have

$$(a + b + c)/2 = 2 \text{ i.e., } a + b + c = 4. \qquad \ldots(3)$$

Solving (1), (2) and (3), we get $a = 1, \ b = 2, \ c = 1.$

Problem 12: *Find the values of a, b, c so that* $\lim_{x \to 0} \frac{x(a + b \cos x) - c \sin x}{x^5} = 1.$

Solution: We have $\lim_{x \to 0} \frac{x(a + b \cos x) - c \sin x}{x^5}$

[Form 0 /0 so we shall apply L' Hospital's rule]

$$= \lim_{x \to 0} \frac{(a + b \cos x) - bx \sin x - c \cos x}{x^5} \qquad \ldots(1)$$

Now the denominator of (1) $\to 0$ as $x \to 0$.

But if the numerator of (1) does not tend to zero as $x \to 0$, then the given limit becomes infinite. Hence for the given limit to be equal to 1, the numerator of (1) must $\to 0$ as $x \to 0$

$$\therefore \qquad a + b - c = 0 \qquad \ldots(2)$$

Now if $a + b - c = 0$, then (1) takes the form 0/0.

Hence by applying L' Hospital's rule to (1), the given limit is equal to

$$\lim_{x \to 0} \frac{-(b - c) \sin x - b \sin x - bx \cos x}{20 x^3} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{-(2b - c) \cos x - b \cos x + bx \sin x}{60 x^2} \qquad \text{[By L' Hospital's rule]}$$

$$= \lim_{x \to 0} \frac{-(3b - c) \cos x + bx \sin x}{60 x^2}$$

$\therefore$ For the given limit to be equal to 1, we must have

$$-3b + c = 0 \qquad \ldots(3)$$

Now again applying L' Hospital's rule for the form 0/0, we have the given limit

$$= \lim_{x \to 0} \frac{(3b - c) \sin x + b \sin x + bx \cos x}{120 x} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{(4b - c) \cos x + b \cos x - bx \sin x}{120} \qquad \text{[By L' Hospital's rule]}$$

$$= \frac{5b - c}{120}.$$

$\therefore$ For the given limit to be equal to 1, we must have

$$\frac{5b-c}{120}=1, \quad i.e., \quad 5b-c=120 \quad \text{...(4)}$$

Solving (2), (3) and (4), we get $a=120,\ b=60,\ c=180$.

Comprehensive Problems 2

Problem 1(i): *Evaluate* $\lim_{x\to\infty} \frac{x}{e^x}$

Solution: We have $\lim_{x\to\infty} \frac{x}{e^x}$ [Form ∞/∞]

$$= \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0 .$$

Problem 1(ii): *Evaluate* $\lim_{x\to\infty} \frac{e^x}{x^3}$

Solution: We have $\lim_{x\to\infty} \frac{e^x}{x^3}$ [Form ∞/∞]

Apply L' Hospital's Rule (3 times)

$$= \lim_{x\to\infty} \frac{e^x}{6} = \frac{e^\infty}{6} = \frac{\infty}{6} = \infty .$$

Problem 2(i): *Evaluate* $\lim_{x\to 1} \frac{\log(1-x)}{\cot \pi x}$.

Solution: We have $\lim_{x\to 1} \frac{\log(1-x)}{\cot \pi x}$ [Form ∞/∞]

$$= \lim_{x\to 1} \frac{\frac{-1}{1-x}}{-\operatorname{cosec}^2 \pi x} = \lim_{x\to 1} \frac{\sin^2 \pi x}{1-x} = \lim_{x\to 1} \frac{2\sin \pi x \cos \pi x.\ \pi}{-1} = 0 .$$

Problem 2(ii): *Evaluate* $\lim_{x\to\infty} \frac{\log x}{a^x},\ a>1$. **(Garhwal 2003)**

Solution: We have $\lim_{x\to\infty} \frac{\log x}{a^x}$ [Form ∞/∞]

$$= \lim_{x\to\infty} \frac{1/x}{a^x \log a} = \frac{0}{\infty} = 0 \times 0 = 0.$$

Problem 3(i): *Evaluate* $\lim_{x\to a} \frac{\log(x-a)}{\log(e^x - e^a)}$.

Solution: We have $\lim_{x\to a} \frac{\log(x-a)}{\log(e^x - e^a)}$, [Form ∞/∞]

$$= \lim_{x\to a} \frac{1/(x-a)}{\{1/(e^x - e^a)\}\, e^x} = \lim_{x\to a} \frac{e^x - e^a}{e^x (x-a)}, \quad \text{[Form 0/0]}$$

$$= \lim_{x \to a} \frac{e^x}{e^x (x-a) + e^x} = \lim_{x \to a} \frac{e^x}{e^x [(x-a)+1]} = \lim_{x \to a} \frac{1}{x-a+1} = 1.$$

Problem 3(ii): *Evaluate* $\lim_{x \to \frac{1}{2}} \frac{\sec \pi x}{\tan 3 \pi x}$.

Solution: We have $\lim_{x \to \frac{1}{2}} \frac{\sec \pi x}{\tan 3 \pi x}$ $\left[\text{Form } \frac{\infty}{\infty}\right]$

$$= \lim_{x \to \frac{1}{2}} \frac{\cos 3 \pi x}{\sin 3 \pi x . \cos \pi x} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x \to \frac{1}{2}} \frac{-\sin 3 \pi x . 3 \pi}{\cos 3 \pi x . 3 \pi \cdot \cos \pi x - \sin 3 \pi x . \sin \pi x . \pi}$$

$$= \frac{1.3 \pi}{0 + 1. \pi} = 3 .$$

Problem 4(i): *Evaluate* $\lim_{x \to \infty} \left\{ \frac{(\log x)^3}{x} \right\}$.

Solution: We have $\lim_{x \to \infty} \frac{(\log x)^3}{x}$ $\left[\text{Form } \frac{\infty}{\infty}\right]$

$$= \lim_{x \to \infty} \frac{3 (\log x)^2 . \frac{1}{x}}{1} = \lim_{x \to \infty} \frac{3 (\log x)^2}{x}$$

$$= \lim_{x \to \infty} \frac{6 (\log x) \frac{1}{x}}{1} = \lim_{x \to \infty} \frac{6 (\log x)}{x} = \lim_{x \to \infty} \frac{6 \cdot \frac{1}{x}}{1} = \frac{6}{\infty} = 0 .$$

Problem 4(ii): *Evaluate* $\lim_{x \to \infty} \frac{x (\log x)^3}{1 + x + x^2}$.

Solution: We have $\lim_{x \to \infty} \frac{x (\log x)^3}{1 + x + x^2}$ $\left[\text{Form } \frac{\infty}{\infty}\right]$

$$= \lim_{x \to \infty} \frac{x . 3 (\log x)^2 . \frac{1}{x} + (\log x)^3}{0 + 1 + 2x}$$

$$= \lim_{x \to \infty} \frac{3 (\log x)^2 + (\log x)^3}{1 + 2x} \qquad \left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x \to \infty} \frac{3 (\log x) \frac{1}{x} + 3 (\log x)^2 . \frac{1}{x}}{2}$$

$$= \lim_{x \to \infty} \frac{3 \log x + 3 (\log x)^2}{2x} \qquad \left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x\to\infty} \frac{3\frac{1}{x} + 6(\log x).\frac{1}{x}}{2} = \lim_{x\to\infty} \frac{3 + 6\log x}{2x} \qquad \left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x\to\infty} \frac{\frac{6}{x}}{2} = \frac{3}{\infty} = 0.$$

Problem 5(i): *Evaluate* $\lim_{x\to\infty} \left\{x \tan\left(\frac{1}{x}\right)\right\}.$ **(Bundelkhand 2001)**

Solution: We have $\lim_{x\to\infty} [\tan(1/x)].x,$ [Form $0 \times \infty$]

$$= \lim_{x\to\infty} [\tan(1/x)]/[1/x], \text{ [form 0/0]}$$

$$= \lim_{x\to\infty} \frac{[(-1/x^2)\sec^2(1/x)]}{[-1/x^2]} = \lim_{x\to\infty} \sec^2(1/x) = 1.$$

Problem 5(ii): *Evaluate* $\lim_{x\to 0} \frac{\log \tan 2x}{\log \tan 3x}.$

Solution: We have $\lim_{x\to 0} \frac{\log \tan 2x}{\log \tan 3x}$ $\left[\text{Form } \frac{\infty}{\infty}\right]$

$$= \lim_{x\to 0} \frac{\frac{2\sec^2 2x}{\tan 2x}}{\frac{3\sec^2 3x}{\tan 3x}} = \lim_{x\to 0} \frac{2}{3} \frac{\sec^2 2x.\tan 3x}{\sec^2 3x.\tan 2x}$$

$$= \lim_{x\to 0} \frac{2}{3}\left[\frac{2\sec^2 2x \tan 2x.2\tan 3x + \sec^2 2x.\sec^2 3x.3}{2\sec^2 3x.\tan 3x.3.\tan 2x + \sec^2 3x.\sec^2 2x.2}\right] = \frac{2}{3}\left[\frac{0+3}{0+2}\right] = 1.$$

Problem 6(i): *Evaluate* $\lim_{x\to\pi/2} \frac{\log(x - \frac{1}{2}\pi)}{\tan x}.$

Solution: We have $\lim_{x\to\pi/2} \frac{\log(x - \frac{1}{2}\pi)}{\tan x},$ $\left[\text{Form } \frac{\infty}{\infty}\right]$

$$= \lim_{x\to\pi/2} \frac{1/(x - \frac{1}{2}\pi)}{\sec^2 x} = \lim_{x\to\pi/2} \frac{\cos^2 x}{x - \frac{1}{2}\pi}, \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x\to\pi/2} \frac{-2\cos x \sin x}{1} = \lim_{x\to\pi/2} (-\sin 2x) = 0.$$

Problem 6(ii): *Evaluate* $\lim_{x\to a} \frac{c\,[e^{1/(x-a)} - 1]}{[e^{1/(x-a)} + 1]}.$

Solution: We have $\lim_{x\to a} \frac{c\,[e^{1/(x-a)} - 1]}{[e^{1/(x-a)} + 1]}$ $\left[\text{Form } \frac{\infty}{\infty}\right]$

$$= \lim_{x \to a} \frac{c\, e^{1/(x-a)}\ \{-1/(x-a)^2\}}{e^{1/(x-a)}\ \{-1/(x-a)^2\}} = c\,.$$

Problem 7(i): *Evaluate* $\lim_{x \to 1} \left[\frac{2}{x^2 - 1} - \frac{1}{x-1}\right]$.

(Garhwal 2002)

Solution: We have $\lim_{x \to 1} \left[\frac{2}{x^2 - 1} - \frac{1}{x-1}\right]$ [Form ∞ – ∞]

$$= \lim_{x \to 1} \left[\frac{2 - (x+1)}{(x^2 - 1)}\right] \quad \text{[Form 0/0]}$$

$$= \lim_{x \to 1} \frac{(1-x)}{(x^2 - 1)} = \lim_{x \to 1} \ -\frac{1}{2x} = -\frac{1}{2}.$$

Problem 7(ii): *Evaluate* $\lim_{x \to \pi/2} (\sec x - \tan x)$.

Solution: We have $\lim_{x \to \pi/2} (\sec x - \tan x)$ [Form ∞ – ∞]

$$= \lim_{x \to \pi/2} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x}, \quad \text{[Form 0/0]}$$

$$= \lim_{x \to \pi/2} \frac{-\cos x}{-\sin x} = \lim_{x \to \pi/2} \cot x = 0.$$

Comprehensive Problems 3

Problem 1(i): *Evaluate* $\lim_{x \to 1} \left(\frac{1}{\log x} - \frac{x}{\log x}\right)$.

Solution: We have $\lim_{x \to 1} \left(\frac{1}{\log x} - \frac{x}{\log x}\right)$ [Form ∞ – ∞]

$$= \lim_{x \to 1} \left(\frac{1 - x}{\log x}\right) \quad \text{[Form 0/0]}$$

$$= \lim_{x \to 1} \frac{-1}{1/x} = -1\,.$$

Problem 1(ii): *Evaluate* $\lim_{x \to 0} \frac{\cot x - (1/x)}{x}$.

Solution: We have $\lim_{x \to 0} \frac{\cot x - (1/x)}{x}$

$$= \lim_{x \to 0} \frac{x \cos x - \sin x}{x^2 \sin x}, \quad \text{[Form 0/0]}$$

$$= \lim_{x \to 0} \left(\frac{x \cos x - \sin x}{x^3} \cdot \frac{x}{\sin x}\right) = \lim_{x \to 0} \frac{x \cos x - \sin x}{x^3}, \quad \text{[Form 0/0]}$$

$$= \lim_{x\to 0} \frac{\cos x - x\sin x - \cos x}{3x^2} = \lim_{x\to 0}\left(-\frac{1}{3}\cdot\frac{\sin x}{x}\right) = -\frac{1}{3}\cdot 1 = -\frac{1}{3}.$$

Problem 2(i): *Evaluate* $\lim_{x\to 0} \frac{\operatorname{cosec} x - \cot x}{x}$.

Solution: We have $\lim_{x\to 0} \frac{\operatorname{cosec} x - \cot x}{x}$

$$= \lim_{x\to 0} \frac{1-\cos x}{x\sin x} \qquad \text{[Form 0/0]}$$

$$= \lim_{x\to 0} \frac{0+\sin x}{x\cos x + \sin x} \qquad \text{[Form 0/0]}$$

$$= \lim_{x\to 0} \frac{\cos x}{-x\sin x + \cos x + \cos x} = \frac{1}{2}.$$

Example 2(ii): *Evaluate* $\lim_{x\to 0}\left[\frac{1}{e^x - 1} - \frac{1}{x}\right]$.

Solution: We have $\lim_{x\to 0}\left[\frac{1}{e^x-1} - \frac{1}{x}\right]$, [Form $\infty - \infty$]

$$= \lim_{x\to 0} \frac{x - e^x + 1}{x(e^x - 1)}, \qquad \text{[Form 0/0]}$$

$$= \lim_{x\to 0} \frac{1-e^x}{e^x - 1 + xe^x}, \qquad \text{[Form 0/0]}$$

$$= \lim_{x\to 0} \frac{-e^x}{e^x + e^x + xe^x} = -\frac{1}{2}.$$

Problem 3(i): *Evaluate* $\lim_{x\to \pi/2}\left(x\tan x - \frac{\pi}{2}\sec x\right)$.

Solution: We have $\lim_{x\to\pi/2}\left(x\tan x - \frac{\pi}{2}\sec x\right)$ [Form ∞/∞]

$$= \lim_{x\to\pi/2} \frac{2x\sin x - \pi}{2\cos x} \qquad \text{[Form 0/0]}$$

$$= \lim_{x\to\pi/2} \frac{2x\cos x + 2\sin x}{-2\sin x} = -1.$$

Problem 3(ii): *Evaluate* $\lim_{x\to 0} x\log x$.

Solution: We have $\lim_{x\to 0} x\log x$, [Form $0\times\infty$]

$$= \lim_{x\to 0} \frac{\log x}{1/x}, \qquad \text{[Form } \infty/\infty\text{]}$$

$$= \lim_{x\to 0} \frac{1/x}{-(1/x^2)} = \lim_{x\to 0}(-x) = 0.$$

Problem 4(i): *Evaluate* $\lim_{x \to 0} \sin x . \log x$.

Solution: We have $\lim_{x \to 1} \sin x . \log x$ [Form $0 \times \infty$]

$$= \lim_{x \to 0} \frac{\log x}{\operatorname{cosec} x} \quad \text{[Form } \infty/\infty]$$

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x} = \lim_{x \to 0} -\frac{\sin^2 x}{\cos x} = \frac{0}{1} = 0 .$$

Problem 4(ii): *Evaluate* $\lim_{x \to \infty} (a^{1/x} - 1)\, x$.

Solution: We have $\lim_{x \to \infty} (a^{1/x} - 1)\, x$ [Form $0 \times \infty$]

$$= \lim_{x \to \infty} \frac{a^{1/x} - 1}{1/x}, \quad \text{[Form } 0/0]$$

$$= \lim_{x \to \infty} \frac{a^{1/x} . \log a . (-1/x^2)}{-1/x^2}$$

$$= \lim_{x \to \infty} a^{1/x} \log a = a^0 \log a = \log a.$$

Problem 5(i): *Evaluate* $\lim_{x \to \infty} 2^x \sin\left(\frac{a}{2^x}\right)$.

(Agra 2003)

Solution: We have $\lim_{x \to \infty} 2^x \sin\left(\frac{a}{2^x}\right)$, [Form $\infty \times 0$]

$$= \lim_{x \to \infty} \frac{\sin (a . 2^{-x})}{2^{-x}}, \quad \text{[Form } 0/0]$$

$$= \lim_{x \to \infty} \frac{\{\cos (a . 2^{-x})\} . a . 2^{-x} (\log 2) . (-1)}{2^{-x} (\log 2) . (-1)}$$

$$= \lim_{x \to \infty} a \cos\left(\frac{a}{2^x}\right) = a \cos 0 = a . 1 = a.$$

Problem 5(ii): *Evaluate* $\lim_{x \to \pi/2} (1 - \sin x) \tan x$.

Solution: We have $\lim_{x \to \pi/2} (1 - \sin x) \tan x$ [Form $0 \times \infty$]

$$= \lim_{x \to \pi/2} \frac{1 - \sin x}{\cot x} \quad \text{[Form } 0/0]$$

$$= \lim_{x \to \pi/2} \frac{-\cos x}{-\operatorname{cosec}^2 x}$$

$$= \lim_{x \to \pi/2} \cos x \sin^2 x = 0 . 1 = 0 .$$

Problem 6: *Evaluate* $\lim_{x\to 0} x^m (\log x)^n$, *where m, n are positive integers.*

Solution: We have $\lim_{x\to 0} x^m (\log x)^n$, [Form $0\times\infty$]

$$= \lim_{x\to 0} \frac{(\log x)^n}{x^{-m}}, \quad \text{[Form } \infty/\infty]$$

$$= \lim_{x\to 0} \frac{n(\log x)^{n-1}(1/x)}{-m\,x^{-m-1}} = \lim_{x\to 0}\left[-\frac{n}{m}\cdot\frac{(\log x)^{n-1}}{x^{-m}}\right],$$

[Form ∞/∞ if $n>1$]

$$= \lim_{x\to 0}\left(-\frac{n}{m}\right)\cdot\frac{(n-1)(\log x)^{n-2}\,.\,(1/x)}{-mx^{-m-1}}$$

$$= \lim_{x\to 0}(-1)^2\,\frac{n(n-1)}{m^2}\cdot\frac{(\log x)^{n-2}}{x^{-m}}, \quad \text{[Form } \infty/\infty \text{ if } n>2]$$

$$= \lim_{x\to 0}(-1)^n\cdot\frac{n(n-1)(n-2)\ldots \text{upto } n \text{ factors}}{m^n}\cdot\frac{(\log x)^{n-n}}{x^{-m}},$$

[By repeated application of the above process]

$$= \lim_{x\to 0}(-1)^n\frac{n!}{m^n}\cdot x^m = (-1)^n\frac{n!}{m^n}\cdot\lim_{x\to 0} x^m = 0.$$

Comprehensive Problems 4

Problem 1(i): *Evaluate* $\lim_{x\to 0} x^x$.

(Agra 2002; Kanpur 04)

Solution: Let $y = \lim_{x\to 0} x^x$. [Form 0^0]

$\therefore$ $\log y = \lim_{x\to 0} x\log x$, [Form $0\times\infty$]

$$= \lim_{x\to 0}\frac{\log x}{(1/x)}, \quad \text{[Form } \infty/\infty]$$

$$= \lim_{x\to 0}\frac{1/x}{(-1/x^2)} = \lim_{x\to 0}(-x) = 0.$$

$\therefore$ $y = e^0 = 1.$

Problem 1(ii): *Evaluate* $\lim_{x\to 0}(\cos x)^{1/x}$.

Solution: Let $y = \lim_{x\to 0}(\cos x)^{1/x}$. [Form 1^∞]

$\therefore$ $\log y = \lim_{x\to 0}\left(\frac{1}{x}\right)(\log\cos x)$, [Form $\infty\times 0$]

$$= \lim_{x\to 0}\frac{\log\cos x}{x}, \quad \text{[Form } 0/0]$$

$$= \lim_{x\to 0}\frac{(1/\cos x)(-\sin x)}{1} = \lim_{x\to 0}(-\tan x) = 0.$$

$\therefore$ $y = e^0 = 1.$

Problem 2(i): *Evaluate* $\lim_{x\to 0} (\cos x)^{1/x^2}$.

(Garhwal 2003)

Solution: Let $y = \lim_{x\to 0} (\cos x)^{1/x^2}$. [Form 1^∞]

$\therefore \quad \log y = \lim_{x\to 0} \left(\frac{1}{x^2}\right)(\log \cos x),$ [Form $\infty \times 0$]

$= \lim_{x\to 0} \frac{\log \cos x}{x^2},$ [Form 0/0]

$= \lim_{x\to 0} \frac{(1/\cos x)(-\sin x)}{2x},$ [By L' Hospital's rule]

$= \lim_{x\to 0} \left(\frac{-\tan x}{2x}\right),$ [Form 0/0]

$= \lim_{x\to 0} \left(\frac{-\sec^2 x}{2}\right),$ [By L' Hospital's Rule]

$= -\frac{1}{2}.$

$\therefore \quad y = e^{-1/2}.$

Problem 2(ii): *Evaluate* $\lim_{x\to \pi/2} (\sin x)^{\tan x}$.

(Garhwal 2003)

Solution: Let $y = \lim_{x\to \pi/2} (\sin x)^{\tan x}$ [Form 1^∞]

$\therefore \quad \log y = \lim_{x\to \pi/2} \tan x \,.\, \log \sin x,$ [form $\infty \times 0$]

$= \lim_{x\to \pi/2} \frac{\log \sin x}{\cot x},$ [Form 0/0]

$= \lim_{x\to \pi/2} \frac{(1/\sin x)\cos x}{-\operatorname{cosec}^2 x},$ [By L' Hospital's rule]

$= \lim_{x\to \pi/2} (-\sin x \cos x) = 0.$

$\therefore \quad y = e^0 = 1.$

Problem 3(i): *Evaluate* $\lim_{x\to \pi/2} (\sec x)^{\cot x}$.

Solution: Let $y = \lim_{x\to \pi/2} (\sec x)^{\cot x}$. [Form ∞^0]

$\therefore \quad \log y = \lim_{x\to \pi/2} (\cot x)\,.\,(\log \sec x),$ [Form $0 \times \infty$]

$= \lim_{x\to \pi/2} \frac{\log \sec x}{\tan x},$ [Form ∞/∞]

$= \lim_{x\to \pi/2} \frac{(1/\sec x)\,.\,\sec x \tan x}{\sec^2 x} = \lim_{x\to \pi/2} \frac{\tan x}{\sec^2 x}$

$$= \lim_{x \to \pi/2} (\sin x \cos x) = 0.$$

$$\therefore \qquad y = e^0 = 1.$$

Problem 3(ii): *Evaluate* $\lim_{x \to \pi/4} (\tan x)^{\tan 2x}$.

Solution: Let $y = \lim_{x \to \pi/4} [(\tan x)^{\tan 2x}]$ [Form 1^∞]

$$\therefore \qquad \log y = \lim_{x \to \pi/4} [(\tan 2x) . (\log \tan x)] \qquad \text{[Form } \infty \times 0]$$

$$= \lim_{x \to \pi/4} \left[\frac{\log \tan x}{\cot 2x}\right], \qquad \text{[Form 0/0]}$$

$$= \lim_{x \to \pi/4} \left[\frac{(1/\tan x) . \sec^2 x}{-2 \operatorname{cosec}^2 (2x)}\right] = \lim_{x \to \pi/4} \left[\frac{\sec^2 x . \sin^2 (2x)}{-2 \tan x}\right]$$

$$= \frac{(\sqrt{2})^2 (1)^2}{-2 . 1} = -1 .$$

$$\therefore \qquad y = e^{-1} = 1/e.$$

Problem 4(i): *Evaluate* $\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan (\pi x/2a)}$ **(Rohilkhand 2012)**

Solution: Let $y = \lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan (\pi x/2a)}$ [Form 1^∞]

$$\therefore \qquad \log y = \lim_{x \to a} \tan \left(\frac{\pi x}{2a}\right)\left[\log \left(2 - \frac{x}{a}\right)\right], \qquad \text{[Form } \infty \times 0]$$

$$= \lim_{x \to a} \left[\left\{\log \left(2 - \frac{x}{a}\right)\right\} \Big/ \cot \left(\frac{\pi x}{2a}\right)\right], \qquad \text{[form } 0 \times 0]$$

$$= \lim_{x \to a} \left[\left(-\frac{1}{a}\right)\left(2 - \frac{x}{a}\right)^{-1}\right] \Big/ \left[\left\{-\operatorname{cosec}^2 \left(\frac{\pi x}{2a}\right)\right\} \frac{\pi}{2a}\right]$$

$$= \lim_{x \to a} \frac{1}{a} \cdot \frac{2a}{\pi} \cdot \frac{1}{\{2 - (x/a)\}} \sin^2 \left(\frac{\pi x}{2a}\right) = \frac{2}{\pi} \cdot$$

$$\therefore \qquad y = e^{2/\pi}.$$

Problem 4(ii): *Evaluate* $\lim_{x \to 1} (1 - x^2)^{1/\log (1-x)}$

Solution: Let $y = \lim_{x \to 1} (1 - x^2)^{1/\log (1-x)}$ [Form 0^0]

$$\log y = \lim_{x \to 1} \frac{\log(1 - x^2)}{\log(1 - x)} \qquad \left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 1} \frac{\frac{1}{1-x^2}(-2x)}{\frac{1}{1-x}(-1)} = \lim_{x\to 1} \frac{2x}{x+1}$$

or $\qquad \log y = 1$

$\therefore \qquad y = e^1 = e\,.$

Problem 5(i): *Evaluate* $\lim\limits_{x\to 1} x^{1/(1-x)}$.

Solution: Let $\qquad y = \lim\limits_{x\to 1} x^{1/(1-x)}$ $\qquad$ [Form 1^{∞}]

$$\log y = \lim_{x\to 1} \frac{1}{1-x} \log x = \lim_{x\to 1} \frac{\log x}{1-x} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 1} \frac{\frac{1}{x}}{-1} = -1.$$

$\therefore \qquad y = e^{-1} = 1/e.$

Problem 5(ii): *Evaluate* $\lim\limits_{x\to\infty} (a_0 x^m + a_1 x^{m-1} + \dots + a_m)^{1/x}$.

Solution: Let $\qquad y = \lim\limits_{x\to\infty} (a_0 x^m + a_1 x^{m-1} + \dots + a_m)^{1/x}$ $\qquad$ [Form ∞^0]

$$\log y = \lim_{x\to\infty} \frac{\log (a_0 x^m + a_1 x^{m-1} + \dots + a_m)}{x} \qquad \left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x\to\infty} \frac{a_0 m x^{m-1} + a_1 (m-1) x^{m-2} \dots)}{a_0 x^m + a_1 x^{m-1} + \dots + a_m}$$

$$= \lim_{x\to\infty} \frac{a_0\, m!}{a_0\, m!\, x} \qquad \text{[Differentiating } (m-1) \text{ times]}$$

$$\log y = \frac{1}{\infty} = 0$$

$\therefore \qquad y = e^0 = 1\,.$

Problem 6(i): *Evaluate* $\lim\limits_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$.

(Garhwal 2001, 03)

Solution: Let $\qquad y = \lim\limits_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$. $\qquad$ [form 1^{∞}]

$\therefore$
$$\log y = \lim_{x\to 0} \frac{1}{x} \log \frac{\tan x}{x}$$

$$= \lim_{x\to 0} \left[\frac{1}{x} \log \left\{\frac{x + (x^3/3) + (2x^5/15) + \dots}{x}\right\}\right],$$

[Writing the expansion for $\tan x$]

$$= \lim_{x\to 0} \frac{1}{x} \log\left[1 + \frac{x^2}{3} + \frac{2x^4}{15} + \ldots\right]$$

$$= \lim_{x\to 0} \frac{1}{x} \log(1 + z), \text{ where } z = \frac{x^2}{3} + \frac{2x^4}{15} + \ldots$$

$$= \lim_{x\to 0} \frac{1}{x} \left[z - \frac{z^2}{2} + \ldots\right], \qquad \text{[Expanding } \log(1+z)\text{]}$$

$$= \lim_{x\to 0} \frac{1}{x} \left[\left(\frac{x^2}{3} + \frac{2x^4}{15} + \ldots\right) - \frac{1}{2}\left(\frac{x^2}{13} + \frac{2x^4}{15} + \ldots\right)^2 + \ldots\right]$$

$$= \lim_{x\to 0} \frac{1}{x} \left[\frac{x^2}{3} + \left(\frac{2}{15} - \frac{1}{18}\right)x^4 + \ldots\right] = \lim_{x\to 0} \frac{1}{x} \left[\frac{x^2}{3} + \frac{7}{90}x^4 + \ldots\right]$$

$$= \lim_{x\to 0} \left[\frac{x}{3} + \frac{7}{90}x^3 + \ldots\right] = 0.$$

$\therefore \qquad y = e^0 = 1.$

Problem 6(ii): *Evaluate* $\lim_{x\to 0+} \left(\frac{\tan x}{x}\right)^{1/x^3}$.

Solution: Proceeding as in problem 6(i), we have

$$\log y = \lim_{x\to 0+} \frac{1}{x^3} \left[\frac{x^2}{3} + \frac{7}{90}x^4 + \ldots\right] = \lim_{x\to 0+} \left[\frac{1}{3x} + \frac{7}{90}x + \ldots\right]$$

$$= +\infty.$$

$\therefore \qquad y = e^{\infty} = \infty.$

Problem 7(i): *Evaluate* $\lim_{x\to 0} \left(\frac{\sinh x}{x}\right)^{1/x^2}$.

(Kumaun 2008)

Solution: Let $y = \lim_{x\to 0} \left(\frac{\sinh x}{x}\right)^{1/x^2}$ [Form 1^∞]

$\therefore \qquad \log y = \lim_{x\to 0} \left[\frac{1}{x^2} \log\left\{\frac{\sinh x}{x}\right\}\right].$

Now $\qquad \sinh x = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

$\therefore \qquad \log y = \lim_{x\to 0} \frac{1}{x^2} \log\left\{\frac{x + (x^3/3!) + (x^5/5!) + \ldots}{x}\right\}$

$$= \lim_{x\to 0} \frac{1}{x^2} \log\left[1 + \frac{x^2}{6} + \frac{x^4}{120} + \ldots\right]$$

$$= \lim_{x\to 0} \frac{1}{x^2} \log(1+z), \qquad \left[\text{Where } z = 1 + \frac{x^2}{6} + \frac{x^4}{120} + \ldots\right]$$

$$= \lim_{x\to 0} \frac{1}{x^2}\left[z - \frac{z^2}{2} + \ldots\right]$$

$$= \lim_{x\to 0} \frac{1}{x^2}\left[\left(\frac{x^2}{6} + \frac{x^4}{120} + \ldots\right) - \frac{1}{2}\left(\frac{x^2}{6} + \frac{x^4}{120} + \ldots\right)^2 + \ldots\right]$$

$$= \lim_{x\to 0} \frac{1}{x^2}\left[\frac{x^2}{6} + \left(\frac{1}{120} - \frac{1}{72}\right)x^4 + \ldots\right]$$

$$\log y = \lim_{x\to 0} \frac{1}{x^2}\left[\frac{x^2}{6} - \frac{1}{180}x^4 + \ldots\right]$$

$$= \lim_{x\to 0}\left[\frac{1}{6} - \frac{1}{180}x^2 + \ldots\right] = \frac{1}{6}.$$

$\therefore$ $$y = e^{1/6}.$$

Problem 7(ii): *Evaluate* $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$. **(Kumaun 2003)**

Solution: Let $$y = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2} \qquad \left[\text{Form } 1^\infty, \text{ since } \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

$\therefore$ $$\log y = \lim_{x\to 0}\left[\frac{1}{x^2}\log\frac{\sin x}{x}\right].$$

$$= \lim_{x\to 0} \frac{1}{x^2}.\log\left\{\frac{x - (x^3/3!) + (x^5/5!) - \ldots}{x}\right\}$$

$$= \lim_{x\to 0} \frac{1}{x^2}.\log\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots\right]$$

$$= \lim_{x\to 0} \frac{1}{x^2}\log\left[1 - \left(\frac{x^2}{6} - \frac{x^4}{120} + \ldots\right)\right]$$

$$= \lim_{x\to 0} \frac{1}{x^2}\log(1-z), \qquad \left[\text{Where } z = \frac{x^2}{6} - \frac{x^4}{120} + \ldots\right]$$

$$= \lim_{x\to 0} \frac{1}{x^2}\left[-z - \frac{z^2}{2} - \ldots\right]$$

$$= \lim_{x\to 0} \frac{1}{x^2}\left[-\left(\frac{x^2}{6} - \frac{x^4}{120} + \ldots\right) - \frac{1}{2}\left(\frac{x^2}{6} - \frac{x^4}{120} + \ldots\right)^2 - \ldots\right]$$

$$= \lim_{x\to 0} \frac{1}{x^2}\left[-\frac{x^2}{6}+\left(\frac{x^4}{120}-\frac{x^4}{72}\right)+\ldots\right]$$

$$= \lim_{x\to 0} \frac{1}{x^2}\left[-\frac{x^2}{6}-\frac{x^4}{180}+\ldots\right] = \lim_{x\to 0}\left[-\frac{1}{6}-\frac{x^2}{180}+\ldots\right] = -\frac{1}{6}.$$

$\therefore$ $y = e^{-1/6}$.

Problem 8(i): *Evaluate* $\lim_{x\to 0}\left[\frac{2(\cosh x - 1)}{x^2}\right]^{1/x^2}$.

Solution: Let $y = \lim_{x\to 0}\left[\frac{2(\cosh x-1)}{x^2}\right]^{1/x^2}$. [Form 1^∞]

$$\therefore \quad \log y = \lim_{x\to 0}\frac{1}{x^2}\log\left\{\frac{2(\cosh x-1)}{x^2}\right\}$$

$$= \lim_{x\to 0}\frac{1}{x^2}\log\left\{\frac{2}{x^2}\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots-1\right)\right\}$$

$$= \lim_{x\to 0}\frac{1}{x^2}\log\left\{\frac{2}{x^2}\left(\frac{x^2}{2}+\frac{x^4}{24}+\ldots\right)\right\}$$

$$= \lim_{x\to 0}\frac{1}{x^2}\log\left[1+\left(\frac{x^2}{12}+\ldots\right)\right]$$

$$= \lim_{x\to 0}\frac{1}{x^2}\left[\left(\frac{x^2}{12}+\ldots\right)-\frac{1}{2}\left(\frac{x^2}{12}+\ldots\right)^2+\ldots\right]$$

$$= \lim_{x\to 0}\frac{1}{x^2}\left[\frac{x^2}{12}+\text{higher powers of } x\right]$$

$$= \lim_{x\to 0}\left[\frac{1}{12}+\text{terms containing } x \text{ and its higher powers}\right] = \frac{1}{12}.$$

$\therefore$ $y = e^{1/12}$.

Problem 8(ii): *Evaluate* $\lim_{x\to\infty}\left\{\frac{\log x}{x}\right\}^{1/x}$.

Solution: Let $y = \lim_{x\to\infty}\log\left(\frac{\log x}{x}\right)^{1/x}$ [Form ∞/∞]

$$\therefore \quad \log y = \lim_{x\to\infty}\frac{1}{x}\log\left(\frac{\log x}{x}\right) = \lim_{x\to\infty}\frac{\log(\log x)-\log x}{x}$$

$$= \lim_{x\to\infty}\frac{\log(\log x)}{x} - \lim_{x\to\infty}\left(\frac{\log x}{x}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{1}{\log x}\cdot\frac{1}{x}}{1} - \lim_{x\to\infty}\left(\frac{\frac{1}{x}}{1}\right) = \lim_{x\to\infty}\frac{1}{x\log x} - \lim_{x\to\infty}\frac{1}{x}$$

or $\log y = 0.$

$\therefore$ $y = e^0 = 1.$

Problem 9(i): *Evaluate* $\lim_{x\to\infty} (\frac{1}{2}\pi - \tan^{-1} x)^{1/x}$.

Solution: Let $y = \lim_{x\to\infty} (\frac{1}{2}\pi - \tan^{-1} x)^{1/x}$. [Form 0^0]

$$\therefore \quad \log y = \lim_{x\to\infty} \left(\frac{1}{x}\right) \log (\frac{1}{2}\pi - \tan^{-1} x)$$

$$= \lim_{x\to\infty} \frac{\log (\frac{1}{2}\pi - \tan^{-1} x)}{x}, \qquad \text{[Form } \infty/\infty]$$

$$= \lim_{x\to\infty} \frac{\{1/(\frac{1}{2}\pi - \tan^{-1} x)\}.\{-1/(1+x^2)\}}{1}$$

$$= \lim_{x\to\infty} \left\{\frac{-1/(1+x^2)}{(\frac{1}{2}\pi - \tan^{-1} x)}\right\}, \qquad \text{[Form 0/0]}$$

$$= \lim_{x\to\infty} \frac{\{1/(1+x^2)^2\}.2x}{-1/(1+x^2)}, \qquad \text{[By L' Hospital's rule]}$$

$$= \lim_{x\to\infty} \left\{\frac{-2x}{1+x^2}\right\}, \qquad \text{[Form } \infty/\infty]$$

$$= \lim_{x\to\infty} \left\{\frac{-2}{2x}\right\} = 0.$$

$\therefore$ $y = e^0 = 1.$

Problem 9(ii): *Evaluate* $\lim_{x\to 0} (\text{cosec } x)^{1/\log x}$.

Solution: Let $y = \lim_{x\to 0} (\text{cosec } x)^{1/\log x}$. [Form ∞^0]

$$\therefore \quad \log y = \lim_{x\to 0} \frac{1}{\log x} (\log \text{cosec } x), \qquad \text{[Form } \infty/\infty]$$

$$= \lim_{x\to 0} \frac{(1/\text{cosec } x)(-\text{cosec } x \cot x)}{1/x}$$

$$= \lim_{x\to 0} \left(\frac{-x}{\tan x}\right), \qquad \text{[Form 0/0]}$$

$$= \lim_{x\to 0} \left(\frac{-1}{\sec^2 x} \right) = -1 .$$

$\therefore \qquad y = e^{-1} = 1/e .$

Problem 10: *Evaluate* $\lim_{x\to 1} (1-x) . \tan \frac{\pi x}{2}$.

Solution: We have $\lim_{x\to 1} (1-x) . \tan \frac{\pi x}{2}$, [Form $0 \times \infty$]

$$= \lim_{x\to 1} \frac{1-x}{\cot (\pi x/2)}, \qquad \text{[Form } 0/0\text{]}$$

$$= \lim_{x\to 1} \frac{-1}{-\frac{\pi}{2} \cdot \operatorname{cosec}^2 \frac{\pi x}{2}} = \lim_{x\to 1} \frac{2}{\pi} \sin^2 \frac{\pi x}{2} = \frac{2}{\pi}.$$

Example 11: *Evaluate* $\lim_{x \to 0} (\sin x)^{\tan x}$. **(Kanpur 2015)**

Solution: Let $y = \lim_{x \to 0} (\sin x)^{\tan x}$. [Form 0^0]

$\therefore \qquad \log y = \lim_{x \to 0} (\tan x) . (\log \sin x),$ [Form $0 \times \infty$]

$$= \lim_{x \to 0} \frac{\log \sin x}{\cot x}, \qquad \text{[Form } \infty/\infty\text{]}$$

$$= \lim_{x \to 0} \frac{(1/\sin x) \cos x}{-\operatorname{cosec}^2 x}, \qquad \text{[Form } \infty/\infty\text{]}$$

$$= \lim_{x \to 0} \left[- \left(\frac{\sin^2 x}{\tan x} \right) \right], \qquad \text{[Form } 0/0\text{]}$$

$$= \lim_{x \to 0} \left[\frac{-2 \sin x \cos x}{\sec^2 x} \right] = 0.$$

$\therefore \qquad y = e^0 = 1.$

Hints to Objective Type Questions

Multiple Choice Questions

1. We have $\lim_{x\to 0} \frac{1-\cos x}{3x^2}$ $\left[\text{Form } \frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{\sin x}{6x} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}.$$

2. We have $1^0 = 1$, which is not an indeterminate form.

3. We have $\lim_{x\to 0} \frac{\log \tan x}{\log x}$ $\left[\text{Form } \frac{\infty}{\infty}\right]$

$$= \lim_{x\to 0} \frac{\sec^2 x/\tan x}{1/x} = \lim_{x\to 0} \left\{\left(\frac{x}{\tan x}\right)\left(\frac{1}{\cos^2 x}\right)\right\} = 1\times 1 = 1.$$

4. We have $\infty + \infty = \infty$, $\infty^{\infty} = \infty$, $0^{\infty} = 0$. These are not indeterminate forms. But, 1^{∞} is an indeterminate form.

5. $\lim_{x\to 0} \left(\frac{a^x - 1 - x\log a}{x^2}\right)$ $\left[\text{Form } \frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{a^x \log a - \log a}{2x} \quad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 0} \frac{a^x (\log a)^2}{2} = \frac{1}{2} (\log a)^2.$$

6. See Problem 4(iv) of Comprehensive Problems 1.

7. See Problem 2(ii) of Comprehensive Problems 2.

8. See Problem 2(i) of Comprehensive Problems 1.

9. See article 2.

10. See Example 11.

11. See Example 6.

Fill in the Blanks

1. We have $\lim_{x\to 0} \frac{\sin ax}{\sin bx}$ $\left[\text{Form } \frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{a\cos ax}{b\cos bx} = \frac{a}{b}.$$

2. We have $\lim_{x\to 0} \frac{\log (1 + kx^2)}{1 - \cos x}$ $\left[\text{Form } \frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{2kx\,(1 + kx^2)}{\sin x} = \lim_{x\to 0} \left(\frac{2k}{1 + kx^2} \cdot \frac{x}{\sin x}\right) = 2k \times 1 = 2k.$$

3. We have $\lim_{x\to \infty} \frac{x^2 + 2x}{5 - 3x^2}$ $\left[\text{Form } \frac{\infty}{\infty}\right]$

$$= \lim_{x\to \infty} \frac{2x + 2}{-6x} \quad \left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x\to \infty} \frac{2}{-6} = -\frac{1}{3}.$$

4. We have $\lim_{x\to 1} \left(\sec \frac{\pi}{2x}\right) \cdot \log x$ $[\text{Form } \infty \times 0]$

$$= \lim_{x\to 1}\left(\frac{\log x}{\cos(\pi/2x)}\right) \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x\to 1} \frac{1/x}{-[\sin(\pi/2x)]\cdot(-\pi/2x^2)} = \lim_{x\to 1} \frac{2x}{\pi} \operatorname{cosec} \frac{\pi}{2x} = \frac{2}{\pi}.$$

5. We have $\lim_{x\to 0} \frac{x - \tan x}{x^3} = \lim_{x\to 0} \frac{1-\sec^2 x}{3x^2}$

$$= \lim_{x\to 0} \frac{-2\sec x \cdot \sec x \tan x}{6x} = \lim_{x\to 0} -\frac{\sec^2 x}{3}\cdot\frac{\tan x}{x} = -\frac{1}{3}\times 1 = -\frac{1}{3}.$$

6. We have $\lim_{x\to 0}\left[\frac{a^x - b^x}{x}\right]$, $\qquad \left[\text{Form } \frac{0}{0}\right]$

$$= \lim_{x\to 0} \frac{a^x \log a - b^x \log b}{1} \qquad \text{[By L'Hospital's rule]}$$

$$= \log a - \log b = \log (a/b).$$

True or False

1. If $\lim_{x\to a} \frac{f(x)}{\phi(x)}$ is of the form $\frac{0}{0}$, then by L'Hospital's rule, we have

$$\lim_{x\to a} \frac{f(x)}{\phi(x)} = \lim_{x\to a} \frac{f'(x)}{\phi'(x)}.$$

Thus, $\frac{f(x)}{\phi(x)}$ is not differentiated as a fraction.

The numerator $f(x)$ and the denominator $\phi(x)$ are differentiated separately.

2. We have $\lim_{x\to a} \frac{f(x)}{\phi(x)}$ $\qquad \left[\text{Form } \frac{\infty}{\infty}\right]$

$$= \lim_{x\to a} \frac{1/\phi(x)}{1/f(x)}, \text{ which is of the form } \frac{0}{0}.$$

Similarly, $\lim_{x\to a} \frac{f(x)}{\phi(x)}$ $\qquad \left[\text{Form } \frac{0}{0}\right]$

$$= \lim_{x\to a} \frac{1/\phi(x)}{1/f(x)}, \text{ which is of the form } \frac{\infty}{\infty}.$$

3. We have $\lim_{x\to a} \frac{f(x)}{\phi(x)}$ $\qquad \left[\text{Form } \frac{\infty}{\infty}\right]$

$$= \lim_{x\to a} \frac{f'(x)}{\phi'(x)}.$$

4. We have $\lim_{x\to 0} (1+x)^{1/x} = e$.

So, $\lim_{x\to 0} \frac{(1+x)^{1/x} - e + \frac{ex}{2}}{x^2}$ is of the form $\frac{0}{0}$.

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Chapter-7

Partial Differentiation

Comprehensive Problems 1

Problem 1: *Find* $\frac{\partial u}{\partial x}$ *and* $\frac{\partial u}{\partial y}$ *when* $u = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$.

Solution: We have $u = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ ...(1)

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{2x}{a^2} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{2y}{b^2}$$

Problem 2: *Prove that* $\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x}$ *in each of the following cases :*

(i) $u = x^4 + x^2 y^2 + y^4$. *(ii)* $u = \log \tan (y/x)$.

(iii) $u = \log \left\{ \frac{x^2 + y^2}{x + y} \right\}$. *(iv)* $u = x^y$.

Solution: (i) We have $u = x^4 + x^2 y^2 + y^4$ ...(1)

Differentiating u partially w. r. to x taking y as constant, we have

$$\therefore \quad \frac{\partial u}{\partial x} = 4x^3 + 2xy^2$$

Now $$\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\{4x^3 + 2xy^2\} = 4xy$$

Again differentiating u partially w. r. to y taking x as constant, we have

$$\frac{\partial u}{\partial y} = 2x^2 y + 4y^3$$

Now $$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\{2x^2 y + 4y^3\} = 4xy$$

Hence $$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial x}$$

(ii) Given $u = \log \tan\left(\frac{y}{x}\right)$ ...(1)

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{\tan(y/x)} . \sec^2 \frac{y}{x} . \left(-\frac{y}{x^2}\right)$$

$$= \frac{-y}{x^2 \sin\frac{y}{x} . \cos\frac{y}{x}} = \frac{-2y}{x^2 . \sin\frac{2y}{x}} = -\frac{2y}{x^2} \operatorname{cosec} \frac{2y}{x}$$

Hence $\quad \dfrac{\partial^2 u}{\partial y\, \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right) = -\dfrac{2}{x^2}\dfrac{\partial}{\partial y}\left(y \operatorname{cosec}\dfrac{2y}{x}\right)$

$$= -\frac{2}{x^2}\left[\operatorname{cosec}\frac{2y}{x} - y \operatorname{cosec}\frac{2y}{x} . \cot\frac{2y}{x} . \frac{2}{x}\right]$$

Again $\quad \dfrac{\partial u}{\partial y} = \dfrac{1}{\tan\frac{y}{x}} . \sec^2\dfrac{y}{x} . \dfrac{1}{x}$

$$= \frac{2}{2x \sin\frac{y}{x} \cos\frac{y}{x}} = \frac{2}{x \sin\frac{2y}{x}} = \frac{2}{x} \operatorname{cosec}\frac{2y}{x}$$

$\therefore \quad \dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{2\partial}{\partial x}\left[\dfrac{1}{x}\operatorname{cosec}\dfrac{2y}{x}\right]$

$$= 2\left[-\frac{1}{x^2}\operatorname{cosec}\frac{2y}{x} - \frac{1}{x}\operatorname{cosec}\frac{2y}{x} . \cot\frac{2y}{x}\left(\frac{-2y}{x^2}\right)\right]$$

$$= -\frac{2}{x^2}\left[\operatorname{cosec}\frac{2y}{x} - y \operatorname{cosec}\frac{2y}{x} \cot\frac{2y}{x} . \frac{2}{x}\right]$$

Here, we note that $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial^2 u}{\partial y\, \partial x}$

(iii) Given $\quad u = \log\left(\dfrac{x^2 + y^2}{x + y}\right) \quad$ ...(1)

$$u = \log(x^2 + y^2) - \log(x + y)$$

$\therefore \quad \dfrac{\partial u}{\partial x} = \dfrac{1}{x^2 + y^2}(2x) - \dfrac{1}{x + y}$

Hence $\quad \dfrac{\partial^2 u}{\partial y\, \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial y}\left[\dfrac{2x}{x^2 + y^2} - \dfrac{1}{x + y}\right]$

$$= -\frac{2x . 2y}{(x^2 + y^2)^2} + \frac{1}{(x + y)^2}$$

Again $\quad \dfrac{\partial u}{\partial y} = \dfrac{2y}{x^2 + y^2} - \dfrac{1}{x + y}$

$\therefore \quad \dfrac{\partial^2 u}{\partial x\, \partial y} = -\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial}{\partial x}\left[\dfrac{2y}{x^2 + y^2} - \dfrac{1}{x + y}\right] = \dfrac{-2y . 2x}{(x^2 + y^2)^2} + \dfrac{1}{(x + y)^2}$

Here, we note that $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial^2 u}{\partial y\, \partial x}$

(iv) Given $u = x^y$ ...(1)

$\therefore$ $$\frac{\partial u}{\partial x} = y\, x^{y-1}$$

Hence $$\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(y\, x^{y-1})$$
$$= y\, x^{y-1} \log x + x^{y-1}$$

Again $$\frac{\partial u}{\partial y} = x^y \log x$$

$\therefore$ $$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(x^y \log x) = x^y \frac{1}{x} + \log x \,.\, y\, x^{y-1}$$
$$= x^{y-1} + y\, x^{y-1} \log x$$

Here, we note that $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial^2 u}{\partial y\, \partial x}$.

Problem 3: *If* $u = \sin^{-1} \dfrac{x}{y} + \tan^{-1} \dfrac{y}{x}$*, show that* $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = 0.$

(Kumaun 2012)

Solution: Here $$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{\{1 - (x/y)^2\}}} \cdot \frac{1}{y} + \frac{1}{1 + (y/x)^2} \cdot \left(-\frac{y}{x^2}\right),$$

[Treating y as constant]

$$= \frac{1}{\sqrt{(y^2 - x^2)}} - \frac{y}{(x^2 + y^2)}.$$

$\therefore$ $$x \frac{\partial u}{\partial x} = \frac{x}{\sqrt{(y^2 - x^2)}} - \frac{x\, y}{x^2 + y^2}. \qquad ...(1)$$

Again $$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{\{1 - (x/y)^2\}}} \left(-\frac{x}{y^2}\right) + \frac{1}{1 + (y/x)^2} \cdot \frac{1}{x},$$

[Treating x as constant]

$$= -\frac{x}{y\sqrt{(y^2 - x^2)}} + \frac{x}{x^2 + y^2}.$$

$\therefore$ $$y \frac{\partial u}{\partial y} = -\frac{x}{\sqrt{(y^2 - x^2)}} + \frac{x\, y}{x^2 + y^2}. \qquad ...(2)$$

Adding (1) and (2), we have $x\,(\partial u/\partial x) + y\,(\partial u/\partial y) = 0.$

Problem 4: *If* $u = xy\, f\left(\dfrac{y}{x}\right)$*, then write the value of the expression* $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y}.$

(Meerut 2001; Kanpur 07)

Solution: Given $u = xy\, f\left(\dfrac{y}{x}\right)$ ...(1)

$\therefore$ $$\frac{\partial u}{\partial x} = y\left[x\, f'\left(\frac{y}{x}\right) . \left(-\frac{y}{x^2}\right) + f\left(\frac{y}{x}\right)\right]$$

So, $x\dfrac{\partial u}{\partial x} = -y^2 f'\left(\dfrac{y}{x}\right) + xy\, f\left(\dfrac{y}{x}\right)$ ...(2)

And $\dfrac{\partial u}{\partial y} = x\left[y\, f'\left(\dfrac{y}{x}\right).\dfrac{1}{x} + f\left(\dfrac{y}{x}\right)\right]$

$\therefore$ $y\dfrac{\partial u}{\partial y} = y^2 f'\left(\dfrac{y}{x}\right) + xy\, f\left(\dfrac{y}{x}\right)$ ...(3)

On adding equations (2) and (3), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2xy\, f\left(\frac{y}{x}\right) = 2u\,.$$

Problem 5: *If* $z = f(x + ay) + \phi(x - ay)$, *prove that* $\partial^2 z/\partial y^2 = a^2\,(\partial^2 z/\partial x^2)$.

(Bundelkhand 2001; Kanpur 05; Meerut 13B)

Solution: We have $z = f(x + ay) + \phi(x - ay)$.

$\therefore$ $\partial z/\partial x = f'(x + ay) + \phi'(x - ay)$, [Diff. partially w.r.t. 'x']

and $\partial^2 z/\partial x^2 = f''(x - ay) + \phi''(x - ay)$. ...(1)

Again, $\partial z/\partial y = a\, f'(x + ay) - a\,\phi'(x - ay)$.

$\therefore$ $\partial^2 z/\partial y^2 = a^2 f''(x + ay) + a^2\,\phi''(x - ay)$. ...(2)

From (1) and (2), we get

$$\partial^2 z/\partial y^2 = a^2\,(\partial^2 z/\partial x^2)\,.$$

Problem 6: *If* $u = f(r)$ *where* $r^2 = x^2 + y^2$, *show that*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$

(Meerut 2001; Agra 01; Avadh 04)

Solution: Differentiating $r^2 = x^2 + y^2$ partially w.r.t. x and y, we get

$$2r\frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}\,;\; 2r\frac{\partial r}{\partial y} = 2y \quad \text{or} \quad \frac{\partial r}{\partial y} = \frac{y}{r}\cdot \qquad ...(1)$$

Now, $u = f(r)$. Therefore

$$\frac{\partial u}{\partial x} = \{f'(r)\}\frac{\partial r}{\partial x} = \frac{x}{r} f'(r), \qquad \text{[From (1)]}$$

and $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial}{\partial x}\left[x\cdot\dfrac{1}{r}\cdot f'(r)\right]$

$$= 1\cdot\frac{1}{r}\cdot f'(r) + \{xf'(r)\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \frac{x}{r}\{f''(r)\frac{\partial r}{\partial x}$$

$$= \frac{1}{r} f'(r) - \frac{x}{r^2}\cdot\frac{x}{r} f'(r) + \frac{x^2}{r^2} f''(r),$$

[$\because$ From (1), $\partial x/\partial r = x/r$]

$$= (1/r)\, f'(r) - (x^2/r^3)\, f'(r) + (x^2/r^2)\, f''(r). \qquad ...(2)$$

Similarly, by symmetry, we have

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{r} f'(r) - \frac{y^2}{r^3} f'(r) + \frac{y^2}{r^2} f''(r). \qquad ...(3)$$

Adding (2) and (3), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{2}{r}f'(r)-\frac{x^2+y^2}{r^3}f'(r)+\frac{x^2+y^2}{r^2}f''(r)$$
$$=(2/r)\,f'(r)-(r^2/r^3)\,f'(r)+(r^2/r^2)\,f''(r),$$
$$[\because\ r^2=x^2+y^2]$$
$$=(2/r)\,f'(r)-(1/r)\,f'(r)+f''(r)$$
$$=f''(r)+(1/r)\,f'(r).$$

Problem 7: *If* $z=x^2\tan^{-1}\frac{y}{x}-y^2\tan^{-1}\frac{x}{y}$, *prove that* $\frac{\partial^2 z}{\partial x\,\partial y}=\frac{x^2-y^2}{x^2+y^2}$.

Solution: We have

$$\frac{\partial z}{\partial y}=x^2\cdot\frac{1}{1+(y/x)^2}\cdot\frac{1}{x}-2y\tan^{-1}\frac{x}{y}-y^2\frac{1}{1+(x/y)^2}\cdot\left(-\frac{x}{y^2}\right)$$
$$=\frac{x^3}{x^2+y^2}-2y\tan^{-1}\frac{x}{y}+\frac{xy^2}{x^2+y^2}$$
$$=\frac{x(x^2+y^2)}{x^2+y^2}-2y\tan^{-1}\frac{x}{y}=x-2y\tan^{-1}\frac{x}{y}.$$

Now, $$\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=1-2y\frac{1}{1+(x/y)^2}\cdot\frac{1}{y}=1-\frac{2y^2}{x^2+y^2}$$
$$=\frac{x^2+y^2-2y^2}{x^2+y^2}=\frac{x^2-y^2}{x^2+y^2}.$$

Problem 8: *(i) If* $u=2(ax+by)^2-(x^2+y^2)$ *and* $a^2+b^2=1$, *prove that* $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

(ii) If $u=e^x(x\cos y-y\sin y)$, *then show that* $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

(Garhwal 2003)

Solution: (i)Given $u=2(ax+by)^2-(x^2+y^2)$ ...(1)

$\therefore$ $$\frac{\partial u}{\partial x}=4(ax+by)a-2x\ \text{and}\ \frac{\partial^2 u}{\partial x^2}=4a^2-2$$

Again from (1) $$\frac{\partial u}{\partial y}=4(ax+by)b-2y\ \text{and}\ \frac{\partial^2 u}{\partial y^2}=4b^2-2$$

Thus $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=(4a^2-2)+(4b^2-2)$$
$$=4(a^2+b^2)-4$$
$$=4-4$$
$$=0\qquad[\because\ a^2+b^2=1]$$

(ii) Given $u = e^x (x\cos y - y\sin y)$

$\therefore$ $$\frac{\partial u}{\partial x} = e^x (x\cos y - y\sin y) + e^x \cos y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x (x\cos y - y\sin y) + 2e^x \cos y$$

Again $$\frac{\partial u}{\partial y} = e^x (-x\sin y - \sin y - y\sin y)$$

$\therefore$ $$\frac{\partial^2 u}{\partial y^2} = e^x (-x\cos y - \cos y - \cos y + y\sin y)$$

$$= -e^x (x\cos y - y\sin y) - 2e^x \cos y.$$

Thus, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Problem 9: *If* $z = (x^2 + y^2)/(x + y)$, *show that*

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$

(Kumaun 2000; Bundelkhand 11; Kanpur 11; Kashi 12, 13; Avadh 14)

Solution: We have $z = (x^2 + y^2)/(x + y)$.

$\therefore$ $$\frac{\partial z}{\partial x} = \frac{(x + y).2x - (x^2 + y^2).1}{(x + y)^2} = \frac{x^2 - y^2 + 2xy}{(x + y)^2},$$

and $$\frac{\partial z}{\partial y} = \frac{(x + y)2y - (x^2 + y^2).1}{(x + y)^2} = \frac{y^2 - x^2 + 2xy}{(x + y)^2}.$$

$\therefore$ $$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left[\frac{(x^2 - y^2 + 2xy) - (y^2 - x^2 + 2xy)}{(x + y)^2}\right]^2$$

$$= \left[\frac{2(x^2 - y^2)}{(x + y)^2}\right]^2 = 4\left[\frac{x - y}{x + y}\right]^2.$$

Also $$1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 1 - \frac{x^2 - y^2 + 2xy}{(x + y)^2} - \frac{y^2 - x^2 + 2xy}{(x + y)^2}$$

$$= \frac{(x^2 + y^2 + 2xy) - x^2 + y^2 - 2xy - y^2 + x^2 - 2xy}{(x + y)^2}$$

$$= \frac{x^2 - 2xy + y^2}{(x + y)^2} = \left(\frac{x - y}{x + y}\right)^2.$$

Hence $$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right).$$

Problem 10: *If* $u = e^{xyz}$, *show that* $\partial^3 u / \partial x\, \partial y\, \partial z = (1 + 3xyz + x^2 y^2 z^2)\, e^{xyz}$.

(Kumaun 2001; Kashi 12; Kanpur 11; Rohilkhand 13)

Solution: Here $u = e^{xyz}$.

$\therefore \qquad \partial u / \partial z = x\, y\, e^{xyz}$.

Now,
$$\frac{\partial^2 u}{\partial y\, \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}(x\, y\, e^{xyz}) = x \frac{\partial}{\partial y}(y\, e^{xyz})$$
$$= x\,[y \,.\, xz\, e^{xyz} + e^{xyz}] = e^{xyz}\,(x^2 yz + x).$$

Again
$$\frac{\partial^3 u}{\partial x\, \partial y\, \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y\, \partial z}\right) = \frac{\partial}{\partial x}[e^{xyz}\,(x^2 yz + x)]$$
$$= e^{xyz}\,(2xz\,y + 1) + yz\, e^{xyz}\,(x^2 yz + x)$$
$$= e^{xyz}\,[2xyz + 1 + x^2 y^2 z^2 + xyz]$$
$$= e^{xyz}\,[1 + 3\,xyz + x^2\, y^2\, z^2].$$

Problem 11: *If* $x^x y^y z^z = c$, *show that at* $x = y = z$, $\partial^2 z / \partial x\, \partial y = -\{x \log (ex)\}^{-1}$.

(Garhwal 2009; Rohilkhand 13; Bundelkhand 14; Purvanchal 14)

Solution: We have $x^x \,.\, y^y \,.\, z^z = c$. From this equation we observe that we can regard z as a function of two independent variables x and y.

Taking logarithms of both sides of the given equation, we get

$$x \log x + y \log y + z \log z = \log c. \qquad \text{...(1)}$$

Now differentiating (1) partially w.r.t. x taking y as constant, we have

$$x \cdot \frac{1}{x} + 1 \,.\, \log x + \left[z \cdot \frac{1}{z} + 1 \,.\, \log z\right] \frac{\partial z}{\partial x} = 0.$$

[Note that z is not a constant but is a function of x and y]

$$\therefore \qquad \frac{\partial z}{\partial x} = -\frac{(1 + \log x)}{(1 + \log z)}. \qquad \text{...(2)}$$

Similarly differentiating (1) partially w.r.t. y, we have

$$\frac{\partial z}{\partial y} = -\frac{(1 + \log y)}{(1 + \log z)}. \qquad \text{...(3)}$$

Now,
$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[-\left(\frac{1 + \log y}{1 + \log z}\right)\right] \qquad \text{[From (3)]}$$
$$= -(1 + \log y) \cdot \frac{\partial}{\partial x}[(1 + \log z)^{-1}]$$
$$= -(1 + \log y) \cdot \left[-(1 + \log z)^{-2} \cdot \frac{1}{z} \cdot \frac{\partial z}{\partial x}\right]$$
$$= \frac{(1 + \log y)}{z\,(1 + \log z)^2} \cdot \left[-\left(\frac{1 + \log x}{1 + \log z}\right)\right]. \qquad \text{[From (2)]}$$

Hence, when $x = y = z$, we have

$$\frac{\partial^2 z}{\partial x\, \partial y} = -\frac{(1 + \log x)^2}{x(1 + \log x)^3}$$

[Putting $y = z = x$ in the value of $(\partial^2 z / \partial x\, \partial y)$]

$$= -\frac{1}{x(1 + \log x)} = -\frac{1}{x(\log e + \log x)} \qquad [\because \log e = 1]$$

$$= -\frac{1}{x \log (e\, x)} = -\{x \log (e\, x)\}^{-1}.$$

Problem 12: *Show that* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ *when*

(i) $u = e^{my} \cos mx.$ **(Agra 2014)**

(ii) $u = \log (x^2 + y^2)$ **(Agra 2014)**

(iii) $u = \tan^{-1} (y/x)$

Solution: (i) Given $u = e^{my} \cos mx$ ...(1)

$\therefore$ $$\frac{\partial u}{\partial x} = e^{my} (-\sin mx) \,.\, m$$

and $$\frac{\partial^2 u}{\partial x^2} = -m^2 e^{my} \cos mx = -m^2 u \qquad \text{[Using (1)]}$$

Again $$\frac{\partial u}{\partial y} = m e^{my} \cos mx$$

and $$\frac{\partial^2 u}{\partial y^2} = m^2 e^{my} \cos mx = m^2 u \qquad \text{[Using (1)]}$$

Thus, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(ii) Given $u = \log (x^2 + y^2)$

$\therefore$ $$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2}$$

and $$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2)2 - 2x \,.\, 2x}{(x^2 + y^2)^2} = \frac{2x^2 + 2y^2 - 4x^2}{(x^2 + y^2)^2} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}$$

Again $$\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2}$$

and $$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2)2 - 2y \,.\, 2y}{(x^2 + y^2)^2} = \frac{2x^2 + 2y^2 - 4y^2}{(x^2 + y^2)^2} = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}$$

Thus $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2y^2 - 2x^2 + 2x^2 - 2y^2}{(x^2 + y^2)^2} = 0.$$

(iii) Proceed as in Example 4.

Problem 13: *If* $V=(x^2+y^2+z^2)^{-1/2}$, *show that*

(i) $x\dfrac{\partial V}{\partial x}+y\dfrac{\partial V}{\partial y}+z\dfrac{\partial V}{\partial z}=-V$ (ii) $\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}+\dfrac{\partial^2 V}{\partial z^2}=0.$

(Kumaun 2008)

Solution: (i) Given $V=(x^2+y^2+z^2)^{-1/2}$

$$\therefore \quad \frac{\partial V}{\partial x}=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}.(2x)=-x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 V}{\partial x^2}=x.\frac{3}{2}(x^2+y^2+z^2)^{-5/2}.(2x)-1(x^2+y^2+z^2)^{-3/2}$$

$$=(x^2+y^2+z^2)^{-5/2}[3x^2-(x^2+y^2+z^2)]=V^5(2x^2-y^2-z^2).$$

Similarly, by symmetry

$$\frac{\partial V}{\partial y}=-y(x^2+y^2+z^2)^{-3/2},\ \frac{\partial^2 V}{\partial y^2}=V^5(2y^2-x^2-z^2).$$

$$\frac{\partial V}{\partial z}=-z(x^2+y^2+z^2)^{-3/2},\ \frac{\partial^2 V}{\partial z^2}=V^5(2z^2-x^2-y^2).$$

$$\therefore \quad x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}$$

$$=-x^2(x^2+y^2+z^2)^{-3/2}-y^2(x^2+y^2+z^2)^{-3/2}-z^2(x^2+y^2+z^2)^{-3/2}$$

$$=-(x^2+y^2+z^2)^{-3/2}(x^2+y^2+z^2)=-(x^2+y^2+z^2)^{-1/2}=-V.$$

Also $$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}$$

$$=V^5(2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2)=V^5.0=0.$$

Problem 14: *If* $u=\tan^{-1}\dfrac{xy}{\sqrt{(1+x^2+y^2)}}$, *show that*

$$\frac{\partial^2 u}{\partial x\,\partial y}=\frac{1}{(1+x^2+y^2)^{3/2}}.$$

Solution: We have

$$\frac{\partial u}{\partial x}=\frac{1}{1+\dfrac{x^2y^2}{1+x^2+y^2}}\cdot y\cdot\frac{1\cdot\sqrt{(1+x^2+y^2)}-x\cdot\frac{1}{2}(1+x^2+y^2)^{-1/2}(-2x)}{1+x^2+y^2}$$

$$=\frac{1+x^2+y^2}{1+x^2+y^2+x^2y^2}\cdot y\cdot\frac{(1+x^2+y^2)-x^2}{(1+x^2+y^2)(1+x^2+y^2)^{1/2}}$$

$$=\frac{y(1+y^2)}{(1+x^2)(1+y^2)(1+x^2+y^2)^{1/2}}$$

$$= \frac{1}{1+x^2} \cdot \frac{y}{(1+x^2+y^2)^{1/2}}.$$

$\therefore$ $$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{1}{1+x^2} \cdot \frac{1.(1+x^2+y^2)^{1/2} - y \cdot \frac{1}{2}(1+x^2+y^2)^{-1/2}.2\,y}{1+x^2+y^2}$$

$$= \frac{1}{1+x^2} \cdot \frac{1+x^2+y^2-y^2}{(1+x^2+y^2)(1+x^2+y^2)^{1/2}}$$

$$= \frac{1}{1+x^2} \cdot \frac{1+x^2}{(1+x^2+y^2)^{3/2}} = \frac{1}{(1+x^2+y^2)^{3/2}}.$$

Problem 15: *If* $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, *prove that*

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right).$$

(Garhwal 2004; Rohilkhand 11B, 12B)

Solution: From the given equation we observe that u is a function of three independent variables x, y and z. Differentiating the given equation partially w.r.t. 'x', we get

$$\frac{2x}{a^2+u} - \left\{\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right\}\frac{\partial u}{\partial x} = 0.$$

$\therefore$ $$\frac{\partial u}{\partial x} = \frac{2\,x/(a^2+u)}{\Sigma\{x^2/(a^2+u)^2\}}.$$

Similarly, by symmetry, we can write the values of $\partial u/\partial y$ and $\partial u/\partial z$.

Now, $$\left(\frac{\partial u}{\partial x}\right)^2 = \frac{4\,x^2/(a^2+u)^2}{[\Sigma\{x^2/(a^2+u)^2\}]^2}.$$

$\therefore$ $$\Sigma(\partial u/\partial x)^2 = \frac{4\Sigma\{x^2/(a^2+u)^2\}}{[\Sigma\{x^2/(a^2+u)^2\}]^2} = \frac{4}{\Sigma\{x^2/(a^2+u)^2\}} \quad ...(1)$$

Again $$2\,x\frac{\partial u}{\partial x} = \frac{4\,x^2/(a^2+u)}{\Sigma\{x^2/(a^2+u)^2\}}.$$

$\therefore$ $$2\Sigma\,x\,(\partial u/\partial x) = \frac{4\Sigma\{x^2/(a^2+u)\}}{\Sigma\{x^2/(a^2+u)^2\}} = \frac{4}{\Sigma\{x^2/(a^2+u)^2\}}. \quad ...(2)$$

[$\because$ $\Sigma\{x^2/(a^2+u)\} = 1$, from the given relation]

Now from (1) and (2), we have

$$\Sigma(\partial u/\partial x)^2 = 2\Sigma\{x(\partial u/\partial x)\}.$$

Problem 16: *If* $x = r\cos\theta,\ y = r\sin\theta$, *prove that*

(i) $\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2 = 1.$

(ii) $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r}\left[\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2\right].$

(Lucknow 2007, 11; Garhwal 11)

Solution: (i) We have $x = r\cos\theta,\ y = r\sin\theta.$

Therefore $\quad r^2 = x^2 + y^2.$ ...(1)

Now $\quad 2r\,(\partial r/\partial x) = 2x\,;$ [Diff. (1) partially w.r.t. 'x']

$\therefore \quad \partial r/\partial x = x/r.$ ...(2)

Again differentiating (1) partially w.r.t. 'y', we get

$$2r\,\frac{\partial r}{\partial y} = 2\,y; \qquad \therefore \quad \frac{\partial r}{\partial y} = \frac{y}{r}\cdot \qquad ...(3)$$

From (2) and (3), on squaring and adding, we get

$$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1.$$

(ii) Differentiating (2) partially w.r.t. x, we get

$$\frac{\partial^2 r}{\partial x^2} = \frac{r\,.\,1 - x\,.\,\partial r/\partial x}{r^2} = \frac{r - x\,.\,x/r}{r^2}, \qquad \text{[Using (2)]}$$

$$= \frac{r^2 - x^2}{r^3} = \frac{(x^2 + y^2) - x^2}{r^3} = \frac{y^2}{r^3}\cdot \qquad ...(4)$$

Differentiating (3) partially w.r.t. y, we get

$$\frac{\partial^2 r}{\partial y^2} = \frac{r\,.\,1 - y\,.\,\partial r/\partial y}{r^2} = \frac{r - y\,.\,y/r}{r^2}, \qquad \left[\because \text{ From (4), } \frac{\partial r}{\partial y} = \frac{y}{r}\right]$$

$$= \frac{r^2 - y^2}{r^3} = \frac{(x^2 + y^2) - y^2}{r^3} = \frac{x^2}{r^3}\cdot \qquad ...(5)$$

Adding (4) and (5), we get

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{y^2}{r^3} + \frac{x^2}{r^3} = \frac{x^2 + y^2}{r^3} = \frac{1}{r}\cdot$$

Also $\quad \dfrac{1}{r}\left[\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2\right] = \dfrac{1}{r}\left[\dfrac{x^2}{r^2} + \dfrac{y^2}{r^2}\right] = \dfrac{x^2 + y^2}{r^3} = \dfrac{r^2}{r^3} = \dfrac{1}{r}\cdot$

$\therefore \quad \dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r}\left[\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2\right].$

Problem 17: *(i) If* $u = x^2 y + y^2 z + z^2 x$*, show that* $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = (x + y + z)^2$.

(ii) If $u = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$*, show that* $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

(Rohilkhand 2013B)

Solution: (i) Given $u = x^2 y + y^2 z + z^2 x$ ...(1)

$\therefore \quad \frac{\partial u}{\partial x} = 2xy + z^2$

and $\quad \frac{\partial u}{\partial y} = x^2 + 2yz,$

and $\quad \frac{\partial u}{\partial z} = y^2 + 2zx$

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 2xy + z^2 + x^2 + 2yz + y^2 + 2zx$$

$$= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = (x + y + z)^2 .$$

(ii) We have
$$u = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ x & y - x & z - x \\ x^2 & y^2 - x^2 & z^2 - x^2 \end{vmatrix} \quad \text{by } C_2 \to C_2 - C_1; \; C_3 \to C_3 - C_1$$

$$= (y - x)(z - x) \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & y + x & z + x \end{vmatrix}$$

$$= (y - x)(z - x)[(z + x) - (y + x)]$$
$$= (y - x)(z - x)(z - y)$$

or $\quad u = yz^2 - xz^2 + x^2 z - y^2 z + xy^2 - x^2 y$

$\therefore \quad \frac{\partial u}{\partial x} = -z^2 + 2xz + y^2 - 2xy$

$\quad \frac{\partial u}{\partial y} = -z^2 - 2yz + 2xy - x^2$

$\quad \frac{\partial u}{\partial z} = 2yz - 2yz + x^2 - y^2$

Now,
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = -z^2 + 2xz + y^2 - 2xy + z^2 - 2yz + 2xy$$
$$-x^2 + 2yz - 2xz + x^2 - y^2$$

or
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Comprehensive Problems 2

Problem 1: *State Euler's theorem on homogeneous functions.* **(Bundelkhand 2001)**

Solution: If u is a homogeneous function of x and y of degree n, then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu.$$

Problem 2: *Verify Euler's theorem in the following cases :*

(i) $u=ax^2+2hxy+by^2$

(ii) $u=\dfrac{x(x^3-y^3)}{x^3+y^3}$,

(iii) $u=axy+byz+czx$,

(iv) $u=x^n\sin(y/x)$

(v) $u=x^n\log(y/x)$,

(vi) $u=1/\sqrt{(x^2+y^2)}$.

Solution: **(i)** Given $u=ax^2+2hxy+by^2$ ...(1)

Here u is a homogeneous function of x and y of degree 2.

Then by Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u \qquad ...(2)$$

Now $\dfrac{\partial u}{\partial x}=2ax+2hy,\ \dfrac{\partial u}{\partial y}=2hx+2by$

$\therefore$ $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=2ax^2+2hxy+2hxy+2by^2$

$$=2(ax^2+2hxy+by^2)=2u$$

This verifies Euler's theorem.

(ii) We have $u=\dfrac{x(x^3-y^3)}{x^3+y^3}$ which is obviously a homogeneous function of x and y of degree $4-3$ *i.e.*, 1. Note that each term in the numerator is of degree 4 while each term in the denominator is of degree 3.

In order to verify Euler's theorem we are to show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=1\cdot u=u$

Now, $\log u=\log x+\log(x^3-y^3)-\log(x^3+y^3)$ ...(1)

Differentiating (1) partially w.r.t. x and y respectively, we get

$$\frac{1}{u}\frac{\partial u}{\partial x}=\frac{1}{x}+\frac{3x^3}{x^3-y^3}-\frac{3x^2}{x^3+y^3} \qquad ...(2)$$

and $$\frac{1}{u}\frac{\partial u}{\partial y}=0-\frac{3y^2}{x^3-y^3}-\frac{3y^2}{x^3+y^3} \qquad ...(3)$$

Multiplying (2) by x and (3) by y and adding, we get

$$\frac{1}{u}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 1 + \frac{3(x^3 - y^3)}{x^3 - y^3} - \frac{3(x^3 + y^3)}{x^3 + y^3} = 1 + 3 - 3 = 1$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

This verifies Euler's theorem.

(iii) Given $u = axy + byz + czx$ ...(1)

Here u is a homogeneous function of x, y and z of degree 2.

Then by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2u \qquad \text{...(2)}$$

Now, $$\frac{\partial u}{\partial x} = ay + cz, \frac{\partial u}{\partial y} = ax + bz, \frac{\partial u}{\partial z} = by + cx$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x(ay + cz) + y(ax + bz) + z(by + cx)$$

$$= 2(axy + byz + czx) = 2u.$$

This verifies Euler's theorem.

(iv) Given $$u = x^n \sin\left(\frac{y}{x}\right) \qquad \text{...(1)}$$

Here u is a homogeneous function of x and y of degree n.

Then by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \qquad \text{...(2)}$$

Now, $$\frac{\partial u}{\partial x} = x^n \cos\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + nx^{n-1}\sin\left(\frac{y}{x}\right)$$

and $$\frac{\partial u}{\partial y} = x^n \cos\left(\frac{y}{x}\right) \cdot \frac{1}{x}$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n \sin\left(\frac{y}{x}\right) = nu$$

This verifies Euler's theorem.

(v) Here u is a homogeneous function of x and y of degree n. So by Euler's theorem we must have

$$x(\partial u/\partial x) + y(\partial u/\partial y) = nu.$$

Now do the verification yourself.

(vi) Here $$u = \frac{1}{\sqrt{(x^2 + y^2)}} = \frac{1}{x\sqrt{[1 + (y/x)^2]}} = x^{-1} \cdot \frac{1}{\sqrt{[1 + (y/x)^2]}}$$

is a homogeneous function of x and y of degree -1. Now proceed yourself.

Problem 3(i): *If* $u = \tan^{-1}\left(\dfrac{x^3 + y^3}{x - y}\right)$, *show that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u.$

(Rohilkhand 2012B; Kumaun 15)

Solution: We have $\tan u = (x^3 + y^3)/(x - y) = v$, say. Then v is a homogeneous function of x and y of degree $3 - 1$ *i.e.*, 2. Therefore by Euler's theorem, we have

$$x\,(\partial v/\partial x) + y\,(\partial v/\partial y) = 2v. \qquad \ldots(1)$$

Now, $v = \tan u.$

$\therefore$ $\dfrac{\partial v}{\partial x} = \sec^2 u\,\dfrac{\partial u}{\partial x}$

and $\dfrac{\partial v}{\partial y} = \sec^2 u\,\dfrac{\partial u}{\partial y}\cdot$

Putting these values in (1), we get

$$x\sec^2 u\,\frac{\partial u}{\partial x} + y\sec^2 u\,\frac{\partial u}{\partial y} = 2v$$

or $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{2v}{\sec^2 u} = \dfrac{2\tan u}{\sec^2 u} = 2\sin u\cos u = \sin 2u.$

Problem 3(ii): *If* $u = \sin^{-1}\left(\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right)$, *show that* $\dfrac{\partial u}{\partial x} = -\dfrac{y}{x}\dfrac{\partial u}{\partial y}\cdot$

(Garhwal 2002; Gorakhpur 05)

Solution: We have $\sin u = (\sqrt{x} - \sqrt{y})/(\sqrt{x} + \sqrt{y}) = v$, say. Then v is a homogeneous function of x and y of degree $\frac{1}{2} - \frac{1}{2}$ *i.e.*, 0.

Therefore by Euler's theorem, we have

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0\,.\,v = 0. \qquad \ldots(1)$$

Now, $v = \sin u.$

$\therefore$ $\dfrac{\partial v}{\partial x} = \cos u\,\dfrac{\partial u}{\partial x},\ \dfrac{\partial v}{\partial y} = \cos u\,\dfrac{\partial u}{\partial y}\cdot$

Putting these values in (1), we get

$$x\cos u\,\frac{\partial u}{\partial x} + y\cos u\,\frac{\partial u}{\partial y} = 0$$

or $\cos u\left(x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}\right) = 0$

or $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$, since $\cos u \neq 0.$

Hence, $\dfrac{\partial u}{\partial x} = -\dfrac{y}{x}\dfrac{\partial u}{\partial y}\cdot$

Problem 4(i): *If* $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, *show that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\tan u.$

(ii) *If* $u = \cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, *show that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \dfrac{1}{2}\cot u = 0$.

(Rohilkhand 2013B)

(iii) *If* $u = \sin^{-1}\left(\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)$, *show that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{12}\tan u.$

(Garhwal 2014)

(iv) *If* $u = \cot^{-1}\dfrac{x+y}{x^{1/2}+y^{1/2}}$, *show that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{4}\sin 2u.$

(Kumaun 2015)

Solution: (i) We have $\sin u = (x+y)/(\sqrt{x}+\sqrt{y}) = v$, say.

Then v is a homogeneous function of x and y of degree $(1-\frac{1}{2})$ *i.e.* $\frac{1}{2}\cdot$ Applying Euler's theorem for v, we have

$$x\,(\partial v/\partial x) + y\,(\partial v/\partial y) = \frac{1}{2}v$$

or $$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u \qquad [\because v = \sin u]$$

or $$x\cos u\,(\partial u/\partial x) + y\cos u\,(\partial u/\partial y) = \frac{1}{2}\sin u$$

or $$x\,(\partial u/\partial x) + y\,(\partial u/\partial y) = \frac{1}{2}\tan u.$$

(ii) Proceed exactly as in part (i).
(iii) Proceed exactly as in part (i).
(iv) Proceed exactly as in part (i).

Problem 5(i): *If* $u = \log\dfrac{x^3+y^3}{x+y}$, *show that* $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2.$

Solution: We have $e^u = \dfrac{x^3+y^3}{x+y} = v$, say.

Obviously $v = (x^3+y^3)/(x+y)$ is a homogeneous function of x and y of degree $3-1$ *i.e.*, 2. Therefore by Euler's theorem, we have

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 2\,.\,v = 2v \qquad \text{...(1)}$$

Now, $$v = e^u.$$

$\therefore$ $$\frac{\partial v}{\partial x} = e^u\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} = e^u\frac{\partial u}{\partial y}.$$

Putting these values in (i), we get

$$xe^{u}\frac{\partial u}{\partial x}+ye^{u}\frac{\partial u}{\partial y}=2e^{u} \quad \text{or} \quad e^{u}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=2e^{u}$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2.$$

Problem 5(ii): *If* $u=\log\dfrac{x^4+y^4}{x+y}$, *show that* $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=3.$

Solution: Given $u=\log\dfrac{x^4+y^4}{x+y}$.

On taking antilog $e^u=\dfrac{x^4+y^4}{x+y}$.

Here e^u is a homogeneous function of x and y of degree 3.

Then by Euler's theorem

$$x\frac{\partial}{\partial x}(e^u)+y\frac{\partial}{\partial y}(e^u)=3e^u$$

or $$xe^u\frac{\partial u}{\partial x}+ye^u\frac{\partial u}{\partial y}=3e^u$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3$$

Problem 5(iii): *If* $u=\log\dfrac{x^2+y^2}{x+y}$, *show that* $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=1.$

(Kanpur 2006; Avadh 13)

Solution: Proceed exactly as in Problem 5, part (i).

Problem 6: *Use Euler's theorem on homogeneous functions to show that if*

$$u=\tan^{-1}\left(\frac{y}{x}\right) \text{ then } x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

Solution: We have $\tan u=\dfrac{y}{x}=v$, say

Then v is a homogeneous function of x and y of degree $1-1$ *i.e.*, 0,

Applying Euler's theorem for v, we have $x\dfrac{\partial v}{\partial x}+y\dfrac{\partial v}{\partial y}=0$

or $$x\frac{\partial}{\partial x}(\tan u)+y\frac{\partial}{\partial y}(\tan u)=0$$

or $$x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=0$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

Problem 7: *If u be a homogeneous function of x and y of degree n, show that*

(i) $x\,(\partial^2 u/\partial x^2) + y\,(\partial^2 u/\partial x\,\partial y) = (n-1)\,(\partial u/\partial x)$. **(Kumaun 2007)**

(ii) $x\,(\partial^2 u/\partial x\,\partial y) + y\,(\partial^2 u/\partial y^2) = (n-1)\,(\partial u/\partial y)$. **(Kumaun 2007)**

(iii) $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2\,xy\,\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = n\,(n-1)\,u$. **(Kumaun 2007)**

Solution: By Euler's theorem, we have

$$x\,(\partial u/\partial x) + y\,(\partial u/\partial y) = nu \qquad ...(1)$$

Differentiating (1) partially w.r.t. x, we get

$$x\,\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\,\frac{\partial^2 u}{\partial x\,\partial y} = n\,\frac{\partial u}{\partial x}$$

or
$$x\,\frac{\partial^2 u}{\partial x^2} + y\,\frac{\partial^2 u}{\partial x\,\partial y} = n\,\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} = (n-1)\,\frac{\partial u}{\partial x} \qquad ...(2)$$

Similarly, differentiating (1) partially w.r.t. 'y', we get

$$x\,\frac{\partial^2 u}{\partial x\,\partial y} + y\,\frac{\partial^2 u}{\partial y^2} = (n-1)\,\frac{\partial u}{\partial y}. \qquad ...(3)$$

Now multiplying (2) by x and (3) by y and adding, we get

$$x^2\,\frac{\partial^2 u}{\partial x^2} + 2\,xy\,\frac{\partial^2 u}{\partial x\,\partial y} + y^2\,\frac{\partial^2 u}{\partial y^2} = (n-1)\left[x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y}\right]$$

$$= (n-1)\,nu = n\,(n-1)\,u.$$

Problem 8: *If* $u = \dfrac{xy}{x+y}$, *show that* $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\,\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\,\dfrac{\partial^2 u}{\partial y^2} = 0.$ **(Kumaun 2013)**

Solution: Given $u = \dfrac{xy}{x+y}$ ...(1)

Here, u is a homogeneous function of x and y of degree 1

By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u \qquad ...(2)$$

Differentiating (2) partially w.r.to x, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\,\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial u}{\partial x}$$

or
$$x\frac{\partial^2 u}{\partial x^2} + y\,\frac{\partial^2 u}{\partial x\,\partial y} = 0 \qquad ...(3)$$

Again differentiating (2) partially w.r.t. y, we get

$$x\frac{\partial^2 u}{\partial y\,\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=\frac{\partial u}{\partial y}$$

or $$x\frac{\partial^2 u}{\partial x\,\partial y}+y\frac{\partial^2 u}{\partial y^2}=0 \qquad \text{...(4)}$$

Multiplying (3) by x and (4) by y and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$$

Problem 9: *If* $u=x\,\phi\,(y/x)+\psi\,(y/x)$, *prove that*

$$x^2\frac{\partial^2 u}{\partial x^2}+2\,x\,y\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$$

(Kanpur 2008; Kumaun 11)

Solution: Let $u=z_1+z_2$, where $z_1=x\,\phi\,(y/x)$ and $z_2=\psi\,(y/x)$. Obviously z_1 is a homogeneous function of x and y of degree 1 and z_2 is a homogeneous function of x and y of degree zero. Now

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x\frac{\partial}{\partial x}(z_1+z_2)+y\frac{\partial}{\partial y}(z_1+z_2)$$

$$=\left(x\frac{\partial z_1}{\partial x}+y\frac{\partial z_1}{\partial y}\right)+\left(x\frac{\partial z_2}{\partial x}+y\frac{\partial z_2}{\partial y}\right)=1\,.\,z_1+0\,.\,z_2.$$

[By Euler's theorem]

Thus $$x\,(\partial u/\partial x)+y\,(\partial u/\partial y)=z_1 \qquad \text{...(1)}$$

Differentiating (1) partially w.r.t. x and y respectively, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial z_1}{\partial x}, \qquad \text{...(2)}$$

and $$x\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial u}{\partial y}+y\frac{\partial^2 u}{\partial y^2}=\frac{\partial z_1}{\partial y}. \qquad \text{...(3)}$$

Multiplying (2) by x and (3) by y and adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2\,x\,y\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x\frac{\partial z_1}{\partial x}+y\frac{\partial z_1}{\partial y}$$

or $$x^2\frac{\partial^2 u}{\partial x^2}+2\,x\,y\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+z_1=1\,.\,z_1$$

[$\because$ $x\,(\partial u/\partial x)+y\,(\partial u/\partial y)=z_1$ by (1), and $x\,(\partial z_1/\partial x)+y\,(\partial z_1/\partial y)$ $=1\,.\,z_1$ by Euler's theorem]

or $$x^2\frac{\partial^2 u}{\partial x^2}+2\,x\,y\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0.$$

Problem 10: *If $u = \sin^{-1}[(x^2 + y^2)^{1/5}]$, prove that*

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{2}{25} \tan u\,(2 \tan^2 u - 3).$$

Solution: It is given that $u = \sin^{-1}[(x^2 + y^2)^{1/5}]$.

$\therefore$ $\sin u = (x^2 + y^2)^{1/5} = v$, say.

Obviously v is a homogeneous function of x and y of degree $2/5$. Therefore by Euler's theorem, we have

$$x\,(\partial v/\partial x) + y\,(\partial v/\partial y) = \frac{2}{5} v. \qquad \text{...(1)}$$

Now $v = \sin u$.

$\therefore$ $$\frac{\partial v}{\partial x} = \cos u \frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} = \cos u \frac{\partial u}{\partial y}.$$

Putting these values in (1), we get

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \frac{2}{5} v = \frac{2}{5} \sin u$$

or $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2}{5} \tan u. \qquad \text{...(2)}$$

Now differentiating (2) partially w.r.t. x and y respectively, we get

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x\, \partial y} = \frac{2}{5} \sec^2 u \frac{\partial u}{\partial x} \qquad \text{...(3)}$$

and $$x \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y \frac{\partial^2 u}{\partial y^2} = \frac{2}{5} \sec^2 u \frac{\partial u}{\partial y} \qquad \text{...(4)}$$

Multiplying (3) by x, (4) by y and adding, we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2}{5} \sec^2 u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right)$$

or $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + \frac{2}{5} \tan u = \frac{2}{5} \sec^2 u \cdot \frac{2}{5} \tan u \qquad \text{[From (2)]}$$

or $$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{2}{5} \tan u \left(\frac{2}{5} \sec^2 u - 1 \right)$$

$$= \frac{4}{25} \tan u\,(2 \sec^2 u - 5)$$

$$= \frac{4}{25} \tan u\,[2\,(1 + \tan^2 u) - 5]$$

$$= \frac{4}{25} \tan u\,(2 \tan^2 u - 3).$$

Problem 11: *If* $u = \sin^{-1}\left[\dfrac{x^{1/3} + y^{1/3}}{x^{1/3} + y^{1/2}}\right]^{1/2}$ *prove that*

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13 + \tan^2 u).$$

(Kanpur 2015)

Solution: Proceed as in Problem 10.

Comprehensive Problems 3

Problem 1: *Find dy/dx in the following:*

(i) $x^y + y^x = a^b$ (ii) $ax^2 + 2hxy + by^2 = 1.$ **(Kashi 2013)**

Solution: (i) Let $f(x, y) = x^y + y^x - a^b$.

Then we have $f(x, y) = 0.$

$$\therefore \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = \frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}.$$

(ii) Let $f(x, y) \equiv ax^2 + 2hxy + by^2 = 1$

then we have $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}.$ ...(1)

Now, $\dfrac{\partial f}{\partial x} = 2ax + 2hy, \dfrac{\partial f}{\partial y} = 2hx + 2by.$

Therefore (1) gives

$$\frac{dy}{\partial x} = -\frac{(2ax + 2hy)}{2hx + 2by} = -\frac{(ax + hy)}{(hx + by)}.$$

Problem 2: *If* $\sqrt{(1 - x^2)} + \sqrt{(1 - y^2)} = a(x - y)$*, prove that*

$$\frac{dy}{dx} = \frac{\sqrt{(1 - y^2)}}{\sqrt{(1 - x^2)}}.$$

Solution: It is given that

$$\frac{\sqrt{(1 - x^2)} + \sqrt{(1 - y^2)}}{x - y} = a.$$

Let $f(x, y) = \dfrac{\sqrt{(1 - x^2)} + \sqrt{(1 - y^2)}}{x - y} - a.$

Then $f(x, y) = 0.$

$$\therefore \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

$$= -\frac{\dfrac{\{\frac{1}{2}(1-x^2)^{-1/2}.(-2x)\}(x-y)-1.\{\sqrt{(1-x^2)}+\sqrt{(1-y^2)}\}}{(x-y)^2}}{\dfrac{\{\frac{1}{2}(1-y^2)^{-1/2}.(-2y)\}(x-y)-(-1).\{\sqrt{(1-x^2)}+\sqrt{(1-y^2)}\}}{(x-y)^2}}$$

$$= -\frac{\dfrac{-x(x-y)}{\sqrt{(1-x^2)}}-\sqrt{(1-x^2)}-\sqrt{(1-y^2)}}{\dfrac{-y(x-y)}{\sqrt{(1-y^2)}}+\sqrt{(1-x^2)}+\sqrt{(1-y^2)}}$$

$$= -\frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}\cdot\frac{-x^2+xy-(1-x^2)-\sqrt{(1-x^2)}\sqrt{(1-y^2)}}{-yx+y^2+\sqrt{(1-x^2)}\sqrt{(1-y^2)}+1-y^2}$$

$$= -\frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}\cdot\frac{xy-1-\sqrt{(1-x^2)}\sqrt{(1-y^2)}}{-xy+1-\sqrt{(1-x^2)}\sqrt{(1-y^2)}} = \frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}.$$

Problem 3: *If $u = \sin(x^2 + y^2)$, where $a^2x^2 + b^2y^2 = c^2$, find du/dx.*

Solution: We have $\dfrac{du}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dx}$. ...(1)

Now, $u = \sin(x^2 + y^2)$.

$\therefore$ $\dfrac{\partial u}{\partial x} = 2x\cos(x^2 + y^2)$

and $\dfrac{\partial u}{\partial y} = 2y\cos(x^2 + y^2)$.

Since $a^2x^2 + b^2y^2 = c^2$,

therefore $2a^2x + 2b^2y\,(dy/dx) = 0$

or $dy/dx = -(a^2x)/(b^2y)$.

$\therefore$ From (1), $\dfrac{du}{dx} = 2x\cos(x^2 + y^2) - [2y\cos(x^2 + y^2)][(a^2x)/(b^2y)]$

$= [2x\cos(x^2 + y^2)][1 - (a^2/b^2)]$.

Problem 4: *If $u = x^4y^5$, where $x = t^2$ and $y = t^3$, find du/dt.*

Solution: Given $u = x^4y^5$ where $x = t^2$, $y = t^3$.

We have $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\cdot\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dt}$ ...(1)

Now, $\dfrac{\partial u}{\partial x} = 4x^3 y^5$. *and* $\dfrac{\partial u}{\partial y} = 5x^4 y^4$

Also $\dfrac{dx}{dt} = 2t$ and $\dfrac{dy}{dt} = 3t^2$

$\therefore$ From (1), we get

$$\frac{du}{dt} = 4\, x^3 y^5\, .\, 2t + 5\, x^4\, y^4\, .\, 3t^2$$

$$= 4\,(t^2)^3 (t^3)^5\, .\, 2t + 5\,(t^2)^4\,(t^3)^4\, .\, 3t^2$$

$$= 8\, t^{22} + 15\, t^{22} \qquad [\because\ x = t^2,\ y = t^3]$$

$$= 23t^{22}$$

Problem 5: *If $f(x, y) = 0$ and $\phi(y, z) = 0$, show that*

$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}.$$

(Lucknow 2009)

Solution: From $f(x, y) = 0$, we have $\dfrac{dy}{dx} = -\dfrac{\partial f/\partial x}{\partial f/\partial y}$. ...(1)

From $\phi(y, z) = 0$, we have $\dfrac{dz}{dy} = -\dfrac{\partial \phi/\partial y}{\partial \phi/\partial z}$. ...(2)

Multiplying the respective sides of (1) and (2), we have

$$\frac{dy}{dx} \cdot \frac{dz}{dy} = \left(\frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}\right) \Big/ \left(\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z}\right)$$

or $\dfrac{dz}{dx} \cdot \dfrac{\partial f}{\partial y} \cdot \dfrac{\partial \phi}{\partial z} = \dfrac{\partial f}{\partial x} \cdot \dfrac{\partial \phi}{\partial y}$.

Problem 6 : *If $u = \sqrt{(x^2 + y^2)}$ and $x^3 + y^3 + 3axy = 5a^2$, find the value of $\dfrac{du}{dx}$ when $x = a,\ y = a$.*

Solution: Given, $u = \sqrt{x^2 + y^2}$

We have, $\dfrac{du}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dx}$. ...(1)

Now $\dfrac{\partial u}{\partial x} = \dfrac{1}{2}(x^2 + y^2)^{-1/2} .2x = \dfrac{x}{\sqrt{(x^2 + y^2)}}$

and $\dfrac{\partial u}{\partial y} = \dfrac{1}{2}(x^2 + y^2)^{-1/2} .2y = \dfrac{y}{\sqrt{(x^2 + y^2)}}$.

Let $f(x, y) = x^3 + y^3 + 3axy = 5a^2$.

Then $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{3x^2 + 3ay}{3y^2 + 3ax}.$$

$\therefore$ From (1), we get

$$\frac{du}{dx} = \frac{x}{\sqrt{x^2 + y^2}} + \frac{y}{\sqrt{x^2 + y^2}}\left(-\frac{3x^2 + 3ay}{3y^2 + 3ax}\right).$$

At $x = a,\ y = a$, we get

$$\frac{du}{dx} = \frac{a}{\sqrt{2a^2}} + \frac{\alpha}{\sqrt{2a^2}}\left(-\frac{3a^2 + 3a^2}{3a^2 + 3a^2}\right) = \frac{a}{\sqrt{2a^2}} - \frac{a}{\sqrt{2a^2}} = 0.$$

Hints to Objective Type Questions

Multiple Choice Questions

1. Here $z = \tan(y + ax) + (y - ax)^{3/2}$.

$\therefore$ $(\partial z/\partial x) = \{\sec^2(y + ax\} \cdot a + \frac{3}{2}(y - ax)^{1/2} \cdot (-a)$

and $(\partial^2 z/\partial x^2) = 2a^2 \tan(y + ax) \sec^2(y + ax) + \frac{3}{4}a^2(y - ax)^{-1/2}$.

Again $(\partial z/\partial y) = \sec^2(y + ax) + \frac{3}{2}(y - ax)^{1/2}$

and $(\partial^2 z/\partial y^2) = 2\sec^2(y + ax)\tan(y + ax) + \frac{3}{4}(y - ax)^{-1/2}$.

Thus, $(\partial^2 z/\partial x^2) - a^2(\partial^2 z/\partial y^2) = 0$.

2. See article 6.

3. We have $u = \frac{1}{\sqrt{(x^2 + y^2 + z^2)}}$,

which is a homogeneous function of x, y and z of degree $n = -1$.
So, by Euler's theorem on homogeneous functions, we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = (-1)\,u = -u.$$

4. See article 9.

5. See Example 4.

6. See article 6.

7. See Example 6.

8. Refer Example 2.

Fill in the Blanks

1. We have, $u = e^{my} \cos mx$

$$\therefore \quad \frac{\partial u}{\partial x} = -me^{my} \sin mx \qquad \frac{\partial^2 u}{\partial x^2} = -m^2 e^{my} \cos mx$$

$$\frac{\partial u}{\partial y} = me^{my} \cos mx \qquad \frac{\partial^2 u}{\partial y^2} = m^2 e^{my} \cos mx.$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -m^2 e^{my} \cos mx + m^2 e^{my} \cos mx = 0.$$

2. We have, $u = \tan^{-1} \dfrac{x^2 + y^2}{x + y}$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{1 + \left(\dfrac{x^2 + y^2}{x + y}\right)^2} \cdot \frac{2x(x + y) - 1 \cdot (x^2 + y^2)}{(x + y)^2} = \frac{x^2 + 2xy - y^2}{(x + y)^2 + (x^2 + y^2)^2}.$$

3. See article 5

4. See article 8.

5. See Problem 7(i) of Comprehensive Problems 2.

We have, $x \dfrac{\partial^2 u}{\partial x^2} + y \dfrac{\partial^2 u}{\partial x \, \partial y} = (n-1) \dfrac{\partial u}{\partial x}$

$$\Rightarrow \quad x \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + y \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = (n-1) \frac{\partial u}{\partial x}$$

$$\Rightarrow \quad x \frac{\partial}{\partial x}(u_x) + y \frac{\partial}{\partial y}(u_x) = (n-1) u_x.$$

6. See Problem 5 of Comprehensive Problems 1.

7. See Example 2.

True or False

1. See article 4.

2. See Note 2 of article 5.

3. See article 9.

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Chapter-8

Jacobians

Comprehensive Problems 1

Problem 1: *If* $x = r \cos \theta$, $y = r \sin \theta$, *show that*

(*i*) $\dfrac{\partial (x, y)}{\partial (r, \theta)} = r,$

(Kanpur 2005; Meerut 13B; Kashi 13; Gorakhpur 15)

(*ii*) $\dfrac{\partial (r, \theta)}{\partial (x, y)} = \dfrac{1}{r} \cdot$

(Kanpur 2005; Meerut 13)

Solution: (i) We have

$$\frac{\partial (x, y)}{\partial (r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

$$= r \cos^2 \theta + r \sin^2 \theta = r.$$

(ii) From the given relations, we have

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan \theta = y / x.$$

Now differentiating $r^2 = x^2 + y^2$ partially w.r.t. x and y, we get

$$2r \frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

and $$2r \frac{\partial r}{\partial y} = 2y \quad \text{or} \quad \frac{\partial r}{\partial y} = \frac{y}{r} \cdot$$

Again differentiating $\tan \theta = y / x$ partially w.r.t. x and y, we get

$$\sec^2 \theta \cdot \frac{\partial \theta}{\partial x} = -\frac{y}{x^2}$$

or $$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2 \sec^2 \theta} = -\frac{y}{r^2 \cos^2 \theta \sec^2 \theta} = -\frac{y}{r^2}$$

and $$\sec^2 \theta \cdot \frac{\partial \theta}{\partial y} = \frac{1}{x} \quad \text{or} \quad \frac{\partial \theta}{\partial y} = \frac{1}{x \sec^2 \theta} = \frac{\cos^2 \theta}{x} = \frac{x^2}{r^2} \cdot \frac{1}{x} = \frac{x}{r^2} \cdot$$

$\therefore$ $$\frac{\partial (r, \theta)}{\partial (x, y)} = \begin{vmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{x}{r} & \dfrac{y}{r} \\ -\dfrac{y}{r^2} & \dfrac{x}{r^2} \end{vmatrix}$$

$$= \frac{x^2}{r^3} + \frac{y^2}{r^3} = \frac{x^2 + y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r} \cdot$$

Problem 2: *If $x = u(1+v)$, $y = v(1+u)$, find the Jacobian of x, y with respect to u, v.*

(Lucknow 2011; Meerut 13)

Solution: We have $\dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$

$$= (1+v)(1+u) - uv = 1 + u + v + uv - uv = 1 + u + v.$$

Problem 3: *If $x = c \cos u \cosh v$, $y = c \sin u \sinh v$, prove that*

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{2} c^2 (\cos 2u - \cosh 2v).$$

(Rohilkhand 2013)

Solution: We have

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -c \sin u \cosh v & c \cos u \sinh v \\ c \cos u \sinh v & c \sin u \cosh v \end{vmatrix}$$

$$= -c^2 \sin^2 u \cosh^2 v - c^2 \cos^2 u \sinh^2 v$$

$$= -\frac{1}{2} c^2 [(1 - \cos 2u) \cosh^2 v + (1 + \cos 2u) \sinh^2 v]$$

$$= -\frac{1}{2} c^2 [\cosh^2 v + \sinh^2 v - \cos 2u (\cosh^2 v - \sinh^2 v)]$$

$$= -\frac{1}{2} c^2 (\cosh 2v - \cos 2u) = \frac{1}{2} c^2 (\cos 2u - \cosh 2v).$$

Problem 4: *If $u = \dfrac{y^2}{2x}$, $v = \dfrac{x^2 + y^2}{2x}$, find $\dfrac{\partial(u, v)}{\partial(x, y)}$.*

(Meerut 2012)

Solution: We have $u = \dfrac{y^2}{2x}$ and $v = \dfrac{x}{2} + \dfrac{y^2}{2x}$.

$$\therefore \quad \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -\dfrac{y^2}{2x^2} & \dfrac{y}{x} \\ \dfrac{1}{2} - \dfrac{y^2}{2x^2} & \dfrac{y}{x} \end{vmatrix}$$

$$= -\frac{y^2}{2x^3} - \frac{y}{x}\left(\frac{1}{2} - \frac{y^2}{2x^2}\right) = -\frac{y^3}{2x^3} - \frac{y}{2x} + \frac{y^3}{2x^3} = -\frac{y}{2x}.$$

Problem 5: *If $u_1 = x_2 x_3 / x_1$, $u_2 = x_3 x_1 / x_2$, $u_3 = x_1 x_2 / x_3$, prove that*

$$J(u_1, u_2, u_3) = 4.$$

(Kumaun 2011; Bundelkhand 14; Purvanchal 14; Gorakhpur 14, 15)

Solution: We have $J(u_1, u_2, u_3)$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2 x_3 & x_3 x_1 & x_1 x_2 \\ x_2 x_3 & -x_3 x_1 & x_1 x_2 \\ x_2 x_3 & x_3 x_1 & -x_1 x_2 \end{vmatrix}$$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} 0 & 0 & 2x_1 x_2 \\ 2x_2 x_3 & 0 & 0 \\ x_2 x_3 & x_3 x_1 & -x_1 x_2 \end{vmatrix}, \text{ adding } R_2 \text{ to } R_1 \text{ and then } R_3 \text{ to } R_2$$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \cdot (2x_1 x_2 \cdot 2x_2 x_3^2 x_1) = 4.$$

Problem 6: *If* $x = \sin\theta \sqrt{(1 - c^2 \sin^2 \phi)}$, $y = \cos\theta \cos\phi$, *then show that*

$$\frac{\partial (x, y)}{\partial (\theta, \phi)} = -\sin\phi \frac{[(1 - c^2) \cos^2 \theta + c^2 \cos^2 \phi]}{\sqrt{(1 - c^2 \sin^2 \phi)}}.$$

Solution: We have $\frac{\partial (x, y)}{\partial (\theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{vmatrix} = \frac{\partial x}{\partial \theta} \cdot \frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \phi} \cdot \frac{\partial y}{\partial \theta}$

$$= \cos\theta \sqrt{(1 - c^2 \sin^2 \phi)} \,.\, (-\cos\theta \sin\phi)$$

$$- \sin\theta \cdot \frac{1}{2} (1 - c^2 \sin^2 \phi)^{-1/2} . (-2c^2 \sin\phi \cos\phi) \,.\, (-\sin\theta \cos\phi)$$

$$= -\frac{\sin\phi}{\sqrt{(1 - c^2 \sin^2 \phi)}} [\cos^2 \theta (1 - c^2 \sin^2 \phi) + c^2 \sin^2 \theta \cos^2 \phi]$$

$$= -\frac{\sin\phi}{\sqrt{(1 - c^2 \sin^2 \phi)}} [\cos^2 \theta - c^2 \cos^2 \theta \sin^2 \phi + c^2 (1 - \cos^2 \theta) \cos^2 \phi]$$

$$= -\frac{\sin\phi}{\sqrt{(1 - c^2 \sin^2 \phi)}} [\cos^2 \theta - c^2 \cos^2 \theta (\sin^2 \phi + \cos^2 \phi) + c^2 \cos^2 \phi]$$

$$= -\frac{\sin\phi}{\sqrt{(1 - c^2 \sin^2 \phi)}} [\cos^2 \theta - c^2 \cos^2 \theta + c^2 \cos^2 \phi]$$

$$= -\frac{\sin\phi}{\sqrt{(1 - c^2 \sin^2 \phi)}} [(1 - c^2) \cos^2 \theta + c^2 \cos^2 \phi].$$

Problem 7: *If* $u = xyz,\ v = xy + yz + zx,\ w = x + y + z$, *compute* $\dfrac{\partial (u, v, w)}{\partial (x, y, z)}$.

(Rohilkhand 2013; Kumaun 14)

Solution: Here

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = \begin{vmatrix} yz & xz & xy \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} y(z-x) & x(z-y) & xy \\ z-x & z-y & y+x \\ 0 & 0 & 1 \end{vmatrix}, \text{ by } C_1 \to C_1 - C_2,\ C_2 \to C_2 - C_3$$

$$= y(z-x)(z-y) - x(z-y)(z-x)$$

$$= (z-x)(z-y)(y-x)$$

$$= (x-y)(y-z)(z-x).$$

Problem 8: *If* $y_1 = 1 - x_1,\ y_2 = x_1(1 - x_2),\ y_3 = x_1 x_2 (1 - x_3), \ldots,\ y_n = x_1 x_2 \ldots x_{n-1}(1 - x_n)$,

prove that $J(y_1, y_2, \ldots, y_n) = (-1)^n\, x_1^{\,n-1}\, x_2^{\,n-2} \ldots x_{n-1}$.

(Kumaun 2007, 11; Gorakhpur 12, 14)

Solution: Here y_1 is a function of x_1 only, y_2 is a function of x_1, x_2 only, y_3 is a function of x_1, x_2, x_3 only, ..., and y_n is a function of $x_1, x_2, \ldots, x_n$.

$$\therefore \quad \frac{\partial (y_1, y_2, \ldots, y_n)}{\partial (x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & 0 & 0 & \ldots & 0 \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \frac{\partial y_n}{\partial x_3} & \ldots & \frac{\partial y_n}{\partial x_n} \end{vmatrix} = \frac{\partial y_1}{\partial x_1} \cdot \frac{\partial y_2}{\partial x_2} \cdot \frac{\partial y_3}{\partial x_3} \cdot \ldots \cdot \frac{\partial y_n}{\partial x_n}$$

$$= (-1) . (-x_1)(-x_1 x_2) \ldots (-x_1 x_2 x_3 \ldots x_{n-1})$$

$$= (-1)^n\, x_1^{\,n-1}\, x_2^{\,n-2} \ldots x_{n-1}.$$

Problem 9: If $y_1 = \cos x_1,\ y_2 = \sin x_1 \cos x_2,\ y_3 = \sin x_1 \sin x_2 \cos x_3, \ldots,$
$y_n = \sin x_1 \sin x_2 \sin x_3 \ldots \sin x_{n-1} \cos x_n$, *find the Jacobian of* $y_1, y_2, \ldots, y_n$ *with respect to* $x_1, x_2, \ldots, x_n$.

Solution: Here y_1 is a function of x_1 only, y_2 is a function of x_1, x_2 only, y_3 is a function of x_1, x_2, x_3 only, ... and y_n is a function of $x_1, x_2, \ldots, x_n$.

$$\therefore \quad \frac{\partial (y_1, y_2, \ldots, y_n)}{\partial (x_1, x_2, \ldots, x_n)} = \text{the principal diagonal term of the determinant}$$

$$= \frac{\partial y_1}{\partial x_1} \cdot \frac{\partial y_2}{\partial x_2} \cdot \frac{\partial y_3}{\partial x_3} \cdot \ldots \cdot \frac{\partial y_n}{\partial x_n}$$

$$= (-\sin x_1) \, . \, (-\sin x_1 \sin x_2) \, . \, (-\sin x_1 \sin x_2 \sin x_3) \ldots$$
$$. \, (-\sin x_1 \sin x_2 \sin x_3 \ldots \sin x_n)$$
$$= (-1)^n \sin^n x_1 \sin^{n-1} x_2 \sin^{n-2} x_3 \ldots \sin x_n.$$

Comprehensive Problems 2

Problem 1: *If $u^3 + v^3 = x + y, u^2 + v^2 = x^3 + y^3$, show that*

$$\frac{\partial (u, v)}{\partial (x, y)} = \frac{1}{2} \frac{y^2 - x^2}{uv (u - v)} .$$

(Kumaun 2002; Garhwal 2003)

Solution: The given relations can be written as

$$F_1 \equiv u^3 + v^3 - x - y = 0$$

and $F_2 \equiv u^2 + v^2 - x^3 - y^3 = 0.$

Now $$\frac{\partial (u, v)}{\partial (x, y)} = (-1)^2 \frac{\partial (F_1, F_2)}{\partial (x, y)} \Big/ \frac{\partial (F_1, F_2)}{\partial (u, v)} \qquad \ldots(1)$$

We have $$\frac{\partial (F_1, F_2)}{\partial (x, y)} = \begin{vmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix}$$

$$= 3y^2 - 3x^2 = 3 (y^2 - x^2)$$

and $$\frac{\partial (F_1, F_2)}{\partial (u, v)} = \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix} = 6u^2 v - 6uv^2 = 6uv (u - v).$$

$\therefore$ From (1), $$\frac{\partial (u, v)}{\partial (x, y)} = \frac{3 (y^2 - x^2)}{6uv (u - v)} = \frac{1}{2} \frac{y^2 - x^2}{uv (u - v)} .$$

Problem 2: *If $x + y + z = u, y + z = uv, z = uvw$, show that*

$$\frac{\partial (x, y, z)}{\partial (u, v, w)} = u^2 v.$$

(Rohilkhand 2005; Gorakhpur 11; Kashi 14)

Solution: The given relations can be written as

$$F_1 \equiv x + y + z - u = 0$$
$$F_2 \equiv y + z - uv = 0$$

and $F_3 \equiv z - uvw = 0.$

Now $$\frac{\partial (x, y, z)}{\partial (u, v, w)} = (-1)^3 \frac{\partial (F_1, F_2, F_3)}{\partial (u, v, w)} \Big/ \frac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} \qquad \ldots(1)$$

We have $$\frac{\partial (F_1, F_2, F_3)}{\partial (u, v, w)} = \begin{vmatrix} -1 & 0 & 0 \\ -v & -u & 0 \\ -vw & -uw & -uv \end{vmatrix}$$

$$= (-1) \, . \, (-u) \, . \, (-uv) = -u^2 v$$

and $$\frac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

$\therefore$ From (1), $$\frac{\partial (x, y, z)}{\partial (u, v, w)} = -\frac{-u^2 v}{1} = u^2 v.$$

Problem 3: *If* $u^3 = xyz, \frac{1}{v} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}, w^2 = x^2 + y^2 + z^2$, *prove that*

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = \frac{-v (y - z)(z - x)(x - y)(x + y + z)}{3u^2 w (yz + zx + xy)}.$$

(Kumaun 2009, 13; Rohilkhand 12B)

Solution: The given relations can be written as

$$F_1 \equiv u^3 - xyz = 0$$

$$F_2 \equiv \frac{1}{v} - \frac{1}{x} - \frac{1}{y} - \frac{1}{z} = 0$$

and $$F_3 \equiv w^2 - x^2 - y^2 - z^2 = 0.$$

Now $$\frac{\partial (u, v, w)}{\partial (x, y, z)} = (-1)^3 \frac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} \Big/ \frac{\partial (F_1, F_2, F_3)}{\partial (u, v, w)} \qquad ...(1)$$

We have $$\frac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} = \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} & \frac{\partial F_3}{\partial y} & \frac{\partial F_3}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -yz & -zx & -xy \\ 1/x^2 & 1/y^2 & 1/z^2 \\ -2x & -2y & -2z \end{vmatrix} = \frac{2}{x^2 y^2 z^2} \begin{vmatrix} x^2 yz & y^2 zx & z^2 xy \\ 1 & 1 & 1 \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= \frac{2}{xyz} \begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^3 & y^3 & z^3 \end{vmatrix} = -\frac{2}{xyz} \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= -\frac{2}{xyz} \begin{vmatrix} 1 & 0 & 0 \\ x & y - x & z - x \\ x^3 & y^3 - x^3 & z^3 - x^3 \end{vmatrix}, \text{ by } C_2 - C_1 \text{ and } C_3 - C_1$$

$$= -\frac{2}{xyz} (y - x)(z - x) \begin{vmatrix} 1 & 1 \\ y^2 + xy + x^2 & z^2 + zx + x^2 \end{vmatrix}$$

$$= -\frac{2}{xyz}(y-x)(z-x)\begin{vmatrix} 1 & 0 \\ y^2+xy+x^2 & (z-y)(x+y+z) \end{vmatrix},$$

by $C_2 - C_1$

$$= -\frac{2}{xyz}(y-x)(z-x)(z-y)(x+y+z)$$

$$= -\frac{2}{xyz}(x-y)(y-z)(z-x)(x+y+z).$$

Also $$\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} = \begin{vmatrix} 3u^2 & 0 & 0 \\ 0 & -1/v^2 & 0 \\ 0 & 0 & 2w \end{vmatrix} = -6u^2 w / v^2.$$

Hence from (1), $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$

$$= -\frac{-2(x-y)(y-z)(z-x)(x+y+z)}{xyz} \cdot \frac{-v^2}{6u^2 w}$$

$$= -\frac{(x-y)(y-z)(z-x)(x+y+z).v}{3u^2 wxyz} \cdot \frac{xyz}{yz+zx+xy}$$

$$\left[\because \frac{1}{v} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}, \text{ so that } v = \frac{xyz}{yz+zx+xy}\right]$$

$$= -\frac{v(y-z)(z-x)(x-y)(x+y+z)}{3u^2 w(yz+zx+xy)}.$$

Problem 4: *If* $u^3 + v^3 + w^3 = x + y + z$, $u^2 + v^2 + w^2 = x^3 + y^3 + z^3$, $u + v + w = x^2 + y^2 + z^2$, *then prove that* $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = \dfrac{(y-z)(z-x)(x-y)}{(u-v)(v-w)(w-u)}$.

(Purvanchal 2007; Kanpur 12)

Solution: The given relations can be written as

$$F_1 \equiv u^3 + v^3 + w^3 - x - y - z = 0,$$

$$F_2 \equiv u^2 + v^2 + w^2 - x^3 - y^3 - z^3 = 0,$$

and $$F_3 \equiv u + v + w - x^2 - y^2 - z^2 = 0.$$

Now $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} \Big/ \frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} \qquad \ldots(1)$$

We have $$\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} = \begin{vmatrix} -1 & -1 & -1 \\ -3x^2 & -3y^2 & -3z^2 \\ -2x & -2y & -2z \end{vmatrix}$$

$$= -6 \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix} = 6 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= 6 \begin{vmatrix} 1 & 0 & 0 \\ x & y - x & z - x \\ x^2 & y^2 - x^2 & z^2 - x^2 \end{vmatrix}$$

$$= 6 (y - x)(z - x) \begin{vmatrix} 1 & 1 \\ y + x & z + x \end{vmatrix}$$

$$= 6 (y - x)(z - x)(z - y) = 6 (x - y)(y - z)(z - x).$$

Also $$\frac{\partial (F_1, F_2, F_3)}{\partial (u, v, w)} = \begin{vmatrix} 3u^2 & 3v^2 & 3w^2 \\ 2u & 2v & 2w \\ 1 & 1 & 1 \end{vmatrix} = 6 \begin{vmatrix} 1 & 1 & 1 \\ u & v & w \\ u^2 & v^2 & w^2 \end{vmatrix}$$

$$= -6 (u - v)(v - w)(w - u).$$

Hence from (1), $$\frac{\partial (u, v, w)}{\partial (x, y, z)} = -\frac{6 (x - y)(y - z)(z - x)}{-6 (u - v)(v - w)(w - u)}$$

$$= \frac{(y - z)(z - x)(x - y)}{(u - v)(v - w)(w - u)}.$$

Problem 5: *Compute the Jacobian* $\dfrac{\partial (u, v)}{\partial (r, \theta)}$ *where*

$$u = 2xy, v = x^2 - y^2, x = r \cos \theta, y = r \sin \theta.$$

Solution: Here we have case of functions of functions. Using the formula for finding the Jacobian in the case of functions of functions, we have

$$\frac{\partial (u, v)}{\partial (r, \theta)} = \frac{\partial (u, v)}{\partial (x, y)} \cdot \frac{\partial (x, y)}{\partial (r, \theta)}. \qquad ...(1)$$

Now $$\frac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} = -4 (x^2 + y^2).$$

Also $$\frac{\partial (x, y)}{\partial (r, \theta)} = \begin{vmatrix} \partial x / \partial r & \partial x / \partial \theta \\ \partial y / \partial r & \partial y / \partial \theta \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}$$

$$= r (\cos^2 \theta + \sin^2 \theta) = r.$$

Hence from (1),

$$\frac{\partial (u, v)}{\partial (r, \theta)} = -4 (x^2 + y^2) r = -4r^2 . r = -4r^3.$$

[$\because$ from $x = r \cos \theta$, $y = r \sin \theta$, we have $x^2 + y^2 = r^2$]

Problem 6: *If* $u_1 = x_1 + x_2 + x_3 + x_4$, $u_1u_2 = x_2 + x_3 + x_4$, $u_1u_2u_3 = x_3 + x_4$,

$u_1u_2u_3\,u_4 = x_4$, *show that* $\dfrac{\partial(x_1, x_2, x_3, x_4)}{\partial(u_1, u_2, u_3, u_4)} = u_1^3 u_2^2 u_3$.

(Kumaun 2012)

Solution: The given relations can be written as

$$F_1 \equiv u_1 - x_1 - x_2 - x_3 - x_4 = 0, \quad F_2 \equiv u_1u_2 - x_2 - x_3 - x_4 = 0,$$
$$F_3 \equiv u_1u_2u_3 - x_3 - x_4 = 0 \text{ and } \quad F_4 \equiv u_1u_2u_3u_4 - x_4 = 0.$$

Now $$\frac{\partial(x_1, x_2, x_3, x_4)}{\partial(u_1, u_2, u_3, u_4)} = (-1)^4 \frac{\partial(F_1, F_2, F_3, F_4)}{\partial(u_1, u_2, u_3, u_4)} \Big/ \frac{\partial(F_1, F_2, F_3, F_4)}{\partial(x_1, x_2, x_3, x_4)} \quad \ldots(1)$$

We have $\dfrac{\partial(F_1, F_2, F_3, F_4)}{\partial(u_1, u_2, u_3, u_4)}$ = the principal diagonal term of the Jacobian determinant

$$= \frac{\partial F_1}{\partial u_1} \cdot \frac{\partial F_2}{\partial u_2} \cdot \frac{\partial F_3}{\partial u_3} \cdot \frac{\partial F_4}{\partial u_4} = 1.\, u_1 . u_1u_2 . u_1u_2u_3 = u_1^3 u_2^2 u_3.$$

Also $$\frac{\partial(F_1, F_2, F_3, F_4)}{\partial(x_1, x_2, x_3, x_4)} = \begin{vmatrix} -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 1.$$

Hence from (1), $$\frac{\partial(x_1, x_2, x_3, x_4)}{\partial(u_1, u_2, u_3, u_4)} = \frac{u_1^3 u_2^2 u_3}{1} = u_1^3 u_2^2 u_3.$$

Problem 7: *Given* $y_1(x_1 - x_2) = 0$, $y_2(x_1^2 + x_1x_2 + x_2^2) = 0$, *show that*

$$\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = 3\,y_1y_2\,\frac{x_1 + x_2}{x_1^3 - x_2^3}.$$

Solution: The given relations can be written as

$$F_1 \equiv y_1(x_1 - x_2) = 0$$

and $$F_2 \equiv y_2(x_1^2 + x_1x_2 + x_2^2) = 0.$$

Now $$\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = (-1)^2 \frac{\partial(F_1, F_2)}{\partial(x_1, x_2)} \Big/ \frac{\partial(F_1, F_2)}{\partial(y_1, y_2)} \ldots(1)$$

We have $$\frac{\partial(F_1, F_2)}{\partial(x_1, x_2)} = \begin{vmatrix} y_1 & -y_2 \\ y_2(2x_1 + x_2) & y_2(x_1 + 2x_2) \end{vmatrix}$$

$$= y_1y_2(x_1 + 2x_2 + 2x_1 + x_2) = 3\,y_1y_2(x_1 + x_2).$$

Also $$\frac{\partial(F_1, F_2)}{\partial(y_1, y_2)} = \begin{vmatrix} x_1 - x_2 & 0 \\ 0 & x_1^2 + x_1x_2 + x_2^2 \end{vmatrix} = x_1^3 - x_2^3.$$

Hence from (1),

$$\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = \frac{3\,y_1y_2(x_1 + x_2)}{x_1^3 - x_2^3}.$$

Problem 8: *If* $u = x(1 - r^2)^{-1/2}$, $v = y(1 - r^2)^{-1/2}$, $w = z(1 - r^2)^{-1/2}$, *where*

$$r^2 = x^2 + y^2 + z^2, \text{ show that } \frac{\partial(u, v, w)}{\partial(x, y, z)} = (1 - r^2)^{-5/2}.$$

Solution: From the given relations, we have

$$x^2 = u^2(1 - r^2) = u^2(1 - x^2 - y^2 - z^2),$$

$$y^2 = v^2(1 - x^2 - y^2 - z^2)$$

and $$z^2 = w^2(1 - x^2 - y^2 - z^2).$$

The above relations can be written as

$$F_1 \equiv x^2 - u^2(1 - x^2 - y^2 - z^2) = 0,$$

$$F_2 \equiv y^2 - v^2(1 - x^2 - y^2 - z^2) = 0,$$

and $$F_3 \equiv z^2 - w^2(1 - x^2 - y^2 - z^2) = 0.$$

Now $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} \Big/ \frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} \quad \ldots(1)$$

We have $$\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} = \begin{vmatrix} 2x(1 + u^2) & 2yu^2 & 2zu^2 \\ 2xv & 2y(1 + v^2) & 2zv^2 \\ 2xw^2 & 2yw^2 & 2z(1 + w^2) \end{vmatrix}$$

$$= 8xyz \begin{vmatrix} 1 + u^2 & u^2 & u^2 \\ v^2 & 1 + v^2 & v^2 \\ w^2 & w^2 & 1 + w^2 \end{vmatrix}$$

$$= 8xyz(1 + u^2 + v^2 + w^2) \begin{vmatrix} 1 & 1 & 1 \\ v^2 & 1 + v^2 & v^2 \\ w^2 & w^2 & 1 + w^2 \end{vmatrix}$$

$$= 8xyz(1 + u^2 + v^2 + w^2) \begin{vmatrix} 1 & 0 & 0 \\ v^2 & 1 & 0 \\ w^2 & 0 & 1 \end{vmatrix}$$

$$= 8xyz(1 + u^2 + v^2 + w^2)$$

$$= 8xyz\left[1 + \frac{x^2}{1 - r^2} + \frac{y^2}{1 - r^2} + \frac{z^2}{1 - r^2}\right] = 8xyz\left[1 + \frac{x^2 + y^2 + z^2}{1 - r^2}\right]$$

$$= 8xyz\left[1 + \frac{r^2}{1 - r^2}\right] = \frac{8xyz}{1 - r^2}.$$

Also $$\frac{\partial (F_1, F_2, F_3)}{\partial (u, v, w)} = \begin{vmatrix} -2u(1-r^2) & 0 & 0 \\ 0 & -2v(1-r^2) & 0 \\ 0 & 0 & -2w(1-r^2) \end{vmatrix}$$

$$= -8uvw(1-r^2)^3.$$

Hence from (1), $$\frac{\partial (u, v, w)}{\partial (x, y, z)} = -\frac{8xyz}{1-r^2} \cdot \frac{1}{-8uvw(1-r^2)^3}$$

$$= \frac{xyz}{uvw(1-r^2)^4}$$

$$= \frac{xyz}{xyz(1-r^2)^{-3/2}(1-r^2)^4} = (1-r^2)^{-5/2}.$$

Problem 9(a): *Find the Jacobian of* $y_1, y_2, y_3, \ldots, y_n$, *being given* $y_1 = x_1(1-x_2)$, $y_2 = x_1x_2(1-x_3)$, $\ldots$, $y_{n-1} = x_1x_2 \ldots x_{n-1}(1-x_n)$, $y_n = x_1 x_2x_3 \ldots x_n$.

Solution: Adding all the given relations, we get $y_1 + y_2 + \ldots + y_n = x_1$.

$$\left.\begin{aligned} &\frac{\partial y_1}{\partial x_1} + \frac{\partial y_2}{\partial x_1} + \ldots + \frac{\partial y_n}{\partial x_1} = 1 \\ &\frac{\partial y_1}{\partial x_r} + \frac{\partial y_2}{\partial x_r} + \ldots + \frac{\partial y_n}{\partial x_r} = 0, \quad r = 2, 3, \ldots, n. \end{aligned}\right\} \quad \ldots(1)$$

Now $$\frac{\partial (y_1, y_2, \ldots, y_n)}{\partial (x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \frac{\partial y_n}{\partial x_3} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_{n-1}}{\partial x_1} & \frac{\partial y_{n-1}}{\partial x_2} & \frac{\partial y_{n-1}}{\partial x_3} & \cdots & \frac{\partial y_{n-1}}{\partial x_n} \\ 1 & 0 & 0 & \cdots & 0 \end{vmatrix},$$

adding $R_1, R_2, \ldots, R_{n-1}$ to R_n and using the relations (1)

$$= (-1)^{n-1} \begin{vmatrix} \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} & \frac{\partial y_1}{\partial x_4} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} & \frac{\partial y_2}{\partial x_4} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_{n-1}}{\partial x_2} & \frac{\partial y_{n-1}}{\partial x_3} & \frac{\partial y_{n-1}}{\partial x_4} & \cdots & \frac{\partial y_{n-1}}{\partial x_n} \end{vmatrix},$$

expanding the determinant along the nth row

$$= (-1)^{n-1} \begin{vmatrix} \frac{\partial y_1}{\partial x_2} & 0 & 0 & \cdots & 0 \\ \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_{n-1}}{\partial x_2} & \frac{\partial y_{n-1}}{\partial x_3} & \frac{\partial y_{n-1}}{\partial x_4} & \cdots & \frac{\partial y_{n-1}}{\partial x_n} \end{vmatrix}$$

$$= (-1)^{n-1} \cdot \frac{\partial y_1}{\partial x_2} \cdot \frac{\partial y_2}{\partial x_3} \cdot \frac{\partial y_3}{\partial x_4} \cdot \ldots \cdot \frac{\partial y_{n-1}}{\partial x_n}$$

$$= (-1)^{n-1} \cdot (-x_1) . (-x_1 x_2) . (-x_1 x_2 x_3) \ldots . (-x_1 x_2 \ldots x_{n-1})$$

$$= (-1)^{n-1} . (-1)^{n-1} x_1^{n-1} x_2^{n-2} x_3^{n-3} \ldots x_{n-1}$$

$$= (-1)^{2n-2} x_1^{n-1} x_2^{n-2} x_3^{n-3} \ldots x_{n-1}$$

$$= x_1^{n-1} x_2^{n-2} \ldots x_{n-1}.$$

Problem 9(b): *If* $y_1 = r \cos \theta_1$, $y_2 = r \sin \theta_1 \cos \theta_2$, $y_3 = r \sin \theta_1 \sin \theta_2 \cos \theta_3, \ldots,$

$y_{n-1} = r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-2} \cos \theta_{n-1}$ *and* $y_n = r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-1}$,

prove that $\dfrac{\partial (y_1, y_2, \ldots, y_n)}{\partial (r, \theta_1, \ldots, \theta_{n-1})} = r^{n-1} \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \ldots \sin \theta_{n-2}$.

(Kumaun 2010)

Solution: If we square and add, we get $y_1^2 + y_2^2 + \ldots + y_n^2 = r^2$.

$$\left.\begin{aligned} \therefore \quad & y_1 \frac{\partial y_1}{\partial r} + y_2 \frac{\partial y_2}{\partial r} + \ldots + y_n \frac{\partial y_n}{\partial r} = r \\ \text{and} \quad & y_1 \frac{\partial y_1}{\partial \theta_k} + y_2 \frac{\partial y_2}{\partial \theta_k} + \ldots + y_n \frac{\partial y_n}{\partial \theta_k} = 0, \ k = 1, 2, \ldots, n-1. \end{aligned}\right\} \quad \ldots(1)$$

Now

$$\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(r, \theta_1, \ldots, \theta_{n-1})} = \begin{vmatrix} \frac{\partial y_1}{\partial r} & \frac{\partial y_1}{\partial \theta_1} & \frac{\partial y_1}{\partial \theta_2} & \ldots & \frac{\partial y_1}{\partial \theta_{n-1}} \\ \frac{\partial y_2}{\partial r} & \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & \ldots & \frac{\partial y_2}{\partial \theta_{n-1}} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\partial y_n}{\partial r} & \frac{\partial y_n}{\partial \theta_1} & \frac{\partial y_n}{\partial \theta_2} & \ldots & \frac{\partial y_n}{\partial \theta_{n-1}} \end{vmatrix}$$

$$= \frac{1}{y_n} \begin{vmatrix} \frac{\partial y_1}{\partial r} & \frac{\partial y_1}{\partial \theta_1} & \frac{\partial y_1}{\partial \theta_2} & \ldots & \frac{\partial y_1}{\partial \theta_{n-1}} \\ \frac{\partial y_2}{\partial r} & \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & \ldots & \frac{\partial y_2}{\partial \theta_{n-1}} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\partial y_{n-1}}{\partial r} & \frac{\partial y_{n-1}}{\partial \theta_1} & \frac{\partial y_{n-1}}{\partial \theta_2} & \ldots & \frac{\partial y_{n-1}}{\partial \theta_{n-1}} \\ r & 0 & 0 & & 0 \end{vmatrix},$$

Operating $y_n R_n + (y_1 R_1 + y_2 R_2 + \ldots + y_{n-1} R_{n-1})$ and using the results (1)

$$= (-1)^{n-1} \cdot \frac{r}{y_n} \begin{vmatrix} \frac{\partial y_1}{\partial \theta_1} & \frac{\partial y_1}{\partial \theta_2} & \ldots & \frac{\partial y_1}{\partial \theta_{n-1}} \\ \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & \ldots & \frac{\partial y_2}{\partial \theta_{n-1}} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial y_{n-1}}{\partial \theta_1} & \frac{\partial y_{n-1}}{\partial \theta_2} & \ldots & \frac{\partial y_{n-1}}{\partial \theta_{n-1}} \end{vmatrix},$$

[Expanding the determinant along the *n*th row]

$$= (-1)^{n-1} \cdot \frac{r}{y_n} \begin{vmatrix} \frac{\partial y_1}{\partial \theta_1} & 0 & 0 & \ldots & 0 \\ \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\partial y_{n-1}}{\partial \theta_1} & \frac{\partial y_{n-1}}{\partial \theta_2} & \frac{\partial y_{n-1}}{\partial \theta_3} & \ldots & \frac{\partial y_{n-1}}{\partial \theta_{n-1}} \end{vmatrix}$$

$$= (-1)^{n-1} \cdot \frac{r}{y_n} \cdot \frac{\partial y_1}{\partial \theta_1} \cdot \frac{\partial y_2}{\partial \theta_2} \cdot \frac{\partial y_3}{\partial \theta_3} \cdot \ldots \frac{\partial y_{n-1}}{\partial \theta_{n-1}}$$

$$= (-1)^{n-1} \cdot \frac{r}{y_n} \cdot (-r \sin \theta_1)(-r \sin \theta_1 \sin \theta_2) \ldots$$

$$\ldots (-r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-1})$$

$$= (-1)^{n-1} \cdot \frac{r}{y_n} (-1)^{n-1} r^{n-1} \sin^{n-1} \theta_1 \sin^{n-2} \theta_2 \ldots . \sin^2 \theta_{n-2} \sin \theta_{n-1}$$

$$= \frac{r}{r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-2} \sin \theta_{n-1}} \cdot r^{n-1} \sin^{n-1} \theta_1 \sin^{n-2} \theta_2 \ldots . \sin^2 \theta_{n-2} \sin \theta_{n-1}$$

$$= r^{n-1} \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \ldots . \sin \theta_{n-2}.$$

Problem 10: *If λ, μ, ν are the roots of the equation in k,* $\dfrac{x}{a+k} + \dfrac{y}{b+k} + \dfrac{z}{c+k} = 1$,

prove that $$\frac{\partial (x, y, z)}{\partial (\lambda, \mu, \nu)} = -\frac{(\mu - \nu)(\nu - \lambda)(\lambda - \mu)}{(b-c)(c-a)(a-b)}.$$

Solution: The given equation in k can be written as

$$(a+k)(b+k)(c+k) - x(b+k)(c+k) - y(c+k)(a+k) - z(a+k)(b+k) = 0$$

or $$k^3 + k^2 (a+b+c-x-y-z) + k \{ab+bc+ca-x(b+c) - y(c+a) - z(a+b)\} + abc - xbc - yca - zab = 0.$$

Since λ, μ, ν are the roots of this equation, therefore from theory of equations, we have

$$\lambda + \mu + \nu = x + y + z - a - b - c$$

$$\lambda\mu + \mu\nu + \nu\lambda = ab + bc + ca - x(b+c) - y(c+a) - z(a+b)$$

and $$\lambda\mu\nu = xbc + yca + zab - abc.$$

The above relations can be written as

$$F_1 \equiv \lambda + \mu + \nu - x - y - z + a + b + c = 0,$$

$$F_2 \equiv \lambda\mu + \mu\nu + \nu\lambda + x(b+c) + y(c+a) + z(a+b) - ab - bc - ca = 0$$

and $$F_3 \equiv \lambda\mu\nu - xbc - yca - zab + abc = 0.$$

Now $$\frac{\partial (x, y, z)}{\partial (\lambda, \mu, \nu)} = (-1)^3 \frac{\partial (F_1, F_2, F_3)}{\partial (\lambda, \mu, \nu)} \Big/ \frac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)}. \quad \ldots(1)$$

We have $$\frac{\partial (F_1, F_2, F_3)}{\partial (\lambda, \mu, \nu)} = \begin{vmatrix} 1 & 1 & 1 \\ \mu + \nu & \lambda + \nu & \mu + \lambda \\ \mu\nu & \nu\lambda & \lambda\mu \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \mu + \nu & \lambda - \mu & \lambda - \nu \\ \mu\nu & \nu(\lambda - \mu) & \mu(\lambda - \nu) \end{vmatrix} = (\lambda - \mu)(\lambda - \nu) \begin{vmatrix} 1 & 1 \\ \nu & \mu \end{vmatrix}$$

$$= (\lambda - \mu)(\lambda - \nu)(\mu - \nu) = -(\lambda - \mu)(\mu - \nu)(\nu - \lambda).$$

Also
$$\frac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} = \begin{vmatrix} -1 & -1 & -1 \\ b+c & c+a & a+b \\ -bc & -ca & -ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ bc & ca & ab \end{vmatrix} = -(b-c)(c-a)(a-b).$$

Hence from (1),
$$\frac{\partial (x, y, z)}{\partial (\lambda, \mu, \nu)} = -\frac{-(\lambda-\mu)(\mu-\nu)(\nu-\lambda)}{-(b-c)(c-a)(a-b)} = -\frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}{(b-c)(c-a)(a-b)}.$$

Problem 11: *The roots of the equation in* $\lambda, (\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ *are* u, v, w. *Prove that* $\dfrac{\partial (u, v, w)}{\partial (x, y, z)} = -2 \dfrac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$.

(Lucknow 2007; Kumaun 08; Kanpur 10; Gorakhpur 11)

Solution: The given equation in λ can be written as
$$3\lambda^3 - 3\lambda^2 (x + y + z) + 3\lambda (x^2 + y^2 + z^2) - (x^3 + y^3 + z^3) = 0.$$
Since u, v, w are the roots of this equation, therefore from theory of equations, we have
$$u + v + w = x + y + z, \; uv + vw + wu = x^2 + y^2 + z^2,$$
and
$$uvw = \frac{1}{3} (x^3 + y^3 + z^3).$$
The above relations can be written as
$$F_1 \equiv u + v + w - x - y - z = 0,$$
$$F_2 \equiv uv + vw + wu - x^2 - y^2 - z^2 = 0$$
and
$$F_3 \equiv uvw - \frac{1}{3} (x^3 + y^3 + z^3) = 0.$$
Now
$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = (-1)^3 \frac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} \Big/ \frac{\partial (F_1, F_2, F_3)}{\partial (u, v, w)} \qquad \text{...(1)}$$
We have
$$\frac{\partial (F_1, F_2, F_3)}{\partial (x, y, z)} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= -2 (y-z)(z-x)(x-y).$$

Also
$$\frac{\partial (F_1, F_2, F_3)}{\partial (u, v, w)} = \begin{vmatrix} 1 & 1 & 1 \\ v + w & u + w & u + v \\ vw & uw & uv \end{vmatrix}$$
$$= -(v - w)(w - u)(u - v).$$ [Refer problem 10]

Hence from (1), $$\frac{\partial (u, v, w)}{\partial (x, y, z)} = -\frac{-2 (y - z)(z - x)(x - y)}{-(v - w)(w - u)(u - v)}$$
$$= -2 \frac{(y - z)(z - x)(x - y)}{(v - w)(w - u)(u - v)}.$$

Problem 12: *If x, y, z are connected by a functional relation $f(x, y, z) = 0$, show that*
$$\frac{\partial (y, z)}{\partial (x, z)} = \left(\frac{\partial y}{\partial x}\right)_{z = const.}$$

Solution: We have $f(x, y, z) = 0 \Rightarrow y$ is a function of x and z.

Also from the equation, $z = z$, z may be regarded as a function of x and z.

$$\therefore \quad \frac{\partial (y, z)}{\partial (x, z)} = \begin{vmatrix} \left(\frac{\partial y}{\partial x}\right)_{z = \text{const.}} & \left(\frac{\partial y}{\partial z}\right)_{x = \text{const.}} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial z} \end{vmatrix}$$
$$= \begin{vmatrix} \left(\frac{\partial y}{\partial x}\right)_{z = \text{const.}} & \left(\frac{\partial y}{\partial z}\right)_{x = \text{const.}} \\ 0 & 1 \end{vmatrix} \quad \left[\because \frac{\partial z}{\partial x} = 0, \frac{\partial z}{\partial z} = 1\right]$$
$$= \left(\frac{\partial y}{\partial x}\right)_{z = \text{const.}}$$

Problem 13(i): *Prove that* $\dfrac{\partial (u, v, w)}{\partial (x, y, z)} \times \dfrac{\partial (x, y, z)}{\partial (u, v, w)} = 1.$ **(Kanpur 2008, 09, 11)**

Solution: Proceed as in Example 4.

Problem 13(ii): *Prove that*
$$\frac{\partial (y_1, y_2, \ldots, y_n)}{\partial (x_1, x_2, \ldots, x_n)} \cdot \frac{\partial (x_1, x_2, \ldots, x_n)}{\partial (y_1, y_2, \ldots, y_n)} = 1.$$

Solution: Let
$$\left. \begin{array}{l} y_1 = f_1 (x_1, x_2, \ldots, x_n), y_2 = f_2 (x_1, x_2, \ldots, x_n), \\ \ldots, y_n = f_n (x_1, x_2, \ldots, x_n) \end{array} \right\} \quad \ldots(1)$$

These relations may be put in the form
$$x_1 = F_1 (y_1, y_2, \ldots, y_n)$$
$$x_2 = F_2 (y_1, y_2, \ldots, y_n), \ldots$$
$$x_n = F_n (y_1, y_2, \ldots, y_n)$$

Differentiating the relations (1) partially w.r.t. $y_1, y_2, ..., y_n$, we have

$$\left.\begin{aligned}
1 &= \frac{\partial y_1}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_1}+\frac{\partial y_1}{\partial x_2}\cdot\frac{\partial x_2}{\partial y_1}+...+\frac{\partial y_1}{\partial x_n}\cdot\frac{\partial x_n}{\partial y_1}=\Sigma\frac{\partial y_1}{\partial x_r}\cdot\frac{\partial x_r}{\partial y_1}\\
0 &= \frac{\partial y_1}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_2}+\frac{\partial y_1}{\partial x_2}\cdot\frac{\partial x_2}{\partial y_2}+\dots+\frac{\partial y_1}{\partial x_n}\cdot\frac{\partial x_n}{\partial y_2}=\Sigma\frac{\partial y_1}{\partial x_r}\cdot\frac{\partial x_r}{\partial y_2}\\
&\dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\\
&\dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\\
0 &= \frac{\partial y_1}{\partial x_1}\cdot\frac{\partial x_1}{\partial y_n}+\frac{\partial y_1}{\partial x_2}\cdot\frac{\partial x_2}{\partial y_n}+\dots+\frac{\partial y_1}{\partial x_n}\cdot\frac{\partial x_n}{\partial y_n}=\Sigma\frac{\partial y_1}{\partial x_r}\cdot\frac{\partial x_r}{\partial y_n}
\end{aligned}\right\} \quad \dots(A)$$

and similar other relations.

Now $$\frac{\partial(y_1, y_2, \dots, y_n)}{\partial(x_1, x_2, ..., x_n)}\cdot\frac{\partial(x_1, x_2, \dots, x_n)}{\partial(y_1, y_2, \dots, y_n)}$$

$$= \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \dots & \frac{\partial y_1}{\partial x_n}\\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \dots & \frac{\partial y_2}{\partial x_n}\\ \dots & \dots & \dots & \dots\\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \dots & \frac{\partial y_n}{\partial x_n}\end{vmatrix} \times \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \dots & \frac{\partial x_1}{\partial y_n}\\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \dots & \frac{\partial x_2}{\partial y_n}\\ \dots & \dots & \dots & \dots\\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \dots & \frac{\partial x_n}{\partial y_n}\end{vmatrix}$$

$$= \begin{vmatrix} \Sigma\frac{\partial y_1}{\partial x_r}\cdot\frac{\partial x_r}{\partial y_1} & \Sigma\frac{\partial y_1}{\partial x_r}\cdot\frac{\partial x_r}{\partial y_2} & \dots & \Sigma\frac{\partial y_1}{\partial x_r}\cdot\frac{\partial x_r}{\partial y_n}\\ \Sigma\frac{\partial y_2}{\partial x_r}\cdot\frac{\partial x_r}{\partial y_1} & \Sigma\frac{\partial y_2}{\partial x_r}\cdot\frac{\partial x_r}{\partial y_2} & \dots & \Sigma\frac{\partial y_2}{\partial x_r}\cdot\frac{\partial x_r}{\partial y_n}\\ \dots & \dots & \dots & \dots\\ \Sigma\frac{\partial y_n}{\partial x_r}\cdot\frac{\partial x_r}{\partial y_1} & \Sigma\frac{\partial y_n}{\partial x_r}\cdot\frac{\partial x_r}{\partial y_2} & \dots & \Sigma\frac{\partial y_n}{\partial x_r}\cdot\frac{\partial x_r}{\partial y_n}\end{vmatrix},$$

[Applying row-by-column multiplication]

$$= \begin{vmatrix} 1 & 0 & \dots & 0\\ 0 & 1 & \dots & 0\\ \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & 1\end{vmatrix}, \text{ using the relations (A)}$$

$$= 1.$$

Problem 14: *If* $x^2+y^2+u^2-v^2=0$ *and* $uv+xy=0$, *prove that*

$$\frac{\partial(u, v)}{\partial(x, y)}=\frac{x^2-y^2}{u^2+v^2}.$$

(Kumaun 2015)

Solution: The given relations can be written as

$$F_1 \equiv x^2+y^2+u^2-v^2=0$$

$$F_2 \equiv uv+xy=0$$

Now $$\frac{\partial(u, v)}{\partial(x, y)} = (-1)^2 \frac{\partial(F_1, F_2)}{\partial(x, y)} \bigg/ \frac{\partial(F_1, F_2)}{\partial(u, v)} \qquad \ldots(1)$$

Here $$\frac{\partial(F_1, F_2)}{\partial(x, y)} = \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix} = 2x^2 - 2y^2 = 2(x^2 - y^2)$$

and $$\frac{\partial(F_1, F_2)}{\partial(u, v)} = \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = 2u^2 + 2v^2 = 2(u^2 + v^2).$$

$\therefore$ From (1), $$\frac{\partial(u, v)}{\partial(x, y)} = \frac{x^2 - y^2}{u^2 + v^2}.$$

Comprehensive Problems 3

Problem 1: *If $u = x^2 + y^2 + z^2$, $v = x + y + z$, $w = xy + yz + zx$, show that the Jacobian $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ vanishes identically. Also find the relation between u, v and w.*

(Avadh 2014)

Solution: We have $\frac{\partial(u, v, w)}{\partial(x, y, z)}$

$$= \begin{vmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 2 \begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2 \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix}, \text{ by } R_1 + R_3$$

$$= 2(x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix} = 0, \text{ the first two rows being identical.}$$

Since the Jacobian of the functions u, v, w is zero, therefore these functions are not independent and there must exist a relation between them.

We have $$v^2 = (x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = u + 2w.$$

Thus $v^2 = u + 2w$ is the required relation between u, v and w.

Problem 2: *If $u = \frac{x + y}{1 - xy}$ and $v = \tan^{-1} x + \tan^{-1} y$, find $\frac{\partial(u, v)}{\partial(x, y)}$. Are u and v functionally related? If so, find the relationship.*

Solution: We have

$$\frac{\partial u}{\partial x} = \frac{1 \,.\, (1 - xy) - (-\,y)\,(x + y)}{(1 - xy)^2} = \frac{1 + y^2}{(1 - xy)^2},$$

$$\frac{\partial u}{\partial y} = \frac{1 \,.\, (1 - xy) - (-\,x)\,(x + y)}{(1 - xy)^2} = \frac{1 + x^2}{(1 - xy)^2},$$

$$\frac{\partial v}{\partial x} = \frac{1}{1 + x^2}, \frac{\partial v}{\partial y} = \frac{1}{1 + y^2}.$$

Now $$\frac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix} \partial u / \partial x & \partial u / \partial y \\ \partial v / \partial x & \partial v / \partial y \end{vmatrix} = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}$$

$$= \frac{1 + y^2}{(1 - xy)^2} \cdot \frac{1}{1 + y^2} - \frac{1 + x^2}{(1 - xy)^2} \cdot \frac{1}{1 + x^2} = \frac{1}{(1 - xy)^2} - \frac{1}{(1 - xy)^2} = 0.$$

Since the Jacobian of the functions u, v is zero, therefore these functions are not independent and so they must be functionally related.

We have

$$v = \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x + y}{1 - xy} = \tan^{-1} u.$$

Thus $v = \tan^{-1} u$ or $\tan v = u$ is the required relation between u and v.

Problem 3: *If the functions u, v, w of three independent variables x, y, z are not independent, prove that the Jacobian of u, v, w with respect to x, y, z vanishes.*

Solution: Since the functions u, v and w are not independent, therefore there exists a relation between them. Let it be

$$F(u, v, w) = 0. \qquad ...(1)$$

Now we are to prove that the Jacobian of u, v, w with respect to x, y, z must be equal to zero.

Differentiating (1) partially w.r.t. x, y, z, we get

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial x} = 0, \qquad ...(2)$$

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial y} = 0, \qquad ...(3)$$

and $$\frac{\partial F}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial z} = 0. \qquad ...(4)$$

Eliminating $\partial F / \partial u, \partial F / \partial v, \partial F / \partial w$ from (2), (3), (4), we get

$$\begin{vmatrix} \partial u / \partial x & \partial v / \partial x & \partial w / \partial x \\ \partial u / \partial y & \partial v / \partial y & \partial w / \partial y \\ \partial u / \partial z & \partial v / \partial z & \partial w / \partial z \end{vmatrix} = 0,$$

or $\dfrac{\partial (u, v, w)}{\partial (x, y, z)} = 0$, which proves the required result.

Problem 4: *Show that the functions* $u = 3x + 2y - z$, $v = x - 2y + z$ *and* $w = x(x + 2y - z)$ *are not independent and find the relation between them.*

Solution: We have

$$u = 3x + 2y - z, v = x - 2y + z, w = x^2 + 2xy - xz.$$

$$\therefore \quad \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 2 & -1 \\ 1 & -2 & 1 \\ 2x + 2y & 2x & -x \end{vmatrix}$$

$$= -2 \begin{vmatrix} 3 & -1 & -1 \\ 1 & 1 & 1 \\ 2x + 2y & -x & -x \end{vmatrix} = 0,$$

the last two columns being identical.

Since the Jacobian of the functions u, v, w is zero, therefore these functions are not independent and so there must exist a relation between them.

We have $\quad u^2 - v^2 = (3x + 2y - z)^2 - (x - 2y + z)^2$

$$= (3x + 2y - z + x - 2y + z)(3x + 2y - z - x + 2y - z)$$

$$= 4x(2x + 4y - 2z) = 8x(x + 2y - z) = 8w.$$

Thus $u^2 - v^2 = 8w$ is the required relation between u, v and w.

Problem 5: *Show that the functions*

$$u = x + y + z, v = xy + yz + zx, w = x^3 + y^3 + z^3 - 3xyz$$

are not independent. Find the relation between them.

(Meerut 2013B; Kanpur 15)

Solution: We have $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ y + z & z + x & x + y \\ 3(x^2 - yz) & 3(y^2 - zx) & 3(z^2 - xy) \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & 1 & 1 \\ y + z & z + x & x + y \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= 3 \begin{vmatrix} 1 & 0 & 0 \\ y + z & x - y & x - z \\ x^2 - yz & (y - x)(x + y + z) & (z - x)(x + y + z) \end{vmatrix},$$

by $C_2 - C_1$ and $C_3 - C_1$

$$= 3(x-y)(x-z)\begin{vmatrix} 1 & 0 & 0 \\ y+z & 1 & 1 \\ x^2 - yz & -(x+y+z) & -(x+y+z) \end{vmatrix} = 0,$$

the last two columns being identical.

Since the Jacobian of the functions u, v and w is zero, therefore these functions are not independent and so there must exist a relation between them.

We have $\quad w = x^3 + y^3 + z^3 - 3xyz$

$$= (x+y+z)(x^2 + y^2 + z^2 - yz - zx - xy)$$

$$= (x+y+z)[(x+y+z)^2 - 3(yz + zx + xy)]$$

$$= u(u^2 - 3v) = u^3 - 3uv.$$

$\therefore \quad u^3 = 3uv + w$ is the required relation between u, v and w.

Problem 6: *If $u = x + 2y + z$, $v = x - 2y + 3z$ and $w = 2xy - xz + 4yz - 2z^2$, show that they are not independent. Find the relation between u, v and w.*

(Lucknow 2009, 11)

Solution: We have $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$

$$= \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y - z & 2x + 4z & -x + 4y - 4z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & -4 & 2 \\ 2y - z & 2x - 4y + 6z & -x + 2y - 3z \end{vmatrix},$$

by $C_2 - 2C_1$ and $C_3 - C_1$

$$= -2\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 2 \\ 2y - z & -x + 2y - 3z & -x + 2y - 3z \end{vmatrix} = 0,$$

the last two columns being identical.

Since the Jacobian of the functions u, v, w is zero, therefore these functions are not independent and so there must exist a relation between them.

We have $\quad u^2 - v^2 = (x + 2y + z)^2 - (x - 2y + 3z)^2$

$$= (x + 2y + z + x - 2y + 3z)(x + 2y + z - x + 2y - 3z)$$

$$= (2x + 4z)(4y - 2z) = 4(x + 2z)(2y - z)$$

$$= 4(2xy - xz + 4yz - 2z^2) = 4w.$$

Therefore $u^2 - v^2 = 4w$ is the required relation between u, v and w.

Problem 7: *If* $u = \dfrac{x+y}{z}, v = \dfrac{y+z}{x}, w = \dfrac{y(x+y+z)}{xz}$, *show that u,v,w are not independent and find the relation between them.*

Solution: We have $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{1}{z} & \dfrac{1}{z} & \dfrac{-(x+y)}{z^2} \\ -\dfrac{(y+z)}{-x^2} & \dfrac{1}{x} & \dfrac{1}{x} \\ \dfrac{-y^2+yz}{x^2 z} & \dfrac{x+2y+z}{xz} & \dfrac{-xy-y^2}{xz^2} \end{vmatrix}$

$$= \frac{1}{x^4 z^4}\begin{vmatrix} z & z & -(x+y) \\ -(y+z) & x & x \\ -yz(y+z) & xz(x+2y+z) & -xy(x+y) \end{vmatrix},$$

by taking $\dfrac{1}{z^2}, \dfrac{1}{x^2}, \dfrac{1}{x^2 z^2}$ common from first, second and third

$$= \frac{1}{x^4 z^4}\begin{vmatrix} z & 0 & -(x+y+z) \\ -(y+z) & x+y+z & x+y+z \\ -yz(y+z) & z(x+y)(x+y+z) & y(z-x)(x+y+z) \end{vmatrix}$$

by $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$= \frac{(x+y+z)^2}{x^4 z^4}\begin{vmatrix} z & 0 & -1 \\ -(y+z) & 1 & 1 \\ -yz(y+z) & z(x+y) & y(z-x) \end{vmatrix}$$

$$= \frac{(x+y+z)^2}{x^4 z^4}\begin{vmatrix} 0 & 0 & -1 \\ -y & 1 & 1 \\ -yz(y+x) & z(x+y) & y(z-x) \end{vmatrix}$$

by $C_1 \to C_1 + zC_3$

$$= \frac{(x+y+z)^2}{z^4 x^4}(-1)\begin{vmatrix} -y & 1 \\ -yz(x+y) & z(x+y) \end{vmatrix}$$

$$= \frac{(x+y+z)^2}{z^4 x^4}y\begin{vmatrix} 1 & 1 \\ z(x+y) & z(x+y) \end{vmatrix}$$

$$= 0$$

$\therefore$ The functions are not independent.

Also we have $\quad uv = \dfrac{(x+y)(y+z)}{xz} = \dfrac{y(x+y+z)}{xz} + 1$

or $\quad uv = w + 1,$

which is the required relation.

Problem 8: *If* $u = x + y + z + t$, $v = x + y - z - t$, $w = xy - zt$, $r = x^2 + y^2 - z^2 - t^2$, *show that* $\dfrac{\partial (u, v, w, r)}{\partial (x, y, z, t)} = 0$ *and hence find a relation between u, v, w and r.*

Solution: We have $\dfrac{\partial (u, v, w, r)}{\partial (x, y, z, t)}$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ y & x & -t & -z \\ 2x & 2y & -2z & -2t \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 2 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ y & x & -t & -z \\ 2x & 2y & -2z & -2t \end{vmatrix}, \text{ by } R_1 + R_2$$

$$= \begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 \\ y & x - y & -t & -z \\ 2x & 2(y - x) & -2z & -2t \end{vmatrix}, \text{ by } C_2 - C_1$$

$$= 2 \begin{vmatrix} 0 & -1 & -1 \\ x - y & -t & -z \\ -2(x - y) & -2z & -2t \end{vmatrix}$$

$$= 2 \begin{vmatrix} 0 & 0 & -1 \\ x - y & z - t & -z \\ -2(x - y) & 2(t - z) & -2t \end{vmatrix}, \text{ by } C_2 - C_3$$

$$= 2(x - y)(z - t) \begin{vmatrix} 0 & 0 & -1 \\ 1 & 1 & -z \\ -2 & -2 & -2t \end{vmatrix}$$

$= 0$, the first two columns being identical.

Since the Jacobian of the functions u, v, w, r is zero, therefore these functions are not independent and so there must exist a relation between them.

Now let us find a relation between u, v, w, r. We have

$$uv = (x + y + z + t)(x + y - z - t) = (x + y)^2 - (z + t)^2$$

$$= (x^2 + y^2 - z^2 - t^2) + 2(xy - zt) = r + 2w.$$

Hence $uv = r + 2w$ is the required relation between u, v, w and r.

Problem 9: *If* $f(0)=0$ *and* $f'(x)=\dfrac{1}{1+x^2}$ *, prove without using the method of integration, that*

$$f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right).$$

(Meerut 2012B, 13; Gorakhpur 13, 14)

Solution: Let $u=f(x)+f(y),\ v=\dfrac{x+y}{1-xy}$.

We have
$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} f'(x) & f'(y) \\ \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2}\end{vmatrix}$$

$$=\begin{vmatrix} \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \\ \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2}\end{vmatrix}=\frac{1}{(1-xy)^2}-\frac{1}{(1-xy)^2}=0.$$

Hence there must exist a functional relation between u and v.

Let $\quad u=\phi(v)$.

i.e., $\quad f(x)+f(y)=\phi\left(\dfrac{x+y}{1-xy}\right)$.

Putting $y=0$, we obtain $f(x)=\phi(x)$. $\quad [\because f(0)=0]$

Thus the functions f and ϕ are equal.

Hence $\quad f(x)+f(y)=f\left(\dfrac{x+y}{1-xy}\right)$.

Hints to Objective Type Questions

Multiple Choice Questions

1. See Problem 1 of Comprehensive Problems 1.
2. See Problem 4 of Comprehensive Problems 1 .
3. We have $\dfrac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} e^x\sin y & e^x\cos y \\ 1 & \dfrac{\cos y}{\sin y}\end{vmatrix}=e^x\cos y-e^x\cos y=0.$
4. We have $\dfrac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix} \sin y & x\cos y \\ y\cos x & \sin x\end{vmatrix}=\sin x\sin y-xy\cos x\cos y.$
5. We have $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}=\begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1\end{vmatrix}=r(\cos^2\theta+\sin^2\theta)=r.$

6. See article 4.
7. See Example 4.
8. See Problem 1 of Comprehensive Problems 1.

Fill in the Blanks

1. See article 1.
2. We have $\dfrac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$

   But $\dfrac{\partial(x, y)}{\partial(r, \theta)} \cdot \dfrac{\partial(r, \theta)}{\partial(x, y)} = 1.$

   $\therefore$ $\dfrac{\partial(r, \theta)}{\partial(x, y)} = \dfrac{1}{r}.$
3. See Example 4.
4. See Problem 2 of Comprehensive Problems 1.
5. See Problem 7 of Comprehensive Problems 1.
6. Refer Problem 5 of Comprehensive Problems 1.
7. See Problem 5 of Comprehensive Problems 1.

True or False

1. See Problem 5 of Comprehensive Problems 1.
2. See article 2, Theorem
3. See Example 1.
4. We have $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} \cdot \dfrac{\partial(x, y, z)}{\partial(u, v, w)} = 1$. Refer Example 4.
5. See article 3.
6. See Problem 1 of Comprehensive Problems 3.
7. See Problem 2 of Comprehensive Problems 2.
8. From article 4 we know that u, v, w are not independent of each other if

   $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0.$$
9. We have $uvw^2 = \left\{\dfrac{3x^2}{2(y+z)}\right\}\left\{\dfrac{2(y+z)}{3(x-y)^2}\right\}\left\{\dfrac{(x-y)^2}{x^2}\right\} = 1.$

❍❍❍

Chapter-9

Tangents and Normals

Comprehensive Problems 1

Problem 1: *Show that in the equiangular spiral $r = ae^{\theta \cot \alpha}$, the tangent is inclined at a constant angle α to the radius vector.* **(Avadh 2014)**

Solution: The curve is

$$r = ae^{\theta \cot \alpha}. \qquad \text{...(1)}$$

$$\therefore \quad \frac{dr}{d\theta} = ae^{\theta \cot \alpha} . \cot \alpha = r \cot \alpha. \qquad [\because \text{ From (1), } r = ae^{\theta \cot \alpha}]$$

Now $$\tan \phi = r\frac{d\theta}{dr} = \frac{r}{dr/d\theta} = \frac{r}{r \cot \alpha} = \tan \alpha.$$

$$\therefore \quad \phi = \alpha = \text{constant}.$$

Problem 2: *Find the angle at which the radius vector cuts the curve $l/r = 1 + e \cos \theta$.*

Solution: The curve is $\frac{e}{r} = (1 + e \cos \theta)$

Taking logarithm, $\log e - \log r = \log(1 + e \cos \theta)$

Differentiating, $$-\frac{1}{r}\frac{dr}{d\theta} = \frac{(-e \sin \theta)}{1 + e \cos \theta}$$

or $$r\frac{d\theta}{dr} = \frac{1 + e \cos \theta}{e \sin \theta} = \tan \phi$$

Now required angle $\phi = \tan^{-1} (\tan \phi) = \tan^{-1}\left(r\frac{d\theta}{dr}\right) = \tan^{-1}\left[\frac{1 + e \cos \theta}{e \sin \theta}\right].$

Problem 3: *Find the angle ϕ for the curve $a\theta = (r^2 - a^2)^{1/2} - a \cos^{-1}(a/r)$.* **(Rohilkhand 2013)**

Solution: Given $a\theta = (r^2 - \alpha^2)^{1/2} - a \cos^{-1}\left(\frac{a}{r}\right).$

Differentiating with respect to θ, we get

$$a = \frac{1}{2}(r^2 - a^2)^{-1/2}\, 2r\frac{dr}{d\theta} + \frac{a}{\sqrt{\{1 - (a/r)^2\}}}\left(-\frac{a}{r^2}\right)\frac{dr}{d\theta}$$

or $$a\frac{d\theta}{dr} = \frac{r}{\sqrt{(r^2 - a^2)}} - \frac{a^2}{r\sqrt{(r^2 - a^2)}} = \frac{r^2 - a^2}{r\sqrt{(r^2 - a^2)}}$$

or $$r\frac{d\theta}{dr} = \frac{\sqrt{(r^2 - a^2)}}{a} \text{ or } \tan \phi = \frac{\sqrt{(r^2 - a^2)}}{a}.$$

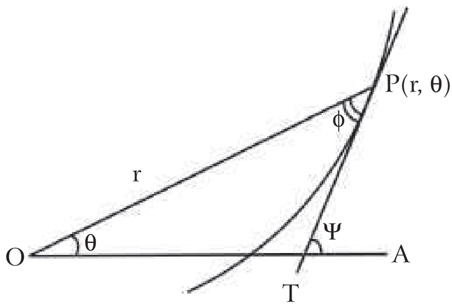
$\therefore \quad \cos\phi = \dfrac{1}{\sec\phi} = \dfrac{1}{\sqrt{(1+\tan^2\phi)}} = \dfrac{1}{\sqrt{\left\{1+\left(\dfrac{r^2-a^2}{a^2}\right)\right\}}} = \dfrac{a}{r}.$

$\therefore \quad \phi = \cos^{-1}\left(\dfrac{a}{r}\right).$

Problem 4: *If ϕ be the angle between tangent to a curve and the radius vector drawn from the origin of coordinates to the point of contact, prove that*

$$\tan\phi = \left(x\frac{dy}{dx} - y\right) \Big/ \left(x + y\frac{dy}{dx}\right).$$

Solution: Let $P(r, \theta)$ be any point on the curve. Then OP is the radius vector and PT is the tangent to the curve at P inclined at an angle ψ to the initial line OA.



Now we know if the coordinates of P be (x, y) in the cartesian form, then

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \quad ...(1)$$

and $\quad dy/dx = \tan\psi \quad ...(2)$

Also from the figure it is evident that

$\psi = \theta + \phi$ ... as exterior angle = sum of opposite interior angles.

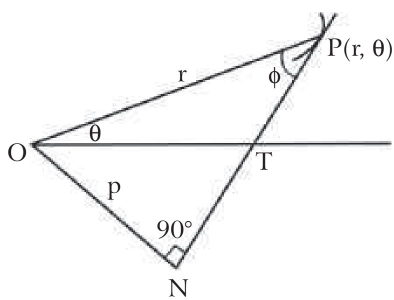
$\therefore \quad \phi = \psi - \theta \quad$ or $\quad \tan\phi = \tan(\psi - \theta)$

or $\quad \tan\phi = \dfrac{\tan\psi - \tan\theta}{1 + \tan\psi\tan\theta} = \dfrac{(dy/dx) - (y/x)}{1 + (dy/dx)(y/x)}, \quad$ [From (1) and (2)]

$$= \left(x\frac{dy}{dx} - y\right) \Big/ \left(x + y\frac{dy}{dx}\right).$$ **Hence proved**

Problem 5: *Prove* $\dfrac{1}{p^2} = u^2 + \left(\dfrac{du}{d\theta}\right)^2$*, where* $u = \dfrac{1}{r}$ *and p is the length of perpendicular from pole to the tangent of the curve at any point* $P(r, \theta)$. **(Bundelkhand 2001)**

Solution: (i) Suppose $ON = p$ is the perpendicular from the pole O on the tangent at P to the curve.



$\therefore \quad \angle OPN = \phi.$

Hence, from the right-angled triangle ONP, we have clearly

$$p = r\sin\phi.$$

(ii) Since $\tan\phi = r\dfrac{d\theta}{dr}.$

$\therefore \qquad \cot \phi = \dfrac{1}{r} \dfrac{dr}{d\theta} \cdot$ ...(1)

Now from $p = r \sin \phi$, we have

$$\frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2 \phi = \frac{1}{r^2} (1 + \cot^2 \phi) = \frac{1}{r^2} \left\{ 1 + \frac{1}{r^2} \left(\frac{dr}{d\theta} \right)^2 \right\}, \text{ by (1)}$$

or $\qquad \dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4} \left(\dfrac{dr}{d\theta} \right)^2.$ ...(2)

(iii) Since $\qquad u = \dfrac{1}{r} \cdot$

$\therefore \qquad \dfrac{du}{d\theta} = - \dfrac{1}{r^2} \left(\dfrac{dr}{d\theta} \right) \quad$ or $\quad \left(\dfrac{du}{d\theta} \right)^2 = \dfrac{1}{r^4} \left(\dfrac{dr}{d\theta} \right)^2.$

Hence, from equation (2)

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta} \right)^2, \text{ where } \quad r = \frac{1}{u}.$$

Problem 6: *Show that the spirals $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$ intersect orthogonally.*

(Garhwal 2000; Agra 03; Kumaun 08, 11)

Solution: The curves are $\quad r^n = a^n \cos n\theta$ ...(1)

and $\qquad r^n = b^n \sin n\theta$ ...(2)

Taking logarithm of both sides of (1), we get

$$n \log r = n \log a + \log \cos n\theta$$

Differentiating, $\quad \dfrac{n}{r} \dfrac{dr}{d\theta} = \dfrac{1}{\cos n\theta} . (-\sin n\theta) \,.\, n$

or $\qquad \tan \theta_1 = r \, (d\theta / dr) = - \cot n\theta = \tan \left(\dfrac{1}{2} \pi + n\theta \right)$

or $\qquad \phi_1 = \dfrac{1}{2} \pi + n\theta.$ ...(3)

Similarly from (2), taking logarithm of both sides, we get

$$n \log r = n \log b + \log \sin n\theta.$$

Differentiating, $\quad \dfrac{n}{r} \dfrac{dr}{d\theta} = \dfrac{1}{\sin n\theta} \cos n\theta \,.\, n$

or $\qquad \tan \phi_2 = r \, (d\theta / dr) = \tan n\theta \quad$ or $\quad \phi_2 = n\theta$ ...(4)

$\therefore$ The angle between (1) and (2)

$$= \phi_1 - \phi_2 = \left(\frac{1}{2} \pi + n\theta \right) - n\theta, \text{ from (3) and (4)}$$

$$= \frac{\pi}{2} \text{ i.e., the curves intersect orthogonally.}$$

Problem 7: *Show that the cardioids $r = a(1 + \cos\theta)$ and $r = b(1 - \cos\theta)$ intersect at right angles.* **(Meerut 2000; Kanpur 07, 11)**

Solution: The curves are

$$r = a(1 + \cos\theta) \quad ...(1)$$

and $$r = b(1 - \cos\theta) \quad ...(2)$$

From (1), $dr/d\theta = -a\sin\theta.$

$$\therefore \quad \tan\phi_1 = \frac{r\,d\theta}{dr} = \frac{a(1 + \cos\theta)}{-a\sin\theta} = -\frac{2\cos^2\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}$$

or $$\tan\phi_1 = -\cot\frac{1}{2}\theta = \tan\left(\frac{1}{2}\pi + \frac{1}{2}\theta\right) \Rightarrow \theta_1 = \frac{1}{2}\pi + \frac{1}{2}\theta. \quad ...(3)$$

From (2), $dr/d\theta = b\sin\theta$.

$$\therefore \quad \tan\phi_2 = \frac{r\,d\theta}{dr} = \frac{b(1 - \cos\theta)}{b\sin\theta} = 2\frac{2\sin^2\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \tan\frac{1}{2}\theta$$

or $$\theta_2 = \frac{1}{2}\theta. \quad ...(4)$$

$\therefore$ Required angle of intersection

$$= \theta_1 - \theta_2 = \left(\frac{1}{2}\pi + \frac{1}{2}\theta\right) - \frac{1}{2}\theta, \text{ from (3) and (4)}$$

$$= \frac{1}{2}\pi$$ *i.e.*, curves intersect orthogonally.

Problem 8: *Show that the curves $r = a(1 + \sin\theta)$ and $r = a(1 - \sin\theta)$ cut orthogonally.*

Solution: The curves are $r = a(1 + \sin\theta)$ ...(1)

and $r = a(1 - \sin\theta)$ ...(2)

From (1) we get $dr/d\theta = a\cos\theta$

$$\therefore \quad \tan\theta_1 = \frac{r\,d\theta}{dr} = \frac{a(1 + \sin\theta)}{a\cos\theta} = \frac{1 + \cos\left(\frac{1}{2}\pi - \theta\right)}{\sin\left(\frac{1}{2}\pi - \theta\right)}, \quad \textbf{(Note)}$$

$$= \frac{2\cos^2\left(\frac{1}{4}\pi - \frac{1}{2}\theta\right)}{2\sin\left(\frac{1}{4}\pi - \frac{1}{2}\theta\right)\cos\left(\frac{1}{4}\pi - \frac{1}{2}\theta\right)}$$

$$= \cot\left(\frac{1}{4}\pi - \frac{1}{2}\theta\right) = \tan\left[\frac{1}{2}\pi - \left(\frac{1}{4}\pi - \frac{1}{2}\theta\right)\right] = \tan\left(\frac{1}{4}\pi + \frac{1}{2}\theta\right)$$

or $$\phi_1 = \frac{1}{4}\pi + \frac{1}{2}\theta \quad ...(3)$$

Also from (2), we get $dr/d\theta = -a\cos\theta$

$$\therefore \qquad \tan\theta_2 = \frac{r\,d\theta}{dr} = \frac{a(1-\sin\theta)}{-a\cos\theta} = \frac{1-\cos\left(\frac{1}{2}\pi-\theta\right)}{-\sin\left(\frac{1}{2}\pi-\theta\right)}, \qquad \textbf{(Note)}$$

$$= \frac{2\sin^2\left(\frac{1}{4}\pi-\frac{1}{2}\theta\right)}{-2\sin\left(\frac{1}{4}\pi-\frac{1}{2}\theta\right)\cos\left(\frac{1}{4}\pi-\frac{1}{2}\theta\right)}$$

$$= -\tan\left(\frac{1}{4}\pi-\frac{1}{2}\theta\right) = \tan\left(\frac{1}{2}\theta-\frac{1}{4}\pi\right). \qquad \text{...(4)}$$

$$\therefore \qquad \phi_2 = \frac{1}{2}\theta - \frac{1}{4}\pi.$$

$\therefore$ Required angle of intersection of the curves

$$= \phi_1 - \phi_2 = \left(\frac{1}{4}\pi+\frac{1}{2}\theta\right) - \left(\frac{1}{2}\theta-\frac{1}{4}\pi\right), \qquad \text{[From (3) and (4)]}$$

$$= \frac{1}{2}\pi$$ *i.e.*, curves intersect orthogonally.

Problem 9: *Find the angle of intersection of the curves $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta$.*

(Lucknow 2009)

Solution: The curves are $r = \sin\theta + \cos\theta$ ...(1)

$r = 2\sin\theta$ ...(2)

From (1), we get $\dfrac{dr}{d\theta} = \cos\theta - \sin\theta$

$$\therefore \qquad \tan\phi_1 = r\frac{d\theta}{dr} = \frac{\sin\theta+\cos\theta}{\cos\theta-\sin\theta} = \frac{1+\tan\theta}{1-\tan\theta} = \tan\left(\frac{\pi}{4}+\theta\right).$$

$$\therefore \qquad \phi_1 = \frac{\pi}{4} + \theta.$$

Also from (2), we get $\dfrac{dr}{d\theta} = 2\cos\theta$

$$\therefore \qquad \tan\phi_2 = r\frac{d\theta}{dr} = \frac{2\sin\theta}{2\cos\theta} = \tan\theta.$$

$$\therefore \qquad \phi_2 = \theta.$$

$\therefore$ Required angle of intersection of the curves

$$= \phi_1 - \phi_2 = \left(\frac{\pi}{4}+\theta\right) - \theta = \frac{\pi}{4}.$$

Problem 10: *Show that the curves $r = 2\sin\theta$ and $r = 2\cos\theta$ intersect at right angles.*

Solution: The curves are $r = 2\sin\theta$ ...(1)

and $r = 2\cos\theta.$ ...(2)

From (1), we get $\dfrac{dr}{d\theta} = 2\cos\theta.$

$\therefore \qquad \tan\phi_1 = r\dfrac{d\theta}{dr} = \dfrac{2\sin\theta}{2\cos\theta} = \tan\theta.$

Also from (2), we get $\dfrac{dr}{d\theta} = -2\sin\theta$

$\therefore \qquad \tan\phi_2 = r\dfrac{d\theta}{dr} = \dfrac{2\cos\theta}{-2\sin\theta} = -\cot\theta = \tan\left(\dfrac{\pi}{2}+\theta\right)$

or $\qquad \phi_2 = \dfrac{\pi}{2}+\theta$

$\therefore$ Required angle of intersection of the curves

$$\phi_1 \sim \phi_2 = \left(\frac{\pi}{2}+\theta\right) - \theta = \frac{\pi}{2}$$

i.e., curves intersect orthogonally.

Problem 11: *Find the angle between the tangent and the radius vector in the case of the curve* $r^n = a^n \sec(n\theta+\alpha)$, *and prove that this curve is intersected by the curve* $r^n = b^n \sec(n\theta+\beta)$ *at an angle which is independent of a and b.*

Solution: The curves are $r^n = a^n \sec(n\theta+\alpha)$ ...(1)

and $\qquad r^n = b^n \sec(n\theta+\beta).$ ...(2)

From (1), taking logarithm of both sides, we get

$$n\log r = n\log a + \log\sec(n\theta+\alpha).$$

Differentiating, $\dfrac{n}{r}\dfrac{dr}{d\theta} = \dfrac{1}{\sec(n\theta+\alpha)}\sec(n\theta+\alpha)\tan(n\theta+\alpha)\,n$

or $\qquad \tan\phi_1 = r(d\theta/dr) = \cot(n\theta+\alpha) = \tan\left[\dfrac{1}{2}\pi - (n\theta+\alpha)\right]$

or $\qquad \phi_1 = \dfrac{1}{2}\pi - (n\theta+\alpha).$ ...(3)

Similarly from (2), we can get $\phi_2 = \dfrac{1}{2}\pi - (n\theta+\beta).$ ...(4)

$\therefore$ Angle of intersection of curves (1) and (2)

$$= \phi_1 \sim \phi_2 = \left[\frac{1}{2}\pi - (n\theta+\alpha)\right] \sim \left[\frac{1}{2}\pi - (n\theta+\beta)\right], \text{ from (3) and (4)}$$

$= \beta \sim \alpha$, which is independent of a and b.

Problem 12: *Find the angle of intersection between the pair of curves* $r = 6\cos\theta$ *and* $r = 2(1+\cos\theta)$.

Solution: The given curves are $r = 6\cos\theta$ ...(1)

and $\qquad r = 2(1+\cos\theta).$ ...(2)

From (1), we get $\dfrac{dr}{d\theta} = -6\sin\theta.$

$\therefore \qquad \tan\phi_1 = r\dfrac{d\theta}{dr} = \dfrac{6\cos\theta}{-6\sin\theta} = -\cot\theta = \tan\left(\dfrac{\pi}{2}+\theta\right)$

or $$\phi_1 = \frac{\pi}{2} + \theta.$$

Also from (2), we get $\dfrac{dr}{d\theta} = -2\sin\theta.$

$\therefore$ $$\tan\phi_2 = r\frac{d\theta}{dr} = \frac{2(1+\cos\theta)}{-2\sin\theta} = \frac{2.2\cos^2(\theta/2)}{-2.2\sin(\theta/2)\cos(\theta/2)}$$
$$= -\cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

or $$\theta_2 = \frac{\pi}{2} + \frac{\theta}{2}.$$

Also at the point of intersection
$$6\cos\theta = 2(1+\cos\theta) \quad \text{or} \quad 6\cos\theta = 2 + 2\cos\theta$$
or $$4\cos\theta = 2 \quad \text{or} \quad \cos\theta = \frac{1}{2} \quad \text{or} \quad \theta = \frac{1}{3}\pi.$$

$\therefore$ Required angle $= \phi_1 - \phi_2 = \left(\dfrac{1}{2}\pi + \theta\right) - \left(\dfrac{1}{2}\pi + \dfrac{1}{2}\theta\right)$, at $\theta = \dfrac{1}{3}\pi = \dfrac{1}{2}\theta = \dfrac{1}{2}\left(\dfrac{1}{3}\pi\right) = \dfrac{1}{6}\pi.$

Comprehensive Problems 2

Problem 1: *Find the polar subtangent of the ellipse $l/r = 1 + e\cos\theta$.*

Solution: Given $\dfrac{l}{r} = 1 + e\cos\theta.$

Taking logarithm $\log l - \log r = \log(1 + e\cos\theta).$

Differentiating, $$-\frac{1}{r}\frac{dr}{d\theta} = \frac{(-e\sin\theta)}{1 + e\cos\theta} \quad \text{or} \quad r\frac{d\theta}{dr} = \frac{1 + e\cos\theta}{e\sin\theta}.$$

$\therefore$ $$\text{Polar subtangent} = r^2\frac{d\theta}{dr} = r\left(r\frac{d\theta}{dr}\right) = \frac{l}{1 + e\cos\theta}\cdot\frac{1 + e\cos\theta}{e\sin\theta} = \frac{l}{e\sin\theta}.$$

Problem 2: *For the parabola $2a/r = 1 - \cos\theta$, prove that*

(i) $\phi = \pi - \dfrac{1}{2}\theta$, *(ii)* $p = a\,\text{cosec}\,\dfrac{1}{2}\theta$,

(iii) $p^2 = ar$, *(iv) the polar subtangent* $= 2a\,\text{cosec}\,\theta$.

(Lucknow 2006; Purvanchal 14)

Solution: The curve is
$$2a/r = 1 - \cos\theta = 2\sin^2\frac{1}{2}\theta$$
or $$r = a\,\text{cosec}^2\frac{1}{2}\theta. \qquad \text{...(1)}$$

Differentiating, $$\frac{dr}{d\theta} = -2a\,\text{cosec}^2\frac{1}{2}\theta\cot\frac{1}{2}\theta\cdot\frac{1}{2} = -a\,\text{cosec}^2\frac{1}{2}\theta\cot\frac{1}{2}\theta.$$

(i) $\therefore$ $$\tan\phi = r\frac{d\theta}{dr} = \frac{a\,\text{cosec}^2\frac{1}{2}\theta}{-a\,\text{cosec}^2\frac{1}{2}\theta\cot\frac{1}{2}\theta} = -\tan\frac{1}{2}\theta$$

or $$\tan\phi = \tan\left(\pi - \frac{1}{2}\theta\right) \text{ or } \phi = \pi - \frac{1}{2}\theta. \qquad ...(2)$$

(ii) $p = r\sin\phi = a\,\text{cosec}^2\frac{1}{2}\theta\sin\left(\pi - \frac{1}{2}\theta\right)$, from (1) and (2)

$$= a\,\text{cosec}^2\frac{1}{2}\theta\sin\frac{1}{2}\theta = a\,\text{cosec}\frac{1}{2}\theta.$$

(iii) $p = a\,[\sqrt{(r/a)}]$, from (1)

or $$p = \sqrt{(ar)} \quad \text{or} \quad p^2 = ar.$$

(iv) polar subtangent $= r^2\dfrac{d\theta}{dr} = a^2\,\text{cosec}^4\dfrac{1}{2}\theta \cdot \dfrac{1}{a\,\text{cosec}^2\frac{1}{2}\theta\cot\frac{1}{2}\theta}$

$$= \frac{a}{\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \frac{2a}{\sin\theta} = 2a\,\text{cosec}\,\theta.$$

Problem 3: *For the cardioid $r = a(1 - \cos\theta)$, prove that*

(i) $\phi = \frac{1}{2}\theta$, **(Meerut 2001; Lucknow 11)**

(ii) $p = 2a\sin^3\frac{1}{2}\theta$,

(iii) the pedal equation is $2ap^2 = r^3$, **(Meerut 2001; Rohilkhand 12B)**

(iv) the polar subtangent $= 2a\sin^2(\theta/2)\tan(\theta/2)$.

Solution: The given curve is $r = a(1 - \cos\theta)$. ...(1)

Differentiating, $dr/d\theta = a\sin\theta$.

(i) We have $$\tan\phi = r\frac{d\theta}{dr} = \frac{a(1-\cos\theta)}{a\sin\theta} = \frac{2a\sin^2\frac{1}{2}\theta}{2a\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \tan\frac{\theta}{2}.$$

$\therefore$ $\phi = \theta/2$.

(ii) We have $$p = r\sin\phi = r\sin\frac{\theta}{2} = a(1-\cos\theta)\sin\frac{\theta}{2} = 2a\sin^2\frac{\theta}{2}\sin\frac{\theta}{2} = 2a\sin^3\frac{\theta}{2}.$$

(iii) We have $p = r\sin\phi = r\sin(\theta/2)$. $[\because \phi = \theta/2]$

Now from (1), $r = 2a\sin^2\frac{1}{2}\theta = 2a(p^2/r^2)$, $[\because p = r\sin\frac{1}{2}\theta]$

$\therefore$ Pedal equation is $2ap^2 = r^3$.

(iv) Polar subtangent $= r^2\dfrac{d\theta}{dr} = a^2(1-\cos\theta)^2 \cdot \dfrac{1}{(a\sin\theta)}$

$$= \frac{4\,a\sin^4 \theta/2}{2\sin\theta/2\,\cos\theta/2} = 2a\sin^2\frac{\theta}{2}\tan\frac{\theta}{2}.$$

Problem 4: *Show that the pedal equation*

(i) of the lemniscate $r^2 = a^2\cos 2\theta$ *is* $r^3 = a^2 p$,

(ii) of the hyperbola $r^2\cos 2\theta = a^2$ *is* $pr = a^2$,

(iii) of the cosine spiral $r^n = a^n \cos n\theta$ *is* $pa^n = r^{n+1}$, **(Lucknow 2005)**

(iv) of the curve $r = a\theta$ *is* $p^2 = r^4/(r^2 + a^2)$.

Solution: (i) Given $r^2 = a^2\cos 2\theta$ ...(1)

Differentiating,

$$2r\frac{dr}{d\theta} = -2a^2\sin 2\theta \quad\text{or}\quad \frac{dr}{d\theta} = -\frac{a^2\sin 2\theta}{r}. \qquad ...(2)$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{a^4\sin^2 2\theta}{r^2}\right), \text{ from (2)}$$

$$= \frac{1}{r^2} + \frac{a^4}{r^6}(1-\cos^2 2\theta) = \frac{1}{r^2} + \frac{a^4}{r^6}\left(1 - \frac{r^4}{a^4}\right), \text{ from (1)}$$

$$= \frac{1}{r^2} + \frac{a^4}{r^6} - \frac{1}{r^2} = \frac{a^4}{r^6}$$

or $\quad p^2 a^4 = r^6 \quad$ or $\quad a^2 p = r^3$.

(ii) The given curve is $\quad r^2\cos 2\theta = a^2 \quad$ or $\quad r^2 = a^2\sec 2\theta$. ...(1)

Differentiating, $2r\,(dr/d\theta) = 2a^2\sec 2\theta\tan 2\theta = 2r^2\tan 2\theta$

or $\quad dr/d\theta = r\tan 2\theta$. ...(2)

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} + \frac{1}{r^4}(r\tan^2 2\theta), \text{ from (2)}$$

$$= \frac{1}{r^2}[1+\tan^2 2\theta] = \frac{1}{r^2}\sec^2 2\theta = \frac{1}{r^2}\left(\frac{r^2}{a^2}\right)^2, \text{ from (1)}$$

or $\quad 1/p^2 = r^2/a^4 \quad$ or $\quad r^2 p^2 = a^4 \quad$ or $\quad pr = a^2$.

(iii) The given curve is $r^n = a^n\cos n\theta$.

Taking logarithm, $n\log r = n\log a + \log\cos n\theta$. ...(1)

Differentiating, $\quad \dfrac{n}{r}\dfrac{dr}{d\theta} = \dfrac{1}{\cos n\theta}(-\sin n\theta)\,n$

or $\quad dr/d\theta = -r\tan n\theta$. ...(2)

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} + \frac{1}{r^4}(r^2\tan^2 n\theta), \text{ from (2)}$$

or $$\frac{1}{p^2}=\frac{1}{r^2}(1+\tan^2 n\theta)=\frac{1}{r^2}(\sec^2 n\theta)=\frac{1}{r^2}\left(\frac{a^n}{r^n}\right)^2, \text{ from (1)}$$

or $$1/p^2=a^2n/r^{2n+2} \quad \text{or} \quad r^{n+1}=pa^n.$$

(iv) The given curve is $r=a\theta$. ...(1)

Differentiating, $\dfrac{dr}{d\theta}=a.$

$$\therefore \quad \frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{1}{r^2}+\frac{1}{r^4}.a^2=\frac{r^2+a^2}{r^4}.$$

$$\therefore \quad p^2=r^4/(r^2+a^2).$$

Problem 5: *Show that the pedal equation of the conic* $\dfrac{l}{r}=1+e\cos\theta$ *is*

$$\frac{1}{p^2}=\frac{1}{l^2}\left(\frac{2l}{r}-1+e^2\right).$$

Solution: The curve is $(l/r)=1+e\cos\theta.$ ...(1)

Differentiating, we get

$$\frac{-l}{r^2}\frac{dr}{d\theta}=-e\sin\theta \quad \text{or} \quad \frac{1}{r^2}\frac{dr}{d\theta}=\frac{e\sin\theta}{l}. \qquad ...(2)$$

$\therefore$ From $\dfrac{1}{p^2}=\dfrac{1}{r^2}+\dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$, we get

$$\frac{1}{p^2}=\frac{1}{r^2}+\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2=\frac{1}{r^2}+\frac{e^2\sin^2\theta}{l^2}, \qquad \text{[From (2)]}$$

$$=\frac{1}{r^2}+\frac{e^2}{l^2}(1-\cos^2\theta)=\frac{1}{r^2}+\frac{e^2}{l^2}-\frac{e^2}{l^2}\cos^2\theta$$

$$=\frac{1}{r^2}+\frac{e^2}{l^2}-\frac{e^2}{l^2}\left(\frac{l-r}{er}\right)^2, \qquad \left[\because \text{From (1) } \cos\theta=\frac{l-r}{er}\right]$$

$$=\frac{1}{r^2}+\frac{e^2}{l^2}-\frac{1}{l^2r^2}(l^2-2lr+r^2)=\frac{1}{r^2}+\frac{e^2}{l^2}-\frac{1}{r^2}+\frac{2}{lr}-\frac{1}{l^2}$$

or $$\frac{1}{p^2}=\frac{1}{l^2}\left(e^2+\frac{2l}{r}-1\right), \text{ is the required pedal equation.}$$

Problem 6: *Show that the pedal equation of the spiral* $r=a\,\text{sech}\,n\theta$ *is of the form* $\dfrac{1}{p^2}=\dfrac{A}{r^2}+B$.

Solution: The curve is $r=a\,\text{sech}\,n\theta.$ ...(1)

Differentiating, $dr/d\theta=-an\,\text{sech}\,n\theta\tanh n\theta=-nr\tanh n\theta.$ ...(2)

$$\therefore \quad \frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{1}{r^2}+\frac{1}{r^4}(n^2r^2\tanh^2 n\theta), \qquad \text{[From (2)]}$$

$$= (1/r^2)\,[1 + n^2 \tanh^2 n\theta] = (1/r^2)\,[1 + n^2\,(1 - \text{sech}^2\, n\theta)]$$

or $$\frac{1}{p^2} = \frac{1}{r^2}[(1+n^2) - n^2 \text{sech}^2\, n\theta] = \frac{1}{r^2}\left[(1+n^2) - n^2\left(\frac{r^2}{a^2}\right)\right], \quad \text{[From (1)]}$$

$$= \frac{(1+n^2)}{r^2} - \frac{n^2}{a^2} = \frac{A}{r^2} + B, \text{ where } A = 1 + n^2 \text{ and } B = -n^2/a^2.$$

Hence the pedal equation of (1) is of the form $\dfrac{1}{p^2} = \dfrac{A}{r^2} + B.$

Problem 7: *Show that the pedal equation of the cardioid $r = a\,(1 + \cos\theta)$ is $r^3 = 2ap^2$.*

Solution: Given $r = a\,(1 + \cos\theta)$. ...(1)

Differentiating, we get $(dr/d\theta) = -a\sin\theta$.

$$\therefore \quad \tan\phi = \frac{r\,d\theta}{dr} = \frac{a\,(1+\cos\theta)}{-a\sin\theta} = \frac{2\cos^2\frac{1}{2}\theta}{-2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}$$

$$= -\cot\frac{1}{2}\theta = \tan\left(\frac{1}{2}\pi + \frac{1}{2}\theta\right)$$

or $$\phi = \frac{1}{2}\pi + \frac{1}{2}\theta.$$

Now we know $p = r\sin\phi = r\sin\left(\dfrac{1}{2}\pi + \dfrac{1}{2}\theta\right)$

or $p = r\cos\dfrac{1}{2}\theta$ or $p^2 = r^2\cos^2\left(\dfrac{1}{2}\theta\right)$

or $2p^2 = r^2\,(1 + \cos\theta)$, $\left[\because 1 + \cos\theta = 2\cos^2\left(\frac{1}{2}\theta\right)\right]$

or $2p^2 = r^2\,(r/a)$, [From (1)]

or $2ap^2 = r^3$.

Problem 8: *Show that the pedal equation of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $r^2 = a^2 - 3p^2$.*

Solution: Given curve is $x^{2/3} + y^{2/3} = a^{2/3}$.

The parametric equations are

$$x = a\cos^3 t \quad \text{...(1)}$$

and $$y = a\sin^3 t. \quad \text{...(2)}$$

From (1) we get $dx/dt = -3a\cos^2 t\sin t$.

From (2) we get $dy/dt = 3a\sin^2 t\cos t$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t.$$

$\therefore$ Equation of the tangent to the given curve at 't' is

$$Y - a\sin^3 t = (-\tan t)\,(X - a\cos^3 t)$$

or $$X\tan t + Y - (a\cos^3 t\tan t + a\sin^3 t) = 0. \quad \text{...(3)}$$

$\therefore$ $\qquad p$ = length of perpendicular from (0, 0) on (3)

$$= \frac{a\cos^3 t \tan t + a\sin^3 t}{\sqrt{(\tan^2 t + 1)}} = \frac{a\sin t\,(\cos^2 t + \sin^2 t)}{\sec t}$$

or $\qquad p = a\sin t \cos t.$ ...(4)

Also $\qquad r^2 = x^2 + y^2 = a^2 \cos^6 t + a^2 \sin^6 t$, from (1) and (2)

$$= a^2 [(\cos^2 t + \sin^2 t)^3 - 3\cos^2 t \sin^2 t\,(\cos^2 t + \sin^2 t)],$$

$\because$ $\qquad a^3 + b^3 = (a+b)^3 - 3ab\,(a+b)$

or $\qquad r^2 = a^2 [1 - 3\cos^2 t \sin^2 t] = a^2[1 - 3p^2/a^2]$, [From (4)]

or $\qquad r^2 = a^2 - 3p^2$ is the required pedal equation.

Problem 9: *Show that the locus of the extremity of the polar subnormal of the curve* $r = f(\theta)$ *is*

$$r = f'\left(\theta - \frac{1}{2}\pi\right).$$

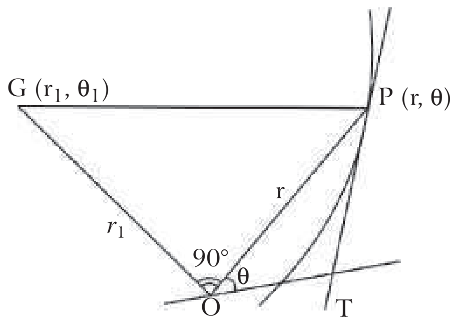
(Gorakhpur 2006)

Hence show that the locus of the extremity of the polar subnormal of the equiangular spiral $r = ae^{m\theta}$ *is another equiangular spiral.*

(Lucknow 2011)

Solution: Let (r, θ) be a point on the curve.

Let the line drawn through O perpendicular to the radius vector OP meet the normal at P in G. Let G be (r_1, θ_1).



Then $\qquad r_1 = OG$ = polar subnormal

$$= dr/d\theta = f'(\theta),$$

since $\qquad r = f(\theta)$

$\therefore$ $\qquad r_1 = f'(\theta)$ ...(1)

Also $\qquad \theta_1 = \angle GOA = 90° + \theta$

or $\qquad \theta = \theta_1 - \frac{1}{2}\pi.$

$\therefore$ From (1), $r_1 = f'(\theta_1 - \frac{1}{2}\pi).$

$\therefore$ Locus of $G(r_1, \theta_1)$ is $r = f'(\theta - \frac{1}{2}\pi).$ ...(2)

(ii) Given $\qquad r = ae^{m\theta}$

Here $\qquad f(\theta) = ae^{m\theta}$ and $f'(\theta) = ame^{m\theta}.$

$\therefore$ $\qquad f'(\theta - \frac{1}{2}\pi) = ame^{m(\theta - \pi/2)} = ame^{m\theta} \,.\, e^{-m\pi/2}$

$$= ke^{m\theta}, \text{ where } k = ame^{-m\pi/2}.$$

$\therefore$ From (2), the required locus is $r = ke^{m\theta}$, which is also an equiangular spiral.

Problem 10: *Prove that the normal at any point* (r, θ) *to the curve* $r^n = a^n \cos n\theta$ *makes an angle* $(n+1)\theta$ *with the initial line.* **(Agra 2002; Kumaun 10)**

Solution: The given curve is

$$r^n = a^n \cos n\theta. \qquad \text{...(1)}$$

Taking logarithm of both sides of (1), we get

$$n \log r = n \log a + \log \cos n\theta.$$

Differentiating with respect to θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta} \cdot (-n \sin n\theta) = -n \tan n\theta$$

or $$\cot \phi = \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta = \cot\left(\frac{1}{2}\pi + n\theta\right).$$

$\therefore$ $$\phi = \frac{1}{2}\pi + n\theta.$$

If ψ is the angle which the tangent at any point (r, θ) to the curve (1) makes with the initial line, then

$$\psi = \theta + \phi = \theta + \frac{1}{2}\pi + n\theta = \frac{1}{2}\pi + (n+1)\theta.$$

The slope of the tangent at (r, θ)

$$= \tan \psi = \tan\left[\frac{1}{2}\pi + (n+1)\theta\right] = -\cot(n+1)$$

$\therefore$ The slope of the normal to (1) at the point (r, θ)

$$= -\frac{1}{-\cot(n+1)\theta} = \tan(n+1)\theta.$$

Hence the normal to (1) at the point (r, θ) makes an angle $(n+1)\theta$ with the axis of x *i.e.*, with the initial line.

Comprehensive Problems 3

Problem 1: *Calculate* ds/dx *for the following curves :*

(i) $y^2 = 4ax$; **(Lucknow 2009)** *(ii)* $y = a \cosh(x/a)$;

(iii) $x^{2/3} + y^{2/3} = a^{2/3}$.

Solution: **(i)** The curve is $y^2 = 4ax$. ...(1)

Differentiating, $2y\,(dy/dx) = 4a$ or $(dy/dx) = 2a/y$.

$\therefore$ $$\frac{ds}{dx} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} = \sqrt{\left(1 + \frac{4a^2}{y^2}\right)} = \sqrt{\left(1 + \frac{4a^2}{4ax}\right)}, \qquad \text{[From (1)]}$$

$$= \sqrt{\{1 + (a/x)\}}.$$

(ii) The curve is $y = a \cosh(x/a)$.

Differentiating, $dy/dx = a \sinh(x/a)(1/a) = \sinh(x/a)$.

$$\therefore \qquad \frac{ds}{dx} = \sqrt{\left[1 + \left\{\frac{dy}{dx}\right\}^2\right]} = \sqrt{\left[1 + \sinh^2\left\{\frac{x}{a}\right\}\right]} = \cosh\left\{\frac{x}{a}\right\}.$$

(iii) The curve is $x^{2/3} + y^{2/3} = a^{2/3}$. ...(1)

Differentiating, $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = -\left\{\frac{y}{x}\right\}^{1/3}$.

$$\therefore \qquad \frac{ds}{dx} = \sqrt{\left[1 + \left\{\frac{dy}{dx}\right\}^2\right]} = \sqrt{\left[1 + \left\{\frac{y}{x}\right\}^{2/3}\right]} = \sqrt{\left\{\frac{x^{2/3} + y^{2/3}}{x^{2/3}}\right\}}$$

or $\qquad ds/dx = \sqrt{(a^{2/3}/x^{2/3})} = (a/x)^{1/3}$, from (1).

Problem 2: *Calculate ds/dt for the following curves :*

(i) $x = a(1 - \cos t),\ y = a(t + \sin t)$; *(ii)* $x = a\cos^3 t,\ y = a\sin^3 t$;

(iii) $x = 2\sin t, y = \cos 2t$.

Solution: **(i)** Given $x = a(1 - \cos t),\ y = a(t + \sin t)$.

$\therefore \qquad dx/dt = a\sin t;\ dy/dt = a(1 + \cos t)$.

$\therefore \qquad ds/dt = \sqrt{\{(dx/dt)^2\} + (dy/dt)^2\}} = \sqrt{\{a^2\sin^2 t + a^2(1 + \cos t)^2\}}$

$\qquad = a\sqrt{\{\sin^2 t + 1 + \cos^2 t + 2\cos t\}} = a\sqrt{\{2(1 + \cos t)\}}$

$\qquad = a\sqrt{\{2(2\cos^2 \frac{1}{2}t)\}} = 2a\cos\frac{1}{2}t.$

(ii) Given $\qquad x = a\cos^3 t,\ y = a\sin^3 t$.

$$\therefore \qquad \frac{dx}{dt} = -3a\cos^2 t\sin t,\ \frac{dy}{dt} = 3a\sin^2 t\cos t.$$

$$\therefore \qquad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= 3a\cos t\sin t\sqrt{(\cos^2 t + \sin^2 t)} = 3a\cos t\sin t.$$

(iii) Given $\qquad x = 2\sin t,\ y = \cos 2t$.

$$\therefore \qquad \frac{dx}{dt} = 2\cos t,\ \frac{dy}{dt} = -2\sin 2t.$$

$$\therefore \qquad \frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} = 2\sqrt{(\cos^2 t + \sin^2 2t)}$$

$$= 2\sqrt{(\cos^2 t + 4\cos^2 t\sin^2 t)} = 2\cos t\sqrt{(1 + 4\sin^2 t)}.$$

Problem 3: *Calculate ds/dθ for the following curves :*

(i) $r = \log\sin 3\theta$; *(ii)* $r = a(1 - \cos\theta)$.

Solution: **(i)** The curve is $r = \log\sin 3\theta$.

Differentiating, $\frac{dr}{d\theta} = \frac{1}{\sin 3\theta} . 3\cos 3\theta = 3\cot 3\theta.$

$\therefore \qquad \dfrac{ds}{d\theta} = \sqrt{\left\{\left(\dfrac{dr}{d\theta}\right)^2 + r^2\right\}} = \sqrt{(9 \cot^2 3\theta + r^2)}.$

(ii) The curve is $r = a(1 - \cos\theta)$.

Differentiating, $\dfrac{dr}{d\theta} = a \sin\theta$.

$$\therefore \qquad \frac{ds}{d\theta} = \sqrt{\left\{\left(\frac{dr}{d\theta}\right)^2 + r^2\right\}} = \sqrt{(a^2 \sin^2\theta + r^2)}$$

$$= \sqrt{\{(a^2 \sin^2\theta + a^2 (1 - \cos\theta)^2)\}}$$

$$= a\sqrt{\{(2 \sin \frac{\theta}{2} \cos \frac{\theta}{2})^2 + (2 \sin^2 \frac{\theta}{2})^2\}}$$

$$= 2a \sin \frac{\theta}{2} \sqrt{(\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2})} = 2a \sin \frac{\theta}{2}.$$

Problem 4: *For the curve $r = ae^{\theta \cot \alpha}$, prove that s/r = constant, s being measured from the pole.* **(Lucknow 2008)**

Solution: The curve is $r = ae^{\theta \cot \alpha}$.

Differentiating, $\dfrac{dr}{d\theta} = ae^{\theta \cot \alpha} . \cot \alpha = r \cot \alpha$.

$$\therefore \qquad \frac{ds}{dt} = \sqrt{\left\{1 + \left(r\frac{d\theta}{dr}\right)^2\right\}} = \left\{1 + \left(\frac{r}{r \cot \alpha}\right)^2\right\} = \sqrt{(1 + \tan^2 \alpha)} = \sec \alpha$$

or $\qquad ds = \sec \alpha . dr$.

Integrating, $\qquad s = r \sec \alpha + C$, where C is constant of integration.

At the origin $r = 0$ and $s = 0$ (given).

$\therefore \qquad 0 = 0 + C \quad$ or $\quad C = 0$.

$\therefore \qquad s = r \sec \alpha \quad$ or $\quad s/r = \sec\alpha$ = constant.

Problem 5: *In any curve, prove that*

(i) $\dfrac{ds}{d\theta} = \dfrac{r^2}{p}$ **(Lucknow 2007, 10)** *(ii)* $\dfrac{ds}{dr} = \dfrac{r}{\sqrt{(r^2 - p^2)}}$.

Solution: **(i)** We know $r\,(d\theta/ds) = \sin \phi$

and $\qquad p = r \sin \phi$ or $\sin \phi = p/r$.

$\therefore \qquad r\dfrac{d\theta}{ds} = \dfrac{p}{r}$ or $\dfrac{ds}{d\theta} = \dfrac{r^2}{p}$. **Hence proved**

(ii) We know $\quad dr/ds = \cos \phi$.

$\therefore \qquad \dfrac{ds}{dr} = \dfrac{1}{\cos \phi} = \dfrac{1}{\sqrt{(1 - \sin^2 \phi)}} = \dfrac{r}{\sqrt{(r^2 - r^2 \sin^2 \phi)}}$ **(Note)**

or $\qquad \dfrac{ds}{dr} = \dfrac{r}{\sqrt{(r^2 - p^2)}}$. $\qquad [\because \; p = r \sin\phi]$

Problem 6: *For the curve $r^n = a^n \cos n\theta$, prove that $a^{2n}\dfrac{d^2r}{ds^2} + nr^{2n-1} = 0$.*

Solution: Note use $m = n$ or $n = m$.

The curve is $\quad r^m = a^m \cos m\theta.$

Taking logarithms, $\quad m \log r = m \log a + \log \cos m\theta. \qquad ...(1)$

Differentiating, $\quad \dfrac{m}{r}\dfrac{dr}{d\theta} = \dfrac{1}{\cos m\theta}(-m\sin m\theta)$

or $\quad \dfrac{dr}{r\,d\theta} = -\tan m\theta \quad$ or $\quad \dfrac{dr}{d\theta} = -r\tan m\theta. \qquad ...(2)$

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{\left\{\left(\frac{dr}{d\theta}\right)^2 + r^2\right\}} = \sqrt{(r^2\tan^2 m\theta + r^2)}, \text{ from (2)}$$

$$= \sqrt{(r^2\sec^2 m\theta)} = r\sec m\theta$$

$$= a(\cos m\theta)^{1/m}(\cos m\theta)^{-1}, \text{ from (i) } r = a(\cos m\theta)^{1/m}$$

$$= a(\cos m\theta)^{(1/m)-1} = a(\cos m\theta)^{(1-m)/m}$$

$$= a(\sec m\theta)^{(m-1)/m} \qquad [\because \cos m\theta = (\sec m\theta)^{-1}]$$

or $\quad ds/d\theta = a(\sec m\theta)^{(m-1)/m}. \qquad ...(3)$

From (1), $\quad \sec m\theta = \left(\dfrac{a}{r}\right)^m.$

$$\therefore \text{ From (3), } \quad \frac{ds}{d\theta} = a\left\{\left(\frac{a}{r}\right)^m\right\}^{(m-1)/m} = a\left(\frac{a}{r}\right)^{m-1}$$

or $\quad \dfrac{ds}{d\theta} = \dfrac{a^m}{r^{m-1}}. \qquad ...(4)$

$$\therefore \quad \frac{dr}{ds} = \frac{dr}{d\theta}\cdot\frac{d\theta}{ds} = \frac{dr}{d\theta}\Big/\frac{ds}{d\theta} = \frac{-r\tan m\theta}{a^m/r^{m-1}} \quad \text{or} \quad \frac{dr}{ds} = -a^m r^m \tan m\theta.$$

Differentiating both sides w.r.to s, we get

$$\frac{d^2r}{ds^2} = -a^{-m}\left[mr^{m-1}\tan m\theta\frac{dr}{ds} + r^m\,.\,m\sec^2 m\theta\,\frac{d\theta}{ds}\right]$$

$$= -\frac{m}{a^m}r^{m-1}\left[\tan m\theta\frac{dr}{ds} + r\sec^2 m\theta\,\frac{d\theta}{ds}\right]$$

$$= \frac{-m}{a^m}r^{m-1}\left[-\frac{r^m\tan^2 m\theta}{a^m} + r\sec^2 m\theta\,.\,\frac{r^{m-1}}{a^m}\right]$$

$$= mr^{2m-1}/a^{2m}\,[\tan^2 m\theta - \sec^2 m\theta]$$

$$= -mr^{2m-1}/a^{2m} \qquad [\because -\sec^2 m\theta + \tan^2 m\theta = -1]$$

or $\quad a^{2m}\dfrac{d^2r}{ds^2} + mr^{2m-1} = 0.$

Problem 7: *For the cycloid* $x = a(1 - \cos t)$, $y = a(t + \sin t)$, *find*

(i) $\frac{ds}{dt}$ (ii) $\frac{ds}{dx}$ (iii) $\frac{ds}{dy}$.

Solution: Given $x = a(1 - \cos t)$, $y = a(t + \sin t)$

(i) Proceed as in Problem 2(i).

(ii) $\frac{ds}{dx} = \frac{ds}{dt} \Big/ \frac{dy}{dt} = \frac{2a \cot(t/2)}{a \sin t} = \frac{2a \cos(t/2)}{2a \sin(t/2) \cos(t/2)} = \operatorname{cosec} t/2.$

(iii) $\frac{ds}{dy} = \frac{ds}{dt} \Big/ \frac{dy}{dt} = \frac{2a \cos(t/2)}{a(1 + \cos t)} = \frac{2a \cos(t/2)}{2a \cos^2(t/2)} = \sec(t/2).$

Hints to Objective Type Questions

Multiple Choice Questions

1. See Problem 3(i) of Comprehensive Problems 2.
2. See article 10.
3. See article 17.
4. See articles 12 and 17.
5. See article 17.
6. See Example 11.
7. The given curve is $r = ae^{\theta \cot \alpha}$.

   Differentiating w.r.t. to θ, we get

   $$\frac{dr}{d\theta} = ae^{\theta \cot \alpha} \cot\alpha = r\cot\alpha$$

   $\therefore \quad \cot\phi = \frac{1}{r}\frac{dr}{d\theta} = \cot\alpha \quad$ or $\quad \phi = \alpha.$

   $\therefore \quad p = r\sin\alpha.$
8. See Example 8.

Fill in the Blanks

1. See article 9.
2. See article 16.
3. See Problem 2(i) of Comprehensive Problems 2.
4. See Problem 7(i) of Comprehensive Problems 3.
5. We have $r^2 = a^2 \cos 2\theta$

   $\therefore \quad 2r\frac{dr}{d\theta} = -2a^2 \sin 2\theta.$

Now, $$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^2 + \frac{a^4}{r^2}\sin^2 2\theta}$$

$$= \frac{1}{r}\sqrt{r^4 + a^4 \sin^2 2\theta} = \frac{1}{r}\sqrt{a^4 \cos^2 2\theta + \alpha^4 \sin^2 2\theta} = \frac{a^2}{r}.$$

6. We know that polar subnormal $= \dfrac{dr}{d\theta}$.

Given $\dfrac{d\theta}{dr} = \dfrac{7}{3}$.

$\therefore$ $\dfrac{dr}{d\theta} = \dfrac{3}{7}$.

True or False

1. See article 12.

2. See article 8.

3. See article 17.

4. See article 13.

❍❍❍

Chapter-10

Curvature

Comprehensive Problems 1

Problem 1: *Find the radius of curvature at the point (s, ψ) on the following curves :*

(i) $s = c \tan \psi$ *(Catenary)* *(ii)* $s = 8a \sin^2 \frac{1}{6}\psi$ *(Cardioid)*

(iii) $s = 4a \sin \psi$ *(Cycloid)* **(Bundelkhand 2001; Rohilkhand 08; Kashi 11)**

(iv) $s = c \log \sec \psi$ *(Tractrix)* **(Kashi 2012)**

Solution: **(i)** We have $s = c \tan \psi$.

$$\therefore \qquad \rho = ds/d\psi = c \sec^2 \psi.$$

(ii) We have $s = 8a \sin^2 \frac{1}{6}\psi$.

$$\therefore \qquad \rho = \frac{ds}{d\psi} = 8a \,.\, (2 \sin \frac{1}{6}\psi \cos \frac{1}{6}\psi) \,.\, \frac{1}{6} = \left(\frac{4a}{3}\right) \sin \frac{1}{3}\psi.$$

(iii) We have $s = 4a \sin \psi$.

$$\therefore \qquad \rho = ds/d\psi = 4a \cos \psi.$$

(iv) We have $s = c \log \sec \psi$.

$$\therefore \qquad \rho = \frac{ds}{d\psi} = c \cdot \frac{1}{\sec \psi} \cdot \sec \psi \tan \psi = c \tan \psi.$$

Problem 2: *Find the radius of curvature at the point (x, y) on the following curves :*

(i) $a^2 y = x^3 - a^3$ *(ii)* $y^2 = 4ax$

(iii) $xy = c^2$ *(iv)* $ay^2 = x^3$

(v) $y = \frac{1}{2} a (e^{x/a} + e^{-x/a})$

(Agra 2007)

(vi) $y = c \log \sec (x/c)$ **(Kanpur 2007; Purvanchal 09)**

(vii) $x^{1/2} + y^{1/2} = a^{1/2}$

(viii) $x^{2/3} + y^{2/3} = a^{2/3}$ **(Rohilkhand 2009B; Kashi 12)**

(ix) $x^m + y^m = 1.$

Solution: **(i)** We have $a^2 y = x^3 - a^3$.

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{a^2}(3x^2); \quad \dfrac{d^2y}{dx^2} = \dfrac{6x}{a^2}.$

$\therefore \quad \rho = \dfrac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \dfrac{[1 + (3x^2/a^2)^2]^{3/2}}{6x/a^2}$

$= \dfrac{(a^4 + 9x^4)^{3/2}}{a^6} \cdot \dfrac{a^2}{6x} = \dfrac{(a^4 + 9x^4)^{3/2}}{6a^4 x}.$

(ii) The given curve is $y^2 = 4ax.$ ...(1)

Differentiating (1) w.r.t. x, we get

$2y\dfrac{dy}{dx} = 4a \quad \text{or} \quad \dfrac{dy}{dx} = \dfrac{2a}{y}.$

$\therefore \quad \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{2a}{y}\right) = -\dfrac{2a}{y^2}\dfrac{dy}{dx} = -\dfrac{2a}{y^2}\cdot\dfrac{2a}{y} = -\dfrac{4a^2}{y^3}.$

Now $\quad \rho = \dfrac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2} = \dfrac{\left(1 + \dfrac{4a^2}{y^2}\right)^{3/2}}{-4a^2/y^3} = -\dfrac{(y^2 + 4a^2)^{3/2}}{y^3}\cdot\dfrac{y^3}{4a^2}$

$= -\dfrac{1}{4a^2}(4ax + 4a^2)^{3/2},$ [From (1)]

$= -\dfrac{1}{4a^2}\cdot a^{3/2} \,.\, 8 \,.\, (x + a)^{3/2} = \dfrac{2}{\sqrt{a}}(x + a)^{3/2},$

neglecting the $-$ ive sign because ρ is a length.

(iii) We have $xy = c^2$ or $y = c^2/x.$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{c^2}{x^2} \quad \text{and} \quad \dfrac{d^2y}{dx^2} = \dfrac{2c^2}{x^3}.$

$\therefore \quad \rho = \dfrac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \dfrac{[1 + (c^4/x^4)]^{3/2}}{2c^2/x^3}$

$= \dfrac{(x^4 + c^4)^{3/2}}{2c^2x^3} = \dfrac{(x^4 + x^2y^2)^{3/2}}{2c^2x^3},$ $[\because xy = c^2]$

$= \dfrac{(x^2 + y^2)^{3/2}}{2c^2} = \dfrac{r^3}{2c^2}$, where $r^2 = x^2 + y^2.$

(iv) The equation of the given curve is $ay^2 = x^3.$ ...(1)

Differentiating (1) w.r.t. x, we get

$2ay\dfrac{dy}{dx} = 3x^2$

or $\quad \dfrac{dy}{dx} = \dfrac{3x^2}{2ay} = \dfrac{3x^2}{2a(x^3/a)^{1/2}},$ [From (1)]

$= (3/2\sqrt{a})\,x^{1/2}.$

Differentiating again, $\dfrac{d^2y}{dx^2}=\dfrac{3}{2\sqrt{a}}\dfrac{1}{2}x^{-1/2}=\dfrac{3}{4\sqrt{a}\,.\,x^{1/2}}\cdot$

Now $$\rho=\frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2}=\frac{\{1+(9x/4a)\}^{3/2}}{3/(4\sqrt{a}x^{1/2})}$$

$$=\frac{(4a+9x)^{3/2}}{8a^{3/2}}\cdot\frac{4\sqrt{a}x^{1/2}}{3}=\frac{x^{1/2}(4a+9x)^{3/2}}{6a}\cdot$$

(v) The curve $y=\dfrac{1}{2}a\,(e^{x/a}+e^{-x/a})=a\cosh\dfrac{x}{a}\cdot$ ...(1)

$\therefore$ Differentiating $\dfrac{dy}{dx}=\sinh\dfrac{x}{a}$

and $$\frac{d^2y}{dx^2}=\frac{1}{a}\cosh\frac{x}{a}\cdot$$

$$\rho=\frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}{\dfrac{d^2y}{dx^2}}=\frac{\left(1+\sinh^2\dfrac{x}{a}\right)^{3/2}}{\dfrac{1}{a}\cosh\dfrac{x}{a}}$$

$$=\frac{a\left(\cosh^2\dfrac{x}{a}\right)^{3/2}}{\cosh\dfrac{x}{a}}=a\cosh^2\frac{x}{a}=y^2/a.\qquad\text{[By (1)]}$$

(vi) We have $y=c\log\sec(x/c)$.

$\therefore$ $$\frac{dy}{dx}=\frac{c}{\sec(x/c)}\cdot\sec\left(\frac{x}{c}\right)\cdot\tan\left(\frac{x}{c}\right)\cdot\frac{1}{c}=\tan\left(\frac{x}{c}\right)\cdot$$

Also $$d^2y/dx^2=(1/c)\sec^2(x/c).$$

$\therefore$ $$\rho=\frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2}=\frac{[1+\tan^2(x/c)]^{3/2}}{(1/c)\sec^2(x/c)}=\frac{c\sec^3(x/c)}{\sec^2(x/c)}=c\sec\left(\frac{x}{c}\right).$$

(vii) The curve is $x^{1/2}+y^{1/2}=a^{1/2}$. ...(1)

Differentiating, $\dfrac{1}{2}\dfrac{1}{\sqrt{x}}+\dfrac{1}{2}\dfrac{1}{\sqrt{y}}\dfrac{dy}{du}=0$

or $$\frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}\qquad\text{...(2)}$$

and $$\frac{d^2y}{dx^2}=-\left[\frac{\sqrt{x}\dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx}-\sqrt{y}\dfrac{1}{2\sqrt{x}}}{x}\right]-\frac{1}{2x}\left[\sqrt{\frac{x}{y}}\left(-\sqrt{\frac{y}{x}}\right)-\sqrt{\frac{y}{x}}\right]$$

$$=+\frac{1}{2x}\left[1+\frac{\sqrt{y}}{\sqrt{x}}\right]=\frac{1}{2x}\left[\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}\right]=\frac{1}{2x}\sqrt{\frac{a}{x}}$$

$$\therefore \qquad \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2 y/dx^2} = \frac{\left(1 + \frac{y}{x}\right)}{\sqrt{a}} . 2x\sqrt{x}$$

$$= \frac{(x + y)^{3/2}\ 2x\sqrt{x}}{x\sqrt{x}\sqrt{a}} = \frac{2\,(x + y)^{3/2}}{\sqrt{a}}.$$

(viii) The curve is $x^{2/3} + y^{2/3} = a^{2/3}$. ...(1)

Differentiating $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$ or $\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$

and $$\frac{d^2 y}{dx^2} = -\left[\frac{x^{1/3}\frac{1}{3}y^{-2/3}\frac{dy}{dx} - y^{1/3}\frac{1}{3}x^{-2/3}}{x^{2/3}}\right]$$

$$= -\frac{1}{3x^{2/3}}\left[\frac{x^{1/3}}{y^{2/3}}\left(-\frac{y^{1/3}}{x^{1/3}}\right) - \frac{y^{1/3}}{x^{2/3}}\right]$$

$$= +\frac{1}{3x^{2/3}}\left[\frac{x^{2/3} + y^{2/3}}{y^{1/3}x^{2/3}}\right] = \frac{a^{2/3}}{3x^{4/3}y^{1/3}}.$$

$$\therefore \qquad \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2 y/dx^2} = \frac{\left[1 + \left(\frac{y}{x}\right)^{3/2}\right]}{a^{2/3}}\, 3x^{4/3}y^{1/3}$$

$$= \frac{(x^{2/3} + y^{2/3})^{3/2}\ 3x^{4/3}\ y^{1/3}}{x a^{2/3}}$$

$$= \frac{3a\,x^{4/3}\,y^{1/3}}{x a^{2/3}} = 3\,a^{1/3}\,x^{1/3}\,y^{1/3}.$$

(ix) The curve is $x^m + y^m = 1$. ...(1)

Differentiating , $mx^{m-1} + my^{m-1}\frac{dy}{dx} = 0$

or $$\frac{dy}{dx} = -\left(\frac{x}{y}\right)^{m-1}$$

and $$\frac{d^2 y}{dx^2} = -\left[\frac{y^{m-1}\,(m-1)x^{m-2} - x^{m-1}\,(m-1)\,y^{m-2}\,\frac{dy}{dx}}{y^{2m-2}}\right]$$

$$= \frac{-(m-1)}{y^{2m-2}}\left[y^{m-1}\,x^{m-2} - x^{m-1}\,y^{m-2}.\left\{\frac{-x^{m-1}}{y^{m-1}}\right\}\right]$$

$$= \frac{-(m-1)}{y^{2m-2}}\left[y^{m-1}\,x^{m-2} + \frac{x^{2m-2}}{y}\right]$$

$$= \frac{-(m-1)}{y^{2m-1}}\,[y^m x^{m-2} + x^{2m-1}]$$

$$= \frac{-(m-1)}{y^{2m-1}}\,[y^m + x^m]\,x^{m-2} = \frac{-(m-1)}{y^{2m-1}}\,x^{m-2}\,.1. \qquad \text{[By (1)]}$$

$\therefore$
$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2} = \frac{\left[1+\frac{x^{2m-2}}{y^{2m-2}}\right]^{(3/2)}}{-(m-1)\,x^{m-2}}\,y^{2m-1}$$

$$= \frac{(y^{2m-2}+x^{2m-2})^{3/2}\,y^{2m-1}}{-(m-1)\,x^{m-2}\,y^{(m-1)3}} = \frac{(x^{2m-2}+y^{2m-2})^{3/2}}{(1-m)\,x^{m-2}\,y^{m-2}}\,.$$

Problem 3(i): *Find the radius of curvature of the curve $y = e^x$ at the point where it crosses the y-axis.* **(Agra 2014)**

Solution: We have $y = e^x$.

Therefore $dy/dx = e^x$, and $d^2y/dx^2 = e^x$.

$\therefore$
$$\rho \text{ at } (x, y) = \frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{(1+e^{2x})^{3/2}}{e^x}\cdot$$

The curve $y = e^x$ crosses the y-axis (*i.e.*, the straight line $x = 0$) at the point $(0, 1)$.

$\therefore$
$$\rho \text{ at } (0,1) = \frac{(1+e^0)^{3/2}}{e^0} = \frac{(1+1)^{3/2}}{1} = 2\sqrt{2}.$$

Problem 3(ii): *Find the radius of curvature of curve $\sqrt{x} + \sqrt{y} = 1$ at the point $\left(\frac{1}{4}, \frac{1}{4}\right)$.*

Solution: The curve is

$$\sqrt{x} + \sqrt{y} = 1. \qquad \text{...(1)}$$

Differentiating, we get

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0 \qquad \text{or} \qquad \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

and
$$\frac{d^2y}{dx^2} = -\left[\frac{\sqrt{x}\,\frac{1}{2\sqrt{y}}\frac{dy}{dx} - \sqrt{y}\,\frac{1}{2\sqrt{x}}}{x}\right]$$

$$= -\frac{1}{2x}\left[\frac{\sqrt{x}}{\sqrt{y}}\left(-\frac{\sqrt{y}}{\sqrt{u}}\right) - \frac{\sqrt{y}}{\sqrt{x}}\right] = \frac{1}{2x}\left[\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}\right] = \frac{1}{2x\sqrt{x}}\cdot \qquad \text{[By (1)]}$$

$$\therefore \qquad \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2} = \frac{\left(1+\frac{y}{x}\right)^{3/2}}{1/2\sqrt{x}} = 2x\sqrt{x}\left(1+\frac{y}{x}\right)^{3/2}.$$

At point $\left(\frac{1}{4}, \frac{1}{4}\right)$, we have

$$\rho = 2.\frac{1}{4}.\sqrt{\frac{1}{4}}\,(1+1)^{3/2} = \frac{1}{2}\cdot\frac{1}{2}\,2^{3/2} = \frac{1}{\sqrt{2}}.$$

Problem 4(i): *Prove that at the point* $x = \frac{1}{2}\pi$ *of the curve* $y = 4\sin x - \sin 2x, \rho = \frac{5\sqrt{5}}{4}$.

Solution: We have $y = 4\sin x - \sin 2x$.

$$\therefore \qquad dy/dx = 4\cos x - 2\cos 2x \text{ and } d^2y/dx^2 = -4\sin x + 4\sin 2x.$$

Therefore at $x = \frac{1}{2}\pi$, we have

$$dy/dx = 4\cos\frac{1}{2}\pi - 2\cos\pi = 2$$

and $$d^2y/dx^2 = -4\sin\frac{1}{2}\pi + 4\sin\pi = -4.$$

$$\therefore \qquad \rho \text{ at } (x = \pi/2) = \frac{(1+4)^{3/2}}{-4} = (5\sqrt{5}/4), \text{ neglecting the – ive sign.}$$

Problem 4(ii): *Prove that for the curve*

$s = a\log\cot\left(\frac{\pi}{4} - \frac{\psi}{2}\right) + a\sin\psi\sec^2\psi, \rho = 2a\sec^3\psi$; *and hence that* $\frac{d^2y}{dx^2} = \frac{1}{2a}$.

(Lucknow 2008)

Solution: We have $s = a\log\cot\left(\frac{\pi}{4} - \frac{\psi}{2}\right) + a\sin\psi\sec^2\psi$.

$$\therefore \qquad \rho = \frac{ds}{d\psi} = a\frac{1}{\cot(\frac{1}{4}\pi - \frac{1}{2}\psi)}\cdot\left\{-\operatorname{cosec}^2\left(\frac{\pi}{4} - \frac{\psi}{2}\right)\right\}\cdot\left(-\frac{1}{2}\right)$$

$$+ a\sin\psi\,.\,2\sec\psi\sec\psi\tan\psi + a\cos\psi\sec^2\psi$$

$$= \frac{a}{2\sin(\frac{1}{4}\pi - \frac{1}{2}\psi)\cos(\frac{1}{4}\pi - \frac{1}{2}\psi)} + \frac{2a\sin^2\psi}{\cos^3\psi} + \frac{a}{\cos\psi}$$

$$= \frac{a}{\sin(\frac{1}{2}\pi - \psi)} + \frac{2a\sin^2\psi}{\cos^3\psi} + \frac{a}{\cos\psi}$$

$$= \frac{2a}{\cos\psi} + \frac{2a\sin^2\psi}{\cos^3\psi} = \frac{2a}{\cos^3\psi}(\cos^2\psi + \sin^2\psi) = 2a\sec^3\psi.$$

or $$1/\rho = \cos^3\psi\,(d^2y/dx^2).$$

Problem 5: *In the curve* $y = ae^{x/a}$, *prove that* $\rho = a \sec^2 \theta \operatorname{cosec} \theta$, *where*

$$\theta = \tan^{-1}(y/a).$$

Solution: We have $dy/dx = a(e^{x/a})(1/a) = e^{x/a} = y/a$;

and $$d^2y/dx^2 = (1/a)(dy/dx) = (1/a)(y/a) = y/a^2.$$

$\therefore$ $$\rho = \frac{[1 + (y^2/a^2)]^{3/2}}{y/a^2} = \frac{(a^2 + y^2)^{3/2}}{ay} = \frac{(a^2 + a^2 \tan^2 \theta)^{3/2}}{a \cdot a \tan \theta},$$

[$\because$ Given $y = a \tan \theta$]

$$= a \sec^3 \theta \cot \theta = a \sec^2 \theta \operatorname{cosec} \theta.$$

Problem 6: *Show that the radius of curvature at a point* $(a \cos^3 \theta, a \sin^3 \theta)$ *on the curve* $x^{2/3} + y^{2/3} = a^{2/3}$ *is* $3a \sin \theta \cos \theta$. **(Meerut 2000, 05; Kashi 13)**

Solution: The given curve is $x^{2/3} + y^{2/3} = a^{2/3}$. ...(1)

The co-ordinates (x, y) of any point on (1) may be taken as

$$x = a \cos^3 \theta, \quad y = a \sin^3 \theta, \text{ where } \theta \text{ is the parameter.}$$

$\therefore$ $$dx/d\theta = -3a \cos^2 \theta \sin \theta \text{ and } dy/d\theta = 3a \sin^2 \theta \cos \theta.$$

Now $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a \sin^2 \theta \cos \theta}{-3a \cos^2 \theta \sin \theta} = -\tan \theta.$$

Also $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-\tan \theta) = \left\{\frac{d}{d\theta}(-\tan \theta)\right\} \cdot \frac{d\theta}{dx}$$

$$= -\sec^2 \theta \cdot (d\theta/dx)$$

$$= -\sec^2 \theta \cdot \frac{1}{-3a \cos^2 \theta \sin \theta}, \qquad \left[\because \frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta\right]$$

$$= -(1/3a) \sec^4 \theta \operatorname{cosec} \theta.$$

$\therefore$ $$\rho \text{ at the point '}\theta\text{'} = \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{(1 + \tan^2 \theta)^{3/2}}{(1/3a) \sec^4 \theta \operatorname{cosec} \theta}$$

$$= \frac{3a \sec^3 \theta}{\sec^4 \theta \operatorname{cosec} \theta} = 3a \cos \theta \sin \theta.$$

But $\cos^3 \theta = x/a$ and $\sin^3 \theta = y/a$. Therefore $\cos \theta = (x/a)^{1/3}$ and $\sin \theta = (y/a)^{1/3}$.

Hence ρ at the point $(x, y) = 3a(x/a)^{1/3}(y/a)^{1/3} = 3a^{1/3} x^{1/3} y^{1/3}$.

Problem 7: *In the ellipse* $(x^2/a^2) + (y^2/b^2) = 1$, *show that the radius of curvature at an end of the major axis is equal to the semi-latus rectum of the ellipse.* **(Agra 2001)**

Solution: The ellipse is $(x^2/a^2) + (y^2/b^2) = 1$. ...(1)

Differentiating , $\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{b^2}{a^2}\left(\frac{x}{y}\right)$.

Differentiating again, $\frac{d^2y}{dx^2}=-\frac{b^2}{a^2}\left[\frac{y.1-x(dy/dx)}{y^2}\right]=-\frac{b^2}{a^2y^2}\left[y-x\left(-\frac{b^2x}{a^2y}\right)\right]$

$$=-\frac{b^2}{a^2y^2}\left(\frac{a^2y^2+b^2x^2}{a^2y}\right)=-\frac{b^2}{a^2y^2}\left(\frac{a^2b^2}{a^2y}\right),$$

[$\because$ From (1), $a^2y^2+b^2x^2=a^2b^2$]

$$=-b^4/a^2y^3.$$

$\therefore$ ρ at the point $(x, y)=\frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2}=\frac{[1+(-b^2x/a^2y)^2]^{3/2}}{b^4/(a^2y^3)}$,

neglecting the – ive sign

$$=\frac{(a^4y^2+b^4x^2)^{3/2}}{a^4b^4}.$$

Now the co-ordinates of one end of major axis are $(a, 0)$.

$\therefore$ $$[\rho]_{\text{at }(a,0)}=\frac{[a^4.0+b^4a^2]^{3/2}}{a^4b^4}=\frac{b^6a^3}{a^4b^4}=\frac{b^2}{a}$$

= semi latus-rectum of the ellipse.

Problem 8: *Prove that for the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\rho=\frac{a^2b^2}{p^3}$, *p being the perpendicular from the centre upon the tangent at* (x, y).

(Garhwal 2002; Meerut 02, 04B, 07; Avadh 05, 09)

Solution: Proceeding as in Problem 7, we get

$$\rho \text{ at the point } (x, y)=\frac{(b^4x^2+a^4y^2)^{3/2}}{a^4b^4}. \quad ...(1)$$

Now the equation of the tangent to the given ellipse at (x, y) is

$$\frac{Xx}{a^2}+\frac{Yy}{b^2}=1.$$

$\therefore$ p = the length of the perpendicular from the centre $(0, 0)$ to the tangent

$$=\frac{1}{\sqrt{(x^2/a^4+y^2/b^4)}}=\frac{a^2b^2}{\sqrt{(b^4x^2+a^4y^2)}}.$$

$\therefore$ $$p^3=\frac{a^6b^6}{(b^4x^2+a^4y^2)^{3/2}} \quad \text{or} \quad (b^4x^2+a^4y^2)^{3/2}=\frac{a^6b^6}{p^3}.$$

$\therefore$ Substituting it in (1), we get $\rho=\frac{a^6b^6/p^3}{a^4b^4}=\frac{a^2b^2}{p^3}$.

Problem 9: *If ρ and ρ′ be the radii of curvature at the extremities of two conjugate diameters of an ellipse, prove that* $\{(\rho)^{2/3}+(\rho')^{2/3}\}(ab)^{2/3}=(a^2+b^2)$.

(Meerut 2001, 03, 04, 06, 11; Bundelkhand 06; Lucknow 07; Kanpur 11; Rohilkhand 13B; Kashi 14)

Solution: Let the equation of the ellipse be $x^2/a^2+y^2/b^2=1$. ...(1)

Let CP and CQ be a pair of conjugate semi-diameters of (1), where C is the centre of the ellipse. Let 't' be the eccentric angle of the point P. Then by co-ordinate geometry, the eccentric angle of the point Q is $t+\frac{1}{2}\pi$.

Now in terms of the eccentric angle 't' the co-ordinates (x, y) of the point P are given by

$$x=a\cos t \quad \text{and} \quad y=b\sin t.$$

$\therefore$ $$dx/dt=-a\sin t,\ dy/dt=b\cos t.$$

Hence $$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{b\cos t}{-a\sin t}=-\frac{b}{a}\cot t.$$

Also $$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(-\frac{b}{a}\cot t\right)=\left\{\frac{d}{dt}\left(-\frac{b}{a}\cot t\right)\right\}\cdot\frac{dt}{dx}$$

$$=\frac{b}{a}\operatorname{cosec}^2 t\cdot\frac{1}{-a\sin t}=-\frac{b}{a^2}\operatorname{cosec}^3 t.$$

$\therefore$ If ρ be the radius of curvature at the point P, *i.e.*, at the point 't', we have

$$\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}=\frac{\left(1+\frac{b^2\cos^2 t}{a^2\sin^2 t}\right)^{3/2}}{(-b/a^2)\operatorname{cosec}^3 t}$$

$$=-\frac{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}{ab}.$$

Neglecting the negative sign, we have

$$\rho=\frac{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}{ab}$$

or $$ab\,.\,\rho=(a^2\sin^2 t+b^2\cos^2 t)^{3/2}$$

or $$(ab)^{2/3}\rho^{2/3}=a^2\sin^2 t+b^2\cos^2 t. \qquad ...(2)$$

If ρ′ be the radius of curvature at the point Q, *i.e.*, at the point $t+\frac{1}{2}\pi$, then replacing ρ by ρ′ and t by $t+\frac{1}{2}\pi$ in (2), we get

$$(ab)^{2/3}\rho'^{\,2/3}=a^2\sin^2\left(\frac{1}{2}\pi+t\right)+b^2\cos^2\left(\frac{1}{2}\pi+t\right)$$

$$=a^2\cos^2 t+b^2\sin^2 t. \qquad ...(3)$$

Adding (2) and (3), we get

$$(ab)^{2/3}\rho^{2/3}+(ab)^{2/3}\rho'^{2/3}=a^2+b^2$$

or $$(ab)^{2/3}(\rho^{2/3}+\rho'^{2/3})=a^2+b^2.$$

Problem 10: *Prove that if* ρ *be the radius of curvature at any point P on the parabola* $y^2=4ax$ *and S be its focus, then* ρ^2 *varies as* $(SP)^3$. **(Kumaun 2003)**

Solution: We have $y^2=4ax$.

$$\therefore \quad 2y\frac{dy}{dx}=4a \quad \text{or} \quad \frac{dy}{dx}=\frac{2a}{y}=\frac{2a}{(4ax)^{1/2}}=\frac{\sqrt a}{\sqrt x}.$$

Also $$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\sqrt a\,.\left(-\frac{1}{2}\right)x^{-3/2}=\frac{-\sqrt a}{2x^{3/2}}.$$

$\therefore$ At any point $P(x, y)$, we have

$$\rho=\frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2}=-\frac{[1+(a/x)]^{3/2}}{-\sqrt a/(2x^{3/2})}$$

$$=(2/\sqrt a)(x+a)^{3/2}, \text{ neglecting the – ive sign.}$$

Now the equation of the normal to the given parabola at the point $P(x, y)$ is

$$\frac{dy}{dx}(Y-y)+(X-x)=0$$

or $$\frac{2a}{y}(Y-y)+(X-x)=0, \qquad \left[\because \frac{dy}{dx}=\frac{2a}{y}\right]$$

or $$yX+2aY=xy+2ay. \qquad ...(1)$$

Also directrix of the parabola is $X=-a$. ...(2)

Now the co-ordinates of the point of intersection of (1) and (2) are obtained by solving (1) and (2) for X and Y.

From equations (1) and (2), we get

$$-ay+2aY=xy+2ay, \qquad [\text{Putting } X=-a \text{ in (1)}]$$

or $$Y=(1/2a)(xy+3ay)=(y/2a)(x+3a).$$

$\therefore$ The point of intersection of the normal and the directrix is $[-a, (y/2a)(x+3a)]$.

Therefore the length of the normal intercepted between the curve and the directrix .

$$= \text{the distance between } (x, y) \text{ and } \{-a, (y/2a)(x+3a)\}$$

$$=\sqrt{\left[(x+a)^2+\left\{y-\frac{y}{2a}(x+3a)\right\}^2\right]}$$

$$=\frac{1}{2a}\sqrt{[4a^2(x+a)^2+y^2(x+a)^2]}=\frac{1}{2a}\cdot(x+a)\,.\sqrt{(4a^2+y^2)}$$

$$=\frac{1}{2a}\cdot(x+a)\sqrt{(4a^2+4ax)}, \qquad [\because y^2=4ax]$$

$$=(x+a)^{3/2}/a^{1/2}.$$

As already proved, ρ at $(x, y) = 2(x+a)^{3/2}/\sqrt{a}$

$= 2 \times$ the part of normal intercepted between the curve and directrix.

The focus S is $(a, 0)$.

$$\therefore \quad SP = \sqrt{[(x-a)^2 + (y-0)^2]} = \sqrt{\{(x-a)^2 + 4ax\}}, \qquad [\because y^2 = 4ax]$$

$$= \sqrt{\{(x+a)^2\}} = x + a.$$

But $\quad \rho = \dfrac{2}{\sqrt{a}} \cdot (x+a)^{3/2} = \dfrac{2}{\sqrt{a}} \cdot (SP)^{3/2}. \quad \rho^2 \propto (SP)^3.$

Problem 11: *If the co-ordinates of a point on a curve be given by the equations*

$$x = c \sin 2\theta\,(1 + \cos 2\theta),\ y = c \cos 2\theta\,(1 - \cos 2\theta),$$

show that the radius of curvature at the point is $4c \cos 3\theta$.

Solution: The curve is

$$x = c \sin 2\theta\,(1 + \cos 2\theta);\quad y = c \cos 2\theta\,(1 - \cos 2\theta).$$

Differentiating w.r.to θ, we get

$$x' = c \sin 2\theta\,(-2 \sin 2\theta) + 2c \cos 2\theta\,(1 + \cos 2\theta)$$

$$= -2c \sin^2 2\theta + 2c \cos 2\theta + 2c \cos^2 2\theta$$

$$= 2c \cos 4\theta + 2c \cos 2\theta$$

and $\quad y' = -2c \sin 2\theta\,(1 - \cos 2\theta) + c \cos 2\theta\,(2 \sin 2\theta)$

$$= -2c \sin 2\theta + 2c \sin 2\theta \cos 2\theta + 2 \sin 2\theta \cos 2\theta$$

$$= 2c \sin 4\theta - 2c \sin 2\theta.$$

Again differentiating,we get

$$x'' = -8c \sin 4\theta - 4c \sin 2\theta \quad \text{and} \quad y'' = 8c \cos 4\theta - 4c \cos 2\theta.$$

Now $\quad (x')^2 + (y')^2 = 4c^2 \cos^2 4\theta + 4c^2 \cos^2 2\theta + 8c^2 \cos 4\theta \cos 2\theta$

$$+ 4c^2 \sin^2 4\theta + 4c^2 \sin^2 2\theta - 8c^2 \sin 4\theta \sin 2\theta$$

$$= 4c^2 + 4c^2 + 8c^2 (\cos 4\theta \cos 2\theta - \sin 4\theta \sin 2\theta)$$

$$= 8c^2 + 8c^2 \cos 6\theta = 8c^2 (1 + \cos 6\theta) = 8c^2 . 2\cos^2 3\theta = 16c^2 \cos^2 3\theta$$

$$\therefore \quad \rho = \frac{[(x')^2 + (y'')^2]^{3/2}}{x'y'' - y'x''}$$

$$= \frac{(16c^2 \cos^2 3\theta)^{3/2}}{2c(\cos 4\theta + \cos 2\theta)\,4c\,(2\cos 4\theta - \cos 2\theta) - 2c(\sin 4\theta - \sin 2\theta) - (4c)(2 \sin 4\theta + \sin 2\theta)}$$

$$= \frac{64c^3 \cos^3 3\theta}{8c^2 [2\cos^2 4\theta - \cos 4\theta \cos 2\theta + 2\cos 4\theta \cos 2\theta - \cos^2 2\theta + 2\sin^2 4\theta + \sin 4\theta \sin 2\theta - 2\sin 4\theta \sin 2\theta - \sin^2 2\theta]}$$

$$= \frac{8c\cos^3 3\theta}{2 + \cos 4\theta \cos 2\theta - \sin 4\theta \sin 2\theta - 1}$$

$$= \frac{8c\cos^3 3\theta}{1+\cos 6\theta} = \frac{8c\cos^3 3\theta}{2\cos^2 3\theta} = 4c\cos 3\theta .$$

Problem 12: *If the co-ordinates of a point on a curve be given by the equations*

$$x = a\sin t - b\sin(at/b),\ y = a\cos t - b\cos(at/b)$$

show that the radius of curvature at the point is $\dfrac{4ab}{a+b}\sin\dfrac{a-b}{2b}t$.

Solution: The curve is

$$x = a\sin t - b\sin(at/b);\ y = a\cos t - b\cos(at/b).$$

Differentiating , we get

$$x' = a\cos t - b\cos(at/b)\frac{a}{b} = a\cos t - a\cos(at/b);$$

$$y' = -a\sin t + b\sin(at/b).\frac{a}{b} = -a\sin t + a\sin(at/b)$$

and $x'' = -a\sin t + \dfrac{a^2}{b}\sin(at/b);\quad y'' = -a\cos t + \dfrac{a^2}{b}\cos\left(\dfrac{at}{b}\right).$

Now $(x')^2 + (y')^2 = a^2\cos^2 t + a^2\cos^2(at/b) - 2a^2\cos t\cos(at/b)$

$$+ a^2\sin^2 t + a^2\sin^2(at/b) - 2a^2\sin t\sin(at/b)$$

$$= a^2 + a^2 - 2a^2\cos\left(t - \frac{at}{b}\right)$$

$$\therefore \quad \rho = \frac{[(x')^2 + (y')^2]^{3/2}}{x'y'' - y'x''}$$

$$= \frac{\left[2a^2 - 2a^2\cos\left(1-\frac{a}{b}\right)t\right]^{3/2}}{a^2\left(\cos t - \cos\frac{at}{b}\right)\left(-\cos t + \frac{a}{b}\cos\frac{at}{b}\right) - a^2\left(-\sin t + \frac{at}{b}\right)\left(-\sin t + \frac{a}{b}\sin\frac{at}{b}\right)}$$

$$= \frac{(2a^2)^{3/2}\left[1 - \cos\left(1-\frac{a}{b}\right)t\right]^{3/2}}{a^2\left[-\cos^2 t + \frac{a}{b}\cos t\cos\frac{at}{b} + \cos\frac{at}{b}\cos t - \frac{a}{b}\cos^2\frac{at}{b} - \sin^2 t + \frac{a}{b}\sin t\sin\frac{at}{b} + \sin\frac{at}{b}\sin t - \frac{a}{b}\sin^2\frac{at}{b}\right]}$$

$$= \frac{2\sqrt{2}\,a^3\left[1 - \cos\left(1-\frac{a}{b}\right)t\right]^{3/2}}{a^2\left[-1 - \frac{a}{b} + \frac{a}{b}\cos\left(1-\frac{a}{b}\right)t + \cos\left(1-\frac{a}{b}\right)t\right]}$$

$$= \frac{2\sqrt{2}a\left[1-\cos\left(1-\frac{a}{b}\right)t\right]^{3/2}}{-\left(1-\frac{a}{b}\right)+\left(1+\frac{a}{b}\right)\left\{\cos\left(1-\frac{a}{b}\right)t\right\}} = \frac{2\sqrt{2}a\left[1-\cos\left(1-\frac{a}{b}\right)t\right]^{3/2}}{-\left(1+\frac{a}{b}\right)\left[1-\cos\left(1-\frac{a}{b}\right)t\right]}$$

$$= -\frac{2\sqrt{2}\,ab}{a+b}\left[1-\cos\left(1-\frac{a}{b}\right)t\right]^{1/2} = -\frac{2\sqrt{2}\,ab}{a+b}\left[2\sin^2\frac{1}{2}\left(1-\frac{a}{b}\right)t\right]^{1/2}$$

$$= -\frac{4ab}{a+b}\sin\left(\frac{b-a}{2b}\right)t = \frac{4ab}{a+b}\sin\left(\frac{b-a}{2b}\right)t. \quad \text{[Neglecting the –ive sign]}$$

Problem 13(i): *Prove that for the curve* $s = ae^{x/a}, a\rho = s(s^2 - a^2)^{1/2}$.

Solution: We have $s = ae^{x/a}$.

Therefore $ds/dx = ae^{x/a} . 1/a = e^{x/a} = s/a$.

$\therefore$ $s = a\,(ds/dx) = a \sec\psi$. $[\because \cos\psi = dx/ds \Rightarrow \sec\psi = ds/dx]$

Now $\rho = \dfrac{ds}{d\psi} = a\sec\psi\tan\psi = s\sqrt{(\sec^2\psi - 1)}$

$$= s\sqrt{\left(\frac{s^2}{a^2}-1\right)} = \frac{s}{a}(s^2-a^2)^{1/2}.$$

Hence $a\rho = s(s^2 - a^2)^{1/2}$.

Problem 13(ii): *Show that for the curve* $s^2 = 8ay, \rho = 4a\sqrt{\left(1-\frac{y}{2a}\right)}$. **(Kanpur 2009)**

Solution: The curve is $s^2 = 8ay$. ...(1)

Differentiating with respect to s, we get

$$2s = 8a.\frac{dy}{ds} \quad \text{or} \quad s = 4a\frac{dy}{ds} = 4a\sin\psi. \qquad ...(2)$$

$$\left[\because \frac{dy}{ds} = \sin\psi\right]$$

$\therefore$ $\rho = \dfrac{ds}{d\psi} = 4a\cos\psi = 4a\sqrt{(1-\sin^2\psi)}$

$= 4a\sqrt{(1 - s^2/16a^2)}$ [By (2)]

$= 4a\sqrt{\left(1-\dfrac{8ay}{16a^2}\right)}$ [By (1)]

$= 4a\sqrt{\left(1-\dfrac{y}{2a}\right)}$.

Problem 14: *Find the radius of curvature at the origin of the following curves*

(i) $y = x^4 - 4x^3 - 18x^2$ *(ii)* $y = x^3 + 5x^2 + 6x$.

Solution: (i) Here $dy/dx = 4x^3 - 12x^2 - 36x, d^2y/dx^2 = 12x^2 - 24x - 36.$

$\therefore$ At $(0,0)$, $dy/dx = 0$ and $d^2y/dx^2 = -36.$

$\therefore$ ρ at $(0,0) = \dfrac{\{1+(dy/dx)^2\}^{3/2}}{d^2y/dx^2}$ at $(0,0) = \dfrac{(1+0)^{3/2}}{-36} = \dfrac{1}{36}$, (numerically)

(ii) The given curve is $y = 6x + 5x^2 + x^3,$ ...(1)

which obviously passes through the origin.

Let $$\left(\frac{dy}{dx}\right)_{(0,0)} = p \text{ and } \left(\frac{d^2y}{dx^2}\right)_{(0,0)} = q.$$

Then by Maclaurin's expansion, we get for this curve $y = px + \dfrac{1}{2}qx^2 + \dots$ ...(2)

Comparing (1) and (2), we get $p = 6, q/2 = 5, i.e., q = 10.$

Hence ρ at the origin $= \dfrac{(1+p^2)^{3/2}}{q} = \dfrac{(1+36)^{3/2}}{10} = \dfrac{1}{10} 37\sqrt{37}.$

Problem 15: *Show that the radii of curvature of the curve $a(y^2 - x^2) = x^3$ at the origin are $\pm 2a\sqrt{2}$.*

Solution: The given curve passes through the origin. The tangents at origin are $y^2 - x^2 = 0$, *i.e.*, $y = \pm x$. Thus neither of the co-ordinate axes is a tangent at the origin. So we cannot apply Newton's formula for finding ρ at origin. From the equation of the curve, we have

$$y^2 = x^2 + \frac{x^3}{a} = x^2\left(1 + \frac{x}{a}\right).$$

$$\therefore \quad y = \pm x\left(1+\frac{x}{a}\right)^{1/2} = \pm x\left(1 + \frac{1}{2}\cdot\frac{x}{a} + \dots\right). \quad \text{...(1)}$$

[By binomial theorem]

Let $$\left(\frac{dy}{dx}\right)_{(0,0)} = p \text{ and } \left(\frac{d^2y}{dx^2}\right)_{(0,0)} = q.$$

Then by Maclaurin's expansion, we get for this curve

$$y = px + \frac{1}{2}qx^2 + \dots \quad \text{...(2)}$$

Comparing (1) and (2), we get $p = 1, q = 1/a$; or $p = -1, q = -1/a$.

Now $$\rho \text{ (at origin)} = \frac{(1+p^2)^{3/2}}{q}.$$

$\therefore$ When $p = 1, q = 1/a$, we have ρ (at origin) $= \dfrac{(1+1)^{3/2}}{1/a} = (2\sqrt{2})a$

and when $p = 1, q = -1/a$, we have ρ (at the origin) $= \dfrac{(1+1)^{3/2}}{-1/a} = -(2\sqrt{2})a.$

Comprehensive Problems 2

Problem 1: *Find the radius of curvature at the point* (p, r) *on the following curves:*

(i) $p^2 = ar$, (Parabola) (ii) $r^3 = 2ap^2$, (Cardioid)

(iii) $r^3 = a^2 p$, (Lemniscate) (iv) $p^2 = \dfrac{r^4}{(r^2 + a^2)}$.

Solution: **(i)** We have $p^2 = ar$. ...(1)

Differentiating (1) w.r.t. p, we get

$$2p = a\frac{dr}{dp} \quad \text{or} \quad \frac{dr}{dp} = \frac{2p}{a}.$$

Now $$\rho = r\frac{dr}{dp} = \frac{2pr}{a} = \frac{2p}{a}\left(\frac{p^2}{a}\right), \qquad \text{[From (1)]}$$

$$= 2p^3/a^2.$$

(ii) We have $r^3 = 2ap^2$. ...(1)

Differentiating (1) w.r.t. p, we have

$$3r^2\frac{dr}{dp} = 4ap \quad \text{or} \quad r\frac{dr}{dp} = \frac{4ap}{3r}.$$

$\therefore$ $$\rho = r\frac{dr}{dp} = \frac{4ap}{3r} = \frac{4a}{3r}\sqrt{\left(\frac{r^3}{2a}\right)}, \qquad \text{[From (1)]}$$

$$= \frac{2}{3}\sqrt{(2ar)}.$$

(iii) We have $r^3 = a^2 p$. ...(1)

Differentiating (1) w.r.t. p, we get

$$3r^2(dr/dp) = a^2; \quad \therefore \ \rho = r(dr/dp) = a^2/3r.$$

(iv) The equation of the curve is $p^2 = r^4/(r^2 + a^2)$. ...(1)

$\therefore$ $$\frac{1}{p^2} = \frac{r^2 + a^2}{r^4} = \frac{r^2}{r^4} + \frac{a^2}{r^4} = \frac{1}{r^2} + \frac{a^2}{r^4}.$$

Now differentiating both sides w.r.t. r, we have

$$-\frac{2}{p^3}\frac{dp}{dr} = -\frac{2}{r^3} - \frac{4a^2}{r^5} = -2\cdot\frac{r^2 + 2a^2}{r^5}.$$

$\therefore$ $$\frac{dp}{dr} = p^3\frac{(r^2 + 2a^2)}{r^5} = (p^2)^{3/2}\frac{(r^2 + 2a^2)}{r^5} \qquad \textbf{(Note)}$$

$$= \left(\frac{r^4}{r^2 + a^2}\right)^{3/2}\frac{(r^2 + 2a^2)}{r^5}, \qquad \text{[From (1)]}$$

$$= \frac{r(r^2 + 2a^2)}{(r^2 + a^2)^{3/2}}.$$

Hence $$\rho = r\frac{dr}{dp} = r \cdot \frac{(r^2 + a^2)^{3/2}}{r(r^2 + 2a^2)} = \frac{(r^2 + a^2)^{3/2}}{r^2 + 2a^2}.$$

Problem 2: *Prove that for any curve* $\frac{r}{\rho} = \sin\phi\left(1 + \frac{d\phi}{d\theta}\right)$, *where* ρ *is the radius of curvature and* $\tan\phi = r\frac{d\theta}{dr}.$ **(Gorakhpur 2004, 05; Lucknow 10)**

Solution: We know that $\psi = \theta + \phi$. ...(1)

Differentiating (1) w.r.t. s, we get

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta}\frac{d\theta}{ds} = \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right).$$

$\therefore$ $$\frac{1}{\rho} = \frac{\sin\phi}{r}\left(1 + \frac{d\phi}{d\theta}\right), \qquad \left[\because \rho = \frac{ds}{d\psi} \text{ and } \sin\phi = r\frac{d\theta}{ds}\right]$$

or $$\frac{r}{\rho} = \sin\phi\left(1 + \frac{d\phi}{d\theta}\right).$$

Problem 3: *In the curve* $p = r^{m+1}/a^m$; *show that the radius of curvature varies inversely as the* $(m-1)^{th}$ *power of the radius vector.*

Solution: We have $a^m p = r^{m+1}$. ...(1)

Differentiating (1) w.r.t. p, we get

$$a^m = (m+1)r^m\frac{dr}{dp}.$$

$\therefore$ $$\frac{dr}{dp} = \frac{a^m}{(m+1)r^m}.$$

Now $$\rho = r\frac{dr}{dp} = \frac{ra^m}{(m+1)r^m} = \frac{a^m}{(m+1)r^{m-1}}.$$

$\therefore$ $\rho \propto \frac{1}{r^{m-1}}$ *i.e.*, the radius of curvature varies inversely as the $(m-1)^{th}$ power of the radius vector.

Problem 4 : *Find the radius of curvature at the point* (r, θ) *on each of the following curves:*

(i) $r = a\cos\theta$. **(Kanpur 2006)**

(ii) $r(1 + \cos\theta) = 2a$

(iii) $r^n = a^n\cos n\theta$ **(Rohilkhand 2005; Garhwal 12; Kumaun 15)**

(iv) $r^n = a^n\sin\theta$ **(Agra 2006; Rohilkhand 12; Avadh 12)**

(v) $r = a(1 - \cos\theta)$ **(Avadh 2010; Garhwal 11)**

(vi) $r^2 = a^2\cos 2\theta$

Solution: **(i)** The curve is $r = a \cos \theta$. ...(1)

Differentiating $\dfrac{dr}{d\theta} = -4\sin\theta$ and $\dfrac{d^2 r}{d\theta^2} = -a \cos \theta$.

$$\therefore \qquad \rho = \frac{\left[r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2 r}{d\theta^2}} = \frac{(a^2 \cos^2 \theta + a^2 \sin^2 \theta)^{3/2}}{a^2 \cos^2 \theta + 2a^2 \sin^2 \theta + ra \cos \theta}$$

$$= \frac{a^3}{a^2 \cos^2 \theta + 2a^2 \sin^2 \theta + a^2 \cos^2 \theta}$$

$$= \frac{a^3}{2a^2 (\cos^2 \theta + \sin^2 \theta)} = \frac{a}{2}.$$

(ii) The given curve is $2a/r = 1 + \cos \theta$, ...(1)

which is a parabola with focus as pole.

Proceeding as in the chapter on tangents and normals, we get pedal equation of the curve as

$$p^2 = ar. \qquad ...(2)$$

Differentiating (2) w.r.t. p, we get

$$2p = a\,(dr/dp); \quad \therefore \quad dr/dp = 2p/a.$$

Now $\rho = r\,(dr/dp) = r\,(2p/a) = (2r/a)\,p = (2r/a)\,.\sqrt{(ar)}$, [From (2)]

$$= (2/\sqrt{a})\,r^{3/2}.$$

$$\therefore \qquad \rho^2 = (4/a)\,r^3.$$

Thus $\rho^2 \propto r^3$, where r is the distance of the point from the pole (*i.e.* focus). Hence ρ^2 varies as the cube of the focal distance.

(iii) The given curve is $r^n = a^n \cos n\theta$. ...(1)

Taking logarithm and differentiating w.r.t. θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{-n \sin n\theta}{\cos n\theta} = -n \tan n\theta \quad \text{or} \quad \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta.$$

$$\therefore \qquad \cot \phi = \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta = \cot\left(\frac{1}{2}\pi + n\theta\right); \quad \text{so that } \phi = \frac{1}{2}\pi + n\theta.$$

Now $p = r \sin \phi = r \sin\left(\frac{1}{2}\pi + n\theta\right) = r \cos n\theta = r\,(r^n / a^n)$. [From (1)]

$\therefore$ The pedal equation of (1) is $p = r^{n+1}/a^n$.

$$\therefore \qquad \frac{dp}{dr} = \frac{1}{a^n}(n+1)\,r^n.$$

Hence $$\rho = r\frac{dr}{dp} = r \cdot \frac{a^n}{(n+1)\,r^n} = \frac{a^n}{(n+1)\,r^{n-1}}.$$

(iv) Proceed exactly as in part (iii).

Ans. $\rho = a^n / [(n+1) r^{n-1}]$.

(v) The given curve is $r = a(1 - \cos\theta)$. ...(1)

$\therefore$ $dr/d\theta = a \sin\theta$ and $d^2 r/d\theta^2 = a\cos\theta$.

Hence
$$\rho = \frac{[r^2 + (dr/d\theta)^2]^{3/2}}{r^2 + 2(dr/d\theta)^2 - r(d^2r/d\theta^2)}$$
$$= \frac{[a^2(1-\cos\theta)^2 + a^2\sin^2\theta]^{3/2}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a^2\cos\theta(1-\cos\theta)}$$
$$= \frac{a^2[1+\cos^2\theta - 2\cos\theta + \sin^2\theta]^{3/2}}{a^2[1+\cos^2\theta - 2\cos\theta + 2\sin^2\theta - \cos\theta + \cos^2\theta]}$$
$$= \frac{a[2(1-\cos\theta)]^{3/2}}{3(1-\cos\theta)} = \frac{2\sqrt{2}}{3} a (1-\cos\theta)^{1/2}$$
$$= \frac{2\sqrt{2}}{3} a \sqrt{\left(\frac{r}{a}\right)}, \qquad \text{[From (1)]}$$
$$= \frac{2}{3}\sqrt{(2ar)}.$$

(vi) The given curve is $r^2 = a^2 \cos 2\theta$. ...(1)

Taking logarithm, $2\log r = \log\cos 2\theta + 2\log a$.

Differentiating w.r.t. θ, we get $\dfrac{2}{r}\dfrac{dr}{d\theta} = -\dfrac{2\sin 2\theta}{\cos 2\theta}$.

$\therefore$ $\cot\phi = \dfrac{1}{r}\dfrac{dr}{d\theta} = -\tan 2\theta = \cot(\tfrac{1}{2}\pi + 2\theta)$; so that $\phi = \tfrac{1}{2}\pi + 2\theta$.

Now $p = r\sin\phi = r\sin(\tfrac{1}{2}\pi + 2\theta) = r\cos 2\theta = r(r^2/a^2)$. [From (1)]

Hence the pedal equation of the curve (1) is

$$a^2 p = r^3. \qquad \text{...(2)}$$

Differentiating (2) w.r.t. r, we get

$$a^2 (dp/dr) = 3r^2 \quad \text{or} \quad (dr/dp) = a^2/3r^2.$$

Hence $\rho = r(dr/dp) = r \cdot (a^2/3r^2) = a^2/3r$.

Problem 5: *Forming the pedal equation of the curve* $\theta = a^{-1}(r^2 - a^2)^{1/2} - \cos^{-1}(a/r)$, *show that* $\rho = \sqrt{(r^2 - a^2)}$. **(Meerut 2006B, 08; Rohilkhand 06; Kashi 11)**

Solution: We have $\dfrac{d\theta}{dr} = \dfrac{1}{a}\cdot\dfrac{1}{2}\dfrac{2r}{\sqrt{(r^2-a^2)}} + \dfrac{1}{\sqrt{\{1-(a/r)^2\}}}\left(-\dfrac{a}{r^2}\right)$

$$= \frac{r}{a\sqrt{(r^2-a^2)}} - \frac{a}{r\sqrt{(r^2-a^2)}} = \frac{r^2-a^2}{ar\sqrt{(r^2-a^2)}} = \frac{\sqrt{(r^2-a^2)}}{ar}.$$

$\therefore \qquad \dfrac{dr}{d\theta} = \dfrac{ar}{\sqrt{(r^2 - a^2)}}.$

Now $\qquad \dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2 = \dfrac{1}{r^2} + \dfrac{1}{r^4} \cdot \dfrac{a^2 r^2}{r^2 - a^2} = \dfrac{1}{r^2} + \dfrac{a^2}{r^2 (r^2 - a^2)}$

$$= \frac{r^2 - a^2 + a^2}{r^2 (r^2 - a^2)} = \frac{1}{r^2 - a^2}.$$

$\therefore$ The pedal equation of the given curve is

$$\frac{1}{p^2} = \frac{1}{r^2 - a^2} \quad \text{or} \quad p^2 = r^2 - a^2. \qquad ...(1)$$

Differentiating (1) w.r.t. p, we get $2p = 2r\,(dr/dp)$.

$\therefore \qquad \rho = r\,(dr/dp) = p = (r^2 - a^2)^{1/2}.$ [From (1)]

Problem 6: *For the rectangular hyperbola* $xy = c^2$, *prove that* $\rho = \dfrac{r^3}{2c^2}$, *r being the central radius vector of the point considered.*

Solution: Given $xy = c^2$. ...(1)

Changing this equation in to polar form

$$r \cos\theta \,.\, r \sin\theta = c^2$$

or $\qquad r^2 \cos\theta \sin\theta = c^2.$ ...(2)

On taking log of both sides

$$2 \log r + \log \cos\theta + \log \sin\theta = \log c^2.$$

Differentiating with respect to θ, we get

$$\frac{2}{r}\frac{dr}{d\theta} - \tan\theta + \cot\theta = 0$$

or $\qquad \dfrac{2}{r}\dfrac{dr}{d\theta} = \tan\theta - \cot\theta = \dfrac{\sin^2\theta - \cos^2\theta}{\sin\theta \cos\theta}$

or $\qquad \dfrac{1}{r}\dfrac{dr}{d\theta} = -\dfrac{(\cos^2\theta - \sin^2\theta)}{2 \sin\theta \cos\theta} = -\dfrac{\cos 2\theta}{\sin 2\theta} = -\cot 2\theta$

or $\qquad \cot\phi = -\cot 2\theta = \cot(\pi - 2\theta).$

$\therefore \qquad \phi = \pi - 2\theta$

but $\qquad p = r \sin\phi = r \sin(\pi - 2\theta) = r \sin 2\theta = 2r \sin\cos\theta$

or $\qquad p = 2r \,.\, \dfrac{c^2}{r^2} = \dfrac{2c^2}{r}.$

Differentiating w.r.to r, we get

$$\frac{dp}{dr} = -\frac{2c^2}{r^2} \quad \text{or} \quad \frac{dr}{dp} = -\frac{r^2}{2c^2}.$$

$$\therefore \qquad \rho = r\frac{dr}{dp} = -\frac{r^3}{2c^2} \quad \text{or} \quad \rho = \frac{r^3}{2c^2}. \qquad \text{[Neglecting the –ive sign]}$$

Problem 7: *Show that at any point on the equiangular spiral* $r = ae^{\theta \cot \alpha}$, $\rho = r \operatorname{cosec} \alpha$, *and that it subtends a right angle at the pole.*

Solution: The equation of the given curve is

$$r = ae^{\theta \cot \alpha}. \qquad \ldots(1)$$

Differentiating (1) w.r.t. θ, we have

$$dr/d\theta = ae^{\theta \cot \alpha} . \cot \alpha = r \cot \alpha.$$

$\therefore$ $(1/r)\, dr/d\theta = \cot \alpha$ or $\cot \phi = \cot \alpha$ or $\phi = \alpha$.

Now $p = r \sin \phi = r \sin \alpha$. Thus the pedal equation of (1) is $p = r \sin \alpha$.

Therefore $dp/dr = \sin \alpha$.

Now $\rho = r\dfrac{dr}{dp} = \dfrac{r}{\sin \alpha} = r \operatorname{cosec} \alpha$. **(First part proved.)**

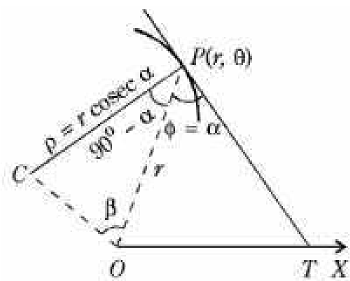
Second part: Let $P\,(r, \theta)$ be any point on the given curve; PT is the tangent and PC is the normal to the curve at P. Let C be the centre of curvature of the point P of the curve. Then PC = the radius of curvature of the curve at $P = r \operatorname{cosec} \alpha$.

Join OP and OC, where O is the pole. Let $\angle POC = \beta$. Then to prove that $\beta = 90^\circ$.

We have $\angle OPT = \phi = \alpha$, [$\because$ for this curve $\phi = \alpha$, as already proved]

$\therefore$ $\angle OPC = 90^\circ - \alpha$, since PC is normal at P

i.e., perpendicular to the tangent PT.



Now in $\Delta\, OPC$, we have

$$\angle OCP = 180^\circ - \{(90^\circ - \alpha) + \beta\}$$

$$= (90^\circ + \alpha - \beta).$$

Hence applying the sine theorem for $\Delta\, OPC$, we get

$$\frac{OP}{\sin \angle OCP} = \frac{PC}{\sin \beta};$$

or $$\frac{r}{\sin(90^\circ + \alpha - \beta)} = \frac{\rho}{\sin \beta}$$

or $$\frac{r}{\cos(\alpha - \beta)} = \frac{r \operatorname{cosec} \alpha}{\sin \beta}, \qquad [\because \rho = r \operatorname{cosec} \alpha]$$

or $\sin \alpha \sin \beta = \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$, [$\because r \neq 0$]

or $\cos \alpha \cos \beta = 0$ or $\cos \beta = 0$, [$\because \cos \alpha \neq 0$]

$\therefore$ $\beta = 90^\circ$.

Problem 8: *If* ρ_1, ρ_2 *be the radii of curvature at the extremities of any chord of the cardioid* $r = a(1 + \cos \theta)$ *which passes through the pole, then show that*

$$\rho_1^2 + \rho_2^2 = 16a^2/9.$$

(Kanpur 2008)

Solution: The given curve is $r = a(1 + \cos\theta)$. ...(1)

Let PQ be any chord of the curve (1) passing through the pole, and let P be the point (r_1, θ_1) and Q be the point (r_2, θ_2). Then $\theta_2 = \pi + \theta_1$.

[***Note :*** To understand this point draw the figure of a chord passing through the pole.]

Since both the points (r_1, θ_1) and (r_2, θ_2) lie on the given cardioid (1), therefore

$$r_1 = a(1 + \cos\theta_1) \quad \text{and} \quad r_2 = a(1 + \cos\theta_2) \qquad ...(2)$$

Now let ρ be the radius of curvature of (1) at the point (r, θ). Then proceeding as in Example 1, after article 9, we get

$$\rho = \frac{2}{3}\sqrt{(2ar)} \quad \text{or} \quad \rho^2 = \frac{8}{9}ar.$$

If ρ_1 and ρ_2 be the radii of curvature at the points P and Q, we have

$$\rho_1^2 = \frac{8}{9}ar_1 \quad \text{and} \quad \rho_2^2 = \frac{8}{9}ar_2.$$

$$\therefore \quad \rho_1^2 + \rho_2^2 = \frac{8}{9}a(r_1 + r_2) = \frac{8}{9}a[a(1 + \cos\theta_1) + a(1 + \cos\theta_2)],$$

[From (2)]

$$= (8a^2/9)[1 + \cos\theta_1 + 1 + \cos(\pi + \theta_1)], \quad [\because \theta_2 = \pi + \theta_1]$$

$$= (8a^2/9)[1 + \cos\theta_1 + 1 - \cos\theta_1] = (16a^2/9).$$

Problem 9: *Show that the radius of curvature at any point on the curve $r = a(1 \pm \cos\theta)$ varies as square root of the radius vector.*

Solution: Proceed as in Problem 4(v).

We get $\rho = \frac{2}{3}\sqrt{2ar}$

or $\rho \propto \sqrt{r}$ $\left\{\text{where } \frac{2}{3}\sqrt{2a} \text{ is constant.}\right.$ **Hence Proved**

Problem 10: *Find the radius of curvature of the cardioid $r = a(1 - \cos\theta)$ at the pole (origin).*

Solution: Proceed as in Problem 4(v).

We get $p = \frac{2}{3}\sqrt{2ar}$.

Therefore, at pole (origin) $(r = 0, \theta = 0)$, we have $\rho = 0$.

Comprehensive Problems 3

Problem 1: *In the parabola $x^2 = 4ay$, prove that the co-ordinates of the centre of curvature are* $\left(-\frac{x^3}{4a^2}, 2a + \frac{3x^2}{4a}\right)$.

Solution: We have $x^2 = 4ay$ ...(1)

Differentiating , $2x = 4a\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{x}{2a}$

and $$\frac{d^2y}{dx^2} = \frac{1}{2a}\cdot$$

If (α, β) be the centre of curvature for the point (x, y), then

$$\alpha = x - \frac{\dfrac{dy}{dx}\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}{\dfrac{d^2y}{dx^2}} = x - \frac{\dfrac{x}{2a}\left(1 + \dfrac{x^2}{4a^2}\right)}{\dfrac{1}{2a}} = x - x - \frac{x^3}{4a^2} = \frac{-x^3}{4a^2}\cdot$$

Also $$\beta = y + \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]}{d^2y/dx^2} = y + \frac{\left(1 + \dfrac{x^2}{4a^2}\right)}{\dfrac{1}{2a}}$$

$$= 2a + y + \frac{x^2}{2a} = 2a + \frac{x^2}{4a} + \frac{x^2}{2a} \qquad \text{[By (1)]}$$

$$= 2a + \frac{3x^2}{4a}\cdot$$

$\therefore$ Centre of curvature is $\left(-\dfrac{x^3}{4a^2}, 2a + \dfrac{3x^2}{4a}\right)$.

Problem 2: *In the catenary $y = c \cosh(x/c)$, show that the centre of curvature (α, β) is given by* $\alpha = x - y\{(y^2/c^2) - 1\}^{1/2}, \beta = 2y$.

Solution: Given $y = c\cosh\dfrac{x}{c}\cdot$ ...(1)

Differentiating, $\dfrac{dy}{dx} = \sinh\dfrac{x}{c}$

and $$\frac{d^2y}{dx^2} = \frac{1}{c}\cosh\frac{x}{c}\cdot$$

$\therefore$ $$\alpha = x - \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]\dfrac{dy}{dx}}{\dfrac{d^2y}{dx^2}} = x - \frac{[1 + \sinh^2\frac{x}{c}]\sinh\frac{x}{c}}{\frac{1}{c}\cosh\frac{x}{c}}$$

$$= x - c\sinh\frac{x}{c}c\cosh\frac{x}{c}c = x - y\sinh\frac{x}{c}$$

$$= x - y\sqrt{\left(\cosh^2\frac{x}{c} - 1\right)} = x - y\sqrt{\left(\frac{y^2}{c^2} - 1\right)}$$

and $$\beta = y + \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = y + \frac{(1+\sinh^2 \frac{x}{c})}{\frac{1}{c}\cosh\frac{x}{c}}$$

$$= y + c\cosh\left(\frac{x}{c}\right) = y + y = 2y.$$

$\therefore$ Centre of curvature is $\left(x - y\sqrt{(y^2/c^2-1)}\, 2y\right)$.

Problem 3: *For the curve $a^2 y = x^3$, show that the centre of curvature (α, β) is given by*

$$\alpha = \frac{x}{2}\left(1 - \frac{9x^4}{a^4}\right), \beta = \frac{5x^3}{2a^2} + \frac{a^2}{6x}.$$

Solution: Here $a^2 y = x^3$;

$$\therefore \quad \frac{dy}{dx} = \frac{3x^2}{a^2} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{6x}{a^2}.$$

$$\therefore \quad \alpha = x - \frac{[1+(dy/dx)^2](dy/dx)}{d^2y/dx^2} = x - \frac{1}{2}x\left(1+\frac{9x^4}{a^4}\right) = \frac{x}{2}\cdot\left(1-\frac{9x^4}{a^4}\right).$$

Also $$\beta = y + \frac{[1+(dy/dx)^2]}{d^2y/dx^2} = \frac{x^3}{a^2} + \frac{[1+(9x^4/a^4)]}{6x/a^2} = \frac{5x^3}{2a^2} + \frac{a^2}{6x}.$$

$\therefore$ The required centre of curvature is

$$\left[\frac{x}{2}\left(1-\frac{9x^4}{a^4}\right), \left(\frac{5x^3}{2a^2}+\frac{a^2}{6x}\right)\right].$$

Problem 4: *Show that the centre of curvature (α, β) at the point determined by 't' on the ellipse $x = a\cos t$, $y = b\sin t$, is given by* $\alpha = \frac{a^2-b^2}{a}\cos^3 t$, $\beta = -\frac{a^2-b^2}{b}\sin^3 t$.

(Lucknow 2007, 10; Kanpur 15)

Solution: Differentiating, $\frac{dx}{dt} = -a\sin t, \frac{dy}{dt} = b\cos t$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b\cos t}{-a\sin t} = -\frac{b}{a}\cot t,$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left\{\frac{d}{dt}\left(-\frac{b}{a}\cot t\right)\right\}\cdot\frac{dt}{dx}$$

$$= \frac{b}{a}\operatorname{cosec}^2 t\left(\frac{1}{-a\sin t}\right) = -\frac{b}{a^2}\operatorname{cosec}^3 t.$$

If (α, β) be the centre of curvature for the point 't', then

$$\alpha = x - \frac{[1 + (dy/dx)^2](dy/dx)}{d^2y/dx^2}$$

$$= x - \frac{\{1 + (b^2/a^2)\cot^2 t\}\{-(b/a)\cot t\}}{-(b/a^2)\operatorname{cosec}^3 t}$$

$$= a\cos t - (1/a)(a^2\sin^2 t + b^2\cos^2 t)\cos t$$

$$= \frac{a^2(\cos t - \sin^2 t\cos t) - b^2\cos^3 t}{a} = \frac{a^2 - b^2}{a}\cos^3 t.$$

Also $$\beta = y + \frac{[1 + (dy/dx)^2]}{d^2y/dx^2} = y + \frac{\{1 + (b^2/a^2)\cot^2 t\}}{-(b/a^2)\operatorname{cosec}^3 t}$$

$$= b\sin t - \frac{\sin t}{b}\cdot(a^2\sin t + b^2\cos^2 t)$$

$$= \frac{1}{b}\sin t\,[b^2 - a^2\sin^2 t - b^2\cos^2 t]$$

$$= \frac{b^2 - a^2}{b}\sin^3 t = -\frac{a^2 - b^2}{b}\sin^3 t.$$

∴ The required centre of curvature is the point $\left(\frac{a^2 - b^2}{a}\cos^3 t, -\frac{a^2 - b^2}{b}\sin^3 t\right)$.

Problem 5: *Prove that the centre of curvature* (α, β) *for the curve*

$$x = 3t,\ y = t^2 - 6 \quad \textit{is} \quad \alpha = -\frac{4}{3}t^3,\ \beta = 3t^2 - \frac{3}{2}.$$

Solution: We have $x = 3t,\ y = t^2 - 6.$

Differentiating $\frac{dx}{dt} = 3, \frac{dy}{dt} = 2t.$

∴ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{2}{9}.$$

∴ $$\alpha = x - \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]\frac{dy}{dx}}{d^2y/dx^2} = 3t - \frac{\left(1 + \frac{4t^2}{9}\right)\frac{2t}{3}}{2/9} = -4t^3/3$$

and $$\beta = y + \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{d^2y/dx^2} = t^2 - 6 + \frac{1 + \left(\frac{2t}{3}\right)^2}{2/9} = 3t^2 - \frac{3}{2}.$$

Problem 6: *Show that in any curve the chord of curvature perpendicular to the radius vector is* $2\rho\sqrt{(r^2 - p^2)}/r$.

Solution: The chord of curvature perpendicular to the radius vector

$$= 2\,\rho \cos \phi = 2\,\rho\,\sqrt{(1 - \sin^2 \phi)} = 2\,\rho\,\sqrt{[1 - (p/r)^2]}, \qquad [\because p = r \sin \phi]$$
$$= [2\,\rho\,\sqrt{(r^2 - p^2)}]/r.$$

Problem 7: *Show that the chord of curvature through the pole of the equiangular spiral* $r = a\,e^{m\,\theta}$ *is* $2\,r$.

Solution: We have $r = a\,e^{m\,\theta}$. Therefore $dr/d\theta = ae^{m\,\theta}\,.\,m = mr.$

$\therefore \qquad \tan \phi = r\,\dfrac{d\theta}{dr} = r \cdot \dfrac{1}{mr} = \dfrac{1}{m} = \tan \alpha$, say. **(Note)**

$\therefore \qquad \phi = \alpha.$

Now $p = r \sin \phi = r \sin \alpha$. Therefore $dp/dr = \sin \alpha$.

$\therefore \qquad \rho = r \cdot \dfrac{dr}{dp} = r \cdot \dfrac{1}{\sin \alpha} = r \operatorname{cosec} \alpha.$

Hence the chord of curvature through the pole $= 2\,\rho \sin \phi$

$$= 2\,r \operatorname{cosec} \alpha \sin \alpha = 2\,r\,.$$

Problem 8: *Show that the chord of curvature, through the pole, for the cardioid,* $r = a\,(1 + \cos \theta)$ *is* $\dfrac{4}{3}r$.

Solution: We have $r = a\,(1 + \cos \theta)$.

Differentiating, $dr/d\theta = -\,a \sin \theta.$

$\therefore \qquad \tan \phi = r\,\dfrac{d\theta}{dr} = \dfrac{a\,(1 + \cos \theta)}{-\,a \sin \theta} = -\cot \dfrac{1}{2}\theta = \tan (\dfrac{1}{2}\pi + \dfrac{1}{2}\theta)$;

so that $\qquad \phi = \dfrac{1}{2}\pi + \dfrac{1}{2}\theta$.

Now $\qquad p = r \sin \phi = r \sin (\dfrac{1}{2}\pi + \dfrac{1}{2}\theta) = r \cos \dfrac{1}{2}\theta.$

$\therefore \qquad 2\,p^2 = r^2\,(2 \cos^2 \dfrac{1}{2}\theta) = r^2\,(1 + \cos \theta) = r^2\,.\,(r/a) = r^3/a.$

Thus $2p^2\,a = r^3$ is the pedal equation of the curve.

Differentiating w.r.t. r, we have $4ap\,\dfrac{dp}{dr} = 3r^2$.

$\therefore \qquad \rho = r\,\dfrac{dr}{dp} = r \cdot \dfrac{4ap}{3r^2} = \dfrac{4ap}{3r}\cdot$

Hence the chord of curvature through the pole

$$= 2\rho \sin \phi = 2 \cdot \frac{4ap}{3r} \cdot \frac{p}{r}, \qquad [\because p = r \sin \phi]$$
$$= \frac{8ap^2}{3r^2} = \frac{8}{3r^2} \cdot \frac{r^3}{2}, \qquad [\because 2ap^2 = r^3]$$
$$= 4r/3.$$

Problem 9: *Show that the circle of curvature at the point* $(am^2, 2am)$ *of the parabola* $y^2 = 4ax$ *has for its equation*

$$x^2 + y^2 - 6am^2x - 4ax + 4am^3y - 3a^2m^4 = 0.$$

Solution: We know that the equation of the circle of curvature is

$$(x-\alpha)^2 + (y-\beta)^2 = \rho^2$$

where, (α, β) is the centre of curvature of parabola $y^2 = 4ax$
and ρ is the radius of curvature.
First find the centre of curvature of parabola $y^2 = 4ax$ at $x = am^2$, $y = 2am$.

Here $\quad y^2 = 4ax \Rightarrow 2y\dfrac{dy}{dx} = 4a$ or $\dfrac{dy}{dx} = \dfrac{2a}{y}$ or $\dfrac{dy}{dx} = a^{1/2}x^{-1/2} = \sqrt{\dfrac{a}{x}}.$

$\therefore \quad \dfrac{d^2y}{dx^2} = -\dfrac{1}{2}a^{1/2}x^{-3/2}.$

If (α, β) be the centre of curvature for the point (x, y), then

$$\alpha = x - \frac{\dfrac{dy}{dx}\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}{\dfrac{d^2y}{dx^2}} = x - \frac{\sqrt{\dfrac{a}{x}}\left\{1 + \dfrac{a}{x}\right\}}{-\dfrac{1}{2x}\sqrt{\dfrac{a}{x}}} = x + 2x\left(1 + \frac{a}{x}\right)$$

$\therefore \quad \alpha = 3x + 2a.$

But $\quad x = am^2 \quad \Rightarrow \quad \alpha = 3am^2 + 2a$

and $$\beta = y + \frac{1 + \left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}} = y + \frac{1 + \left(\dfrac{a}{x}\right)}{\dfrac{-1}{2x}\sqrt{\dfrac{a}{x}}} = 2a^{1/2}x^{1/2} - 2a^{-1/2}x^{3/2}(1 + a/x).$$

$\therefore \quad \beta = -2x\sqrt{\dfrac{x}{a}}.$

Putting $x = am^2 \Rightarrow \beta = -2am^3.$

Also radius of curvature of $y^2 = 4ax$ is

$$\rho = \frac{2}{\sqrt{a}}(x+a)^{3/2}.$$

But $x = am^2 \quad \Rightarrow \quad \rho = \dfrac{2}{\sqrt{a}}(am^2 + a)^{3/2} = \dfrac{2}{\sqrt{a}}a\sqrt{a}(m^2+1)^{3/2} = 2a\,(m^2+1)^{3/2}.$

Putting α, β and ρ in

$$(x-\alpha)^2 + (y-\beta)^2 = \rho^2$$

we get $\quad (x - 3am^2 - 2a)^2 + (y + 2am^3)^2 = \{2a\,(m^2+1)^{3/2}\}^2$

$$(x - 3am^2 - 2a)^2 + (y + 2am^3)^2 = 4a^2(m^2+1)^3$$

or $\quad y^2 + x^2 - 6am^2x - 4ax + 4am^3y - 3a^2m^4 = 0.$

Problem 10: *If C_x, C_y be the chords of curvature parallel to the axes at any point of the curve $y = ae^{x/a}$, prove that*

$$\frac{1}{C_x^2} + \frac{1}{C_y^2} = \frac{1}{2\,a\,C_x}.$$

(Agra 2007; Rohilkhand 07; Purvanchal 07)

Solution: We have, C_x = the chord of curvature parallel to the x-axis

$$= 2\rho \sin\psi = 2\rho\frac{1}{\operatorname{cosec}\psi} = 2\rho\frac{1}{\sqrt{(1+\cot^2\psi)}}$$

$$= 2\rho\frac{\tan\psi}{\sqrt{(1+\tan^2\psi)}} = 2\frac{(1+y_1^2)^{3/2}}{y_2}\cdot\frac{y_1}{\sqrt{(1+y_1^2)}}$$

$$= \frac{2y_1}{y_2}(1+y_1^2). \qquad \text{...(1)}$$

Here $y = ae^{x/a}$; $y_1 = dy/dx = e^{x/a}$;

and $y_2 = d^2y/dx^2 = (1/a)\,e^{x/a}$.

$\therefore$ From (1), we get

$$C_x = \frac{2e^{x/a}}{(1/a)\,e^{x/a}}[1+e^{2x/a}] = 2a\,(1+e^{2x/a}).$$

Again C_y = the chord of curvature parallel to y-axis

$$= 2\rho\cos\psi = \frac{2\rho}{\sec\psi} = \frac{2\rho}{\sqrt{(1+\tan^2\psi)}}$$

$$= 2\frac{(1+y_1^2)^{3/2}}{y_2}\cdot\frac{1}{\sqrt{(1+y_1^2)}}$$

$$= \frac{2(1+y_1^2)}{y_2} = \frac{2}{(1/a)\,e^{x/a}}[1+e^{2x/a}] = \frac{2a}{e^{x/a}}(1+e^{2x/a}).$$

$\therefore$

$$\frac{1}{C_x^2} + \frac{1}{C_y^2} = \frac{1}{4a^2(1+e^{2x/a})^2}[1+e^{2x/a}]$$

$$= \frac{1}{4a^2(1+e^{2x/a})} = \frac{1}{2a\,C_x}.$$

Hints to Objective Type Questions

Multiple Choice Questions

1. See article 7.
2. See article 8.

3. See article 15.
4. See Problem 1(i) of Comprehensive Problems 2.
5. See Example 4.
6. See Example 11.
7. See article 4.
8. See Problem 2(v) of Comprehensive Problems 1.
9. See Example 15.
10. See article 10.
11. See Problem 13(i) of Comprehensive Problems 1.
12. See Corollary of article 4.

Fill in the Blanks

1. See article 4.
2. See Corollary of article 4.
3. See article 5.
4. See article 4
5. See Example 4.
6. See Example 3.
7. See Problem 2(v) of Comprehensive Problems 1.
8. See Example 10.

True or False

1. The circle of curvature of a curve at any point on it is the circle whose centre is the centre of curvature at that point and radius is radius of curvature at that point. By the curvature of a circle, we mean the curvature of the curve which is a circle.
2. See article 4. Corollary.
3. See article 9.
4. See article 10.
5. See article 8.

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Chapter-11

Envelopes, Evolutes and Involutes

Comprehensive Problems 1

Problem 1: *Find the envelope of the straight lines $(x/a)\cos\theta + (y/b)\sin\theta = 1$, the parameter being θ and interpret the result geometrically.* **(Kashi 2012)**

Solution: The equation of the given family of straight lines is

$$(x/a)\cos\theta + (y/b)\sin\theta = 1, \qquad \text{...(1)}$$

the parameter being θ.

Differentiating (1) partially with respect to θ, we get

$$-(x/a)\sin\theta + (y/b)\cos\theta = 0. \qquad \text{...(2)}$$

Eliminating θ between (1) and (2), we get the envelope of the family of straight lines (1). So squaring and adding (1) and (2), we get

$$(x^2/a^2)(\cos^2\theta + \sin^2\theta) + (y^2/b^2)(\sin^2\theta + \cos^2\theta) = 1$$

or
$$x^2/a^2 + y^2/b^2 = 1, \qquad \text{...(3)}$$

which is the required envelope of the family of straight lines (1).

Geometrical interpretation: The equation (3) represents an ellipse whose centre is origin. Whatever may be the value of θ, the straight line (1) always touches the ellipse (3) and the ellipse (3) is also touched at each point by some straight line belonging to the family (1).

Problem 2(i): *Find the envelope of the straight lines $y = m^2 x + 1/m^2$, where m is the parameter.*

Solution: The equation of the given family of straight lines can be written as

$$m^2 y = m^4 x + 1 \quad \text{or} \quad m^4 x - m^2 y + 1 = 0. \qquad \text{...(1)}$$

The equation (1) is a quadratic in m^2. So the required envelope is obtained by equating to zero the discriminant of (1). Hence the required envelope is

$$(-y)^2 - 4x \,.\, 1 = 0 \quad \text{or} \quad y^2 = 4x.$$

Problem 2(ii): *Find the envelope of the straight lines $y = mx + a\sqrt{(1 + m^2)}$, the parameter being m.* **(Kumaun 2000)**

Solution: Given $y = mx + a\sqrt{1 + m^2}$

or $(y - mx)^2 = a^2 (1 + m^2)$

or $y^2 + m^2 x^2 - 24\, mx - a^2 - a^2 m^2 = 0$

or $m^2 (x^2 - a^2) - 2xy\, m + y^2 - a^2 = 0$

which is a quadratic equation in parameter m. Hence its envelope is $B^2 = 4AC$

i.e., $(-2xy)^2 = 4(x^2 - a^2)(y^2 - a^2)$

or $4x^2 y^2 = 4(x^2 y^2 - a^2 y^2 - a^2 x^2 + a^4)$

or $a^2 x^2 + a^2 y^2 = a^4$

or $x^2 + y^2 = a^2$.

Problem 2(iii): *Find the envelope of the straight lines $y = mx + am^3$, the parameter being m.*

Solution: We have $y = mx + am^3$. ...(1)

Differentiating (1) partially w.r.t. 'm', we have

$$0 = x + 3am^2 \quad \text{or} \quad m^2 = -x/(3a). \qquad ...(2)$$

From (1), $y^2 = m^2 (x + am^2)^2$

$= (-x/3a)(x - x/3)^2$, substituting for m^2 from (2)

$= -4x^3/27a$.

Hence $27ay^2 + 4x^3 = 0$, is the required envelope.

Problem 2(iv): *Find the envelope of the family of straight lines $y = mx + am^p$, the parameter being m.*

Solution: The equation of the given family of straight lines is

$$y = mx + am^p, \qquad ...(1)$$

the parameter being m.

Differentiating (1) partially with respect to m, we get

$$0 = x + pam^{p-1}. \qquad ...(2)$$

Eliminating m between (1) and (2), we get the envelope of the family of straight lines (1).

From (2), $m^{p-1} = -\dfrac{x}{pa}$. ...(3)

Now the equation (1) can be written as

$$y = m(x + am^{p-1}).$$

Raising both sides to the power $p - 1$, we get

$$y^{p-1} = m^{p-1}(x + am^{p-1})^{p-1}$$

or $$y^{p-1} = -\frac{x}{pa}\left[x - a \cdot \frac{x}{pa}\right]^{p-1}, \text{ substituting for } m^{p-1} \text{ from (3)}$$

or $$y^{p-1} = -\frac{x}{pa} \cdot \frac{x^{p-1}(p-1)^{p-1}}{p^{p-1}}$$

or $$(p-1)^{p-1} x^p + p^p a y^{p-1} = 0, \text{ which is the required envelope.}$$

Problem 2(v): *Find the envelope of the family of straight lines $x \operatorname{cosec} \theta - y \cot \theta = c$, the parameter being θ.*

Solution: The equation of the given family of straight lines is

$$x \operatorname{cosec} \theta - y \cot \theta = c$$

or $$\frac{x}{\sin \theta} - \frac{y \cos \theta}{\sin \theta} = c$$

or $$x - y \cos \theta = c \sin \theta$$

or $$y \cos \theta + c \sin \theta = x, \qquad \text{...(1)}$$

the parameter being θ.

Differentiating (1) partially with respect to θ, we get

$$-y \sin \theta + c \cos \theta = 0. \qquad \text{...(2)}$$

Squaring and adding (1) and (2), we get

$$y^2 + c^2 = x^2 \quad \text{or} \quad x^2 - y^2 = c^2,$$

which is the required envelope of the given family of straight lines.

Problem 2(vi): *Find the envelope of the family of straight lines $x \cos^3 \alpha + y \sin^3 \alpha = a$, the parameter being α.*

Solution: Proceed as in problem 6 of Comprehensive Problems 1.

The required envelope is $a^2(x^2 + y^2) = x^2 y^2$

or $$\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{a^2}.$$

Problem 3: *Find the envelope of the family of circles*

$$x^2 + y^2 - 2ax \cos \alpha - 2ay \sin \alpha = c^2, \quad (a^2 > c^2)$$

where α is the parameter, and interpret the result.

Solution: The equation of the given family of circles can be written as

$$2ax \cos \alpha + 2ay \sin \alpha = x^2 + y^2 - c^2. \qquad \text{...(1)}$$

[Note that we have brought the terms containing $\cos \alpha$ and $\sin \alpha$ to one side and the rest of the terms to the other side].

Differentiating (1) partially with respect to α, we get

$$-2ax \sin \alpha + 2ay \cos \alpha = 0. \qquad ...(2)$$

Squaring and adding (1) and (2), we get

$$4a^2 x^2 + 4a^2 y^2 = (x^2 + y^2 - c^2)^2$$

or $$(x^2 + y^2 - c^2)^2 = 4a^2 (x^2 + y^2), \qquad ...(3)$$

which is the required envelope.

Interpretation: The equation (3) can be written as

$$(x^2 + y^2)^2 - 2(2a^2 + c^2)(x^2 + y^2) + c^4 = 0.$$

Solving it as a quadratic in $(x^2 + y^2)$, we get

$$x^2 + y^2 = \frac{2(2a^2 + c^2) \pm \sqrt{\{4(2a^2 + c^2)^2 - 4c^4\}}}{2}$$

$$= 2a^2 + c^2 \pm 2a\sqrt{(c^2 + a^2)}$$

$$= (a^2 + c^2) \pm 2a\sqrt{(a^2 + c^2)} + a^2 = [\sqrt{(a^2 + c^2)} \pm a]^2.$$

Therefore the required envelope consists of the two circles

$$x^2 + y^2 = [\sqrt{(a^2 + c^2)} + a]^2$$

and $$x^2 + y^2 = [\sqrt{(a^2 + c^2)} - a]^2.$$

These are the circles with centre at origin and radii $\sqrt{(a^2 + c^2)} \pm a$.

Problem 4: *Find the envelope of the following systems of circles :*

(i) $(x - \alpha)^2 + y^2 = 4\alpha, \alpha$ *being the parameter,*

(ii) $(x - \alpha)^2 + (y - \alpha)^2 = 2\alpha$, *where* α *is the parameter.*

(Lucknow 2009; Rohilkhand 13)

(iii) $(x - c)^2 + y^2 = R^2$, *where c is the parameter.*

Solution: **(i)** We have $(x - \alpha)^2 + y^2 = 4\alpha$ or $x^2 - 2\alpha x + \alpha^2 + y^2 = 4\alpha$

or $$\alpha^2 - 2\alpha(x + 2) + (x^2 + y^2) = 0.$$

This equation is a quadratic in α. Hence the required envelope is

$$4(x + 2)^2 - 4.(x^2 + y^2) = 0$$

or $$x^2 + 4x + 4 - x^2 - y^2 = 0 \text{ or } y^2 - 4x - 4 = 0.$$

(ii) We have $(x - \alpha)^2 + (y - \alpha)^2 = 2\alpha$

or $$x^2 - 2\alpha x + \alpha^2 + y^2 - 2\alpha y + \alpha^2 = 2\alpha$$

or $$2\alpha^2 - 2(x + y + 1)\alpha + (x^2 + y^2) = 0.$$

This equation is a quadratic in α. Hence the required envelope is

$$4(x + y + 1)^2 - 4.2(x^2 + y^2) = 0$$

or $$(x + y + 1)^2 = 2(x^2 + y^2).$$

(iii) The given family of circles is

$$(x-c)^2 + y^2 = R^2 \qquad \text{...(1)}$$

where c is the parameter.

Differentiating (1) partially with respect 'c', we get

$$-2(x-c)=0 \quad \text{or} \quad x-c=0. \qquad \text{...(2)}$$

Eliminating c between (1) and (2), we get the envelope of the family (1).

Putting $x-c=0$ in (1), we get $y^2 = R^2$ as the required envelope of the family (1).

Hence the envelope of the family (1) consists of the straight lines $y = \pm R$.

Problem 4(iv): *Find the envelope of the following family of curves :*
$y^2 = m^2 (x-m)$, the parameter being m.

Solution: Here $\quad y^2 = m^2 (x-m) = m^2 x - m^3. \qquad \text{...(1)}$

Differentiating (1) partially w.r.t. the parameter 't', we get

$$0 = 2mx - 3m^2 \quad \text{or} \quad m = 2x/3.$$

Substituting this value of t in (1), we get

$$y^2 = (2x/3)^2 (x - 2x/3) = 4x^3/27$$

or $\qquad 4x^3 = 27y^2$, which is the required envelope.

Problem 4(v): *Find the envelope of the family of parabolas $tx^3 + t^2 y = a$, the parameter being t.*

Solution: Given $t^2 y + tx^3 - a = 0$

which is a quadratic equation in parameter t hence its envelope is $B^2 = 4AC$

$$x^6 = 4.y.(-a) \quad \text{or} \quad x^6 + 4ay = 0$$

Problem 4(vi): *Obtain the envelope of the family of curves given by*

$$\frac{x^2}{\alpha^2} + \frac{y^2}{k^2 - \alpha^2} = 1,$$

where α is the parameter. **(Kanpur 2008)**

Solution: The given equation is

$$(x^2/\alpha^2) + y^2/(k^2 - \alpha^2) = 1$$

or $\qquad x^2k^2 - \alpha^2 x^2 + y^2\alpha^2 = k^2\alpha^2 - \alpha^4,$

or $\qquad \alpha^4 + \alpha^2 (y^2 - x^2 - k^2) + x^2k^2 = 0,$

which is a quadratic equation in the parameter α^2.

$\therefore$ The envelope is given by $(y^2 - x^2 - k^2)^2 = 4k^2x^2$

or $\qquad y^2 - x^2 - k^2 = \pm 2kx$

or $\qquad y^2 = x^2 + k^2 \pm 2kx \qquad$ or $\qquad y^2 = (x \pm k)^2$

or $\qquad y = \pm (x \pm k)$.

Hence the required envelope consists of the four straight lines $x \pm y = \pm k$.

Problem 5: *Find the envelope of the family of curves*

$$(a^2 / x) \cos \theta - (b^2 / y) \sin \theta = c^2 / a,$$

θ being the parameter.

Solution: The equation of the given family of curves is

$$(a^2 / x) \cos \theta - (b^2 / y) \sin \theta = c^2 / a, \qquad ...(1)$$

the parameter being θ.

Differentiating (1) partially with respect to θ, we get

$$-(a^2 / x) \sin \theta - (b^2 / y) \cos \theta = 0. \qquad ...(2)$$

Squaring and adding (1) and (2), we get

$$\frac{a^4}{x^2} + \frac{b^4}{y^2} = \frac{c^4}{a^2},$$

which is the required envelope of the family of curves (1).

Problem 6: *Find the envelope of the family of straight lines*

$$x \cos^n \theta + y \sin^n \theta = a, \text{ for different values of } \theta.$$

Solution: The equation of the given family of straight lines is

$$x \cos^n \theta + y \sin^n \theta = a, \qquad ...(1)$$

the parameter being θ.

Differentiating (1) partially with respect to θ, we get

$$-nx \cos^{n-1} \theta \sin\theta + ny \sin^{n-1} \theta \cos \theta = 0$$

or $\qquad \dfrac{\cos^{n-1} \theta \sin\theta}{\sin^{n-1} \theta \cos \theta} = \dfrac{y}{x} \qquad$ or $\qquad \cot^{n-2} \theta = \dfrac{y}{x}$

or $\qquad \dfrac{1}{\cot^{n-2} \theta} = \dfrac{x}{y} \qquad$ or $\qquad \tan^{2-n} \theta = \dfrac{x}{y}$

or $\qquad \tan\theta = \dfrac{x^{1/(2-n)}}{y^{1/2(2-n)}}$.

$\therefore \qquad \cos \theta = \dfrac{x^{1/(2-n)}}{[x^{2/(2-n)} + y^{2/(2-n)}]^{1/2}}$

and $\qquad \sin\theta = \dfrac{y^{1/(2-n)}}{[x^{2/(2-n)} + y^{2/(2-n)}]^{1/2}}$.

Putting the values of $\cos \theta$ and $\sin \theta$ in (1), the required envelope of the family of straight lines (1) is

$$x \cdot \frac{x^{n/(2-n)}}{[x^{2/(2-n)} + y^{2/(2-n)}]} + y \cdot \frac{y^{n/(2-n)}}{[x^{2/(2-n)} + y^{2/(2-n)}]^{n/2}} = a$$

or $$\frac{x^{2/(2-n)} + y^{2/(2-n)}}{[x^{2/(2-n)} + y^{2/(2-n)}]^{n/2}} = a$$

or $$[x^{2/(2-n)} + y^{2/(2-n)}]^{1-(n/2)} = a$$

$$[x^{2/(2-n)} + y^{2/(2-n)}]^{(2-n)/2} = a$$

or $$x^{2/(2-n)} + y^{2/(2-n)} = a^{2/(2-n)},$$

raising both sides to the power $2/(2-n)$.

Problem 7: *Find the envelope of the ellipse*

$$x = a \sin(\theta - \alpha), \ y = b \cos\theta,$$

where α *is the parameter.* **(Kanpur 2009)**

Solution: The given ellipse is

$$x = a\sin(\theta - \alpha), \ y = b\cos\theta, \text{ where } \alpha \text{ is the parameter}$$

or $$x = a(\sin\theta\cos\alpha - \cos\theta\sin\alpha), \ y = b\cos\theta$$

or $$x = a\{\sqrt{(1 - y^2/b^2)}\cos\alpha - (y/b)\sin\alpha\}, \text{ eliminating } \theta. \quad \text{...(1)}$$

Differentiating (1) partially w.r.t. the parameter α, we have

$$0 = a\{-\sqrt{(1 - y^2/b^2)}\sin\alpha - (y/b)\cos\alpha\}. \quad \text{...(2)}$$

Squaring and adding (1) and (2), we get

$$x^2 = a^2\{1 - y^2/b^2 + y^2/b^2\},$$

or $$x^2 = a^2 \quad \text{or} \quad x = \pm a \text{ as the required envelope.}$$

Problem 8: *Projectiles are fired from a gun with a constant initial velocity* V_0. *Supposing the gun can be given any elevation and is kept always in the same vertical plane, what is the envelope of the possible trajectories, assuming their equation to be*

$$y = x \tan\alpha - \frac{gx^2}{2V_0^2 \cos^2\alpha}?$$

(Lucknow 2007, 11)

Solution: The given equation is

$$y = x\tan\alpha - gx^2/(2V_0^2\cos^2\alpha),$$

where α is parameter and V_0 is constant

or $$y = x\tan\alpha - (\tfrac{1}{2}gx^2/V_0^2)\sec^2\alpha$$

or $$y = x\tan\alpha - (\tfrac{1}{2}gx^2/V_0^2)(1 + \tan^2\alpha)$$

or $$(\tfrac{1}{2}gx^2/V_0^2)\tan^2\alpha - x\tan\alpha + (y + \tfrac{1}{2}gx^2/V_0^2) = 0,$$

which is a quadratic equation in $\tan\alpha$.

∴ The envelope is given by

$$x^2 - 4\left(\frac{1}{2}gx^2 / V_0^2\right)\left(y + \frac{1}{2}gx^2 / V_0^2\right) = 0$$

or $$1 - (2g/V_0^2)\left(y + \frac{1}{2}gx^2 / V_0^2\right) = 0$$

or $$V_0^2 / 2g = y + \frac{1}{2}gx^2 / V_0^2, \text{ multiplying by } V_0^2 / 2g$$

or $$V_0^4 = g^2 x^2 + 2V_0^2\, gy, \text{ which is the required envelope.}$$

Problem 9: *Find the envelope of the straight lines*

$$x \cos \alpha + y \sin \alpha = l \sin \alpha \cos \alpha,$$

where the parameter is the angle α *. Give the geometrical interpretation.*

(Avadh 2007, 12; Rohilkhand 2012)

Solution: Here $x \cos \alpha + y \sin \alpha = l \sin \alpha \cos \alpha$,

or $$x \operatorname{cosec} \alpha + y \sec \alpha = l. \qquad \textbf{(Note)} \quad ...(1)$$

Differentiating (1) partially w.r.t. the parameter α, we have

$$x(-\operatorname{cosec} \alpha \cot \alpha) + y \sec \alpha \tan \alpha = 0$$

or $$\tan \alpha = x^{1/3} / y^{1/3}.$$

∴ $$\operatorname{cosec} \alpha = \sqrt{(1 + \cot^2 \alpha)} = \sqrt{\{1 + (y^{2/3} / x^{2/3})\}}$$

$$= \sqrt{(x^{2/3} + y^{2/3})} / x^{1/3}, \qquad ...(2)$$

and $$\sec \alpha = \sqrt{(1 + \tan^2 \alpha)} = \sqrt{(x^{2/3} + y^{2/3})} / y^{1/3}. \qquad ...(3)$$

Eliminating α between (1), (2) and (3), we have

$$\frac{x(x^{2/3} + y^{2/3})^{1/2}}{x^{1/3}} + y \cdot \frac{(x^{2/3} + y^{2/3})^{1/2}}{y^{1/3}} = l$$

or $$(x^{2/3} + y^{2/3})^{(3/2)} = l, \quad \text{or} \quad x^{2/3} + y^{2/3} = l^{2/3},$$

which is the required envelope.

Geometrical Interpretation: The equation (1) may be written as $x/(l \sin \alpha) + y/(l \cos \alpha) = 1$, which shows that the intercepts on the axes made by the line are $l \sin \alpha$ and $l \cos \alpha$. Hence the length of the line between the axes is $\sqrt{(l^2 \sin^2 \alpha + l^2 \cos^2 \alpha)}$ *i.e.*, l which is constant. Hence the result may be interpreted geometrically as follows :

If a straight line of constant length l slides between the axes, the envelope of the straight line is the astroid

$$x^{2/3} + y^{2/3} = l^{2/3}.$$

Problem 10: *Find the envelope of the family of the straight lines*

$$x \cos m\theta + y \sin m\theta = a(\cos n\theta)^{m/n},$$

where θ *is the parameter.*

Solution: We have $x \cos m\theta + y \sin m\theta = a(\cos n\theta)^{m/n}$. ...(1)

Differentiating (1), partially w.r.t. θ, we have

$$-mx \sin m\theta + my \cos m\theta = a\,(m/n)\,(\cos n\theta)^{(m/n)-1}\,(-n \sin n\theta)$$

or $$-x \sin m\theta + y \cos m\theta = -a\,(\cos n\theta)^{(m/n)-1}\,(\sin n\theta). \qquad ...(2)$$

Squaring and adding (1) and (2), we have

$$(\cos^2 m\theta + \sin^2 m\theta)\,(x^2 + y^2)$$
$$= a^2\,(\cos n\theta)^{2\,(m/n-1)}\,(\cos^2 n\theta + \sin^2 n\theta)$$

or $$x^2 + y^2 = a^2\,(\cos n\theta)^{2(m-n)/n}. \qquad ...(3)$$

Multiplying (1) by $\sin n\theta$ and (2) by $\cos n\theta$ and adding, we get

$$-x\,(\sin m\theta \cos n\theta - \cos m\theta \sin n\theta)$$
$$+ y\,(\cos m\theta \cos n\theta + \sin m\theta \sin n\theta) = 0$$

or $$-x \sin (m-n)\,\theta + y \cos (m-n)\,\theta = 0$$

or $$\tan (m-n)\,\theta = y/x \qquad ...(4)$$

Now let (r, ϕ) denote the polar coordinates of the point (x, y) so that $x = r \cos \phi$ and $y = r \sin \phi$. Then from (4), we have $\tan (m-n)\,\theta = \tan \phi$ or $(m-n)\,\theta = \phi$ or $\theta = \phi/(m-n)$. Substituting the value of θ in (3), we get

$$(r^2 \cos^2 \phi + r^2 \sin^2 \phi) = a^2\,[\cos \{n\,\phi/(m-n)\}]^{2\,(m-n)/n}$$

or $$r^2 = a^2\,[\cos \{n\,\phi/(m-n)\}]^{2\,(m-n)/n}$$

or $$r^{n/(m-n)} = a^{n/(m-n)} \cos \{n\,\phi/(m-n)\},$$

which is the required envelope where (r, ϕ) are the polar coordinates of the point (x, y).

Problem 11: *Show that the radius of curvature of the envelope of the family of lines* $x \cos \alpha + y \sin \alpha = f(\alpha)$ *is* $f(\alpha) + f''(\alpha)$.

Solution: The given equation of the family of lines is

$$x \cos \alpha + y \sin \alpha = f(\alpha), \qquad ...(1)$$

where α is the parameter.

Differentiating (1) partially w.r.t. α, we have

$$-x \sin \alpha + y \cos \alpha = f'(\alpha). \qquad ...(2)$$

Squaring (1) and (2) and adding, we get

$$x^2 + y^2 = \{f(\alpha)\}^2 + \{f'(\alpha)\}^2,$$

or $$r^2 = \{f(\alpha)\}^2 + \{f'(\alpha)\}^2, \text{ changing to polars.} \qquad ...(3)$$

Now the envelope of the family of lines (1) touches each member of the family (1). Therefore (1) is a tangent to the envelope of (1).

Hence if p be the length of the perpendicular from the pole (*i.e.*, origin) upon the tangent (1) to the envelope of (1), then

$$p = f(\alpha). \qquad ...(4)$$

Therefore (3) may be regarded as the pedal equation of the envelope where α is given by (4).

Differentiating (4) and (3) w.r.t. 'p', we have

$$1 = f'(\alpha)\,(d\alpha/dp) \qquad \text{...(5)}$$

and $$2r\,(dr/dp) = 2f(\alpha)\,f'(\alpha)\,.(d\alpha/dp) + 2f'(\alpha)\,f''(\alpha).\,(d\alpha/dp).$$

$\therefore$ ρ (*i.e.*, the radius of curvature of the envelope)

$$= r\,\frac{dr}{dp} = f(\alpha)\,f'(\alpha)\left(\frac{d\alpha}{dp}\right) + f'(\alpha)\,f''(\alpha)\left(\frac{d\alpha}{dp}\right)$$

$$= \{f(\alpha) + f''(\alpha)\}\,\{f'(\alpha)\,(d\alpha/dp)\}$$

$$= f(\alpha) + f''(\alpha), \text{ from (5).}$$

Alternative method: To find the radius of curvature of the envelope of the given family of lines we can also proceed as follows :

The equations (1) and (2) are parametric equations of the envelope of the family of lines (1), the parameter being α.

First we shall solve (1) and (2) for x and y to express x and y as functions of the parameter α.

Multiplying (1) by $\cos\alpha$, (2) by $\sin\alpha$ and subtracting, we get

$$x = f(\alpha)\cos\alpha - f'(\alpha)\sin\alpha. \qquad \text{...(6)}$$

Again multiplying (1) by $\sin\alpha$, (2) by $\cos\alpha$ and adding, we get

$$y = f(\alpha)\sin\alpha + f'(\alpha)\cos\alpha. \qquad \text{...(7)}$$

Now from (6), $$\frac{dx}{d\alpha} = f'(\alpha)\cos\alpha - f(\alpha)\sin\alpha - f''(\alpha)\sin\alpha - f'(\alpha)\cos\alpha$$

$$= -[f(\alpha) + f''(\alpha)]\sin\alpha \qquad \text{...(8)}$$

and from (7), $$\frac{dy}{d\alpha} = f'(\alpha)\sin\alpha + f(\alpha)\cos\alpha + f''(\alpha)\cos\alpha - f'(\alpha)\sin\alpha$$

$$= [f(\alpha) + f''(\alpha)]\cos\alpha. \qquad \text{...(9)}$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/d\alpha}{dx/d\alpha} = -\cot\alpha.$$

$\therefore$ $$\frac{d^2y}{dx^2} = \frac{d}{dx}(-\cot\alpha) = \frac{d}{d\alpha}(-\cot\alpha)\cdot\frac{d\alpha}{dx}$$

$$= \operatorname{cosec}^2\alpha\cdot\frac{-1}{[f(\alpha) + f''(\alpha)]\sin\alpha} = \frac{-\operatorname{cosec}^3\alpha}{f(\alpha) + f''(\alpha)}.$$

Now the required radius of curvature ρ

$$= \frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = [1 + \cot^2\alpha]^{3/2}\cdot\frac{f(\alpha) + f''(\alpha)}{-\operatorname{cosec}^3\alpha}$$

$$= -[f(\alpha) + f''(\alpha)].$$

Neglecting the negative sign, $\rho = f(\alpha) + f''(\alpha)$.

Problem 12: *If* $x^{2/3} + y^{2/3} = k^{2/3}$ *is the envelope of the lines*

$$x/a + y/b = 1,$$

then find the necessary relation between a, b *and* k.

Solution: The equation of the given family of straight lines is

$$\frac{x}{a} + \frac{y}{b} = 1, \qquad \text{...(1)}$$

where a and b are parameters.

If the envelope of the family of straight lines (1) is the curve

$$x^{2/3} + y^{2/3} = k^{2/3}, \qquad \text{...(2)}$$

then each member of the family of straight lines (1) touches the curve (2).

So to find the necessary relation between a, b and k we have to find the condition that the straight line (1) touches the curve (2).

The coordinates (x, y) of any point on the curve (2) may be taken as

$$x = k \cos^3 \theta, \; y = k \sin^3 \theta, \text{ where } \theta \text{ is a parameter.}$$

We have $\quad dx/d\theta = -3k \cos^2 \theta \sin \theta$

and $\quad dy/d\theta = 3k \sin^2 \theta \cos \theta.$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3k \sin^2 \theta \cos \theta}{-3k \cos^2 \theta \sin \theta} = -\frac{\sin \theta}{\cos \theta}$$

= slope of the tangent to the curve (2) at the point $(k \cos^3 \theta, k \sin^3 \theta)$.

$\therefore$ The equation of the tangent to the curve (2) at the point $(k \cos^3 \theta, k \sin^3 \theta)$ is

$$y - k \sin^3 \theta = -\frac{\sin \theta}{\cos \theta}(x - k \cos^3 \theta)$$

or $\quad y \cos \theta - k \sin^3 \theta \cos \theta = -x \sin \theta + k \sin \theta \cos^3 \theta$

or $\quad x \sin \theta + y \cos \theta = k \sin \theta \cos \theta$

or $$\frac{x}{k \cos \theta} + \frac{y}{k \sin \theta} = 1. \qquad \text{...(3)}$$

Now suppose the straight line (1) touches the curve (2) at the point $(k \cos^3 \theta, k \sin^3 \theta)$. Then (1) and (3) are the equations of the same straight line. So comparing the coefficients of like terms in (1) and (3), we get

$$a = k \cos \theta, b = k \sin \theta.$$

Squaring and adding, we get

$$a^2 + b^2 = k^2,$$

which is the required relation between a, b and k so that the line (1) touches the curve (2).

Problem 13: *Find the envelope of the family of curves $x^2 \sin\alpha + y^2 \cos\alpha = a^2$, where α is the parameter.*

Solution: The equation of the given family of curves is

$$x^2 \sin\alpha + y^2 \cos\alpha = a^2, \quad \ldots(1)$$

where α is the parameter.

Differentiating (1) partially with respect to 'α', we get

$$x^2 \cos\alpha - y^2 \sin\alpha = 0. \quad \ldots(2)$$

Squaring and adding (1) and (2), we get

$$x^4 + y^4 = a^4$$

which is the required envelope of the family of curves (1).

Problem 14: *Find the envelope of the family of curves $(y-c)^2 - \frac{2}{3}(x-c)^3 = 0$, where c is the parameter.*

Solution: The equation of the given family of curves is

$$(y-c)^2 - \frac{2}{3}(x-c)^3 = 0, \quad \ldots(1)$$

where c is the parameter.

Differentiating (1) partially with respect to 'c', we get

$$2(y-c)(-1) - \frac{2}{3}\cdot 3(x-c)^2(-1) = 0$$

or $$(y-c) - (x-c)^2 = 0$$

or $$y - c = (x-c)^2. \quad \ldots(2)$$

Eliminating c between (1) and (2), we get the envelope of the family (1).

Putting $y - c = (x-c)^2$ in (1), we get

$$(x-c)^4 - \frac{2}{3}(x-c)^3 = 0$$

or $$(x-c)^3 \left[x - c - \frac{2}{3}\right] = 0.$$

$\therefore$ $$x - c = 0 \quad \text{or} \quad x - c = \frac{2}{3}\cdot$$

Putting $x - c = 0$ in (2), we get

$$y - c = 0.$$

Eliminating c between $x - c = 0$ and $y - c = 0$, we get $x = y$.

Again putting $x - c = \frac{2}{3}$ in (2), we get $y - c = \frac{4}{9}$.

Eliminating c between $x - c = \frac{2}{3}$ and $y - c = \frac{4}{9}$, we get

$$x - y = \frac{2}{3} - \frac{4}{9} = \frac{2}{9}.$$

Hence the required envelope of the family of curves (1) consists of the straight lines

$$x - y = 0 \quad \text{and} \quad x - y = \frac{2}{9}.$$

Comprehensive Problems 2

Problem 1: *Find the envelope of the circles drawn upon the radii vectors of the ellipse $x^2/a^2 + y^2/b^2 = 1$ as diameter.*

Solution: Any point on the given ellipse is $(a \cos \theta, b \sin \theta)$.

So the equation of the circle on the radius vector to this point as diameter is

$$(x - 0)(x - a \cos \theta) + (y - 0)(y - b \sin \theta) = 0$$

or $$x^2 + y^2 - ax \cos \theta - by \sin \theta = 0$$

or $$ax \cos \theta + by \sin \theta = x^2 + y^2. \qquad \text{...(1)}$$

We have to find the envelope of (1) where θ is the parameter.

Differentiating (1) partially w.r.t. θ, we get

$$- ax \sin \theta + by \cos \theta = 0. \qquad \text{...(2)}$$

Squaring and adding (1) and (2), we get

$$a^2 x^2 + b^2 y^2 = (x^2 + y^2)^2 \text{ as the required envelope.}$$

Problem 2: *Show that the envelope of the circles whose centres lie on the parabola $y^2 = 4ax$ and which pass through its vertex is the cissoid $y^2 (2a + x) + x^3 = 0$.*

Solution: Let any point P on the parabola $y^2 = 4ax$ be $(at^2, 2at)$.

$\therefore$ The centre of the circle is $(at^2, 2at)$ and it passes through origin (0, 0).

Equation of any circle with centre $(-g, -f)$ and passing through the origin is

$$x^2 + y^2 + 2gx + 2fy = 0.$$

$\therefore$ Equation of the circle whose centre is $(at^2, 2at)$, and high passes through the origin is

$$x^2 + y^2 - 2axt^2 - 4ayt = 0$$

or $$2axt^2 + 4ayt - (x^2 + y^2) = 0$$

This equation is quadratic in the parameter t. Hence the required envelope is

$$(4ay)^2 + 4.2ax.(x^2 + y^2) = 0$$

or $$2ay^2 + x^3 + xy^2 = 0$$

or $$x^3 + y^2 (x + 2a) = 0.$$

Problem 3: *Show that the envelope of the circles whose centres lie on the rectangular hyperbola $x^2 - y^2 = a^2$ and which pass through the origin is the lemniscate $r^2 = 4a^2 \cos 2\theta$.*

Solution: Any point on the rectangular hyperbola $x^2 - y^2 = a^2$ is $(a \cosh \theta, a \sinh \theta)$. Its distance from the origin is

$$\sqrt{(a^2 \cosh^2 \theta + a^2 \sinh^2 \theta)}.$$

Therefore the equation of the circle whose centre is the point $(a \cosh \theta, a \sinh \theta)$ and which passes through the origin is

$$(x - a \cosh \theta)^2 + (y - a \sinh \theta)^2 = a^2 \cosh^2 \theta + a^2 \sinh^2 \theta$$

or $$x^2 + y^2 - 2ax \cosh \theta - 2ay \sinh \theta = 0$$

or $$2ax \cosh \theta + 2ay \sinh \theta = x^2 + y^2. \quad \text{...(1)}$$

We are to find the envelope of the family of circles (1), where θ is the parameter. Differentiating (1) partially with respect to θ, we get

$$2ax \sinh \theta + 2ay \cosh \theta = 0. \quad \text{...(2)}$$

Squaring and subtracting (1) and (2), we get

$$4a^2x^2 (\cosh^2 \theta - \sinh^2 \theta) + 4a^2 y^2 (\sinh^2 \theta - \cosh^2 \theta) = (x^2 + y^2)^2$$

or $$4a^2x^2 - 4a^2 y^2 = (x^2 + y^2)^2 \qquad [\because \cosh^2 \theta - \sinh^2 \theta = 1]$$

or $$4a^2 (r^2 \cos^2 \theta - r^2 \sin^2 \theta) = (r^2)^2, \text{ changing to polars}$$

or $$4a^2r^2 \cos 2\theta = r^4$$

or $$r^2 = 4a^2 \cos 2\theta, \text{ which is the required envelope.}$$

Problem 4: *Show that the envelope of circles described on the central radii of a rectangular hyperbola is a lemniscate $r^2 = a^2 \cos 2\theta$.*

Solution: Referred to the centre as origin, let the equation of the rectangular hyperbola be $x^2 - y^2 = a^2$.

Any point on it is $(a \cosh \phi, a \sinh \phi)$.

Then the equation of the circle having $(0, 0)$ and $(a \cosh \phi, a \sinh \phi)$ as the ends of the diameter is

$$(x - 0)(x - a \cosh \phi) + (y - 0)(y - a \sinh \phi) = 0$$

or $$ax \cosh \phi + ay \sinh \phi = x^2 + y^2. \quad \text{...(1)}$$

Differentiating (1) partially w.r.t. ϕ, we have

$$ax \sinh \phi + ay \cosh \phi = 0. \quad \text{...(2)}$$

Squaring (2) and subtracting from the square of (1), we get

$$a^2x^2 (\cosh^2 \phi - \sinh^2 \phi) + a^2 y^2 (\sinh^2 \phi - \cosh^2 \phi) = (x^2 + y^2)^2$$

or $$a^2 (x^2 - y^2) = (x^2 + y^2)^2, \qquad [\because \cosh^2 \phi - \sinh^2 \phi = 1]$$

or $$a^2 (r^2 \cos^2 \theta - r^2 \sin^2 \theta) = (r^2)^2$$ [Changing to polars]

or $$a^2 r^2 \cos 2\theta = r^4$$

or $$r^2 = a^2 \cos 2\theta,$$

which is the required envelope.

Problem 5: *Show that the envelope of the polars of points on the ellipse $x^2/h^2 + y^2/k^2 = 1$ with respect to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is $\dfrac{h^2 x^2}{a^4} + \dfrac{k^2 y^2}{b^4} = 1.$* **(Bundelkhand 2011)**

Solution: Any point on the ellipse $x^2/h^2 + y^2/k^2 = 1$ is

$$(h \cos \theta, k \sin \theta),$$

where θ is the parameter. And polar of this point with respect to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is

$$\frac{xh \cos \theta}{a^2} + \frac{yk \sin \theta}{b^2} = 1. \quad \text{...(1)}$$

We have to find the envelope of the family (1) where θ is the parameter.

Differentiating (1) partially w.r.t. θ, we have

$$-(xh/a^2) \sin \theta + (yk/b^2) \cos \theta = 0. \quad \text{...(2)}$$

Squaring and adding (1) and (2), we have

$$\left(\frac{xh}{a^2}\right)^2 + \left(\frac{yk}{b^2}\right)^2 = 1$$

or $$\frac{x^2 h^2}{a^4} + \frac{y^2 k^2}{b^4} = 1,$$

which is the required envelope.

Problem 6: *Show that the envelope of the straight line joining the extremities of a pair of conjugate diameters of the ellipse*

$$x^2/a^2 + y^2/b^2 = 1 \textit{ is the ellipse } x^2/a^2 + y^2/b^2 = \frac{1}{2}.$$

Solution: Let the equation of the given ellipse be

$$x^2/a^2 + y^2/b^2 = 1$$

Let the extremity of a diameter be $P(a \cos \theta, b \sin \theta)$. Then the extremity of the conjugate diameter is

$$Q\left\{a \cos\left(\frac{1}{2}\pi + \theta\right), b\left(\frac{1}{2}\pi + \theta\right)\right\} \textit{ i.e.}, (-a\sin\theta, b \cos\theta).$$

$\therefore$ Equation of the straight line PQ joining the extremities P and Q is

$$y - b \sin \theta = \frac{b (\sin \theta - \cos \theta)}{a(\cos\theta + \sin \theta)} (x - a \cos \theta)$$

or $$ay(\cos\theta + \sin\theta) - ab(\sin\theta\cos\theta + \sin^2\theta)$$

$$= bx(\sin\theta - \cos\theta) - ab(\sin\theta\cos\theta - \cos^2\theta)$$

or $$(ay + bx)\cos\theta + (ay + bx)\sin\theta = ab(\cos^2\theta + \sin^2\theta) = ab \qquad ...(1)$$

Differentiating (1) partially w.r.t. θ, we have

$$-(ay + bx)\sin\theta + (ay - bx)\cos\theta = 0 \qquad ...(2)$$

Squaring and adding (1) and (2), we get

$$(ay + bx)^2 + (ay - bx)^2 = a^2 b^2$$

or $$\frac{x^2}{a^2/2} + \frac{y^2}{b^2/2} = 1,$$

which is the required envelope and is a similar ellipse.

Problem 7: *Find the envelopes of circles described on the radii vectors of the following curves as diameters :*

(i) $l/r = 1 + e\cos\theta$ *(ii)* $r^3 = a^3\cos 3\theta$

(iii) $r\cos^n(\theta/n) = a.$

Solution: **(i)** For figure see Example 3. Let $P(OP, \alpha)$ be any point on the curve $l/r = 1 + e\cos\theta$. Then

$$l/OP = 1 + e\cos\alpha. \qquad ...(1)$$

Let (r, θ) be any point Q on the circle described on OP as diameter. Now from the figure we have the equation of the circle as

$$r = OP\cos(\theta - \alpha)$$

or $$r = \{l/(1 + e\cos\alpha)\}\cos(\theta - \alpha), \qquad \text{[From (1)]}$$

or $$r\{1 + e\cos\alpha\} = l(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$

or $$r(l\cos\theta - re)\cos\alpha + l\sin\theta\sin\alpha = r. \qquad ...(3)$$

We have to find the envelope of the family of circles (3), where α is the parameter. Partially differentiating (3) w.r.t. α, we have

$$-(l\cos\theta - re)\sin\alpha + l\sin\theta\cos\alpha = 0. \qquad ...(4)$$

Squaring (3) and (4) and adding, we have

$$(l\cos\theta - re)^2 + l^2\sin^2\theta = r^2$$

or $$l^2\cos^2\theta - 2ler\cos\theta + r^2e^2 + l^2\sin^2\theta = r^2$$

or $$l^2 - 2ler\cos\theta + r^2e^2 = r^2,$$ which is the required envelope.

(ii) The equation of the given curve is

$$r^3 = a^3\cos 3\theta. \qquad ...(1)$$

Let (r_1, θ_1) be any point P on (1).

Then $$r_1^3 = a^3\cos 3\theta_1. \qquad ...(2)$$

Equation of the circle drawn on OP as diameter is

$$r = r_1 \cos(\theta - \theta_1).$$ [Proceed as in part (i)]

Substituting for r_1 from (2), this equation becomes

$$r = a(\cos 3\theta_1)^{1/3} \cos(\theta - \theta_1), \quad ...(3)$$

where θ_1 is the parameter.

Taking log, $\log r = \log a + \frac{1}{3} \log \cos 3\theta_1 + \log \cos(\theta - \theta_1)$.

Differentiating partially w.r.t. θ_1, we have

$$0 = 0 + \frac{1}{3}(-3 \sin 3\theta_1 / \cos 3\theta_1) + \sin(\theta - \theta_1)/\cos(\theta - \theta_1)$$

or $\tan 3\theta_1 = \tan(\theta - \theta_1)$ *i.e.*, $3\theta_1 = \theta - \theta_1$ or $\theta_1 = \theta/4$.

Substituting this value of θ_1 in (3), we have

$$r = a\{\cos(3\theta/4)\}^{1/3} \cos(\theta - \theta/4) = a\{\cos(3\theta/4)\}^{4/3}$$

or $r^3 = a^3 \cos^4(3\theta/4)$, which is the required envelope.

(iii) The given equation is $r \cos^n(\theta/n) = a$. ...(1)

Let (r_1, θ_1) be any point P on (1); then

$$r_1 \cos^n(\theta_1/n) = a. \quad ...(2)$$

Equation of the circle drawn on OP as diameter is

$$r = r_1 \cos(\theta - \theta_1).$$

Substituting for r_1 from (2), this equation becomes

$$r = a \cos(\theta - \theta_1)/\cos^n(\theta_1/n), \quad ...(3)$$

where θ_1 is the parameter.

Taking log,

$$\log r = \log a + \log \cos(\theta - \theta_1) - n \log \cos(\theta_1/n).$$

Differentiating partially w.r.t. θ_1, we have

$$0 = 0 + \frac{\sin(\theta - \theta_1)}{\cos(\theta - \theta_1)} - n\left\{-(1/n)\frac{\sin(\theta_1/n)}{\cos(\theta_1/n)}\right\}$$

or $\tan(\theta - \theta_1) + \tan(\theta_1/n) = 0$

or $\tan(\theta - \theta_1) = -\tan(\theta_1/n) = \tan(-\theta_1/n)$

or $\theta - \theta_1 = -\theta_1/n$

or $\theta = \theta_1(1 - 1/n) = \theta_1(n-1)/n$

or $\theta_1 = n\theta/(n-1)$.

Substituting this value of θ_1 in (3), we have

$$r = \frac{n \cos\{\theta - n\theta/(n-1)\}}{\cos^n\{\theta/(n-1)\}} = \frac{a \cos\{-\theta/(n-1)\}}{\cos^n\{\theta/(n-1)\}} = \frac{a}{\cos^{n-1}\{\theta/(n-1)\}}$$

or $r \cos^{n-1}\{\theta/(n-1)\} = a$, which is the required envelope.

Problem 8: *Find the envelopes of the straight lines drawn at right angles to the radii vectors of the following curves through their extremities .*

(i) $r = ae^{\theta \cot \alpha}$, *(ii)* $r^n = a^n \cos n\theta$, **(Lucknow 2009)**

(iii) $r = a + b \cos \theta$.

Solution: (i) The equation of the given curve is

$$r = ae^{\theta \cot \alpha}. \qquad \text{...(1)}$$

Let (r_1, θ_1) be any point P on (1). [See fig. of Example 10]

Then $$r_1 = ae^{\theta_1 \cot \alpha}, \qquad \text{...(2)}$$

r_1 and θ_1 being parameters.

Let Q be any point (r, θ) on the straight line drawn through P and perpendicular to OP. From the right angled triangle OPQ, we have

$$OP = OQ \cos \angle POQ$$

or $$r_1 = r \cos (\theta - \theta_1)$$

or $$r \cos (\theta - \theta_1) = ae^{\theta_1 \cot \alpha}, \qquad \text{...(3)}$$

substituting for the parameter r_1 from (2).

Now (3) is the equation of the straight line through P and perpendicular to OP. We are to find the envelope of the family of straight lines (3), where θ_1 is the parameter.

Taking logarithm of both sides of (3), we get

$$\log r + \log \cos (\theta - \theta_1) = \log a + \theta_1 \cot \alpha. \qquad \text{...(4)}$$

Differentiating (4) partially with respect to θ_1, we get

$$0 + \frac{1}{\cos (\theta - \theta_1)} \cdot [-\sin (\theta - \theta_1)].(-1) = 0 + \cot \alpha$$

or $$\tan (\theta - \theta_1) = \cot \alpha = \tan (\tfrac{1}{2} \pi - \alpha).$$

$\therefore$ $$\theta - \theta_1 = \frac{1}{2} \pi - \alpha \quad \text{or} \quad \theta_1 = \theta + \alpha - \frac{1}{2} \pi.$$

Substituting this value of θ_1 in (3), the required envelope of the family of straight lines (3) is

$$r \cos (\tfrac{1}{2} \pi - \alpha) = ae^{(\theta + \alpha - \frac{1}{2} \pi) \cot \alpha}$$

or $$r \sin \alpha = ae^{(\alpha - \frac{1}{2} \pi) \cot \alpha} e^{\theta \cot \alpha}.$$

(ii) The equation of the given curve is

$$r^n = a^n \cos n\theta. \qquad \text{...(1)}$$

Let (r_1, θ_1) be any point P on (1). Then r_1 and θ_1 are parameters connected by

$$r_1{}^n = a^n \cos n\theta_1. \qquad ...(2)$$

Let Q be any point (r, θ) on the straight line drawn through P and perpendicular to OP. From the right angled triangle OPQ, we have

$$OP = OQ \cos \angle POQ$$

or $$r_1 = r \cos (\theta - \theta_1)$$

or $$r \cos (\theta - \theta_1) = a (\cos n\theta_1)^{1/n}, \qquad ...(3)$$

substituting for the parameter r_1 from (2).

Now (3) is the equation of the straight line through P and perpendicular to OP. We are to find the envelope of the family of straight lines (3), where θ_1 is the parameter. Taking logarithm of both sides of (3), we get

$$\log r + \log \cos (\theta - \theta_1) = \log a + (1/n) \log \cos n\theta_1. \qquad ...(4)$$

Differentiating (4) partially with respect to θ_1, we get

$$0 + \frac{1}{\cos (\theta - \theta_1)} \sin (\theta - \theta_1) = 0 + \frac{1}{n} \cdot \frac{-n \sin n\theta_1}{\cos n\theta_1}$$

or $$\tan (\theta - \theta_1) = -\tan n\theta_1 = \tan (-n\theta_1).$$

$\therefore$ $$\theta - \theta_1 = -n\theta_1 \quad \text{or} \quad \theta_1 (1 - n) = \theta \quad \text{or} \quad \theta_1 = \theta/(1 - n).$$

Substituting this value of θ_1 in (3), the required envelope of the family of straight lines (3) is

$$r \cos \left[\theta - \frac{\theta}{1 - n}\right] = a \left[\cos \left(\frac{n\theta}{1 - n}\right)\right]^{1/n}$$

or $$r \cos \left(\frac{\theta - n\theta - n}{1 - n}\right) = a \left[\cos \left(\frac{n\theta}{1 - n}\right)\right]^{1/n}$$

or $$r \cos \left(\frac{n\theta}{1 - n}\right) = a \left[\cos \left(\frac{n\theta}{1 - n}\right)\right]^{1/n}$$

or $$r = a \left[\cos \left(\frac{n\theta}{1 - n}\right)\right]^{(1/n) - 1} = a \left[\cos \left(\frac{n\theta}{1 - n}\right)\right]^{(1 - n)/n}$$

or $$r^{n/(1 - n)} = a^{n/(1 - n)} \cos \{n\theta/(1 - n)\},$$

raising both sides to the power $n/(1 - n)$.

(iii) The equation of the given curve is

$$r = a + b \cos \theta. \qquad ...(1)$$

Let (r_1, θ_1) be any point P on (1). Then r_1 and θ_1 are parameters connected by

$$r_1 = a + b \cos \theta_1. \qquad ...(2)$$

Let Q be any point (r, θ) on the straight line drawn through P and perpendicular to OP. From the right angled triangle OPQ, we have

$$OP = OQ \cos \angle POQ \quad \text{or} \quad r_1 = r\cos(\theta - \theta_1)$$

or $$r\cos(\theta - \theta_1) = a + b\cos\theta_1, \qquad ...(3)$$

substituting for the parameter r_1 from (2).

Now (3) is the equation of the straight line through P and perpendicular to OP. We are to find the envelope of the family of straight lines (3), where θ_1 is the parameter.

The equation (3) may be written as

$$r\cos\theta\cos\theta_1 + r\sin\theta\sin\theta_1 = a + b\cos\theta_1$$

or $$(r\cos\theta - b)\cos\theta_1 + r\sin\theta\sin\theta_1 = a. \qquad ...(4)$$

Differentiating (4) partially with respect to θ_1, we get

$$-(r\cos\theta - b)\sin\theta_1 + r\sin\theta\cos\theta_1 = 0. \qquad ...(5)$$

To eliminate θ_1, squaring and adding (4) and (5), we get

$$(r\cos\theta - b)^2 + r^2\sin^2\theta = a^2$$

or $$r^2 - 2br\cos\theta + b^2 - a^2 = 0,$$

which is the required envelope of the family of straight lines (3).

Problem 9(i): *Find the envelope of the straight lines*

$$x/a + y/b = 1, \qquad ...(1)$$

where the parameters a and b are related by the equation

$$a^n + b^n = c^n, c \text{ being a constant}. \qquad ...(2)$$

(Garhwal 2003; Gorakhpur 06; Lucknow 06; Kashi 11; Meerut 13B)

Solution: Let us regard a and b as functions of some other parameter t. Differentiating (1) and (2) w.r.t. t, taking x and y as constants, we have

$$\frac{x}{a^2}\frac{da}{dt} + \frac{y}{b^2}\frac{db}{dt} = 0 \quad \text{and} \quad a^{n-1}\frac{da}{dt} + b^{n-1}\frac{db}{dt} = 0.$$

Equating the values of $\dfrac{da/dt}{db/dt}$ obtained from these equations, we have

$$\frac{(x/a^2)}{a^{n-1}} = \frac{(y/b^2)}{b^{n-1}} \qquad ...(3)$$

or $$\frac{(x/a)}{a^n} = \frac{(y/b)}{b^n} = \frac{(x/a)+(y/b)}{a^n + b^n} = \frac{1}{c^n}, \text{ from (1) and (2).}$$

$$\left[\textbf{Note}: \frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}\right]$$

Thus $$a^{n+1} = c^n x \text{ and } b^{n+1} = c^n y.$$

Substituting in (2), we have

$$(c^n x)^{n/(n+1)} + (c^n y)^{n/(n+1)} = c^n$$

or $$x^{n/(n+1)} + y^{n/(n+1)} = c^{n/(n+1)},$$

which is the required equation of the envelope.

Problem 9(ii): *Find the envelope of the straight line* $x/a + y/b = 1$, *where the parameters a and b are connected by the relation* $a^m b^n = c^{m+n}$, *c being a constant.*

Solution: The equation of the given family of straight lines is

$$x/a + y/b = 1, \quad \text{...(1)}$$

where the parameters a and b are connected by the relation

$$a^m b^n = c^{m+n} \quad \text{...(2)}$$

Taking log of (2), we get

$$m \log a + n \log b = (m+n) \log c \quad \text{...(3)}$$

Let us regard a and b as functions of some other parameter t.

Differentiating (1) and (3) w.r.t. t taking x and y as constants, we have

$$-\frac{x}{a^2}\frac{da}{dt} - \frac{y}{b^2}\frac{db}{dt} = 0 \text{ or } \frac{x}{a^2}\frac{da}{dt} + \frac{y}{b^2}\frac{db}{dt} = 0 \quad \text{...(4)}$$

and $$\frac{m}{a}\cdot\frac{da}{dt} + \frac{n}{b}\frac{db}{dt} = 0 \quad \text{...(5)}$$

From (4) and (5) comparing the coefficients of (da/dt) and (db/dt); we get

$$\frac{(x/a^2)}{(m/a)} = \frac{(y/b^2)}{(n/b)} \text{ or } \frac{x/a}{m} = \frac{y/b}{n} = \frac{x/a + y/b}{m+n} = \frac{1}{m+n}. \quad \textbf{(Note)}$$

$\therefore$ $$a = x(m+n)/m, \; b = y(m+n)/n.$$

Substituting these values of a and b in (2), we get

$$\{x^m (m+n)^m / m^m\} \times \{y^n (m+n)^n / n^n\} = c^{m+n}$$

or $$(m+n)^{m+n} x^m y^n = m^m n^n c^{m+n},$$

which is the required envelope.

Problem 9: *Find the envelope of the family of straight lines*

$$x/a + y/b = 1,$$

where the parameters a, b are connected by the relations

(iii) $a + b = c$, **(Garhwal 2001, 03)**

(iv) $a^2 + b^2 = c^2$,

(v) $ab = c^2$, *c is a constant.*

Solution: Given straight line is

$$\frac{x}{a} + \frac{y}{b} = 1. \quad \text{...(1)}$$

Let us regard a and b as function of same other parameter t

Differentiating w.r.t. t, we get

$$\frac{x}{a^2}\frac{da}{dt} + \frac{y}{b^2}\frac{db}{dt} = 0. \quad \text{...(2)}$$

(iii) Given $a + b = c$ ...(3)

Differentiating w.r.to t, we get

$$\frac{da}{dt} + \frac{db}{dt} = 0. \quad ...(4)$$

Equating the value of $\frac{da}{dt}\Big/\frac{db}{dt}$ from equations (2) and (3), we get

$$\frac{x/a^2}{1} = \frac{y/b^2}{1}$$

or $$\frac{\frac{x}{a}}{a} = \frac{\frac{y}{b}}{b} = \frac{\frac{x}{a}+\frac{y}{b}}{a+b} = \frac{1}{c}$$ [By (1) and (3)]

Thus $a^2 = cx, b^2 = cy$ or $a = \sqrt{cx}, b = \sqrt{cy}$.

Substituting in (3), we have

$$\sqrt{cx} + \sqrt{cy} = c \quad \text{or} \quad \sqrt{x} + \sqrt{y} = \sqrt{c}.$$

(iv) $a^2 + b^2 = c^2$ (Given) ...(5)

Differentiating w.r.t. t, we get

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 0 \quad \text{or} \quad a\frac{da}{dt} + b\frac{db}{dt} = 0. \quad ...(6)$$

Equating the value of $\frac{da}{dt}\Big/\frac{db}{dt}$ from (2) and (4)

$$\frac{x/a^2}{a} = \frac{y/b^2}{a} \quad \text{or} \quad \frac{\frac{x}{a}}{a^2} = \frac{\frac{y}{b}}{b^2} = \frac{\frac{x}{a}+\frac{y}{b}}{a^2+b^2} = \frac{1}{c^2}.$$

Thus $a^3 = c^2 x, b^3 = c^2 y$ or $a = (c^2 x)^{1/3}, b = (c^2 y)^{1/3}$.

Substituting in (5), we have

$$(c^2 x)^{2/3} + (c^2 y)^{2/3} = c^2 \quad \text{or} \quad x^{2/3} + y^{2/3} = c^{2/3}.$$

(v) $ab = c^2$ (Given) ...(7)

Differentiating w.r. to t

$$b\frac{da}{dt} + a\frac{db}{dt} = 0. \quad ...(8)$$

Equating the values of $\frac{da}{dt}\Big/\frac{db}{dt}$ from (2) and (8), we get

$$\frac{x/a^2}{b} = \frac{y/b^2}{a} \quad \text{or} \quad \frac{x/a}{ab} = \frac{y/b}{ab} = \frac{x/a + y/b}{2ab} = \frac{1}{2c^2}$$

Thus $2c^2 x = a^2 b$ and $2c^2 y = b^2 a$

Hence $4c^4 xy = a^3 b^3 = c^6$ [By (7)]

or $4xy = c^2$.

Problem 10: *Prove that the envelope of the ellipses* $x^2/a^2 + y^2/b^2 = 1$ *having the sum of their semi-axes constant and equal to c, is the astroid* $x^{2/3} + y^{2/3} = a^{2/3}$.

Solution: Let the equation of the family of ellipses be

$$x^2/a^2 + y^2/b^2 = 1, \qquad \text{...(1)}$$

where a and b are parameters.

Given that the sum of their semi-axes is equal to c, we have

$$a + b = c.$$

Let us regard a and b as functions of some other parameter t.

Differentiating (1) and (2) w.r.t. t, we have

$$-\frac{2x^2}{a^3}\cdot\frac{da}{dt} - \frac{2y^2}{b^3}\cdot\frac{db}{dt} = 0$$

and $$\frac{da}{dt} + \frac{db}{dt} = 0.$$

Comparing the coefficients of (da/dt) and (db/dt), we get

$$\frac{x^3/a^3}{1} = \frac{y^2/b^3}{1} \text{ i.e., } \frac{x^2/a^2}{a} = \frac{y^2/b^2}{b} = \frac{x^2/a^2 + y^2/b^2}{a+b} = \frac{1}{c}$$

$\therefore$ $$a^3 = cx^2 \text{ and } b^3 = cy^2.$$

Substituting these values of a and b in (2), we have

$$c^{1/3}x^{2/3} + c^{1/3}y^{2/3} = c \text{ or } x^{2/3} + y^{2/3} = c^{2/3}.$$

Hence the required envelope is the astroid $x^{2/3} + y^{2/3} = c^{2/3}$.

Problem 11: *Find the envelope of a system of concentric and co-axial ellipses of constant area.* **(Gorakhpur 2005; Kanpur 10)**

Solution: Let the equation of a system of concentric and coaxial ellipses be

$$x^2/a^2 + y^2/b^2 = 1, \text{ where } a \text{ and } b \text{ are parameters} \qquad \text{...(1)}$$

Also given that the area of each ellipse is constant, we have

$$\pi ab = \pi c^2 \quad \text{or} \quad ab = c^2, \text{ where } c \text{ is a constant.} \qquad \text{...(2)}$$

Let us regard a and b as functions of t.

Differentiating (1) and (2) partially w.r.t. t, we get

$$-\frac{2x^2}{a^3}\cdot\frac{da}{dt} - \frac{2y^2}{b^3}\cdot\frac{db}{dt} = 0 \quad \text{and} \quad b\frac{da}{dt} + a\frac{db}{dt} = 0.$$

Comparing the coefficients of (da/dt) and (db/dt), we get

$$\frac{x^2/a^3}{b} = \frac{y^2/b^3}{a} \text{ i.e., } \frac{x^2/a^2}{1} = \frac{y^2/b^2}{1} = \frac{x^2/a^2 + y^2/b^2}{1+1} = \frac{1}{2}.$$

$\therefore \qquad \dfrac{x^2}{a^2} = \dfrac{y^2}{b^2} = \dfrac{1}{2}$

or $\qquad a^2 = 2x^2 \quad \text{and} \quad b^2 = 2y^2.$

$\therefore$ From (2), $\quad 2x^2 . 2y^2 = c^4 \quad \text{or} \quad 4x^2 y^2 = c^4,$

which is the required envelope.

Problem 12: *Show that the envelope of the family of parabolas*

$$(x/a)^{1/2} + (y/b)^{1/2} = 1,$$

under the condition $ab = c^2$, *is a hyperbola whose asymptotes coincide with the axes.*

(Garhwal 2000; Kumaun 02)

Solution: The given equation of the family of parabolas is

$$(x/a)^{1/2} + (y/b)^{1/2} = 1, \qquad ...(1)$$

where $\qquad ab = c^2. \qquad ...(2)$

Let us regard the parameters a and b as functions of t.

Differentiating (1) and (2) w.r.t. t taking x and y as constants, we have

$$\{-x^{1/2}/(2a^{3/2})\}(da/dt) + \{-y^{1/2}/(2b^{3/2})\}(db/dt) = 0$$

i.e., $\qquad (x^{1/2}/a^{3/2})(da/dt) + (y^{1/2}/b^{3/2})(db/dt) = 0 \qquad ...(3)$

and $\qquad b\,(da/dt) + a\,(db/dt) = 0. \qquad ...(4)$

Comparing the coefficients of (da/dt) and (db/dt) from (3) and (4), we have

$$\frac{x^{1/2}/a^{3/2}}{b} = \frac{y^{1/2}/b^{3/2}}{a} \quad \text{or} \quad (x/a)^{1/2} = (y/b)^{1/2}. \qquad \textbf{(Note)}$$

$\therefore$ From (1), $\quad (x/a)^{1/2} = (y/b)^{1/2} = \dfrac{1}{2}$

or $\qquad x/a = y/b = \dfrac{1}{4} \quad \text{or} \quad a = 4x, b = 4y.$

Substituting these values of a and b in (2), we have

$$4x . 4y = c^2 \quad \text{or} \quad 16xy = c^2.$$

Hence the required envelope is a hyperbola $16xy = c^2$, whose asymptotes are $x = 0, y = 0$ *i.e.*, the coordinate axes.

Problem 13: *A straight line of given length slides with its extremities on two fixed straight lines at right angles. Find the envelope of the circle drawn on the sliding line as diameter.*

Solution: Take the two fixed straight lines at right angles as the coordinate axes OX and OY. Let the equation of the sliding line AB be $x/a + y/b = 1$ whose end A remains on the x-axis and B on the y-axis. Thus a and b are the intercepts OA and OB of the line AB on the axes of x and y respectively.

We have $\qquad a^2 + b^2 = OA^2 + OB^2 = AB^2 = l^2, \qquad ...(1)$

where l is the given length of the line AB.

The coordinates of the ends A and B are $(a, 0)$ and $(0, b)$ respectively. Hence the equation of the circle described on the line AB as diameter is

$$(x-a)(x-0)+(y-0)(y-b)=0$$

or $$ax+by=x^2+y^2. \qquad ...(2)$$

We have to find the envelope of (2) where the parameters a and b are connected by (1).

Let us regard a and b as functions of some other parameter t.

Differentiating (1) and (2) w.r.t. t taking x and y as constants, we have

$$2a\frac{da}{dt}+2b\frac{db}{dt}=0 \text{ and } x\frac{da}{dt}+y\frac{db}{dt}=0.$$

Comparing the coefficients of da/dt and db/dt from these, we have $\dfrac{x}{a}=\dfrac{y}{b}$

or $$\frac{ax}{a^2}=\frac{by}{b^2}=\frac{ax+by}{a^2+b^2}=\frac{x^2+y^2}{l^2}. \qquad \textbf{(Note)}$$

$\therefore$ $$a=l^2x/(x^2+y^2), \text{ and } b=l^2y/(x^2+y^2).$$

Substituting these values of a and b in (1), we have

$$l^4x^2/(x^2+y^2)^2+l^4y^2/(x^2+y^2)^2=l^2$$

or $$l^4(x^2+y^2)=l^2(x^2+y^2)^2$$

or $$(x^2+y^2)=l^2,$$

which is the required envelope and is a circle.

Comprehensive Problems 3

Problem 1: *Show that the equation of the normal to the ellipse* $x^2/a^2+y^2/b^2=1$ *at the point* $(a\cos\theta, b\sin\theta)$ *is* $\dfrac{ax}{\cos\theta}-\dfrac{by}{\sin\theta}=a^2-b^2$. *Hence find the evolute of the above ellipse.*

(Avadh 2006, 10)

Solution: The given ellipse is

$$x^2/a^2+y^2/b^2=1. \qquad ...(1)$$

The evolute of the ellipse (1) is the envelope of the family of normals to the ellipse (1). The coordinates (x, y) of any point P on the ellipse (1) may be taken as

$$x=a\cos\theta,\ y=b\sin\theta, \text{ where } \theta \text{ is parameter.}$$

We have $dx/d\theta=-a\sin\theta$, $dy/d\theta=b\cos\theta$.

$\therefore$ $$\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{b\cos\theta}{-a\sin\theta}.$$

$\therefore$ Slope of the normal to the ellipse (1) at the point $(a\cos\theta, b\sin\theta)$

$$=-\frac{dx}{dy}=\frac{a\sin\theta}{b\cos\theta}.$$

$\therefore$ Equation of the normal to the ellipse (1) at the point $(a\cos\theta, b\sin\theta)$ is

$$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$$

or $$by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$$

or $$ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$$

or $$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2. \quad \ldots(2)$$

Now the evolute of the ellipse (1) is the envelope of the family (2) of normals to the ellipse (1), where θ is the parameter.

To find the envelope of the family of straight lines (2), proceed as in Example 2.

Thus the envelope of the family of straight lines (2) is the curve

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3},$$

which is the required evolute of the ellipse (1).

Problem 2: *Find the equation of the evolute of the parabola $y^2 = 2px$.*

Solution: Equation of any normal to the parabola $y^2 = 2px$ is

$$y = mx - pm - \frac{1}{2}pm^3,$$

where m is the parameter.

Now proceed as in Example 1. The required evolute is

$$27py^2 = 8(x - p)^3.$$

Problem 3: *Show that the evolute of the tractrix $x = a(\cos t + \log\tan\frac{1}{2}t)$, $y = a\sin t$ is the catenary $y = a\cosh(x/a)$.*

Solution: The given tractrix is $x = a(\cos t + \log\tan\frac{1}{2}t)$, $y = a\sin t$.

Differentiating these equations w.r.t. t, we get

$$\frac{dx}{dt} = a\left\{-\sin t + \frac{\frac{1}{2}\sec^2\frac{1}{2}t}{\tan\frac{1}{2}t}\right\} = a\left\{-\sin t + \frac{1}{2\sin\frac{1}{2}t\cos\frac{1}{2}t}\right\}$$

$$= a\{-\sin t + (1/\sin t)\} = a(1 - \sin^2 t)/\sin t$$

$$= a\cos^2 t/\sin t$$

and $$\frac{dy}{dt} = a\cos t.$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos^2 t/\sin t}{a\cos t} = \tan t.$$

$\therefore$ The slope of the normal at the point 't'

$$= -(dx/dy) = -\cot t.$$

Hence the equation of the normal at the point 't' of the tractrix is

$$y - a\sin t = -\cot t\,(x - a\cos t - a\log\tan\tfrac{1}{2}t)$$

or $$\tan t\,(y - a\sin t) + x - a\cos t - a\log\tan\tfrac{1}{2}t = 0$$

or $$x + y\tan t - a\sin^2 t/\cos t - a\cos t - a\log\tan\tfrac{1}{2}t = 0$$

or $$x + y\tan t - a(\sin^2 t + \cos^2 t)/\cos t - a\log\tan\tfrac{1}{2}t = 0$$

or $$x + y\tan t - a\sec t - a\log\tan\tfrac{1}{2}t = 0. \qquad ...(1)$$

Now (1) is the equation of the family of normals of the given tractrix, the parameter being t. The envelope of the family (1) is the evolute of the given tractrix.

Differentiating (1) partially w.r.t. 't', we have

$$y\sec^2 t - a\sec t\tan t - \frac{1}{2}a(\sec\frac{1}{2}t/\tan\frac{1}{2}t) = 0$$

or $$y\sec^2 t - a\sec t\tan t - a\,\mathrm{cosec}\,t = 0$$

or $$\frac{y}{\cos^2 t} - \frac{a\sin t}{\cos^2 t} - \frac{a}{\sin t} = 0$$

or $$y\sin t - a(\sin^2 t + \cos^2 t) = 0$$

or $$y\sin t = a \quad\text{or}\quad y = a/\sin t. \qquad ...(2)$$

Substituting this value of y in (1), we have

$$x + a\tan t/\sin t - a\sec t - a\log\tan\tfrac{1}{2}t = 0$$

or $$x + a\sec t - a\sec t - a\log\tan\tfrac{1}{2}t = 0$$

or $$x = a\log\tan\tfrac{1}{2}t \quad\text{or}\quad x/a = \log\tan\tfrac{1}{2}t$$

or $$\tan\tfrac{1}{2}t = e^{x/a}. \qquad ...(3)$$

Now from (2), we have

$$y = \frac{a}{\sin t} = \frac{a(1 + \tan^2\frac{1}{2}t)}{2\tan\frac{1}{2}t}$$

$$= \frac{1}{2}a(\cot\tfrac{1}{2}t + \tan\tfrac{1}{2}t) = \frac{1}{2}a(e^{x/a} + e^{-x/a}), \qquad \text{[From (3)]}$$

$$= a \cosh (x/a).$$

Hence the envelope of the family of normals (1) *i.e.*, the evolute of the given tractrix is

$$y = a \cosh (x/a).$$

Problem 4: *Prove that the evolute (i.e., the locus of the centre of curvature) of the hyperbola* $2xy = a^2$ *is* $(x+y)^{2/3} - (x-y)^{2/3} = 2a^{2/3}$. **(Lucknow 2006)**

Solution: The given curve is

$$xy = a^2/2 = c^2, \text{ (say)}. \qquad ...(1)$$

The evolute of (1) is the envelope of the normals of (1).

Let $P(ct, c/t)$ be any point on the hyperbola (1).

Also from (1), differentiating we get $dy/dx = -c^2/x^2$.

$\therefore$ The slope of the normal to (1) at P

$$= -dx/dy = x^2/c^2 = t^2.$$

Hence the equation of the normal at P is

$$y - c/t = t^2 (x - ct) \qquad ...(2)$$

Differentiating (2) partially w.r.t. t, we get

$$c/t^2 = 2tx - 3ct^2 \quad \text{or} \quad 2x = 3ct + c/t^3. \qquad ...(3)$$

Substituting this value of x in (2), we get

$$2y = 3ct + ct^3. \qquad ...(4)$$

Adding (3) and (4), we have

$$2(x+y) = c(t + 1/t)^3 \qquad ...(5)$$

Subtracting (4) from (3), we have

$$2(x-y) = c\{(1/t) - t\}^3 \qquad ...(6)$$

Eliminating t from (5) and (6), we have

$$2^{2/3}[(x+y)^{2/3} - (x-y)^{2/3}] = 4c^{2/3},$$

[Raising (5) and (6) to the power 2/3 and subtracting]

or $$(x+y)^{2/3} - (x-y)^{2/3} = (4c)^{2/3} = (4 \,.\, a/\sqrt{2})^{2/3}$$

$$= (2\sqrt{2}\, a)^{2/3} = 2a^{2/3},$$

which is the required evolute.

Problem 5: *Prove that the evolute of the ellipses* $b^2x^2 + a^2y^2 = a^2b^2$ *is the envelope of the family of ellipses given by*

$$a^2x^2 \sec^4 \alpha + b^2y^2 \operatorname{cosec}^4 \alpha = (a^2 - b^2)^2,$$

α *being the variable parameter.* **(Lucknow 2007)**

Solution: The given ellipse is

$$b^2x^2 + a^2y^2 = a^2b^2 \text{ or } x^2/a^2 + y^2/b^2 = 1.$$

The evolute of this ellipse is

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3} \qquad \text{[See problem 1]}$$

The given family of ellipses is

$$a^2x^2 \sec^4\alpha + b^2y^2 \operatorname{cosec}^4\alpha = (a^2 - b^2)^2, \qquad \text{...(1)}$$

where α is the parameter.

Differentiating (1) partially w.r.t. α, we have

$$4a^2x^2 \sec^4\alpha \tan\alpha - 4b^2y^2 \operatorname{cosec}^4\alpha \cot\alpha = 0$$

or $$\frac{4a^2x^2 \sin\alpha}{\cos^5\alpha} - \frac{4b^2y^2 \cos\alpha}{\sin^5\alpha} = 0$$

or $$a^2x^2 \sin^6\alpha - b^2y^2\cos^6\alpha = 0$$

or $$\tan^6\alpha = (b^2y^2/a^2x^2)$$

or $$\tan\alpha = (by)^{1/3}/(ax)^{1/3}.$$

$\therefore$ $$\sec\alpha = \frac{\{(by)^{2/3} + (ax)^{2/3}\}^{1/2}}{(ax)^{1/3}},$$

$$\operatorname{cosec}\alpha = \frac{\{(by)^{2/3} + (ax)^{2/3}\}^{1/2}}{(by)^{1/3}}.$$

Substituting these values in (1), we have

$$\frac{(ax)^2\{(by)^{2/3} + (ax)^{2/3}\}^2}{(ax)^{4/3}} + \frac{(by)^2\{(by)^{2/3} + (ax)^{2/3}\}^2}{(by)^{4/3}} = (a^2 - b^2)^2$$

or $$\{(ax)^{2/3} + (by)^{2/3}\}^2\{(ax)^{2/3} + (by)^{2/3}\} = (a^2 - b^2)^2$$

or $$\{(ax)^{2/3} + (by)^{2/3}\}^3 = (a^2 - b^2)^2$$

or $$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$$

which is the envelope of the family of ellipses (1). We see that this envelope is the same as the evolute of the given ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

Problem 6: *Find the evolute of the curve* $x^{2/3} + y^{2/3} = a^{2/3}$. **(Lucknow 2008)**

Solution: The given curve is

$$x^{2/3} + y^{2/3} = a^{2/3}. \qquad \text{...(1)}$$

The evolute of the curve (1) is the envelope of the family of the normals to the curve (1). The coordinates (x, y) of any point P on the curve (1) may be taken as

$x = a\cos^3\theta,\ y = a\sin^3\theta$, where θ is parameter.

We have $\quad dx/d\theta = -3a\cos^2\theta\sin\theta,\ dy/d\theta = 3a\sin^2\theta\cos\theta.$

$\therefore$ Slope of the normal to the curve (1) at the point $(a\cos^3\theta, a\sin^3\theta)$

$$= -\frac{dx}{dy} = -\frac{dx/d\theta}{dy/d\theta} = -\frac{-3a\cos^2\theta\sin\theta}{3a\sin^2\theta\cos\theta} = \frac{\cos\theta}{\sin\theta}.$$

$\therefore$ Equation of the normal to the curve (1) at the point $(a\cos^3\theta, a\sin^3\theta)$ is

$$y - a\sin^3\theta = \frac{\cos\theta}{\sin\theta}(x - a\cos^3\theta)$$

or $\quad x\cos\theta - y\sin\theta = a(\cos^4\theta - \sin^4\theta)$

$$= a(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$$

or $\quad x\cos\theta - y\sin\theta = a(\cos^2\theta - \sin^2\theta).$ ...(2)

Now the evolute of the curve (1) is the envelope of the family of straight lines (2), where θ is parameter.

Differentiating (2) partially with respect to θ, we get

$$-x\sin\theta - y\cos\theta = a(-2\cos\theta\sin\theta - 2\sin\theta\cos\theta)$$

or $\quad x\sin\theta + y\cos\theta = 4a\sin\theta\cos\theta.$ ...(3)

Now to find the envelope of the family of straight lines (2), we have to eliminate θ between (2) and (3).

Solving (2) and (3) for x and y, we get

$$x = a(\cos^3\theta + 3\cos\theta\sin^2\theta)$$

and $\quad y = a(3\sin\theta\cos^2\theta + \sin^3\theta).$

$\therefore \quad x + y = a(\cos^3\theta + 3\cos^2\theta\sin\theta + 3\cos\theta\sin^2\theta + \sin^3\theta)$

$$= a(\cos\theta + \sin\theta)^3$$

and $\quad x - y = a(\cos^3\theta - 3\cos^2\theta\sin\theta + 3\cos\theta\sin^2\theta - \sin^3\theta)$

$$= a(\cos\theta - \sin\theta)^3.$$

$\therefore \quad (x+y)^{2/3} + (x-y)^{2/3} = a^{2/3}[(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2]$

or $\quad (x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3},$

which is the required evolute of the curve (1).

Problem 7: *Show that the evolute of the curve whose pedal equation is $r^2 - a^2 = mp^2$ is the curve whose pedal equation is*

$$r^2 - (1-m)a^2 = mp^2.$$

Solution: Draw figure as in article 10. The pedal equation of the given curve is

$r^2 - a^2 = mp^2.$ ...(1)

Differentiating both sides of (1) with respect to p, we have

$$2r\frac{dr}{dp}=2mp.$$

$\therefore$ $$\rho=r\frac{dr}{dp}=mp. \quad ...(2)$$

Corresponding to the point (p, r) on the given curve (1), let the point on the evolute be (p', r') the coordinates in each case being expressed in pedal form.

Then $r'^2=r^2+\rho^2-2\rho p$

$=r^2+m^2p^2-2mpp$, substituting for ρ from (2)

$=a^2+mp^2+m^2p^2-2mp^2$, substituting for r^2 from (1).

$\therefore$ $$r'^2-a^2=m^2p^2-mp^2=mp^2(m-1). \quad ...(3)$$

Also $p'^2=r^2-p^2=a^2+mp^2-p^2$

$\therefore$ $$p'^2-a^2=(m-1)p^2. \quad ...(4)$$

To eliminate p, dividing (3) by (4), we get

$$\frac{r'^2-a^2}{p'^2-a^2}=m \quad \text{or} \quad r'^2-a^2=mp'^2-ma^2.$$

or $$r'^2-(1-m)a^2=mp'^2$$

Hence the locus of the point (p', r') is

$$r^2-(1-m)a^2=mp^2.$$

This is the pedal equation of the evolute of the curve (1).

Problem 8: *Prove that the normals to a given curve are always tangent lines to its evolute.*

Solution: We know that the evolute of a given curve is the envelope of the family of normals to that curve.

Again we know that the envelope of a family of curves touches each member of the family.

Hence the normals to a given curve are always tangent lines to its evolute.

Hints to Objective Type Questions

Multiple Choice Questions

1. See Example 1.
2. See Problem 1 of Comprehensive Problems 1.
3. See Problem 2(i) of Comprehensive Problems 1.
4. See article 4.

5. See article 8.
6. See Example 5.
7. See Problem 4(i) of Comprehensive Problems 1.
8. See Example 8.
9. See Problem 6 of Comprehensive Problems 3.
10. See article 11, Theorem 1.

Fill in the Blanks

1. See working rule of article 3.
2. See article 4.
3. See Problem 4(v) of Comprehensive Problems 1.

True or False

1. See article 5.
2. See Problem 13 of Comprehensive Problems 1.
3. The evolute of a curve is the envelope of the normals to that curve. See article 8.
4. See article 11.
5. The evolute of an equi-angular spiral is an equi-angular spiral. See Example 15.

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Chapter-12

Asymptotes

Comprehensive Problems 1

Problem 1: *Find the asymptotes of the curve* $a^2/x^2 + b^2/y^2 = 1$.

(Meerut 2007B; Purvanchal 14)

Solution: The curve is

$$\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1 \qquad \text{or} \qquad a^2 y^2 + b^2 x^2 = x^2 y^2$$

or $$x^2 y^2 - a^2 y^2 - b^2 x^2 = 0.$$

Here the highest power of y is y^2 and its coefficient is $x^2 - a^2$.

Hence the asymptotes parallel to y-axis are

$$x^2 - a^2 = 0 \;\; i.e., \;\; x = \pm a.$$

Also the highest power of x is x^2 and its coefficient is $y^2 - b^2$.

Hence the asymptotes parallel to x-axis are

$$y^2 - b^2 = 0 \; i.e., \; y = \pm b.$$

Problem 2: *Find the asymptotes of the curve* $y^2 (x^2 - a^2) = x$.

(Meerut 2010; Rohilkhand 14)

Solution: Since the given curve is of degree 4, therefore it cannot have more than four asymptotes.

Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2) the asymptotes parallel to y-axis are given by

$$x^2 - a^2 = 0 \quad i.e., \quad x = \pm a.$$

Again equating to zero the coefficient of the highest power of x (*i.e.*, of x^2) the asymptotes parallel to x-axis are given by

$$y^2 = 0$$

i.e, $$y = 0, y = 0 \qquad \text{(two coincident asymptotes).}$$

Thus all the four asymptotes have been found.

$\therefore$ The asymptotes are $x = \pm a$, $y = 0$.

Problem 3: *Find the asymptotes of the curve* $xy^2 = 4a^2 (2a - x)$.

Solution: The equation of the curve is $xy^2 + 4a^2 x - 8a^3 = 0$.

Since the curve is of degree 3, therefore it cannot have more than three asymptotes. Equating to zero the coefficient of the highest power of y (*i.e.*, y^2) the asymptotes parallel to y-axis are given by $x = 0$.

Also equating to zero the coefficient of the highest power of x (*i.e.*, of x^2) the asymptotes parallel to x-axis are given by $y^2 + 4a^2 = 0$, which gives two imaginary asymptotes.

Thus all the three possible asymptotes of the curve have been found and the only real asymptotes is $x = 0$.

Problem 4: *Find the asymptotes of the curve* $x^2 y^2 = a^2 (x^2 + y^2)$.

(Bundelkhand 2001, 05, 08; Garhwal 13)

Solution: The given curve is of degree 4, so it cannot have more than four asymptotes. Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2) the asymptotes parallel to y-axis are given by

$$x^2 - a^2 = 0$$

i.e., $$x = \pm a.$$

Again equating to zero the coefficient of the highest power of x (*i.e.*, of x^2) the asymptotes parallel to x-axis are given by

$$y^2 - a^2 = 0$$

i.e., $$y = \pm a.$$

Thus all the four asymptotes have been found and they are

$$x = \pm a, \; y = \pm a.$$

Problem 5: *Find the asymptotes of the curve* $y^2 (x^2 - a^2) = x^2 (x^2 - 4a^2)$.

(Meerut 2003, 06; Agra 05; Rohilkhand 05, 06)

Solution: The given curve is

$$y^2 (x^2 - a^2) = x^2 (x^2 - 4a^2)$$

or $$y^2 x^2 - x^4 - a^2 y^2 + 4a^2 x^2 = 0. \qquad ...(1)$$

Since the curve (1) is of degree 4, therefore it cannot have more than four asymptotes. Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2) the asymptotes parallel to y-axis are given by

$$x^2 - a^2 = 0 \text{ i.e., } x = \pm a. \qquad \text{(Two asymptotes)}$$

The coefficient of the highest power of x *i.e.*, of x is simply a constant and so there is no asymptote parallel to x-axis.

To find the remaining oblique asymptotes, we put $y = m$ and $x = 1$ in the highest *i.e.*, fourth degree terms in the equation (1) and we get

$$\phi_4(m) = m^2 - 1.$$

The slopes of the asymptotes are given by the equation

$$\phi_4(m) = 0 \quad i.e., \quad m^2 - 1 = 0.$$

$$\therefore \qquad m = \pm 1.$$

Again putting $y = m$ and $x = 1$ in the next highest *i.e.*, third degree terms, we get

$$\phi_3(m) = 0.$$

[Note that there are no terms of degree 3 in the equation of the curve and so $\phi_3(m) = 0$.]

Now c is given by the equation

$$c\,\phi_4'(m) + \phi_3(m) = 0$$

i.e., $c(2m) + 0 = 0$ *i.e.*, $2cm = 0$.

When $m = 1, c = 0$; and when $m = -1, c = 0$.

$\therefore$ the asymptotes are $y = 1\,x + 0$ and $y = (-1)\,x + 0$

i.e., $y = x$ and $y = -x$.

Hence all the four asymptotes of the curve are $x = \pm a$, $y = \pm x$.

Problem 6: *Find the asymptotes of the curve* $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1.$

(Meerut 2002; Gorakhpur 05; Bundelkhand 11)

Solution: The given curve may be written as $y^2 = \dfrac{b^2}{a^2}(x^2 - a^2)$.

$$\therefore \qquad y = \pm \frac{b}{a} x \left(1 - \frac{a^2}{x^2}\right)^{1/2}$$

$$= \pm \frac{b}{a} x \left[1 - \frac{a^2}{2x^2} - \frac{a^4}{8x^4} - \dots\right], \qquad \text{(By binomial theorem)}$$

$$= \pm \frac{b}{a}\left[x - \frac{a^2}{2x} - \frac{a^4}{8x^3} - \dots\right].$$

Hence, by article 10, the asymptotes of the curve are $\dfrac{y}{b} = \pm \dfrac{x}{a}$.

Problem 7: *Find all the asymptotes of the curve* $x^2y^2 - x^2y - xy^2 + x + y + 1 = 0.$

(Meerut 2007)

Solution: The curve is $x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$.

Here the highest power of x is x^2 and its coefficient is $y^2 - y$. Hence the required equation of the asymptote parallel to x-axis are

$$y^2 - y = 0 \quad i.e., \quad y = 0,\ y = 1.$$

Also the highest power of y is y^2 and its coefficient is $x^2 - x$. Hence the required equation of the asymptotes parallel to y-axis are $x^2 - x = 0$ *i.e.*, $x = 0$, $x = 1$

Problem 8: *Find the asymptotes parallel to the axes of the curve*

$$x^2 y^2 - x^2 - y^2 - x - y + 1 = 0.$$

(Bundelkhand 2001)

Solution: Asymptotes parallel to x-axis, equating the coefficient of highest power of x to zero

i.e., $y^2 - 1 = 0$ or $y = \pm 1$.

Also, asymptotes parallel to y-axis, equating the coefficient of highest power of y to zero

i.e., $x^2 - 1 = 0$ or $x = \pm 1$.

$\therefore$ The asymptotes are $x = \pm 1$, $y = \pm 1$.

Problem 9: *Find the asymptotes of the curve*

$$x^2 y^2 - a^2 (x^2 + y^2) - a^3 (x + y) + a^4 = 0.$$

(Meerut 2001; Garhwal 08)

Show that these asymptotes form a square through two of whose angular points the curve passes.

Solution: Since the curve is of degree 4, therefore it cannot have more than four asymptotes.

Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2), we get the asymptotes parallel to y-axis as

$$x^2 - a^2 = 0, \text{ i.e., } x = \pm a.$$

Also equating to zero the coefficient of the highest power of x (*i.e.*, of x^2), we get the asymptotes parallel to x-axis as

$$y^2 - a^2 = 0 \text{ i.e., } y = \pm a.$$

Thus all the four asymptotes of the curve have been found and they are $x = \pm a$, $y = \pm a$. These lines form a square whose sides are parallel to the axes each of length $2a$.

The angular points of the square are $(a, a), (a, -a), (-a, a)$ and $(-a, -a)$. It is evident that the curve passes through two angular points $(a, -a)$ and $(-a, a)$.

Comprehensive Problems 2

Problem 1: *Find all the asymptotes of the curve*

$$x^3 + 2x^2 y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0.$$

(Meerut 2007)

Solution: The given curve is $x^3 + 2x^2 y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0$.

Let $y = mx + c$ be an asymptote of this curve.

Then putting $y = mx + c$, we have

$$x^3 + 2x^2 (mx + c) - x(mx + c)^2 - 2(mx + c)^3 + 4(mx + c)^2 + 2x(mx + c) + (mx + c) - 1 = 0$$

or $$(1 + 2m - m^2 - 2m^3)x^3 + 2(c - mc - 3m^2c + 2m^2 + m)x^2 + \dots + \dots + \dots = 0.$$

Equating to zero the coefficient of two highest degree terms in x, we get

$$1 + 2m - m^2 - 2m^3 = 0 \quad \dots(1)$$

and $$c - mc - 3m^2c + 2m^2 + m = 0$$

i.e., $$c(1 - m - 3m^2) + 2m^2 + m = 0. \quad \dots(2)$$

Now, from (1) we have $(1 + m)(1 + 2m)(1 - m) = 0$

i.e., $m = -1, 1, -1/2$ [From (2)]

$$c = -\frac{2m^2 + m}{1 - m - 3m^2}$$

for $m = -1, \quad c = 1$

$m = 1, \quad c = 1$

$m = -1/2, c = 0.$

Hence the asymptotes are

$y = -x + 1$ or $x + y - 1 = 0,$

$y = x + 1$ or $x - y + 1 = 0$

and $y = -\frac{1}{2}x + 0$ or $x + 2y = 0.$

Problem 2: *Find the asymptotes of the curve*

$$2x^3 - x^2 y - 2xy^2 + y^3 - 4x^2 + 8xy - 4x + 1 = 0. \qquad \textbf{(Kumaun 2014)}$$

Solution: The curve is $2x^3 - x^2 y - 2xy^2 + y^3 - 4x^2 + 8xy - 4x + 1 = 0.$

Putting $x = 1$ and $y = m$ in the highest degree term, we get

$$\phi_3(m) = 2 - m - 2m^2 + m^3$$

$$\phi_3'(m) = -1 - 4m + 3m^2.$$

Equating $\phi_3(m) = 0$

$$2 - m - 2m^2 + m^3 = 0$$

or $(m^2 - 1)(m - 2) = 0.$

$\therefore$ $m = -1, 1, 2.$

Putting $x = 1$, $y = m$ in the second degree term of the given equation, we get

$$\phi_2(m) = 8m - 4.$$

Also, we have $$c = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{-(8m - 4)}{3m^2 - 4m - 1}.$$

When $m = -1, c = 2$

$m = 1, \quad c = 2$

$m = 2, \quad c = -4.$

The required asymptotes are $y = -x + 2;\ y = x + 2;\ y = 2x - 4$.

Problem 3: *Find all the asymptotes of the curve*

$$x^3 + 2x^2 y - xy^2 - 2y^3 + xy - y^2 - 1 = 0.$$ **(Bundelkhand 2006)**

Solution: The curve is $x^3 + 2x^2 y - xy^2 - 2y^3 + xy - y^2 - 1 = 0$.

Here the highest degree is 3, so putting $x = 1$ and $y = m$ in third degree terms, we see that

$$\phi_n(m) = 1 + 2m - m^2 - 2m^3.$$

Now, $\phi_n(m) = 0$ gives

$\Rightarrow \quad 1 + 2m - m^2 - 2m^3 = 0 \quad \Rightarrow \quad 1 - m^2 + 2m - 2m^3 = 0$

$\Rightarrow \quad 1 - m^2 + 2m(1 - m^2) = 0 \quad \Rightarrow \quad (1 - m^2)(1 + 2m) = 0$

$\Rightarrow \quad (1 - m)(1 + m)(1 + 2m) = 0$

$\Rightarrow \quad m = 1, -1, -\frac{1}{2}.$

Now, putting $x = 1$ and $y = m$ in second degree terms, we see that

$$\phi_n - 1(m) = m - m^2.$$

$$c = -\frac{\phi_n - 1(m)}{\phi_n'(m)} = \frac{m^2 - m}{2 - 2m - 6m^2}$$

when $m = 1, \quad c = 0$

when $m = -1, \quad c = -1$

when $m = -\frac{1}{2}, c = \frac{1}{2}.$

Hence, the required asymptotes are :

$$y = 1 . x + 0,\ y = (-1)x - 1 \text{ and } y = -\frac{1}{2}x + \frac{1}{2}$$

$$y = x,\ y = -x - 1 \text{ and } 2y = -x + 1.$$

Problem 4: *Find the asymptotes of the curve* $x^2 y + xy^2 + xy + y^2 + 3x = 0$.

(Purvanchal 2006; Kashi 11)

Solution: The curve is $x^2 y + xy^2 + xy + y^2 + 3x = 0$.

Equating the coefficient of highest powered by y (*i.e.*, of y^2) to zero, the asymptote parallel to y axis is $x - 1 = 0$.

Similarly equating to zero the coefficient of the highest power of x (*i.e.*, of x^2) the asymptotes parallel to x axis is $y = 0$.

Now to determine the third asymptote.

Putting $x = 1$ and $y = m$ in the highest degree term we get

$$\phi_3(m) = m + m^2.$$

Again from second degree term

$$\phi_2(m) = m + m^2.$$

Putting $\phi_3(m) = 0$, we have $m = 0, -1$.

Now $$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{m + m^2}{1 + 2m}.$$

Hence, when $m = 0$, $c = 0$

when $m = -1$, $c = 0$.

So, asymptotes are

$$y = 0, \ y = -x.$$

Thus, the required asymptotes are

$$x - 1 = 0, \ y = 0, \ y + x = 0.$$

Problem 5: *Find all the asymptotes of the curve*

$$y^3 - 5xy^2 + 8x^2y - 4x^3 - 3y^2 + 9xy - 6x^2 + 2y - 2x + 1 = 0.$$

Solution: Here $\phi_3(m) = m^3 - 5m^2 + 8m - 4 = 0$.

Putting $\phi_3(m) = 0$, we get

$$(m - 1)(m - 2)^2 = 0$$

or $$m = 1, 2, 2$$

and $$\phi_2(m) = -3m^2 + 9m - 6$$

$$\phi_1(m) = 2m - 2.$$

For $m = 1$, $$c = \frac{-\phi_2(m)}{\phi_3'(m)} \Rightarrow c = 0.$$

Therefore asymptote is $y = x$.

For $m = 2$, we have

$$\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0$$

or $$c^2 - 3c + 2 = 0 \quad \text{or} \quad c = 1, 2.$$

Hence there are two parallel asymptotes

$$y = 2x + 1 \text{ and } y = 2x + 2$$

Thus, the required asymptotes are

$$y = x, \ y = 2x + 1, \ y = 2x + 2.$$

Problem 6: *Find all the asymptotes of the curve* $2x(y - 3)^2 = 3y(x - 1)^2$.

Solution: The curve is

$$2x(y - 3)^2 = 3y(x - 1)^2.$$

or $\qquad 2x(y^2+9-6y)=3y(x^2+1-2x)$

or $\qquad 2xy^2-3yx^2-12xy+6xy+18x-3y=0$

or $\qquad 2xy^2-3yx^2-6xy+18x-3y=0.$

This equation is of three degree, it can not have more than three asymptotes.

Equating the coefficient of highest power of y (*i.e.*, of y^2) to zero, the asymptotes parallel to y-axis are given by $2x=0$ *i.e*,, $x=0$.

Again equating to zero the coefficient of the highest power of x (*i.e.* of x^2), the asymptotes parallel to x-axis are given by

$$-3y=0 \text{ i.e., } y=0.$$

Now to determine the third asymptotes putting $x=1$, $y=m$, in given curve, we get

$$\phi_3(m)=2m^2-3m$$

$$\phi_2(m)=-6m.$$

Putting $\phi_3(m)=0$, we have $2m^2-3m=0$.

$\therefore \qquad m=0, 3/2.$

We have $\qquad c=-\dfrac{\phi_2(m)}{\phi_3'(m)}=\dfrac{6m}{4m-3}.$

For $\qquad m=0,\ c=0.$

For $\qquad m=3/2,\ c=+3.$

$\therefore$ Asymptotes are

$$y=0 \text{ and } y=\frac{3}{2}x+3 \quad \text{or} \quad 2y=3x+6.$$

Thus, the asymptotes of the given curve are

$$x=0,\ y=0,\ 2y=3x+6.$$

Problem 7: *Find all the asymptotes of the curve* $y^2(x-2a)=x^3-a^3$.

(Bundelkhand 2008)

Solution: The curve is

$$y^2(x-2a)=x^3-a^3 \qquad ...(1)$$

or $\qquad xy^2-x^3-2ay^2+a^3=0$

or $\qquad x(y-x)(y+x)-2ay^2+a^3=0. \qquad ...(2)$

By (1), equation of asymptote parallel to y-axis is

$$x-2a=0. \qquad ...(3)$$

To get the equation (2) asymptotes corresponding to the factor $y-x$.

Let $\qquad y-x=k \quad$ or $\quad y=x+k.$

Substituting this value of y in (2), we get

$$xk(2x+k)-2a(x+k)^2+a^3=0$$

or $\qquad 2x^2(k-a)+xk(k-ya)+a^3-2ak^2=0.$

Equating to zero the coefficient of x^3, we get

$$k - a = 0 \quad \text{or} \quad k = a.$$

Hence corresponding asymptote is $y - x = a$.

Now to find asymptote corresponding to $x + y$, let $x + y = k$ or $x = k - y$ putting in (2) , we get

$$k(k - y)(2y - k) - 2ay^2 + a^3 = 0.$$

Equating to zero the coefficient of y^2, we get

$$k = -a.$$

Hence the third asymptote is $x + y = -a$.

Thus, the required asymptotes are

$$x = 2a, \; x - y + a = 0 \text{ and } x + y + a = 0.$$

Problem 8: *Find all the asymptotes of the curve*

$$y^3 - 2y^2x - yx^2 + 2x^3 + y^2 - 6xy + 5x^2 - 2y + 2x + 1 = 0.$$

Solution: Putting $x = 1$, $y = m$ in the highest degree term, we get

$$\phi_3(m) = m^3 - 2m^2 - m + 2$$

and $\qquad \phi_3'(m) = 3m^2 - 4m - 1.$

Equating $\phi_3(m)$ to zero, we get

$$m^3 - 2m^2 - m + 2 = 0$$

or $\qquad (m^2 - 1)(m - 2) = 0 \quad \text{or} \quad m = -1, 1, 2.$

Also $\qquad \phi_2(m) = 1 - 6m + 5m^2.$

Then $\qquad c = \dfrac{-\phi_2(m)}{\phi_3'(m)} = \dfrac{-(1 - 6m + 5m^2)}{3m^2 - 4m - 1}$

when $\qquad m = -1, c = -2$

$$m = 1, c = 0$$

$$m = 2, c = -3.$$

$\therefore$ The required asymptotes are

$$y = -x - 2, \; y = x, \; y = 2x - 3.$$

Problem 9: *Find all the asymptotes of the curve* $(x^2 - y^2)^2 - 4y^2 + y = 0$.

(Kanpur 2007)

Solution: The equation is of 4^{th} degree.

Putting $x = 1$, $y = m$, we have

$$\phi_4(m) = (1 - m^2)^2.$$

Similarly $\qquad \phi_3(m) = 0, \; \phi_2(m) = -4m^2 \text{ and } \phi_1(m) = m.$

Solving $\qquad \phi_4(m) = 0$, we get $m = 1, 1, -1, -1.$

The values of c for repeated values of m are given by

$$\frac{c^2}{2}\phi_4''(m) + c\,\phi_3'(m) + \phi_2\,(m) = 0$$

or $$\frac{c^2}{2}(-4 + 12m^2) + c.0 - 4m^2 = 0$$

or $$c^2\,(3m^2 - 1) - 2m^2 = 0.$$

$\therefore$ $$c^2 = 1 \text{ when } m = 1 \;\; i.e., c = \pm 1$$

and $$c^2 = 1 \text{ when } m = -1 \; i.e., c = \pm 1.$$

Hence the four asymptotes are

$$y = x \pm 1,\; y = -x \pm 1.$$

Problem 10: *Find the asymptotes of the curve* $x^3 + x^2 y - xy^2 - y^3 - 3x - y - 1 = 0.$

(Meerut 2001, 04, 06B; Gorakhpur 06)

Solution: The given curve is of degree 3, so it cannot have more than three asymptotes.

There are no asymptotes parallel to y-axis because the coefficient of the highest power of y *i.e.*, of y^3 is merely a constant. Similarly there are no asymptotes parallel to x-axis.

Now putting $y = m$ and $x = 1$ in the highest *i.e.*, third degree terms in the equation of the curve, we get

$$\phi_3\,(m) = 1 + m - m^2 - m^3.$$

The slopes of the asymptotes are given by the equation

$$\phi_3\,(m) = 0 \quad i.e., \; 1 + m - m^2 - m^3 = 0 \quad \text{or} \;\; (1 + m) - m^2\,(1 + m) = 0$$

or $$(1 + m)\,(1 - m^2) = 0 \quad \text{or} \quad (1 + m)^2\,(1 - m) = 0.$$

$\therefore$ $$m = 1, -1, -1.$$

Again putting $y = m$ and $x = 1$ in the second degree terms in the equation of the curve, we get

$\phi_2\,(m) = 0$, since there are no second degree terms.

To determine c, we have the equation

$$c\,\phi_3'\,(m) + \phi_2\,(m) = 0 \;\; i.e., \; c\,(1 - 2m - 3m^2) + 0 = 0. \qquad ...(1)$$

For $m = 1$, the equation (1) gives $-4c = 0$ or $c = 0$ and the corresponding asymptote is

$$y = 1.\,x + 0 \;\; i.e., \; y = x.$$

For $m = -1$, the equation (1) reduces to the identity $c\,.\,0 = 0$ and thus it fails to give c. In this case c is to be determined by the equation

$$\frac{c^2}{2\,!}\,\phi_3''(m) + \frac{c}{1\,!}\,\phi_2'\,(m) + \phi_1\,(m) = 0.$$

Now putting $y = m$ and $x = 1$ in the first degree terms in the equation of the curve, we get

$$\phi_1(m) = -3 - m.$$

So for $m = -1, c$ is to be given by the equation

$$\frac{1}{2}c^2 . (-2 - 6m) + 0 . c - 3 - m = 0$$

or $$c^2(1 + 3m) + 3 + m = 0.$$

Putting $m = -1$ in this equation, we get

$$-2c^2 + 2 = 0 \quad \text{or} \quad c^2 = 1 \quad \text{or} \quad c = \pm 1.$$

$\therefore$ The asymptotes corresponding to $m = -1$ are

$$y = -1\,x + 1 \quad \text{and} \quad y = -1\,x - 1.$$

Hence all the three asymptotes of the given curve are

$$y = x, y = -x + 1 \text{ and } y = -x - 1.$$

Problem 11: *Find the asymptotes of* $y^3 + x^2 y + 2xy^2 - y + 1 = 0.$ **(Lucknow 2011)**

Solution: Putting $y = m$ and $x = 1$ in the third degree and second degree terms separately, we get

$$\phi_3(m) = m^3 + m + 2m^2, \quad \text{and} \quad \phi_2(m) = 0.$$

$\therefore$ $$\phi_3'(m) = 3m^2 + 4m + 1.$$

Now the slopes of the asymptotes are given by the equation $\phi_3(m) = 0$

i.e., $$m^3 + 2m^2 + m = 0 \quad i.e., \quad m(m^2 + 2m + 1) = 0$$

i.e., $$m(m+1)^2 = 0.$$

$\therefore$ $$m = 0, -1, -1.$$

Again, c, is given by the equation

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{0}{3m^2 + 4m + 1}. \quad \text{...(1)}$$

When $m = 0$, we have $c = -\frac{0}{1} = 0$, and the corresponding asymptote is

$$y = 0 . x + 0 \quad i.e., \quad y = 0.$$

When $m = -1$, the equation (1) gives $c = -0/0$ which is indeterminate form. So the equation (1) fails to give c when $m = -1$. In this case c is to be determined from the equation

$$\frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0.$$

Putting $y = m$ and $x = 1$ in the first degree terms of the equation of the curve, we get $\phi_1(m) = -m$. Also $\phi_3''(m) = 6m + 4, \phi_2'(m) = 0.$

Therefore for $m = -1, c$ is given by the equation

$$\frac{1}{2}c^2(6m + 4) + c . 0 - m = 0 \quad i.e., \quad (3m + 2)c^2 - m = 0.$$

Putting $m = -1$ in this equation, we get

$$-c^2+1=0 \quad \text{or} \quad c^2=1 \quad \text{or} \quad c=\pm 1.$$

Hence $y=-x+1$ and $y=-x-1$ are two parallel asymptotes corresponding to the slope $m=-1$.

$\therefore$ All the three asymptotes of the curve are

$$y=0,\quad y+x-1=0 \text{ and } y+x+1=0.$$

Problem 12: *Find the asymptotes of the curve* $x^2y^3+x^3y^2=x^3+y^3$.

(Rohilkhand 2007; Kumaun 10)

Solution: The given curve is

$$x^2y^3+x^3y^2-x^3-y^3=0 \qquad \text{...(1)}$$

The curve (1) is of degree 5, so it cannot have more than five asymptotes.

The asymptotes parallel to y-axis are given by

$$x^2-1=0 \quad \textit{i.e.,} \quad x=\pm 1.$$

The asymptotes parallel to x-axis are given by

$$y^2-1=0 \quad \textit{i.e.,} \quad y=\pm 1.$$

To find the remaining oblique asymptotes, we put $y=m$ and $x=1$ in the highest i.e., fifth degree terms in the equation (1) and we get

$$\phi_5(m)=m^3+m^2.$$

The slopes of the asymptotes are given by the equation

$$\phi_5(m)=0 \quad \textit{i.e.,} \quad m^3+m^2=0 \quad \textit{i.e.,} \quad m^2(m+1)=0.$$

$$\therefore \qquad m=0,0,-1.$$

Again putting $y=m$ and $x=1$ in the next highest *i.e.,* fourth degree terms in the equation (1), we get

$$\phi_4(m)=0.$$ [Note that there are no fourth degree terms in the equation (1) and so $\phi_4(m)=0$.]

Now c is given by the equation

$$c\,\phi_5'(m)+\phi_4(m)=0$$

i.e., $$c(3m^2+2m)+0=0. \qquad \text{...(2)}$$

The asymptotes corresponding to $m=0$ are parallel to x-axis and have already been found and so no need of finding c for $m=0$.

When $m=-1$, we have from (2), $c=0$ and so the corresponding asymptote is $y=-1x+0$ *i.e.,* $y=-x$.

Hence all the five asymptotes of the given curve are

$$x=\pm 1,\ y=\pm 1,\ y=-x.$$

Problem 13: *Find the asymptotes of the curve* $x^3-2x^2y+xy^2+x^2-xy+2=0$.

(Garhwal 2010; Kanpur 14)

Solution: The given curve is of degree 3. So it cannot have more than three asymptotes.

Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2), we get the asymptote parallel to y-axis as $x = 0$. Also there is no asymptote parallel to x-axis because the coefficient of x^2 is merely a constant.

Now we proceed to find the remaining oblique asymptotes.

Putting $y = m$ and $x = 1$ in the third degree and second degree terms separately, we get

$$\phi_3(m) = 1 - 2m + m^2 \quad \text{and} \quad \phi_2(m) = 1 - m.$$

The slopes of the asymptotes are given by the equation

$$\phi_3(m) = 0 \quad i.e., \quad 1 - 2m + m^2 = 0 \quad i.e., \quad (1 - m)^2 = 0.$$

$\therefore \qquad m = 1, 1.$

To determine c, we have the equation

$$c\,\phi_3'(m) + \phi_2(m) = 0 \quad i.e., \quad c(-2 + 2m) + (1 - m) = 0. \qquad ...(1)$$

For $m = 1$, the equation (1) reduces to the identity $c \,.\, 0 + 0 = 0$ and thus it fails to give c. In this case c is to be determined by the equation

$$\frac{c^2}{2!}\phi_3''m + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0.$$

Now $\phi_3''(m) = 2, \phi_2'(m) = -1$ and $\phi_1(m) = 0$ because there are no first degree terms in the equation of the curve. So for $m = 1, c$ is to be given by

$$\frac{1}{2}c^2 \cdot (2) + c \cdot (-1) + 0 = 0 \quad i.e., \quad c^2 - c = 0 \quad i.e., \quad c(c - 1) = 0.$$

$\therefore \qquad c = 0, \; 1.$

Hence $y = x + 0$ and $y = x + 1$ are two parallel asymptotes corresponding to the slope $m = 1$.

$\therefore$ The required asymptotes are $x = 0, y = x$ and $y = x + 1$.

Problem 14: *Find all the asymptotes of the curve*

$$x^3 - 5x^2y + 8xy^2 - 4y^3 + x^2 - 3xy + 2y^2 - 1 = 0.$$

Solution: Putting $x = 1$, $y = m$; we have

$$\phi_3(m) = 1 - 5m + 8m^2 - 4m^3.$$

Similarly $\qquad \phi_2(m) = 1 - 3m + 2m^2, \phi_1(m) = 0.$

Solving $\phi_3(m) = 0$, we have

$$m = \frac{1}{2}, \frac{1}{2}.$$

For $m = 1$, we have

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = \frac{-(1 - 3m + 2m^2)}{-5 + 16m - 12m^2} = 0.$$

$\therefore \quad$ Asymptote is $y = x + 0$.

For $m = \frac{1}{2}$, we have

$$\frac{c^2}{2}\phi_3''(m) + c\,\phi_2'(m) + \phi_1(m) = 0$$

or $$\frac{c^2}{2}(16 - 24m) + c\,(-3 + 4m) + 0 = 0$$

or $$c^2(8 - 12m) + c\,(-3 + 4m) = 0$$

or $$c^2.2 - c = 0 \text{ when } m = 1/2$$

or $$c = 0, 1/2.$$

∴ Asymptotes are

$$y = \frac{1}{2}x + 0,\ y = \frac{1}{2}x + \frac{1}{2}.$$

∴ The required asymptotes are

$$2y - x = 0, 2y - x - 1 = 0 \text{ and } y = x.$$

Problem 15: *Find all the asymptotes of the curve* $(2x - 3y + 1)^2 (x + y) = 8x - 2y + 9.$

Solution: The asymptote corresponding to the linear factor $x + y$ is given by

$$x + y = \text{Lim}\frac{8x - 2y + 9}{(2x - 3y + 1)^2}.$$

When $x \to \infty$ and $\frac{y}{x} \to -1$

$$x + y = \text{Lim}\,\frac{8 - 2y/x + 9/x}{x\,(2 - 3y/x + 1/x)^2}$$

or $$x + y = 0.$$

The asymptotes corresponding to the repeated factor $(2x - 3y + 1)^2$ are

$$2x - 3y + 1 = \pm\sqrt{\text{Lim}\left\{\frac{8x - 2y + 9}{x + y}\right\}}.$$

When $x \to \infty$, $\frac{y}{x} \to \frac{2}{3}$

$$2x - 3y + 1 = \pm\sqrt{\text{Lim}\frac{8 - 2y/x + 9/x}{1 + y/x}}$$

$$= \pm\sqrt{\frac{8 - 4/3}{1 + 2/3}} = \pm 2.$$

∴ $$2x - 3y + 3 = 0 \text{ and } 2x - 3y = 1.$$

Hence the asymptotes are

$$2x - 3y = 1, 2x - 3y + 3 = 0, x + y = 0.$$

Comprehensive Problems 3

Problem 1: *Find the asymptotes of the curve*

$$(x^2 - y^2)(y^2 - 4x^2) - 6x^3 + 5x^2 y + 3xy^2 - 2y^3 - x^2 + 3xy - 1 = 0.$$

Solution: Factorizing the highest degree terms, the given equation can be re-written as

$$(x + y)(x - y)(y + 2x)(y - 2x) = 6x^3 - 5x^2 y - 3xy^2 + 2y^3 + x^2 - 3xy + 1.$$

$\therefore$ The asymptotes are parallel to the lines $x + y = 0$, $x - y = 0$, $y + 2x = 0$ and $y - 2x = 0$.

Asymptote parallel to $x + y = 0$ is

$$x + y = \lim_{\substack{x \to \infty,\\ y/x \to -1}} \left[\frac{6x^3 - 5x^2 y - 3xy^2 + 2y^3 + x^2 - 3xy + 1}{(x - y)(y + 2x)(y - 2x)}\right], \quad [\because m = -1]$$

$$= \lim \left[\frac{6 - 5(y/x) - 3(y^2/x^2) + 2(y^3/x^3) + (1/x) - 3(y/x^2) + (1/x^3)}{(1 - y/x)(2 + y/x)(-2 + y/x)}\right],$$

on dividing the numerator and denominator by x^3

$$= \frac{6 + 5 - 3 - 2}{2 \,.\, 1 \,.\, (-3)} = \frac{6}{-6} = -1.$$

Hence $x + y + 1 = 0$ is one asymptote of the curve.

The second asymptote parallel to $x - y = 0$ is

$$x - y = \lim_{\substack{x \to \infty,\\ y/x \to 1}} \left[\frac{6x^3 - 5x^2 y - 3xy^2 + 2y^3 + x^2 - 3xy + 1}{(x + y)(y + 2x)(y - 2x)}\right].$$

Taking limits as above, we get $x - y = 0$ as the second asymptote of the curve.

The third asymptote parallel to $y + 2x = 0$ is

$$y + 2x = \lim_{\substack{x \to \infty,\\ y/x \to -2}} \left[\frac{6x^3 - 5x^2 y - 3xy^2 + 2y^3 + x^2 - 3xy + 1}{(x - y)(x + y)(y - 2x)}\right].$$

Taking limits as above, we get $y + 2x = -1$, as the third asymptote of the curve.

The fourth asymptote parallel to $(y - 2x)$ is

$$y - 2x = \lim_{\substack{x \to \infty, \\ y/x \to 2}} \left[\frac{6x^3 - 5x^2 y - 3xy^2 + 2y^3 + x^2 - 3xy + 1}{(x - y)(x + y)(y + 2x)} \right].$$

Taking limits as above, we get $y - 2x = 0$, as the fourth asymptote of the curve.

Hence the required asymptotes are

$$x + y + 1 = 0, \; x - y = 0, \; 2x + y + 1 = 0 \text{ and } 2x - y = 0.$$

Problem 2: *Find the asymptotes of the curve* $(y - x)^2 x - 3y(y - x) + 2x = 0.$

Solution: The given equation is of third degree. So there are at most three asymptotes.

Dividing each term by x and taking limits, the asymptotes corresponding to the factor $(y - x)^2$ are given by

$$(y - x)^2 - 3(y - x) \lim_{\substack{x \to \infty, \\ y/x \to 1}} \left(\frac{y}{x} \right) + \lim_{\substack{x \to \infty, \\ y/x \to 1}} \left(\frac{2x}{x} \right) = 0$$

or $\qquad (y - x)^2 - 3(y - x) + 2 = 0.$

Hence $\qquad y - x = \frac{1}{2} [3 \pm \sqrt{(9 - 8)}]$ *i.e.*, $= 2$, or 1.

Therefore, $\qquad y - x = 2$ and $y - x = 1$ are the two asymptotes.

Also equating to zero the coefficient of the highest power of y, we get the asymptote parallel to y-axis as $x = 3$.

Hence the required asymptotes are $x = 3$, $y - x = 2$ and $y - x = 1$.

Problem 3: *Find the asymptotes of the curve*

$$(y - x)(y - 2x)^2 + (y + 3x)(y - 2x) + 2x + 2y - 1 = 0.$$

(Kanpur 2010; Meerut 12B; Bundelkhand 14; Purvanchal 14)

Solution: The asymptotes corresponding to the factor $(y - 2x)^2$ are given by

$$(y - 2x)^2 + (y - 2x) \lim_{\substack{x \to \infty, \\ y/x \to 2}} \left[\frac{y + 3x}{y - x} \right] + \lim_{\substack{x \to \infty, \\ y/x \to 2}} \left[\frac{2x + 2y - 1}{y - x} \right] = 0$$

or

$$(y - 2x)^2 + (y - 2x) \lim_{\substack{x \to \infty, \\ y/x \to 2}} \left[\frac{(y/x) + 3}{(y/x) - 1} \right] + \lim_{\substack{x \to \infty, \\ y/x \to 2}} \left[\frac{2 + 2(y/x) - (1/x)}{(y/x) - 1} \right] = 0$$

or $\quad (y-2x)^2 + 5(y-2x) + 6 = 0$

i.e., $\quad y - 2x = \frac{1}{2}[-5 \pm \sqrt{(25-24)}] = \frac{1}{2}(-5 \pm 1)$

i.e., $\quad y - 2x = \frac{1}{2}(-5+1)$ and $y - 2x = \frac{1}{2}(-5-1)$.

$\therefore$ $y = 2x - 2$ and $y = 2x - 3$ are the asymptotes corresponding to the factor $(y-2x)^2$.

Also the asymptote corresponding to the factor $(y - x)$ is

$$y - x = \lim_{x\to\infty,\ y/x\to 1}\left[\frac{-(y+3x)(y-2x)-2x-2y+1}{(y-2x)^2}\right]$$

$$= \lim_{x\to\infty,\ y/x\to 1}\left[\frac{-\{(y/x)+3\}\{(y/x)-2\} - 2(1/x) - 2(y/x^2) + (1/x^2)}{\{(y/x)-2\}^2}\right]$$

$= 4$, on taking limits.

Hence $y = x + 4$ is another asymptote of the given curve.

Problem 4: *Find all the asymptotes of the curve*

$$(x-2y)^2(x-y) - 4y(x-2y) - (8x+7y) = 0.$$

(Meerut 2005B; Bundelkhand 07)

Solution: The curve is $(x-2y)^2(x-y) - 4y(x-2y) - (8x+7y) = 0$.

The asymptote corresponding to the factor $(x - y)$ is

$$x - y = \lim_{x\to\infty,\ y/x\to 1}\frac{4y(x-2y)+(8x+7y)}{(x-2y)^2}$$

$$= \lim_{x\to\infty,\ y/x\to 1}\frac{4\left(\frac{y}{x}\right)\left\{1-2\left(\frac{y}{x}\right)\right\}+\frac{1}{x}\left(8+7\frac{y}{x}\right)}{\left\{1-2\left(\frac{y}{x}\right)\right\}^2} = \frac{4.1(1-2)+0}{[1-2(1)]^2} = -4.$$

Again the asymptotes corresponding to the factor $(x - 2y)$ are

$$(x-2y)^2 - (x-2y)\lim_{x\to\infty,\ y/x\to 1/2}\frac{4y}{x-y} - \lim_{x\to\infty,\ y/x\to 1/2}\frac{8x+7y}{x-y} = 0$$

or $\quad (x-2y)^2 - (x-2y).4 - 23 = 0$

or $\quad x - 2y = \frac{1}{2}[4 \pm \sqrt{16+4.23}] = 2 \pm 3\sqrt{3}$

Hence, the required asymptotes are

$$x - y = -4 \text{ and } x - 2y = 2 \pm 3\sqrt{3}.$$

Problem 5: *Find all the asymptotes of the curve* $(y-a)^2(x^2-a^2)=x^4+a^4.$

Solution: The given curve is

$$y^2(x^2-a^2)-2ay(x^2-a^2)+a^2(x^2-a^2)=x^4+a^4$$

or $$x^2(y-x)(y+x)-2ax^2y+a^2(x^2-y^2)+2a^3y-2a^4=0.$$

Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2), we get $x^2-a^2=0$ or $x=\pm a$ as the asymptotes parallel to y-axis.

The other asymptotes are

$$y-x=\lim_{\substack{x\to\infty,\\ y/x\to 1}}\left[\frac{2ax^2y-a^2(x^2-y^2)-2a^3y+2a^4}{x^2(y+x)}\right]$$

$$=\lim_{\substack{x\to\infty,\\ y/x\to 1}}\left[\frac{2a(y/x)-a^2\{(1/x)-(y^2/x^3)\}-2a^3(y/x^3)+2a^4(1/x^3)}{[1+(y/x)]}\right]=a,$$

and $$y+x=\lim_{\substack{x\to\infty,\\ y/x\to -1}}\left[\frac{2ax^2y-a^2(x^2-y^2)-2a^3y+2a^4}{x^2(y-x)}\right]$$

$$=\lim_{\substack{x\to\infty,\\ y/x\to -1}}\left[\frac{2a(y/x)-a^2\{(1/x)-(y^2/x^3)\}-2a^3(y/x^3)+2a^4(1/x^3)}{(y/x)-1}\right]=a.$$

$\therefore$ The required asymptotes are $x=\pm a$ and $y=\pm x+a$.

Problem 6: *Find the asymptotes of the curve* $x(y-3)^3=4y(x-1)^3.$

Solution: The equation of the given curve can be written as

$$xy(y-2x)(y+2x)=9xy^2-12yx^2-15xy+27x-4y.$$

Here the coefficient of the highest power of $y=x$.

Therefore $x=0$ is an asymptote parallel to y-axis.

Again the coefficient of the highest power of $x=4y$.

Therefore $y=0$ is an asymptote parallel to x-axis.

The other asymptotes are given by

$$y-2x=\lim_{\substack{x\to\infty,\\ y/x\to 2}}\left[\frac{9xy^2-12yx^2-15xy+27x-4y}{xy(y+2x)}\right]=\frac{3}{2};$$

i.e., $2y-4x=3;$

and $$y + 2x = \lim_{\substack{x \to \infty, \\ y/x \to -2}} \left[\frac{9xy^2 - 12yx^2 - 15xy + 27x - 4y}{xy(y - 2x)} \right] = \frac{15}{2}$$

i.e., $$2y + 4x = 15.$$

Problem 7: *Find the asymptotes of the curve*

$$(x - y)^2 (x^2 + y^2) - 10(x - y)x^2 + 12y^2 + 2x + y = 0.$$

(Kumaun 2013)

Solution: The given curve is

$$(x - y)^2 (x^2 + y^2) - 10(x - y)x^2 + 12y^2 + 2x + y = 0.$$

The equation of the curve is of fourth degree and so there are at most four asymptotes. Dividing each term by the coefficient $x^2 + y^2$ of $(x - y)^2$ and taking limits, the asymptotes corresponding to the factor $(x - y)^2$ are given by

$$(x - y)^2 - 10(x - y) \lim_{\substack{x \to \infty, \\ y/x \to 1}} \frac{x^2}{x^2 + y^2} + \lim_{\substack{x \to \infty, \\ y/x \to 1}} \frac{12y^2 + 2x + y}{x^2 + y^2} = 0$$

or $$(x - y)^2 - 10(x - y) \lim_{\substack{x \to \infty, \\ y/x \to 1}} \frac{1}{1 + (y/x)^2} + \lim_{\substack{x \to \infty, \\ y/x \to 1}} \frac{12(y/x)^2 + (2/x) + (y/x)(1/x)}{1 + (y/x)^2} = 0,$$

on dividing the Nr. and Dr. by x^2

or $$(x - y)^2 - 10(x - y) \cdot (1/2) + (12/2) = 0$$

or $$(x - y)^2 - 5(x - y) + 6 = 0$$

i.e., $$x - y = \frac{5 \pm \sqrt{(25 - 24)}}{2} = \frac{5 \pm 1}{2} = 2 \text{ or } 3$$

i.e., $x - y = 2$ and $x - y = 3$.

Again $x^2 + y^2 = (x + iy)(x - iy)$ implies that the remaining linear factors of the fourth degree terms in the equation of the curve are imaginary and so the other two asymptotes are imaginary.

Problem 8: *Find all the asymptotes of the curve*

$$x^2 (x + y)(x - y)^2 + ax^3 (x - y) - a^2 y^3 = 0.$$

(Kumaun 2011)

Solution: The given equation is of fifth degree. So there are at most five asymptotes. Dividing each term by the coefficient of $(x - y)^2$ and taking the limits, the asymptotes corresponding to the factor $(x - y)^2$ are given by

$$(x-y)^2+(x-y)\lim_{\substack{x\to\infty,\\ y/x\to 1}}\left[\frac{a}{x+y}\right]-\lim_{\substack{x\to\infty,\\ y/x\to 1}}\left[\frac{a^2y^3}{x^2(x+y)}\right]=0$$

or $$(x-y)^2+(x-y)\lim\left(\frac{a/x}{1+y/x}\right)-\lim a^2\left(\frac{y^3/x^2}{1+y/x}\right)=0.$$

Taking limits, we get

$$(x-y)^2+(x-y)\,.\,0-\frac{a^2}{2}=0$$

or $$x-y=\pm\frac{a}{\sqrt{2}} \text{ be two asymptotes.}$$

Also the asymptote corresponding to the factor $(x+y)$ is

$$(x+y)=\lim_{\substack{x\to\infty,\\ y/x\to -1}}\left[\frac{ax^2(y-x)+a^2y^3}{x^2(x-y)^2}\right]$$

$$=\lim_{\substack{x\to\infty,\\ y/x\to -1}}\left[\frac{a\left\{\frac{y}{x}-1\right\}\left(\frac{1}{x}\right)+a^2\left(\frac{y^3}{x^3}\right)\left(\frac{1}{x}\right)}{\left\{1-\left(\frac{y}{x}\right)\right\}^2}\right].$$

Taking limits, we have $x+y=0$ is the asymptote.

Also asymptotes parallel to y-axis is $x^2-a^2=0$ or $x=\pm a$.

Hence, the required asymptotes are $x=\pm a,\ x-y=\pm\frac{a}{\sqrt{2}},\ x+y=0.$

Problem 9: *Find all the asymptotes of the curve* $(x-y+1)(x-y-2)(x+y)=8x-1$.

Solution: The given curve may be written as

$$(x+y)(x-y)^2-(x^2-y^2)-10x-2y+1=0.$$

The asymptote corresponding to the factor $(x+y)$ is

$$x+y=\lim_{\substack{x\to\infty,\\ y/x\to -1}}\left[\frac{8x-1}{(x-y+1)(x-y-2)}\right]$$

$$=0.$$ [Dividing the numerator and the denominator by x^2 and taking the limits.]

Also the asymptotes corresponding to the factor $(x-y)^2$ are given by

$$(x-y)^2-(x-y)=\lim_{\substack{x\to\infty,\\ y/x\to 1}}\left[\frac{10x+2y-1}{x+y}\right]=6$$

i.e., $$x-y=\frac{1}{2}[1\pm\sqrt{(1+24)}]=3 \text{ or } -2$$

i.e., $$y=x-3 \text{ and } y=x+2.$$

$\therefore$ The required asymptotes are $y+x=0$, $y=x-3$ and $y=x+2$.

Problem 10: *Find the asymptotes of the curve*

$$xy(x^2 - y^2)(x^2 - 4y^2) + xy(x^2 - y^2) + x^2 + y^2 - 7 = 0.$$

Solution: The given equation can be put in the form

$$[xy(x^2 - y^2)(x^2 - 4y^2)] + [xy(x^2 - y^2) + x^2 + y^2 - 7] = 0.$$

This equation is of the form $F_n + F_{n-2} = 0$, where F_n can be broken up into n linear factors so as to represent n straight lines no two of which are parallel or coincident.

$\therefore$ All the asymptotes are given by $F_n = 0$

or $\quad xy(x - y)(x + y)(x - 2y)(x + 2y) = 0.$

Then by inspection, the asymptotes are

$$x = 0, y = 0, x - y = 0, x + y = 0, x - 2y = 0 \text{ and } x + 2y = 0.$$

Comprehensive Problems 4

Problem 1: *Find the asymptotes of the curve* $x^2y - xy^2 + xy + y^2 + x - y = 0$ *and show that they cut the curve again in three points which lie on the straight line* $x + y = 0$.

Solution: The equation of the curve is

$$x^2y - xy^2 + xy + y^2 + x - y = 0.$$

By putting $x = 1, y = m$ we have

$$\phi_3(m) = m - m^2, \quad \phi_3'(m) = 1 - 2m.$$

Similarly $\quad \phi_2(m) = m + m^2.$

Also $\quad \phi_3(m) = 0$

or $\quad m(1 - m) = 0 \text{ or } m = 0, 1.$

Also $\quad c = \dfrac{-\phi_2(m)}{\phi_3'(m)} = \dfrac{-(m + m^2)}{1 - 2m}.$

For $\quad m = 0, c = 0$

$\quad m = 1, c = 2.$

$\therefore$ The asymptotes are $y = 0, y = x + 2$.

Also equating the coefficient of highest power of y to zero, we have $x = 1$ as the asymptote.

Thus the required asymptotes are

$$x = 1, y = 0, y = x + 2.$$

Their combined equation is

$$(x - 1)y(y - x - 2) = 0$$

or $\quad x^2y + xy + xy^2 - 2y + y^2 = 0.$

Subtracting this equation from the given equation, we find that the point of intersection of the curve and asymptote lies on $x + y = 0$.

Problem 2: *Show that asymptotes of the cubic*

$$x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0$$

cut the curve in three points which lie on the straight line $x - y + 1 = 0$.

(Kumaun 2007; Kanpur 09; Avadh 10)

Solution: The given curve is

$$x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0. \quad \text{...(1)}$$

$\therefore$ $\phi_3(m) = 1 - 2m^3 + 2m - m^2 = 0$ gives $m = 1, -1, -\frac{1}{2}$.

Also $\phi_3'(m) = -6m^2 + 2 - 2m$ and $\phi_2(m) = m - m^2$.

$$\therefore \quad c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\frac{(m - m^2)}{(2 - 2m - 6m^2)}.$$

When $m = 1, c = 0$; when $m = -1, c = -1$; and when $m = -\frac{1}{2}$, $c = \frac{1}{2}$.

$\therefore$ The asymptotes of (1) are $y = x$, $y = -x - 1$ and $y = -\frac{1}{2}x + \frac{1}{2}$.

Hence the combined equation of the asymptotes of (1) is

$$(x - y)(x + y + 1)(x + 2y - 1) = 0$$

or $$x^3 - 2y^3 + 2x^2y - xy^2 + xy - y^2 - x + y = 0. \quad \text{...(2)}$$

Subtracting (2) from (1), we get $x - y + 1 = 0$,

which shows that the points of intersection of the curve and its asymptotes lie on the straight line $x - y + 1 = 0$.

Also the three asymptotes cut the cubic in $n(n - 2)$, *i.e.*, $3(3 - 2) = 3$ points. As shown above, these, three points lie on the straight line $x - y + 1 = 0$.

Problem 3: *Find the equation of the straight line on which lie the three points of intersection of the curve* $(x + a)y^2 = (y + b)x^2$ *and its asymptotes.* **(Garhwal 2011)**

Solution: The given curve is

$$(x + a)y^2 = (y + b)x^2 \quad \text{or} \quad x^2y + x^2b - xy^2 - ay^2 = 0.$$

By putting $x = 1$, $y = m$, we have

$$\phi_3(m) = m^2 - m, \quad \phi_3'(m) = 2m - 1$$

and $$\phi_2(m) = am^2 - b.$$

Also $\phi_3(m) = 0$, we get $m = 0, 1$

$$\therefore \quad c = \frac{-\phi_2(m)}{\phi_3'(m)} = -\frac{(am^2 - b)}{2m - 1}.$$

For $m = 0$, $c = -b$.

For $m = 1$, $c = -(a - b)$.

$\therefore$ Asymptotes are $y = -b$, $y = x - (a - b)$.

Also equating the coefficient of highest power of y to zero, we have $x + a = 0$ as the asymptotes

Thus the required asymptotes are

$$x + a = 0,\ y + b = 0,\ y = x + b - a.$$

The combined equation is

$$(x + a)(y + b)(y - x - b + a) = 0.$$

Subtracting this equation from the given equation, we find that the point of intersection of the curve and asymptotes lie on

$$a^2 (y + b) = b^2 (x + a).$$

Problem 4: *Show that the eight points of intersection of the curve*

$$xy (x^2 - y^2) + x^2 + y^2 = a^2$$

and its asymptotes lie on a circle whose centre is at the origin.

Solution: The curve is

$$xy (x^2 - y^2) + x^2 + y^2 = a^2$$

or $$x^3 y - y^3 x + x^2 + y^2 - a^2 = 0.$$

Putting $x = 1,\ y = m$, we have

$$\phi_4 (m) = m - m^3,\ \phi_4'(m) = 1 - 3m^2$$

and $$\phi_3 (m) = 0.$$

Also $\phi_4 (m) = 0$, we get $m = 0, 1, -1$.

$$\therefore \qquad c = \frac{-\phi_3 (m)}{\phi_4'(m)} = 0.$$

For $m = 0, 1, -1 \Rightarrow c = 0$.

$\therefore$ Asymptotes are $y = 0,\ y = x,\ y = -x$.

The combined equation is

$$y (y - x)(y + x) = 0 \qquad \text{or} \qquad x^3 y - x y^3 = 0.$$

Subtracting this equation from given equation we find the point of intersection of the curve and the asymptotes lie on

$$x^2 + y^2 = a^2.$$

Problem 5: *Show that the four asymptotes of the curve*

$$(x^2 - y^2)(y^2 - 4x^2) + 6x^3 - 5x^2 y - 3xy^2 + 2y^3 - x^2 + 3xy - 1 = 0$$

cut the curve in eight points which lie on the circle $x^2 + y^2 = 1$.

Solution: The given curve is

$$P \equiv (x^2 - y^2)(y^2 - 4x^2) + 6x^3 - 5x^2 y - 3xy^2 + 2y^3 - x^2 + 3xy - 1 = 0. \qquad \text{...(1)}$$

Here $$\phi_4 (m) = (1 - m^2)(m^2 - 4) = -4 + 5m^2 - m^4.$$

The slopes of the asymptotes are given by the equation

$$\phi_4(m) = (1 - m^2)(m^2 - 4) = 0.$$

$\therefore$ $m = \pm 1, \pm 2.$

Also $\phi_3(m) = 6 - 5m - 3m^2 + 2m^3$

and $\phi'(m) = 10m - 4m^3.$

Now c is given by the equation

$$c\,\phi_4'(m) + \phi_3(m) = 0$$

i.e., $c(10m - 4m^3) + 6 - 5m - 3m^2 + 2m^3 = 0$

i.e., $c = \dfrac{6 - 5m - 3m^2 + 2m^3}{4m^3 - 10m}.$

When $m = 1, c = 0$; when $m = -1, c = 1$;
when $m = 2, c = 0$; and when $m = -2, c = 1.$
Thus the asymptotes of the curve (1) are

$$y = x,\ y = -x + 1,\ y = 2x \text{ and } y = -2x + 1.$$

The combined equation of the asymptotes is

$$(y - x)(y + x - 1)(y - 2x)(y + 2x - 1) = 0$$

or $[(y^2 - x^2) - y + x][(y^2 - 4x^2) - y + 2x] = 0$

or $(y^2 - x^2)(y^2 - 4x^2) - y^3 + 2xy^2 + x^2y - 2x^3 - y^3 + 4x^2y + xy^2 - 4x^3 + y^2 - 2xy - xy + 2x^2 = 0$

or $Q \equiv (y^2 - x^2)(y^2 - 4x^2) - 6x^3 + 5x^2y + 3xy^2 - 2y^3 + y^2 - 3xy + 2x^2 = 0.$ ...(2)

Now each asymptote of (1) will cut it in $4 - 2$ *i.e.*, 2 points. Therefore the four asymptotes will cut it in 4×2 *i.e.*, 8 points.

Now taking $\lambda = 1$, $P + \lambda Q = 0$ gives $x^2 + y^2 - 1 = 0$ *i.e.*, $x^2 + y^2 = 1$, which is the equation of a circle.

Hence the eight points of intersection of (1) and (2) lie on the circle $x^2 + y^2 = 1.$

Problem 6: *Show that the eight points of intersection of the curve*

$$x^4 - 5x^2y^2 + 4y^4 + x^2 - y^2 + x + y + 1 = 0$$

and its asymptotes lie on a rectangular hyperbola.

Solution: The equation of the given curve is

$$x^4 - 5x^2y^2 + 4y^4 + x^2 - y^2 + x + y + 1 = 0 \qquad ...(1)$$

or $(x^2 - y^2)(x^2 - 4y^2) + x^2 - y^2 + x + y + 1 = 0$

or $(x - y)(x + y)(x - 2y)(x + 2y) + x^2 - y^2 + x + y + 1 = 0.$

$\therefore$ By inspection, the combined equation of the asymptotes of (1) is

$$(x - y)(x + y)(x - 2y)(x + 2y) = 0$$

or $$x^4 - 6x^2 y^2 + 4y^4 = 0. \qquad ...(2)$$

Now each asymptote of (1) will cut it in $4 - 2$ *i.e.*, 2 points. Therefore the four asymptotes will cut it in 4×2 *i.e.*, 8 points.

Subtracting (2) from (1), we get

$$x^2 - y^2 + x + y + 1 = 0. \qquad ...(3)$$

The curve (3) passes through the eight points of intersection of (1) and (2). Also the conic (3) is a rectangular hyperbola because in its equation the sum of the coefficients of x^2 and y^2 is zero.

Hence the eight points of intersection of (1) and (2) lie on a rectangular hyperbola.

Problem 7: *Find the equation of the quadratic curve which has $x = 0$, $y = 0$, $y = x$ and $y = -x$ for asymptotes and which passes through the point (a, b) and cuts its asymptotes again in eight points that lie on a circle whose centre is origin and radius a.*

Solution: The combined equation of the asymptotes is

$$xy(y^2 - x^2) = 0. \qquad ...(1)$$

The required curve passes through the points of intersection of the asymptotes and the given circle $x^2 + y^2 - a^2 = 0$.

Let the equation of the curve be

$$xy(y^2 - x^2) + \lambda(x^2 + y^2 - a^2) = 0. \qquad ...(2)$$

Now this curve also passes through the point (a, b).

$\therefore$ $$ab(b^2 - a^2) + \lambda(a^2 + b^2 - a^2) = 0,$$

i.e., $$\lambda = (a/b)(a^2 - b^2).$$

Substituting this value of λ in (2), we have

$$bxy(y^2 - x^2) + a(a^2 - b^2)(x^2 + y^2 - a^2) = 0$$

as the required equation of the curve.

Problem 8: *Find the equation of the cubic which has the same asymptotes as the curve $x^3 - 6x^2 y + 11xy^2 - 6y^3 + x + y + 1 = 0$ and which touches the axis of y at the origin and passes through the point $(3, 2)$.* **(Agra 2007)**

Solution: The given curve is

$$(x^3 - 6x^2 y + 11xy^2 - 6y^3) + (x + y) + 1 = 0.$$

Here $$\phi_3(m) = 1 - 6m + 11m^2 - 6m^3 = (1 - m)(1 - 2m)(1 - 3m).$$

Therefore $$\phi_3(m) = 0 \text{ gives } m = 1, \frac{1}{2}, \frac{1}{3}.$$

Also $$\phi_2(m) = 0.$$

$\therefore \qquad c = -\dfrac{\phi_2(m)}{\phi_3'(m)} = 0$, for all the three values of m.

$\therefore$ The asymptotes of the given curve are

$$y = x;\ y = \frac{1}{2}x \quad \text{and} \quad y = \frac{1}{3}x.$$

Hence the combined equation of the asymptotes is

$$(x - y)(x - 2y)(x - 3y) = 0. \qquad \ldots(1)$$

Now the most general equation of any curve, having these asymptotes, is of the form

$$(x - y)(x - 2y)(x - 3y) + F_1 = 0, \qquad \text{[See article 12, Cor.]}$$

where F_1 is of the first degree in x and y (say $F_1 = ax + by + c$).

If the curve $(x - y)(x - 2y)(x - 3y) + ax + by + c = 0$, passes through the origin (0, 0), then $c = 0$ and the equation of the curve becomes

$$(x - y)(x - 2y)(x - 3y) + ax + by = 0. \qquad \ldots(2)$$

Equating to zero, the lowest degree terms in (2), we get $ax + by = 0$, as the equation of the tangent at origin.

But $x = 0$ *i.e.*, y-axis is given to be the tangent at origin.

Therefore $b = 0$ and hence the equation of the curve is

$$(x - y)(x - 2y)(x - 3y) + ax = 0 \qquad \ldots(3)$$

It passes through the point (3, 2), (given).

$\therefore \qquad (3 - 2)(3 - 4)(3 - 6) + 3a = 0$

or $\qquad a = -1$.

Therefore the required equation of the curve is

$$(x - y)(x - 2y)(x - 3y) - x = 0, \qquad \text{[Putting } a = -1 \text{ in (3)]}$$

or $\qquad x^3 - 6x^2y + 11xy^2 - 6y^3 - x = 0.$

Comprehensive Problems 5

Problem 1: *If* α *be a root of the equation* $f(\theta) = 0$, *then write the equation of asymptote of the polar curve* $1/r = f(\theta)$ *corresponding to the root* α. **(Meerut 2001)**

Solution: Let P be any point (r, θ) on the curve

$$1/r = f(\theta). \qquad \ldots(1)$$

As $P \to \infty, r \to \infty$ and consequently $f(\theta) \to 0$.

Let OT be the perpendicular to the radius vector OP.

Then OT (*i.e.*, the polar subtangent of the curve at P) is given by

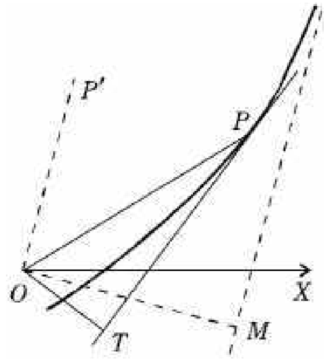
$$OT = r^2 \frac{d\theta}{dr} = -\frac{1}{f'(\theta)}. \qquad \left[\because \text{ from (1), } -\frac{1}{r^2}\frac{dr}{d\theta} = f'(\theta)\right]$$

Now let $\theta \to \alpha$. Then $f(\theta) \to 0$. $\quad [\because f(\alpha) = 0]$

Therefore $r \to \infty, PT \to$ to an asymptote,

and $\quad OT \to \left(r^2 \dfrac{d\theta}{dr}\right)_{\theta = \alpha}$

or $\quad OT \to -\dfrac{1}{f'(\alpha)}$ if $f'(\alpha) \neq 0$.



Also OP and PT will tend to become parallel, and the angle OTP will tend to a right angle and OT will tend to OM where OM is perpendicular to the asymptote. Hence, the asymptote is the straight line parallel to the radius vector $\theta = \alpha$ and situated at a distance $\left(r^2 \dfrac{d\theta}{dr}\right)_{\theta = \alpha}$ from O.

Now the polar equation of the straight line, the perpendicular p on which from the origin makes an angle β with the initial line, is $r \cos(\theta - \beta) = p$. **[Remember]**

Here, for the asymptote,

$$p = OM = \left(r^2 \frac{d\theta}{dr}\right)_{\theta = \alpha} = -\frac{1}{f'(\alpha)},$$

and $\quad \beta = -\left(\dfrac{1}{2}\pi - \alpha\right) = \alpha - \dfrac{1}{2}\pi.$

Hence the equation of the asymptote is

$$r \cos\{\theta - (\alpha - \tfrac{1}{2}\pi)\} = -\frac{1}{f'(\alpha)} \quad i.e., \quad r \sin(\theta - \alpha) = \frac{1}{f'(a)}.$$

Problem 2: *Find the asymptotes of the curve $y = \tan x$.*

Solution: The curve is $\quad y = \tan x$.

$\therefore \quad \dfrac{dy}{dx} = \sec^2 x.$

Therefore the tangent at (x, y) to the curve is

$$Y - \tan x = \sec^2 x\,(X - x)$$

or $\quad Y \cos^2 x - \tan x \,.\, \cos^2 x = (X - x). \qquad \ldots(1)$

Now as $x \to \dfrac{\pi}{2}$, $y \to \infty$ and the distance of (x, y) from the origin tends to infinity.

Therefore taking line of (1) as $x \to \dfrac{\pi}{2}$, we have

$$y \,.\, 0 - 0 = \left(x - \frac{\pi}{2}\right) \quad \text{or} \quad x = \frac{\pi}{2}.$$

This is one asymptote.

The other asymptotes are $x = \pm \dfrac{\pi}{2}, \pm \dfrac{3\pi}{2} \pm \dfrac{5\pi}{2}, \ldots\ldots$

Problem 3: *Find the asymptotes of the curve $r \sin m\theta = a$.* **(Meerut 2000, Garhwal 11)**

Solution: The equation of the curve can be written as

$$1/r = (1/a) \sin m\theta = f(\theta), \text{ say.}$$

Now $f(\theta) = 0$ if $\sin m\theta = 0$ *i.e.*, $m\theta = n\pi$ (where n is any integer).

$\therefore \quad \theta = (n\pi / m) = \alpha$, say.

Also $\quad f'(\theta) = (1/a) \,.\, m \cos m\theta.$

$\therefore \quad f'(\alpha) = (1/a) \,.\, m \cos m\alpha = (1/a) \,.\, m \cos \left(m \cdot \dfrac{n\pi}{m} \right) = (1/a) \,.\, m \,.\, \cos n\pi.$

$\therefore$ The asymptotes are $r \sin \left(\theta - \dfrac{n\pi}{m} \right) = \dfrac{a}{m \cos n\pi}$,

where n is any integer.

Problem 4: *Find the asymptotes of the curve $r\theta = a$.* **(Meerut 2008, 12B)**

Solution: The equation of the given curve can be written as $1/r = \theta/a = f(\theta)$, say.

Now $\quad f(\theta) = 0$ if $\theta/a = 0$

i.e., $\quad \theta = 0 = \alpha$, say.

Also $\quad f'(\theta) = 1/a$, so that $f'(\alpha) = f'(0) = 1/a$,

or $\quad 1/f'(\alpha) = a.$

Now the asymptote corresponding to $\theta = \alpha$ is given by

$$r \sin (\theta - \alpha) = 1/f'(\alpha)$$

i.e., $\quad r \sin (\theta - 0) = a, \qquad [\because \text{ Here } \alpha = 0]$

i.e., $\quad r \sin \theta = a.$

Problem 5: *Find the asymptotes of the curve $2/r = 1 + 2 \sin \theta$.*

Solution: The curve is

$$\frac{2}{r} = 1 + 2 \sin\theta.$$

$\therefore \quad \dfrac{1}{r} = \dfrac{1 + 2 \sin\theta}{2} = f(\theta)$, say.

Now $\quad f(\theta) = 0$ *i.e.*, $1 + 2 \sin \theta = 0 \Rightarrow \sin \theta = -\dfrac{1}{2} = \sin \left(-\dfrac{\pi}{6} \right)\cdot$

Also $\quad f'(\theta) = \cos \theta.$

$\therefore \quad f' = \left(\dfrac{-\pi}{6} \right) = \cos \left(\dfrac{-\pi}{6} \right) = \dfrac{\sqrt{3}}{2}\cdot$

$\therefore$ The asymptotes are given by $r \sin \left(\theta \pm \dfrac{\pi}{6} \right) = \dfrac{2}{\sqrt{3}}\cdot$

Problem 6: *Find the asymptotes of the following curves :*

(i) $r \sin \theta = 2 \cos 2\theta.$ (ii) $r \sin \theta = a \cos 2\theta.$

Solution: **(i)** The equation to the curve can be written as

$$\frac{1}{r}=\frac{\sin\theta}{2\cos 2\theta}=f(\theta), \text{ say.}$$

Now $\quad f(\theta)=0 \Rightarrow \sin\theta=0 \;\; i.e., \theta=k\pi=\alpha$ (say).

Here k is any integer.

Also $\quad f'(\theta)=\frac{1}{2}\,\frac{\cos 2\theta\cos\theta-\sin\theta\,.\,(-2\sin 2\theta)}{\cos^2 2\theta}.$

$$\therefore \quad \frac{1}{f'(\alpha)}=\frac{2\cos^2(2k\pi)}{\cos 2k\pi\,.\,\cos k\pi+2\sin k\pi.\sin 2k\pi}=\frac{2}{\cos k\pi}, \quad [\because \cos 2k\pi=1]$$

$$=2/(-1)^k. \qquad [\because \cos k\pi=(-1)^k]$$

$\therefore$ The required asymptotes are given by

$$r\sin(\theta-k\pi)=2/(-1)^k \quad \text{or} \quad -r\sin(k\pi-\theta)=2/(-1)^k$$

or $\quad -r\{(-1)^{k-1}\sin\theta\}=2/(-1)^k \quad$ or $\quad r\sin\theta=2.$

(ii) We have $\quad \frac{1}{r}=\frac{\sin\theta}{a\cos 2\theta}=f(\theta)$ (say)

Now $\quad f(\theta)=0 \;\Rightarrow\; \sin\theta=0 \;\Rightarrow\; \theta=0$ or π etc.

Also $\quad f'(\theta)=\frac{a\cos 2\theta\,.\cos\theta+\sin\theta\,.\,a\sin 2\theta\,.\,2}{a^2\cos^2 2\theta}.$

$\therefore \quad f'(0)=\frac{a}{a^2}=\frac{1}{a}.$

$\therefore$ The asymptotes are given by

$$r\sin(\theta-0)=a \quad \text{or} \quad r\sin\theta=a.$$

Problem 7: *Find the asymptotes of the following curves :*

(i) $r\cos\theta=a\sin\theta$ **(Meerut 2004B; Lucknow 05)**

(ii) $r\sin\theta=2\cos\theta.$

Solution: **(i)** We have $\frac{1}{r}=\frac{\cot\theta}{a}=f(\theta)$ (say).

Now $\quad f(\theta)=0 \;\Rightarrow\; \cot\theta=0 \;\Rightarrow\; \theta=n\pi+\frac{\pi}{2}$ where $n=0,1,2,\ldots.$

Now $f'(\theta)=-\frac{1}{a}\operatorname{cosec}^2\theta=-\frac{1}{a}$ for all the above value of θ.

$\therefore$ The asymptotes are given by

$$r\sin\left(\theta-\frac{\pi}{2}\right)=-a \quad \text{or} \quad r\cos\theta=a$$

$$r\sin\left(\theta-\frac{3\pi}{2}\right)=-a \quad \text{or} \quad r\cos\theta=-a$$

and $\quad r\sin\left(\theta-\frac{5\pi}{2}\right)=-a \quad$ or $\quad r\cos\theta=a$

... ...

... ...

... ...

So asymptotes are $r \cos\theta = \pm a$.

(ii) $\dfrac{1}{r} = \dfrac{1}{2}\tan\theta = \dfrac{1}{2}\dfrac{\sin\theta}{\cos\theta} = f(\theta)$ (say).

Now $f(\theta) = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = n\pi.$

$\therefore$ $f'(\theta) = \dfrac{1}{2}\sec^2\theta = \dfrac{1}{2\cos^2\theta}\cdot$

Also $f'(n\pi) = \dfrac{1}{2\cos^2 n\pi}\cdot$

$\therefore$ Asymptotes are $r\sin(\theta - n\pi) = 2\cos^2 n\pi$

or $r(\sin\theta\cos n\pi - \cos\theta\sin n\pi) = 2\cos^2 n\pi$

or $r\sin\theta = 2\cos n\pi$ $[\because \sin n\pi = 0]$

or $r\sin\theta = \pm 2.$

Problem 8: *Find the asymptotes of the curve $r = 4(\sec\theta + \tan\theta)$.*

Solution: $\dfrac{1}{r} = \dfrac{1}{4(\sec\theta + \tan\theta)} = \dfrac{1}{4}\dfrac{\cos\theta}{1+\sin\theta} = f(\theta)$ (say).

Now $f(\theta) = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \dfrac{\pi}{2}\cdot$

Also $f'(\theta) = \dfrac{1}{4}\left[\dfrac{-(1+\sin\theta)\sin\theta - \cos\theta\,.\,\cos\theta}{(1+\sin\theta)^2}\right] = -\dfrac{1}{4}\dfrac{(\sin\theta + 1)}{(1+\sin\theta)^2}\cdot$

$\therefore$ $f'\left(\dfrac{\pi}{2}\right) = -\dfrac{1}{4}\left(\dfrac{1}{2}\right) = -\dfrac{1}{8}\cdot$

$\therefore$ Asymptote is $r\sin\left(\theta - \dfrac{\pi}{2}\right) = -8$ or $r\cos\theta = 8.$

Problem 9: *Find the asymptotes of the curve $r\sin 2\theta = a$.*

Solution: We have $\dfrac{1}{r} = \dfrac{\sin 2\theta}{a} = f(\theta)$ (say).

Now $f(\theta) = 0$ *i.e.*, $\sin 2\theta \Rightarrow 2\theta = n\pi \Rightarrow \theta = \dfrac{n\pi}{2}\cdot$

Also $f'(\theta) = \dfrac{2}{a}\cos 2\theta.$

$\therefore$ $f'\left(\dfrac{n\pi}{2}\right) = \dfrac{2}{a}\cos n\pi.$

$\therefore$ The asymptotes are given by

$$r\sin\left(\theta - \frac{n\pi}{2}\right) = \frac{a}{2}\cos n\pi \quad \text{or} \quad r\sin\theta = \pm\frac{1}{2}a,\ r\cos\theta = \pm\frac{1}{2}a.$$

Problem 10: *Find the asymptotes of the curve $r \cos\theta = 4 \sin^2\theta$.* **(Meerut 2005)**

Solution: We have

$$\frac{1}{r} = \frac{\cos\theta}{4\sin^2\theta} = f(\theta) \text{ (say)}.$$

Now $\quad f(\theta) = 0 \quad i.e., \quad \cos\theta = 0 \quad i.e., \quad \theta = (2k+1)\frac{\pi}{2}\cdot$

Also $\quad f'(\theta) = \dfrac{-4\sin^3\theta - 8\cos^2\theta\sin\theta}{16\sin^2\theta}\cdot$

$\therefore \quad f'\left\{(2k+1)\dfrac{\pi}{2}\right\} = \dfrac{-4\sin^3\left\{(2k+1)\dfrac{\pi}{2}\right\}}{16\sin^4\left\{(2k+1)\dfrac{\pi}{2}\right\}} = -\dfrac{1}{4\sin(2k+1)\dfrac{\pi}{2}}\cdot$

$\therefore$ The asymptotes are

$$r\sin\left\{\theta - (2k+1)\frac{\pi}{2}\right\} = -4\sin(2k+1)\frac{\pi}{2}$$

or $\quad r\left\{\sin\theta\cos(2k+1)\dfrac{\pi}{2} - \cos\theta\sin(2k+1)\dfrac{\pi}{2}\right\} = -4\sin(2k+1)\dfrac{\pi}{2}$

or $\quad r\cos\theta\sin(2k+1)\dfrac{\pi}{2} = 4\sin(2k+1)\dfrac{\pi}{2}$

or $\quad r\cos\theta = 4.$

Problem 11: *Find the asymptotes of the curve $r\theta\cos\theta = a\cos 2\theta$.*

Solution: The equation of the given curve can be written as

$$\frac{1}{r} = \frac{\theta\cos\theta}{a\cos 2\theta} = f(\theta), \text{ say}.$$

$\therefore \quad f'(\theta) = \dfrac{1}{a}\left[\dfrac{\cos 2\theta\,.\,\{\cos\theta - \theta\,.\,\sin\theta\} + 2\theta\cos\theta\,.\,\sin 2\theta}{\cos^2 2\theta}\right]\cdot$

Now $\quad f(\theta) = 0 \;\Rightarrow\; \theta = 0 \text{ or } \cos\theta = 0 \quad i.e., \quad \theta = (2k+1)\cdot\dfrac{1}{2}\pi.$

If $\quad \theta = 0 = \alpha$ (say), then $f'(\alpha) = 1/a.$

$\therefore \quad r\sin(\theta - 0) = a$ or $r\sin\theta = a$ is the corresponding asymptote.

When $\theta = (2k+1)\,.\,\dfrac{1}{2}\pi = \alpha$ (say), we have

$$\cos\alpha = 0,\ \sin 2\alpha = 0,\ \sin\alpha = (-1)^k$$

and $\quad \cos 2\alpha = \cos(2k\pi + \pi) = \cos\pi = -1.$

$\therefore \quad f'(\alpha) = \dfrac{1}{a}\left[\dfrac{-\{(-1)^k(2k+1)\,.\,\frac{1}{2}\pi\}}{1}\right] = (-1)^{k+1}(2k+1)\cdot\dfrac{\pi}{2a}\cdot$

The corresponding asymptote is

$$r \sin\left\{\theta - (2k+1)\cdot\frac{\pi}{2}\right\} = \frac{2a}{(-1)^{k+1}.(2k+1)\pi}$$

or $\quad (-1)^k\, r\{\cos(k\pi - \theta)\}\,(2k+1)\,\pi = 2a,$

or $\quad (-1)^{2k}\, r\cos\theta = 2a/\{(2k+1)\,\pi\}.$

∴ The required asymptotes are

$$r\sin\theta = a \quad\text{and}\quad r\cos\theta = 2a/\{(2k+1)\,\pi\}.$$

Problem 12: *Find the asymptotes of the curve* $r(e^\theta - 1) = a(e^\theta + 1)$.

Solution: We have $\frac{1}{r} = \frac{e^\theta - 1}{a(e^\theta + 1)} = f(\theta)$ (say).

Now $\quad f(\theta) = 0$ gives $e^\theta - 1 = 0$ or $\theta = 0$.

Also $\quad f'(\theta) = \frac{1}{a}\left[\frac{e^\theta(e^\theta+1) - e^\theta(e^\theta - 1)}{(e^\theta+1)^2}\right] = \frac{1}{a}\,\frac{2e^\theta}{(e^\theta+1)^2}.$

∴ $\quad f'(0) = \frac{1}{2a}.$

Hence the asymptote is $r\sin(\theta - 0) = 2a$ or $r\sin\theta = 2a.$

Problem 13: *Find the asymptotes of the curve* $r = \frac{2\theta}{\sin\theta}$.

Solution: We have $\frac{1}{r} = \frac{\sin\theta}{2\theta} = f(\theta)$ (say).

Now $\quad f(\theta) = 0 \quad$ *i.e.*, $\quad \sin\theta = 0 \quad \Rightarrow \quad \theta = n\pi.$

Also $\quad f'(\theta) = \frac{2\theta\cos\theta - 2\sin\theta}{4\theta^2}.$

∴ $\quad f'(n\pi) = \frac{\cos n\pi}{2n\pi}.$

∴ The asymptote is

$$r\sin(\theta - n\pi) = \frac{2n\pi}{\cos n\pi} \quad\text{or}\quad r\{\sin\theta\cos n\pi - \cos\theta\sin n\pi\} = \frac{2n\pi}{\cos n\theta}$$

or $\quad r\sin\theta\cos n\pi = \frac{2n\pi}{\cos n\pi} \qquad [\because \sin n\theta = 0]$

or $\quad r\sin\theta\cos^2 n\pi = 2n\pi$

or $\quad r\sin\theta = 2n\pi. \qquad [\because \cos^2 n\pi = 1]$

Problem 14: *Find the circular asymptote of the curve* $r(e^\theta - 1) = a(e^\theta + 1)$.

Solution: The given equation is $r = \frac{a(e^\theta + 1)}{(e^\theta - 1)} = f(\theta)$, say.

∴ The circular asymptote is given by

$$r = a \lim_{\theta \to \infty} \frac{e^{\theta} + 1}{e^{\theta} - 1}, \qquad \left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= a \lim_{\theta \to \infty} \frac{1 + e^{-\theta}}{1 - e^{-\theta}} = a.$$

Hence $r = a$ is the circular asymptote.

Problem 15: *Find the circular asymptote of the curve* $r = \dfrac{3\theta^2 + 2\theta + 1}{2\theta^2 + \theta + 1}$.

Solution: The given equation is

$$r = \frac{3 + (2/\theta) + (1/\theta^2)}{2 + (1/\theta) + (1/\theta^2)},$$

dividing numerator and denominator by θ^2.

Taking limits when $\theta \to \infty$, we see that the circular asymptote is

$$r = \frac{3}{2}.$$

Hints to Objective Type Questions

Multiple Choice Questions

1. See Example 1.
2. Equating to zero the coefficient of the highest power of x *i.e.*, of x^2 in the equation of the given curve, the asymptotes parallel to x-axis are given by $y^2 = 0$ *i.e.*, $y = 0, y = 0$.
3. See Problem 5 of Comprehensive Problems 1.
4. See Example 3.
5. We know that a closed curve has no asymptotes.
6. Since the curve is of degree 4, therefore it has four asymptotes.
7. See Problem 8 of Comprehensive Problems 5.
8. See Problem 5 of Comprehensive Problems 1.
9. See Problem 6(ii) of Comprehensive Problems 5.

Fill in the Blanks

1. See article 7.
2. See article 8.
3. See Example 4.
4. See article 15.

5. The circular asymptote of the curve $r(\theta^2+1)=a\theta^2-1$ is

$$r=\lim_{\theta\to\infty}\frac{a\theta^2-1}{\theta^2+1}=\lim_{\theta\to\infty}\frac{\theta^2\,[a-(1/\theta^2)]}{\theta^2\,[1+(1/\theta^2)]}=\lim_{\theta\to\infty}\frac{a-(1/\theta^2)}{1+(1/\theta^2)}=\frac{1-0}{1+0}=a.$$

True or False

1. See article 8.

2. The asymptotes of the given curve parallel to y-axis are given by $x^2-a^2=0$. Thus, $x=\pm a$ are asymptotes parallel to y-axis.

3. The equation to the curve cane be written as

$$\frac{1}{r}=\frac{1}{a}(1-\cos\theta)=f(\theta),\text{ say}$$

Now, $f(\theta)=\Rightarrow 1-\cos\theta=0$ or $\cos\theta=1$ *i.e.*, $\theta=2k\pi$, where k is any integer.

Also, $f'(\theta)=\frac{1}{a}\sin\theta$

$\therefore$ $f'(2k\pi)=\frac{1}{a}\sin(2k\pi)=0$.

4. In rough language, an asymptote of a curve touches the curve at infinity.

5. See article 1.

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Chapter-13

Singular Points : Curve Tracing

Comprehensive Problems 1

Problem 1: *Show that the points of inflexion upon the curve* $x^2 y = a^2 (x - y)$ *or* $(a^2 + x^2) y = a^2 x$ *are given by* $x = 0, x = \pm a\sqrt{3}$. **(Meerut 2013B; Kanpur 15)**

Solution: The curve is $y = a^2 x/(x^2 + a^2)$. Differentiating the equation of the curve w.r.t. x, we get

$$\frac{dy}{dx} = a^2 \cdot \frac{(x^2 + a^2) . 1 - x . 2x}{(x^2 + a^2)^2} = \frac{a^2 (a^2 - x^2)}{(x^2 + a^2)^2}$$

and
$$\frac{d^2 y}{dx^2} = a^2 \cdot \frac{(x^2 + a^2)^2 (-2x) - (a^2 - x^2) . 2 (x^2 + a^2) . 2x}{(x^2 + a^2)^4}$$

$$= a^2 \cdot \frac{-2x (x^2 + a^2) - 4x (a^2 - x^2)}{(x^2 + a^2)^3}$$

$$= a^2 \cdot \frac{2 (x^3 - a^2 x)}{(x^2 + a^2)^3} = \frac{2a^2 x (x^2 - 3a^2)}{(x^2 + a^2)^3}.$$

For the points of inflexion, putting $d^2 y / dx^2 = 0$, we have $2a^2 x (x^2 - 3a^2) = 0$.

$\therefore$ $x = 0$ or $\pm \sqrt{3}a$.

Now
$$\frac{d^3 y}{dx^3} = \frac{2a^2 [(x^2 + a^2)^3 . (3x^2 - 3a^2) - (x^3 - 3a^2 x) \{3 . (a^2 + x^2)^2 2x\}]}{(x^2 + a^2)^6}$$

$$= 6a^2 \cdot \frac{(x^2 + a^2)(x^2 - a^2) - 2x (x^3 - 3a^2 x)}{(x^2 + a^2)^4}$$

$$= \frac{6a^2 [x^4 - a^4 - 2x^4 + 3a^2 x^2]}{(x^2 + a^2)^4} = \frac{6a^2 [3a^2 x^2 - x^4 - a^4]}{(x^2 + a^2)^4}.$$

When $x = 0$, $d^3 y / dx^3 = -6/a^2 \neq 0$,

when $x = \sqrt{3}a$, $d^3 y / dx^3 = 3/4a^2 \neq 0$,

and when $x = -\sqrt{3}a, d^3 y / dx^3 = 3/4a^2 \neq 0$.

Hence $x = 0, \pm \sqrt{3}a$ give us points of inflexion.

From the equation of the curve, we have

when $x = 0$, $y = 0$; when $x = \sqrt{3}a$, $y = (\sqrt{3}/4)\, a$
and when $x = -\sqrt{3}a$, $y = -(\sqrt{3}/4)\, a$.
Hence the points of inflexion are

$$(0, 0), \left(\sqrt{3}a, \frac{\sqrt{3}}{4}a\right) \text{ and } \left(-\sqrt{3}a, -\frac{\sqrt{3}}{4}a\right).$$

Problem 2: *Find the points of inflexion of the curve* $y(a^2 + x^2) = x^3$.

(Lucknow 2008; Purvanchal 10)

Solution: Differentiating the equation of the curve w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(a^2 + x^2) \cdot 3x^2 - x^3 \cdot 2x}{(a^2 + x^2)^2} = \frac{3a^2x^2 + x^4}{(a^2 + x^2)^2}$$

and

$$\frac{d^2y}{dx^2} = \frac{(a^2 + x^2)^2 \cdot (6a^2x + 4x^3) - (3a^2x^2 + x^4) \cdot 2(a^2 + x^2) \cdot 2x}{(a^2 + x^2)^4}$$

$$= \frac{x[6a^4 + 10a^2x^2 + 4x^4 - 12a^2x^2 - 4x^4]}{(a^2 + x^2)^3} = \frac{2a^2x(3a^2 - x^2)}{(a^2 + x^2)^3}.$$

For the points of inflexion, we must have $d^2y/dx^2 = 0$.

$$\therefore \quad \frac{2a^2x(3a^2 - x^2)}{(a^2 + x^2)^3} = 0 \quad \text{or} \quad 2a^2x(3a^2 - x^2) = 0$$

i.e., $x = 0$ or $x = \pm\sqrt{3}a$. [For the points of inflexion]

Now

$$\frac{d^3y}{dx^3} = 2a^2 \cdot \frac{[(a^2 + x^2)^3 (3a^2 - 3x^2) - (3a^2x - x^3) . 3(a^2 + x^2)^2 . 2x]}{(a^2 + x^2)^6}$$

$$= \frac{6a^2[(a^2 + x^2)(a^2 - x^2) - 2x(3a^2x - x^3)]}{(a^2 + x^2)^4}$$

$$= \frac{6a^2(a^4 - x^4 - 6a^2x^2 + 2x^4)}{(a^2 + x^2)^4} = \frac{6a^2(a^4 - 6a^2x^2 + x^4)}{(a^2 + x^2)^4}.$$

When $x = 0, d^3y/dx^3 = 6/a^2 \neq 0$, when $x = \sqrt{3}a$, $d^3y/dx^3 = -3/4a^2 \neq 0$

and when $x = -\sqrt{3}a, d^3y/dx^3 = -3/4a^2 \neq 0$.

Therefore $x = 0, \pm\sqrt{3}a$ give us points of inflexion.

From the equation of the curve, we have

when $x = 0$, $y = 0$;
when $x = \sqrt{3}a$, $y = (3\sqrt{3}/4)\, a$
and when $x = -\sqrt{3}a$, $y = -(3\sqrt{3}/4)\, a$.

Hence the points of inflexion are

$$(0, 0), (\sqrt{3}a, 3\sqrt{3}a/4), (-\sqrt{3}a, -3\sqrt{3}a/4).$$

Problem 3: *Find the points of inflexion of the curve* $xy = a^2 \log(y/a)$.

Solution: The given curve is $x = (a^2/y) \log(y/a)$. ...(1)

Differentiating (1) taking y as independent variable and x dependent variable, we have

$$\frac{dx}{dy} = -\frac{a^2}{y^2} \log\left(\frac{y}{a}\right) + \frac{a^2}{y} \cdot \frac{1}{y/a} \cdot \frac{1}{a} = \frac{a^2}{y^2}\left(1 - \log\frac{y}{a}\right),$$

and
$$\frac{d^2x}{dy^2} = \frac{a^2}{y^2}\left(-\frac{1}{y/a} \cdot \frac{1}{a}\right) + \left(1 - \log\frac{y}{a}\right)\left(-2 \cdot \frac{a^2}{y^3}\right)$$

$$= -\frac{a^2}{y^3}\left(3 - 2\log\frac{y}{a}\right) \cdot$$

For the points of inflexion, putting $d^2x/dy^2 = 0$, we have

$$3 - 2\log(y/a) = 0 \quad \text{or} \quad y = ae^{3/2}.$$

Now
$$\frac{d^3x}{dy^3} = -\frac{a^2}{y^3} \cdot \left(-2 \cdot \frac{1}{y/a} \cdot \frac{1}{a}\right) + \left(3 - 2\log\frac{y}{a}\right) \cdot \frac{3a^2}{y^4}$$

$$= \frac{a^2}{y^4}\left(5 - 2\log\frac{y}{a}\right).$$

When $y = a\,e^{3/2}$ or $\log(y/a) = 3/2$, we have

$$\frac{d^3x}{dy^3} = \frac{a^2}{a^4 e^6}\left(5 - 2 \cdot \frac{3}{2}\right) = \frac{2}{a^2 e^6} \neq 0.$$

Hence $y = a\,.e^{3/2}$ gives a point of inflexion.

From (1), putting $y = a\,e^{3/2}$, we have $x = \dfrac{a^2}{a\,e^{3/2}} \cdot \dfrac{3}{2} = \dfrac{3}{2}\,a\,e^{-3/2}$.

Hence the point of inflexion is $\left(\dfrac{3}{2}\,a\,e^{-3/2}, a\,e^{3/2}\right)$.

Problem 4: *Find the points of inflexion of the curve* $x = (\log y)^3$. **(Purvanchal 2009)**

Solution: Differentiating the given equation w.r.t. y, we get

$$\frac{dx}{dy} = 3(\log y)^2 \times \frac{1}{y} = \frac{3(\log y)^2}{y}$$

and
$$\frac{d^2x}{dy^2} = 3\,\frac{y\,[2(\log y).1/y] - (\log y)^2}{y^2}$$

$$= \frac{3\log y}{y^2}[2 - \log y] = 3 \cdot \frac{2\log y - (\log y)^2}{y^2}$$

$$= 3y^{-2}\,[2\log y - (\log y)^2].$$

For the points of inflexion, putting $d^2x/dy^2 = 0$, we have

$$2 \log y - (\log y)^2 = 0 \quad \text{or} \quad \log y\,[2 - \log y] = 0.$$

$\therefore$ $\log y = 0$ *i.e.*, $y = 1$ or $\log y = 2$ *i.e.*, $y = e^2$.

Now
$$\frac{d^3x}{dy^3} = 3y^{-2}\left[\frac{2}{y} - (2 \log y)\cdot\frac{1}{y}\right] + [2 \log y - (\log y)^2](-6y^{-3})$$
$$= 6y^{-3}\,[1 - \log y - 2 \log y + (\log y)^2]$$
$$= 6y^{-3}\,[(\log y)^2 - 3 \log y + 1].$$

When $y = 1, d^3x/dy^3 = 6 \neq 0$, and when $y = e^2, \dfrac{d^3x}{dy^3} = \dfrac{-6}{e^6} \neq 0.$

Hence $y = 1, e^2$ give us the points of inflexion.

From the curve when $y = 1$, $x = 0$ and when $y = e^2$, $x = 8$.

Hence the points of inflexion are $(0, 1)$ and $(8, e^2)$.

Problem 5: *Investigate the points of inflexion of the curve* $y = (x-1)^4 (x-2)^3$.

(Agra 2014)

Solution: The given curve is

$$y = (x-1)^4 (x-2)^3. \qquad \ldots(1)$$

$\therefore$
$$\frac{dy}{dx} = 4(x-1)^3 (x-2)^3 + 3(x-1)^4 (x-2)^2$$
$$= (x-1)^3 (x-2)^2 [4(x-2) + 3(x-1)]$$
$$= (x-1)^3 (x-2)^2 (7x - 11)$$

and
$$\frac{d^2y}{dx^2} = 3(x-1)^2 (x-2)^2 (7x-11)$$
$$+ 2(x-1)^3 (x-2)(7x-11) + 7(x-1)^3 (x-2)^2$$
$$= (x-1)^2 (x-2)[3(x-2)(7x-11)$$
$$+ 2(x-1)(7x-11) + 7(x-1)(x-2)]$$
$$= (x-1)^2 (x-2)[21x^2 - 75x + 66 + 14x^2 - 36x$$
$$+ 22 + 7x^2 - 21x + 14]$$
$$= (x-1)^2 (x-2)(42x^2 - 132x + 102)$$
$$= 6(x-1)^2 (x-2)(7x^2 - 22x + 17).$$

For the points of inflexion of the given curve, we must have

$d^2y/dx^2 = 0$ *i.e.*, $(x-1)^2 (x-2)(7x^2 - 22x + 17) = 0$, which gives

$$x = 1,\ 2, \frac{22 \pm \sqrt{[(22)^2 - 4.7.17]}}{14} \quad \text{i.e.,} \quad x = 1,\ 2, \frac{11 \pm \sqrt{2}}{7}.$$

Thus the given curve may have points of inflexion where

$$x = 1,\ 2, (11 \pm \sqrt{2})/7.$$

Since $(x - 1)$ occurs as a factor of second degree in $d^2 y/dx^2$, therefore the sign of $d^2 y/dx^2$ does not change as x passes through 1. Consequently $x = 1$ does not give a point of inflexion. Thus there is a point of undulation at $x = 1$.

Again each of the factors $x - 2, [x - \frac{1}{7}(11 + \sqrt{2})]$ and $[x - \frac{1}{7}(11 - \sqrt{2})]$ occurs in first degree in $d^2 y/dx^2$ and so the sign of $d^2 y/dx^2$ changes as x passes through each of the values $2, (11 + \sqrt{2})/7$ and $(11 - \sqrt{2})/7$. Hence the given curve has points of inflexion at $x = 2, (11 \pm \sqrt{2})/7$.

Problem 6: *Show that every point in which the sine curve $y = c \sin (x/a)$ meets the axis of x is a point of inflexion.*

Solution: The given curve meets x-axis where $y = 0$ *i.e.,* where

$$\sin (x/a) = 0 = \sin n\pi$$

or $\qquad x/a = n\pi$ or $x = an\pi$, where n is any integer.

Differentiating the given equation of the curve w.r.t. x, we get

$$\frac{dy}{dx} = \frac{c}{a} \cos \frac{x}{a}, \frac{d^2 y}{dx^2} = -\frac{c}{a^2} \sin \frac{x}{a}.$$

For points of inflexion, putting $d^2 y/dx^2 = 0$, we have

$$\sin (x/a) = 0 \text{ i.e., } x = an\pi.$$

Now $\qquad \dfrac{d^3 y}{dx^3} = -\dfrac{c}{a^3} \cos \dfrac{x}{a}.$

When $\qquad x = an\pi, \dfrac{d^3 y}{dx^3} = -\dfrac{c}{a^3} \cos n\pi = -(-1)^n \cdot \dfrac{c}{a^3} \neq 0.$

Hence the points of inflexion are given by $x = an\pi$. These are the points where the curve cuts the x-axis.

Problem 7: *Show that points of inflexion of the curve $y^2 = (x - a)^2 (x - b)$ lie on the line $3x + a = 4b$.* **(Agra 2006; Avadh 11; Kashi 12)**

Solution: The given curve is $y^2 = (x - a)^2 (x - b)$

or $\qquad y = \pm (x - a) \sqrt{(x - b)}. \qquad \ldots(1)$

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = \pm \left[(x - a) \cdot \frac{1}{2\sqrt{(x - b)}} + \sqrt{(x - b)} \right]$$

$$= \pm \frac{x - a + 2x - 2b}{2\sqrt{(x - b)}} = \pm \frac{3x - 2b - a}{2\sqrt{(x - b)}}$$

$$= \pm \frac{1}{2}(3x - 2b - a).(x - b)^{-1/2}$$

and $$\frac{d^2y}{dx^2} = \pm \frac{1}{2}[(3x - 2b - a).(-\frac{1}{2})(x - b)^{-3/2} + (x - b)^{-1/2}.3]$$

$$= \pm \frac{1}{2}\left[\frac{a + 2b - 3x}{2(x - b)^{3/2}} + \frac{3}{(x - b)^{1/2}}\right]$$

$$= \pm \frac{1}{2\sqrt{(x-b)}}\left[\frac{a + 2b - 3x}{2(x - b)} + 3\right] = \pm \frac{1}{2\sqrt{(x-b)}} \cdot \frac{a - 4b + 3x}{2(x - b)}.$$

For the points of inflexion, we must have $d^2y/dx^2 = 0$

i.e., $a - 4b + 3x = 0$ or $3x + a = 4b$.

Hence the points of inflexion lie on the straight line

$$3x + a = 4b.$$

Problem 8: *Find the points of inflexion on the curve* $y^2 = x(x + 1)^2$ *and also obtain the equations of the inflexional tangents.*

Solution: The given curve is $y^2 = x(x + 1)^2$ (symmetry about x-axis)

or $$y = \pm (x + 1)x^{1/2}.$$

Let us take $$y = (x + 1)x^{1/2}.$$

Then $$\frac{dy}{dx} = 1.x^{1/2} + (x + 1).\frac{1}{2}x^{-1/2} = x^{1/2} + \frac{x + 1}{2x^{1/2}} = \frac{2x + x + 1}{2x^{1/2}} = \frac{3x + 1}{2x^{1/2}}$$

and $$\frac{d^2y}{dx^2} = \frac{1}{2} \cdot \frac{3.x^{1/2} - (3x + 1).\frac{1}{2}x^{-1/2}}{x} = \frac{6x - 3x - 1}{4x.x^{1/2}} = \frac{3x - 1}{4x^{3/2}}.$$

For the points of inflexion of the given curve, we must have $d^2y/dx^2 = 0$.

$$\therefore \quad \frac{3x - 1}{4x^{3/2}} = 0 \quad \text{or} \quad 3x - 1 = 0 \quad \text{or} \quad x = \frac{1}{3}.$$

Now there will be points of inflexion where $x = \frac{1}{3}$ if we have

$$d^3y/dx^3 \neq 0 \text{ at } x = \frac{1}{3}.$$

We have $$\frac{d^3y}{dx^3} = \frac{1}{4} \cdot \frac{3.x^{3/2} - (3x - 1).(3/2)x^{1/2}}{x^3}$$

$$= \frac{1}{4} \cdot \frac{3x^{1/2}[2x - 3x + 1]}{2.x^3} = \frac{3}{8x^{5/2}}(1 - x) \neq 0, \text{ at } x = \frac{1}{3}.$$

$\therefore$ The given curve has points of inflexion where $x = \frac{1}{3}$. Putting $x = \frac{1}{3}$ in the equation of the given curve, we get

$$y = \pm \frac{4}{3} \cdot \left(\frac{1}{3}\right)^{1/2} = \pm \frac{4}{3\sqrt{3}}.$$

Hence the points of inflexion on the given curve are $\left(\frac{1}{3}, \pm \frac{4}{3\sqrt{3}}\right)$.

Now tangents at the points of inflexion are called inflexional tangents.
The equation of the given curve can be written as

$$y^2 = x(x^2 + 2x + 1) = x^3 + 2x^2 + x.$$

Differentiating with respect to x, we get

$$2y\frac{dy}{dx} = 3x^2 + 4x + 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{3x^2 + 4x + 1}{2y}.$$

$\therefore$ At the point $\left(\frac{1}{3}, \frac{4}{3\sqrt{3}}\right)$, $\frac{dy}{dx} = \frac{(1/3) + (4/3) + 1}{2.(4/3\sqrt{3})} = \sqrt{3}$

and at the point $\left(\frac{1}{3}, \frac{-4}{3\sqrt{3}}\right)$, $\frac{dy}{dx} = -\sqrt{3}$.

$\therefore$ Inflexional tangent at the point $\left(\frac{1}{3}, \frac{4}{3\sqrt{3}}\right)$ is

$$y - \frac{4}{3\sqrt{3}} = \sqrt{3}\left(x - \frac{1}{3}\right) \quad \text{or} \quad \sqrt{3}x - y - \frac{\sqrt{3}}{3} + \frac{4}{3\sqrt{3}} = 0$$

or
$$\sqrt{3}x - y + \frac{1}{3\sqrt{3}} = 0 \quad \text{or} \quad 9x - 3\sqrt{3}\,y + 1 = 0$$

and inflexional tangent at the point $\left(\frac{1}{3}, \frac{-4}{3\sqrt{3}}\right)$ is

$$y + \frac{4}{3\sqrt{3}} = -\sqrt{3}\left(x - \frac{1}{3}\right) \quad \text{or} \quad \sqrt{3}x + y + \frac{4}{3\sqrt{3}} - \frac{\sqrt{3}}{3} = 0$$

or
$$\sqrt{3}x + y + \frac{1}{3\sqrt{3}} = 0 \quad \text{or} \quad 9x + 3\sqrt{3}\,y + 1 = 0.$$

Hence the inflexional tangents are $9x \pm 3\sqrt{3}\,y + 1 = 0$.

Problem 9: *Show that origin is a point of inflexion of the curve* $a^{m-1}.\ y = x^m$ *if m is odd and greater than* 2.

Solution: The given curve is

$$a^{m-1}.\ y = x^m \quad \text{or} \quad y = x^m / a^{m-1}. \qquad \text{...(1)}$$

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{m}{a^{m-1}} \cdot x^{m-1}; \frac{d^2y}{dx^2} = \frac{m(m-1)}{a^{m-1}} \cdot x^{m-2}.$$

For the points of inflexion, we must have

$$d^2y/dx^2 = 0$$

or $\qquad \dfrac{m(m-1)}{a^{m-1}} \cdot x^{m-2} = 0 \qquad$ or $\qquad x^{m-2} = 0$

or $\qquad x = 0$, if $m > 2$.

Also $\qquad \dfrac{d^3 y}{dx^3} = \dfrac{m(m-1)(m-2)}{a^{m-1}} \cdot x^{m-3};\ldots\ldots; \dfrac{d^m y}{dx^m} = \dfrac{m!}{a^{m-1}}$.

At $\qquad x = 0, \dfrac{d^2 y}{dx^2} = \dfrac{d^3 y}{dx^3} = \ldots = \dfrac{d^{m-1} y}{dx^{m-1}} = 0$ and $\dfrac{d^m y}{dx^m} \neq 0$.

Hence there is a point of inflexion at $x = 0$ (*i.e.*, at origin) if m is odd and greater than 2 and no point of inflexion if m is even.

Problem 10: *Show that the abscissae of the points of inflexion on the curve $y^2 = f(x)$ satisfy the equation $[f'(x)]^2 = 2f(x) f''(x)$.*

Solution: The curve is $y^2 = f(x)$, or $y = [f(x)]^{1/2}$. ...(1)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2} \cdot [f(x)]^{-1/2} . f'(x)$$

and
$$\frac{d^2 y}{dx^2} = \frac{1}{2} [\{f(x)\}^{-1/2} . f''(x) + f'(x) . \left(-\frac{1}{2}\right) \{f(x)\}^{-3/2} . f'(x)]$$
$$= \frac{1}{2} [\{f(x)\}^{-1/2} f''(x) - \frac{1}{2} \{f(x)\}^{-3/2} \{f'(x)\}^2]$$
$$= \frac{1}{2} \{f(x)\}^{-3/2} [f(x) . f''(x) - \frac{1}{2} \{f(x)\}^2].$$

For the points of inflexion, we must have $d^2 y / dx^2 = 0$.

$\therefore \qquad 2f(x) . f''(x) - [f'(x)]^2 = 0$

or $\qquad [f'(x)]^2 = 2f(x) . f''(x)$.

Problem 11: *Show that the line joining the points of inflexion of the curve $y^2 (x - a) = x^2 (x + a)$ subtends an angle of $\pi/3$ at the origin.*

Solution: The given curve is

$$y^2 = \frac{x^2 (x + a)}{(x - a)} \quad \text{or} \quad y = \pm \frac{x \sqrt{(x + a)}}{\sqrt{(x - a)}}. \qquad \ldots(1)$$

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = \pm \frac{\sqrt{(x - a)} . \{x \cdot \frac{1}{2} (x + a)^{-1/2} + (x + a)^{1/2}\} - x \sqrt{(x + a)} \cdot \frac{1}{2} (x - a)^{-1/2}}{(x - a)}$$

$$= \pm \frac{[\sqrt{(x - a)} (3x + 2a)/\{2 \sqrt{(x + a)}\}] - [x \sqrt{(x + a)}/\{2 \sqrt{(x - a)}\}]}{(x - a)}$$

$$= \pm \frac{(x-a)(3x+2a) - x(x+a)}{2(x-a)^{3/2}(x+a)^{1/2}} = \pm \frac{x^2 - ax - a^2}{(x-a)^{3/2}(x+a)^{1/2}},$$

and

$$\frac{d^2y}{dx^2} = \pm \frac{(x-a)^{3/2}(x+a)^{1/2}.[2x-a] - [x^2 - ax - a^2]\left[\frac{3}{2}(x-a)^{1/2}.(x+a)^{1/2} + (x-a)^{3/2}.\frac{1}{2}(x+a)^{-1/2}\right]}{(x-a)^3.(x+a)}$$

$$= \pm \frac{2(x^2-a^2)(2x-a) - (x^2-ax-a^2)(4x+2a)}{2(x-a)^{5/2}.(x+a)^{3/2}},$$

[On simplification]

$$= \pm \frac{a^2(x+2a)}{(x-a)^{5/2}.(x+a)^{3/2}}.$$

For the points of inflexion, we must have $d^2y/dx^2 = 0$.

$\therefore$ $x = -2a$, and hence from (1), $y = \pm 2a/\sqrt{3}$.

Hence the points of inflexion are

$P(-2a, 2a/\sqrt{3})$ and $Q(-2a, -2a/\sqrt{3})$.

Let the line joining the points of inflexion (*i.e.*, the line PQ) subtend an angle θ at the origin. We have

$$m_1 = \text{slope of } OP = \frac{-2a/\sqrt{3}}{2a} = -1/\sqrt{3}$$

and $$m_2 = \text{slope of } O = \frac{2a/\sqrt{3}}{2a} = \frac{1}{\sqrt{3}}.$$

$\therefore$ $$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{(1/\sqrt{3}) + (1/\sqrt{3})}{1 - (1/3)} = \frac{2/\sqrt{3}}{2/3} = \sqrt{3}.$$

$\therefore$ $\theta = \pi/3$ *i.e.*, the required angle is $\pi/3$.

Problem 12: *Prove that the curve* $y = \dfrac{1-x}{1+x^2}$ *has three points of inflexion which lie in a straight line.*

Solution: The given curve is

$$y = \frac{1-x}{1+x^2}. \qquad ...(1)$$

$\therefore$ $$\frac{dy}{dx} = \frac{-1.(1+x^2) - 2x(1-x)}{(1+x^2)^2} = \frac{-1-2x+x^2}{(1+x^2)^2}$$

and $$\frac{d^2y}{dx^2} = \frac{(-2+2x)(1+x^2)^2 - (-1-2x+x^2).2(1+x^2).2x}{(1+x^2)^4}$$

$$= \frac{2[(1+x^2)(x-1) - 2x(-1-2x+x^2)]}{(1+x^2)^3}$$

$$= \frac{2\,[x - 1 + x^3 - x^2 + 2x + 4x^2 - 2x^3]}{(1 + x^2)^3} = \frac{2\,[-x^3 + 3x^2 + 3x - 1]}{(1 + x^2)^3}$$

$$= \frac{-2\,(x^3 - 3x^2 - 3x + 1)}{(1 + x^2)^3} = \frac{-2\,[(x^3 + 1) - 3x\,(x + 1)]}{(x^2 + 1)^3}$$

$$= \frac{-2\,[(x + 1)\,(x^2 - x + 1) - 3x\,(x + 1)]}{(x^2 + 1)^3} = \frac{-2\,(x + 1)\,(x^2 - 4x + 1)}{(x^2 + 1)^3}.$$

For the points of inflexion of the given curve, we must have $d^2 y/dx^2 = 0$.

$\therefore$ $(x + 1)\,(x^2 - 4x + 1) = 0$, which gives

$$x = -1, \frac{4 \pm \sqrt{(16 - 4)}}{2} \quad i.e., x = -1,\ 2 \pm \sqrt{3}.$$

Now we observe that each of the factors $x + 1$, $x - (2 + \sqrt{3})$ and $x - (2 - \sqrt{3})$ occurs in first degree in $d^2 y/dx^2$. Therefore the sign of $d^2 y/dx^2$ changes when x passes through each of the values $-1, 2 + \sqrt{3}$ and $2 - \sqrt{3}$; or we also conclude that $d^3 y/dx^3 \neq 0$ at any of the points $x = -1, 2 \pm \sqrt{3}$.

Hence there are points of inflexion where $x = -1, 2 \pm \sqrt{3}$.

From the equation of the curve, we have

when $x = -1,\ y = 1$

when $x = 2 + \sqrt{3},\ y = \dfrac{1 - 2 - \sqrt{3}}{1 + 4 + 3 + 4\sqrt{3}} = \dfrac{-1 - \sqrt{3}}{8 + 4\sqrt{3}} = -\dfrac{1}{4} \cdot \dfrac{1 + \sqrt{3}}{2 + \sqrt{3}}$

$$= -\frac{1}{4} \cdot \frac{(1 + \sqrt{3})\,(2 - \sqrt{3})}{(2 + \sqrt{3})\,(2 - \sqrt{3})} = -\frac{-1 + \sqrt{3}}{4} = -\frac{1 - \sqrt{3}}{4}$$

and when $x = 2 - \sqrt{3},\ y = \dfrac{1 - 2 + \sqrt{3}}{1 + 4 + 3 - 2\sqrt{3}} = \dfrac{-1 + \sqrt{3}}{4\,(2 - \sqrt{3})} = \dfrac{1 + \sqrt{3}}{4}.$

Hence the given curve has three points of inflexion

$$(-1, 1), \left(2 + \sqrt{3}, \frac{1 - \sqrt{3}}{4}\right) \text{ and } \left(2 - \sqrt{3}, \frac{1 + \sqrt{3}}{4}\right).$$

Let us name these points as A, B and C respectively.

The slope of the line

$$AB = \frac{\frac{1}{4}(1 - \sqrt{3}) - 1}{2 + \sqrt{3} + 1} = \frac{1}{4} \cdot \frac{1 - \sqrt{3} - 4}{3 + \sqrt{3}} = \frac{1}{4} \cdot \frac{-3 - \sqrt{3}}{3 + \sqrt{3}} = -\frac{1}{4}$$

and the slope of the line

$$AC = \frac{\frac{1}{4}(1 + \sqrt{3}) - 1}{2 - \sqrt{3} + 1} = \frac{1}{4} \cdot \frac{1 + \sqrt{3} - 4}{3 - \sqrt{3}} = \frac{1}{4} \cdot \frac{-3 + \sqrt{3}}{3 - \sqrt{3}} = -\frac{1}{4}.$$

Since the slope of the line $AB =$ the slope of the line AC, therefore the points A, B and C lie in a straight line.

Problem 13: *Show that the points of inflexion on the curve* $y = b\, e^{-(x/a)^2}$ *are given by* $x = \pm\, a/\sqrt{2}$. **(Agra 2005)**

Solution: The given curve is $y = b\, e^{-x^2/a^2}$. ...(1)

$$\therefore \quad \frac{dy}{dx} = b\, e^{-x^2/a^2} \cdot \left(\frac{-2x}{a^2}\right) = y \cdot \left(\frac{-2x}{a^2}\right),$$

$$\frac{d^2 y}{dx^2} = \frac{dy}{dx} \cdot \left(\frac{-2x}{a^2}\right) + y \cdot \frac{-2}{a^2} = y \cdot \left(\frac{-2x}{a^2}\right) \cdot \left(\frac{-2x}{a^2}\right) - \frac{2y}{a^2}$$

$$= y \left[\frac{4x^2}{a^4} - \frac{2}{a^2}\right]$$

and $$\frac{d^3 y}{dx^3} = \frac{dy}{dx}\left[\frac{4x^2}{a^4} - \frac{2}{a^2}\right] + y \cdot \frac{8x}{a^4} = y \cdot \left(\frac{-2x}{a^2}\right)\left[\frac{4x^2}{a^4} - \frac{2}{a^2}\right] + y \cdot \frac{8x}{a^4}.$$

For the points of inflexion of (1), we must have $d^2 y/dx^2 = 0$.

$$\therefore \quad y\left[\frac{4x^2}{a^4} - \frac{2}{a^2}\right] = 0 \quad \text{or} \quad \frac{4x^2}{a^4} - \frac{2}{a^2} = 0$$

[$\because$ $y = b\, e^{-x^2/a^2}$ cannot be zero for any real x]

or $$x^2 = \frac{a^2}{2} \quad \text{or} \quad x = \pm \frac{a}{\sqrt{2}}.$$

The curve (1) will have points of inflexion where $x = \pm\, a/\sqrt{2}$, provided $d^3 y/dx^3$ is not zero at these points.

Now for $x = \pm \dfrac{a}{\sqrt{2}}$, we have $\dfrac{d^3 y}{dx^3} = 0 + y \cdot \dfrac{8}{a^4} \cdot \left(\pm \dfrac{a}{\sqrt{2}}\right) \neq 0.$

Hence the points of inflexion of (1) are given by $x = \pm\, a/\sqrt{2}$.

Problem 14: *Show that the points of inflexion of the curve* $r = b\theta^n$ *are given by*

$$r = b\,\{-n(n+1)\}^{n/2}.$$

Solution: Differentiating the given equation of the curve w.r.t. θ, we get

$$dr/d\theta = nb\,\theta^{n-1} \text{ and } d^2 r/d\theta^2 = n(n-1)\, b\theta^{n-2}.$$

We know that at the point of inflexion, the radius of curvature is infinite. Hence at the point of inflexion, we have

$$r^2 + 2\,(dr/d\theta)^2 - r(d^2 r/d\theta) = 0 \qquad \textbf{(Note)}$$

or $$(b\theta^n)^2 + 2\,(nb\theta^{n-1})^2 - (b\theta^n)\{n(n-1)\, b\theta^{n-2}\} = 0,$$

substituting the values of r, $dr/d\theta$, $d^2 r/d\theta^2$

or $$b^2 \theta^{2n}\left[1 + \frac{2n^2}{\theta^2} - \frac{n(n-1)}{\theta^2}\right] = 0$$

or $\qquad b^2\,\theta^{2n-2}\,[\theta^2 + n^2 + n] = 0$ giving $\theta^2 = -\,n\,(n+1)$.

Now from the equation of the curve, we have

$$r = b\theta^{\,n} = b\,(\theta^2)^{n/2}. \qquad ...(1)$$

Putting $\theta^2 = -\,n\,(n+1)$ in (1), we see that the points of inflexion are given by

$$r = b\,\{-\,n\,(n+1)\,\}^{n/2}.$$

Comprehensive Problems 2

Problem 1: *Write down the equations to the tangent at the origin for the following curves :*

(i) $y^2\,(a-x) = x^2\,(a+x)$, $\qquad$ *(ii)* $x^4 + 3x^3\,y + 2xy - y^2 = 0$,

(iii) $(x^2 + y^2)(2a - x) = b^2 x.$

Solution: **(i)** The curve is

$$y^2\,(a-x) = x^2\,(a+x) \quad \text{or} \quad y^2 a - y^2 x - x^2 a - x^3 = 0.$$

Equating to zero the lowest degree term in the curve, we get

$$y^2 a - x^2 a = 0 \quad \text{or} \quad y^2 = x^2 \quad \text{or} \quad y = \pm\, x$$

(ii) The curve can be written as

$$x^4 + 3x^3\,y + 2xy - y^2 = 0.$$

Equating to zero the lowest degree term in the curve, we get

$$2xy - y^2 = 0 \quad \text{or} \quad y\,(2x - y) = 0 \quad \text{or} \quad y = 0,\ y = 2x.$$

(iii) The curve can be written as

$$2ax^2 - x^3 + 2ay^2 - xy^3 - b^2 x = 0.$$

Equating to zero the lowest degree term in the curve, we get

$$x = 0.$$

Problem 2: *For the curve* $y^2\,(a^2 + x^2) = x^2\,(a^2 - x^2)$, *show that the origin is a node.*

Solution: Equating to zero, the lowest degree terms in the equation of the curve, the tangents at the origin are

$$a^2\,y^2 - a^2 x^2 = 0 \quad \text{or} \quad y = \pm\, x,$$

which are **real and distinct.** Therefore the origin is a **node.**

Problem 3: *Show that the origin is a conjugate point on the curve* $y^2 = 2x^2\,y + x^4\,y - 2\,x^4$.

Solution: Equating to zero, the lowest degree terms in the given curve, the tangents at the origin are given by $y^2 = 0$ *i.e.*, $y = 0,\ y = 0$.

$\therefore$ Origin is either a *cusp or a conjugate point .*

Now the equation of the given curve can be written as

$$y^2 - x^2 y (2 + x^2) + 2x^4 = 0.$$

Solving it for y, we have

$$y = \frac{x^2 (2 + x^2) \pm \sqrt{\{x^4 (2 + x^2)^2 - 8x^4\}}}{2}$$

$$= \frac{x^2 (x^2 + 2) + x^2 \sqrt{(x^4 + 4x^2 - 4)}}{2}.$$

Now for small values of $x \neq 0$, $x^4 + 4x^2 - 4$ is negative.

$\therefore$ y is imaginary in the neighbourhood of origin.

Hence origin is a *conjugate point.*

Problem 4: *Show that the curve* $x^3 + x^2 y = ay^2$ *has the cusp at the origin.*

Solution: The equation of the given curve may be written as

$$ay^2 - x^2 y - x^3 = 0. \qquad ...(1)$$

Equating to zero, the lowest degree terms in (1), the tangents at the origin are $y^2 = 0$ *i.e.*, $y = 0$ and $y = 0$.

$\therefore$ Origin is either a *cusp or a conjugate point.*

Now from (1), $y = [x^2 \pm \sqrt{(x^4 + 4ax^3)}]/(2a)$.

For small values of $x \neq 0$, $x^4 + 4ax^3$ has the same sign as $4ax^3$, which is +ive when x is +ive and – ive when x is – ive.

$\therefore$ When x is +ive, y has two real values one +ive and the other –ive, whereas y is imaginary when x is negative.

Hence there is a single cusp of the first kind at the origin on the right side of the y-axis.

Problem 5: *Show that the curve* $y^3 = (x - a)^2 (2x - a)$ *has a single cusp of the first species at the point* $(a, 0)$. **(Garhwal 2006)**

Solution: Here $f(x, y) \equiv y^3 - (x - a)^2 (2x - a) = 0. \qquad ...(1)$

$$\therefore \quad (\partial f / \partial x) = -2(x - a)^2 - 2(x - a)(2x - a)$$

$$= -2(x - a)(3x - 2a) = -6x^2 + 10ax - 4a^2$$

and $\quad (\partial f / \partial y) = 3y^2$.

Now for double points $(\partial f / \partial x) = 0, (\partial f / \partial y) = 0$ and $f = 0$.

Here $\quad (\partial f / \partial x) = 0$ gives $(x - a)(3x - 2a) = 0$ or $x = a, 2a/3$

and $\quad (\partial f / \partial y) = 0$ gives $y = 0$.

Thus $(a, 0)$ and $(2a/3, 0)$ are the possible double points.

Out of these only $(a, 0)$ satisfies the given curve. Hence $(a, 0)$ is the only double point.

Shifting the origin to $(a, 0)$ by putting $x = X + a$ and $y = Y + 0$ in (1), the equation of the curve becomes

$$Y^3 = (X + a - a)^2 (2X + 2a - a) = X^2 (2X + a). \qquad ...(2)$$

Now equating to zero, the lowest degree terms in (2), the tangents at the new origin are given by $aX^2 = 0$ *i.e.*, $X = 0$, $X = 0$ are two real coincident tangents at the new origin. Therefore the new origin may be a cusp or a conjugate point.

Now for the curve (2), the tangent at the origin is $X = 0$. Let $X = p$. Putting $X = p$ in (2) and neglecting powers of p greater than 2, we get

$$ap^2 = Y^3. \qquad \text{...(3)}$$

From (3), we see that for sufficiently small positive values of Y, p is real and the two values of p are of opposite signs. Also for numerically small negative values of Y, p is imaginary. Hence for the curve (2) there is a single cusp of the first species at the origin.

Thus for the given curve the point $(a, 0)$ is a single cusp of the first kind.

Problem 6(i): *Find the position and nature of double points on the curve* $y^3 = x^3 + ax^2$.

(Meerut 2013B)

Solution: Proceed as in problem 5 of this problem set. Here for the given curve $(0, 0)$ is the only double point and is a single cusp of the first kind.

Problem 6(ii): *Find the position and nature of double points of the curve* $y^2 + 3ax^2 + x^3 = 0$.

Solution: The curve is

$$y^2 + 3ax^2 + x^3 = 0. \qquad \text{...(1)}$$

The curve is $\quad y^2 + 3ax^2 + x^3 = 0. \qquad$ ...(2)

Here $\quad f(x, y) \equiv y^2 + 3ax^2 + x^3 = 0$

$\therefore \quad \dfrac{\partial f}{\partial x} = 2y + 6ax + 3x^2, \dfrac{\partial f}{\partial y} = 2y$

Now at a double point $f = 0$,

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

i.e., $\quad 2y + 6ax + 3x^2 = 0, \quad y = 0.$

At $y = 0$, we have

$$6ax + 3x^2 = 0 \quad \text{or} \quad x(6a + 3x) = 0 \quad x = 0, x = -2a.$$

Thus $(0, 0)$, $(-2a, 0)$ are the possible double points.

But out of these only $(0, 0)$ satisfies the given equation (1).

Hence (0, 0) is the only double point.

Here $\quad \dfrac{\partial^2 f}{\partial x^2} = 6a + 6x, \ \dfrac{\partial^2 f}{\partial y^2} = 2, \ \dfrac{\partial^2 f}{\partial x \, \partial y} = 0.$

At $(0,0)$, we have $\dfrac{\partial^2 f}{\partial x^2}=6a, \dfrac{\partial^2 f}{\partial y^2}=2, \dfrac{\partial^2 f}{\partial x\,\partial y}=0.$

So $$\frac{\partial^2 f}{\partial x\,\partial y}<\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}.$$

$\therefore$ There is a conjugate point at the origin.

Problem 6(iii): *Find the position and nature of double points on the curve* $x^3+y^3=3axy$.

(Meerut 2003, 12B, 13; Agra 06; Rohilkhand 07; Kumaun 08, 09; Lucknow 11)

Solution: Here $f(x, y)\equiv x^3+y^3-3axy=0$. ...(1)

$\therefore$ $$(\partial f/\partial x)=3x^2-3ay,\ (\partial f/\partial y)=3y^2-3ax. \quad ...(2)$$

Now at a double point $f=0, (\partial f/\partial x)=0$ and $(\partial f/\partial y)=0$.

$\therefore$ From (2), by equating each of $\partial f/\partial x$ and $\partial f/\partial y$ to zero, we get

$$x^2=ay \text{ and } y^2=ax.$$

Solving, we get $x(x^3-a^3)=0$ or $x=0, a$.

When $x=0, y=0$ and when $x=a, y=a$.

Out of these points only $(0,0)$ satisfies the equation. $f(x, y)=0$ of the given curve. Therefore $(0,0)$ is the only double point.

Now equating the lowest degree terms in (1) to zero, we get $xy=0$ or $x=0, y=0$ as the tangents at the origin. These being real and distinct implies that origin is a node.

Problem 6(iv): *Find the position and nature of the curve* $x^3+y^3=3xy$, *at the origin.*

(Meerut 2001, 05; Agra 14)

Solution: The curve is $x^3+y^3=3xy$.

Equating to zero the lowest degree term in the equation of the curve, we get equation of the tangent at the origin as

$$3xy=0 \quad i.e., \quad x=0, y=0$$

Since the tangents are real and distinct, the origin is node.

Problem 6(v): *Find the position and nature of the double points of the curve*

$$a^4y^2=x^4(2x^2-3a^2).$$

(Meerut 2007B)

Solution: Here $f(x, y)\equiv 2x^6-3a^2x^4-a^4y^2=0$. ...(1)

$\therefore$ $$(\partial f/\partial x)=12x^5-12a^2x^3,\ (\partial f/\partial y)=-2a^4y. \quad ...(2)$$

Now at a double point $f=0, (\partial f/\partial x)=0$ and $(\partial f/\partial y)=0$.

Here $(\partial f/\partial x)=0$ gives $12x^3(x^2-a^2)=0$ or $x=0, a, -a$

and $(\partial f/\partial y)=0$ gives $-2a^4y=0$ or $y=0$.

Thus $(0,0), (a,0), (-a,0)$ are the *possible double points*.

But out of these only $(0, 0)$ satisfies the given equation (1). Hence $(0, 0)$ is the only double point.

Now equating to zero the lowest degree terms in the equation of the curve we get the tangents at the origin as $a^2 y^2 = 0$ *i.e.*, $y = 0$, $y = 0$ are two coincident tangents at the origin.

Thus the origin may be a cusp or a conjugate point.

Now from (1), $\quad y = \pm \dfrac{x^2}{a^2} \sqrt{(2x^2 - 3a^2)}$.

For small values of $x \neq 0$, +ive or –ive, $(2x^2 - 3a^2)$ is – ive *i.e.*, y is imaginary. Hence no portion of the curve lies in the neighbourhood of the origin.

Hence origin is a *conjugate point and not a cusp*.

Problem 6(vi): *Find the position and nature of the double points on the following curve :*

$$f(x, y) \equiv x^4 - 2y^3 - 3y^2 - 2x^2 + 1 = 0.$$

(Kumaun 2011)

Solution: At a double point, we have

$$(\partial f / \partial x) = 4x^3 - 4x = 0 \text{ and } (\partial f / \partial y) = -6y^2 - 6y = 0.$$

These give $\qquad x(x^2 - 1) = 0$ and $y(y + 1) = 0$

i.e., $\qquad x = 0, \pm 1;\ y = 0, -1.$

Out of these points only $(0, -1)$, $(1, 0)$ and $(-1, 0)$ satisfy the equation of the curve, which are double points.

Also $\qquad \dfrac{\partial^2 f}{\partial x^2} = 12x^2 - 4,\ \dfrac{\partial^2 f}{\partial x\, \partial y} = 0,\ \dfrac{\partial^2 f}{\partial y^2} = -12y - 6.$

At the point $(0, -1)$, $\quad \dfrac{\partial^2 f}{\partial x^2} = -4,\ \dfrac{\partial^2 f}{\partial x\, \partial y} = 0,\ \dfrac{\partial^2 f}{\partial y^2} = 6.$

$$\therefore \qquad \left(\frac{\partial^2 f}{\partial x\, \partial y}\right)^2 > \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right).$$

The above relation also holds at the points $(1, 0)$ and $(-1, 0)$.

Hence there are nodes at the points $(0, -1)$, $(1, 0)$ and $(-1, 0)$.

Problem 6(vii): *Find the position and nature of origin on the curve*

$$x^4 + y^3 + 2x^2 + 3y^2 = 0.$$

(Bundelkhand 2001; Meerut 07; Avadh 13)

Solution: The curve is

$$x^4 + y^3 + 2x^2 + 3y^2 = 0.$$

Equating to zero the lowest degree term in the equation of the curve, we get equation of the tangent at the origin as

$$2x^2 + 3y^2 = 0 \quad i.e., \quad \sqrt{3}\, y = \pm i\sqrt{2}\, x.$$

Thus, both the tangents at the origin are imaginary.

Hence, the origin is a conjugate point.

Problem 7: *Show that the curve $y^2 = bx \tan (x/a)$ has a node or a conjugate point at the origin according as a and b have like or unlike signs.*

Solution: Here $f(x, y) \equiv y^2 - bx \tan (x/a) = 0$. ...(1)

At a double point, we must have

$$\frac{\partial f}{\partial x} = -b \tan \frac{x}{a} - \frac{b}{a} x \sec^2 \left(\frac{x}{a}\right) = 0 \text{ and } \frac{\partial f}{\partial y} = 2y = 0.$$

These give $x = 0$, $y = 0$. Also $(0, 0)$ satisfies the given curve. Thus $(0, 0)$ is a double point.

Also $$\frac{\partial^2 f}{\partial x \partial y} = 0, \frac{\partial^2 f}{\partial y^2} = 2, \text{ at } (0, 0)$$

and $$\frac{\partial^2 f}{\partial x^2} = -\frac{b}{2} \sec^2 \frac{x}{a} - \frac{b}{a} \sec^2 \frac{x}{a} - \frac{2bx}{a^2} \sec \frac{x}{a} \cdot \tan \frac{x}{a} = -2(b/a), \text{ at } (0, 0).$$

Also we know that (see article 11) the double point will be a node, cusp or conjugate point according as

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 >, = \text{ or } < \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right). \qquad \textbf{(Remember)}$$

Here $$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 0 \quad \text{and} \quad \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) = -4 \frac{b}{a}.$$

Now $0 > -4 b/a$ if a and b have like signs

and $0 < -4 b/a$ if a and b have unlike signs.

$\therefore$ Origin is a node or conjugate point according as a and b have like or unlike signs.

Problem 8: *Prove that the curve $y^2 = (x - a)^2 (x - b)$ has at $x = a$, a conjugate point if $a < b$, a node if $a > b$ and a cusp if $a = b$.*

Solution: Here $f(x, y) \equiv (x - a)^2 (x - b) - y^2 = 0$. ...(1)

At $x = a$, $y = 0$; therefore the point is $(a, 0)$.

Shifting the origin to $(a, 0)$ by putting $x = X + a$ and $y = Y + 0$ in (1), the given equation of the curve changes to

$$Y^2 = X^2 (X + a - b). \qquad ...(2)$$

Equating to zero, the lowest degree terms in (2), the tangents at the new origin are given by

$$Y^2 = (a - b) X^2 \quad \text{or} \quad Y = \pm \sqrt{(a - b)}\, X. \qquad ...(3)$$

When $b > a$, the two tangents given by (3) become imaginary. Hence the new origin *i.e.*, the point $(a, 0)$ on the given curve is a conjugate point.

When $b = a$, the two tangents given by (3) are $Y = 0, Y = 0$ *i.e.*, they are real and coincident. Therefore the new origin is a cusp or a conjugate point.

In this case from (2), we get

$$Y^2 = X^3 \quad \text{or} \quad Y = \pm X\sqrt{X}. \qquad [\because a = b]$$

When X is small and +ive, the two values of Y are real.

$\therefore$ The curve has real branches at the new origin.

$\therefore$ The point $(a, 0)$ on the given curve is a cusp.

When $b < a$, the two tangents given by (3) are real and distinct. Hence the new origin *i.e.*, the point $(a, 0)$ on the given curve is a node.

Problem 9: *Examine the curve* $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$ *for singular points and show that it has a cusp of first kind at the point* $(-1, -2)$.

Solution: Here $f(x, y) \equiv x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$. ...(1)

$\therefore$ $(\partial f / \partial x) = 3x^2 + 4x + 2y + 5$; $(\partial f / \partial y) = 2x - 2y - 2$.

For double points we must have $\dfrac{\partial f}{\partial x} = 0$, $\dfrac{\partial f}{\partial y} = 0$ and $f = 0$.

Here $\partial f / \partial x = 0$ gives $3x^2 + 4x + 2y + 5 = 0$

and $\partial f / \partial y = 0$ gives $2x - 2y - 2 = 0$ or $2y = 2x - 2$.

Solving these equations, we get $x = -1, -1$ and $y = -2$. Thus $(-1, -2)$ is the only possible double point.

Since $(-1, -2)$ satisfies the equation of the given curve, therefore it is a double point.

Shifting the origin to $(-1, -2)$ (by putting $x = X - 1$ and $y = Y - 2$), the equation of the curve changes to

$$(X-1)^3 + 2(X-1)^2 + 2(X-1)(Y-2) - (Y-2)^2 + 5(X-1) - 2(Y-2) = 0$$

or $$X^3 - X^2 + 2XY - Y^2 = 0 \quad \text{or} \quad (Y - X)^2 = X^3. \qquad ...(2)$$

Equating to zero the lowest degree terms in (2), the tangents at the new origin are $(Y - X)^2 = 0$ *i.e.*, $Y - X = 0, Y - X = 0$ are two coincident tangents at the new origin.

Therefore the new origin may be a cusp or a conjugate point.

Now the tangent to the curve (2) at the origin is $Y - X = 0$.

Let $Y - X = p$.

Putting $Y - X = p$ in (2), we get

$$p^2 = X^3. \qquad ...(3)$$

From (3), we see that for sufficiently small positive values of X, p is real and the two values of p are of opposite signs. Also for negative values of X, p is imaginary. Hence for the curve (2) there is a single cusp of the first kind at the origin.

Hence the point $(-1, -2)$ on the given curve is a single cusp of first species.

Problem 10 (i): *Determine the positions and character of the double points on*

$$y(y-6) = x^2(x-2)^3 - 9.$$ **(Rohilkhand 2008, 09)**

Solution: Here $f(x, y) \equiv x^2(x-2)^3 - y(y-6) - 9 = 0$. ...(1)

$\therefore$ $(\partial f/\partial x) = 2x(x-2)^3 + 3(x-2)^2 . x^2$

$= x(x-2)^2 [2(x-2)+3x] = x(x-2)^2 (5x-4)$

and $(\partial f/\partial y) = -2y + 6.$

For double points, $(\partial f/\partial x) = 0, (\partial f/\partial y) = 0$ and $f = 0$.

Here $(\partial f/\partial x) = 0$ gives $x = 0, 2, 4/5$ and $(\partial f/\partial y) = 0$ gives $y = 3$.

$\therefore$ $(0, 3), (2, 3)$ and $(4/5, 3)$ are the possible double points.

Out of these only $(0, 3)$ and $(2, 3)$ satisfy (1) *i.e.*, these are the only two double points on the curve.

Nature of the double point (0, 3) : Shifting the origin to $(0, 3)$ by putting $x = X + 0$ and $y = Y + 3$, the equation (1) changes to

$$(Y+3)(Y+3-6) = X^2 (X-2)^3 - 9$$

or $Y^2 - 9 = X^2 (X-2)^3 - 9$ or $Y^2 = X^2 (X-2)^3$. ...(2)

Equating to zero, the lowest degree terms in (2), the tangents at the new origin are given by $Y^2 = -8X^2$, which are imaginary.

$\therefore$ The new origin *i.e.*, the point $(0, 3)$ on the given curve is a conjugate point.

Nature of the double point (2, 3) : Shifting the origin to $(2, 3)$ by putting $x = X + 2$ and $y = Y + 3$, the equation (1) changes to

$$(Y+3)(Y+3-6) = (X+2)^2 X^3 - 9$$

or $Y^2 = X^3 (X+2)^2$. ...(3)

Equating to zero the lowest degree terms in (3), the tangents at the new origin are given by $Y^2 = 0$ or $Y = 0$, $Y = 0$ which are real and coincident.

$\therefore$ The new origin *i.e.*, the point $(2, 3)$ on the given curve is either a cusp or a conjugate point.

Now from (3), $Y = \pm X(X+2)\sqrt{X}$.

When X is small and +ive, Y is real and the values of Y are of opposite signs. When X is small and –ive, Y is imaginary. Thus the curve has real branches at the new origin only on one side of the line $X = 0$ and the two branches of the curve lie on opposite sides of their common tangent $Y = 0$. Hence the new origin *i.e.*, the point $(2, 3)$ on the curve (1) is a single cusp of the first kind.

Problem 10 (ii): *Determine the position and character of the double points on the curve*

$$(x-2)^2 = y(y-1)^2.$$

(Meerut 2000, 02; Gorakhpur 05; Rohilkhand 12)

Solution: Here $f(x, y) \equiv (x-2)^2 - y(y-1)^2 = 0$. ...(1)

$\therefore$ $(\partial f/\partial x) = 2(x-2)$ and $(\partial f/\partial y) = -(y-1)^2 - 2y(y-1)$.

For double points, $\partial f/\partial x = 0$, $\partial f/\partial y = 0$ and $f(x, y) = 0$.

Here $(\partial f/\partial x) = 0$ gives $x = 2$ and $(\partial f/\partial y) = 0$ gives $y = 1, \frac{1}{3}$.

Thus $(2, 1)$ and $(2, \frac{1}{3})$ are the possible double points.

Out of these only $(2,1)$ satisfies the equation of the curve. Hence $(2,1)$ is the only double point.

Shifting the origin to $(2,1)$ (by putting $x = X + 2$, $y = Y + 1$), the equation of the curve changes to

$$X^2 = (Y + 1) Y^2. \quad \text{...(2)}$$

Now equating to zero, the lowest degree terms in (2), the tangents at the new origin are given by $Y^2 = X^2$ or $Y = \pm X$. The two tangents being real and distinct, the new origin is a node. Hence the double point $(2,1)$ is a node.

Problem 10(iii): *Determine the position and character of the double points on the curve*

$$x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0.$$

Solution: The equation of the given curve is

$$f(x, y) \equiv x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0. \quad \text{...(1)}$$

We have $\partial f / \partial x = 3x^2 - 14x + 15$ and $\partial f / \partial y = -2y + 4$.

For the double points,

$$\partial f / \partial x = 0, \partial f / \partial y = 0 \quad \text{and} \quad f(x, y) = 0.$$

Here $\partial f / \partial x = 0$ gives $3x^2 - 14x + 15 = 0$

i.e., $3x^2 - 9x - 5x + 15 = 0$

i.e., $(x - 3)(3x - 5) = 0$ *i.e.*, $x = 3, 5/3$

and $\partial f / \partial y = 0$ gives $-2y + 4 = 0$ *i.e.*, $y = 2$.

$\therefore$ The possible double points are $(3,2)$ and $(5/3, 2)$. Out of these only $(3,2)$ satisfies the equation of the curve. Hence $(3,2)$ is the only double point of the given curve.

Nature of the double point (3, 2).

Now $\partial^2 f / \partial x^2 = 6x - 14$, $\partial^2 f / \partial x \partial y = 0$, $\partial^2 f / \partial^2 y = -2$.

At the point $(3,2)$, we have $\dfrac{\partial^2 f}{\partial x^2} = 4, \dfrac{\partial^2 f}{\partial x \partial y} = 0, \dfrac{\partial^2 f}{\partial y^2} = -2.$

$\therefore$ At the point $(3,2)$, we have $\left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2 = 0$ and $\dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} = -8.$

Thus at the point $(3,2)$, we have $\left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2 > \dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2}.$

Therefore there is a node at the point $(3,2)$.

Alternative method for finding the nature of the double point (3, 2).

Shifting the origin to the point $(3,2)$, the equation of the curve becomes

$$(x + 3)^3 - (y + 2)^2 - 7(x + 3)^2 + 4(y + 2) + 15(x + 3) - 13 = 0$$

or

$$x^3 + 9x^2 + 27x + 27 - y^2 - 4y - 4 - 7x^2 - 42x - 63$$
$$+ 4y + 8 + 15x + 45 - 13 = 0$$

or $$x^3 + 2x^2 - y^2 = 0. \qquad ...(2)$$

Equating to zero the lowest degree terms in (2), the tangents at the new origin are

$$y^2 - 2x^2 = 0 \quad i.e., \quad y^2 = 2x^2 \quad i.e., \quad y = \pm x\sqrt{2}.$$

Thus there are two real and distinct tangents at the new origin. Therefore the new origin is a node.

Hence there is a node at the point $(3, 2)$ on the given curve.

Problem 10 (iv): *Determine the position and character of the double points on the curve*

$$y^2 - x(x-a)^2 = 0, (a > 0).$$

Solution: The equation of the given curve is

$$f(x, y) \equiv y^2 - x(x-a)^2 = 0. \qquad ...(1)$$

We have $\partial f / \partial x = -(x-a)^2 - 2x(x-a) = -(x-a)(3x-a)$

and $\partial f / \partial y = 2y$.

For double points, $\partial f / \partial x = 0$, $\partial f / \partial y = 0$ and $f(x, y) = 0$.

Here $\partial f / \partial x = 0$ gives $(x-a)(3x-a) = 0$ *i.e.* $x = a, a/3$

and $\partial f / \partial y = 0$ gives $y = 0$.

$\therefore$ The possible double points are $(a, 0)$, $(a/3, 0)$.

Out of these only $(a, 0)$ satisfies the equation of the curve. Therefore $(a, 0)$ is the only double point on the given curve.

Nature of the double point at $(a, 0)$. Shifting the origin to the point $(a, 0)$, the equation of the curve becomes

$$y^2 - (x+a)x^2 = 0. \qquad ...(2)$$

Equating to zero the lowest degree terms in (2), the tangents at the new origin are

$$y^2 - ax^2 = 0 \quad i.e., \quad y^2 = ax^2 \quad i.e., \quad y = \pm\sqrt{a}x.$$

Thus there are two real and distinct tangents at the new origin. Therefore the new origin is a node.

Hence there is a node at the point $(a, 0)$ on the given curve.

Problem 10 (v): *Determine the position and character of the double points on the curve*

$$y^2 - x^3 = 0.$$

Solution: The equation of the given curve is

$$f(x, y) \equiv y^2 - x^3 = 0. \qquad ...(1)$$

We have $\partial f / \partial x = -3x^2$ and $\partial f / \partial y = 2y$.

For double points, $\partial f / \partial x = 0$, $\partial f / \partial y = 0$ and $f(x, y) = 0$.

$\therefore$ The only double point of (1) is (0, 0).

Equating to zero the lowest degree terms in (1), the tangents to (1) at the origin are

$$y^2 = 0 \quad i.e., \quad y = 0 \text{ and } y = 0.$$

$\therefore$ Origin is either a cusp or a conjugate point.

Now from (1), we have $y = \pm\sqrt{(x^3)}$.

For small values of $x \neq 0$, the values of y are real when x is +ive and are imaginary when x is –ive.

Also when x is +ive, the two values of y are of opposite signs.

Hence there is a single cusp of the first kind at the origin (0, 0) on the right side of the y-axis.

Problem 10 (vi): *Determine the position and character of the double points on the curve*

$$a^4 y^2 = x^4 (a^2 - x^2).$$

Solution: The equation of the given curve is

$$f(x, y) \equiv a^4 y^2 - x^4 (a^2 - x^2) = 0. \qquad \ldots(1)$$

We have $\partial f / \partial x = -4x^3 (a^2 - x^2) + 2x^5 = 2x^3 (3x^2 - 2a^2)$

and $\partial f / \partial y = 2a^4 y$.

For double points, $\partial f / \partial x = 0, \partial f / \partial y = 0$ and $f(x, y) = 0$.

Here $\partial f / \partial x = 0$ gives $x = 0, \pm\sqrt{(2/3)}\, a$ and $\partial f / \partial y = 0$ gives $y = 0$.

∴ The possible double points are (0, 0), $(\sqrt{(2/3)}\, a, 0)$, $(-\sqrt{(2/3)}\, a, 0)$.

Out of these only (0, 0) satisfies the equation of the curve. Therefore (0, 0) is the only double point on the given curve.

Nature of the double point at (0, 0). Equating to zero the lowest degree terms in (1), the tangents to (1) at (0, 0) are given by

$$a^4 y^2 = 0 \quad i.e., \quad y = 0 \quad \text{and} \quad y = 0.$$

∴ Origin is either a cusp or a conjugate point.

Now from (1), we have $y = \pm \frac{1}{a^2} \sqrt{(a^2 x^4 - x^6)} = \pm \frac{1}{a^2} \sqrt{(a^2 x^4)}$,

keeping only the lowest degree terms in x under the radical sign.

For small values of $x \neq 0$, the values of y are real both when x is +ive and x is –ive.

Also the two values of y are of opposite signs both when x is +ive and x is –ive.

Hence there is a double cusp of the first kind at (0, 0).

Problem 10 (vii): *Determine the position and character of the double points on the curve*

$$y^2 = x^2 (9 - x^2).$$

Solution: The equation of the given curve is

$$f(x, y) \equiv y^2 - x^2 (9 - x^2) = 0. \qquad \ldots(1)$$

We have $\partial f / \partial x = -2x(9 - x^2) + 2x^3 = 2x(2x^2 - 9)$ and $\partial f / \partial y = 2y$.

For double points $\partial f / \partial x = 0, \partial f / \partial y = 0$ and $f(x, y) = 0$.

Here $\partial f / \partial x = 0$ gives $x = 0, \pm 3/\sqrt{2}$ and $\partial f / \partial y = 0$ gives $y = 0$.

∴ The possible double points are (0, 0), $(3/\sqrt{2}, 0)$, $(-3/\sqrt{2}, 0)$. Out of these only $(0, 0)$ satisfies the equation of the curve. Therefore (0, 0) is the only double point on the given curve.

Equating to zero the lowest degree terms in (1), the tangents to (1) at (0, 0) are

$$y^2 - 9x^2 = 0 \quad i.e., \quad y^2 = 9x^2 \quad i.e., \quad y = \pm 3x.$$

Thus there are two real and distinct tangents at (0, 0). Hence origin is a node on the given curve.

Problem 11: *Find the position and nature of double points on the curve*

$$x^2 y^2 = (a + y)^2 (b^2 - y^2)$$

if (i) $b > a$, (ii) $b = a$, (iii) $b < a$.

Solution: Here $f(x, y) \equiv x^2 y^2 - (a + y)^2 (b^2 - y^2) = 0.$ ...(1)

$\therefore$ $(\partial f / \partial x) = 2xy^2,$

$$(\partial f / \partial y) = 2x^2 y - 2(a + y)(b^2 - y^2) + 2y(a + y)^2.$$

For the double points, $(\partial f / \partial x) = 0, (\partial f / \partial y) = 0$ and $f = 0$.

Now $(\partial f / \partial x) = 0$ gives $2xy^2 = 0$ or $x = 0, y = 0$ and $(\partial f / \partial y) = 0$ gives

$$2x^2 y - 2(a + y)(b^2 - y^2) + 2y(a + y)^2 = 0$$

or $(a + y)[y(a + y) - (b^2 - y^2)] = 0,$ when $x = 0$

or $(y + a)(2y^2 + ay - b^2) = 0.$ ...(2)

$\therefore$ When $x = 0, y = -a$ or $y = [-a \pm \sqrt{(a^2 + 8b^2)}]/4.$

For $y = 0, \partial f / \partial y = 0$ gives no value of x.

$\therefore$ The possible double points are $(0, -a), (0, \{-a \pm \sqrt{(a^2 + 8b^2)}\}/4).$

Out of these only $(0, -a)$ satisfies the given equation of the curve.

$\therefore$ $(0, -a)$ is the only double point on the given curve.

Nature of $(0, -a)$: Shifting the origin to $(0, -a)$ by putting $x = X + 0$ and $y = Y - a$ in (1), the given equation becomes

$$X^2 (Y - a)^2 = \{a + Y - a\}^2 \{b^2 - (Y - a)^2\}$$

or $(Y^2 - 2aY + a^2) X^2 = Y^2 (b^2 - a^2 + 2aY - Y^2)$ ...(3)

Equating to zero, the lowest degree terms in (3), the tangents at the new origin are given by

$$(a^2 - b^2) Y^2 + a^2 X^2 = 0 \text{ or } \sqrt{(b^2 - a^2)}\, Y = \pm aX. \quad ...(4)$$

When $b > a$, the two tangents given by (4) are real and distinct.

$\therefore$ The new origin is a node. Hence when $b > a$, the point $(0, -a)$ is a node on the given curve.

When $b = a$, the two tangents given by (4) become $X = 0, X = 0$ which are real and coincident.

$\therefore$ The new origin is a cusp or a conjugate point.

Putting $b = a$, the equation (3) becomes

$$(Y^2 - 2aY + a^2) X^2 = Y^2 (2aY - Y^2). \quad ...(5)$$

Now for the curve (5) the tangent at the origin is $X = 0$, Let $X = p$. Putting $X = p$ in (5), we get

$$p^2 = \frac{2aY^3 - Y^4}{Y^2 - 2aY + a^2} = \frac{2\,aY^3}{a^2},$$

neglecting $-Y^4$ in the numerator and $Y^2 - 2\,aY$ in the denominator.

Now for small +ive values of Y, p is real and the two values of p are of opposite signs. Also for numerically small negative values of Y, p is imaginary. Thus, in the neighbourhood of origin the curve lies only on one side of the line $Y = 0$ and the two branches of the curve lie on opposite sides of the common tangent $X = 0$. Hence for the curve (5) there is a single cusp of the first kind at the origin. Hence when $b = a$, the point $(0, -a)$ is a single cusp of the first kind on the given curve.

When $b < a$, the two tangents given by (4) become imaginary. Hence the new origin *i.e.*, the point $(0, -a)$ on the given curve is a conjugate point.

Problem 12: *Discuss the nature of double points of the curve*

$$(x + y)^3 = \sqrt{2}\,(y - x + 2)^2.$$

Solution: Here $f(x, y) \equiv (x + y)^3 - \sqrt{2}\,(y - x + 2)^2 = 0$. ...(1)

$\therefore$ $\quad (\partial f/\partial x) = 3\,(x + y)^2 + 2\sqrt{2}\,(y - x + 2),$

and $\quad (\partial f/\partial y) = 3\,(x + y)^2 - 2\sqrt{2}\,(y - x + 2).$

For double points we must have $\dfrac{\partial f}{\partial x} = 0, \dfrac{\partial f}{\partial y} = 0$

and $\quad f = 0.$

Now $\partial f/\partial x = 0$ gives $3\,(x + y)^2 + 2\sqrt{2}\,(y - x + 2) = 0$ ...(2)

and $\partial f/\partial y = 0$ gives $3\,(x + y)^2 - 2\sqrt{2}\,(y - x + 2) = 0$. ...(3)

Adding (2) and (3), we get $x + y = 0$.

Subtracting (3) from (2), we get $y - x + 2 = 0$.

Solving these equations, we get $x = 1$, $y = -1$.

Also $(1, -1)$ satisfies the equation of the given curve. Hence $(1, -1)$ is a double point.

Shifting the origin to $(1, -1)$, the given equation of the curve reduces to

$$(X + 1 + Y - 1)^3 = \sqrt{2}\,(Y - 1 - X - 1 + 2)^2$$

or $\quad (X + Y)^3 = \sqrt{2}\,(Y - X)^2.$ ...(4)

Equating to zero, the lowest degree terms in (4), the tangents at the new origin $(1, -1)$ are given by $(Y - X)^2 = 0$, *i.e.*, $Y - X = 0, Y - X = 0$ are two real and coincident tangents at the new origin.

$\therefore$ The new origin $(1, -1)$ is either a cusp or a conjugate point. In equation (4), putting $Y - X = p$ *i.e.*, $Y = p + X$, we get

$$(2X + p)^3 = \sqrt{2}\,p^2$$

or $$p^3 + p^2 (6X - \sqrt{2}) + 12X^2 p + 8X^3 = 0$$

or $$(6X - \sqrt{2}) p^2 + 12X^2 p + 8X^3 = 0. \qquad [\text{neglecting } p^3 \text{ as } p \to 0]$$

$\therefore$ $$p = \frac{-12X^2 + \sqrt{\{144X^4 - 32X^3 (6X - \sqrt{2})\}}}{2 (6X - \sqrt{2})}$$

$$= \frac{-6X^2 \pm \sqrt{(8\sqrt{2}X^3 - 12X^4)}}{(6X - \sqrt{2})}$$

$$= \frac{-6X^2 \pm \sqrt{(8\sqrt{2}X^3)}}{(6X - \sqrt{2})}, \text{ neglecting } -12X^4.$$

When X is +ive and small p is real and the two values of p are of opposite signs.
When X is –ive and small p is imaginary.
Thus there is a single cusp of the first kind at the new origin.
Hence on the given curve the double point (1, – 1) is a single cusp of the first species.

Problem 13: *Show that the curve* $(xy + 1)^2 + (x - 1)^3 (x - 2) = 0$ *has a single cusp of the first species at the point* (1, – 1).

Solution: The given curve is

$$(xy + 1)^2 + (x - 1)^3 (x - 2) = 0. \qquad \dots(1)$$

The point (1, – 1) satisfies the equation (1) and so it lies on (1).
Shifting the origin to (1, – 1), the equation (1) becomes

$$[(x + 1)(y - 1) + 1]^2 + x^3 (x - 1) = 0$$

or $$[xy - (x - y)]^2 + x^3 (x - 1) = 0$$

or $$x^2 y^2 - 2xy (x - y) + (x - y)^2 + x^3 (x - 1) = 0 \qquad \dots(2)$$

Equating to zero the lowest degree terms in (2), the tangents to (2) at the new origin are

$$(x - y)^2 = 0 \quad \textit{i.e.,} \quad x - y = 0, x - y = 0.$$

$\therefore$ The new origin is either a cusp or a conjugate point.
In equation (2), putting $x - y = p$ *i.e.,* $y = x - p$, we get

$$x^2 (x - p)^2 - 2x (x - p) p + p^2 + x^3 (x - 1) = 0$$

or $$x^4 - 2x^3 p + x^2 p^2 - 2x^2 p + 2xp^2 + p^2 + x^3 (x - 1) = 0$$

or $$p^2 (x^2 + 2x + 1) - (2x^3 + 2x^2) p + 2x^4 - x^3 = 0$$

or $$p^2 (x + 1)^2 - 2x^2 (x + 1) p + 2x^4 - x^3 = 0.$$

$\therefore$ $$p = \frac{2x^2 (x + 1) \pm \sqrt{[4x^4 (x + 1)^2 - 4 (x + 1)^2 (2x^4 - x^3)]}}{2 (x + 1)^2}$$

$$= \frac{2x^2 (x + 1) \pm 2(x + 1) \sqrt{(x^4 - 2x^4 + x^3)}}{2(x + 1)^2}$$

$$= \frac{x^2 (x+1) \pm (x+1) \sqrt{(x^3)}}{(x+1)^2},$$

keeping only the lowest degree terms in x under the radical sign.

Also $$p_1 p_2 = \frac{2x^4 - x^3}{(x+1)^2} = \frac{-x^3}{(x+1)^2},$$

keeping only the lowest degree terms in x in the numerator.

For small values of $x \neq 0$, the values of p are real when x is + ive and are imaginary when x is –ive.

Also when x is positive, $p_1 p_2$ is –ive *i.e.*, the two values of p are of opposite signs.

Hence the new origin on the curve (2) is a single cusp of the first kind.

Hence the given curve (1) has a single cusp of the first species at the point $(1, -1)$.

Comprehensive Problems 3

Problem 1: *Trace the curve* $x^3 y = x + 1$.

Solution: The given curve may be written as

$$y = (x+1)/x^3 = (1/x^2) + (1/x^3).$$

(i) No symmetry.

(ii) The curve does not pass through the origin.

(iii) When $y = 0$, $x = -1$; when $x = 0$, we do not get any value of y. Hence the curve cuts the coordinate axes only at the point $(-1, 0)$.

(iv) Asymptotes parallel to x-axis : $y = 0$ *i.e.*, x-axis. Asymptotes parallel to y-axis : $x^3 = 0$ or $x = 0$ *i.e.*, y-axis.

(v) When x is positive, y is positive ; when $x \to \infty$, $y \to 0$; when $x \to 0$ from the right, $y \to \infty$; when $x \to 0$ from the left, $y \to -\infty$. At $x = -1$, $y = 0$.

When $x < -1$, y is positive ; when $x \to -\infty$, $y \to 0$.

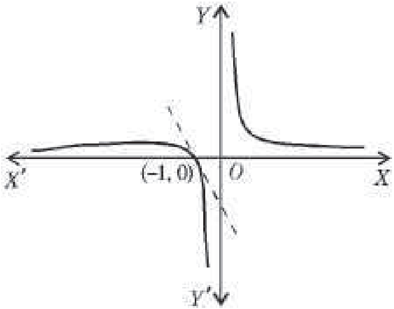
Hence when $x > 0$, the curve is in the first quadrant, when $-1 < x < 0$, the curve is in the third quadrant and when $x < -1$, the curve is in the second quadrant.

(vi) Special points :

x	$-\infty$	-2	-1	$-\frac{1}{2}$	-0
y	0	$\frac{1}{8}$	0	-4	$-\infty$

x	$+0$	1	2	∞
y	∞	2	$\frac{3}{8}$	0

Hence the curve is as shown in the figure.



Problem 2: *Trace the curve* $x = (y-1)(y-2)(y-3)$.

(Purvanchal 2006; Kanpur 09)

Solution: (i) The curve is not symmetrical about the coordinate axes or about the line $x = y$ or in opposite quadrants.

(ii) The curve does not pass through the origin.

(iii) Taking y as the independent variable, we have

when $y = 0$, $x = -6$; when $y = 1$, $x = 0$.

When $0 < y < 1$, x is negative, as then all the three factors are negative.

When $1 < y < 2$, x is positive as one factor is +ive and two are –ive. Also x becomes zero at $y = 2$.

When $2 < y < 3$, x is negative and x becomes zero at $y = 3$.

When $y > 3$, x is positive. As $y \to \infty$, $x \to \infty$.

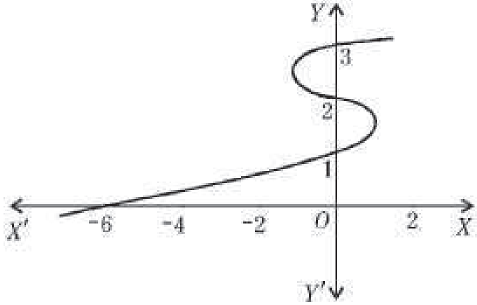
When $y < 0$, x is negative and when $y \to -\infty$, $x \to -\infty$.

(iv) When $y \to \pm\infty$, $x \to \pm\infty$. For very large values of y, x is approximately equal to y^3.

Hence there are no linear asymptotes.

(v) When $y = \dfrac{3}{2}$, $x = \dfrac{3}{8}$; when $y = \dfrac{5}{2}$, $x = -\dfrac{3}{8}$.

Hence the shape of the curve is as shown in the figure.



Problem 3: *Trace the curve* $y = x(x^2 - 1)$. **(Garhwal 2002)**

Solution: (i) There is symmetry in opposite quadrants.

[$\because$ By putting $-x$ for x and $-y$ for y, the equation of the curve remains unchanged]

(ii) The curve passes through $(0, 0)$. Tangent at the origin is $y = -x$.

(iii) When $y = 0$, $x = 0, \pm 1$; when $x = 0$, $y = 0$.

Hence the curve intersects the coordinate axes at the points $(0, 0), (\pm 1, 0)$.

From the equation of the curve, $dy/dx = 3x^2 - 1$. At $(\pm 1, 0)$, $dy/dx = 2$ *i.e.*, tangents at the points $(\pm 1, 0)$ make an angle lying between 60° and 90° with the positive direction of x-axis.

(iv) No asymptotes.

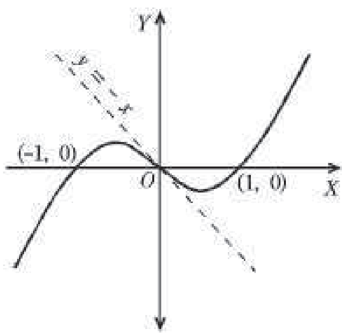
(v) When x lies between 0 and 1, y is negative.

When $x > 1$, y is positive and when $x \to \infty$, $y \to \infty$.

(vi) Special points :

x	0	$\frac{1}{2}$	1	2	3	∞
y	0	$-\frac{3}{8}$	0	6	24	∞

Trace the curve first on the right hand side of the x-axis (*i.e.*, in I and IV quadrants) and then by symmetry on the left hand side of the x-axis (*i.e.*, in II and III quadrants). The curve is as shown in the figure.



Problem 4: *Trace the curve* $y^2 = 4ax$. ***(Parabola)***

Solution: The curve $y^2 = 4ax$

(i) The power of y in the equation is even so the curve is symmetric about the x-axis.

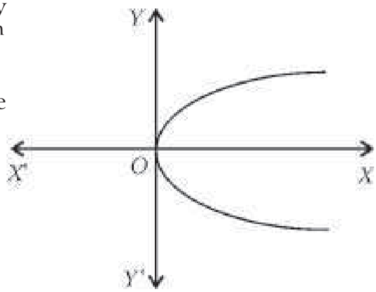
(ii) the curve passes through the origin and the tangent at the origin and the tangent at the origin are $4ax = 0$ *i.e.*, $x = 0$.

(iii) The curve meets the co-ordinate axes only at the origin .

(iv) The curve has no asymptotes.

(v) For the negative value of x, y is imaginary then no part of curve in III[rd] and IV[th] quadrant at $x \to \infty \Rightarrow y \to \infty$.

Taking all these facts into consideration, the shape of the curve is as shown in figure.



Problem 5: *Trace the curve $xy^2 = 4a^2 (2a - x)$.* ***(Witch of Agnesi)***

Solution: (i) The curve is symmetrical about the axis of x.

(ii) The curve does not pass through the origin.

(iii) When $y = 0, 2a - x = 0$ or $x = 2a$.

$\therefore$ The curve crosses the x-axis at $(2a, 0)$.

When $x = 0$, we do not get any value of y and so the curve does not meet the y-axis.

Shifting the origin to $(2a, 0)$ the equation to the curve becomes

$$(x + 2a)\, y^2 = 4a^2 (2a - x - 2a) \quad \text{or} \quad y^2 x + 2ay^2 + 4a^2 x = 0. \quad ...(1)$$

Equating to zero, the lowest degree terms in (1), the equation of tangent at the new origin is $x = 0$ *i.e.*, the new y-axis.

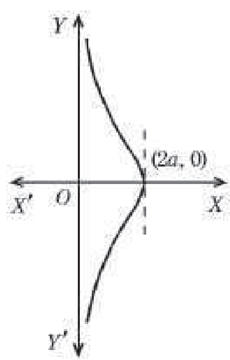
Thus the tangent to the given curve at the point $(2a, 0)$ is parallel to y-axis.

(iv) The asymptote of the curve parallel to y-axis is $x = 0$ *i.e.*, the y-axis and we note that the curve has no other real asymptotes.

(v) Solving the equation of the curve for y, we have

$$y^2 = \{4a^2 (2a - x)\}/x.$$

When $x \to 0$ from the right $y^2 \to \infty$ showing that the line $x = 0$ is an asymptote. When $x = 2a$, $y = 0$. When $0 < x < 2a$, y is real and so the curve exists in this region. When $x > 2a$, y is imaginary and the curve does not exist for $x > 2a$. When $x < 0$, y is again imaginary and so the curve does not exist in the region where x is –ive. When x decreases from $2a$ to 0, y^2 increases from 0 to ∞. Thus the shape of the curve is as shown in the figure.

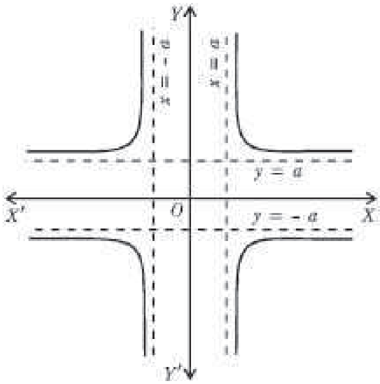


Problem 6: *Trace the curve* $x^2 y^2 = a^2 (x^2 + y^2)$. **(Gorakhpur 2006)**

Solution: (i) The curve is symmetrical about both the axes.

(ii) The curve passes through the origin.

Equating to zero, the lowest degree terms of the given curve, the tangents at the origin are $x^2 + y^2 = 0$ or $y^2 = -x^2$ *i.e.*, two imaginary tangents. Hence $(0, 0)$ is a conjugate point *i.e.*, an isolated point.



(iii) The curve does not meet the coordinate axes.

(iv) Equating to zero the coefficients of highest powers of x and y, the asymptotes parallel to the coordinate axes are $x = \pm a$ and $y = \pm a$.

(v) Solving the given equation for y, we get

$$y^2 = a^2 x^2 / (x^2 - a^2).$$

When $0 < x < a$, y^2 is negative *i.e.*, y is imaginary and so the curve does not exist in this region. When $x \to a$ from the right, $y^2 \to \infty$.

When $x > a$, y is real and so the curve exists in this region. When $x \to \infty$, $y^2 \to a^2$ showing the fact that the lines $y = \pm a$ are asymptotes of the curve.

Similarly, we find that the curve does not exist in the region $0 < y < a$. It exists in the region where $y > a$ and as $y \to \infty$, $x^2 \to a^2$ showing the fact that the lines $x = \pm a$ are asymptotes of the curve. Combining all these facts we see that the shape of the curve is as shown in the figure.

Problem 7: *Trace the curve* $y(x^2 - 1) = (x^2 + 1)$.

(Bundelkhand 2001; Garhwal 02, 08, 11, 14; Kashi 11)

Solution: We have $y = (x^2 + 1)/(x^2 - 1)$

or $$y = 1 + 2/(x^2 - 1)$$

or $$y = -1 + 2x^2/(x^2 - 1).$$

(i) The curve is symmetrical about y-axis.

(ii) The curve does not pass through the origin.

(iii) When $y = 0$, x is imaginary; when $x = 0$, $y = -1$.

Hence the curve cuts the coordinate axes only at the point $(0, -1)$.

From the equation of the curve, $dy/dx = -4x/(x^2 - 1)^2$.

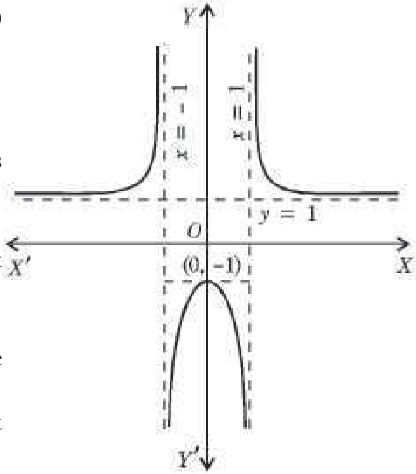
$\therefore$ At the point $(0, -1)$, $dy/dx = 0$ *i.e.*, the tangent to the curve at this point is parallel to the x-axis.

(iv) The curve is $y(x^2-1)-x^2-1=0$.

$\therefore$ Asymptote parallel to x-axis is $y-1=0$ or $y=1$.

Asymptotes parallel to y-axis are

$$x^2-1=0 \quad \text{or} \quad x=\pm 1.$$



(v) When $0<x<1$, y is negative and less than -1.

At $x=0$, $y=-1$. When $x\to 1$ from the left, $y\to-\infty$ and when $x\to 1$ from the right $y\to\infty$.

When $x>1$, y is positive and greater than 1.

Hence when $0<x<1$, the curve is in the IVth quadrant below the line $y=-1$ and when $x>1$, the curve is in the Ist quadrant above the line $y=1$.

(iv) Special points :

$x=$	0	1/2	2	3	$\to\infty$
$y=$	-1	$-5/3$	$5/3$	$5/4$	$\to 1$

$\therefore$ The curve is as shown in the figure.

Problem 8: *Trace the curve* $y(x^2+4a^2)=8a^3$. **(Agra 2008)**

Solution: We have $y=8a^3/(x^2+4a^2)$. ...(1)

(i) The curve is symmetrical about y-axis.

(ii) The curve does not pass through the origin.

(iii) When $y=0$, we get no value of x; when $x=0$, $y=2a$.

Hence the curve cuts the coordinate axes only at the point $(0,2a)$.

Shifting the origin to the point $(0,2a)$ the equation of the curve becomes

$$(y+2a)(x^2+4a^2)=8a^3$$

or

$$x^2y+4a^2y+2ax^2=0.$$

Equating to zero the lowest degree terms in it, we get the tangent at the new origin as $4a^2y=0$ *i.e.*, $y=0$. Thus the new x-axis is tangent at the new origin.

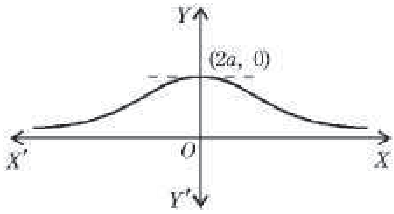
(iv) Asymptote parallel to x-axis is $y=0$.

Asymptotes parallel to y-axis are given by $x^2+4a^2=0$ or $x^2=-4a^2$ *i.e.*, imaginary asymptotes.

(v) For all values of x, y is positive *i.e.*, the curve lies in I and II quadrants. Also y is greatest when x^2 is 0. Thus the greatest value of y is $2a$.

(vi) Special points :

x	0	a	$2a$	$\to \infty$
y	$2a$	$8a/5$	a	$\to 0.$



First trace the curve on the right hand side of the y-axis and then by symmetry on its left hand side. The curve is as shown in the figure.

Problem 9: *Trace the curve* $y^2(1-x^2)=x^2(1+x^2)$.

(Agra 2001; Meerut 07B; Bundelkhand 07, 10)

Solution: (i) The given curve is symmetrical about both the axes.

(ii) The curve passes through the origin. The tangents at origin are

$$y^2 - x^2 = 0 \quad i.e., \quad y = \pm x.$$

Since there are two real and distinct tangents at the origin, the origin is a node on the curve.

(iii) The curve cuts the x-axis where $y=0$. Putting $y=0$ in the equation of the curve we get $x^2(1+x^2)=0$. The only real value of x satisfying this equation is $x=0$. So the curve cuts the x-axis only at the origin. Similarly we observe that the curve cuts the y-axis only at the origin.

(iv) Solving the equation of the curve for y, we get $y^2 = \dfrac{x^2(1+x^2)}{1-x^2}$.

When $x=0$, $y^2=0$ *i.e.*, $y=0$. When $x \to 1$, $y^2 \to \infty$.

Therefore $x=1$ is an asymptote of the curve. When $0<x<1$, y^2 is positive and so y is real.

Therefore the curve exists in this region.

When $x>1$, y^2 is negative and so y is imaginary. Therefore the curve does not exist in the region where $x>1$.

We need not consider the negative values of x as the curve is symmetrical about the y-axis and so it can be drawn by symmetry in the region where $x<0$.

(v) The asymptotes of the curve parallel to y-axis are given by $1-x^2=0$. Thus $x=\pm 1$ are two asymptotes of the curve.

To find the other asymptotes, if there are any, the equation of the curve can be written as

$$y^2x^2 + x^4 + x^2 - y^2 = 0.$$

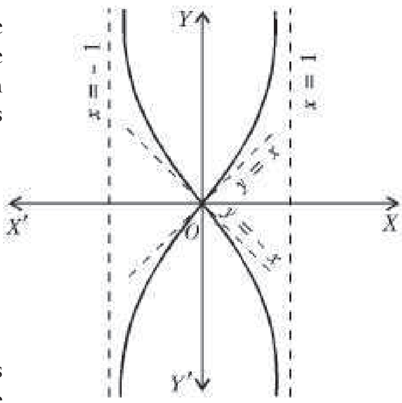
Putting $y = m$ and $x = 1$ in the highest degree *i.e.*, 4 degree terms of the equation of the curve, we get $\phi_4(m) = m^2 + 1$. The equation $\phi_4(m) = 0$ *i.e.*, $m^2 + 1 = 0$ gives no real values of m. So the curve has no other asymptotes.

(vi) In the positive quadrant, we have

$$y^2 = x^2 \frac{1 + x^2}{1 - x^2}, 0 < x < 1 \quad \text{or}$$

$$y = x \sqrt{\left(\frac{1 + x^2}{1 - x^2}\right)}.$$

When $0 < x < 1$, $y > x$. Therefore the curve lies above the line $y = x$ which is tangent at the origin.



Combining all these facts the shape of the curve is as shown in the figure.

Problem 10(i): *Trace the curve* $a^2 y^2 = x^2 (a^2 - x^2)$. **(Kumaun 2002; Garhwal 03)**

Solution: (i) The curve is symmetrical about both the axes.

(ii) The curve passes through the origin.

Equating to zero the lowest degree terms in the given equation, tangents at the origin are given by $a^2 y^2 - a^2 x^2 = 0$ or $y = \pm x$. These being real and distinct, origin is a node.

(iii) When $y = 0$, $x = 0, \pm a$ *i.e.*, the curve crosses the x-axis at $(0, 0)$, $(a, 0)$ and $(-a, 0)$. When $x = 0$, $y = 0$ and so the curve meets the y-axis only at the origin. Shifting the origin to $(a, 0)$, the equation of the curve becomes

$$a^2 y^2 = (x + a)^2 [a^2 - (x + a)^2]$$

or

$$a^2 y^2 = (x + a)^2 [-2ax - x^2].$$

$\therefore$ The tangent at the new origin $(a, 0)$ is $x = 0$ *i.e.*, the new y-axis.

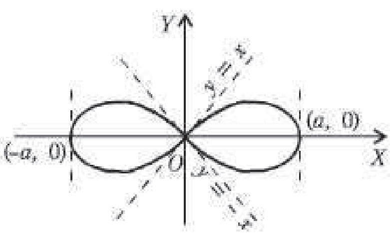
(iv) Solving the equation of the curve for y, we have

$$y^2 = \{x^2 (a^2 - x^2)\}/a^2.$$

When $x = 0$, $y = 0$ and when $x = a$, $y = 0$. When $0 < x < a$, y is real and so the curve exists in this region. When $x > a$, y^2 is negative or y is imaginary and so the curve does not exist in the region where $x > a$.

(v) No asymptotes.

Thus the shape of the curve is as shown in the figure.



Problem 10(ii): *Trace the curve* $ay^2 = x^2 (a - x)$.

Solution: We have $x^3 + a(y^2 - x^2) = 0$. ...(1)

(i) The curve is symmetrical about x-axis.

(ii) The curve passes through $(0, 0)$.

Tangents at the origin are $y^2 - x^2 = 0$ or $y = \pm x$.

(iii) When $x = 0$, $y = 0$; when $y = 0$, $x = 0, a$.

Hence the curve cuts the coordinates axes at the points $(0, 0), (a, 0)$.

Differentiating (1), $2ay (dy/dx) = 2ax - 3x^2$.

At $(a, 0)$, $dy/dx = -\infty$;

$\therefore$ tangent at $(a, 0)$ is parallel to y-axis.

(iv) No asymptotes.

(v) From (1), $y = \pm x \sqrt{\{1 - (x/a)\}}$.

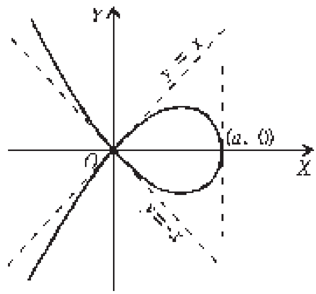
At $x = 0$, $y = 0$.

When $0 < x < a$, y is real and is numerically less than x. Hence the curve exists in the region $0 < x < a$ and in the first quadrant it lies below the line $y = x$.

When $x > a$, y is imaginary and therefore the curve does not lie on the right hand side of the line $x = a$.

When $x < 0$, y is real and is numerically greater than x. Hence the curve exists when $x < 0$ and in the third quadrant it lies below the line $y = x$.

When $x \to -\infty$, $y \to \pm\infty$.



First trace the curve in the Ist and IIIrd quadrants and then by symmetry about x-axis in IInd and IVth quadrants. The shape is as shown in the figure.

Problem 11: *Trace the curve* $a^2 y^2 = x^3 (2a - x)$.

Solution: (i) The curve is symmetrical about x-axis.

(ii) The curve passes through the origin and tangents at the origin are $y^2 = 0$ *i.e.*, $y = 0$, $y = 0$. So a cusp is expected at the origin.

(iii) When $y = 0$, $x = 0, 2a$ *i.e.*, the curve crosses the x-axis at $(0, 0)$ and $(2a, 0)$. When $x = 0$, $y = 0$ and so the curve meets the y-axis only at the origin. Shifting the origin to $(2a, 0)$, the equation of the curve transforms to

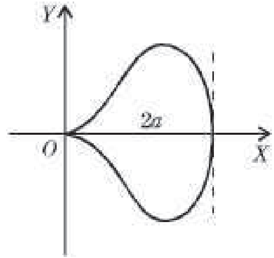
$$a^2 y^2 = (x + 2a)^3 (2a - x - 2a) = -x(x + 2a)^3.$$

$\therefore$ Tangent at the new origin $(2a, 0)$ is $x = 0$ *i.e.*, the new y-axis.

(iv) No asymptotes.

(v) Solving for y, the equation of curve is

$$y^2 = x^3 (2a - x)/a^2.$$



When $x = 0, y = 0$ and when $x = 2a, y = 0$. When $0 < x < 2a$, y^2 is +ive *i.e.*, y is real and so the curve exists in this region.

When $x > 2a$, y^2 is negative *i.e.*, y is imaginary and so the curve does not exist for $x > 2a$.

When $x < 0$, y^2 is negative *i.e.*, y is imaginary and so the curve does not exist in the region where x is –ive.

Thus the curve exists only for values of x from $x = 0$ to $x = 2a$.

Thus the shape of the curve is as shown in the figure.

Problem 12: *Trace the curve* $y^2 (a^2 + x^2) = x^2 (a^2 - x^2)$

(Agra 2000, 01; Meerut 01, 03, 12; Kumaun 08, 14; Garhwal 10)

or $$x^2 (x^2 + y^2) = a^2 (x^2 - y^2). \quad ...(1)$$

Solution: (i) The curve is symmetrical about both the axes.

(ii) The curve passes through the origin. The tangents at the origin are $y^2 = x^2$ or $y = \pm x$. These being real and distinct the origin is a node.

(iii) When $y = 0, x = 0, \pm a$ *i.e.*, the curve crosses the x-axis at $(0, 0), (a, 0)$ and $(-a, 0)$. When $x = 0$, $y = 0$ and so the curve cuts the y-axis only at the origin.

Shifting the origin to $(a, 0)$, the equation of the curve transforms to

$$y^2 \{a^2 + (x + a)^2\} = (x + a)^2 \{a^2 - (x + a)^2\}$$

or $$y^2 (x^2 + 2ax + 2a^2) = (x + a)^2 (-x^2 - 2ax). \quad ...(2)$$

Equating to zero, the lowest degree terms in (2), the tangent at the new origin is given by $x = 0$ *i.e.*, the new y-axis.

(iv) Solving for y, we get $y^2 = x^2 (a^2 - x^2)/(a^2 + x^2)$.

When $x = 0, y = 0$ and when $x = a, y = 0$. When $0 < x < a$, y is real and so the curve exists in this region.

When $x > a$, y^2 is negative *i.e.*, y is imaginary and so the curve does not exist for $x > a$.

(v) No asymptotes.

Combining all these facts, we see that the shape of the curve is as shown in the figure of Problem 10 (i).

Problem 13: *Trace the curve* $y^2 x = a^2 (x - a)$. ...(1)

Solution: (i) The curve is symmetrical about x-axis.

(ii) The curve does not pass through the origin.

(iii) When $y = 0, x = a$ *i.e.*, the curve crosses the x-axis at $(a, 0)$.

When $x = 0$, we do not get any value of y and so the curve does not meet the y-axis.

Shifting the origin to $(a, 0)$ the equation of the curve transforms to

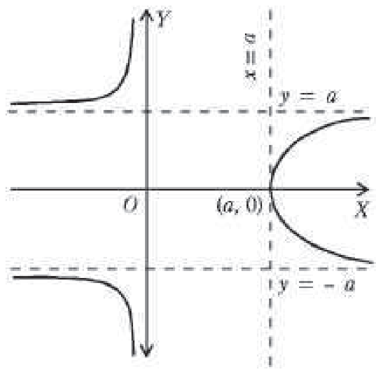
$$y^2 (x + a) = a^2 x. \qquad ...(2)$$

Equating to zero the lowest degree terms in (2), the tangent at the new origin is $x = 0$ *i.e.*, the new y-axis.

(iv) Equating to zero, the coefficients of highest powers of x and y in (1), we get $y = \pm a$ and $x = 0$ as the asymptotes parallel to the coordinate axes. Also these are the only asymptotes of the curve.

(v) From (1), solving for y, we get $y^2 = a^2 (x - a) / x$.

When $x = a$, $y^2 = 0$ and when $0 < x < a$, y^2 is – ive *i.e.*, y is imaginary and so the curve does not exist in this region. When $x > a$, y^2 is +ive *i.e.*, y is real and so the curve exists in this region and when $x \to \infty$, $y^2 \to a^2$ showing that the lines $y = \pm a$ are asymptotes of the curve. Also when $x > a$, we observe that $y^2 < a^2$. When $x \to 0$ (from the left), $y^2 \to \infty$ showing that the line $x = 0$ (on its left side) is asymptote of the curve. When $x < 0$, y^2 is +ive *i.e.*, y is real and so the curve exists in this region and when $x \to -\infty$, $y^2 \to a^2$ showing that the lines $y = \pm a$ are asymptotes of the curve. Also when $x < 0$, we observe that $y^2 > a^2$.



Combining all these facts we see that the shape of the curve is as shown in the figure.

Problem 14: *Trace the curve* $9ay^2 = x(x - 3a)^2$.

Solution: We have

$$9ay^2 = x(x - 3a)^2 = x(x^2 - 6ax + 9a^2). \qquad ...(1)$$

(i) The curve is symmetrical about x-axis.

(ii) The curve passes through $(0, 0)$. Tangent at origin is $x = 0$ *i.e.*, the y-axis.

(iii) When $y = 0$, $x = 0$ and $3a$; when $x = 0$, $y = 0$.

Hence the curve cuts the coordinate axes at the points $(0, 0)$ and $(3a, 0)$.

Transferring the origin to $(3a, 0)$ the equation of the curve becomes

$$9ay^2 = (x + 3a)(x + 3a - 3a)^2,$$

(putting $x + 3a$ for x and $y + 0$ for y)

or $$9ay^2 = (x + 3a) x^2.$$

Tangents at the new origin are $9ay^2 = 3ax^2$ or $y = \pm (1/\sqrt{3}) x$.

Hence there is a node at the new origin *i.e.*, at the point $(3a, 0)$ on (1) and the two branches of the curve cross at this point.

Tangents at $(3a, 0)$ are parallel to the lines $y = \pm\, x/\sqrt{3}$ each of which is inclined at an angle of 30° to the x-axis.

Again, the curve is

$$y = (x - 3a)\, x^{1/2} / (3a^{1/2}).$$

$$\therefore \qquad dy/dx = \{1/(3a^{1/2})\}\,\{x^{1/2} + (x - 3a)\,(\tfrac{1}{2}\, x^{-1/2})\}$$

$$= (3x - 3a)/(6x^{1/2}\, a^{1/2}).$$

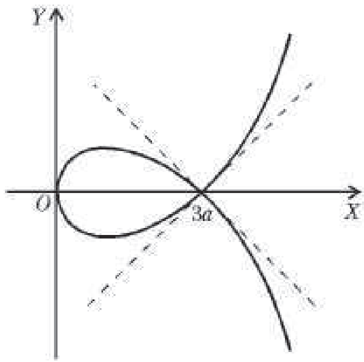
Therefore at $x = a$, $(dy/dx) = 0$ *i.e.*, the tangent to the curve is parallel to the x-axis.

(iv) No asymptotes.

(v) Solving the equation of the curve for y, we have $y^2 = \{x\,(x - 3a)^2\}/9a$.

When $x = 0$, $y^2 = 0$ and when $x = 3a$, $y^2 = 0$. When $0 < x < 3a$, y^2 is +ive *i.e.*, y is real and so the curve exists in this region. When $x > 3a$, y^2 is +ive *i.e.*, y is real and so the curve exists in this region and when $x \to \infty$, $y^2 \to \infty$.

When $x < 0$, y^2 is –ive *i.e.*, y is imaginary and so the curve does not exist in the region $x < 0$. Thus the curve does not lie on the left hand side of the y-axis.



(vi) Special points :

x	0	a	$3a$	$4a$	$9a$	$\to \infty$
y	0	$\pm \frac{2}{3} a$	0	$\pm \frac{2}{3} a$	$\pm\, 6a$	$\to \pm\, \infty$

The curve is as shown in the figure.

Problem 15: *Trace the curve* $y^2 (x + a) = (x - a)^3$. **(Meerut 2004, 06B)**

Solution: (i) The curve is symmetrical about x-axis.

(ii) The curve does not pass through $(0, 0)$.

(iii) When $x = 0$, y is imaginary, and when $x = a$, $y = 0$.
The point $(a, 0)$ is a single cusp of the first kind.

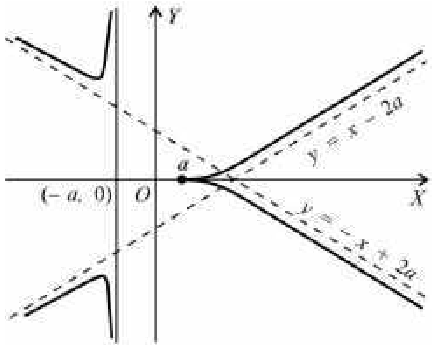
(iv) Asymptotes parallel to the y-axis is $x + a = 0$.
The other asymptotes are

$$y = x - 2a, \; y = -\,x + 2a.$$

(v) When x lies between $(-a)$ and a, y is imaginary,
but for all other values of x, y is real.
(vi) Special points.

x	$-\infty$	$-a$	a	$3a$	∞
y	$-\infty$	∞	0	$a\sqrt{2}$	∞

The curve lies above the line $y = x - 2a$ in I quadrant and below this line in III quadrant, the approximate shape of the curve is as shown in the figure .



Problem 16: *Trace the curve* $x^2 y^2 = (1 + y)^2 (4 - y^2)$. ...(1)

(Agra 2004)

Solution: (i) The curve is symmetrical about y-axis.

(ii) The curve does not pass through $(0, 0)$.

(iii) When $x = 0$, $y = -1, \pm 2$; when $y = 0$ we get no value of x.

Hence the curve cuts the coordinate axes at the points $(0, -1)$, $(0, \pm 2)$.

Shifting the origin to the point $(0, -1)$ the equation of the curve becomes

$$x^2 (y - 1)^2 = \{1 + (y - 1)\}^2 \{4 - (y - 1)^2\}$$

or $$x^2 (y^2 - 2y + 1) = y^2 (3 + 2y - y^2).$$

Therefore the tangents at this new origin are $x^2 = 3y^2$ *i.e.*, $y = \pm (1/\sqrt{3}) x$. Thus the given curve has a node at the point $(0, -1)$ and the tangents to the two branches of the curve at this point make angles $\pm \frac{1}{6} \pi$ with x-axis.

Now shifting the origin to the point $(0, 2)$ the given equation of the curve becomes

$$x^2 (y + 2)^2 = \{1 + (y + 2)\}^2 \{4 - (y + 2)^2\}$$

or $$x^2 (y^2 + 4y + 4) = (3 + y)^2 (-y^2 - 4y).$$

Therefore the tangent at this new origin is $y = 0$. Thus at the point $(0, 2)$ on the given curve the tangent is parallel to x-axis.

Again shifting the origin to the point $(0, -2)$ the given equation of the curve becomes

$$x^2 (y - 2)^2 = \{1 + (y - 2)\}^2 \{4 - (y - 2)^2\}$$

or $$x^2 (y - 2)^2 = (y - 1)^2 (-y^2 + 4y).$$

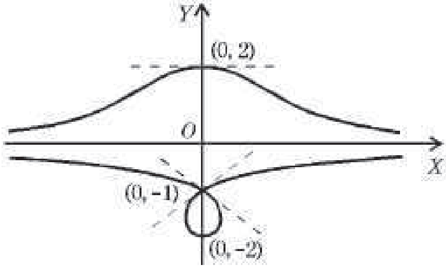
Therefore the tangent at this new origin is $y = 0$. Thus at the point $(0, -2)$ on the given curve the tangent is parallel to x-axis.

(iv) Asymptotes parallel to x-axis are given by $y^2 = 0$ *i.e.*, x-axis is an asymptote and we note that the curve has no other asymptotes.

(v) Solving the equation of the curve for x, we get $x^2 = \{(1+y)^2 (4-y^2)\}/y^2$. When $y \to 0$, $x^2 \to \infty$. When $y=2$, $x=0$. When $0<y<2$, x is real and so the curve exists in this region. When $y>2$, x is imaginary and so the curve does not exist in this region.

When $y=-1$, $x=0$ and when $y=-2$ again $x=0$. When $-2<y<0$, x is real and so the curve exists in this region. But when $y<-2$, x is imaginary and so the curve does not exist in this region.

Tracing the curve from the above data the curve is as shown in the figure.



Problem 17: *Trace the curve* $y^2 (x+3a) = x(x-a)(x-2a)$. **(Meerut 2005B)**

Solution: The given curve is $y^2 (x+3a) = x(x-a)(x-2a)$. ...(1)

(i) The curve is symmetrical about the x-axis.

(ii) The curve passes through the origin and the tangent at the origin is $x=0$ *i.e.*, the y-axis.

(iii) When $y=0$, $x=0, a, 2a$; when $x=0$, $y=0$. Therefore the curve meets the x-axis at the points $(0,0)$, $(a,0)$, $(2a,0)$ and it meets the y-axis only at the origin.

Shifting the origin to the point $(a,0)$, the equation of the curve becomes

$$y^2 (x+4a) = (x+a)\,x\,(x-a) \quad i.e., \quad y^2 (x+4a) = x(x^2-a^2).$$

Therefore the tangent at the new origin is $x=0$ *i.e.*, the new y-axis.

Again shifting the origin to the point $(2a,0)$ the equation of the curve becomes

$$y^2 (x+5a) = (x+2a)(x+a)\,x.$$

Therefore the tangent at this new origin is $x=0$ *i.e.*, the new y-axis.

(iv) Solving the equation of the curve for y, we have

$$y^2 = \frac{x(x-a)(x-2a)}{x+3a}.$$

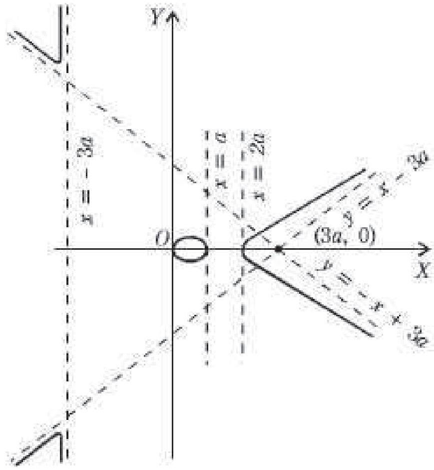
When $x=0$, $y^2=0$ and when $x=a$, $y^2=0$. When $0<x<a$, y^2 is +ive *i.e.*, y is real and so the curve exists in this region and it has a loop between the lines $x=0$ and $x=a$. When $x=2a$, $y^2=0$ and when $a<x<2a$, y^2 is –ive *i.e.*, y is imaginary and so the curve does not exist in this region.

When $x>2a$, y^2 is +ive *i.e.*, y is real and so the curve exists in this region and when $x\to\infty$, $y^2\to\infty$.

When $x\to -3a$ (from the left), $y^2\to\infty$ and so on its left side the line $x=-3a$ is an asymptote of the curve.

When $-3a < x < 0$, y^2 is –ive *i.e.*, y is imaginary and so the curve does not exist in this region.
When $x < -3a$, y^2 is +ive *i.e.*, y is real and so the curve exists in this region and when $x \to -\infty$, $y^2 \to \infty$.

(v) Asymptotes. The curve has an asymptote parallel to x-axis *i.e.*, the line $x + 3a = 0$. Putting $y = m$ and $x = 1$ in the third degree terms in the equation of the curve, we get $\phi_3(m) = m^2 - 1$. The equation $\phi_3(m) = 0$ gives $m^2 - 1 = 0$ *i.e.*, $m = \pm 1$. Also $\phi_2(m) = 3am^2 + 3a$.



Now c is given by

$$c\,\phi'_3(m) + \phi_2(m) = 0$$

i.e., $$c\,(2m) + 3am^2 + 3a = 0.$$

When $m = 1$, $c = -3a$ and when

$$m = -1, c = 3a.$$

Hence the oblique asymptotes of the curve are $y = x - 3a$ and $y = -x + 3a$.
Both these lines pass through the point $(3a, 0)$ and they make angles $\pm 45^\circ$ with the x-axis.
Combining all these facts the shape of the curve is as shown in the above figure.

Problem 18: *Trace the curve* $a^3 y^2 = (x - a)^4 (x - b), a > b.$

Solution: **(i).** The curve is symmetrical about x-axis.
(ii) The curve does not pass through the origin.
(iii) When $y = 0, x = a, b$; when $x = 0$, y is imaginary.
Hence the curve cuts the coordinate axes at the points $(a, 0)$ and $(b, 0)$.
Shifting the origin to the point $(a, 0)$ the equation of the curve becomes

$$a^3 y^2 = x^4 (x + a - b).$$

Therefore the tangents at this new origin are

$$a^3 y^2 = 0 \text{ i.e., } y = 0,\ y = 0.$$

Thus the new x-axis is tangent at this new origin and this new origin may be a cusp.
Again shifting the origin to the point $(b, 0)$ the equation of the curve becomes

$$a^3 y^2 = (x + b - a)^4\, x.$$

Therefore the tangent at this new origin is $x = 0$. Thus the new y-axis is tangent at this new origin.
(iv) The curve has no asymptotes.

(v) Solving the equation of the curve for y, we have

$$y^2 = \frac{(x-a)^4 (x-b)}{a^3}.$$

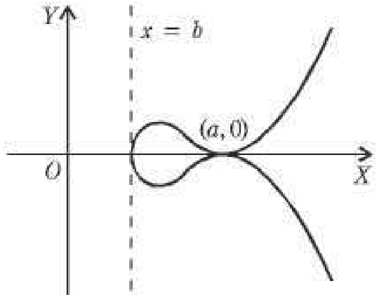
When $x = 0$, y is imaginary and when $x = b$, $y = 0$.

When $0 < x < b$, y^2 is –ive *i.e.*, y is imaginary and so the curve does not exist in this region.

When $x = a$, $y = 0$ and when $b < x < a$, y is real and so the curve exists in this region. When $x > a$, y is real and so the curve exists in this region and when $x \to \infty$, $y^2 \to \infty$.

When $x < 0$, y^2 is –ive *i.e.*, y is imaginary and so the curve does not exist in this region.

Hence the curve is as shown in the figure.



Problem 19: *Trace the curve* $y^2 (x^2 - 1) = x$. ...(1)

Solution: (i) The curve is symmetrical about the x-axis.

(ii) The curve passes through the origin. Equating to zero the lowest degree terms in (1), the equation of tangent at the origin is $x = 0$ *i.e.*, y-axis.

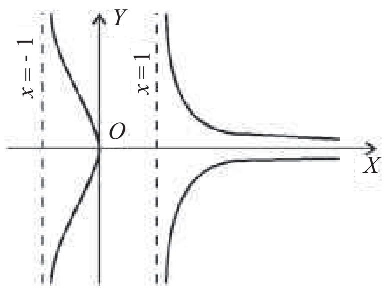
(iii) The curve meets the coordinate axes only at $(0, 0)$.

(iv) Asymptotes parallel to coordinate axes are obtained by equating to zero the coefficients of the highest powers of x and y in (1). Thus $y = 0$, $y = 0$ and $x = \pm 1$ are the asymptotes of the given curve and these are the only asymptotes of the curve because the curve is of degree 4 and so it cannot have more than four asymptotes.

(v) Solving the equation of the curve for y, we have

$$y^2 = x/(x^2 - 1)$$
$$= x/\{(x-1)(x+1)\}.$$

When $0 < x < 1$, y^2 is –ive *i.e.*, y is imaginary and so the curve does not exist in this region. When $x > 1$, y^2 is +ive *i.e.*, y is real and so the curve exists in this region and when $x \to \infty$, $y^2 \to 0$ showing that the line $y = 0$ is an asymptote of the curve. When $x \to 1$ (from the right) $y^2 \to \infty$ showing that the line $x = 1$ (on its right side) is an asymptote of the curve. When $x \to -1$ (but from the right), $y^2 \to \infty$ showing that the line $x = -1$ is an asymptote of the curve.



When $-1 < x < 0$, y^2 is +ive *i.e.*, y is real and so the curve exists in this region. When $x < -1$, y^2 is –ive *i.e.*, y is imaginary and so the curve does not exist in this region.

Combining all these facts we see that the shape of the curve is as shown in the figure.

Problem 20: *Trace the curve* $x(x-2a)y^2 = a^2(x-a)(x-3a)$.

Solution: (i) The curve is symmetrical about the axis of x.

(ii) The curve does not pass through the origin.

(iii) When $y=0$, $x=a, 3a$; when $x=0$, we do not get any value of y.

Hence the curve cuts the coordinate axes at the points $(a,0),(3a,0)$.

Transferring the origin to the points $(a,0),(3a,0)$ respectively, we find that the tangents at these points are parallel to the y-axis.

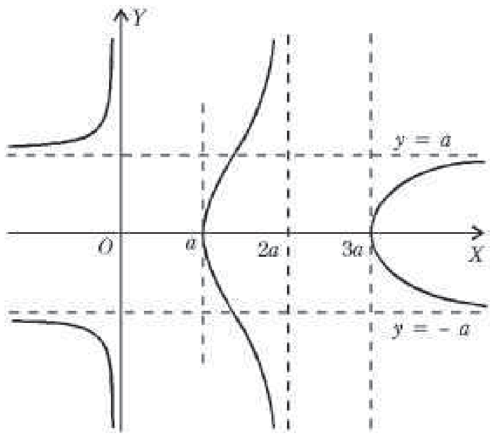
(iv) Asymptotes parallel to x-axis : $y^2 - a^2 = 0$ or $y = \pm a$.

Asymptotes parallel to y-axis : $x(x-2a)=0$ or $x=0, x=2a$.

(v) Solving the equation of the curve for y, we have

$$y^2 = \frac{a^2(x-a)(x-3a)}{x(x-2a)}.$$

When $0<x<a$, y^2 is –ive *i.e.*, y is imaginary and so the curve does not exist in this region. When $x=a$, $y^2=0$.



When $a<x<2a$, y^2 is +ive *i.e.*, y is real and so the curve exists in this region. When $x \to 2a$ (from the left), $y^2 \to \infty$ showing that on its left side the line $x=2a$ is an asymptote of the curve. When $x=3a$, $y^2=0$ and when $2a<x<3a$, y^2 is –ive and so the curve does not exist in this region. When $x>3a$, y^2 is +ive and so the curve exists in this region. When $x\to\infty$, $y^2\to a^2$ showing that the lines $y=\pm a$ are asymptotes of the curve. Also when $x>3a$, we observe that $y^2<a^2$. When $x<0$, y^2 is +ive *i.e.*, y is real and so the curve exists in this region. When $x\to 0$ (from the left), $y^2\to\infty$ and so on its left side the line $x=0$ is an asymptote of the curve. When $x\to-\infty$, $y^2\to a^2$ showing that the lines $y=\pm a$ are asymptotes of the curve. Also when $x<0$, we observe that $y^2>a^2$. Combining all these facts the shape of the curve is as drawn in the figure.

Problem 21: *Trace the curve* $y^2=(x-a)(x-b)(x-c)$, *where* $a>b>c$. ...(1)

Solution: (i) The curve is symmetrical about x-axis.

(ii) The curve does not pass through the origin.

(iii) From the equation of the curve when $y=0$, we get $x=a, b, c$ *i.e.*, the curve crosses the x-axis at $(a, 0)$, $(b, 0)$ and $(c, 0)$. Again putting $x=0$ in the equation of the curve we get $y^2 = -abc$ which gives imaginary values of y and so the curve does not cut the y-axis.

Shifting the origin to $(a, 0)$ the equation of the curve transforms to

$$y^2 = x(x+a-b)(x+a-c). \quad ...(2)$$

Equating to zero, the lowest degree terms in (2) we find that the tangent at the new origin is $x=0$ *i.e.*, the new y-axis. Thus on the given curve the tangent at $(a, 0)$ is parallel to the y-axis.

Similarly the tangents at $(b, 0)$ and $(c, 0)$ are the lines through them parallel to the y-axis.

(iv) The curve has no asymptotes.

(v) Solving the equation of the curve for y, we have

$$y^2 = (x-a)(x-b)(x-c).$$

When $x<c$, y^2 is –ive, *i.e.*, y is imaginary and so the curve does not exist in the region $x<c$ *i.e.*, to the left of the line $x=c$.

When $x=c$, $y=0$.

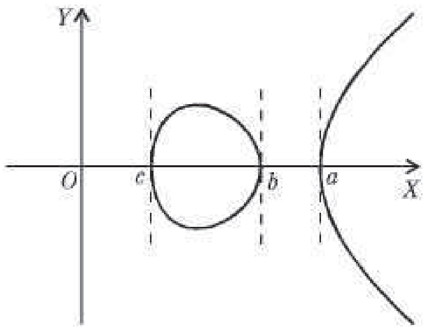
When $c<x<b$ (where $b>c$), y^2 is positive *i.e.*, y is real and so the curve exists in this region.

When $x=b$, $y=0$.

This implies that there is a loop between $x=c$ and $x=b$.

When $b<x<a$ (where $a>b$), y^2 is negative *i.e.*, y is imaginary and so the curve does not exist between the lines $x=b$ and $x=a$.

At $x=a$, $y=0$. When $x>a$, y^2 is +ive and so the curve exists in the region $x>a$. Also as $x \to \infty$, $y^2 \to \infty$.



(vi) As $x \to \infty$, $(dy/dx) \to \infty$

i.e., the curve ultimately becomes parallel to y-axis.

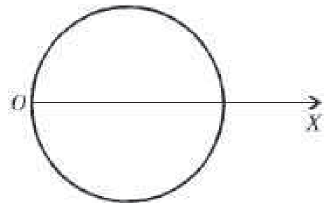
Combining all these facts we see that the shape of the curve is as shown in the figure.

Comprehensive Problems 4

Problem 1: *Trace the curve $r = 2a\cos\theta$.* ***(Circle)***

Solution: We have $r = 2a\cos\theta$. ...(1)

Multiplying both sides by r, we have $r^2 = 2ar\cos\theta$.



Changing to cartesians, we get

$$x^2 + y^2 = 2ax.$$

$$[\because r^2 = x^2 + y^2 \text{ and } x = r \cos\theta]$$

This is the equation of a circle with centre $(a, 0)$ and the radius a. Thus $r = 2a \cos\theta$ is the equation of a circle passing through the pole and the diameter through the pole as initial line.

Problem 2: *Trace the curve $r = a(1 - \cos\theta)$.* ***(Cardioid)***

(Meerut 2001; Lucknow 09)

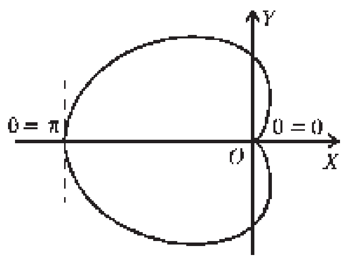
Solution: (i) The curve is symmetrical about the initial line.

(ii) We have $r = 0$ when $1 - \cos\theta = 0$ or $\cos\theta = 1$ or $\theta = 0$. Therefore the line $\theta = 0$ is tangent to the curve at the pole.

(iii) We have $dr/d\theta = a \sin\theta$.

Therefore $\cot\phi = \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{a \sin\theta}{a(1 - \cos\theta)} = \cot\dfrac{1}{2}\theta.$

Now $\phi = 90^\circ$ when $\frac{1}{2}\theta = \frac{1}{2}\pi$ *i.e.*, $\theta = \pi$. Thus at the point $\theta = \pi$ the tangent to the curve is perpendicular to the radius vector.



(iv) Table showing corresponding values of θ and r :

$\theta = 0$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	$\frac{2}{3}\pi$	π
$r = 0$	$\frac{1}{2}a$	a	$\frac{3}{2}a$	$2a$

Thus as θ increases from 0 to π, r also increases from 0 to $2a$.

Hence the curve is as shown in the figure.

Problem 3: *Trace the curve $r = a + b\cos\theta$, when $a > b$.* ***(Limacon)***

(Meerut 2000; Agra 01)

Solution: (i) The curve is symmetrical about the initial line.

(ii) We have $r = 0$ when $a + b\cos\theta = 0$

i.e., $\theta = \cos^{-1}(-a/b)$.

Since $a > b$, therefore $(a/b) > 1$ and so $\cos^{-1}(-a/b)$ gives no real values of θ. Thus in the given curve, r cannot be equal to zero.

r is maximum when $\cos\theta = 1$ *i.e.*, $\theta = 0$.

Then $r = a + b$.

Also r is minimum when $\cos\theta = -1$ *i.e.*, $\theta = \pi$. Then $r = a - b$, which is positive because $a > b$.

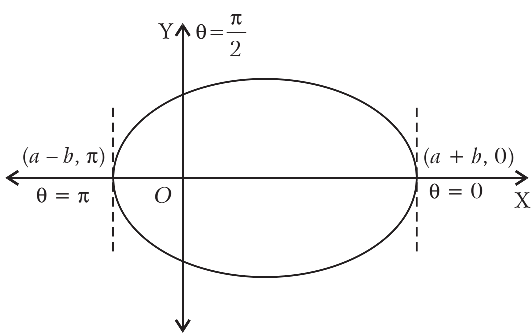
(iii) $\dfrac{dr}{d\theta} = -b\sin\theta$. When $0 < \theta < \pi$, $dr/d\theta$ is throughout negative. Therefore r decreases continuously as θ increases from 0 to π.

Also $$\tan\phi = r\frac{d\theta}{dr} = -\frac{a + b\cos\theta}{\sin\theta}.$$

We have $\phi = 90^\circ$ when $\theta = 0$ and π. Therefore at the points $\theta = 0$ and $\theta = \pi$, the tangent to the curve is perpendicular to the radius vector.

(iv) The following table gives the corresponding values of r and θ.

θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
r	$a + b$	$a + \frac{1}{2}b$	a	$a - \frac{1}{2}b$	$a - b$



The variation of θ from π to 2π need not be considered because of the symmetry about the initial line. Hence the curve is as shown in the figure.

Problem 4: *Trace the curve $r^2 = a^2\cos 2\theta$.* ***(Lemniscate of Bernouli)***

(Meerut 2002, 08; Lucknow 05, 11; Agra 07; Kumaun 13)

Solution: (i) The curve is symmetrical about the initial line and it is also symmetrical about the pole.

(ii) When $r = 0$, $\cos 2\theta = 0$ or $2\theta = \pm\frac{1}{2}\pi$

i.e., $\theta = \pm\frac{1}{4}\pi$.

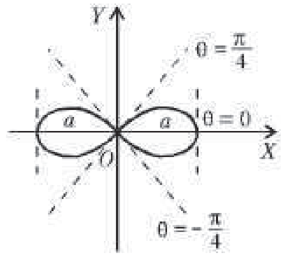
Therefore the lines $\theta = \pm\frac{1}{4}\pi$ are the tangents to the curve at the pole.

When $\theta = 0$, $r = a$. Also the greatest value of the radius vector of this curve is a.

(iii) Table showing variation of r as θ varies from 0 to π :

$\theta = 0$	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{4}\pi < \theta < \frac{3}{4}\pi$	$\frac{3}{4}\pi$	$\frac{5}{6}\pi$	π
$r^2 = a^2$	$\frac{1}{2}a^2$	0	–ive	0	$\frac{1}{2}a^2$	a^2
$r = \pm a$	$\pm a/\sqrt{2}$	0	imaginary	0	$\pm a/\sqrt{2}$	$\pm a.$

Hence the shape of the curve is as shown in the figure.



Problem 5: *Trace the curve* $r^2 = a^2 \sin 2\theta$. ***(Lemniscate)***

Solution: (i) The given curve is not symmetrical about the initial line, but it is symmetrical about the pole.

(ii) We have $r = 0$ when $\sin 2\theta = 0$ *i.e.*, $2\theta = 0, \pi$ *i.e.*, $\theta = 0, \pi/2$.

Thus two consecutive values of θ for which r is zero are 0 and $\pi/2$. Therefore one loop of the curve lies between the lines $\theta = 0$ and $\theta = \pi/2$ and both these lines are tangents to the curve at the pole.

(iii) r^2 is maximum when $\sin 2\theta = 1$ *i.e.*, $2\theta = \pi/2$ *i.e.*, $\theta = \pi/4$.

When $\theta = \pi/4, r^2 = a^2$ or $r = \pm a$.

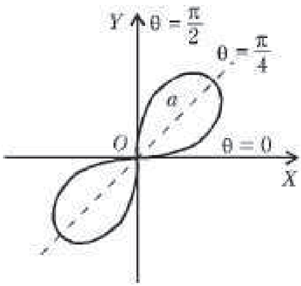
Thus when $0 < \theta < \pi/2$, the greatest value of the radius vector of this curve is a and it occurs at $\theta = \pi/4$.

(iv) We have $2r\dfrac{dr}{d\theta} = 2a^2 \cos 2\theta$; so that

$$\frac{dr}{d\theta} = \frac{a^2 \cos 2\theta}{r}.$$

$$\therefore \qquad \cot\phi = \frac{1}{r}\frac{dr}{d\theta} = \frac{a^2 \cos 2\theta}{r^2}$$

$$= \frac{a^2 \cos 2\theta}{a^2 \sin 2\theta} = \cot 2\theta.$$



When $\theta = \pi/4$, $\cot\phi = \cot\frac{1}{2}\pi = 0$ so that $\phi = 90°$.

Thus at the point $\theta = \pi/4$ the tangent to the curve is perpendicular to the radius vector.

(v) When $\pi < 2\theta < 2\pi$ *i.e.*, $\frac{1}{2}\pi < \theta < \pi$, $\sin 2\theta$ is < 0.

Thus when $\frac{1}{2}\pi < \theta < \pi$, r^2 is –ive *i.e.*, r is imaginary and so the given curve does not exist in the region $\frac{1}{2}\pi < \theta < \pi$.

Taking into consideration all the above facts the shape of the curve is as shown in the figure.

Problem 6: *Trace the curve $r = a \sin 2\theta$.* ***(Four leaved rose)*** **(Bundelkhand 2009)**

Solution: (i) The given curve is neither symmetrical about the initial line nor it is symmetrical about the pole.

(ii) We have $r = 0$ when $\sin 2\theta = 0$

i.e., $2\theta = 0, \pi, 2\pi, 3\pi, 4\pi$ etc.

i.e., $\theta = 0, \pi/2, \pi, 3\pi/2, 2\pi$ etc.

Thus two consecutive values of θ for which r is zero are 0 and $\pi/2$. Therefore one loop of the curve lies between the lines $\theta = 0$ and $\theta = \pi/2$ and both these lines are tangents to the curve at the pole.

(iii) r is maximum when $\sin 2\theta = 1$. Therefore when $0 < \theta < \pi/2$, r is maximum when $2\theta = \pi/2$ *i.e.*, $\theta = \pi/4$. Thus when $0 < \theta < \pi/2$, the greatest value of the radius vector of this curve is a and it occurs at $\theta = \pi/4$.

(iv) We have $\frac{dr}{d\theta} = 2a \cos 2\theta$.

$$\therefore \quad \cot\phi = \frac{1}{r}\frac{dr}{d\theta} = \frac{2a\cos 2\theta}{a \sin 2\theta} = 2\cot 2\theta.$$

When $\theta = \pi/4$, $\cot\phi = 2\cot\frac{1}{2}\pi = 0$ so that $\phi = 90^\circ$.

Thus at the point $\theta = \pi/4$ the tangent to the curve is perpendicular to the radius vector.

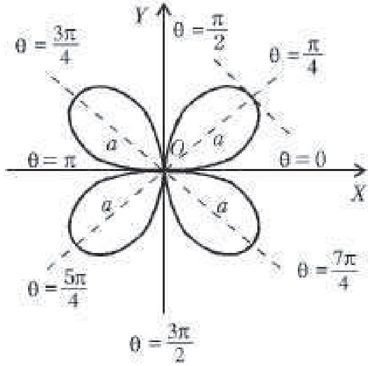
(v) The following table gives the corresponding values of 2θ, θ and r.

2θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$	3π	$\frac{7\pi}{2}$	4π
θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	0	a	0	$-a$	0	a	0	$-a$	0

We observe that the given curve consists of four similar loops which lie in the regions $0 < \theta < \pi/2$, $\pi/2 < \theta < \pi$, $\pi < \theta < 3\pi/2$ and $3\pi/2 < \theta < 2\pi$.

All these four loops lie within a circle of radius a and centre at the pole.

When $\pi/2 < \theta < \pi$, we have $\pi < 2\theta < 2\pi$ so that $\sin 2\theta$ is negative. Thus when $\pi/2 < \theta < \pi$, r is – ive and so for these values of θ the points of the curve lie on the opposite side of the pole. Similarly when θ takes values between $3\pi/2$ and 2π, r is again negative and consequently for these values of θ also the curve lies on the opposite side of the pole. Taking into consideration all the above facts the shape of the curve is as shown in the figure.



Problem 7: *Trace the curve $r = a \cos 3\theta$.* ***(Three leaved rose)***

(Avadh 2010; Kashi 13)

Solution: (i) The curve is symmetrical about the initial line.

(ii) We have $r = 0$ when $\cos 3\theta = 0$

i.e., $$3\theta = \pm \frac{1}{2}\pi, \pm \frac{3}{2}\pi, \pm \frac{5}{2}\pi, \text{etc.}$$ **(Note)**

or $$\theta = \pm \frac{1}{6}\pi, \pm \frac{1}{2}\pi, \pm \frac{5}{6}\pi, \text{etc.}$$

Thus the lines $\theta = \pm \frac{1}{6}\pi, \pm \frac{1}{2}\pi$ etc. are the tangents to the curve at the pole.

(iii) Differentiating the equation of the curve, we get $(dr/d\theta) = -3a \sin 3\theta$.

Therefore $$\cot \phi = \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{a \cos 3\theta} \cdot (-3a \sin 3\theta) = -3 \tan 3\theta.$$

Now $\phi = 90^{\circ}$, when $\tan 3\theta = 0$

i.e., $3\theta = 0 \quad \pi \quad 2\pi \quad 3\pi$

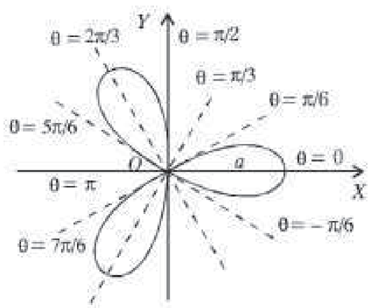
or $\theta = 0 \quad \frac{1}{3}\pi \quad \frac{2}{3}\pi \quad \pi.$

At all these points the tangent to the curve is perpendicular to the radius vector. Also at each of these points the numerical value of r is a which is the greatest value of the radius vector for this curve.

(iv) Table showing corresponding values of θ and r :

$\theta = 0$	$\frac{1}{6}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$	π
$r = a$	0	$-a$	0	a	0	$-a$

Hence the curve is as shown in the figure.



Problem 8: *Trace the curve* $\frac{2a}{r} = 1 + \cos 2\theta$. ***(Parabola)***

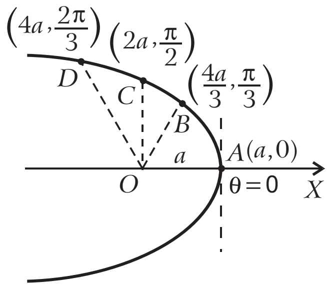
Solution: The given equation of the curve can be written as

$$r = \frac{2a}{1 + \cos 2\theta}.$$

The given curve is symmetrical about the initial line.

The following table gives the corresponding values of r and θ.

θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
$\cos\theta$	1	$1/2$	0	$-1/2$	-1
$1+\cos\theta$	2	$3/2$	1	$1/2$	0
r	a	$4a/3$	$2a$	$4a$	∞



Hence the curve is as shown in the figure.

Problem 9: *Trace the curve* $r = \frac{1}{2} + \cos 2\theta$. **(Meerut 2004B)**

Solution: The given curve is symmetrical about the initial line.

In the given equation of the curve $r = \frac{1}{2} + \cos 2\theta$ putting $r = 0$,

we get $\cos 2\theta = -\frac{1}{2}$ *i.e.,* $2\theta = \pm 2\pi/3$ or $\pm 4\pi/3$

i.e., $\theta = \pm \pi/3$ or $\pm 2\pi/3$.

The following table gives the corresponding values of r and θ.

θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
r	$\frac{3}{2}$	0	$-\frac{1}{2}$	0	$\frac{3}{2}$

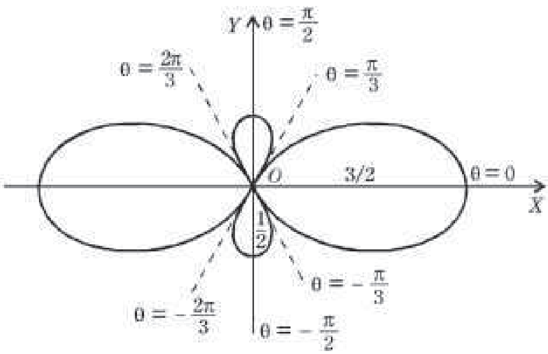
The greatest radius vector of the loop lying between $\theta = -\frac{1}{3}\pi$ and $\theta = \frac{1}{3}\pi$ is given by $\theta = 0$ and it is equal to $3/2$. The greatest radius vector of the loop lying between $\theta = \frac{1}{3}\pi$ and $\theta = \frac{2}{3}\pi$ is given by $\theta = \frac{1}{2}\pi$ and its absolute value is $\frac{1}{2}$.

Thus we observe that the larger loop lies between $\theta = -\pi/3$ and $\theta = \pi/3$ and is symmetrical about the initial line $\theta = 0$.

Also the smaller loop lies between $\theta = \pi/3$ and $\theta = 2\pi/3$.

We first trace the curve from $\theta = 0$ to $\theta = \pi$. The variation of θ from π to 2π need not be considered because of the symmetry about the initial line.

Hence the curve has four loops and is as shown in the figure.



Problem 10(i): *Trace the curve $r = a(\cos\theta + \sec\theta)$.* **(Lucknow 2007)**

Solution: We have $r = a(\cos\theta + \sec\theta)$. Multiplying both sides by r, we get

$$r^2 = a\left(r\cos\theta + \frac{r^2}{r\cos\theta}\right).$$

Changing to cartesian form, the equation becomes

$$x^2 + y^2 = a\left\{x + \frac{(x^2 + y^2)}{x}\right\}$$

or $$y^2(x - a) = x^2(2a - x). \qquad ...(1)$$

Now we shall trace the curve (1) by the method we used for tracing the curves whose cartesian equations were given.

Tracing: (i) The curve is symmetrical about x-axis.

(ii) The curve passes through the origin. Tangents at the origin are given by

$$-ay^2 - 2ax^2 = 0$$

or $$y^2 + 2x^2 = 0$$

i.e., the tangents at $(0, 0)$ are imaginary and so origin is a conjugate point.

(iii) When $y = 0$, $x = 0, 2a$ *i.e.*, the curve meets the x-axis at the points $(0, 0)$ and $(2a, 0)$ out of which $(0, 0)$ is an isolated point. When $x = 0$, $y = 0$ *i.e.*, the curve meets the y-axis only at the origin and that too is an isolated point.

Shifting the origin to $(2a, 0)$ the equation of the curve becomes

$$y^2(x + a) = (x + 2a)^2(-x).$$

Therefore the tangent at the new origin is $x = 0$ *i.e.*, the new y-axis.

(iv) $x = a$ is an asymptote parallel to y-axis and the curve has no other real asymptotes.

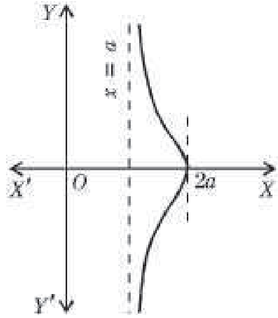
(v) Solving the equation of the curve for y, we have

$$y^2 = \{x^2(2a - x)\}/(x - a).$$

When $x = 0$, $y^2 = 0$. When $0 < x < a$, y^2 is –ive *i.e.*, y is imaginary and so the curve does not exist in this region. When $a < x < 2a$, y^2 is +ive *i.e.*, y is real and so the curve exists in this region. When $x \to a$ (from the right) $y^2 \to \infty$ showing that the line $x = a$ (on its right side) is an asymptote of the curve. When $x = 2a$, $y^2 = 0$ and when $x > 2a$, y^2 is –ive *i.e.*, y is imaginary and so the curve does not exist in the region where $x > 2a$.

When $x < 0$, y^2 is –ive *i.e.*, y is imaginary and so the curve does not exist in this region.

(vi) As x decreases from $2a$ to a, y^2 increases from 0 to ∞.



Taking all these facts into consideration, the shape of the curve is as shown in the figure.

Problem 10(ii): *Trace the curve $r \cos\theta = 2a \sin^2\theta$.* ***(Cissoid)***

Solution: We have $r\cos\theta = 2a\sin^2\theta$. Multiplying both sides by r^2, we get

$$r^2(r\cos\theta) = 2ar^2\sin^2\theta$$

Changing to cartesian form, the equation becomes

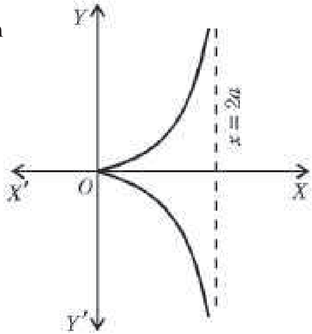
$$(x^2 + y^2)x = 2ay^2$$

or $$x^3 + xy^2 = 2ay^2$$

or $$x^3 = y^2(2a - x)$$

Now, proceed as in Example 19.

The shape of the curve is as shown in the figure.



Comprehensive Problems 5

Problem 1: *Trace the curve $x = a(t + \sin t)$, $y = a(1 + \cos t)$, $-\pi \le t \le \pi$.* ***(Cycloid)***

Solution: (i) Differentiating, we get

$$(dx/dt) = a(1 + \cos t) \quad \text{and} \quad (dy/dt) = -a\sin t.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-a\sin t}{a(1+\cos t)} = \frac{-2a\sin\frac{1}{2}t\cos\frac{1}{2}t}{2a\cos^2\frac{1}{2}t} = -\tan\frac{1}{2}t.$$

(ii) We have $y = 0$, when $\cos t = -1$ *i.e.*, $t = -\pi, \pi$.

When $t = \pi$, $x = a\pi$, $dy/dx = -\infty$. Thus at the point $(a\pi, 0)$ the tangent to the curve is perpendicular to the x-axis. Again when $t = -\pi$, $x = -a\pi$, $dy/dx = \infty$.

(iii) y is maximum when $\cos t = 1$ *i.e.*, $t = 0$.

When $t = 0$, $x = 0$, $y = 2a$ and $dy/dx = 0$. Thus at the point $(0, 2a)$ the tangent to the curve is parallel to the x-axis.

(iv) In this curve y cannot be negative. Also no portion of the curve lies in the region $y > 2a$.

(v) Corresponding values of x, y and (dy/dx) for different values of t are given in the following table :

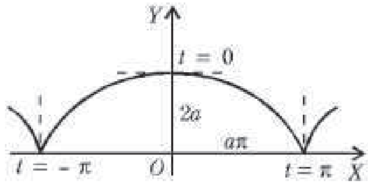
t	$-\pi$	$-\frac{1}{2}\pi$	0	$\frac{1}{2}\pi$	π
x	$-a\pi$	$-a(\frac{1}{2}\pi+1)$	0	$a(\frac{1}{2}\pi+1)$	$a\pi$
y	0	a	$2a$	a	0
dy/dx	∞	1	0	-1	$-\infty$

From above, $(-a\pi, 0)$ is a point on the curve with tangent inclined to x-axis at the angle

$$\psi = \pi/2. \quad [\because \tan\psi = dy/dx = \infty]$$

Arguing as in Example 30, the curve is symmetrical about the y-axis.

Hence the shape of the curve is as shown in the figure.



Problem 2: *Trace the curve $x = a(t - \sin t)$, $y = a(1 - \cos t)$.* **(Meerut 2005)**

Solution: The parametric equations of the given cycloid are

$$x = a(t - \sin t),\ y = a(1 - \cos t).$$

We have $\quad dx/dt = a(1 - \cos t),\ dy/dt = a \sin t.$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a(1 - \cos t)} = \frac{2 \sin \frac{1}{2}t \cos \frac{1}{2}t}{2 \sin^2 \frac{1}{2}t} = \cot \frac{1}{2}t.$$

In this curve $y = 0$ when $\cos t = 1$ *i.e.*, $t = 0, 2\pi$.

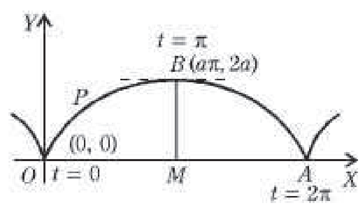
When $t = 0$, $x = a(0 - \sin 0) = 0$, $y = 0$ and $dy/dx = \cot 0 = \infty$. Thus the curve passes through the point $(0, 0)$ and the axis of y is tangent to the curve at this point.

In this curve y is maximum when $\cos t = -1$ *i.e.*, $t = \pi$.

When $\quad t = \pi$, $x = a(\pi - \sin \pi) = a\pi$, $y = 2a$,

$$dy/dx = \cot \frac{1}{2}\pi = 0.$$

Thus at the point $t = \pi$, whose cartesian coordinates are $(a\pi, 2a)$, the tangent to the curve is parallel to x-axis. This curve does not exist in the region $y > 2a$.



In this curve y cannot be –ive because $\cos t$ cannot be greater than 1. Thus one complete arch of the given cycloid lying between $0 \le t \le 2\pi$ is as shown in the figure.

Problem 3: *Trace the curve* $x = a \cos t + \frac{1}{2} a \log \tan^2 \frac{1}{2} t,\ y = a \sin t.$ **(Tractrix)**

(Garhwal 2006)

Solution: (i) If we put $-t$ in place of t in the equation of the curve, we get

$$x = a \cos t + \frac{1}{2} a \log \tan^2 \frac{1}{2} t, \quad \text{and} \quad y = -a \sin t.$$

Thus for every value of x, there are two equal and opposite values of y. Therefore the curve is symmetrical about the x-axis.

Again if we put $\pi - t$ in place of t in the equation of the curve, we get

$$x = -a \cos t + \frac{1}{2} a \log \cot^2 \frac{1}{2} t = -a \cos t - \frac{1}{2} a \log \tan^2 \frac{1}{2} t,$$

and $\qquad y = a \sin t.$

Thus for every value of y, there are two equal and opposite values of x. Therefore the curve is symmetrical about the y-axis.

(ii) Differentiating the equations of the curve w.r.t. 't', we get

$$\frac{dx}{dt} = -a \sin t + \frac{1}{2} a \frac{1}{\tan^2 \frac{1}{2} t} \cdot \left(2 \tan \frac{1}{2} t \sec^2 \frac{1}{2} t\right) \cdot \frac{1}{2}$$

$$= -a \sin t + \frac{a}{2 \sin \frac{1}{2} t \cos \frac{1}{2} t} = -a \sin t + \frac{a}{\sin t}$$

$$= \frac{a(1 - \sin^2 t)}{\sin t} = \frac{a \cos^2 t}{\sin t}, \quad \text{and} \quad dy/dt = a \cos t.$$

Therefore $dy/dx = \tan t$.

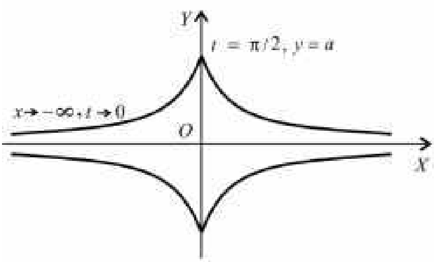
(iii) We have $y = 0$ when $\sin t = 0$ *i.e.*, $t = 0$. When $t \to 0$, $x \to -\infty$. Thus $x \to -\infty$ when $y \to 0$, showing that the line $y = 0$ is an asymptote of the curve.

(iv) y is maximum when $\sin t = 1$ *i.e.*, $t = \frac{1}{2} \pi$.

When $\qquad t = \frac{1}{2} \pi, x = 0, y = a \quad$ and $\quad dy/dx = \tan \frac{1}{2} \pi = \infty.$

Thus the curve passes through the point $(0, a)$ and the tangent at this point is the y-axis.

(v) In this curve the numerical value of y cannot be greater than a. Thus the curve does not exist in the regions $y > a$ and $y < -a$. The shape of the curve is as shown in the figure. The curve has four infinite branches and for the branch in the second quadrant t varies from 0 to $\frac{1}{2} \pi$ while x varies from $-\infty$ to 0.



Hints to Objective Type Questions

Multiple Choice Questions

1. The given curve is $x^3 + y^3 - 3axy = 0$, which clearly passes through the origin. Equating to zero the lowest degree terms *i.e.*, the second degree terms in the equation of the curve, the tangents at origin are given by $-3axy = 0$. Thus, $x = 0$, $y = 0$ are the tangents at the origin.
2. The equation of the curve $y = x^3$ remains unchanged on replacing x by $-x$ and y by $-y$. So, the curve is symmetrical in opposite quadrants.
3. The number of loops in the curve $r = a \cos n\theta$ is n if n is odd and is $2n$ if n is even. Here, $n = 2$ which is even. So, the number of loops in the given curve $= 2 \times 2 = 4$.
4. The equation of the given curve changes on changing the sign of θ, so it is not symmetrical about the initial line. Again, the equation of the given curve changes if we put $-r$ in place of r. So, the given curve is not symmetrical about the pole. Now, let $\left(r, \frac{\pi}{2} - \alpha\right)$ and $\left(r, \frac{\pi}{2} + \alpha\right)$ be two points equidistant from the line $\theta = \frac{\pi}{2}$ and on opposite sides of it.
   Putting $\theta = \frac{\pi}{2} - \alpha$ in the equation of the given curve $r = a \sin 3\theta$, we get
   $$r = a \sin 3\left(\frac{\pi}{2} - \alpha\right) = a \sin\left(\frac{3\pi}{2} - 3\alpha\right) = -a \cos 3\alpha. \qquad ...(1)$$
   Again, putting $\theta = \frac{\pi}{2} + \alpha$ in the equation of the given curve, we get
   $$r = a \sin 3\left(\frac{\pi}{2} + \alpha\right) = a \sin\left(\frac{3\pi}{2} + 3\alpha\right) = -a \cos 3\alpha. \qquad ...(2)$$
   We see that the values of r given by (1) and (2) are the same. It means that if the point $\left(r, \frac{\pi}{2} - \alpha\right)$ lies on the given curve, then its mirror reflection in the line $\theta = \frac{\pi}{2}$ *i.e.*, the point $\left(r, \frac{\pi}{2} + \alpha\right)$ also lies on it. So, the given curve is symmetrical about the line $\theta = \frac{\pi}{2}$.
5. See article 4, corollary .
6. See Problem 4 of Comprehensive Problems 4.
7. See Problem 4 of Comprehensive Problems 4.
8. See article 4, corollary.
9. See Example 10.
10. See Example 14.

11. Here $n = 5$, which is odd. So the number of loops in the given curve is 5.
12. See Problem 3 of Comprehensive Problems 5.
13. See Problem 3 of Comprehensive Problems 4.
14. See article 8, point (ii).
15. See article 8, point (iii).

Fill in the Blanks

1. We know that a point (x, y) on the curve $y = f(x)$ is a point of inflexion if at that point $\frac{d^2 y}{dx^2} = 0$ and $\frac{d^3 y}{dx^3} \neq 0$.
2. See article 8(i).
3. See Problem 6(iii) of Comprehensive Problems 2.
4. Since in the equation of the given curve, the powers of x and y both are all even, therefore the curve is symmetrical about both the axes.
5. It is given that the equation of the curve $r = f(\theta)$ does not change by changing the sign of θ. It means that if the point (r, θ) lies on the curve, then its reflection in the initial line *i.e.*, the point $(r, -\theta)$ also lies on the curve. Hence, the curve is symmetrical about the initial line.
6. See Example 23.

True or False

1. Under the given conditions, the double point is a cusp and not a node.
2. It is given that the equation of the given curve remains unchanged on changing the signs of r and θ both. It means that if the point (r, θ) lies on the given curve, then its reflection in the line $\theta = \frac{\pi}{2}$ *i.e.*, the point $(-r, -\theta)$ also lies on it. So, the curve is symmetrical about the line $\theta = \frac{\pi}{2}$.
3. See article 16 .
4. See article 2 .

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