

Krishna's

Differential Equations

*(For M.A./M.Sc. Ist Year; Ist Sem., C.C.S. University Meerut and for Competitive Examinations
viz. I.A.S., P.C.S., C.S.I.R. (N.E.T.) etc.)*

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to
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Preface to the Revised Edition

The book entitled "**DIFFERENTIAL EQUATIONS**" has been written to meet the requirements of the students of **M.A./M.Sc. First Semester of C.C.S. University, Meerut** and other universities around the globe. The book covers almost all the topics prescribed in the syllabus of differential equations of various universities according to **U.G.C. Curriculum**. The book is also useful for various **Competitive Examinations** like **I.A.S., P.C.S. and C.S.I.R. (N.E.T.)** etc.

The subject matter is prescribed in such a clear and easy way that the reader will have no difficulty to understand and grasp it. Sufficient number of solved examples have been given in each chapter so that the reader may gain confidence in the techniques of solving problems. Each chapter contains exercises to provide the reader an opportunity to test their knowledge of the subject.

The authors wish to express their thanks to **Mr. S.K. Rastogi, Managing Director, Mr. Sugam Rastogi, Executive Director, Mrs. Kanupriya Rastogi Director** and **entire team of KRISHNA Prakashan Media (P) Ltd., Meerut** for bringing out this book in the present nice form.

The authors will feel amply rewarded if the book serves the purpose for which it is meant. Suggestions for the improvement of the book are always welcome.

July, 2014



— Authors

Syllabus

Differential Equations

M.A./M.Sc. Ist Semester
C.C.S. University

M-103

Unit-I

Ordinary Differential Equations: Qualitative properties of solution: Oscillation, Wronskian, Sturm separation and comparison theorems.

Unit-II

Ordinary points, Regular singular points, Frobenius series solution.

Unit-III

Gauss hyper geometric equation, the point at infinity, Gamma functions, Hermite polynomials.

Unit-IV

Partial Differential Equations: Origin of first order partial differential equations, Linear equations of the first order, Integral surfaces passing through a given curve, Surface orthogonal to a given system of surface, Non-linear partial differential equations of the first order, Charpits method, Special type of first order equations, Jacobi method. Origin of second order partial differential equation, Linear partial differential equations with constant coefficients, Equations with variable coefficients.

Unit-V

Problems of Laplace, wave and diffusion equations by the method of separation of variables, Reduction of second order partial differential equation into its canonical form.

Non-linear equation of second order.

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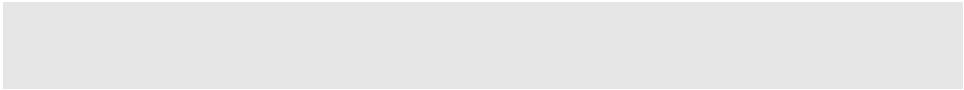
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UNIT- 1

CHAPTER 1 : Ordinary Differential Equations and Wronskian

CHAPTER 2 : Qualitative Properties of Solutions
(Oscillation, Adjoint and Sturm Theory)



Unit - 1

Chapter

1

Ordinary Differential Equations and Wronskian

1.1 Introduction

The theory of **differential equations** is an important branch of science and engineering. To understand and to investigate problems involving the motion of fluids, the flow of current in electric circuits, transmission of electric waves, conduction of heat in solid objects, the propagation detection of seismic waves, the law of mass action, the increase or decrease of population and in many other, the study of differential equations is essential.

1.2 Differential Equations

“An equation containing an independent variable (or variables), dependent variable and differential coefficients of dependent variable with respect to the independent variable or variables is called a differential equation.”

For Example: The following equations are the examples of differential equations :

(i) $\frac{dy}{dx} = x + 2y$	(ii) $\left(\frac{dy}{dx}\right)^2 - 4\frac{dy}{dx} - 3y^2 + 7x = 0$
(iii) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 7y = \log x$	(iv) $\frac{d^3y}{dx^3} + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = k\frac{dy}{dx}$
(v) $\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = kz$	(vi) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

The differential equations can be classified as follows :

1. **Ordinary Differential Equation** : A differential equation involving the derivatives of the dependent variable with respect to only one independent variable is called an **ordinary differential equation**.

For Example: All the above four differential equations given in (i) to (iv) are examples of ordinary differential equations.

2. **Partial Differential Equation** : A differential equation involving the derivatives of the dependent variable with respect to more than one independent variable is called a **partial differential equation**.

For Example: Above two differential equations given in (v) and (vi) are examples of partial differential equations.

In the partial differential equation given in (v) there are two independent variables x and y , while in that given in (vi) there are three independent variables x , y and z .

Note : (i) We shall confine ourselves to the study of ordinary differential equations only in chap 1.

(ii) Now onward we will use the term 'differential equation' for ordinary differential equation.

1.3 Order and Degree of a Differential Equation

1. **Order of a Differential Equation** : The **order** of a differential equation is the order of the highest order derivative appearing in the equation.

For Example: The orders of the differential equations given in (i), (ii), (iii) and (iv) of 1.2 are 1, 1, 2 and 3 respectively.

Note : The order of a differential equation is a positive integer.

2. **Degree of a Differential Equation** : The **degree** of a differential equation is the power of the highest order derivative appearing in it, when the equation has been made free from radicals and fractions as far as the powers of the derivatives are concerned.

For Example: Consider the differential equation

$$\frac{d^3 y}{dx^3} + \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2} = k \frac{dy}{dx} \text{ given in (iv) of 1.2}$$

Here we make the powers of derivatives free from fractions as follows :

$$\text{Here} \quad \left[\frac{d^3 y}{dx^3} - k \frac{dy}{dx} \right]^2 = \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^3$$

$$\Rightarrow \left(\frac{d^3 y}{dx^3} \right)^2 - 2k \frac{d^3 y}{dx^3} \cdot \frac{dy}{dx} - \left(\frac{dy}{dx} \right)^6 - 3 \left(\frac{dy}{dx} \right)^4 - (3 - k^2) \left(\frac{dy}{dx} \right)^2 - 1 = 0$$

Which is free from fractional powers of the derivatives.

Here the power of highest order derivative $\frac{d^3 y}{dx^3}$ is 2, so the degree of the differential equation is 2.

Note : The Definition of degree of a diff. equation does not require the variables (dependent and independent both) to be free from radicals and fractions.

1.4 Linear and Non-linear Differential Equations

An ordinary differential equation

$$F(x, y, y', y'', \dots, y^n) = 0$$

is said to be **linear** if F is a linear function of the variables $y, y', y'', \dots, y^n$.

Thus the most general **linear differential equation (L.D.E) of order n** is

$$\frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + a_2(x) \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_{n-1}(x) \frac{dy}{dx} + a_n(x) y = f(x) \quad \dots(1)$$

where $a_1(x), \dots, a_n(x), f(x)$ are constants or functions of independent variable x .

A differential equation which is not of the form of (1) is called a **non-linear differential equation**.

In the above differential equation (1), if $f(x) = 0$ then it is called **homogeneous linear differential equation** otherwise (*i.e.*, if $f(x) \neq 0$) it is called **non-homogeneous linear differential equation**.

Similar definitions applies to *partial differential equations*.

For Example: (i) $\frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + 4y = \sin x$ is non-homogeneous L.D.E. of order 2.

(ii) $x^3 \frac{d^3 y}{dx^3} + x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$ is homogeneous L.D.E. of order 3.

(iii) $a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ and $a^2 \frac{\partial^2 u}{\partial x^2} = b^2 \frac{\partial^2 u}{\partial t^2}$ are linear partial differential equations

(iv) $\frac{d^3 y}{dx^3} + 2e^x \frac{d^2 y}{dx^2} + y \frac{dy}{dx} = x^2$ is non-linear differential equation of order 3, because of the term $y \frac{dy}{dx}$

(v) $\frac{d^2 y}{dx^2} + \sin(x + y) = \sin x$ is non-linear differential equation of order 2, because of the term $\sin(x + y)$.

1.5 Solution of a Differential Equation

The **solution** of a differential equation is a relation between the dependent and independent variables (not containing the derivatives) involved in the equation which satisfies the differential equation or reduces it to an identity.

For Example: $y = a \cos mx + b \sin mx$ is the solution of the differential equation

$$\frac{d^2 y}{dx^2} + m^2 y = 0, \text{ as it satisfies the equation.}$$

1. **General solution (or Complete Solution or Complete Primitive) of a Differential Equation :** The solution of a differential equation is said to be a **general solution (or Complete Primitive)** of the differential equation if the number of arbitrary constants in the solution is exactly equal to the order of the differential equation.

For Example: $y = a \sin x + b \cos x + x \cos x$

is the general solution of the differential equation

$$\frac{d^2 y}{dx^2} + y + 2 \sin x = 0, \text{ as it contains two arbitrary constants } a \text{ and } b \text{ whose number is equal to the order of the equation.}$$

2. **Particular Solution (or Particular Integral) of a Differential Equation :** A solution of a differential equation obtained by giving some particular values to the arbitrary constants in its general solution is called a **particular solution (or particular integral)** of the differential equation.

For Example: $y = 3 \cos mx + 7 \sin mx$ is the particular solution of the differential equation $\frac{d^2 y}{dx^2} + m^2 y = 0$ as it is obtained by assigning particular values 3 and 7 to the arbitrary constants c_1 and c_2 respectively in the general solution $y = c_1 \cos mx + c_2 \sin mx$ of the equation.

Note : In many cases the complete primitive of the differential equation consists of the sum of two parts, one part contains the arbitrary constants while the other part is a definite function of the independent variable. The first part which contains the arbitrary constants is known as the **complementary function** abbreviated as **C.F.** and the remaining second part which can be obtained by giving the value zero to each of the arbitrary constants is called the **particular integral** abbreviated as **P.I.**

The P.I. corresponds to the function $f(x)$ on the R.H.S. of the differential equation $F(x, y, y', y'', \dots, y^n) = f(x)$. If $f(x) = 0$, then P.I. = 0.

1.6 Properties of Solution of a Differential Equation

Theorems

Theorem 1: If $y = y_1, y = y_2, \dots, y = y_n$ are n solutions of an ordinary linear differential equation of order n , of the form

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_{n-1} \frac{dy}{dx} + P_n y = 0 \quad \dots(1)$$

where $P_1, P_2, \dots, P_n$ are functions of x or constants, then $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$, where $c_1, c_2, \dots, c_n$ are arbitrary constants, is also a solution of the equation.

Proof: Since $y_1, y_2, \dots, y_n$ are n solutions of the diff. equation (1),

$$\therefore \frac{d^n y_1}{dx^n} + P_1 \frac{d^{n-1} y_1}{dx^{n-1}} + P_2 \frac{d^{n-2} y_1}{dx^{n-2}} + \dots + P_{n-1} \frac{dy_1}{dx} + P_n y_1 = 0 \quad \dots(A_1)$$

$$\frac{d^n y_2}{dx^n} + P_1 \frac{d^{n-1} y_2}{dx^{n-1}} + P_2 \frac{d^{n-2} y_2}{dx^{n-2}} + \dots + P_{n-1} \frac{dy_2}{dx} + P_n y_2 = 0 \quad \dots(A_2)$$

.....

$$\frac{d^n y_n}{dx^n} + P_1 \frac{d^{n-1} y_n}{dx^{n-1}} + P_2 \frac{d^{n-2} y_n}{dx^{n-2}} + \dots + P_{n-1} \frac{dy_n}{dx} + P_n y_n = 0 \quad \dots(A_n)$$

If $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n \dots(2)$ is also a solution of (1) then it must satisfy (1).

Substituting from (2) in (1), we get

$$\begin{aligned} & \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_{n-1} \frac{dy}{dx} + P_n y \\ &= \left[c_1 \frac{d^n y_1}{dx^n} + c_2 \frac{d^n y_2}{dx^n} + \dots + c_n \frac{d^n y_n}{dx^n} \right] \\ & \quad + P_1 \left[c_1 \frac{d^{n-1} y_1}{dx^{n-1}} + c_2 \frac{d^{n-1} y_2}{dx^{n-1}} + \dots + c_n \frac{d^{n-1} y_n}{dx^{n-1}} \right] \\ & \quad + \dots + P_n [c_1 y_1 + c_2 y_2 + \dots + c_n y_n] \\ &= c_1 \left[\frac{d^n y_1}{dx^n} + P_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \dots + P_n y_1 \right] + c_2 \left[\frac{d^n y_2}{dx^n} + P_1 \frac{d^{n-1} y_2}{dx^{n-1}} + \dots + P_n y_2 \right] \\ & \quad + \dots + c_n \left[\frac{d^n y_n}{dx^n} + P_1 \frac{d^{n-1} y_n}{dx^{n-1}} + \dots + P_n y_n \right] = 0 \end{aligned}$$

Since the value of expression in each bracket [] is 0 due to $(A_1), (A_2), \dots, (A_n)$.

Hence $y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ is also a solution of (1).

Theorem 2: *The complete solution of a differential equation of the n -th order contains n arbitrary independent constants.*

Proof: First, let us consider a differential equation of first order.

$$\text{i.e.,} \quad f(y', y, x) = 0 \quad \dots(1)$$

$$\text{Let} \quad y(x) = C_0 + C_1 x + C_2 x^2 + \dots \quad \dots(2)$$

be the solution of (1), when expanded in ascending powers of x .

Also from Maclaurin's Theorem, we have

$$y(x) = y(0) + y'(0) \cdot x + y''(0) \cdot \frac{x^2}{2!} + y'''(0) \cdot \frac{x^3}{3!} + \dots \quad \dots(3)$$

where $y', y'', \dots$ denote the first, second, ... order derivatives of y w.r.t. ' x ' and $y(0), y'(0), \dots$ denote the values of $y, y', y'', \dots$ when $x = 0$.

Comparing (2) and (3), we get

$$C_0 = y(0), C_1 = y'(0), C_2 = \frac{y''(0)}{2!}, C_3 = \frac{y'''(0)}{3!}, \dots$$

Now $C_0 = y(0)$ cannot be expressed in terms of any thing known or determinate.

Putting $x = 0$ in (1), we have

$$\begin{aligned} & f[y'(0), y(0), 0] = 0 \\ \Rightarrow & f(C_1, C_0, 0) = 0 \Rightarrow C_1 \text{ can be determined in terms of } C_0 \end{aligned}$$

Also solving (1) for y' , we get $y' = \phi(y, x)$.

$$\Rightarrow y'' = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} y' \quad \dots(4)$$

$\therefore$ Putting $x = 0$ in (4), we get,

$$\begin{aligned} y''(0) &= \left(\frac{\partial \phi}{\partial x} \right)_{x=0} + \left(\frac{\partial \phi}{\partial y} \right)_{x=0} y'(0) \\ \Rightarrow 2! C_2 &= \left(\frac{\partial \phi}{\partial x} \right)_{x=0} + \left(\frac{\partial \phi}{\partial y} \right)_{x=0} C_1 \end{aligned}$$

$$\Rightarrow C_2 \text{ is determined in terms of } C_1$$

$$\Rightarrow C_2 \text{ is determined in terms of } C_0. \text{ (Since } C_1 \text{ can be determined in terms of } C_0 \text{)}$$

Proceeding similarly by differentiating (4) and substituting $x = 0$, we can find $C_3, C_4, \dots$ in terms of C_0 .

Thus all the constants in the solution (2) of (1) i.e., $C_1, C_2, \dots$ are determined in terms of C_0 which is undetermined.

Hence substituting in (2), the general solution of a differential equation of the first order is obtained which has only one arbitrary constant.

Next, we consider a differential equation of second order

$$\text{i.e., } F(y'', y', y, x) = 0 \quad \dots(5)$$

Let the solution of (5) be taken as

$$y(x) = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots \quad \dots(6)$$

Comparing (6) and (3), we get

$$C_0 = y(0), C_1 = y'(0), C_2 = \frac{y''(0)}{2!}, C_3 = \frac{y'''(0)}{3!}, \dots$$

i.e., C_0 cannot be determined.

Now solving (5), for y'' , we get

$$y'' = \psi(y', y, x) \quad \dots(7)$$

$$\Rightarrow y''(0) = \psi[y'(0), y(0), 0] = \psi(C_1, C_0, 0)$$

$$\Rightarrow 2!C_2 = \psi(C_1, C_0, 0)$$

$$\Rightarrow C_2 \text{ can be determined in terms of } C_0 \text{ and } C_1.$$

Again differentiating (7), we get

$$y'''(x) = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} + \frac{\partial \psi}{\partial y'} \frac{dy'}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} y' + \frac{\partial \psi}{\partial y'} y'' \quad \dots(8)$$

Putting $x = 0$ in (8), we get

$$y'''(0) = \left(\frac{\partial \psi}{\partial x} \right)_{x=0} + \left(\frac{\partial \psi}{\partial y} \right)_{x=0} y'(0) + \left(\frac{\partial \psi}{\partial y'} \right)_{x=0} y''(0)$$

$$\Rightarrow 3!C_3 = \left(\frac{\partial \psi}{\partial x} \right)_{x=0} + \left(\frac{\partial \psi}{\partial y} \right)_{x=0} C_1 + \left(\frac{\partial \psi}{\partial y'} \right)_{x=0} 2!C_2$$

From which C_3 can be determined in terms of C_1 and C_2 . But C_2 has already been determined in terms of C_0 and C_1 . So C_3 can be determined in terms of C_0 and C_1 .

Proceeding similarly by differentiating (8) and substituting $x=0$, we can find $C_4, C_5, \dots$ in terms of C_0 and C_1 .

Thus all the constants in the solution (6) of (5) i.e., $C_2, C_3, C_4, \dots$ are determined in terms of C_0 and C_1 which are undetermined.

Hence substituting in (6), the general solution of a differential equation of second order is obtained, which have exactly two arbitrary constants.

Proceeding similarly in this way it can be shown that the general solution of differential equation of n -th order contains exactly n arbitrary constants.

1.7 Independence of Constants of Integration

The constants of integration C_1, C_2 in the solution $y = \phi(x, C_1, C_2)$ of an equation of second order are said to be *independent* if this solution is not reducible to the form $y = \psi[x, f(C_1, C_2)]$, in which there is really only one arbitrary constant $f(C_1, C_2)$.

1.8 Necessary and Sufficient Condition for the Independence of Constants of Integration

Theorem *The necessary and sufficient condition that the parameters (i.e., constants of integration) in the solution $y = \phi(x, C_1, C_2)$ of the second order differential equation are independent is that the determinant*

$$\begin{vmatrix} \frac{\partial \phi}{\partial C_1} & \frac{\partial \phi}{\partial C_2} \\ \frac{\partial^2 \phi}{\partial C_1 \partial x} & \frac{\partial^2 \phi}{\partial C_2 \partial x} \end{vmatrix}$$

be not equal to zero.

[Meerut 2010 (Sem. I)]

Proof: **Necessary Condition :** Let the parameters C_1 and C_2 in the solution $\phi(x, C_1, C_2)$ be dependent, then one of these can be expressed in terms of the other. In this case the solution $y = \phi(x, C_1, C_2)$ can be written as $y = \psi[x, f(C_1, C_2)]$, in which there is only one arbitrary constant $f(C_1, C_2)$.

Now
$$\frac{\partial \phi}{\partial C_1} = \frac{\partial \psi}{\partial C_1} = \frac{\partial \psi}{\partial f} \cdot \frac{\partial f}{\partial C_1} \text{ and } \frac{\partial \phi}{\partial C_2} = \frac{\partial \psi}{\partial C_2} = \frac{\partial \psi}{\partial f} \cdot \frac{\partial f}{\partial C_2}$$

and
$$\frac{\partial^2 \phi}{\partial C_1 \partial x} = \frac{\partial}{\partial C_1} \left(\frac{\partial \psi}{\partial x} \right) = \frac{\partial}{\partial C_1} \left(\frac{\partial \psi}{\partial f} \cdot \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial f} \left(\frac{\partial \psi}{\partial x} \right) \cdot \frac{\partial f}{\partial C_1} = \frac{\partial^2 \psi}{\partial f \partial x} \cdot \frac{\partial f}{\partial C_1}$$

Similarly,
$$\frac{\partial^2 \phi}{\partial C_2 \partial x} = \frac{\partial^2 \psi}{\partial f \partial x} \cdot \frac{\partial f}{\partial C_2}$$

$$\begin{aligned} \therefore \begin{vmatrix} \frac{\partial \phi}{\partial C_1} & \frac{\partial \phi}{\partial C_2} \\ \frac{\partial^2 \phi}{\partial C_1 \partial x} & \frac{\partial^2 \phi}{\partial C_2 \partial x} \end{vmatrix} &= \begin{vmatrix} \frac{\partial \psi}{\partial f} \cdot \frac{\partial f}{\partial C_1} & \frac{\partial \psi}{\partial f} \cdot \frac{\partial f}{\partial C_2} \\ \frac{\partial^2 \psi}{\partial f \partial x} \cdot \frac{\partial f}{\partial C_1} & \frac{\partial^2 \psi}{\partial f \partial x} \cdot \frac{\partial f}{\partial C_2} \end{vmatrix} \\ &= \frac{\partial \psi}{\partial f} \cdot \frac{\partial f}{\partial C_1} \cdot \frac{\partial^2 \psi}{\partial f \partial x} \cdot \frac{\partial f}{\partial C_2} - \frac{\partial \psi}{\partial f} \cdot \frac{\partial f}{\partial C_2} \cdot \frac{\partial^2 \psi}{\partial f \partial x} \cdot \frac{\partial f}{\partial C_1} = 0 \end{aligned}$$

i.e., if in $\phi(x, C_1, C_2)$, C_1 and C_2 are dependent then the determinant is equal to zero.

Hence if in $\phi(x, C_1, C_2)$, C_1 and C_2 are independent then the determinant is not equal to zero.

1. Sufficient Condition : i.e., if the determinant is not equal to zero, then the parameters C_1 and C_2 are independent.

To prove this, we suppose that the determinant is equal to zero and prove that C_1 and C_2 are dependent i.e., the solution $\phi(x, C_1, C_2)$ can be written as $\psi(x, f(C_1, C_2))$ i.e., $\phi(x, C_1, C_2)$ will not vary, no matter how C_1 and C_2 are varied as long as $f(C_1, C_2)$ is assigned some particular constant value.

Let
$$\frac{\partial \phi}{\partial C_1} = p \text{ and } \frac{\partial \phi}{\partial C_2} = q$$

$$\begin{aligned} \therefore \begin{vmatrix} \frac{\partial \phi}{\partial C_1} & \frac{\partial \phi}{\partial C_2} \\ \frac{\partial^2 \phi}{\partial C_1 \partial x} & \frac{\partial^2 \phi}{\partial C_2 \partial x} \end{vmatrix} &= 0 \Rightarrow \begin{vmatrix} p & q \\ \frac{\partial p}{\partial x} & \frac{\partial q}{\partial x} \end{vmatrix} = 0 \\ \Rightarrow p \frac{\partial q}{\partial x} - q \frac{\partial p}{\partial x} &= 0 \Rightarrow \frac{1}{p} \frac{\partial p}{\partial x} = \frac{1}{q} \frac{\partial q}{\partial x} \end{aligned}$$

Integrating, $\log p = \log q + \log a \Rightarrow p/q = a$ (constant)

$\Rightarrow p$ and q are independent of x and hence can involve C_1 and C_2 .

Let us assume $\frac{p}{q} = \frac{L}{M}$, where L and M are functions of C_1 and C_2 .

$$\Rightarrow Mp = Lq \Rightarrow M \frac{\partial \phi}{\partial C_1} = L \frac{\partial \phi}{\partial C_2}$$

$$\begin{aligned} \therefore d\phi &= \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial C_1} dC_1 + \frac{\partial \phi}{\partial C_2} dC_2 \because \phi \text{ is a function of } x, C_1 \text{ and } C_2 \\ &= \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial C_1} dC_1 + \frac{M}{L} \frac{\partial \phi}{\partial C_1} dC_2 \end{aligned}$$

$$= \frac{\partial \phi}{\partial x} dx + (L dC_1 + M dC_2) \frac{1}{L} \frac{\partial \phi}{\partial C_1}$$

But $L dC_1 + M dC_2$ has an integrating factor say λ such that $\lambda L dC_1 + \lambda M dC_2$ is complete differential of some function of C_1 and C_2 i.e.,

$$\lambda L dC_1 + \lambda M dC_2 = d f(C_1, C_2)$$

$$\therefore d\phi = \frac{\partial \phi}{\partial x} dx + \frac{1}{\lambda L} \frac{\partial \phi}{\partial C_1} d f(C_1, C_2)$$

Thus, $\phi(x, C_1, C_2)$ will not vary, no matter how C_1 and C_2 are various provided only that they satisfy the condition $f(C_1, C_2) = a$ (constant).

Hence if the determinant is not equal to zero, then the parameters C_1 and C_2 are independent.

Hence, the necessary and sufficient condition that the parameters C_1 and C_2 in the solution $y = \phi(x, C_1, C_2)$ are independent is that the above determinant is not equal to zero.

2. **In General :** The necessary and sufficient condition for the n parameters in the solution $y = \phi(x, C_1, C_2, \dots, C_n)$ of the n -th order differential equation to be independent is that the determinant

$$\begin{vmatrix} \frac{\partial \phi}{\partial C_1} & \frac{\partial \phi}{\partial C_2} & \dots & \dots & \frac{\partial \phi}{\partial C_n} \\ \frac{\partial^2 \phi}{\partial C_1 \partial x} & \frac{\partial^2 \phi}{\partial C_2 \partial x} & \dots & \dots & \frac{\partial^2 \phi}{\partial C_n \partial x} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \frac{\partial^n \phi}{\partial C_1 \partial x^{n-1}} & \frac{\partial^n \phi}{\partial C_2 \partial x^{n-1}} & \dots & \dots & \frac{\partial^n \phi}{\partial C_n \partial x^{n-1}} \end{vmatrix}$$

be not equal to zero.

1.9 Wronskian

The **Wronskian** of n functions $y_1(x), y_2(x), \dots, y_n(x)$ each of which has $(n-1)$ derivatives on a real interval $a \leq x \leq b$ is denoted by $W(y_1, y_2, \dots, y_n)$ or $W(x)$ and is defined by the determinant

$$W(y_1, y_2, \dots, y_n)(x) = W(x) = \begin{vmatrix} y_1 & y_2 & \dots & \dots & y_n \\ y_1' & y_2' & \dots & \dots & y_n' \\ y_1'' & y_2'' & \dots & \dots & y_n'' \\ \vdots & \vdots & \dots & \dots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & \dots & y_n^{(n-1)} \end{vmatrix}$$

[Meerut 2000, 01, 13 (Sem. I); Delhi 2000, 03, 05, 07; Kurukshetra 2002]

Note : The Wronskian is named after **Hoene Wronski**, one of the early writers on determinants.

1.10 Linear Dependence and Independence of Solutions of an Equation

Let $y_1, y_2, \dots, y_n$ be n solutions of the linear differential equation

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_n y = 0$$

where $P_1, P_2, \dots, P_n$ are functions of x or constants.

These n solutions $y_1, y_2, \dots, y_n$ are said to be **linearly dependent** (abbreviated as L.D.) if there exists a set of n constants $a_1, a_2, \dots, a_n$ at least one of which is different from zero, such that

$$a_1 y_1 + a_2 y_2 + \dots + a_n y_n = 0$$

If no such set of n constants exists then the solutions $y_1, y_2, \dots, y_n$ are said to be **linearly independent** (abbreviated as L.I.).

i.e., if $y_1, y_2, \dots, y_n$ are L.I. and $a_1 y_1 + a_2 y_2 + \dots + a_n y_n = 0$

then $a_1 = 0 = a_2 = \dots = a_n$

1.11 Necessary and Sufficient Condition for the Independence of n Solutions of a Linear Differential Equation of Order n .

Theorem 3: *The necessary and sufficient condition that n integrals or solutions $y_1, y_2, \dots, y_n$ of the linear differential equation of order n i.e.,*

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_n y = 0$$

where $P_1, P_2, \dots, P_n$ are functions of x or constants form a system of linearly independent integrals (or solutions) is that the determinant (i.e., Wronskian of $y_1, y_2, \dots, y_n$)

$$W(x) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ y_1'' & y_2'' & \dots & y_n'' \\ \vdots & \vdots & \dots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

does not vanish identically

[Delhi 2000, 01, 02, 04, 07; Kolkata 2000, 05, Kurukshetra 2001; Lucknow 2000]

Proof: Necessary Condition : If $y_1, y_2, \dots, y_n$ are not linearly independent then there exists constants $a_1, a_2, \dots, a_n$ not all equal to zero (*i.e.*, at least one differential from zero), such that

$$a_1 y_1 + a_2 y_2 + \dots, a_n y_n = 0 \quad \dots(1)$$

If $a_1 \neq 0$, then the determinant

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1 & y_2 & \dots & \dots & y_n \\ y_1' & y_2' & \dots & \dots & y_n' \\ \dots & \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & \dots & y_n^{(n-1)} \end{vmatrix} \\ &= \frac{1}{a_1} \begin{vmatrix} a_1 y_1 & + & a_2 y_2 & + \dots + & a_n y_n & y_2 & \dots & \dots & y_n \\ a_1 y_1' & + & a_1 y_2' & + \dots + & a_1 y_n' & y_2' & \dots & \dots & y_n' \\ \dots & & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_1 y_1^{(n-1)} & + & a_2 y_2^{(n-1)} & + \dots + & a_n y_n^{(n-1)} & y_2^{(n-1)} & \dots & \dots & y_n^{(n-1)} \end{vmatrix} \\ &= \frac{1}{a_1} \begin{vmatrix} 0 & y_2 & \dots & \dots & y_n \\ 0 & y_2' & \dots & \dots & y_n' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & y_2^{(n-1)} & \dots & \dots & y_n^{(n-1)} \end{vmatrix} = 0 \end{aligned}$$

(Using (1) and its $(n-1)$ differentials)

i.e., if $y_1, y_2, \dots, y_n$ are not linearly independent then the determinant $W(x)$ will identically be equal to zero.

$\Rightarrow$ if $y_1, y_2, \dots, y_n$ are linearly independent then the determinant $W(x)$ does not vanish identically.

1. Sufficient Condition : *i.e.*, if $W(x) \neq 0$, then the solutions $y_1, y_2, \dots, y_n$ are linearly independent.

For this we shall prove that if $W(x) = 0$, then the solutions $y_1, y_2, \dots, y_n$ are linearly dependent *i.e.*, there will be a linear relation of the form (1), between them.

If $W(x) = 0$, the determinant must be reducible to zero where in all the elements of one column (or row) are zero. Therefore, there exists certain multipliers $\lambda_1, \lambda_2, \dots, \lambda_n$ such that

$$\left[\begin{array}{cccccc} \lambda_1 y_1 & + & \lambda_2 y_2 & + & \dots & \dots & + \lambda_n y_n = 0 \\ \lambda_1 y_1' & + & \lambda_2 y_2' & + & \dots & \dots & + \lambda_n y_n' = 0 \\ \dots & & \dots & & \dots & \dots & \dots \\ \dots & & \dots & & \dots & \dots & \dots \\ \lambda_1 y_1^{(n-1)} & + & \lambda_2 y_2^{(n-1)} & + & \dots & \dots & + \lambda_n y_n^{(n-1)} = 0 \end{array} \right] \quad \dots(2)$$

Differentiating first $(n-1)$, equations in (2) and subtracting each from the next, we get

$$\left[\begin{array}{cccccc} \lambda'_1 y_1 & + & \lambda'_2 y_2 & + & \dots & \dots & + \lambda'_n y_n = 0 \\ \lambda'_1 y'_1 & + & \lambda'_2 y'_2 & + & \dots & \dots & + \lambda'_n y'_n = 0 \\ \dots & & \dots & & \dots & \dots & \dots \\ \dots & & \dots & & \dots & \dots & \dots \\ \lambda'_1 y_1^{(n-2)} & + & \lambda'_2 y_2^{(n-2)} & + & \dots & \dots & + \lambda'_n y_n^{(n-2)} = 0 \end{array} \right] \quad \dots(3)$$

Which are $(n-1)$ equations in n -unknowns $\lambda'_1, \lambda'_2, \dots, \lambda'_n$.

If one of the determinants formed by omitting one column, say r -th column, of the $(n-1) \times n$ matrix of coefficients of $\lambda'_1, \lambda'_2, \dots, \lambda'_n$

$$i.e., \quad \begin{vmatrix} y_1 & y_2 & \dots & \dots & y_n \\ y'_1 & y'_2 & \dots & \dots & y'_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \dots & \dots & y_n^{(n-2)} \end{vmatrix}$$

vanishes, then there exist a relation

$$C_1 y_1 + C_2 y_2 + \dots + C_{r-1} y_{r-1} + C_{r+1} y_{r+1} + \dots + C_n y_n = 0 \quad \dots(4)$$

But if no one of these determinants vanishes, then from equations in (3) and first $(n-1)$ equations in (2), we have

$$\frac{\lambda'_1}{\lambda_1} = \frac{\lambda'_2}{\lambda_2} = \dots = \frac{\lambda'_n}{\lambda_n} = \mu \text{ (say)}$$

$$\text{Now} \quad \frac{\lambda'_1}{\lambda_1} = \mu \Rightarrow \log \lambda_1 = \int \mu \, dx + \log a_1 \text{ (on integration)}$$

$$\Rightarrow \quad \lambda_1 = a_1 e^{\int \mu \, dx}$$

$$\text{Similarly,} \quad \lambda_2 = a_2 e^{\int \mu \, dx}, \dots, \lambda_n = a_n e^{\int \mu \, dx}$$

where $a_1, a_2, \dots, a_n$ are constants.

Substituting these in the first equation in (2), we get

$$a_1 y_1 + a_2 y_2 + \dots + a_n y_n = 0 \quad \dots(5)$$

Hence if the condition holds (*i.e.*, relation exists) for $(n-1)$ functions in (4), then it also holds (*i.e.*, relation exists) for n functions in (5). Also as in 1.8 it can be shown that the condition holds for $n=2$ *i.e.*, the relation exists between two functions and so between 3 functions and so on for any number.

Thus if $W(x) = 0$, then a linear relation exists between the solutions $y_1, y_2, \dots, y_n$ i.e., they are linearly dependent. So if $W(x) \neq 0$, then $y_1, y_2, \dots, y_n$ are linearly independent.

Hence the necessary and sufficient condition that $y_1, y_2, \dots, y_n$ form a system of linearly independent integrals (solutions) of linear differential equation of order n is that $W(x) \neq 0$.

1.12 Fundamental set of Solution

A set of linearly independent solutions of a linear differential equation is called its **fundamental set of solutions**.

Solved Examples

Example 1: What are the order and degree of the differential equation

$$\frac{d^2}{dx^2} \left(\frac{d^2 y}{dx^2} \right)^{-7/2} = 0$$

Solution: The given equation can be written as

$$\frac{d}{dx} \left[\frac{d}{dx} \left(\frac{d^2 y}{dx^2} \right)^{-7/2} \right] = 0$$

$$\Rightarrow \frac{d}{dx} \left[-\frac{7}{2} \left(\frac{d^2 y}{dx^2} \right)^{-9/2} \cdot \frac{d^3 y}{dx^3} \right] = 0$$

$$\Rightarrow -\frac{7}{2} \left[\left(\frac{d^2 y}{dx^2} \right)^{-9/2} \frac{d^4 y}{dx^4} - \frac{9}{2} \left(\frac{d^2 y}{dx^2} \right)^{-11/2} \left(\frac{d^3 y}{dx^3} \right)^2 \right] = 0$$

$$\Rightarrow \left(\frac{d^2 y}{dx^2} \right) \cdot \frac{d^4 y}{dx^4} - \frac{9}{2} \left(\frac{d^3 y}{dx^3} \right)^2 = 0$$

Which is a differential equation of order 4 and degree 1.

Example 2: Show that $y_1(x) = \sin x$ and $y_2(x) = \sin x - \cos x$ are linearly independent solutions of $y'' + y = 0$ [Meerut 2003]

Determine the constants C_1 and C_2 so that the solution

$$\sin x + 3 \cos x = C_1 y_1(x) + C_2 y_2(x)$$

[Delhi 2002]

Solution: $y_1(x) = \sin x \Rightarrow y_1''(x) = -\sin x = -y_1(x) \Rightarrow y_1''(x) + y_1(x) = 0$

$$y_2(x) = \sin x - \cos x \Rightarrow y_2''(x) = -\sin x + \cos x = -y_2(x) \Rightarrow y_2''(x) + y_2(x) = 0$$

i.e., both $y_1(x)$ and $y_2(x)$ are solutions of $y'' + y = 0$

$$\text{Now } W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin x & \sin x - \cos x \\ \cos x & \cos x + \sin x \end{vmatrix}$$

$$= \sin x (\cos x + \sin x) - \cos x (\sin x - \cos x) = 1 \neq 0$$

Hence $y_1(x)$ and $y_2(x)$ are linearly independent solutions of $y'' + y = 0$

II part : Given $\sin x + 3 \cos x = C_1 y_1(x) + C_2 y_2(x) = C_1 \sin x + C_2 (\sin x - \cos x)$

$$= (C_1 + C_2) \sin x - C_2 \cos x$$

$$\Rightarrow C_1 + C_2 = 1 \text{ and } -C_2 = 3 \Rightarrow C_1 = 4 \text{ and } C_2 = -3$$

Example 3: Show that the Wronskian of the functions x^a, x^b, x^c ($x > 0$) is equal to $(a-b)(b-c)(c-a)x^{a+b+c-3}$. Also prove that the functions are linearly independent if a, b, c are distinct.

Solution: Let $y_1 = x^a, y_2 = x^b, y_3 = x^c$.

$$\therefore W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} x^a & x^b & x^c \\ ax^{a-1} & bx^{b-1} & cx^{c-1} \\ a(a-1)x^{a-2} & b(b-1)x^{b-2} & c(c-1)x^{c-2} \end{vmatrix}$$

$$= x^{a+b+c} \begin{vmatrix} 1 & 1 & 1 \\ ax^{-1} & bx^{-1} & cx^{-1} \\ a(a-1)x^{-2} & b(b-1)x^{-2} & c(c-1)x^{-2} \end{vmatrix}$$

$$= x^{a+b+c-3} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a(a-1) & b(b-1) & c(c-1) \end{vmatrix}$$

$$= x^{a+b+c-3} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = x^{a+b+c-3} \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$$\text{By } R_3 \rightarrow R_3 + R_2$$

$$\text{By } C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1$$

$$= x^{a+b+c-3} \{(b-a)(c^2-a^2) - (c-a)(b^2-a^2)\}$$

$$= x^{a+b+c-3}(b-a)(c-a)(c+a-b-a)$$

$$= (a-b)(b-c)(c-a)x^{a+b+c-3}$$

II Part : The functions will be linearly independent if $W(x) \neq 0 \Rightarrow a, b, c$ are distinct.

Example 4: Use Wronskian to show that the functions x, x^2, x^3 are independent. Determine the differential equation with these as independent solution.

[Meerut 2002, 02 (BP), 03, 10 (Sem. I), 11 (Sem. I)]

Solution: Let $y_1 = x, y_2 = x^2, y_3 = x^3$

Wronskian of these functions is

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$= x(12x^2 - 6x^2) - 1(6x^3 - 2x^3) = 2x^3 \neq 0$$

$\Rightarrow$ The given functions are independent.

To determine the diff. equation.

$$\text{Let } y = C_1 y_1 + C_2 y_2 + C_3 y_3 = C_1 x + C_2 x^2 + C_3 x^3 \quad \dots(1)$$

where C_1, C_2, C_3 are arbitrary constants.

Differentiating (1), 3 times, we get

$$\frac{dy}{dx} = C_1 + 2C_2 x + 3C_3 x^2 \quad \dots(2)$$

$$\frac{d^2 y}{dx^2} = 2C_2 + 6C_3 x \quad \dots(3)$$

$$\text{and } \frac{d^3 y}{dx^3} = 6C_3 \quad \dots(4)$$

Required diff. equation is obtained by eliminating C_1, C_2, C_3 , between (1) to (4).

$$\text{From (4), } C_3 = \frac{1}{6} \frac{d^3 y}{dx^3},$$

$$\therefore \text{ From (3) } C_2 = \frac{1}{2} \frac{d^2 y}{dx^2} - \frac{1}{2} x \frac{d^3 y}{dx^3}$$

Now subtracting x times of (2) from (1), we get

$$\begin{aligned}
 y - x \frac{dy}{dx} &= -C_2 x^2 - 2C_3 x^3 \\
 &= -\left(\frac{1}{2} \frac{d^2 y}{dx^2} - \frac{1}{2} x \frac{d^3 y}{dx^3} \right) x^2 - 2 \left(\frac{1}{6} \frac{d^3 y}{dx^3} \right) x^3
 \end{aligned}$$

or

$$x^3 \frac{d^3 y}{dx^3} - 3x^2 \frac{d^2 y}{dx^2} + 6x \frac{dy}{dx} - 6y = 0$$

Which is the required differential equation.

Example 5: Show that x^4 and $x^3|x|$ are linearly independent on $[-1, 1]$ but are linearly dependent on $[-1, 0]$ and $[0, 1]$.

Solution: Let there exist constants C_1 and C_2 such that

$$C_1 x^4 + C_2 x^3 |x| = 0 \quad \dots(1)$$

$$\Rightarrow C_1 x^4 + C_2 x^4 = 0 \quad \text{for } x \geq 0 \quad \dots(2)$$

and $C_1 x^4 - C_2 x^4 = 0 \quad \text{for } x < 0 \quad \dots(3)$

$\therefore$ In order that (1) hold, (2) and (3) both must hold for $x \in [-1, 1]$.

This is possible only when $C_1 + C_2 = 0$ and $C_1 - C_2 = 0$

$$\Rightarrow C_1 = 0 \text{ and } C_2 = 0$$

$$\Rightarrow x^4 \text{ and } x^3 |x| \text{ are L.I. in } [-1, 1].$$

In the interval, $[-1, 0]$, $-1 \leq x \leq 0$ i.e., $|x| = -x$

$$\therefore \text{ equation (1)} \Rightarrow \text{only } C_1 x^4 - C_2 x^4 = 0 \Rightarrow C_1 - C_2 = 0$$

$$\Rightarrow C_2 = C_1 \Rightarrow \text{If } C_1 \neq 0 \text{ then } C_2 \neq 0$$

$$\Rightarrow x^4 \text{ and } x^3 |x| \text{ are L.D. in } [-1, 0].$$

Also in the interval $[0, 1]$, $0 \leq x \leq 1$ i.e., $|x| = x$

$$\therefore \text{ equation (1)} \Rightarrow \text{only } C_1 x^4 + C_2 x^4 = 0 \Rightarrow C_2 = -C_1$$

$$\Rightarrow \text{If } C_1 \neq 0, \text{ then } C_2 \neq 0$$

$$\Rightarrow x^4 \text{ and } x^3 |x| \text{ are L.D. in } [0, 1].$$

Example 6: Show that $\sin 3x$, $\cos 3x$ and $\sin 3x + \cos 3x$ are solutions of differential equation $y'' + 9y = 0$. Are these solutions linearly dependent? Use the idea of Wronskian.

[Delhi 2008]

Solution: Let $y_1 = \sin 3x$, $y_2 = \cos 3x$ and $y_3 = \sin 3x + \cos 3x$

Now $y_1' = 3 \cos 3x$ and $y_1'' = -9 \sin 3x = -9y_1 \Rightarrow y_1'' + 9y_1 = 0$

$\Rightarrow y_1$ satisfy $y'' + 9y = 0$... (1)

$\Rightarrow y_1$ is solution of (1)

Similarly, it can be shown that y_2 and y_3 are also solutions of (1).

Since equation (1) is of order 2. So these three solutions cannot be L.I. i.e., these are L.D. solutions.

Also by the idea of Wronskian.

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} \sin 3x & \cos 3x & \sin 3x + \cos 3x \\ 3 \cos 3x & -3 \sin 3x & 3(\cos 3x - \sin 3x) \\ -9 \sin 3x & -9 \cos 3x & -9(\sin 3x + \cos 3x) \end{vmatrix}$$

$$= \begin{vmatrix} \sin 3x & \cos 3x & 0 \\ 3 \cos 3x & -3 \sin 3x & 0 \\ -9 \sin 3x & -9 \cos 3x & 0 \end{vmatrix} = 0 \quad \text{By } C_3 \rightarrow C_3 - C_1 - C_2$$

$\Rightarrow$ The solutions y_1, y_2, y_3 are L.D.

Example 7: Show that e^{2x} and e^{3x} are linearly independent solutions of $y'' - 5y' + 6y = 0$. Find the solutions $y(x)$ with the property that $y(0) = 0$ and $y'(0) = 1$.

[Delhi 2006]

Solution: Given equation is $y'' - 5y' + 6y = 0$

... (1)

Let $y_1 = e^{2x}$ and $y_2 = e^{3x}$

Now $y_1' = 2e^{2x} = 2y_1$ and $y_1'' = 4e^{2x} = 4y_1$

$\therefore y_1'' - 5y_1' + 6y_1 = (4 - 5 \cdot 2 + 6)y_1 = 0$

$\Rightarrow y_1$ is the solution of (1).

Similarly y_2 is also a solution of (1).

Now
$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x} = e^{5x} \neq 0$$

$\Rightarrow y_1(x)$ and $y_2(x)$ are L.I. solutions of (1)

$\therefore$ The general solution of (1) can be given as

$$y(x) = C_1 y_1(x) + C_2 y_2(x) = C_1 e^{2x} + C_2 e^{3x} \quad \dots(2)$$

where C_1 and C_2 are arbitrary constants.

$$(2) \Rightarrow y'(x) = 2C_1 e^{2x} + 3C_2 e^{3x}$$

$$\therefore y(0) = 0 \Rightarrow C_1 + C_2 = 0$$

$$\text{and } y'(0) = 1 \Rightarrow 2C_1 + 3C_2 = 1$$

Solving we get $C_1 = -1, C_2 = 1$

$\therefore$ Solution of (1), satisfying the given conditions is

$$y(x) = e^{3x} - e^{2x}$$

Example 8: Show that $\phi_1(x) = e^{2x}$, $\phi_2(x) = x e^{2x}$ and $\phi_3(x) = x^2 e^{2x}$ are linearly independent solutions of $y''' - 6y'' + 12y' - 8y = 0$ on the interval $0 \leq x \leq 1$.

Solution: Given equation is

$$y''' - 6y'' + 12y' - 8y = 0 \quad \dots(1)$$

$$\text{Also } \phi_1(x) = e^{2x}, \phi_2(x) = x e^{2x} \text{ and } \phi_3(x) = x^2 e^{2x}$$

$$\therefore \phi_1'(x) = 2e^{2x}, \phi_1''(x) = 4e^{2x} \text{ and } \phi_1'''(x) = 8e^{2x}$$

$$\begin{aligned} \text{Now } \phi_1'''(x) - 6\phi_1''(x) + 12\phi_1'(x) - 8\phi_1(x) \\ = (8 - 6 \cdot 4 + 12 \cdot 2 - 8) e^{2x} = 0 \end{aligned}$$

$\Rightarrow \phi_1(x)$ is a solution of (1).

Similarly, $\phi_2(x)$ and $\phi_3(x)$ are also solutions of (1)

$$\begin{aligned} \text{Now } W(\phi_1, \phi_2, \phi_3)(x) &= \begin{vmatrix} \phi_1(x) & \phi_2(x) & \phi_3(x) \\ \phi_1'(x) & \phi_2'(x) & \phi_3'(x) \\ \phi_1''(x) & \phi_2''(x) & \phi_3''(x) \end{vmatrix} \\ &= \begin{vmatrix} e^{2x} & x e^{2x} & x^2 e^{2x} \\ 2e^{2x} & (1+2x)e^{2x} & (2x+2x^2)e^{2x} \\ 4e^{2x} & (4+4x)e^{2x} & (2+8x+4x^2)e^{2x} \end{vmatrix} \end{aligned}$$

$$\begin{aligned}
 &= e^{2x} \cdot e^{2x} \cdot e^{2x} \begin{vmatrix} 1 & x & x^2 \\ 2 & 1+2x & 2x+2x^2 \\ 4 & 4+4x & 2+8x+4x^2 \end{vmatrix} \\
 &= e^{6x} \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 4 & 2+8x \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 2x \\ 4 & 2+8x \end{vmatrix} = 2e^{6x} \neq 0
 \end{aligned}$$

By $R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - 4R_1$

Hence, $\phi_1(x), \phi_2(x)$ and $\phi_3(x)$ are L.I. solutions of (1).

Example 9: Show that the functions $1, x, x^2, x^3, \dots, x^n, -\infty < x < \infty$ are linearly independent.

Solution: Let there exist constants $C_1, C_2, \dots, C_{n+1}$ such that

$$C_1 \cdot 1 + C_2 x + C_3 x^2 + C_4 x^3 + \dots + C_{n+1} x^n = 0$$

Equating coefficients of like powers of x from the two sides, we get

$$C_1 = 0 = C_2 = C_3 = \dots = C_{n+1}$$

$\Rightarrow$ The functions are L.I.

2nd Method By Wronskian Idea :

Now Wronskian of given functions say $y_1, y_2, y_3, \dots, y_n$

$$\begin{aligned}
 &= \begin{vmatrix} y_1 & y_2 & y_3 & \dots & y_n \\ y_1' & y_2' & y_3' & \dots & y_n' \\ y_1'' & y_2'' & y_3'' & \dots & y_n'' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 & \dots & x^n \\ 0 & 1! & 2x & \dots & nx^{n-1} \\ 0 & 0 & 2! & \dots & n(n-1)x^{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & n! \cdot x^0 \end{vmatrix} \\
 &= 1 \cdot 1! \cdot 2! \cdot 3! \cdot \dots \cdot n!
 \end{aligned}$$

(Since all elements below the leading diagonal are zero)

Hence the functions are L.I.

Example 10: If the functions $p(x)$ and $q(x)$ are continuous on $\alpha < x < \beta$ and if the function $y_1(x)$ and $y_2(x)$ are L.I. solutions of

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0 \quad \dots(1)$$

then show that the Wronskian $W(y_1, y_2)$ is non-vanishing on $\alpha < x < \beta$

[Madras 2000, 01, 04, 06; Lucknow 2000, 01, 02, 06; Kolkata 2005; Meerut 2005 (B.P.)]

Solution: If possible let $y_1(x)$ and $y_2(x)$ be L.D.

$\Rightarrow$ there exists scalars C_1, C_2 (not both zero) such that

$$C_1 y_1 + C_2 y_2 = 0 \quad \dots(2)$$

$$\text{Diff.} \quad C_1 y_1' + C_2 y_2' = 0 \quad \dots(3)$$

Eliminating C_1, C_2 between (2) and (3), we get

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$$

$\Rightarrow$ If y_1 and y_2 are L.I. then $W(y_1, y_2)$ must not vanish.

Alternative Method : If possible let the Wronskian $W(y_1, y_2)$ be zero

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0 \Rightarrow y_1 y_2' = y_1' y_2 \Rightarrow \frac{y_1}{y_2} = \frac{y_1'}{y_2'} = k \text{ (say)}$$

$$\Rightarrow y_1 = k y_2 \Rightarrow y_1 - k y_2 = 0$$

i.e., there exists scalars $C_1 = 1$ and $C_2 = -k$ such that

$$C_1 y_1 + C_2 y_2 = 0$$

Obviously $C_1 = 1 \neq 0$, so y_1, y_2 are L.D.

Which is a contradiction as the functions y_1 and y_2 are given L.I.

Hence $W(y_1, y_2) \neq 0$

Example 11: If $f_1, f_2, \dots, f_m$ is a set of m solutions of the n -th order homogeneous linear differential equation

$$a_0(t) \frac{d^n y}{dt^n} + a_1(t) \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_n(t) y = 0 \quad (m > n) \quad \dots(1)$$

on $a \leq t \leq b$, prove that $f_1, f_2, \dots, f_m$ are L.D. on $a \leq t \leq b$.

[Kurukshetra 2002, 04; GNDU Amritsar 2002, 07]

OR The above differential equation does not have more than n L.I. solutions.

Solution: Since $f_1, f_2, \dots, f_m$ ($m > n$) are the solutions of differential equation (1)

$$\therefore a_0(t) \frac{d^n f_r}{dt^n} + a_1(t) \frac{d^{n-1} f_r}{dt^{n-1}} + \dots + a_n(t) f_r = 0 \quad \dots(2)$$

for $r = 1, 2, 3, \dots, m$

$$\text{Now} \quad W(f_1, f_2, \dots, f_m) = \begin{vmatrix} f_1 & f_2 & \dots & \dots & f_m \\ f_1' & f_2' & \dots & \dots & f_m' \\ f_1'' & f_2'' & \dots & \dots & f_m'' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_1^{(n)} & f_2^{(n)} & \dots & \dots & f_m^{(n)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_1^{(m-1)} & f_2^{(m-1)} & \dots & \dots & f_m^{(m-1)} \end{vmatrix}$$

$$\because m > n \Rightarrow m - 1 \geq n$$

$$\begin{aligned}
 &= \frac{1}{a_n(t)} \begin{vmatrix} a_n(t)f_1 & a_n(t)f_2 & \dots & \dots & a_n(t)f_m \\ f'_1 & f'_2 & \dots & \dots & f'_m \\ f''_1 & f''_2 & \dots & \dots & f''_m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_1^{(n)} & f_2^{(n)} & \dots & \dots & f_m^{(n)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_1^{(m-1)} & f_2^{(m-1)} & \dots & \dots & f_m^{(m-1)} \end{vmatrix} \\
 &= \frac{1}{a_n(t)} \begin{vmatrix} 0 & 0 & \dots & \dots & 0 \\ f'_1 & f'_2 & \dots & \dots & f'_m \\ f''_1 & f''_2 & \dots & \dots & f''_m \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_1^{(n)} & f_2^{(n)} & \dots & \dots & f_m^{(n)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_1^{(m-1)} & f_2^{(m-1)} & \dots & \dots & f_m^{(m-1)} \end{vmatrix} \quad \text{Using (2)} \\
 &= 0
 \end{aligned}$$

$\Rightarrow f_1, f_2, \dots, f_m$ are L.D.

Exercise

1. Prove that the Wronskian of the functions $e^{m_1 x}, e^{m_2 x}, e^{m_3 x}$ is equal to

$$(m_1 - m_2)(m_2 - m_3)(m_3 - m_1) e^{(m_1 + m_2 + m_3)x}.$$

Also show that the functions are L.I. if m_1, m_2, m_3 are distinct.

2. Test the linear independence of the following sets of functions :

(i) $\sin x, \cos x, \sin 2x$

(ii) $\sin bx, e^x, e^{-x}$

(iii) $e^x, x e^x, \sinh x$

(iv) $\sin x, \sin 3x, \sin^3 x$

(v) $x^2 - x + 1, x^2 - 1, 3x^2 - x - 1$

[Kurukshetra 2002]

(vi) $x^2, 1 + x, 1 + 2x$

3. Evaluate the Wronskian of the functions $e^x, x e^x$. Hence conclude whether or not they are linearly independent. If they are independent, set up the differential equation having them as its independent solutions. [Lucknow 2001]

4. Find the interval for x , so that the functions x^2, e^x, e^{-x} are linearly independent. Hence, show that these functions form a fundamental system of solutions of the

diff. equation $\frac{d^3 y}{dx^3} - 2 \frac{d^2 y}{dx^2} - \frac{dy}{dx} + 2y = 0$

Also find its general solution.

5. Define Wronskian. Prove that the functions $1, x, x^2$ are linearly independent. Hence form the differential equation whose roots are $1, x, x^2$. [Meerut 2002]
6. Show that linearly independent solutions of $y'' - 3y' + 2y = 0$ are e^x and e^{2x} . [Delhi Maths (Hons.) 2008]
7. Show that x^2 and $x|x|$ are linearly independent on $-\infty < x < \infty$.
8. Show that $\sin x, \cos x$ and $\sin x - \cos x$ are solutions of the differential equation $y'' + y = 0$ where $y' = \frac{dy}{dx}$. Prove that these solutions are linearly dependent. (Use the idea of Wronskian). [Delhi 2007]
9. Show that the Wronskian of the functions $x^2, x^2 \log x$ is non-zero. Can these functions be independent solutions of an ordinary differential equation, if so determine this equation. [Meerut 2001 (BP)]
10. Show that e^x, e^{-2x}, e^{2x} are linearly independent solutions of $\frac{d^3 y}{dx^3} - \frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 4y = 0$. Hence or otherwise solve the given equation.
11. Show that $y_1 = e^x \cos x, y_2 = e^x \sin x$ are linearly independent solutions of $y'' - 2y' + 2y = 0$. What is the general solution? Find the solution $y(x)$ with the property $y(0) = 2, y'(0) = 3$. [Delhi Maths (Hons.) 2002; General 2006]
12. Using the idea of Wronskian, show that $e^x \cos x$ and $e^x \sin x$ are linearly independent solutions of $y'' - 2y' + 2y = 0$. Find the solution with the property that $y(0) = 1$ and $y'(0) = 2$. [Delhi Maths (Hons.) 2004]
13. Show that e^x, e^{-x} and e^{2x} are L.I. solutions of $\frac{d^3 y}{dx^3} - 2 \frac{d^2 y}{dx^2} - \frac{dy}{dx} + 2y = 0$ on every real interval.
14. Show that $e^x, e^x(x-1)$ and $2e^x - e^{2x}$ are linearly independent solutions of $\frac{d^3 y}{dx^3} - 4 \frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} - 2y = 0$
15. Show that the general solution of a differential equation of first order has only one arbitrary constant. [Hint : See theorem 2, of 1.6]
16. Show that the general solution of a differential equation of second order has two arbitrary constants. [Hint : See theorem 2, of 1.6]
17. If $y = y_1$ and $y = y_2$ are solutions of a differential equation of second order $\frac{d^2 y}{dx^2} + p_1(x) \frac{dy}{dx} + p_2(x)y = 0$ then $y = C_1 y_1 + C_2 y_2$ where C_1 and C_2 are arbitrary constants, is also the solution of the equation. [Hint : See theorem 1, of 1.6]

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter (a), (b), (c) or (d).

- The order of the differential equation $\frac{d^2 y}{dx^2} + k \frac{dy}{dx} + y = 0$ is :
 (a) 1 (b) 2 (c) 0 (d) None of these
- The order of the differential equation $\frac{d^3 y}{dx^3} + 2 \left(\frac{d^2 y}{dx^2} \right)^2 + 3 \left(\frac{dy}{dx} \right)^3 + y = \log x$ is :
 (a) 0 (b) 1 (c) 2 (d) 3
- The degree of the differential equation : $\frac{d^2 y}{dx^2} + 2 \left(\frac{dy}{dx} \right)^3 + y = \sin x$ is :
 (a) 0 (b) 1 (c) 2 (d) 3
- The degree of the differential equation $\frac{d^3 y}{dx^3} + \left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\}^{5/2} = 0$ is :
 (a) 1 (b) 2 (c) 10 (d) None of these
- The order and degree respectively of the differential equation $\frac{d^2}{dx^2} \left(\frac{d^2 y}{dx^2} \right)^{-3/2} = 0$ are
 (a) 2, 2 (b) 3, 1 (c) 4, 1 (d) 3, 2
- The degree of the differential equation $\left(\frac{d^2 y}{dx^2} \right)^{2/3} + \left(\frac{d^3 y}{dx^3} \right)^{3/2} = 0$ is :
 (a) $3/2$ (b) 6 (c) 1 (d) 9
- Which one of the following equations has the same order and degree :
 (a) $\frac{d^4 y}{dx^4} + 8 \left(\frac{dy}{dx} \right)^4 + 5y = e^x$ (b) $5 \left(\frac{d^3 y}{dx^3} \right)^4 + 8 \left(\frac{dy}{dx} + 1 \right)^2 + 5y = x^3$
 (c) $\left\{ 1 + \left(\frac{dy}{dx} \right)^3 \right\}^{2/3} = 4 \left(\frac{d^3 y}{dx^3} \right)$ (d) $y = x^2 \frac{dy}{dx} + \left\{ \left(\frac{dy}{dx} \right)^2 + 1 \right\}^{1/2}$ [IAS (pre) 2005]
- The Wronskian of the functions $1 + x$, $1 + 2x$ and x^2 is :
 (a) 0 (b) 1 (c) 2 (d) $2x$

9. The Wronskian of the functions $e^{ax} \sin bx$ and $e^{ax} \cos bx$, ($b \neq 0$) is :
 (a) 0 (b) be^{ax} (c) $2be^{ax}$ (d) be^{2ax}
10. If $y_1 = x^m$ and $y_2 = x^n$, where y_1 and y_2 are solutions of a second order differential equation with constant coefficients. Then the condition under which $y = C_1 y_1 + C_2 y_2$ is the general solution of the same equation is :
 (a) $m + n = 1$ only (b) $m \neq n$
 (c) $m = n$ (d) $m = -n$ only [IAS 2008]
- [Hint : $y = C_1 y_1 + C_2 y_2$ will be general solutions if y_1 and y_2 are L.I.
 $\therefore W(x) = e^{m+n-1} (n-m) \neq 0 \Rightarrow m \neq n$]
11. Two linearly independent solutions of the differential equation
 $4 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 5y = 0$ are
 (a) $e^{-x/2} \cos x, e^{-x/2} \sin x$ (b) $e^{x/2} \cos x, e^{x/2} \sin x$
 (c) $e^{x/2} \cos x, e^{-x/2} \sin x$ (d) $e^{-x/2} \cos x, e^{x/2} \sin x$
12. Let $y = \phi(x) = 1$ and $y = \psi(x)$ be solutions of $y'' - 2xy' + (\sin x^2) y = 0$, such that $\phi(0) = 1, \phi'(0) = 1$ and $\psi(0) = 1, \psi'(0) = 2$. The value of Wronskian $W(\phi, \psi)$ at $x = 0$ is:
 (a) 0 (b) 1 (c) e (d) e^2 [Gate 2004]

Fill in the Blank

Fill in the blank's "...." so that the following statements are complete and correct.

- The differential equation $\frac{d^2 y}{dx^2} + 3y \cdot \frac{dy}{dx} + y = \sin x$ is
- The differential equation $2 \frac{d^3 y}{dx^3} + x \frac{d^2 y}{dx^2} + e^x \frac{dy}{dx} = \log x$ is
- For the differential equation $y = x \frac{dy}{dx} + k \left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\}^{3/2}$ the order and degree are and respectively.
- $y = \cos(mx + b)$ is a solution of the differential equation $\frac{d^2 y}{dx^2} + a^2 y = 0$ if $a^2 = \dots\dots\dots$
- The Wronskian of the functions $\sin x, \cos x$ is
- The functions e^{ax}, e^{bx}, e^{cx} , (a, b, c distinct) is linearly for all x .
- The functions $\cos at, \sin at, \sin(at + b)$ are linearly
- A set of linearly independent solutions of a differential equation is called a set of solutions.

True / False

Write 'T' for true and 'F' for false statement.

1. $y = \frac{A}{x} + B$ is a solution of $\frac{d^2 y}{dx^2} + \frac{2}{x} \cdot \frac{dy}{dx} = 0$
2. The differential equation $y = x \frac{dy}{dx} + \frac{k}{(dy/dx)}$ is linear.
3. The Wronskian of the function $e^{2x} \cos 3x$ and $e^{2x} \sin 3x$ is $3e^{4x}$.
4. The number of linearly independent solutions of a differential equation of order 3 is less than 3.
5. The complete solution of a differential equation of n th order contains at least n arbitrary independent constants.

Answers

Exercise

2.	(i) L.I. (ii) L.D. (iii) L.I. (iv) L.D. (v) L.D. (vi) L.I.	3.	$W(x) = e^{2x}$, L.I., $\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = 0$
4.	$R - \{-\sqrt{2}, \sqrt{2}\}$, $y = C_1 x^2 + C_2 e^x + C_3 e^{-x}$	5.	$\frac{d^3 y}{dx^3} = 0$
9.	Yes, $x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 4y = 0$	10.	$y = C_1 e^x + C_2 e^{-2x} + C_3 e^{2x}$
11.	$y = C_1 e^x \cos x + C_2 e^x \sin x$, $y = e^x (2 \cos x - 5 \sin x)$	12.	$y = e^x (\sin x + \cos x)$

Objective Type Questions

Multiple Choice Questions

1.	(b)	2.	(d)	3.	(b)	4.	(b)
5.	(c)	6.	(d)	7.	(c)	8.	(c)
9.	(d)	10.	(b)	11.	(a)	12.	(b)

Fill in the Blank

1.	non-linear	2.	linear	3.	1, 6	4.	m^2
5.	-1	6.	Independent	7.	Dependent	8.	Fundamental

True / False

1.	True	2.	False	3.	True	4.	False
5.	False						

Unit - 1

Chapter

2

Qualitative Properties of Solutions (Oscillation, Adjoint and Sturm Theory)

2.1 Introduction

In this chapter we shall study the qualitative properties of solutions of linear differential equations. This chapter is divided into three sections namely : Oscillatory theory, the adjoint theory and the Sturm theory. Some definitions and theorems with proofs related to these topics are discussed here to facilitate readers to study some advanced topics in science, engineering and technology.

SECTION 1 : OSCILLATORY THEORY

2.2 Oscillatory Equation

Oscillatory Equation (Def.) : A differential equation is called an **oscillatory equation** if all non-trivial solutions of this equation have an infinite number of zeroes.”

For Example: The differential equation $u'' + a^2u = 0$ where a is a real constant is an **oscillatory equation** as all its non-trivial solutions i.e., $\sin at$ and $\cos at$ have infinite number of zeroes.

$\sin at$, have infinite number of zeroes $t = \frac{n\pi}{a}$, $n \in I$

(i.e., $t = 0, \pm \pi/a, \pm 2\pi/a, \pm 3\pi/a, \dots$)

and $\cos at$, have infinite number of zeroes, $t = \frac{(2n+1)\pi}{2a}$, $n \in I$

(i.e., $t = \pm \pi/2a, \pm 3\pi/2a, \pm 5\pi/2a, \dots$)

2.3 Non-Oscillatory Equation

“A differential equation is called **non-oscillatory equation** if some of its non-trivial solutions have infinite number of zeroes while some non-trivial solutions does not have infinite number of zeroes or A differential equation is called **non-oscillatory equation** if it is not oscillatory.”

For Example: The differential equation $u''' - u'' + u' - u = 0$ is non-oscillatory equation. As all non-trivial solutions of this equation are e^t , $\sin t$ and $\cos t$, and out of these solutions $\sin t$ and $\cos t$ have infinite number of zeroes while the solution e^t does not have infinite number of zeroes.

2.4 Oscillatory and Non-Oscillatory Solutions

“For any differential equation, the non-trivial solutions which have an infinite number of zeroes are called **oscillatory solutions**. And other non-trivial solutions of the equation which does not have an infinite number of zeroes are called **non-oscillatory solutions**.”

For Example: All the non-trivial solutions of the differential equation $u''' - u'' + u' - u = 0$ are e^t , $\sin t$ and $\cos t$.

The solutions $\sin t$ and $\cos t$ are oscillatory solutions of the equation as they both have infinite number of zeroes, while the solution e^t is non-oscillatory solution as it does not have infinite number of zeroes.

Hence all the solutions of an oscillatory equation are oscillatory solutions while for a non-oscillatory equation some solutions are oscillatory and some non-oscillatory.

2.5 Some Theorems on Oscillation

Theorem 1: If all the non-trivial solutions of $u'' + a(t)u = 0$... (1)

are oscillatory, $b(t)$ is continuous and $b(t) \geq a(t)$, $t_0 \leq t < \infty$, then some non-trivial solutions of

$$v'' + b(t)v = 0 \quad \dots (2)$$

are oscillatory.

On the other hand, if some non-trivial solutions of (1) are non-oscillatory and $b(t) \geq a(t)$, then some non-trivial solutions of (2) must be non-oscillatory.

Proof: Let $u(t)$ be an oscillatory solution of (1) i.e., $u(t)$ is a non-trivial solution of (1) having infinite number of zeroes. Also let $v(t)$ be a non-trivial solution of (2).

Multiplying (2) by u and (1) by v and subtracting, we get

$$uv'' - u''v + \{b(t) - a(t)\}uv = 0$$

$$\Rightarrow d(uv' - u'v) + \{b(t) - a(t)\}uv = 0 \quad \dots(3)$$

Now let t_1 and t_2 be two consecutive zeroes of $u(t)$ and assume $t_0 \leq t_1 < t_2$ for some t_0 and let $u(t) \geq 0$ on the interval $t_1 \leq t \leq t_2$ (See above fig. 2.1)

Integrating (3) from t_1 to t_2 , we get

$$\begin{aligned} \{u(t_2)v'(t_2) - u'(t_2)v(t_2)\} - \{u(t_1)v'(t_1) - u'(t_1)v(t_1)\} \\ + \int_{t_1}^{t_2} \{b(t) - a(t)\} \cdot u(t)v(t) dt = 0 \end{aligned}$$

$$\Rightarrow u'(t_1)v(t_1) - u'(t_2)v(t_2) + \int_{t_1}^{t_2} \{b(t) - a(t)\}u(t)v(t) dt = 0 \quad \dots(4)$$

$\because u(t_1) = 0 = u(t_2)$ as t_1, t_2 are zeroes of $u(t)$.

Now we assume that $v(t)$ has no zero on $[t_1, t_2]$. So $v(t)$ does not change its sign on $[t_1, t_2]$. But $u'(t_1) > 0$ and $u'(t_2) < 0$ (see figure 2.1, slope at $t_1 > 0$ and slope at $t_2 < 0$)

$$\therefore u'(t_1)v(t_1) \geq 0 \text{ and } u'(t_2)v(t_2) \leq 0 \Rightarrow u'(t_1)v(t_1) - u'(t_2)v(t_2) \geq 0$$

$$\text{Also } \int_{t_1}^{t_2} \{b(t) - a(t)\}u(t)v(t) dt \geq 0 \text{ as } b(t) \geq a(t)$$

So equation (4) is not true. i.e., assuming that $v(t)$ has no zero on $[t_1, t_2]$, equation (4) leads to a contradiction. Hence $v(t)$ changes its sign in $[t_1, t_2]$ i.e., $v(t)$ has a zero in $[t_1, t_2]$.

Thus between two consecutive zeroes of $u(t)$, there is a zero of $v(t)$. Since $u(t)$ have infinite number of zeroes, so $v(t)$ also have infinite number of zeroes.

Hence if all the non-trivial solutions of (1) are oscillatory then some non-trivial solutions of (2) are oscillatory.

The second part can be proved with similar arguments.

Note : This theorem is a special case of **Sturm Comparison Theorem**. (See 2.16)

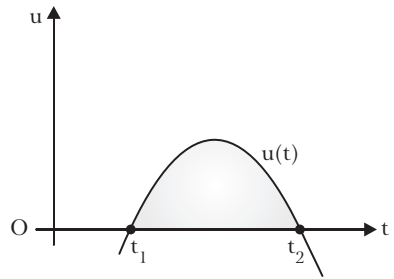


Fig. 2.1

Corollary 1: *The non-trivial solutions of $u'' + \phi(t)u = 0$ are oscillatory if $\phi(t) \geq m^2 > 0$ for all t .*

Proof: Since non-trivial solutions of $u'' + m^2u = 0$ i.e. $\sin mt$ and $\cos mt$ are oscillatory and $\phi(t) \geq m^2$, so by above theorem non-trivial solutions of $u'' + \phi(t)u = 0$ are oscillatory.

Corollary 2: *If $\lim_{t \rightarrow \infty} a(t) = \infty$ monotonically, then all the non-trivial solutions of $u'' + a(t)u = 0$ are oscillatory.*

Proof: $\lim_{t \rightarrow \infty} a(t) = \infty \Rightarrow$ there exists $\epsilon > 0$ such that $a(t) > \epsilon > 0$ for all t greater than some t_0 .

Now since all the non-trivial solutions of $u'' + \epsilon u = 0$, $\epsilon > 0$ are oscillatory, so by above theorem 1 all the non-trivial solutions of $u'' + a(t)u = 0$ are oscillatory.

Corollary 3: *The non-trivial solutions of $u'' + \{1 + \phi(t)\}u = 0$, where $\phi(t) \rightarrow 0$ as $t \rightarrow \infty$ are oscillatory.*

Proof: Given $\lim_{t \rightarrow \infty} \phi(t) = 0$ for sufficiently large t_0

$$\begin{aligned} \Rightarrow & |\phi(t)| \leq \epsilon, \text{ for } t \geq t_0 \\ \Rightarrow & -\epsilon \leq \phi(t) \leq \epsilon \\ \Rightarrow & 1 - \epsilon \leq 1 + \phi(t) \leq 1 + \epsilon \\ \Rightarrow & 1 - 1/2 \leq 1 + \phi(t) \leq 1 + 1/2, \text{ choosing } \epsilon = 1/2 \text{ (small)} \\ \Rightarrow & 1 + \phi(t) \geq 1/2 \end{aligned}$$

Since all non-trivial solutions of $u'' + (1/2)u = 0$ are oscillatory, so by theorem 1 non-trivial solutions of $u'' + \{1 + \phi(t)\}u = 0$ are oscillatory.

Corollary 4: *Any non-trivial solution of the equation $u'' + k(t)u = 0$ in the interval $a < t < b$ cannot vanish more than once in this interval if $k(t) \leq 0$ for all $t \in (a, b)$*

Proof: Let $u(t)$ be a non-trivial solution of $u'' + k(t)u = 0$. If $u(t)$ vanishes more than once say at $t = t_1$ and $t = t_2$ of the interval (a, b) .

$$\text{i.e.,} \quad a < t_1 < t_2 < b$$

If $k(t) \leq 0$ i.e., $0 \geq k(t)$ for all $t \in (a, b)$, then by the theorem all non-trivial solutions of $u'' + 0 \cdot u = 0$ will vanish at least once in the closed interval $t_1 < t < t_2$, which is impossible.

Hence any non-trivial solution of $u'' + k(t)u = 0$ cannot vanish more than once in the interval (a, b) .

Theorem 2: Let $a(t)$ be continuously differential and $a(t), a'(t) \geq 0$ on $0 \leq t < \infty$. If $u(t)$ is a non-trivial solution of $u'' + a(t)u = 0$ and t_1 and t_2 are two consecutive zeroes of its derivative, then

$$|u(t_2)| \leq |u(t_1)|$$

Proof: We have, $u'' + a(t)u = 0$... (1)

Multiplying (1) by $2u'(t)$ and integrating from t_1 and t_2 ,

$$\text{we get} \quad [u'^2(t)]_{t_1}^{t_2} + 2 \int_{t_1}^{t_2} a(t) u(t) u'(t) dt = 0$$

$$\Rightarrow u'^2(t_2) - u'^2(t_1) + 2 \int_{t_1}^{t_2} a(t) u(t) u'(t) dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} a(t) u(t) u'(t) dt = 0 \quad \dots (2)$$

$\therefore u'(t_1) = 0 = u'(t_2)$, as t_1 and t_2 are zeroes of $u'(t)$

Now integrating (2) by parts taking $a(t)$ as the first function, we get

$$[a(t) \cdot \frac{1}{2} u^2(t)]_{t_1}^{t_2} - \int_{t_1}^{t_2} a'(t) \cdot \frac{1}{2} u^2(t) dt = 0$$

$$\Rightarrow a(t_2) u^2(t_2) - a(t_1) u^2(t_1) = \int_{t_1}^{t_2} a'(t) u^2(t) dt \quad \dots (3)$$

Since t_1 and t_2 are two consecutive zeroes of $u'(t)$, so $u'(t)$ does not change its sign in $t_1 \leq t \leq t_2$ i.e., in $t_1 \leq t \leq t_2$, either $u'(t) \geq 0$ or $u'(t) \leq 0$

$\Rightarrow u(t)$ is strictly monotonic in $t_1 \leq t \leq t_2$

Now when $a'(t) = 0 \Rightarrow a(t) = c$ is constant for $t_1 \leq t \leq t_2$

$\therefore$ from (3), we get $c[u^2(t_2) - u^2(t_1)] = 0 \Rightarrow u^2(t_2) = u^2(t_1)$

Again when $a'(t) \neq 0$, we assume that $u^2(t_2) > u^2(t_1)$... (4)

$$\text{So} \quad \int_{t_1}^{t_2} a'(t) u^2(t) dt < [a(t_2) - a(t_1)] u^2(t_2)$$

$\therefore$ from (3), we get

$$a(t_2) u^2(t_2) - a(t_1) u^2(t_1) \leq [a(t_2) - a(t_1)] u^2(t_2)$$

$$\Rightarrow a(t_1) [u^2(t_2) - u^2(t_1)] \leq 0$$

$$\Rightarrow u^2(t_2) \leq u^2(t_1) \quad \because a(t_1) > 0$$

$\Rightarrow$ Which contradicts the assumption (4).

Thus we have $u^2(t_2) \leq u^2(t_1) \Rightarrow |u(t_2)| \leq |u(t_1)|$

Theorem 3: If $a(t) \geq (1 + \epsilon)(4t^2)$, $\epsilon > 0$ for all $t \geq t_0$, then all the non-trivial solutions of $u''(t) + a(t)u = 0$ are oscillatory.

[Kolkata 2001, 02, 04, 05, 07;
Rohilkhand 2007, Rajasthan 2003, 07]

Proof: Consider the equation $u'' + m^2u = 0$... (1)

where $m > 0$ is a real constant. All the non-trivial solutions of this equation are oscillatory.

Now let $t_1 = e^t$, then $\frac{dt_1}{dt} = e^t = t_1$

$$\therefore \frac{du}{dt} = \frac{du}{dt_1} \cdot \frac{dt_1}{dt} = t_1 \frac{du}{dt_1}$$

$$\text{and} \quad \frac{d^2u}{dt^2} = \frac{d}{dt} \left(\frac{du}{dt} \right) = \frac{d}{dt_1} \left(t_1 \frac{du}{dt_1} \right) \frac{dt_1}{dt} = \left(t_1 \frac{d^2u}{dt_1^2} + \frac{du}{dt_1} \right) t_1$$

Substituting in (1), it reduces to

$$t_1^2 \frac{d^2u}{dt_1^2} + t_1 \frac{du}{dt_1} + m^2u = 0 \quad \dots (2)$$

Again substituting

$$u = v/\sqrt{t_1} = vt_1^{-1/2}$$

$$\frac{du}{dt_1} = t_1^{-1/2} \frac{dv}{dt_1} - \frac{1}{2} t_1^{-3/2} v, \quad \frac{d^2u}{dt_1^2} = t_1^{-1/2} \frac{d^2v}{dt_1^2} - t_1^{-3/2} \frac{dv}{dt_1} + \frac{3}{4} t_1^{-5/2} v$$

equation (2), reduces to

$$t_1^2 \left[t_1^{-1/2} \frac{d^2v}{dt_1^2} - t_1^{-3/2} \frac{dv}{dt_1} + \frac{3}{4} t_1^{-5/2} v \right] + t_1 \left(t_1^{-1/2} \frac{dv}{dt_1} - \frac{1}{2} t_1^{-3/2} v \right) + m^2 \cdot vt_1^{-1/2} = 0$$

$$\text{or} \quad \frac{d^2v}{dt_1^2} + \left(\frac{1 + 4m^2}{4t_1^2} \right) v = 0 \quad \dots (3)$$

where $v = 4\sqrt{t_1}$

This equation (3) is oscillatory as the transformation considered do not effect the zeroes of (1).

So by theorem 1, all non-trivial solutions of $u''(t) + a(t)u = 0$ are oscillatory, if

$$a(t) \geq \frac{1+4m^2}{4t^2} = \frac{1+\epsilon}{4t^2}, \quad \epsilon = 4m^2 > 0 \quad \forall t \geq t_0$$

which is given.

Hence all non-trivial solutions of $u''(t) + a(t)u = 0$ are oscillatory.

SECTION 2 : THE ADJOINT THEORY

2.6 Adjoint Operator and Adjoint Equation

[Kurukshetra 2005, 06]

Consider the n -th order linear differential operator

$$L_n \equiv a_0(x) \frac{d^n}{dx^n} + a_1(x) \frac{d^{n-1}}{dx^{n-1}} + \dots + a_{n-1}(x) \frac{d}{dx} + a_n(x) \quad \dots(1)$$

where $a_0(x), a_1(x), a_2(x), \dots, a_n(x)$ are continuous functions on a closed interval $I = [a, b]$ and $a_0(x) \neq 0$ for all $x \in I$

Then the corresponding n -th order homogeneous linear differential equation is

$$L_n(y) = a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1}(x) \frac{dy}{dx} + a_n(x) y = 0 \quad \dots(2)$$

Then the n -th order linear differential operator, called the **adjoint operator** of the operator L_n , given in 1, denoted by $\bar{L}_n$ or L_n^* is given by

$$\begin{aligned} \bar{L}_n \equiv L_n^* &= (-1)^n \frac{d^n}{dx^n} [a_0(x)] + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}} [a_1(x)] + \dots \\ &+ (-1)^1 \frac{d}{dx} [a_{n-1}(x)] + a_n(x) \end{aligned} \quad \dots(3)$$

and the differential equation

$$\begin{aligned} L_n^*(y) &= (-1)^n \frac{d^n}{dx^n} [a_0(x) \cdot y] + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}} [a_1(x) \cdot y] + \dots \\ &+ (-1)^1 \frac{d}{dx} [a_{n-1}(x) \cdot y] + a_n(x) \cdot y = 0 \end{aligned} \quad \dots(4)$$

is called the **adjoint equation** to the equation $L_n(y) = 0$, given by (2).

Note : In operator (1) and equation (2), the coefficients $a_0(x), a_1(x), a_2(x), \dots, a_n(x)$ are all continuous functions on a closed interval $I = [a, b]$ and $a_0(x) \neq 0$ for all $x \in I$. In the adjoint equation (4) all the coefficients are also continuous function on the interval I with $a_0(x) \neq 0$ for all $x \in I$, and also the coefficients a_k must have continuous $(n-k)$ th derivatives $[k = 0, 1, 2, \dots, (n-1)]$ on the interval $I = [a, b]$.

2.6.1 Particular Case (Second Order Linear Differential Equation)

Now we shall consider a special case for $n=2$ i.e., 2nd order linear differential operator, and equation and its adjoint operator and equation.

The second order linear differential operator is given by,

$$L_2 \equiv a_0(x) \frac{d^2}{dx^2} + a_1(x) \frac{d}{dx} + a_2(x) \quad \dots(1)$$

Where $a_0(x) (\neq 0)$, $a_1(x)$ and $a_2(x)$ are continuous function on a closed interval $I = [a, b]$ and the corresponding second-order differential equation is

$$L_2(y) = a_0(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = 0 \quad \dots(2)$$

Then **second order adjoint operator** of L_2 is given by

$$L_2^* \equiv (-1)^2 \frac{d^2}{dx^2} [a_0(x)] + (-1)^1 \frac{d}{dx} [a_1(x)] + a_2(x) \quad \dots(3)$$

and the **adjoint equation** to the equation (3) is given by

$$L_2^*(y) = (-1)^2 \frac{d^2}{dx^2} [a_0(x) \cdot y] + (-1)^1 \frac{d}{dx} [a_1(x) \cdot y] + a_2(x) \cdot y = 0$$

which can be written as

$$a_0(x) \frac{d^2 y}{dx^2} + [2a_0'(x) - a_1(x)] \frac{dy}{dx} + [a_0''(x) - a_1'(x) + a_2(x)] y = 0 \quad \dots(4)$$

$$\therefore L_2^* \equiv a_0(x) \frac{d^2}{dx^2} + [2a_0'(x) - a_1(x)] \frac{d}{dx} + [a_0''(x) - a_1'(x) + a_2(x)] \quad \dots(5)$$

For Example: Consider the second order differential equation

$$(2x+1) \frac{d^2 y}{dx^2} + x^3 \frac{dy}{dx} + y = 0$$

Then its adjoint equation is given by

$$(-1)^2 \frac{d^2}{dx^2} [(2x+1) y] + (-1)^1 \frac{d}{dx} [x^3 y] + y = 0$$

or
$$\left[(2x+1) \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} \right] - \left[x^3 \frac{dy}{dx} + 3x^2 y \right] + y = 0$$

or
$$(2x+1) \frac{d^2 y}{dx^2} + (4-x^3) \frac{dy}{dx} + (1-3x^2) y = 0$$

Aliter : By use of formula (5)

Here $a_0(x) = 2x + 1$, $a_1(x) = x^3$, $a_2(x) = 1$

$\therefore$ Adjoint equation is

$$a_0(x) \frac{d^2 y}{dx^2} + [2a_0'(x) - a_1(x)] \frac{dy}{dx} + [a_0''(x) - a_1'(x) + a_2(x)] y = 0$$

or
$$(2x + 1) \frac{d^2 y}{dx^2} + (2 \cdot 2 - x^3) \frac{dy}{dx} + (0 - 3x^2 + 1) y = 0$$

or
$$(2x + 1) \frac{d^2 y}{dx^2} + (4 - x^3) \frac{dy}{dx} + (1 - 3x^2) y = 0$$

2.7 Lagrange Identity

Statement: Let the coefficient a_k in the linear differential operator

$$L_n \equiv a_0(x) \frac{d^n}{dx^n} + a_1(x) \frac{d^{n-1}}{dx^{n-1}} + \dots + a_{n-1}(x) \frac{d}{dx} + a_n(x)$$

have continuous $(n - k)$ th derivatives ($k = 0, 1, 2, \dots, n$) on the interval $a \leq x \leq b$.

Further, let u and v be any two functions having n -th derivatives on $a \leq x \leq b$.

Then $v L_n(u) - u L_n^*(v) = \frac{d}{dx} [P(u, v)]$

where $P(u, v) = \sum_{k=1}^n \left[\sum_{j=1}^k (-1)^{j-1} u^{(k-j)} \cdot (v a_{n-k})^{(j-1)} \right]$

$(*)^{(t)}$ denotes the t -th derivative of $*$ w.r.t. ' x '

[Bhopal 2002; Lucknow 2002, 04, 05;

Allahabad 2002, 04, 07; Gorakhpur 2003, 05, 06; Kurukshetra 2004, 05;

Gwalior 2004, 06; Jabalpur 2006; Meerut 2006]

Proof: We have

$$\begin{aligned} & \frac{d}{dx} [U^{(k-1)}V - U^{(k-2)}V' + U^{(k-3)}V'' - \dots \\ & \quad + (-1)^{k-2}U'V^{(k-2)} + (-1)^{k-1}UV^{(k-1)}] \\ &= \{U^{(k)}V + U^{(k-1)}V'\} - \{U^{(k-1)}V' + U^{(k-2)}V''\} + \{U^{(k-2)}V'' + U^{(k-3)}V'''\} \\ & \quad + \dots + (-1)^{k-2} \{U''V^{(k-2)} + U'V^{(k-1)}\} + (-1)^{k-1} \{U'V^{(k-1)} + UV^{(k)}\} \\ &= U^{(k)}V + (-1)^{k-1}UV^{(k)} \text{ (Other terms with +, - sign are cancelled)} \end{aligned}$$

$$\Rightarrow VU^{(k)} = (-1)^k V^{(k)} \cdot U + \frac{d}{dx} [U^{(k-1)} \cdot V - U^{(k-2)} V' + U^{(k-3)} V'' - \dots + (-1)^{k-2} U' \cdot V^{(k-2)} + (-1)^{k-1} U \cdot V^{(k-1)}] \quad \dots(1)$$

$$\text{Now } {}^v L_n(u) = va_0 \frac{d^n u}{dx^n} + va_1 \frac{d^{n-1} u}{dx^{n-1}} + va_2 \frac{d^{n-2} u}{dx^{n-2}} + \dots + va_{n-1} \frac{du}{dx} + va_n u \quad \dots(2)$$

Applying the above formula (1), with $U = u$ and

$V = va_0, va_1, va_2, \dots, va_{n-1}$ in (1), and substituting in (2), we get

$$\begin{aligned} {}^v L_n(u) &= (-1)^n (va_0)^n \cdot u + \frac{d}{dx} [u^{(n-1)} va_0 - u^{(n-2)} (va_0)' + u^{(n-3)} (va_0)'' - \dots \\ &\quad + (-1)^{n-2} u' (va_0)^{(n-2)} + (-1)^{n-1} u (va_0)^{(n-1)}] \\ &\quad + (-1)^{n-1} (va_1)^{(n-1)} \cdot u + \frac{d}{dx} [u^{(n-2)} \cdot va_1 - u^{(n-3)} (va_1)' + u^{(n-4)} (va_1)'' - \dots \\ &\quad + (-1)^{n-3} u' (va_1)^{(n-3)} + (-1)^{n-2} u (va_1)^{(n-2)}] \\ &\quad + (-1)^{n-2} (va_2)^{(n-2)} \cdot u + \frac{d}{dx} [u^{(n-3)} \cdot va_2 - u^{(n-4)} (va_2)' + u^{(n-4)} (va_2)'' - \dots \\ &\quad + (-1)^{n-4} u' (va_2)^{(n-3)} + (-1)^{n-3} u (va_2)^{(n-3)}] + \dots \\ &\quad + (-1)^1 (va_{n-1})' u + \frac{d}{dx} [u \cdot va_{n-1}] + va_n \cdot u \\ &= u \cdot \sum_{k=0}^n (-1)^{n-k} (va_k)^{(n-k)} + \frac{d}{dx} \left[\sum_{j=1}^n (-1)^{j-1} u^{(n-j)} (va_0)^{(j-1)} \right] \\ &\quad + \frac{d}{dx} \left[\sum_{j=1}^{n-1} (-1)^{j-1} u^{(n-1-j)} (va_1)^{(j-1)} \right] + \dots \\ &\quad + \frac{d}{dx} \left[\sum_{j=1}^1 (-1)^{j-1} u^{(1-j)} (va_{n-1})^{(j-1)} \right] \\ &= u \cdot L_n^*(v) + \sum_{k=1}^n \frac{d}{dx} \left[\sum_{j=1}^k (-1)^{j-1} u^{(k-j)} (va_{n-k})^{(j-1)} \right] \\ &\Rightarrow {}^v L_n(u) - u L_n^*(v) = \frac{d}{dx} \left\{ \sum_{k=1}^n \left[\sum_{j=1}^k (-1)^{j-1} u^{(k-j)} \cdot (va_{n-k})^{(j-1)} \right] \right\} \\ &\Rightarrow {}^v L_n(u) - u L_n^*(v) = \frac{d}{dx} P(u, v) \end{aligned}$$

Note : The function $P(u, v)$ is called the **bilinear concomitant** associated with L_n .

2.8 Green's Formula (Corollary to Lagrange Identity)

Statement: Let the coefficients a_k in the linear differential operator

$$L_n \equiv a_0(x) \frac{d^n}{dx^n} + a_1(x) \frac{d^{n-1}}{dx^{n-1}} + \dots + a_{n-1}(x) \frac{d}{dx} + a_n(x)$$

have continuous $(n - k)$ th derivatives ($k = 0, 1, 2, \dots, n$) on the interval $a \leq x \leq b$.

Further, let u and v be any two functions having n -th derivatives on $a \leq x \leq b$.

Then, for any two points x_1 and x_2 of $a \leq x \leq b$

$$\int_{x_1}^{x_2} [v L_n(u) - u L_n^*(v)] dx = [P(u, v)]_{x=x_2} - [P(u, v)]_{x=x_1}$$

where
$$P(u, v) = \sum_{k=1}^n \left[\sum_{j=1}^k (-1)^{j-1} u^{(k-j)} (v a_{n-k})^{(j-1)} \right]$$

($*^{(t)}$ denotes the t -th derivative of $*$ w.r.t. x)

[Meerut 2006; Gwalior 2006]

Proof: By Lagrange's Identity, we have

$$v L_n(u) - u L_n^*(v) = \frac{d}{dx} [P(u, v)]$$

Integrating both sides from x_1 to x_2 w.r.t. ' x '

We get

$$\begin{aligned} \int_{x_1}^{x_2} [v L_n(u) - u L_n^*(v)] dx &= [P(u, v)]_{x=x_1}^{x_2} \\ &= [P(u, v)]_{x=x_2} - [P(u, v)]_{x=x_1} \end{aligned}$$

2.8.1 Particular Case (For Second Order Linear Differential Equation)

Here we shall consider Lagrange Identity, and $P(u, v)$, bilinear concomitant associated with L_2 (L_n for $n=2$) in case of second order linear differential equation.

The second order linear differential equation (hom.) is given by

$$L_2(y) = a_0(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x)y = 0 \quad \dots(1)$$

where, second order linear differential operator L_2 is given by

$$L_2 \equiv a_0(x) \frac{d^2}{dx^2} + a_1(x) \frac{d}{dx} + a_2(x)$$

Its adjoint operator L_2^* can be written as

$$L_2^* \equiv a_0(x) \frac{d^2}{dx^2} + [2a'_0(x) - a_1(x)] \frac{d}{dx} + [a''_0(x) - a'_1(x) + a_2(x)] \quad \dots(2)$$

Also in this case

$$\begin{aligned} P(u, v) &= \sum_{k=1}^2 \left[\sum_{j=1}^k (-1)^{j-1} u^{(k-j)} (va_{2-k})^{(j-1)} \right] \\ &= \sum_{j=1}^1 (-1)^{j-1} u^{(1-j)} (va_1)^{(j-1)} + \sum_{j=1}^2 (-1)^{j-1} u^{(2-j)} (va_0)^{(j-1)} \\ &= u(va_1) + (-1)^0 u' (va_0) + (-1)^1 u (va_0)' \\ &= uva_1 + u' va_0 - u (v' a_0 + v a'_0) \\ &= a_0 u' v + (a_1 - a'_0) uv - a_0 uv' \end{aligned} \quad \dots(3)$$

The **Lagrange Identity** for second order linear differential equation, is

$$v L_2(u) - u L_2^*(v) = \frac{d}{dx} [P(u, v)] \quad \dots(4)$$

Now, $v L_2(u) - u L_2^*(v)$

$$\begin{aligned} &= v \left[a_2(x) \frac{d^2}{dx^2} u + a_1(x) \frac{du}{dx} + a_2(x) u \right] - u \left[a_0(x) \frac{d^2 v}{dx^2} + \{2a'_0(x) - a_1(x)\} \frac{dv}{dx} \right. \\ &\quad \left. + \{a''_0(x) - a'_1(x) + a_2(x)\} v \right] \\ &= v (a_0 u'' + a_1 u' + a_2 u) - u [a_0 v'' + (2a'_0 - a_1) v' + (a''_0 - a'_1 + a_2) v] \\ &= a_0 u'' v + a_1 u' v - (a''_0 - a'_1) uv - (2a'_0 - a_1) uv' - a_0 uv'' \end{aligned} \quad \dots(5)$$

and

$$\begin{aligned} \frac{d}{dx} [P(u, v)] &= \frac{d}{dx} [a_0 u' v + (a_1 - a'_0) uv - a_0 uv'] \\ &= a'_0 u' v + a_0 u'' v + a_0 u' v' + (a'_1 - a''_0) uv + (a_1 - a'_0) (u' v + uv') \\ &\quad - a'_0 uv' - a_0 u' v' - a_0 uv'' \\ &= a_0 u'' v + a_1 u' v - (a''_0 - a'_1) uv - (2a'_0 - a_1) uv' - a_0 uv'' \end{aligned} \quad \dots(6)$$

From (2) and (3) Lagrange Identity given in (4) is verified.

2.9 Some Theorems

Theorem 1: A necessary and sufficient condition that $f(x)$ be a solution of n -th order linear differential equation $L_n(y) = 0$ is that $f(x)$ be a solution of the $(n - 1)$ -th order linear differential equation

$$P[y, g(x)] = c$$

where $P(u, v)$ is the bilinear concomitant associated with L_n , the function $g(x)$ is a non-trivial solution of the adjoint equation $L_n^*(y) = 0$ and c is an arbitrary constant.

Proof: **Necessary Condition :** Since $g(x)$ is a solution of adjoint equation $L_n^*(y) = 0$

$$\therefore L_n^*[g(x)] = 0$$

$\therefore$ The Lagrange identity with $u = y$ and $v = g(x)$, becomes

$$g(x)L_n(y) - y L_n^*[g(x)] = \frac{d}{dx} [P(y, g(x))]$$

$$\text{becomes} \quad g(x) L_n(y) = \frac{d}{dx} P[y, g(x)] \quad \dots(1)$$

If $f(x)$ is a solution of $L_n(y) = 0$, then $L_n[f(x)] = 0$

$\therefore$ Taking $y = f(x)$ in (1), we get

$$\frac{d}{dx} [P(f(x), g(x))] = g(x) L_n[f(x)] = 0$$

Integrating, $P[f(x), g(x)] = c$, where c is an arbitrary constant.

Thus if $f(x)$ is a solution of $L_n(y) = 0$, then $f(x)$ satisfies the equation $P[y, g(x)] = c$, which is an $(n - 1)$ -th order linear differential equation with y as dependent variable.

Sufficient Condition : i.e., if $f(x)$ is a solution of $P[y, g(x)] = c$, where c is an arbitrary constant, then to prove that $f(x)$ is a solution of $L_n(y) = 0$

Since $f(x)$ is a solution of $P[y, g(x)] = c$

$$\Rightarrow P[f(x), g(x)] = c \Rightarrow \frac{d}{dx} [P[f(x), g(x)]] = 0$$

then from (1), we have

$$g(x) L_n\{f(x)\} = 0 \Rightarrow L_n\{f(x)\} = 0, \because g(x) \text{ is a non-trivial solution of } L_n^*(y) = 0$$

$$\Rightarrow f(x) \text{ is a solution of } L_n(y) = 0$$

2.9.1 Particular Case

If $P(u, v)$ is bilinear concomitant associated with the second order linear differential equation

$$L_n(y) = a_0 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0 \quad \dots(1)$$

Then $P(u, v) = a_0 v u' + [(a_1 - a_0') v - a_0 v'] u$

(See equation (3) of 2.8.1 particular case)

$$\Rightarrow P[y, g(x)] = a_0(x) g(x) \frac{dy}{dx} + [\{a_1(x) - a_0'(x)\} g(x) - a_0(x) g'(x)] y$$

Hence $f(x)$ is a solution of second order differential equation (1), if and only if it is a solution of first order differential equation.

$$a_0(x) g(x) \frac{dy}{dx} + [\{a_1(x) - a_0'(x)\} g(x) - a_0(x) g'(x)] y = c$$

where $g(x)$ is a non-trivial solution of $L_n^*(y) = 0$ and c is an arbitrary constant.

Theorem 2: If L_n^* is adjoint operator of L_n , then L_n is the adjoint operator of L_n^* .

[Rohilkhand 2006]

Proof: Since L_n^* is adjoint operator of L_n , so by Lagrange Identity we have

$$v L_n(u) - u L_n^*(v) = \frac{d}{dx} [P(u, v)] \quad \dots(1)$$

$$\text{where } P(u, v) = \sum_{k=1}^n \left[\sum_{j=1}^k (-1)^{j-1} u^{(k-j)} \cdot (v a_{n-k})^{(j-1)} \right] \quad \dots(2)$$

is the linear concomitant associated with the operator L_n .

If L_n^{**} denote the adjoint operator of L_n^* , then

$$u L_n^*(v) - v L_n^{**}(u) = \frac{d}{dx} [P^*(v, u)] \quad \dots(3)$$

where $P^*(v, u)$ is the linear concomitant associated with the operator L_n^* .

Adding (1) and (3), we get

$$v [L_n(u) - L_n^{**}(u)] = \frac{d}{dx} [P(u, v) + P^*(v, u)] \quad \dots(4)$$

$P(u, v)$ given by (4) is a homogeneous linear form in $v, v', v'', \dots, v^{(n-1)}$

$$\therefore P(u, v) + P^*(v, u) = C_1 v^{(n-1)} + C_2 v^{(n-2)} + \dots + C_n v \quad \dots(5)$$

where the coefficients $C_1, C_2, \dots, C_n$ are functions of $u, u', u'', \dots, u^{(n-1)}$ which are various coefficients in the operators under consideration and certain derivatives of these various coefficients.

$\therefore$ From (4), we get

$$v[L_n(u) - L_n^{**}(u)] = \frac{d}{dx} [C_1 v^{(n-1)} + C_2 v^{(n-2)} + C_3 v^{(n-3)} + \dots + C_{n-1} v' + C_n v] \quad \dots(6)$$

$$\Rightarrow v[L_n(u) - L_n^{**}(u)] = C_1 v^{(n)} + (C_1' + C_2) v^{(n-1)} + (C_2' + C_3) v^{(n-2)} + \dots + (C_{n-1}' + C_n) v' + C_n' v \quad \dots(7)$$

Equating coefficients of $v^{(n)}, v^{(n-1)}, v^{(n-2)}, \dots, v'$ from the two sides of (7), we get

$$C_1 = 0, C_1' + C_2 = 0, C_2' + C_3 = 0, \dots, C_{n-1}' + C_n = 0$$

Solving, $C_1 = 0 = C_2 = C_3 = \dots, C_n$

$\therefore$ from (6), $v[L_n(u) - L_n^{**}(u)] = 0$

$$\Rightarrow L_n(u) - L_n^{**}(u) = 0 \because v \neq 0$$

$$\Rightarrow L_n(u) = L_n^{**}(u)$$

$$\Rightarrow L_n \text{ is the adjoint operator of } L_n^*.$$

2.10 Self-Adjoint Equation

An n -th order homogeneous linear differential equation $L_n(y) = 0$ is called **self-adjoint** if it is identical with its adjoint equation $L_n^*(y) = 0$.

2.11 Theorems on Self-Adjoint Equations of the Second Order

Theorem 1: Necessary and Sufficient Condition for a second order linear differential equation to be self-adjoint.

Statement : The necessary and sufficient condition for a second order linear differential equation

$$a_0(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = 0$$

where a_0 has a continuous second order derivative, a_1 has a continuous first derivative, $a_2(x)$ is continuous and $a_0(x) \neq 0$ on $a \leq x \leq b$, to be self-adjoint is that

$$\frac{d}{dx} [a_0(x)] = a_1(x) \text{ i.e., } a'_0(x) = a_1(x) \text{ on } a \leq x \leq b$$

Proof: Necessary Condition : i.e., if the second order linear differential equation

$$L_n(y) = a_0(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = 0 \quad \dots(1)$$

is self-adjoint, then $\frac{d}{dx} [a_0(x)] = a_1(x)$

The adjoint equation to equation (1) is

$$L_n^*(y) = a_0(x) \frac{d^2 y}{dx^2} + [2a'_0(x) - a_1(x)] \frac{dy}{dx} + [a''_0(x) - a'_1(x) + a_2(x)] y = 0 \quad \dots(2)$$

If equation (1) is self-adjoint, then (1) and (2) are identical, so comparing (1) and (2), we get

$$2a'_0(x) - a_1(x) = a_1(x) \Rightarrow a'_0(x) = a_1(x) \quad \dots(3)$$

$$\text{and} \quad a''_0(x) - a'_1(x) + a_2(x) = a_2(x) \Rightarrow a''_0(x) = a'_1(x)$$

$$\text{Integrating,} \quad a'_0(x) = a_1(x) + c \quad \dots(4)$$

where c is an arbitrary constant

From (3) and (4), we get $c = 0$

$$\text{Thus,} \quad a'_0(x) = a_1(x) \text{ i.e., } \frac{d}{dx} [a_0(x)] = a_1(x)$$

which is the required condition.

Sufficient Condition (Converse) : i.e., if $\frac{d}{dx} [a_0(x)] = a_1(x)$, then equation (1) is self-adjoint.

$$\text{we have,} \quad \frac{d}{dx} [a_0(x)] = a_1(x) \Rightarrow a'_0(x) = a_1(x)$$

$$\text{Now coefficient of } \frac{d^2 y}{dx^2} \text{ in (2)} = a_0(x) = \text{coefficient of } \frac{d^2 y}{dx^2} \text{ in (1)}$$

$$\text{In (2), coefficient of } \frac{dy}{dx} = 2a'_0(x) - a_1(x) = 2a_1(x) - a_1(x)$$

$$= a_1(x) = \text{coefficient of } \frac{dy}{dx} \text{ in (1)}$$

In (2), coefficient of $y = a''_0(x) - a'_1(x) + a_2(x) = a'_1(x) - a'_1(x) + a_2(x)$

$$= a_2(x) = \text{coefficient of } y \text{ in (1)}.$$

Thus if $\frac{d}{dx}[a_0(x)] = a_1(x)$, then equation (1) and (2) are identical.

Hence equation (1) is self-adjoint.

Theorem 2: *If the second order linear differential equation*

$$a_0(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = 0$$

is self-adjoint, then it can be written as

$$\frac{d}{dx} \left[a_0(x) \frac{dy}{dx} \right] + a_2(x) y = 0$$

Proof: If second order linear differential equation

$$a_0(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = 0 \quad \dots(1)$$

is self adjoint, then $\frac{d}{dx}[a_0(x)] = a_1(x)$ i.e., $a'_0(x) = a_1(x)$

So (1) can be written as

$$a_0(x) \frac{d^2 y}{dx^2} + a'_0(x) \frac{dy}{dx} + a_2(x) y = 0$$

or
$$\frac{d}{dx} \left[a_0(x) \frac{dy}{dx} \right] + a_2(x) y = 0$$

Theorem 3: *If the coefficients $a_0(x)$, $a_1(x)$ and $a_2(x)$ in the second order linear differential equation*

$$a_0(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = 0 \quad \dots(1)$$

are continuous on $a \leq x \leq b$, then it can be transformed into the equivalent self-adjoint equation

$$\frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + Q(x)y = 0$$

on $a \leq x \leq b$, where

$$P(x) = \exp. \left[\int \frac{a_1(x)}{a_0(x)} dx \right], Q(x) = \frac{a_2(x)}{a_1(x)} \exp. \left[\int \frac{a_1(x)}{a_0(x)} dx \right]$$

by multiplication throughout by the factor

$$\frac{1}{a_0(x)} \exp. \left[\int \frac{a_1(x)}{a_0(x)} dx \right]$$

or Every linear differential equation of second order can be put in self adjoint form.

[Lucknow 2001; Himachal 2001, 02; Purvanchal 2001, 03; Rewa 2002, 04]

Proof: Multiplying (1) throughout by the factor

$$\begin{aligned} \frac{1}{a_0(x)} \exp. \left[\int \frac{a_1(x)}{a_0(x)} dx \right], \text{ we get} \\ \exp. \left[\int \frac{a_1(x)}{a_0(x)} dx \right] \cdot \frac{d^2 y}{dx^2} + \frac{a_1(x)}{a_0(x)} \exp. \left[\int \frac{a_1(x)}{a_0(x)} dx \right] \frac{dy}{dx} \\ + \frac{a_2(x)}{a_0(x)} \exp. \left[\int \frac{a_1(x)}{a_0(x)} dx \right] y = 0 \quad \dots (2) \end{aligned}$$

which is self-adjoint equation,

$$\therefore \frac{d}{dx} \left\{ \text{coeff. of } \frac{d^2 y}{dx^2} \right\} = \frac{d}{dx} \left\{ \exp. \left[\int \frac{a_1(x)}{a_0(x)} dx \right] \right\} = \frac{a_1(x)}{a_0(x)} \exp. \left[\int \frac{a_1(x)}{a_0(x)} dx \right] = \text{coeff. of } \frac{dy}{dx}$$

Which is necessary and sufficient condition for the equation (2) to be self-adjoint. Now equation (2) can be written as

$$\frac{d}{dx} \left\{ \exp. \left[\int \frac{a_1(x)}{a_0(x)} dx \right] \frac{dy}{dx} \right\} + \frac{a_2(x)}{a_0(x)} \exp. \left[\int \frac{a_1(x)}{a_0(x)} dx \right] y = 0$$

or
$$\frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + Q(x) y = 0$$

where
$$P(x) = \exp. \left[\int \frac{a_1(x)}{a_0(x)} dx \right] \text{ and } Q(x) = \frac{a_2(x)}{a_0(x)} \exp. \left[\int \frac{a_1(x)}{a_0(x)} dx \right]$$

SECTION 3 : THE STURM THEORY

Here we consider the equation

$$\frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + Q(x) y = 0$$

where $P(x)$ has a continuous derivative, $Q(x)$ is continuous and $P(x) > 0$ on $a \leq x \leq b$

Now we give some theorems to study above equation.

2.12 Theorem I

Let $f(x)$ be a solution of

$$\frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + Q(x) y = 0 \quad \dots(1)$$

such that (i) $f(x)$ has derivative $f'(x)$ on $a \leq x \leq b$, and (ii) $f(x)$ has an infinite number of zeroes on $a \leq x \leq b$,

then $f(x) = 0$ for all x on $a \leq x \leq b$.

[Kurukshetra 2003; 06]

Proof: Since $f(x)$ has an infinite number of zeroes on $[a, b]$, so by the Bolzano-Weierstrass theorem, the set of zeroes has a limit point x_0 , where $x_0 \in [a, b]$. Thus, there exists a sequence $\langle x_n \rangle$ of zeroes of $f(x)$ which converges to x_0 (where $x_n \neq x_0$).

Since the function $f(x)$ is continuous.

$$\therefore \lim_{x \rightarrow x_0} f(x) = f(x_0)$$

where $x \rightarrow x_0$ through any sequence of points on $[a, b]$.

Let $x \rightarrow x_0$ through the sequence $\langle x_n \rangle$ of zeroes of $f(x)$.

$$\therefore \lim_{x \rightarrow x_0} f(x) = 0 = f(x_0) \quad \dots(2)$$

Now since $f'(x_0)$ exists,

$$\Rightarrow f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

where $x \rightarrow x_0$ through the sequence $\langle x_n \rangle$.

$$\text{For such points} \quad \frac{f(x) - f(x_0)}{x - x_0} = 0 \quad \Rightarrow \quad f'(x_0) = 0$$

Thus $f(x)$ is a solution of (1) such that

$$f(x_0) = f'(x_0) = 0, \text{ where } x_0 \text{ is a point of an interval } a \leq x \leq b$$

Hence, $f(x) = 0$ for all x on $a \leq x \leq b$.

2.13 Theorem II (Abel's Formula)

Let $f(x)$ and $g(x)$ be any two solutions of

$$\frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + Q(x) y = 0 \quad \dots(1)$$

on the interval $a \leq x \leq b$,

Then for all x on $a \leq x \leq b$

$$P(x) \cdot [f(x) g'(x) - f'(x) g(x)] = k,$$

where k is a constant.

[Delhi 2001, 03, 07; Gwalior 2003]

Proof: Since $f(x)$ and $g(x)$ are solutions of (1) on the interval $[a, b]$, so

$$\frac{d}{dx} [P(x) f'(x)] + Q(x) f(x) = 0 \quad \dots(2)$$

and $\frac{d}{dx} [P(x) g'(x)] + Q(x) g(x) = 0 \quad \dots(3)$

for all $x \in [a, b]$

Multiplying (2) by $g(x)$ and (3) by $f(x)$ and subtracting, we get

$$g(x) \frac{d}{dx} [P(x) f'(x)] - f(x) \frac{d}{dx} [P(x) g'(x)] = 0$$

$$\Rightarrow \frac{d}{dx} [P(x) \cdot \{f(x) g'(x) - f'(x) g(x)\}] = 0$$

Integrating, we get

$$P(x) [f(x) g'(x) - f'(x) g(x)] = k$$

where k is a constant.

2.14 Theorem III

Let $f(x)$ and $g(x)$ be two solutions of

$$\frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + Q(x) y = 0 \quad \dots(1)$$

Then (i) If $f(x)$ and $g(x)$ have a common zero on $a \leq x \leq b$.

then $f(x)$ and $g(x)$ are linearly dependent on $a \leq x \leq b$.

(ii) If $f(x)$ and $g(x)$ be non-trivial linearly dependent solutions of (1) on $a \leq x \leq b$ and $f(x_0) = 0$, where $x_0 \in [a, b]$, then $g(x_0) = 0$ [Ujjain 2003; Himachal 2003, 04, 05]

Proof: (i) Since $f(x)$ and $g(x)$ are two solutions of (1),

$\therefore$ by Abel's formula, we have

$$P(x) [f(x) g'(x) - f'(x) g(x)] = K \text{ (constant)} \quad \dots(2)$$

Let $x_0 \in [a, b]$ be the common zero of $f(x)$ and $g(x)$,

i.e., $f(x_0) = 0$ and $g(x_0) = 0$

$\therefore$ Putting $x = x_0$ in (2), we get

$$K = P(x_0) [f(x_0) g'(x_0) - f'(x_0) g(x_0)] = P(x_0) [0 - 0] = 0$$

$\therefore$ from (2), $P(x) [f(x) g'(x) - f'(x) g(x)] = 0 \quad \dots(3)$

$$\forall x \in [a, b]$$

Since we have assumed $P(x) > 0$ in (1), so equation (3)

$$\Rightarrow f(x) g'(x) - f'(x) g(x) = 0, \text{ for all } x \in [a, b]$$

$$\Rightarrow W(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = 0$$

$\Rightarrow f(x)$ and $g(x)$ are linearly dependent on $a \leq x \leq b$.

(ii) If $f(x)$ and $g(x)$ be non-trivial linear dependent solutions of (1) on $a \leq x \leq b$, then there exists scalars C_1 and C_2 (not both zero) such that

$$C_1 f(x) + C_2 g(x) = 0 \quad \dots(4)$$

$$\forall x \in [a, b]$$

If $C_1 = 0$, then (4) $\Rightarrow C_2 g(x) = 0 \Rightarrow C_2 = 0$, $\because g(x)$ is not zero for all $x \in [a, b]$

which is a contradiction, as $f(x)$ and $g(x)$ are linearly dependent.

Thus $C_1 \neq 0$, Similarly $C_2 \neq 0$ i.e., neither C_1 nor C_2 is zero.

Now putting $x = x_0 \in [a, b]$ in (4), we get

$$C_1 f(x_0) + C_2 g(x_0) = 0$$

$$\Rightarrow C_2 g(x_0) = 0, \quad \because f(x_0) = 0 \text{ (Given)}$$

$$\Rightarrow g(x_0) = 0, \quad \because C_2 \neq 0$$

2.15 Theorem IV (Sturm Separation Theorem)

Let $f(x)$ and $g(x)$ be real linearly independent solutions of

$$\frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + Q(x) y = 0 \quad \dots(1)$$

on the interval $a \leq x \leq b$

Then between any two consecutive zeroes of $f(x)$ there is precisely one zero of $g(x)$.

[Kurukshetra 2002, 03; Gwalior 2004; Ujjain 2004;
Himachal 2001, 02, 03, 04, 05; Meerut 2011 (Sem. I)]

Or

The zeroes of one of the two real linearly independent solutions of $f(x)$ separate the zeroes of the other solutions.

Proof: Let x_1 and x_2 be two consecutive zeroes of $f(x)$ on $[a, b]$

i.e., $f(x_1) = 0$ and $f(x_2) = 0$, $a \leq x_1 < x_2 \leq b$

But $f(x)$ and $g(x)$ are L.I., so $g(x_1) \neq 0$ and $g(x_2) \neq 0$

Since by theorem III, if $f(x)$ and $g(x)$ have common zero on $[a, b]$ then they will be L.D.

Now we assume that $g(x)$ has no zero in the open interval $x_1 < x < x_2$.

Since $f(x)$ and $g(x)$ have continuous derivatives on $[a, b]$, so the quotient $\frac{f(x)}{g(x)}$ has a continuous derivative on the interval $x_1 \leq x \leq x_2$.

Also at the end points x_1 and x_2 of the interval $[x_1, x_2]$,

$$\frac{f(x)}{g(x)} = 0 \quad \because \quad f(x_1) = 0 \text{ and } f(x_2) = 0$$

Thus $\frac{f(x)}{g(x)}$ satisfy all conditions of Rolle's theorem in the interval $[x_1, x_2]$, so there exists a point $x = c$ (say) in the open interval $] x_1, x_2 [$, such that

$$\begin{aligned} & \left[\frac{d}{dx} \frac{f(x)}{g(x)} \right]_{x=c} = 0 \\ \Rightarrow & \left[\frac{f'(x) g(x) - f(x) g'(x)}{\{g(x)\}^2} \right]_{x=c} = 0 \\ \Rightarrow & [f'(x) g(x) - f(x) g'(x)]_{x=c} = 0 \quad \because \quad g'(c) \neq 0 \end{aligned}$$

$$\Rightarrow [W(g, f)]_{x=c} = \begin{vmatrix} g(x) & f(x) \\ g'(x) & f'(x) \end{vmatrix}_{x=c} = 0$$

$\Rightarrow$ functions $f(x)$ and $g(x)$ are L.D.

Which is a contradiction, as $f(x)$ and $g(x)$ are given L.I.

This contradiction shows that $g(x)$ has at least one zero in $x_1 < x < x_2$

Now we suppose that $g(x)$ has more than one zero in $x_1 < x < x_2$. Thus let x_3 and x_4 be two consecutive zeroes of $g(x)$ in $x_1 < x < x_2$.

Then proceeding as above by interchanging $f(x)$ and $g(x)$ [i.e., by taking $f(x)$ for $g(x)$ and $g(x)$ for $f(x)$] we can prove that $f(x)$ has at least one zero say $x = x_5$ in the open interval $x_3 < x < x_4$ i.e., $x_3 < x_5 < x_4$.

Now $x_3 < x_5 < x_4 \Rightarrow x_1 < x_5 < x_2$ as $x_3, x_4 \in [x_1, x_2]$ [i.e., $f(x)$ has zero at $x = x_5$ and $x_1 < x_5 < x_2$]

Which contradicts the fact that x_1 and x_2 are two consecutive zeroes of $f(x)$.

Thus, $g(x)$ has precisely one zero in the open interval $x_1 < x < x_2$.

Hence between any two consecutive zeroes of $f(x)$ there is precisely one zero of $g(x)$.

Similarly interchanging the role of $f(x)$ and $g(x)$ we can prove that between any two consecutive zeroes of $g(x)$ there is precisely one zero of $f(x)$.

Hence the zeroes of one of the two real L.I. solutions $f(x)$ and $g(x)$ of (1) separate the zeroes of the other solution.

For example: Consider the equation $\frac{d^2 y}{dx^2} + y = 0$

Obviously $f(x) = \cos x$ and $g(x) = \sin x$ are two L.I. solutions of the equation.

It is obvious from the following diagram of the two functions, that between two consecutive zeroes of one of the functions (solution) lies the zero of the other function (solution).

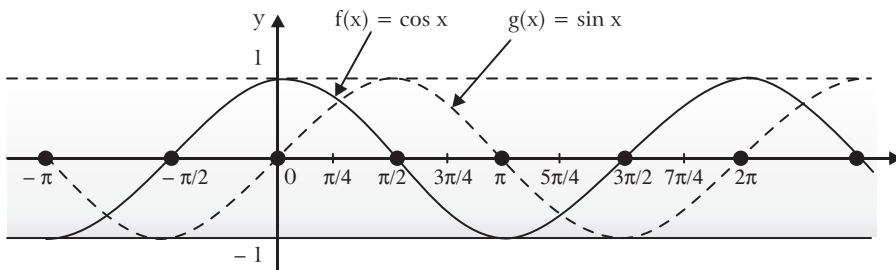


Fig. 2.2

Zeroes of $f(x) = \cos x$ are $x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$

and zeroes of $g(x) = \sin x$ are $x = 0, \pm \pi, \pm 2\pi, \dots$

2.16 Theorem V (Sturm's Fundamental Comparison Theorem)

On the interval $a \leq x \leq b$.

$$(i) \quad \text{let } f(x) \text{ be a real solution of } \frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + Q_1(x) y = 0 \quad \dots(1)$$

$$(ii) \quad \text{let } g(x) \text{ be a real solution of } \frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + Q_2(x) y = 0 \quad \dots(2)$$

and (iii) let P have a continuous derivative and be such that $P(x) > 0$ and let $Q_1(x)$ and $Q_2(x)$ be continuous and such that $Q_2(x) > Q_1(x)$.

Then if x_1 and x_2 are successive zeroes of $f(x)$ on $[a, b]$, then $g(x)$ has at least one zero at some point of the open interval $x_1 < x < x_2$.

[Amritsar 2000, 04; Kolkata 2004, Rohilkhand 2006; Meerut 2010 (Sem. I), 12 (Sem. I), 13 (Sem. I)]

Proof: Given x_1, x_2 are successive zeroes of $f(x)$ on $[a, b]$. Let $g(x)$ does not have a zero on the open interval $x_1 < x < x_2$.

So without loss of generality, we can assume that $f(x) > 0$ and $g(x) > 0$ on $x_1 < x < x_2$.

Since $f(x)$ is solution of (1) and $g(x)$ is solution of (2).

$$\therefore \quad \frac{d}{dx} [P(x) f'(x)] + Q_1(x) f(x) = 0 \quad \dots(3)$$

$$\text{and} \quad \frac{d}{dx} [P(x) g'(x)] + Q_2(x) g(x) = 0 \quad \dots(4)$$

Multiplying (3) by $g(x)$ and (4) by $f(x)$ and subtracting, we get

$$\begin{aligned} g(x) \frac{d}{dx} [P(x) f'(x)] - f(x) \frac{d}{dx} [P(x) g'(x)] &= [Q_2(x) - Q_1(x)] f(x) g(x) \\ \Rightarrow \quad \frac{d}{dx} [P(x) \{f'(x) g(x) - f(x) g'(x)\}] &= [Q_2(x) - Q_1(x)] f(x) g(x) \end{aligned}$$

Integrating both sides, from x_1 to x_2 , we get

$$[P(x) \{f'(x) g(x) - f(x) g'(x)\}]_{x_1}^{x_2} = \int_{x_1}^{x_2} [Q_2(x) - Q_1(x)] f(x) g(x) dx$$

$$\Rightarrow P(x_2) \cdot f'(x_2) g(x_2) - P(x_1) f'(x_1) g(x_1) = \int_{x_1}^{x_2} [Q_2(x) - Q_1(x)] f(x) g(x) dx \quad \dots(5)$$

$\therefore f(x_1) = 0 = f(x_2)$ as x_1 and x_2 are zeroes of $f(x)$.

Given that $P(x_2) > 0$. Since $f(x_2) = 0$ and $f(x) > 0$ on $x_1 < x < x_2$

so we have

$$f'(x_2) = 0$$

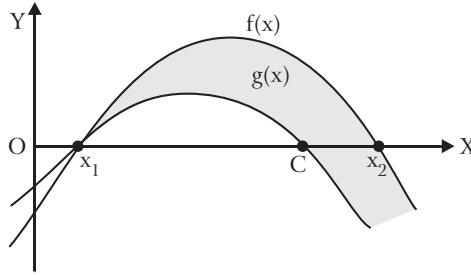


Fig. 2.3

Also $g(x) > 0$ on

$$x_1 < x < x_2,$$

we have

$$g(x_2) \geq 0$$

$$\therefore P(x_2) \cdot f'(x_2) g(x_2) \leq 0$$

Similarly, we have $P(x_1) f'(x_1) g(x_1) \geq 0$

$\therefore$ L.H.S. of (5) is not positive.

Since $Q_2(x) > Q_1(x)$ on $x_1 < x < x_2$, so the R.H.S. of (5) is positive. $\Rightarrow$ Equation (5) is not true.

Thus the assumption that $g(x)$ does not have a zero on the open interval $x_1 < x < x_2$ leads to a contradiction and so $g(x)$ has a zero at some point (say $x = c$) of this open interval.

Hence if x_1 and x_2 are successive zeroes of $f(x)$ on $[a, b]$, then $g(x)$ has at least one zero at some point of the open interval $x_1 < x < x_2$.

Particular Case

If x_1 is a zero of both $f(x)$ and $g(x)$ and x_2 and c are next (consecutive) zeroes of $f(x)$ and $g(x)$ respectively, then $c < x_2$, i.e., $x_1 < c < x_2$ (See above figure 2.3)

Note : If we take $P(x) = 1$ in (1) and (2) both, then the above theorem is stated as in the next article 2.17 and is also known as **Sturm Comparison Theorem** which is given in the next article 2.17.

2.17 Sturm Comparison Theorem

Let $y(x)$ and $z(x)$ be non-trivial solutions of $y'' + q(x)y = 0$ and $z'' + r(x)z = 0$, where $q(x)$ and $r(x)$ are positive functions, such that $q(x) > r(x)$.

Then $y(x)$ vanishes at least once between any two successive zeroes of $z(x)$.

Proof: Let x_1 and x_2 ($x_1 > x_2$) be two successive zeroes of $z(x)$, so that $z(x_1) = z(x_2) = 0$ and $z(x)$ does not vanish on the open interval (x_1, x_2) .

We assume that $y(x)$ does not vanish on the open interval (x_1, x_2) .

So without loss of generality, we can assume that $y(x) > 0$ and $z(x) > 0$ on $x_1 < x < x_2$.

The Wronskian of function y and z i.e., $W(y, z)$ is a function of x , so we can write this as $W(x)$.

$$\begin{aligned} \text{i.e.,} \quad W(x) = W(y, z) &= \begin{vmatrix} y(x) & z(x) \\ y'(x) & z'(x) \end{vmatrix} = y(x)z'(x) - z(x)y'(x) \\ &= yz' - zy' \text{ (say)} \end{aligned} \quad \dots(1)$$

$$\begin{aligned} \Rightarrow \quad \frac{dW}{dx} &= yz'' - zy'' \\ &= y(-rz) - z(-qy) \quad [\text{using the given equation (1)}] \\ &= (q - r)yz \end{aligned}$$

$\therefore$ Integrating both sides from x_1 to x_2 , we get

$$\begin{aligned} W(x_2) - W(x_1) &= \int_{x_1}^{x_2} (q - r) y(x) z(x) dx \\ \Rightarrow \quad W(x_2) - W(x_1) &> 0 \\ &[\because (q - r) y(x) z(x) > 0 \text{ as } q > r_1, y(x) > 0, z(x) > 0 \text{ on } (x_1, x_2)] \\ \Rightarrow \quad W(x_2) &> W(x_1) \end{aligned} \quad \dots(2)$$

Now $z(x) > 0$ on (x_1, x_2) and $z(x_1) = z(x_2) = 0$.

$\Rightarrow$ Slopes at $x = x_1$ is positive and that at $x = x_2$ is negative.

$$\Rightarrow \quad z'(x_1) > 0 \text{ and } z'(x_2) < 0$$

$$\text{Now, from (1)} \quad W(x_1) = y(x_1)z'(x_1) - z(x_1)y'(x_1) = y(x_1)z'(x_1) > 0$$

$$\text{and} \quad W(x_2) y(x_2) z'(x_2) - z(x_2) y'(x_2) = y(x_2) z'(x_2) < 0$$

$$\Rightarrow \quad W(x_2) < W(x_1)$$

Thus, (2) is a contradiction and so our assumption that $y(x)$ does not vanish on the open interval (x_1, x_2) is not true.

Hence, $y(x)$ vanishes at least once between any two successive zeroes of $z(x)$.

Solved Examples

Example 1: Find the adjoint to each of the following equations

$$(i) \quad x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + 3y = 0 \quad [\text{G.N.D.U. Amritsar 2002, 06, 07, 10}]$$

$$(ii) \quad \frac{d^4 y}{dx^4} + x \frac{d^3 y}{dx^3} + x^2 \frac{d^2 y}{dx^2} + x^3 \frac{dy}{dx} + x^3 y = 0$$

Solution: (i) Given, $x^2 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + 3y = 0$... (1)

Comparing (1) with the equation

$$a_0(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = 0 \quad \dots (2)$$

We get $a_0(x) = x^2$, $a_1(x) = 3x$ and $a_2(x) = 3$

The adjoint equation to equation (2) is

$$a_0(x) \frac{d^2 y}{dx^2} + [2a'_0(x) - a_1(x)] \frac{dy}{dx} + [a''_0(x) - a'_1(x) + a_2(x)] y = 0$$

$\therefore$ adjoint equation to equation (1) is

$$x^2 \frac{d^2 y}{dx^2} + [2 \cdot 2x - 3x] \frac{dy}{dx} + (2 - 3 + 3) y = 0$$

or $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + 2y = 0$

which is the required adjoint equation.

Aliter : Adjoint equation to equation (2) is

$$(-1)^2 \frac{d^2}{dx^2} [a_0(x) \cdot y] + (-1)^1 \frac{d}{dx} [a_1(x) \cdot y] + a_2(x) \cdot y = 0$$

$\therefore$ adjoint equation to equation (1) is

$$(-1)^2 \frac{d^2}{dx^2} (x^2 y) + (-1)^1 \frac{d}{dx} (3xy) + 3y = 0$$

or
$$\frac{d}{dx} \left(x^2 \frac{dy}{dx} + 2xy \right) - 3 \left(x \frac{dy}{dx} + y \right) + 3y = 0$$

or
$$\left(x^2 \frac{d^2 y}{dx^2} + 4x \frac{dy}{dx} + 2y \right) - 3 \left(x \frac{dy}{dx} + y \right) + 3y = 0$$

or
$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + 2y = 0$$

(ii) Given
$$\frac{d^4 y}{dx^4} + x \frac{d^3 y}{dx^3} + x^2 \frac{d^2 y}{dx^2} + x^3 \frac{dy}{dx} + x^3 y = 0$$

whose adjoint equation is

$$(-1)^4 \frac{d^4}{dx^4} (1 \cdot y) + (-1)^3 \frac{d^3}{dx^3} (x \cdot y) + (-1)^2 \frac{d^2}{dx^2} (x^2 \cdot y) + (-1)^1 \frac{d}{dx} (x^3 y) + x^3 y = 0$$

$$\begin{aligned} \text{or } \frac{d^4 y}{dx^4} - \left(\frac{d^3 y}{dx^3} \cdot x + {}^3C_1 \frac{d^2 y}{dx^2} \cdot 1 \right) + \left(\frac{d^2 y}{dx^2} \cdot x^2 + {}^2C_1 \frac{dy}{dx} \cdot 2x + {}^2C_2 y \cdot 2 \right) \\ - \left(x^3 \frac{dy}{dx} + 3x^2 y \right) + x^3 y = 0 \end{aligned}$$

$$\text{or } \frac{d^4 y}{dx^4} - x \frac{d^3 y}{dx^3} + (x^2 - 3) \frac{d^2 y}{dx^2} + (4x - x^3) \frac{dy}{dx} + (x^3 - 3x^2 + 2) y = 0$$

Example 2: Show that the Legendre's equation

$$(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n+1) y = 0$$

is self-adjoint.

Solution: Given
$$(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n+1) y = 0 \quad \dots(1)$$

Comparing with $a_0(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = 0$, we get

$$a_0(x) = 1 - x^2, a_1(x) = -2x \text{ and } a_2(x) = n(n+1)$$

∴ Adjoint equation of equation (1) is

$$a_0(x) \frac{d^2 y}{dx^2} + [2a'_0(x) - a_1(x)] \frac{dy}{dx} + [a''_0(x) - a'_1(x) + a_2(x)] y = 0$$

or $(1 - x^2) \frac{d^2 y}{dx^2} + [2 \cdot (-2x) - (-2x)] \frac{dy}{dx} + [-2 - (-2) + n(n+1)] y = 0$

or $(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n+1) y = 0$

Which is identical with (1).

Hence differential equation (1) is self-adjoint.

Aliter : Here $a_0(x) = 1 - x^2$ and $a_1(x) = -2x$

And $\frac{d}{dx} [a_0(x)] = \frac{d}{dx} (1 - x^2) = -2x = a_1(x)$

Which is necessary and sufficient condition for equation (1) to be self-adjoint.

Hence (1) is self-adjoint.

Example 3: Show that the following differential equation

$$\left(\frac{x+1}{x} \right) \frac{d^2 y}{dx^2} - \frac{1}{x^2} \frac{dy}{dx} + \frac{1}{x^3} y = 0 \text{ is self-adjoint and write it in the form}$$

$$\frac{d}{dx} \left[a_0(x) \frac{dy}{dx} \right] + a_2(x) y = 0$$

Solution: Given $\left(\frac{x+1}{x} \right) \frac{d^2 y}{dx^2} - \frac{1}{x^2} \frac{dy}{dx} + \frac{1}{x^3} y = 0$... (1)

Here $a_0(x) = \frac{x+1}{x} = \left(1 + \frac{1}{x} \right)$ and $a_1(x) = -\frac{1}{x^2}$

And $\frac{d}{dx} [a_0(x)] = \frac{d}{dx} \left(\frac{x+1}{x} \right) = -\frac{1}{x^2} = a_1(x)$

Hence equation (1) is self-adjoint.

This equation in the required form can be written as

$$\frac{d}{dx} \left[\left(\frac{x+1}{x} \right) \cdot \frac{dy}{dx} \right] + \frac{1}{x^3} y = 0$$

Example 4: Transform the following equation into an equivalent self-adjoint equation

$$\frac{d^2 y}{dx^2} - \tan x \frac{dy}{dx} + y = 0 \quad [\text{Meerut 2011 (Sem. I)}]$$

Solution: Given, $\frac{d^2 y}{dx^2} - \tan x \frac{dy}{dx} + y = 0 \quad \dots(1)$

Comparing with

$$a_0(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x)y = 0 \quad \dots(2)$$

We get $a_0(x) = 1, a_1(x) = -\tan x, a_2(x) = 1$

Multiplying (1) by

$$\frac{1}{a_0(x)} \exp. \left[\int \frac{a_1(x)}{a_0(x)} dx \right] = \frac{1}{1} \exp. \left[\int \left(\frac{-\tan x}{1} \right) dx \right] = e^{\log \cos x} = \cos x$$

the equivalent self-adjoint equation of (1) is

$$\cos x \left[\frac{d^2 y}{dx^2} - \tan x \frac{dy}{dx} + y \right] = 0$$

or $\cos x \frac{d^2 y}{dx^2} - \sin x \frac{dy}{dx} + \cos x \cdot y = 0$

or $\frac{d}{dx} [\cos x \cdot y] + \cos x \cdot y = 0$

Example 5: Solve $x^2 \frac{d^2 y}{dx^2} + (2x^3 + 7x) \frac{dy}{dx} + (8x^2 + 8)y = 0$ by finding a simple solution of its adjoint equation by inspection.

Solution: Given $x^2 \frac{d^2 y}{dx^2} + (2x^3 + 7x) \frac{dy}{dx} + (8x^2 + 8)y = 0 \quad \dots(1)$

Comparing with $a_0(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x)y = 0$

$$a_0(x) = x^2, a_1(x) = 2x^3 + 7x, a_2(x) = 8x^2 + 8$$

$\therefore$ Adjoint equation of equation (1) is

$$a_0(x) \frac{d^2 y}{dx^2} + [2a'_0(x) - a_1 x] \frac{dy}{dx} + [a''_0(x) - a'_1(x) + a_2(x)] y = 0$$

or
$$x^2 \frac{d^2 y}{dx^2} + [2 \cdot 2x - (2x^3 + 7x)] \frac{dy}{dx} + [2 - (6x^2 + 7) + 8x^2 + 8] y = 0$$

or
$$x^2 \frac{d^2 y}{dx^2} - (2x^3 + 3x) \frac{dy}{dx} + (2x^2 + 3) y = 0$$

or
$$\frac{d^2 y}{dx^2} - \left(2x + \frac{3}{x}\right) \frac{dy}{dx} + \left(2 + \frac{3}{x^2}\right) y = 0 \quad \dots(2)$$

Here $P + Qx = 0$, so $g(x) = x$ is a solution of (2).

$\therefore f(x)$, solution of (1) is a solution of $2 - 1 = 1$ st order linear differential equation

$$P[y, g(x)] = C$$

i.e., of
$$a_0(x) g(x) \frac{dy}{dx} + \{a_1(x) - a'_0(x)\} g(x) - a_0(x) g'(x) y = 0$$

(See equation (3) in particular case 2.8.1)

here $u = y$ and $v = g(x)$

$$\Rightarrow x^2 \cdot x \frac{dy}{dx} + [\{(2x^3 + 7x) - 2x\} x - x^2 \cdot 1] y = C$$

$$\Rightarrow x^3 \frac{dy}{dx} + (2x^4 + 4x^2) y = C$$

$$\Rightarrow \frac{dy}{dx} + (2x + 4/x) y = \frac{C}{x^3}$$

Which is a linear differential equation.

$$\text{I.F.} = e^{\int (2x + 4/x) dx} = e^{x^2 + 4 \log x} = x^4 e^{x^2}$$

$\therefore$ Solution is,
$$y \cdot x^4 e^{x^2} = C_1 + \int x^4 e^{x^2} (C_1 / x^3) dx = C_1 + 2 \int x e^{x^2} dx$$

$$= C_1 + C \cdot \frac{1}{2} e^{x^2}$$

$$\Rightarrow y = C_1 x^{-4} e^{-x^2} + C_2 x^{-4}$$

Hence by Particular case 2.9.1 on page 42, $f(x) = y = C_1 x^{-4} e^{-x^2} + C_2 x^{-4}$ is also the solution of the given differential equation.

Example 6: Prove that $u_1(x) = \sin x + \sqrt{2} \cos x$ and $u_2(x) = \sin x - \cos x$ are solutions of differential equation $u'' + u = 0$ and show that between every pair of consecutive zeroes of $u_1(x)$, there is a zero of $u_2(x)$.

Solution: Given equation is

$$u'' + u = 0 \quad \dots(1)$$

Also $u_1(x) = \sin x + \sqrt{2} \cos x \Rightarrow u_1' = \cos x - \sqrt{2} \sin x$ and $u_1'' = -\sin x - \sqrt{2} \cos x$

$$\Rightarrow u_1'' + u_1 = 0 \Rightarrow u_1 \text{ is solution of (1)}$$

Similarly, we can show that u_2 is also a solution of (1).

Now, Wronskian of u_1 and u_2 is

$$\begin{aligned} W(u_1, u_2) &= \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} \sin x + \sqrt{2} \cos x & \sin x - \cos x \\ \cos x - \sqrt{2} \sin x & \cos x + \sin x \end{vmatrix} \\ &= (\sin x + \sqrt{2} \cos x)(\cos x + \sin x) - (\sin x - \cos x)(\cos x - \sqrt{2} \sin x) \\ &= (\sin^2 x + \cos^2 x) + \sqrt{2}(\sin^2 x + \cos^2 x) = 1 + \sqrt{2} \neq 0 \end{aligned}$$

$\Rightarrow u_1$ and u_2 are linearly independent.

Thus u_1 and u_2 are linearly independent non-trivial solutions of (1). Hence by Sturm separation theorem between energy pair of consecutive zeroes of $u_1(x)$, there is a zero of $u_2(x)$.

Example 7: Show that the zeroes of the function $a \sin x + b \cos x$ and $c \sin x + d \cos x$ are distinct and occur alternatively whenever $ad - bc \neq 0$.

Solution: Let $u_1 = a \sin x + b \cos x$ and $u_2 = c \sin x + d \cos x$

$$\therefore u_1 = 0 \Rightarrow \tan x = -b/a \text{ and } u_2 = 0 \Rightarrow \tan x = -d/c$$

$$\therefore ad - bc \neq 0 \Rightarrow b/a \neq d/c$$

$\Rightarrow$ Zeroes of functions u_1 and u_2 are distinct.

Now, Wronskian of u_1 and u_2 is

$$W(u_1, u_2) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} a \sin x + b \cos x & c \sin x + d \cos x \\ a \cos x - b \sin x & c \cos x - d \sin x \end{vmatrix} = bc - ad \neq 0$$

$\Rightarrow u_1$ and u_2 are independent functions.

$$\begin{aligned} \text{Now let, } y &= C_1 u_1 + C_2 u_2 = C_1 (a \sin x + b \cos x) + C_2 (c \sin x + d \cos x) \quad \dots(1) \\ &= (aC_1 + cC_2) \sin x + (bC_1 + dC_2) \cos x \end{aligned}$$

where C_1 and C_2 are arbitrary constants.

Differentiating (1), we get $y'' = -(aC_1 + cC_2) \sin x - (bC_1 + dC_2) \cos x$

$$\Rightarrow y'' + y = 0. \quad \dots(2)$$

Thus u_1 and u_2 are linearly independent non-trivial solutions of equations (2), whenever $ad - bc \neq 0$. Hence by Sturm separation theorem 9.17, between every pair of zeroes of $u_1(x)$, there is a zero of $u_2(x)$ i.e., the zeroes of two functions occur alternatively.

Exercise

1. Find the adjoint to each of the following equations :

$$(i) \quad x^2 \frac{d^2 y}{dx^2} - 2x(1+x) \frac{dy}{dx} + 2(1+x)y = 0$$

$$(ii) \quad x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$$

$$(iii) \quad \frac{d^2 y}{dx^2} - \tan x \frac{dy}{dx} + y = 0$$

$$(iv) \quad x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0$$

$$(v) \quad (1-x^2) \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = 0$$

$$(vi) \quad x^3 \frac{d^3 y}{dx^3} + x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$$

2. Show that the following equations are self-adjoint and write it in the form

$$\frac{d}{dx} \left[a_0(x) \frac{dy}{dx} \right] + a_2(x) y = 0$$

$$(i) \quad \sin x \frac{d^2 y}{dx^2} + \cos x \frac{dy}{dx} + 2y = 0$$

$$(ii) \quad x^3 \frac{d^2 y}{dx^2} + 3x^2 \frac{dy}{dx} + y = 0$$

[Meerut 2012 (Sem. I)]

$$(iii) \quad (x \sin x + \cos x) \frac{d^2 y}{dx^2} + x \cos x \frac{dy}{dx} + y \cos x = 0$$

3. Transform each of the following equations into an equivalent self-adjoint equation :

$$(i) \quad x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$$

$$(ii) \quad x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0$$

$$(iii) \quad x^3 \frac{d^2 y}{dx^2} + 3x^2 \frac{dy}{dx} + y = 0$$

$$(iv) \quad (x^4 + x^2) \frac{d^2 y}{dx^2} + 2x^3 \frac{dy}{dx} + 3y = 0$$

4. Show that the adjoint equation of the adjoint equation of each of the following differential equations is the equation itself.

$$(i) \quad x^2 \frac{d^2 y}{dx^2} - (x^2 + 2x) \frac{dy}{dx} + (x + 2) y = 0$$

$$(ii) \quad x \frac{d^2 y}{dx^2} - 2(x + 1) \frac{dy}{dx} + (x + 2) y = 0$$

$$(iii) \quad (x + 1) \frac{d^2 y}{dx^2} - 2(x + 3) \frac{dy}{dx} + (x + 5) y = 0$$

5. Find bilinear concomitant $P(u, v)$ associated with L_2 in case of the following equations :

$$(i) \quad \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + (4x^2 - 1) y = 0$$

$$(ii) \quad x^2 \frac{d^2 y}{dx^2} + 9x \frac{dy}{dx} + 12y = 0$$

6. Consider the differential equation $L_2(y) = x \frac{d^2 y}{dx^2} - (x + 2) \frac{dy}{dx} + 2y = 0$

Find the adjoint equation $L_2^*(y) = 0$ and the bilinear concomitant $P(u, v)$ associated with L_2 and verify the truth of the Lagrange Identity.

7. Solve $x^2 \frac{d^2 y}{dx^2} + 7x \frac{dy}{dx} + 8y = 0$ by finding a solution of its adjoint equation by inspection.
8. Prove that $u_1(x) = \sin x + \sqrt{2} \cos x$ and $u_2(x) = \sin x - \cos x$ are solutions of differential equation $u'' + u = 0$ and show that between every pair of consecutive zeroes of $u_1(x)$, there is a zero of $u_2(x)$.
9. Prove that between every pair of consecutive zeroes of $u_1(x)$, there is a zero of $u_2(x)$
- (i) $u_1(x) = (3/2) \sin x - (1/2) \cos x$, $u_2(x) = 2 \sin x - \cos x$
- (ii) $u_1(x) = \sin(\log x)$, $u_2(x) = \cos(\log x)$
- [Hint : From the equation $u'' + u = 0$ whose L.I. solutions are u_1 and u_2]

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter (a), (b), (c) or (d).

1. The adjoint equation to the equation $x^2 \frac{d^2 y}{dx^2} - 3x \frac{dy}{dx} + 3y = 0$ is :

(a) $x^2 \frac{d^2 y}{dx^2} + 7x \frac{dy}{dx} + 8y = 0$

(b) $x^2 \frac{d^2 y}{dx^2} - 7x \frac{dy}{dx} + 8y = 0$

(c) $x^2 \frac{d^2 y}{dx^2} + 7x \frac{dy}{dx} - 8y = 0$

(d) $x^2 \frac{d^2 y}{dx^2} - 7x \frac{dy}{dx} - 8y = 0$

2. The adjoint equation to the equation $x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0$ is :

(a) $\frac{1}{x^2} \frac{d^2 y}{dx^2} + \frac{2}{x^3} \frac{dy}{dx} + \frac{2}{x^4} y = 0$

(b) $\frac{1}{x^2} \frac{d^2 y}{dx^2} - \frac{2}{x^3} \frac{dy}{dx} + \frac{2}{x^4} y = 0$

(c) $\frac{1}{x^2} \frac{d^2 y}{dx^2} - \frac{2}{x^3} \frac{dy}{dx} - \frac{2}{x^4} y = 0$

$$(d) \ x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0$$

3. The adjoint equation of the differential equation $(x-1) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + y = 0$ is :

$$(a) \ (x-1) y'' + (x+1) y' + 2y = 0$$

$$(b) \ (x-1) y'' + (x+2) y' - 2y = 0$$

$$(c) \ (x-1) y'' - (x+2) y' + 2y = 0$$

$$(d) \ (x-1) y'' + (x+2) y' + 2y = 0$$

4. The equivalent self-adjoint form of the equation $(1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - y = 0$ is :

$$(a) \ \frac{d}{dx} \left((1+x^2) \frac{dy}{dx} \right) + y = 0$$

$$(b) \ \frac{d}{dx} \left((1+x^2) \frac{dy}{dx} \right) - y = 0$$

$$(c) \ \frac{d}{dx} \left((1-x^2) \frac{dy}{dx} \right) - y = 0$$

$$(d) \ \frac{d}{dx} \left((1-x^2) \frac{dy}{dx} \right) + y = 0$$

5. For the differential equation $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + y = 0$, the bilinear concomitant

$P(u, v)$ associated with L_2 is :

$$(a) \ x^2 u' v + xuv - x^2 uv'$$

$$(b) \ x^2 u' v - xuv + x^2 uv'$$

$$(c) \ x^2 u' v - xuv - x^2 uv'$$

$$(d) \ x^2 u' v + xuv + x^2 uv'$$

Fill in the Blank

Fill in the blank's "...." so that the following statements are complete and correct.

- The necessary and sufficient condition for a second order linear differential equation to be self-adjoint is that
 - The adjoint equation of a self-adjoint equation is identical with the equation.
 - If the second order linear differential equation $a_0(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = 0$ is self-adjoint then it can be written as
 - Every linear differential equation of second order can be put in form.
-

5. If $f(x)$ and $g(x)$ be real linearly independent solutions of $\frac{d}{dx} \left[P(x) \frac{dy}{dx} \right] + Q(x)y = 0$ on $a \leq x \leq b$, then between any two consecutive zeroes of $f(x)$ there is one zero of $g(x)$.

True / False

Write 'T' for true and 'F' for false statement.

- The equation $x^2 \frac{d^2 y}{dx^2} + 7x \frac{dy}{dx} - 8y = 0$ is self-adjoint equation.
- For $L_2(y) = x^3 \frac{d^2 y}{dx^2} - 3x^2 \frac{dy}{dx} + y = 0$, $L_2^*(y)$ is $x^3 \frac{d^2 y}{dx^2} + 3x^2 \frac{dy}{dx} + y = 0$
- $L_2(y) = x \frac{d^2 y}{dx^2} - (2x+1) \frac{dy}{dx} + (x+1)y$, $L_2^*(y)$ is $x \frac{d^2 y}{dx^2} + (2x+3) \frac{dy}{dx} + (x+3)y$
- The equation $x^3 \frac{d^2 y}{dx^2} + 3x^2 \frac{dy}{dx} + y = 0$ is self-adjoint.

Answers

Exercise

1.	(i) $x^2 y'' + (6x + 2x^2) y' + (6 + 6x) y = 0$ (ii) $x^2 y'' + 3xy' + 2y = 0$ (iii) $y'' + \tan x \cdot y' + (\sec^2 x + 1) y = 0$ (iv) $x^2 y'' + 6xy' + 6y = 0$ (v) $(1 - x^2) y'' - 5xy' - 4y = 0$ (vi) $-x^3 y''' - 8x^2 y'' - 15xy' - 4y = 0$
2.	(i) $\frac{d}{dx} \left(\sin x \frac{dy}{dx} \right) + 2y = 0$ (ii) $\frac{d}{dx} \left(x^3 \frac{dy}{dx} \right) + y = 0$ (iii) $\frac{d}{dx} \left((x \sin x + \cos x) \cdot \frac{dy}{dx} \right) + y \cos x = 0$
3.	(i) $\frac{d}{dx} \left(x \frac{dy}{dx} \right) + \frac{1}{x} = 0$ (ii) $\frac{d}{dx} \left(\frac{1}{x^2} \frac{dy}{dx} \right) + \frac{2}{x^4} y = 0$ (iii) $\frac{d}{dx} \left(x^3 \frac{dy}{dx} \right) + y = 0$ (iv) $\frac{d}{dx} \left[(x^2 + 1) \frac{dy}{dx} \right] + \frac{3}{x^2} y = 0$
5.	(i) $u'v - 4xuv - uv'$ (ii) $x^2 u'v + 7xuv - x^2 uv'$
6.	$L_2^*(y) = x \frac{d^2 y}{dx^2} + (x+4) \frac{dy}{dx} + 3y = 0$, $P(u, v) = xu'v - (x+3)uv - xuv'$
7.	$y = C_1 x^{-2} + C_2 x^{-4}$

Objective Type Questions

Multiple Choice Questions

1.	(a)	2.	(b)	3.	(d)	4.	(c)
5.	(c)						

Fill in the Blank

1.	$\frac{d}{dx} [a_0(x)] = a_1(x)$	2.	Given
3.	$\frac{d}{dx} \left[a_0(x) \frac{dy}{dx} \right] + a_2(x) y = 0$	4.	Self-adjoint
5.	$g(x)$		

True / False

1.	False	2.	False	3.	True	4.	True

○○○



UNIT- 2

CHAPTER 3 : Integration in Series (Ordinary Points, Regular Singular Points and Frobenius Series Solution)



Unit - 2

Chapter

3

Integration in Series (Ordinary Points, Regular Singular Points and Frobenius Series Solution)

3.1 Introduction

Some times we come across a linear differential equation which cannot be solved by Ordinary methods. For solving such equations we assume that it is possible to find an infinite convergent series arranged in ascending or descending powers of the independent variable, which approximately satisfies the equation. The solutions of Legendre equation, Bessel equation, Hermite equation, Laguerre equation, Hypergeometric equation, in the form of convergent infinite series in ascending and descending powers of x play a very important role in applied mathematics.

In the present chapter we shall discuss the solutions of some linear differential equations of second order with variable coefficients in the form of infinite convergent series.

3.2 Some Important Definitions

1. **Power Series** : An infinite series of the form

$$\sum_{n=0}^{\infty} C_n (x-x_0)^n = C_0 + C_1 (x-x_0) + C_2 (x-x_0)^2 + \dots \quad \dots(1)$$

is called a **power series** in $(x-x_0)$

In particular, an infinite series

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + \dots \quad \dots(2)$$

is a power series in x .

2. **Radius and Interval of Convergence of Power Series :** The *radius of convergence* R of power series (1) or (2) is given by

$$R = \lim_{n \rightarrow \infty} \left| \frac{C_n}{C_{n+1}} \right|, \text{ provided the limit exists} \quad \dots(3)$$

i.e., The power series converges (absolutely) for $|x| < R$, where R is given by (3).

The interval $(-R, R)$ is said to be the *interval of convergence* of the power series.

In further discussions we shall use the following results.

- (i) A power series represents a continuous function within its interval of convergence.
- (ii) A power series can be differentiated term by term within its interval of convergence.

3.3 Ordinary and Singular Points

[Meerut 2011 (Sem. I)]

Ordinary Point : Def. 1 : “The point $x = x_0$ is called an **ordinary point** of the differential equation”

$$\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = 0 \quad \dots(1)$$

if both $P(x)$ and $Q(x)$ do not become infinite in a neighbourhood of the point $x = x_0$ and they can be expanded in the form of power series *i.e.*,

$$P(x) = \sum_{n=0}^{\infty} f(x) \cdot (x - x_0)^n \text{ and } Q(x) = \sum_{n=0}^{\infty} g(x) \cdot (x - x_0)^n$$

Def. 2 : “The point $x = x_0$ is called an **ordinary point** of the differential equation (1), if both $P(x)$ and $Q(x)$ are analytic at $x = x_0$ ”

Singular Point (Def.) : “The point $x = x_0$ is called a **singular point** of the differential equation (1) if it is not an ordinary point of this differential equation. Thus $x = x_0$ is a singular point of the differential equation (1), if both $P(x)$ and $Q(x)$ are not defined at $x = x_0$ *i.e.*, if either or both $P(x)$ and $Q(x)$ are not analytic at $x = x_0$.

The singular points are of two types :

1. Regular singular points and
2. Irregular singular points.

1. Regular Singular Point

[Meerut 2012 (Sem. I), 13 (Sem. I)]

A singular point $x = x_0$ is called the **regular singular point** of the differential equation (1) if both $(x - x_0) P(x)$ and $(x - x_0)^2 Q(x)$ can be expanded in series of $x - x_0$ in the neighbourhood of $x = x_0$ i.e., if both $(x - x_0) P(x)$ and $(x - x_0)^2 Q(x)$ are defined and so are analytic at $x = x_0$.

2. Irregular Singular Point

[Meerut 2012 (Sem. I)]

A singular point $x = x_0$ is called **irregular singular point** of the differential equation (1) if it is not a singular point i.e., if either or both $(x - x_0) P(x)$ and $(x - x_0)^2 Q(x)$ does not exist and so are not analytic at $x = x_0$.

Note : (i) Analytic Function : A function $f(x)$ defined on an interval containing the point $x = x_0$ is called **analytic** at x_0 , if its Taylor series about x_0

$\sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!} \cdot (x - x_0)^n$, exist and converges to $f(x)$ for all x in its interval of convergence.

(ii) A rational function $\frac{\phi(x)}{\psi(x)}$ is analytic except at those values of x at which the denominator $\psi(x) = 0$. i.e., the singular points of above rational function are given by $\psi(x) = 0$

For Example: Consider $f(x) = x^2 / (x^2 - 4x + 3) = \phi(x) / \psi(x)$

Here singular points of $f(x)$ are given by

$$\psi(x) = x^2 - 4x + 3 = 0 \Rightarrow x = 1 \text{ and } x = 3$$

(iii) All polynomial functions are analytic everywhere.

(iv) The functions e^x , $\sin x$ and $\cos x$ are analytic everywhere.

Solved Examples

Example 1: Show that $x = 0$ is an ordinary point of $(x^2 + 1)y'' + xy' - xy = 0$.

Solution: Dividing by $(x^2 + 1)$ the coefficient of y'' , the given equation reduces to

$$y'' + \frac{x}{x^2 + 1} y' - \frac{x}{x^2 + 1} y = 0 \quad \dots(1)$$

Comparing (1) with standard equation

$$y'' + P(x) y' + Q(x) y = 0 \quad \dots(2)$$

we get, $P(x) = \frac{x}{x^2 + 1}$ and $Q(x) = \frac{-x}{x^2 + 1}$

Since both $P(x)$ and $Q(x)$ are defined at $x=0$, so they are analytic at $x=0$.

Hence, $x=0$ is an ordinary point of the given equation.

Example 2: Show that $x=0$ is a regular singular point and $x=1$ is an irregular singular point of

$$x(x-1)^3 y'' + 2(x-1)^3 y' + 3y = 0$$

Solution: Dividing by $x(x-1)^3$, the coefficient of y'' , the given equation reduces to

$$y'' + \frac{2}{x} y' + \frac{3}{x(x-1)^3} y = 0 \quad \dots(1)$$

Comparing (1) with standard equation

$$y'' + P(x) y' + Q(x) y = 0 \quad \dots(2)$$

we get, $P(x) = \frac{2}{x}$ and $Q(x) = \frac{3}{x(x-1)^3}$

The point $x=0$. Since both $P(x)$ and $Q(x)$ are not defined at $x=0$ so they are not analytic at $x=0$.

Thus $x=0$ is a singular point.

Also $(x-0)P(x) = 2$ and $(x-0)^2 Q(x) = \frac{3x}{(x-1)^3}$

Since these both are defined at $x=0$, so are analytic at $x=0$.

Hence $x=0$ is a regular singular point.

The point $x=1$. Since $P(x)$ is defined but $Q(x)$ is not defined at $x=1$, so $Q(x)$ is not analytic at $x=1$. Thus $x=1$ is a singular point.

Also $(x-1)P(x) = \frac{2(x-1)}{x}$ and $(x-1)^2 Q(x) = \frac{3}{x(x-1)}$

Since, $(x-1)^2 Q(x) = \frac{3}{x(x-1)}$ is not defined at $x=1$, so it is not analytic at $x=1$.

Hence the point $x=1$ is irregular singular point.

Example 3: Discuss the singularities of the differential equation

$$(x-1) \frac{d^2 y}{dx^2} + (\cot \pi x) \frac{dy}{dx} + (\operatorname{cosec}^2 \pi x) y = 0 \text{ at } x=0 \text{ and } x=1.$$

Solution: Dividing by $(x-1)$, the coefficient of $\frac{d^2 y}{dx^2}$, the given differential equation reduces to

$$\frac{d^2 y}{dx^2} + \frac{\cot \pi x}{x-1} \frac{dy}{dx} + \frac{\operatorname{cosec}^2 \pi x}{(x-1)} y = 0 \quad \dots(1)$$

Comparing (1) with the standard equation

$$\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = 0 \quad \dots(2)$$

$$\text{We get, } P(x) = \frac{\cot \pi x}{x-1}, \quad Q(x) = \frac{\operatorname{cosec}^2 \pi x}{x-1}$$

Point $x=0$: Since both $P(x)$ and $Q(x)$ are not defined at $x=0$ (As $\cot 0 = \infty$ and $\operatorname{cosec} 0 = \infty$), so $P(x)$ and $Q(x)$ are not analytic at $x=0$. Thus $x=0$ is a singular point.

$$\text{Also } (x-0) P(x) = \frac{x \cot \pi x}{x-1} \Rightarrow \lim_{x \rightarrow 0} (x-0) P(x) = \lim_{x \rightarrow 0} \frac{\pi x}{\tan \pi x} \cdot \frac{1}{(x-1)\pi} = 1 \left(\frac{1}{-\pi} \right) = -\frac{1}{\pi}$$

$$\text{and } (x-0)^2 Q(x) = \frac{x^2 \operatorname{cosec}^2 \pi x}{x-1} \Rightarrow \lim_{x \rightarrow 0} \left(\frac{\pi x}{\sin \pi x} \right)^2 \cdot \frac{1}{\pi^2 (x-1)} = 1 \cdot \left(\frac{1}{-\pi^2} \right) = -\frac{1}{\pi^2}$$

i.e., both $(x-0) P(x)$ and $(x-0)^2 Q(x)$ exists and so are analytic at $x=0$

and hence $x=0$ is a regular singular point.

The point $x=1$: Since both $P(x)$ and $Q(x)$ are not defined at $x=1$, so $P(x)$ and $Q(x)$ are not analytic at $x=1$. Thus $x=1$ is a singular point.

$$\text{Also } (x-1) P(x) = (x-1) \cdot \frac{\cot \pi x}{x-1} = \cot \pi x, \text{ which is not defined at } x=1 \text{ so is not analytic at } x=1$$

$$\text{and } (x-1)^2 Q(x) = (x-1) \operatorname{cosec}^2 \pi x$$

$$\begin{aligned} \Rightarrow \lim_{x \rightarrow 1} (x-1) \operatorname{cosec}^2 \pi x &= \lim_{x \rightarrow 1} \frac{x-1}{\sin^2 \pi x}, \left(\frac{0}{0} \text{ form} \right) \\ &= \lim_{x \rightarrow 1} \frac{1}{\pi \sin 2\pi x} = \infty, \text{ i.e., does not exist.} \end{aligned}$$

Thus, both $(x-1)P(x)$ and $(x-1)^2Q(x)$ are not defined at $x=1$.

$\Rightarrow$ the point $x=1$ is irregular singular point.

Example 4: Discuss the singularities of the equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0 \text{ at } x=0 \text{ and } x=\infty.$$

Solution: Dividing by x^2 , the coefficient of $\frac{d^2 y}{dx^2}$, the given differential equation reduces

$$\text{to } \frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \frac{x^2 - n^2}{x^2} y = 0 \quad \dots(1)$$

Comparing (1) with the standard equation

$$\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = 0 \quad \dots(2)$$

$$\text{we get } P(x) = \frac{1}{x} \text{ and } Q(x) = \frac{x^2 - n^2}{x^2} = 1 - \frac{n^2}{x^2}$$

Singularity at the point $x=0$:

Since both $P(x)$ and $Q(x)$ are not defined at $x=0$, so $x=0$ is a singular point.

Also, $(x-0)P(x) = 1$ and $(x-0)^2Q(x) = x^2 - n^2$.

Since both these are defined at $x=0$ so are analytic at $x=0$.

Hence, $x=0$ is a regular singular point.

Singularity at the point $x=\infty$:

To discuss singularity at $x=\infty$, we proceed as follows :

$$\text{Let } x = \frac{1}{t}, \text{ so that } t = \frac{1}{x} \text{ and } \frac{dt}{dx} = -\frac{1}{x^2} = -t^2$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = -t^2 \frac{dy}{dt}$$

$$\begin{aligned} \text{and } \frac{d^2 y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(-t^2 \frac{dy}{dt} \right) = \frac{d}{dt} \left(-t^2 \frac{dy}{dt} \right) \cdot \frac{dt}{dx} \\ &= \left(-t^2 \frac{d^2 y}{dt^2} - 2t \frac{dy}{dt} \right) \cdot (-t^2) = t^4 \frac{d^2 y}{dt^2} + 2t^3 \frac{dy}{dt} \end{aligned}$$

Substituting in (1), it reduces to

$$t^4 \frac{d^2 y}{dt^2} + 2t^3 \frac{dy}{dt} + t \left(-t^2 \frac{dy}{dt} \right) + (1 - n^2 t^2) y = 0$$

or
$$\frac{d^2 y}{dt^2} + \frac{1}{t} \frac{dy}{dt} + \frac{1 - n^2 t^2}{t^4} y = 0 \quad \dots(3)$$

Obviously the nature of point $x = \infty$ for (2) is the same as the nature of the point $t = 0$ for (3).

Comparing (3), with $\frac{d^2 y}{dt^2} + P(t) \frac{dy}{dt} + Q(t) y = 0$

We get, $P(t) = \frac{1}{t}$ and $Q(t) = \frac{1 - n^2 t^2}{t^4}$

Since both $P(t)$ and $Q(t)$ are not defined at $t = 0$, so $t = 0$ is a singular point.

Also $(t - 0) P(t) = 0$ and $(t - 0)^2 Q(t) = \frac{1}{t^2} - n^2$

Since $(t - 0)^2 Q(t) = \frac{1}{t^2} - n^2$ does not exist at $t = 0$ i.e., is not analytic at $t = 0$, so the point $t = 0$ is irregular singular point of the equation (3).

Hence, $x = \infty$ is irregular singular point of the given equation.

3.4 Theorem

If x_0 is an ordinary point of the differential equation

$$P_0(x) \frac{d^2 y}{dx^2} + P_1(x) \frac{dy}{dx} + P_2(x) y = 0$$

then this differential equation has two non-trivial linearly independent power series solutions of the form

$$\sum_{n=0}^{\infty} C_n (x - x_0)^n$$

and these power series converge in some interval $|x - x_0| < R$, where $R > 0$, about x_0 .

Proof: The proof of this theorem is omitted at this stage.

3.5 Method of Series Solution about an Ordinary Point $x = x_0$

Let $x = x_0$ be an ordinary point of the differential equation

$$\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = 0 \quad \dots(1)$$

Since $x = x_0$ is an ordinary point of (1), so $P(x)$ and $Q(x)$ are analytic at $x = x_0$ and thus they can be expanded in non-trivial linearly independent power series in $(x - x_0)$

$$\therefore \quad \text{let} \quad P(x) = \sum_{n=0}^{\infty} f(x) \cdot (x - x_0)^n \quad \dots(2)$$

$$\text{and} \quad Q(x) = \sum_{n=0}^{\infty} g(x) \cdot (x - x_0)^n \quad \dots(3)$$

$$\text{Also let} \quad y = \sum_{n=0}^{\infty} C_n (x - x_0)^n \quad \dots(4)$$

be the series solution of (1) about the point $x = x_0$ (ordinary point).

Differentiating (4), we get

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} n C_n (x - x_0)^{n-1} \quad \dots(5)$$

$$\text{and} \quad \frac{d^2 y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) C_n (x - x_0)^{n-2} \quad \dots(6)$$

Substituting from (2), (3), (4), (5) and (6) in (1), we shall get an equation of the form.

$$A_0 + A_1 (x - x_0) + A_2 (x - x_0)^2 + A_3 (x - x_0)^3 + \dots + A_n (x - x_0)^n + \dots = 0 \quad \dots(7)$$

where $A_0, A_1, A_2, \dots, A_n, \dots$ are some functions of coefficients $C_0, C_1, C_2, \dots$, in (4). Since (7) is an identity, so all the coefficients of various power of $(x - x_0)$ must be zero.

$$\text{i.e., } A_0 = 0, A_1 = 0, A_2 = 0, \dots, A_n = 0, \dots \quad \dots(8)$$

From these equations in (8), we can obtain the values of $C_0, C_1, C_2, \dots$.

Substituting the values of these coefficients in (4) we obtain the required series solution of (1) in powers of $(x - x_0)$.

Solved Examples

Example 1: Find the series solution of the differential equation

$$(1 - x^2) y'' + 2xy' - y = 0 \text{ about } x = 0.$$

Solution: Given differential equation is

$$(1 - x^2) y'' + 2xy' - y = 0 \quad \dots(1)$$

Dividing by $(1 - x^2)$ it reduces to

$$y'' + \frac{2x}{1-x^2} y' - \frac{1}{1-x^2} y = 0$$

$$\Rightarrow P(x) = \frac{2x}{1-x^2} \text{ and } Q(x) = -\frac{1}{1-x^2}$$

Since at $x = 0$, $P(x)$ and $Q(x)$ both exists.

So the point $x = 0$ is an ordinary point of the diff. equation (1).

Let the solution of (1) in the form of a power series about $x = 0$ be given by

$$y = \sum_{n=0}^{\infty} C_n x^n \quad \dots(2)$$

Diff. (2), we get $y' = \sum_{n=1}^{\infty} n C_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$

Putting in (1) and simplifying we get

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) C_n x^n + 2 \sum_{n=1}^{\infty} n C_n x^n - \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\Rightarrow (2 \cdot 1 C_2 + 3 \cdot 2 C_3 x + 4 \cdot 3 C_4 x^2 + 5 \cdot 4 C_5 x^3 + \dots)$$

$$- (2 \cdot 1 C_2 x^2 + 3 \cdot 2 C_3 x^3 + 4 \cdot 3 C_4 x^4 + 5 \cdot 4 C_5 x^5 + \dots)$$

$$+ 2 (1 \cdot C_1 x + 2 C_2 x^2 + 3 C_3 x^3 + 4 C_4 x^4 + 5 C_5 x^5 + \dots)$$

$$- (C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + \dots) = 0$$

$$\Rightarrow (2 C_2 - C_0) + (6 C_3 + 2 C_1 - C_1) x + (12 C_4 - 2 C_2 + 4 C_2 - C_2) x^2$$

$$+ (20 C_5 - 6 C_3 + 6 C_3 - C_3) x^3 + \dots = 0$$

$$\Rightarrow (2 C_2 - C_0) + (6 C_3 + C_1) x + (12 C_4 + C_2) x^2 + (20 C_5 - C_3) x^3 + \dots = 0 \quad \dots(3)$$

Since (3) is an identity, so the coefficients of various powers of x must be zero.

$$\therefore 2C_2 - C_0 = 0, 6C_3 + C_1 = 0, 12C_4 + C_2 = 0, 20C_5 - C_3 = 0, \dots\dots$$

$$\Rightarrow C_2 = \frac{1}{2}C_0, C_3 = -\frac{1}{6}C_1, C_4 = -\frac{1}{12}C_2 = -\frac{1}{24}C_0, C_5 = \frac{1}{20}C_3 = -\frac{1}{120}C_1, \dots\dots$$

$$\therefore \text{From (2), } y = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + \dots$$

$$= C_0 + C_1x + \frac{1}{2}C_0x^2 - \frac{1}{6}C_1x^3 - \frac{1}{24}C_0x^4 - \frac{1}{120}C_1x^5 + \dots$$

$$= C_0 \left(1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \dots \right) + C_1 \left(x - \frac{x^3}{3!} - \frac{x^5}{5!} + \dots \right)$$

Which is the required general solution of (1) in the form of series about $x=0$ as it has two arbitrary constant C_0 and C_1 .

Example 2: Find the solution in series of $\frac{d^2y}{dx^2} + x \frac{dy}{dx} + x^2y = 0$, about $x=0$.

[Delhi Maths (Hons.) 2005, 08, Guwahati 2007]

Solution: Given differential equation is

$$\frac{d^2y}{dx^2} + x \frac{dy}{dx} + x^2y = 0 \quad \dots(1)$$

Here, $P(x) = x$ and $Q(x) = x^2$. Since both $P(x)$ and $Q(x)$ exists at $x=0$, so are analytic at $x=0$. Thus $x=0$ is an ordinary point of (1).

Let the solution of (1), in the form of a power series about $x=0$, be given by

$$y = \sum_{n=0}^{\infty} C_n x^n \quad \dots(2)$$

$$\text{Diff. (2), we get } \frac{dy}{dx} = \sum_{n=1}^{\infty} n C_n x^{n-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

Putting in (1), we get

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + \sum_{n=1}^{\infty} n C_n x^n + \sum_{n=0}^{\infty} C_n x^{n+2} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n + \sum_{n=1}^{\infty} n C_n x^n + \sum_{n=2}^{\infty} C_{n-2} x^n = 0$$

$$\Rightarrow \left[2.1C_2 + 3.2C_3x + \sum_{n=2}^{\infty} (n+2)(n+1) C_{n+2} x^n \right] + \left[C_1x + \sum_{n=2}^{\infty} n C_n x^n \right] + \sum_{n=2}^{\infty} C_{n-2} x^n = 0$$

$$\Rightarrow 2C_2 + (6C_3 + C_1)x + \sum_{n=2}^{\infty} [(n+2)(n+1)C_{n+2} + nC_n + C_{n-2}]x^n = 0 \quad \dots(3)$$

Since (3) is an identity, so the coefficient of various power of x must be zero.

$$\therefore 2C_2 = 0, 6C_3 + C_1 = 0 \text{ and } (n+2)(n+1)C_{n+2} + nC_n + C_{n-2} = 0 \text{ for } n \geq 2$$

$$\Rightarrow C_2 = 0, C_3 = -\frac{1}{6}C_1 \text{ and } C_{n+2} = \frac{-(nC_n + C_{n-2})}{(n+2)(n+1)}; \text{ for } n \geq 2$$

$$\text{For } n = 2, C_4 = \frac{-(2C_2 + C_0)}{4 \cdot 3} = \frac{-C_0}{12}$$

$$\text{For } n = 3, C_5 = \frac{-(3C_3 + C_1)}{5 \cdot 4} = \frac{-(-C_1/2 + C_1)}{20} = -\frac{C_1}{40}$$

$$\text{For } n = 4, C_6 = \frac{-(4C_4 + C_2)}{6 \cdot 5} = \frac{-4C_4}{30} = -\frac{2}{15} \cdot \left(-\frac{C_0}{12}\right) = \frac{C_0}{90}$$

and so on.

$\therefore$ from (2), we have

$$y = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + C_6x^6 + \dots$$

$$\text{or } y = C_0 \left(1 - \frac{1}{12}x^4 + \frac{1}{90}x^6 - \dots\right) + C_1 \left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - \dots\right)$$

Which is the required solution of (1) in the form of a series about $x=0$ as it has two arbitrary constant C_0 and C_1 .

Example 3: Solve in series the Legendre's equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

where n is an arbitrary constant. [Utkal 2003; Delhi Maths (Hons.) 2006; Meerut 2012 (Sem. I)]

Solution: Given differential equation is

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \dots(1)$$

Dividing by $1-x^2$, (1) reduces to

$$y'' + \frac{-2x}{1-x^2}y' + \frac{n(n+1)}{1-x^2}y = 0$$

$$\text{Here } P(x) = -\frac{2x}{1-x^2} \text{ and } Q(x) = \frac{n(n+1)}{1-x^2}$$

Both $P(x)$ and $Q(x)$ exist at $x=0$ i.e., are analytic at $x=0$, so $x=0$ is an ordinary point of (1).

Let the solution of (1) in the form of a power series about $x=0$, be given by

$$y = \sum_{r=0}^{\infty} C_r x^r \quad \dots(2)$$

Diff. (2), we get $y' = \sum_{r=1}^{\infty} r C_r x^{r-1}$ and $y'' = \sum_{r=2}^{\infty} r(r-1) C_r x^{r-2}$

Putting in (1), we get

$$(1-x^2) \sum_{r=2}^{\infty} r(r-1) C_r x^{r-2} - 2x \sum_{r=1}^{\infty} r C_r x^{r-1} + n(n+1) \sum_{r=0}^{\infty} C_r x^r = 0$$

$$\text{or } \sum_{r=2}^{\infty} r(r-1) C_r x^{r-2} - \sum_{r=2}^{\infty} r(r-1) C_r x^r - 2 \sum_{r=1}^{\infty} r C_r x^r + n(n+1) \sum_{r=0}^{\infty} C_r x^r = 0$$

$$\begin{aligned} \text{or } \sum_{r=0}^{\infty} (r+2)(r+1) C_{r+2} x^r - \sum_{r=2}^{\infty} r(r-1) C_r x^r - 2 \left[C_1 x + \sum_{r=2}^{\infty} r C_r x^r \right] \\ + n(n+1) \left[C_0 + C_1 x + \sum_{r=2}^{\infty} C_r x^r \right] = 0 \end{aligned}$$

$$\begin{aligned} \text{or } 2.1C_2 + 3.2C_3x + \sum_{r=2}^{\infty} (r+2)(r+1) C_{r+2} x^r - 2C_1x + n(n+1) C_0 \\ + n(n+1) C_1x - \sum_{r=2}^{\infty} r(r-1) C_r x^r - 2 \sum_{r=2}^{\infty} r C_r x^r + n(n+1) \sum_{r=2}^{\infty} C_r x^r = 0 \end{aligned}$$

$$\begin{aligned} \text{or } [n(n+1) C_0 + 2C_2] + [6C_3 - 2C_1 + n(n+1) C_1] x \\ + \sum_{r=2}^{\infty} [(r+2)(r+1) C_{r+2} - r(r-1) C_r - 2r C_r + n(n+1) C_r] x^r = 0 \end{aligned}$$

$$\begin{aligned} \text{or } [n(n+1) C_0 + 2C_2] + [6C_3 + (n^2 + n - 2) C_1] x \\ + \sum_{r=2}^{\infty} [(r+2)(r+1) C_{r+2} + (n^2 - r^2 + n - r) C_r] x^r = 0 \end{aligned}$$

$$\begin{aligned} \text{or } [n(n+1) C_0 + 2C_2] + [6C_3 + (n-1)(n+2) C_1] x \\ + \sum_{r=2}^{\infty} [(r+2)(r+1) C_{r+2} + (n-r)(n+r+1) C_r] x^r = 0 \quad \dots(3) \end{aligned}$$

Since (3) is an identity, so the coefficients of various powers of x must be zero.

$$\therefore n(n+1)C_0 + 2C_2 = 0 \Rightarrow C_2 = -\frac{n(n+1)}{2!}C_0$$

$$6C_3 + (n-1)(n+2)C_1 = 0 \Rightarrow C_3 = -\frac{(n-1)(n+2)}{3!}C_1$$

$$\text{and } (r+2)(r+1)C_{r+2} + (n-r)(n+r+1)C_r = 0 \text{ for } r \geq 2$$

$$\Rightarrow C_{r+2} = -\frac{(n-r)(n+r+1)}{(r+2)(r+1)}C_r, \text{ for } r \geq 2$$

$$\begin{aligned} \text{For } r=2, C_4 &= -\frac{(n-2)(n+3)}{4 \cdot 3} \cdot C_2 = \left(-\frac{(n-2)(n+3)}{4 \cdot 3}\right) \left(-\frac{n(n+1)}{2!}C_0\right) \\ &= \frac{n(n-2)(n+1)(n+3)}{4!}C_0 \end{aligned}$$

$$\begin{aligned} \text{For } r=3, C_5 &= -\frac{(n-3)(n+4)}{5 \cdot 4} \cdot C_3 = \left(-\frac{(n-3)(n+4)}{5 \cdot 4}\right) \left(-\frac{(n-1)(n+2)}{3!}C_1\right) \\ &= \frac{(n-1)(n-3)(n+2)(n+4)}{5!}C_1 \end{aligned}$$

and so on.

$$\therefore \text{ From (2), we have, } y = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + \dots$$

$$\begin{aligned} \text{or } y &= C_0 \left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \dots \right] \\ &\quad + C_1 \left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \dots \right] \end{aligned}$$

which is the required general solution of (1) in the form of a series about $x=0$, as it has two arbitrary constants C_0 and C_1 .

Example 4: Solve $y'' - xy' - py = 0$ is power of x .

Solution: Given $y'' - xy' - py = 0$... (1)

Obviously $x=0$ is an ordinary point of (1).

Let the solution of (1) in the form of a power series about $x=0$, be given by

$$y = \sum_{r=0}^{\infty} C_r x^r \quad \dots (2)$$

$$\text{Diff. (2) we get } y' = \sum_{r=1}^{\infty} r C_r x^{r-1} \text{ and } y'' = \sum_{r=2}^{\infty} r(r-1) C_r x^{r-2}$$

Putting in (1), we get

$$\sum_{r=2}^{\infty} r(r-1) C_r x^{r-2} - \sum_{r=1}^{\infty} r C_r x^r - p \sum_{r=0}^{\infty} C_r x^r = 0$$

$$\text{or } \sum_{r=0}^{\infty} (r+2)(r+1) C_{r+2} x^r - \sum_{r=1}^{\infty} r C_r x^r - p \sum_{r=0}^{\infty} C_r x^r = 0$$

$$\text{or } 2 \cdot 1 C_2 + \sum_{r=1}^{\infty} (r+2)(r+1) C_{r+2} x^r - \sum_{r=1}^{\infty} r C_r x^r - p \left[C_0 + \sum_{r=1}^{\infty} C_r x^r \right] = 0$$

$$\text{or } (2C_2 - pC_0) + \sum_{r=1}^{\infty} [(r+2)(r+1) C_{r+2} - r C_r - p C_r] x^r = 0 \quad \dots(3)$$

Which is an equality, so we can equal to zero the coefficients of various powers of x .

$$\therefore 2C_2 - pC_0 = 0 \Rightarrow C_2 = \frac{p}{2} C_0 = \frac{p}{2!} C_0$$

$$\text{and } (r+2)(r+1) C_{r+2} - r C_r - p C_r = 0 \text{ for } r \geq 1$$

$$\Rightarrow C_{r+2} = \frac{(p+r)}{(r+1)(r+2)} C_r \text{ for } r \geq 1 \quad \dots(4)$$

Putting $r = 1, 2, 3, \dots$

$$C_3 = \frac{p+1}{2 \cdot 3} C_1 = \frac{p+1}{3!}, C_4 = \frac{p+2}{3 \cdot 4} C_2 = \frac{p+2}{3 \cdot 4} \frac{p}{2} C_0 = \frac{p(p+2)}{4!} C_0$$

$$C_5 = \frac{p+3}{4 \cdot 5} C_3 = \frac{p+3}{4 \cdot 5} \cdot \frac{p+1}{2 \cdot 3} C_1 = \frac{(p+1)(p+3)}{5!} C_1$$

$$C_6 = \frac{p+4}{5 \cdot 6} C_4 = \frac{p(p+2)(p+4)}{5 \cdot 6 \cdot 4!} C_0 = \frac{p(p+2)(p+4)}{6!} C_0 \text{ and so on.}$$

$\therefore$ From (2),

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots$$

$$\begin{aligned} \text{or } y = C_0 & \left[1 + p \cdot \frac{x^2}{2!} + p(p+2) \cdot \frac{x^4}{4!} + p(p+2)(p+4) \cdot \frac{x^6}{6!} + \dots \right] \\ & + C_1 \left[1 + (p+1) \cdot \frac{x^3}{3!} + (p+1)(p+3) \cdot \frac{x^5}{5!} + \dots \right] \end{aligned}$$

Which is the required solution, where C_0 and C_1 are arbitrary constants.

Example 5: Solve $y'' + x^2 y = 2 + x + x^2$ is powers of x .

Solution: Given equation is $y'' + x^2 y = 2 + x + x^2$... (1)

Here $P(x) = 0$ and $Q(x) = x^2$, both exist at $x = 0$. So $x = 0$ is an ordinary point of the equation (1).

Let the solution of (1) in powers of x be given by

$$y = \sum_{n=0}^{\infty} C_n x^n \quad \dots (2)$$

$$\text{Diff. (2),} \quad y' = \sum_{n=1}^{\infty} n C_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

Putting in (1), we get

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+2} = 2 + x + x^2$$

$$\text{or} \quad \sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n + \sum_{n=2}^{\infty} C_{n-2} x^n = 2 + x + x^2$$

$$\begin{aligned} \text{or} \quad & (2 \cdot 1 \cdot C_2 - 2) + (3 \cdot 2 \cdot C_3 - 1)x + (4 \cdot 3 \cdot C_4 + C_0 - 1)x^2 \\ & + \sum_{n=3}^{\infty} [(n+2)(n+1) C_{n+2} + C_{n-2}] x^n = 0 \quad \dots (3) \end{aligned}$$

Since (3) is an identity, so the coefficients of various powers of x must be zero.

$$\therefore 2C_2 - 2 = 0 \Rightarrow C_2 = 1, 6C_3 - 1 = 0 \Rightarrow C_3 = 1/6, 12C_4 + C_0 - 1 = 0 \Rightarrow C_4 = \frac{1}{12} - \frac{C_0}{12}$$

$$\text{and } (n+2)(n+1) C_{n+2} + C_{n-2} = 0 \quad \text{for } n \geq 3$$

$$\Rightarrow C_{n+2} = -\frac{1}{(n+2)(n+1)} C_{n-2}, \quad \text{for } n \geq 3$$

$$\text{for } n=3, \quad C_5 = -\frac{1}{5 \cdot 4} C_1 = -\frac{1}{20} C_1$$

$$\text{for } n=4, \quad C_6 = -\frac{1}{6 \cdot 5} C_2 = \left(-\frac{1}{30}\right)(1) = -\frac{1}{30}$$

$$\text{for } n=5, \quad C_7 = -\frac{1}{7 \cdot 6} C_3 = -\left(-\frac{1}{42}\right)\left(\frac{1}{6}\right) = -\frac{1}{252}$$

for $n=6$, $C_8 = -\frac{1}{8.7} \cdot C_4 = \left(-\frac{1}{56}\right)\left(\frac{1}{12} - \frac{1}{12} C_0\right) = -\frac{1}{672} + \frac{1}{672} C_0$

for $n=7$, $C_9 = -\frac{1}{9.8} C_5 = \left(-\frac{1}{72}\right)\left(-\frac{1}{20} C_1\right) = \frac{1}{1440} C_1$

and so on.

$\therefore$ from (2), we have

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + \dots$$

or
$$y = C_0 + C_1 x + 1. x^2 + \frac{1}{6} x^3 + \left(\frac{1}{12} - \frac{1}{12} C_0\right) x^4 + \left(-\frac{1}{20} C_1\right) x^5 + \left(-\frac{1}{30}\right) x^6$$

$$+ \left(-\frac{1}{252}\right) x^7 + \left(-\frac{1}{672} + \frac{1}{672} C_0\right) x^8 + \left(\frac{1}{1440} C_1\right) x^9 + \dots$$

or
$$y = C_0 \left[1 - \frac{1}{12} x^4 + \frac{1}{672} x^8 \dots\right] + C_1 \left[x - \frac{1}{20} x^5 + \frac{1}{1440} x^9 \dots\right]$$

$$+ x^2 + \frac{1}{6} x^3 + \frac{1}{12} x^4 - \frac{1}{30} x^6 - \frac{1}{252} x^7 - \frac{1}{672} x^8 - \dots$$

Which is the required general solution of (1) in the form of a power series in x , as it has two arbitrary constant C_0 and C_1 .

3.6 Series Solution in Powers of $(x - a)$, where $x = a \neq 0$ is an Ordinary Point

We can also find the series solution of a differential equation about a point other than $x=0$. To find series solution about a point $x=a$ i.e., in power of $x-a$ put $x-a=v$ i.e., $x=a+v$ in the given equation and find the series solution of the transformed equation about the point $v=0$. Finally replace $v=x-a$ to obtain the required solution. For clear understanding of the procedure see the following examples.

Solved Examples

Example 1: Find the power series solution of the initial value problem :

$$xy'' + y' + 2y = 0, y(1) = 1, y'(1) = 2$$

[Purvanchal 2007]

Solution: Given differential equation is

$$xy'' + y' + 2y = 0 \quad \dots(1)$$

Dividing by x , the coeff. of y'' , it reduces to

$$y'' + \frac{1}{x} y' + \frac{2}{x} y = 0$$

$$\Rightarrow P(x) = \frac{1}{x}, Q(x) = \frac{2}{x}$$

Since the values of y and y' are given at $x = 1$, so we shall find the series solution of (1) near the point $x = 1$.

Here both $P(x)$ and $Q(x)$ are finite i.e., analytic at $x = 1$, so $x = 1$ is an ordinary point of (1).

To find the solution of (1) in power of $(x - 1)$, we proceed as follows (Note).

$$\text{Let } x - 1 = v, \quad \therefore \frac{dv}{dx} = 1$$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx} = 1 \cdot \frac{dy}{dv}$$

$$\text{and } \frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{dy}{dv} \right) = \frac{d^2 y}{dv^2} \cdot \frac{dv}{dx} = 1 \cdot \frac{d^2 y}{dv^2}$$

Substituting in (1), it reduces to

$$(v + 1) \frac{d^2 y}{dv^2} + \frac{dy}{dv} + 2y = 0 \quad \dots(2)$$

$$\text{where } v = x - 1$$

Now let the solution of (2) in powers of v be given by

$$y = \sum_{n=0}^{\infty} C_n v^n \quad \dots(3)$$

$$\text{Differentiating, } \frac{dy}{dv} = \sum_{n=1}^{\infty} n C_n v^{n-1} \quad \dots(4)$$

$$\text{and } \frac{d^2 y}{dv^2} = \sum_{n=2}^{\infty} n(n-1) C_n v^{n-2} \quad \dots(5)$$

Substituting in (2), we get

$$\sum_{n=2}^{\infty} n(n-1) C_n v^{n-1} + \sum_{n=2}^{\infty} n(n-1) C_n v^{n-2} + \sum_{n=1}^{\infty} n C_n v^{n-1} + 2 \sum_{n=0}^{\infty} C_n v^n = 0$$

$$\begin{aligned} \text{or } \sum_{n=1}^{\infty} (n+1) n C_{n+1} v^n + \sum_{n=0}^{\infty} (n+2) (n+1) C_{n+2} v^n + \sum_{n=0}^{\infty} (n+1) C_{n+1} v^n \\ + 2 \sum_{n=0}^{\infty} C_n v^n = 0 \end{aligned}$$

$$\text{or } \sum_{n=1}^{\infty} (n+1)n C_{n+1} v^n + \left[2 \cdot 1 C_2 + \sum_{n=1}^{\infty} (n+2)(n+1) C_{n+2} v^n \right] \\ + \left[1 \cdot C_1 + \sum_{n=1}^{\infty} (n+1) C_{n+1} v^n \right] + 2 \left[C_0 + \sum_{n=1}^{\infty} C_n v^n \right] = 0$$

$$\text{or } (2C_2 + C_1 + 2C_0) + \sum_{n=1}^{\infty} [(n+1)n C_{n+1} \\ + (n+2)(n+1) C_{n+2} + (n+1) C_{n+1} + 2C_n] v^n = 0$$

$$\text{or } (2C_2 + C_1 + 2C_0) + \sum_{n=1}^{\infty} [(n+2)(n+1) C_{n+2} + (n+1)(n+1) C_{n+1} + 2C_n] v^n = 0 \quad \dots(6)$$

Equation (6) is an identity, so the coefficients of various powers of v must be zero.

$$\therefore 2C_2 + C_1 + 2C_0 = 0 \quad \dots(7)$$

$$\text{and } (n+2)(n+1) C_{n+2} + (n+1)^2 C_{n+1} + 2C_n = 0 \quad \text{for } n \geq 1$$

$$\Rightarrow C_{n+2} = -\frac{(n+1)^2 C_{n+1} + 2C_n}{(n+2)(n+1)}, \quad \text{for } n \geq 1 \quad \dots(8)$$

Given that $y=1$ and $\frac{dy}{dx}=2$ where $x=1$

i.e., when $v = x - 1 = 1 - 1 = 0$, $y = 1$ and $\frac{dy}{dv} = 2 \therefore \frac{dy}{dx} = \frac{dy}{dv}$

Putting $v=0$ in (3) and (4), we get $1 = C_0$ and $2 = 1 \cdot C_1$

$$\therefore \text{ from (7), } C_2 = -2$$

Putting $n = 1, 2, 3, \dots$, in (8), we get

$$C_3 = -\frac{4C_2 + 2C_1}{3 \cdot 2} = -\frac{4 \cdot (-2) + 2 \cdot 2}{6} = \frac{2}{3}$$

$$C_4 = -\frac{9C_3 + 2C_2}{4 \cdot 3} = -\frac{9 \cdot (2/3) + 2 \cdot (-2)}{12} = -\frac{1}{6}$$

$$C_5 = -\frac{16C_4 + 2C_3}{5 \cdot 4} = -\frac{16 \cdot (-1/6) + 2 \cdot (2/3)}{20} = \frac{1}{15} \text{ and so on.}$$

$\therefore$ from (3), we have

$$y = C_0 + C_1 v + C_2 v^2 + C_3 v^3 + C_4 v^4 + C_5 v^5 + \dots$$

$$\text{or } y = 1 + 2(x-1) - 2(x-1)^2 + (2/3)(x-1)^3 - (1/6)(x-1)^4 + (1/15)(x-1)^5 + \dots$$

Which is the required solution.

Example 2: Find power series solution of $x \frac{dy}{dx} - y - x - 1 = 0$ in powers of $(x - 1)$.

Solution: Given, $x \frac{dy}{dx} - y - x - 1 = 0$... (1)

Putting $x - 1 = v, \frac{dv}{dx} = 1$

$\therefore \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx} = \frac{dy}{dv} \cdot 1$

The equation (1) reduces to

$$(v + 1) \frac{dy}{dv} - y - (1 + v) - 1 = 0$$

or $(v + 1) \frac{dy}{dv} - y - v - 2 = 0$... (2)

Now let the solution of (2) in powers of v be given by

$$y = \sum_{n=0}^{\infty} C_n v^n$$
 ... (3)

Differentiating $\frac{dy}{dv} = \sum_{n=1}^{\infty} C_n n v^{n-1}$

Putting in (2), we get

$$(v + 1) \sum_{n=1}^{\infty} n C_n v^{n-1} - \sum_{n=0}^{\infty} C_n v^n - v - 2 = 0$$

or $\sum_{n=1}^{\infty} n C_n v^n + \sum_{n=1}^{\infty} n C_n v^{n-1} - \sum_{n=0}^{\infty} C_n v^n - v - 2 = 0$

or $\sum_{n=1}^{\infty} n C_n v^n + \sum_{n=0}^{\infty} (n+1) C_{n+1} v^n - \sum_{n=0}^{\infty} C_n v^n - v - 2 = 0$

or $\sum_{n=1}^{\infty} n C_n v^n + C_1 v^0 + \sum_{n=1}^{\infty} (n+1) C_{n+1} v^n - C_0 v^0 - \sum_{n=1}^{\infty} C_n v^n - v - 2 = 0$

or $(C_1 - C_0 - 2) - v + \sum_{n=1}^{\infty} [(n-1) C_n + (n+1) C_{n+1}] v^n = 0$... (4)

Which is an identity, so we can equate to zero the coefficients of various power to v .

$\therefore$ Equating to zero the coefficients of various powers of v , we get,

$$C_1 - C_0 - 2 = 0 \Rightarrow C_1 = C_0 + 2$$

$$-1 + 0 \cdot C_1 + 2 \cdot C_2 = 0 \Rightarrow C_2 = 1/2$$

and $(n-1)C_n + (n+1)C_{n+1} = 0$ for $n \geq 2$

or $C_{n+1} = -\frac{n-1}{n+1}C_n$, for $n \geq 2$... (5)

Putting $n = 2, 3, 4, \dots$ in (5), we get

$$C_3 = -\frac{1}{3}C_2 = -\frac{1}{3} \cdot \frac{1}{2} = -\frac{1}{6}, C_4 = -\frac{2}{4}C_3 = -\frac{1}{2} \cdot \left(-\frac{1}{6}\right) = \frac{1}{12}$$

For $n \geq 2$, replacing n by $n-1, n-2, \dots$ in (5)

$$\begin{aligned} C_n &= \left(-\frac{n-2}{n}\right)C_{n-1} = \left(-\frac{n-2}{n}\right)\left(-\frac{n-3}{n-1}\right)C_{n-2} \\ &= \left(-\frac{n-2}{n}\right)\left(-\frac{n-3}{n-1}\right)\left(-\frac{n-4}{n-2}\right)\dots\left(-\frac{1}{3}\right)C_2 \\ &= \left(-\frac{n-2}{n}\right)\left(-\frac{n-3}{n-1}\right)\left(-\frac{n-4}{n-2}\right)\dots\left(-\frac{1}{3}\right) \cdot \frac{1}{2} \\ &= (-1)^n \frac{1}{n(n-1)}, \text{ for } n \geq 2 \end{aligned}$$

$\therefore$ from (3)

$$y = C_0 + C_1x + \sum_{n=2}^{\infty} C_n x^n = C_0 + (C_0 + 2)x + \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)} x^n$$

or $y = C_0 + (C_0 + 2)(x-1) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)} \cdot (x-1)^n$

or $y = C_0x + 2(x-1) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)} (x-1)^n$

Exercise 3.1

1. Show that $x=0$ is a regular singular point of the differential equation

$$2x^2 y'' + xy' + (x+1)y = 0.$$

2. Determine the nature of the point $x=0$ for the differential equation

$$3xy'' + 2x(x-1)y' + 5y = 0$$

3. Find the regular singular point of the following differential equations

(i) $2x^2 y'' + 7x(1+x)y' - 3y = 0$ (ii) $2x^2 y'' - xy' + (x-5)y = 0$

[Meerut 2013 (Sem. I)]

4. Show that $x=0$ is irregular and $x=-1$ is regular singular point of the differential equation

$$x^2 (x+1)^2 y'' + (x^2 - 1) y' + 2y = 0$$

5. Find singular points of the equation

$$(t^2 + 3t - 4)^2 x'' + 5(t + 4)x' + (t - 1)x = 0 \quad \text{Which of these are regular.}$$

Find the power series solutions in powers of x for each of the following differential equations.

6. $y'' - x^2 y' - y = 0$

7. $y'' + xy' + (2x^2 + 1)y = 0$

8. $y'' + xy' + (x^2 + 2)y = 0$

9. $y'' + xy' + (3x + 2)y = 0$

10. $(x^2 - 1)y'' + 4xy' + 2y = 0$

11. $(2 + x^2)y'' + xy' - (1 + x)y = 0$

[Delhi Maths (Hons.) 2002]

12. $(x^3 - 1)y'' + x^2 y' + xy = 0$

13. $y'' - y = 0$

[Sagar 2004]

14. $y'' - 4y = 0$

[Agra 2007]

15. $y'' - 2x^2 y' + 4xy = x^2 + 2x + 4$

16. $y'' - y = x$

Find the power series solution of each of the following initial value problems.

17. $(x^2 - 1)y'' + 3xy' + xy = 0$, $y(0) = 4$, $y'(0) = 6$

18. $(x^2 + 1)y'' + xy' + 2xy = 0$, $y(0) = 2$, $y'(0) = 3$

19. Find the power series solution in powers of $(x - 2)$ of the differential equation

$$y'' + (x - 3)y' + y = 0$$

20. Find the power series solution in powers of $(x - 2)$ of the differential equation

$$y'' + (x - 1)y' + y = 0$$

21. Find the power series solution in power of $(x - 1)$ of the initial value problem

$$xy'' + y' + 2y = 0, \quad y(1) = 2, \quad y'(1) = 4$$

22. Find power series solution of $xy' - y - x - 1 = 0$ in powers of $(x - 1)$

Answers 3.1

2. Regular singular point
3. (i) $x=0$, (ii) $x=0$
5. $t=1, t=-4$ (Both irregular singular points); No regular point
6. $y = a \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots \right) + b \left(x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \dots \right)$
7. $y = a \left(1 - \frac{x^2}{2} - \frac{x^4}{24} + \dots \right) + b \left(x - \frac{x^3}{3} - \frac{x^5}{30} + \dots \right)$
8. $y = a \left(1 - x^2 + \frac{1}{4}x^4 + \dots \right) + b \left(x - \frac{1}{2}x^3 + \frac{3}{40}x^5 + \dots \right)$
9. $y = a \left(1 - x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^4 + \frac{11}{40}x^5 + \dots \right) + b \left(x - \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 + \dots \right)$
10. $y = a(1 + x^2 + x^4 + \dots) + b(x + x^3 + x^5 + \dots)$
11. $y = a \left(1 + \frac{x^2}{4} + \frac{x^3}{12} - \frac{x^4}{32} + \dots \right) + b \left(x + \frac{x^4}{24} + \dots \right)$
12. $y = a \left(1 + \frac{x^3}{6} + \frac{x^6}{18} + \dots \right) + b \left(x + \frac{x^4}{6} + \frac{17x^7}{252} + \dots \right)$
13. $y = ae^x + be^{-x}$ or $y = a \cosh x + b \sinh x$ or $y = a(1 + x^2/2! + x^4/4! + \dots) + b(x + x^3/3! + x^5/5! + \dots)$
14. $y = ae^{2x} + be^{-2x}$ or $y = a \cosh 2x + b \sinh 2x$ or $y = a\{1 + (2x^2)/2! + (2x)^4/4! + \dots\} + b\{x + 2x^3/3! + 2x^5/5! + \dots\}$
15. $y = a \left(1 - \frac{2}{3}x^3 - \frac{2}{45}x^6 + \dots \right) + b \left(x - \frac{1}{6}x^4 - \frac{1}{63}x^7 + \dots \right) + 2x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \frac{1}{126}x^7.$
16. $y = a \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right) + b \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right) + \left(\frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right)$
17. $y = 4 + 6x + \frac{11}{3}x^3 + \frac{1}{2}x^4 + \frac{11}{4}x^5 + \dots$
18. $y = 2 + 3x - \frac{7}{6}x^3 - \frac{1}{2}x^4 + \frac{21}{40}x^5 + \dots$
19. $y = a \left[1 - \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 - \frac{1}{12}(x-2)^4 + \dots \right] + b \left[(x-2) + \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 - \frac{1}{6}(x-2)^4 + \dots \right]$
20. $y = a \left[1 - \frac{1}{2}(x-2)^2 + \frac{1}{6}(x-2)^3 + \frac{1}{12}(x-2)^4 + \dots \right] + b \left[(x-2) - \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 + \frac{1}{6}(x-2)^4 + \dots \right]$
21. $y = 2 + 4(x-1) - 4(x-1)^2 + \frac{4}{3}(x-1)^3 - \frac{1}{3}(x-1)^4 + \frac{2}{15}(x-1)^5 + \dots$
22. $y = ax + 2(x-1) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)}(x-1)^n$

3.7 Series Solution about a Regular Singular Point $x = 0$ (Frobenius Method)

[Meerut 2013 (Sem. I)]

Let
$$f(x) \frac{d^2 y}{dx^2} + P_1(x) \frac{dy}{dx} + Q_1(x) y = 0 \quad \dots(1)$$

be a linear differential equation of second order.

Recall that if $f(0) = 0$, then $x = 0$ is a singular point of (1).

and if $P(x) = (x - 0) \cdot \frac{P_1(x)}{f(x)}$ and $Q(x) = \frac{(x - 0)^2 Q_1(x)}{f(x)}$

are finite for $x = 0$, then $x = 0$ is a regular singular point of (1).

Now, we give the working rule for the solution of (1) in series about the regular singular point $x = 0$ by **Frobenius method**.

Step 1 : In Frobenius method the series solution of (1) about the regular singular point $x = 0$ is taken as

$$y = \sum_{r=0}^{\infty} C_r x^{m+r}, \quad C_0 \neq 0 \quad \dots(2)$$

Diff. (2)
$$y' = \sum_{r=0}^{\infty} (m+r) C_r x^{m+r-1} \quad \dots(3)$$

and
$$y'' = \sum_{r=0}^{\infty} (m+r)(m+r-1) C_r x^{m+r-2} \quad \dots(4)$$

Putting from (2), (3) and (4) in (1), and simplifying it reduces to an identity in x .

Step 2 : Equate to zero the coefficient of the lowest power of x in the identity obtained in step 1. Thus we obtain a quadratic equation in m . This quadratic equation is called the **indicial equation** of differential equation (1).

Step 3 : Solve the *indicial equation* and determine its roots, which are often called the *exponents* of the differential equation (1). We denote the roots of indicial equation by m_1 and m_2 where $m_1 \geq m_2$.

There arise following cases depending upon the nature of two roots of the **indicial equation**.

Case I : The roots (m_1, m_2) of the indicial equation *unequal* and differing by a quantity *not an integer* i.e., $m_1 - m_2 \neq 0, 1, 2$.

Case II : The roots of the indicial equation *unequal* differing by an integer making a coefficient of y *indeterminate*.

Case III : The roots of the indicial equation *unequal* differing by *an integer* making a coefficient of *y* *infinity*.

Case IV : The roots of the indicial equation equal *i.e.*, $m_1 - m_2 = 0$.

Step 4 : Now we equate to zero the coefficient of the general power term in the identity obtained in *step 1*. Usually we equate to zero the coefficient of the general term having the lowest general power and obtain an equation which connects C_r, C_{r-1} or C_r, C_{r-2} etc. This equation is called as *recurrence relation*. If this recurrence relation connects C_r and C_{r-2} then we also equate to zero the coefficient of next higher power of x , (higher to the power used to obtained indicial equation).

Step 5 : Replacing r by 0, 1, 2, ... (as needed) in the above recurrence relation we obtain the values of various coefficients. Substituting in (2) the series solution of diff. equation (1) is obtained.

3.8 Case I. Roots of Indicial Equation Unequal and Differing by a Quantity not an Integer i.e., $m_1 - m_2 \neq 0, 1, 2, \dots$ (Frobenius Method)

Method : Let $m = m_1$ and m_2 be two roots of the indicial equation which are unequal and their difference is not an integer. Then two independent solutions say u and v are obtained by putting $m = m_1$ and $m = m_2$ in the series for y . Then the general solution of the equation is taken as $y = au + bv$ where a and b are arbitrary constants.

Solved Examples

Example 1: Solve in series $(2x + x^3) \frac{d^2 y}{dx^2} - \frac{dy}{dx} - 6xy = 0$

[Meerut 2011 (Sem. I)]

Solution: Given diff. equation is

$$(2x + x^3) \frac{d^2 y}{dx^2} - \frac{dy}{dx} - 6xy = 0 \quad \dots(1)$$

Comparing with $y'' + P(x)y' + Q(x)y = 0$

we get $P(x) = -\frac{1}{2x + x^3}$ and $Q(x) = \frac{-6x}{2x + x^3} = -\frac{6}{2 + x^2}$

$\therefore P(x)$ does not exist at $x = 0$, so $x = 0$ is not ordinary point.

Also $xP(x) = -\frac{1}{2 + x^2}$ and $x^2Q(x) = -\frac{6x^2}{2 + x^2}$

Both of these exist at $x = 0$.

$\therefore x = 0$ is a regular singular point of (1).

Let the series solution of (1), about $x = 0$, be given by

$$y = \sum_{r=0}^{\infty} C_r x^{m+r}, C_0 \neq 0 \quad \dots(2)$$

$$\text{Diff. } \frac{dy}{dx} = \sum_{r=0}^{\infty} C_r (m+r) x^{m+r-1} \text{ and } \frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) x^{m+r-2}$$

Substituting in (1), we get

$$\sum_{r=0}^{\infty} C_r [(2x + x^3)(m+r)(m+r-1)x^{m+r-2} - (m+r)x^{m+r-1} - 6x^{m+r+1}] = 0$$

$$\text{or } \sum_{r=0}^{\infty} C_r [\{(m+r)^2 - (m+r) - 6\}x^{m+r+1} + \{2(m+r)^2 - 3(m+r)\}x^{m+r-1}] = 0$$

$$\text{or } \sum_{r=0}^{\infty} C_r [(m+r+2)(m+r-3)x^{m+r+1} + (m+r)(2m+2r-3)x^{m+r-1}] = 0 \quad \dots(3)$$

Which is an identity and so we can equate to zero the coefficients of various powers of x .

Equating to zero the coefficient of lowest power of x i.e., of x^{m-1} , the *indicial equation* is

$$C_0 m(2m-3) = 0 \Rightarrow m = 0, m = 3/2$$

$\therefore C_0 \neq 0$ as it is coefficient of first term with which we write the series.

Thus the roots of indicial equation are unequal and their difference is not an integer.

Again equating to zero the coefficient of x^{m+r-1} (lowest power term of x in m and r), we get

$$(m+r)(m+r-5)C_{r-2} + (m+r)(2m+2r-3)C_r = 0$$

$$\Rightarrow C_r = -\frac{(m+r-5)}{(2m+2r-3)}C_{r-2}, r \geq 2 \quad \dots(4)$$

Here difference of suffixes of C is $r - (r-2) = 2$, so we also equate to zero the coefficient of next higher power of x i.e., of x^m , (next to lowest power term in x to get indicial equation) we get

$$C_1(m+1)(2m-1) = 0 \Rightarrow C_1 = 0, \quad \therefore m = 0 \text{ or } 3/2.$$

Putting $r = 3, 5, 7, \dots$ in (4), we get $C_3 = -\frac{(m-2)}{(2m+3)}C_1 = 0$. Similarly $C_5 = 0 = C_7 = \dots$

Again putting $r = 2, 4, 6, \dots$ in (4), we get

$$C_2 = -\frac{(m-3)}{(2m+1)}C_0, C_4 = -\frac{(m-1)}{(2m+5)}C_2 = \frac{(m-3)(m-1)}{(2m+1)(2m+5)}C_0$$

$$C_6 = -\frac{(m+1)}{(2m+9)} C_4 = -\frac{(m-3)(m-1)(m+1)}{(2m+1)(2m+5)(2m+9)} C_0 \text{ and so on.}$$

∴ from (2), we get

$$y = C_0 x^m + C_1 x^{m+1} + C_2 x^{m+2} + C_3 x^{m+3} + \dots$$

$$\text{or } y = C_0 x^m \left[1 - \frac{(m-3)}{(2m+1)} x^2 + \frac{(m-3)(m-1)}{(2m+1)(2m+5)} x^4 - \frac{(m-3)(m-1)(m+1)}{(2m+1)(2m+5)(2m+9)} x^6 + \dots \right] \quad \dots(5)$$

When $m=0$, taking $C_0 = a$, we have

$$y = a \left[1 + 3x^2 + \frac{3}{5}x^4 - \frac{1}{15}x^6 - \dots \right] = au \text{ (say)}$$

Which is one solution of the given equation.

Again when $m=3/2$, taking $C_0 = b$, we have

$$y = bx^{3/2} \left[1 + \frac{3}{8}x^2 - \frac{1.3}{8.16}x^4 + \frac{1.3.5}{8.16.24}x^6 + \dots \right] = bv \text{ (say)}$$

Which is other solution of the given equation.

Hence, the complete primitive (general solution) is $y = au + bv$.

When a and b are arbitrary constants.

Example 2: Solve in series $9x(1-x) \frac{d^2 y}{dx^2} - 12 \frac{dy}{dx} + 4y = 0$ [Delhi Maths (Hons.) 2007, 08]

Solution: Given equation is $9x(1-x) \frac{d^2 y}{dx^2} - 12 \frac{dy}{dx} + 4y = 0 \quad \dots(1)$

Comparing with $y'' + P(x)y' + Q(x)y = 0$, we get

$$P(x) = \frac{-4}{3x(1-x)} \text{ and } Q(x) = \frac{4}{9x(1-x)}$$

Since both $P(x)$ and $Q(x)$ do not exist at $x=0$, so $x=0$ is not ordinary point.

$$\text{Also } xP(x) = \frac{-4}{3(1-x)} \text{ and } x^2Q(x) = \frac{4x}{9(1-x)}$$

Since both of these exist at $x=0$, so the point $x=0$ is a regular singular point of (1).

Let the series solution of (1) about $x=0$ be given by

$$y = \sum_{r=0}^{\infty} C_r x^{m+r}, \quad C_0 \neq 0 \quad \dots(2)$$

$$\text{Diff.} \quad \frac{dy}{dx} = \sum_{r=0}^{\infty} C_r (m+r) x^{m+r-1}$$

$$\text{and} \quad \frac{d^2 y}{dx^2} = \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) x^{m+r-2}$$

Substituting in (1), we get

$$\sum_{r=0}^{\infty} C_r [9x(1-x)(m+r)(m+r-1)x^{m+r-2} - 12(m+r)x^{m+r-1} + 4x^{m+r}] = 0$$

$$\text{or} \quad \sum_{r=0}^{\infty} C_r [\{9(m+r)^2 - 9(m+r) - 12(m+r)\} x^{m+r-1} - \{9(m+r)^2 - 9(m+r) - 4\} x^{m+r}] = 0$$

$$\text{or} \quad \sum_{r=0}^{\infty} C_r [3(m+r)(3m+3r-7)x^{m+r-1} - (3m+3r-4)(3m+3r+1)x^{m+r}] = 0 \quad \dots(3)$$

Which is an identity and so we can equate to zero the coefficients of various powers of x .

Equating to zero the coefficient of lowest power of x i.e., of x^{m-1} , the *indicial equation* is

$$3C_0 m(3m-7) = 0 \Rightarrow m = 0, 7/3$$

$\therefore C_0 \neq 0$ as it is the coefficient of first term with which we write the series.

Thus the roots of the indicial equation are unequal and their difference is not an integer.

Again equating to zero the coefficient of x^{m+r-1} (lowest power term of x is m and r), we get

$$\begin{aligned} & 3(m+r)(3m+3r-7)C_r - \{3m+3(r-1)-4\}\{3m+3(r-1)+1\}C_{r-1} = 0 \\ \Rightarrow & C_r = \frac{(3m+3r-7)(3m+3r-2)}{3(m+r)(3m+3r-7)} C_{r-1} \\ \Rightarrow & C_r = \frac{(3m+3r-2)}{3(m+r)} C_{r-1}, \quad r \geq 1 \quad \dots(4) \end{aligned}$$

Putting $r = 1, 2, 3, \dots$ in (4), we get

$$\begin{aligned} C_1 &= \frac{(3m+1)}{3(m+1)} C_0, \quad C_2 = \frac{(3m+4)}{3(m+2)} C_1 = \frac{3m+4}{3(m+2)} \cdot \frac{(3m+1)}{3(m+1)} C_0 = \frac{(3m+1)(3m+4)}{3^2(m+1)(m+2)} C_0 \\ C_3 &= \frac{(3m+7)}{3(m+3)} C_2 = \frac{(3m+7)}{3(m+3)} \cdot \frac{(3m+1)(3m+4)}{3^2(m+1)(m+2)} C_0 \end{aligned}$$

$$= \frac{(3m+1)(3m+4)(3m+7)}{3^3(m+1)(m+2)(m+3)} C_0$$

and so on.

∴ from (2), we get

$$y = C_0 x^m + C_1 x^{m+1} + C_2 x^{m+2} + C_3 x^{m+3} + \dots$$

$$\text{or } y = C_0 x^m \left[1 + \frac{(3m+1)}{3(m+1)} x + \frac{(3m+1)(3m+4)}{3^2(m+1)(m+2)} x^2 + \frac{(3m+1)(3m+4)(3m+7)}{3^3(m+1)(m+2)(m+3)} x^3 + \dots \right]$$

When $m=0$, taking $C_0 = a$, we have

$$y = a \left[1 + \frac{1}{3} x + \frac{1.4}{3.6} x^2 + \frac{1.4.7}{3.6.7} x^3 + \dots \right] = au \text{ (say)}$$

Which is one solution of the given equation.

Again, when $m = 7/3$, taking $C_0 = b$, we have

$$y = b x^{7/3} \left[1 + \frac{8}{10} x + \frac{8.11}{10.13} x^2 + \frac{8.11.14}{10.13.16} x^3 + \dots \right] = bv \text{ (say)}$$

Which is the other solution of the given equation.

Hence the complete primitive (general solution) of the given equation is $y = au + bv$

when a and b are arbitrary constants.

Example 3: Find the series solution of $4xy'' + 2y' + y = 0$

[Vikram 2004; Ravishankar 2004; Bilaspur 2004; Purvanchal 2005]

Solution: Given $4xy'' + 2y' + y = 0$...(1)

Obviously $x=0$ is a regular singular point of (1).

Let the series solution of (1) about $x=0$ be given by

$$y = \sum_{r=0}^{\infty} C_r x^{m+r}, \quad C_0 \neq 0 \quad \text{...(2)}$$

$$\text{Diff. } y' = \sum_{r=0}^{\infty} C_r (m+r) x^{m+r-1} \text{ and } y'' = \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) x^{m+r-2}$$

Substituting in (1), we get

$$\sum_{r=0}^{\infty} C_r [4(m+r)(m+r-1) x^{m+r-1} + 2(m+r) x^{m+r-1} + x^{m+r}] = 0$$

$$\text{or } \sum_{r=0}^{\infty} C_r [2(m+r)(2m+2r-1)x^{m+r-1} + x^{m+r}] = 0 \quad \dots(3)$$

Which is an identity and so we can equate to zero the coefficients of various powers of x .

Equating to zero the coefficient of lowest power of x i.e., of x^{m-1} the indicial equation is

$$2C_0(m)(2m-1) = 0 \Rightarrow m=0, m=1/2 \quad \because C_0 \neq 0$$

Thus the roots of the indicial equation are unequal and their difference is not an integer

Again equating to zero the coefficient of x^{m+r-1} (lower power term of x in m and r), we get

$$\begin{aligned} 2(m+r)(2m+2r-1)C_r + C_{r-1} &= 0 \\ \Rightarrow C_r &= -\frac{1}{2(m+r)(2m+2r-1)} C_{r-1}, r \geq 1 \end{aligned} \quad \dots(4)$$

Putting $r = 1, 2, 3, \dots$

$$\begin{aligned} C_1 &= -\frac{1}{2(m+1)(2m+1)} C_0, \\ C_2 &= -\frac{1}{2(m+2)(2m+3)} C_1 = \frac{1}{2^2(m+1)(m+2)(2m+1)(2m+3)} C_0 \\ C_3 &= -\frac{1}{2(m+3)(2m+5)} C_2 \\ &= -\frac{1}{2^3(m+1)(m+2)(m+3)(2m+1)(2m+3)(2m+5)} C_0 \end{aligned}$$

and so on.

$\therefore$ from (2), we get

$$y = C_0 x^m + C_1 x^{m+1} + C_2 x^{m+2} + C_3 x^{m+3} + \dots$$

$$\begin{aligned} \text{or } y &= C_0 x^m \left[1 - \frac{x}{2(m+1)(2m+1)} + \frac{x^2}{2^2(m+1)(m+2)(2m+1)(2m+3)} \right. \\ &\quad \left. - \frac{x^3}{2^3(m+1)(m+2)(m+3)(2m+1)(2m+3)(2m+5)} + \dots \right] \dots(5) \end{aligned}$$

when $m=0$, taking $C_0 = a$, we get

$$y = a \left[1 - \frac{x}{2} + \frac{x^2}{2^2(1.2)(1.3)} - \frac{x^3}{2^3(1.2.3)(1.3.5)} + \dots \right]$$

$$\begin{aligned}
 &= a \left[1 - \frac{x}{2} + \frac{x^2}{(2.4)(1.3)} - \frac{x^3}{(2.4.6)(1.3.5)} + \dots \right] \\
 &= a \left[1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \dots \right] \\
 &= a \left[1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \dots \right] = a \cos \sqrt{x}
 \end{aligned}$$

Again, when $m = 1/2$, taking $C_0 = b$ from (5), we have

$$\begin{aligned}
 y &= b x^{1/2} \left[1 - \frac{x}{2 \cdot (3/2) \cdot 2} + \frac{x^2}{2^2 \cdot (3/2 \cdot 5/2) \cdot (2.4)} \right. \\
 &\quad \left. - \frac{x^3}{2^3 \cdot [3/2 \cdot 5/2 \cdot 7/2] \cdot (2.4.6)} + \dots \right] \\
 &= b \left[\sqrt{x} - \frac{(\sqrt{x})^3}{3!} + \frac{(\sqrt{x})^5}{5!} - \frac{(\sqrt{x})^7}{7!} + \dots \right] = b \sin \sqrt{x}
 \end{aligned}$$

Hence the required solution is

$$y = a \cos \sqrt{x} + b \sin \sqrt{x}, \text{ where } a \text{ and } b \text{ are arbitrary constants.}$$

Example 4: Solve the Bessel equation, $x^2 y'' + x y' + (x^2 - n^2) y = 0$ in series, taking $2n$ as non integral.

Solution: Given $x^2 y'' + x y' + (x^2 - n^2) y = 0$... (1)

Obviously $x = 0$ is regular singular point of (1).

Let the series solution of (1) about $x = 0$ be given by

$$y = \sum_{r=0}^{\infty} C_r x^{m+r}, C_0 \neq 0 \quad \dots (2)$$

Diff. $y' = \sum_{r=0}^{\infty} C_r (m+r) x^{m+r-1}$

and $y'' = \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) x^{m+r-2}$

Substituting in (1), we get

$$\sum_{r=0}^{\infty} C_r [(m+r)(m+r-1) x^{m+r} + (m+r) x^{m+r} + (x^2 - n^2) x^{m+r}] = 0$$

or $\sum_{r=0}^{\infty} C_r [x^{m+r+2} + \{(m+r)^2 - n^2\} x^{m+r}] = 0$

$$\text{or } \sum_{r=0}^{\infty} C_r [x^{m+r+2} + (m+r-n)(m+r+n)x^{m+r}] = 0 \quad \dots(3)$$

Which is an identity and so we can equate to zero the coefficients of various powers of x .

Equating to zero the coefficient of lowest power of x i.e., of x^m , the *indicial equation*, is

$$C_0(m-n)(m+n) = 0 \Rightarrow m = -n, n \quad \because C_0 \neq 0$$

Thus, the roots of indicial equation are unequal and their difference $2n$ is non-integral.

Again equating to zero, the coefficient of x^{m+r} (lowest power term of x in m and r), we get

$$\begin{aligned} C_r(m+r-n)(m+r+n) + C_{r-2} &= 0 \\ \Rightarrow C_r &= -\frac{1}{(m+r-n)(m+r+n)} C_{r-2}, \quad r \geq 2 \end{aligned} \quad \dots(4)$$

Since the difference in suffixes of C in (4) is $r - (r-2) = 2$, so we again equate to zero the coefficient of x^{m+1} (next higher power term of x , than lowest power x^m), we get

$$C_1(m+1-n)(m+1+n) = 0 \Rightarrow C_1 = 0 \text{ for both } m = n \text{ and } m = -n.$$

Putting $r = 3, 5, 7, \dots$ in (4), we get

$$C_1 = 0 = C_3 = C_5 = \dots$$

Again putting $r = 2, 4, 6, \dots$ in (4), we get

$$\begin{aligned} C_2 &= -\frac{1}{(m+2-n)(m+2+n)} C_0 \\ C_4 &= -\frac{1}{(m+4-n)(m+4+n)} C_2 = \frac{1}{(m+2-n)(m+4-n)(m+2+n)(m+4+n)} C_0 \end{aligned}$$

and so on.

$\therefore$ from (2), we get

$$\begin{aligned} y &= C_0 x^m + C_1 x^{m+1} + C_2 x^{m+2} + C_3 x^{m+3} + C_4 x^{m+4} + \dots \\ \text{or } y &= C_0 x^m \left[1 - \frac{x^2}{(m+2-n)(m+2+n)} + \frac{x^4}{(m+2-n)(m+4-n)(m+2+n)(m+4+n)} - \dots \right] \end{aligned} \quad \dots(5)$$

When $m = n$ taking $C_0 = a$ in (5), we get

$$y = a x^n \left[1 - \frac{x^2}{2^2 \cdot 1! (n+1)} + \frac{x^4}{2^4 \cdot 2! (n+1)(n+2)} - \dots \right] = au \text{ (say)}$$

Again, when $m = -n$, taking $C_0 = b$ in (5), we get

$$\begin{aligned} y &= b x^{-n} \left[1 - \frac{x^2}{(-2n+2) \cdot 2} + \frac{x^4}{(-2n+2)(-2n+4)(2 \cdot 4)} \dots \right] \\ &= b x^{-1} \left[1 + \frac{x^2}{2^2 \cdot 1! (n-1)} + \frac{x^4}{2^4 \cdot 2! (n-1)(n-2)} + \dots \right] = bv \text{ (say)} \end{aligned}$$

Hence, the required solution is $y = au + bv$

where a and b are arbitrary constants.

Example 5: Using Frobenius method solve the differential equation

$$\frac{d^2 y}{dx^2} + \frac{1}{4x} \cdot \frac{dy}{dx} + \frac{1}{8x^2} y = 0 \quad [\text{Delhi Maths (Hons.) 2006}]$$

Solution: Given, $\frac{d^2 y}{dx^2} + \frac{1}{4x} \cdot \frac{dy}{dx} + \frac{1}{8x^2} y = 0 \quad \dots(1)$

Obviously $x = 0$ is a regular singular point of (1). (Check).

$\therefore$ Let the series solution of (1), about $x = 0$, be given by

$$y = \sum_{r=0}^{\infty} C_r x^{m+r}, \quad C_0 \neq 0 \quad \dots(2)$$

Diff. $y' = \sum_{r=0}^{\infty} C_r (m+r) x^{m+r-1}$ and $y'' = \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) x^{m+r-2}$

Substituting in (1), we get

$$\sum_{r=0}^{\infty} C_r [8x^2 (m+r)(m+r-1) x^{m+r-2} + 2x (m+r) x^{m+r-1} + x^{m+r}] = 0$$

or $\sum_{r=0}^{\infty} C_r [8 \{(m+r)^2 - (m+r)\} + 2(m+r) + 1] x^{m+r} = 0$

or $\sum_{r=0}^{\infty} C_r [8(m+r)^2 - 6(m+r) + 1] x^{m+r} = 0$

or $\sum_{r=0}^{\infty} C_r [\{4(m+r) - 1\} \{2(m+r) - 1\}] x^{m+r} = 0 \quad \dots(3)$

Which is an identity and so we can equate to zero the coefficients of various powers of x .

Equate to zero the coefficient of lowest power of x i.e., of x^m , the *indicial equation* is

$$C_0 \cdot (4m - 1)(2m - 1) = 0 \Rightarrow m = \frac{1}{4}, \frac{1}{2} \quad \because \quad C_0 \neq 0$$

Which are unequal and their difference is not an integer.

Again equating to zero the coefficient of x^{m+r} lowest power term of x in m and r , we get

$$C_r \{4(m+r) - 1\} \{2(m+r) - 1\} = 0 \text{ for all } r \geq 1 \quad \dots(4)$$

Equation (4) is satisfied for both values $1/4$ and $1/2$ of m by choosing $C_r = 0$, for all $r \geq 1$

$$\therefore \text{ from (2), we get } y = C_0 x^m \quad \because \quad C_1 = 0 = C_2 = \dots$$

When $m = 1/4$, taking $C_0 = a$, we get $y = a \cdot x^{1/4}$

and when $m = 1/2$ taking $C_0 = b$, we get $y = b x^{1/2}$

Hence, the required solution is $y = a x^{1/4} + b x^{1/2}$,

where a and b are arbitrary constant.

Example 6: Solve in series $xy'' + (x + n)y' + (n + 1)y = 0$

where n is not an integer.

$$\text{Solution: Given, } xy'' + (x + n)y' + (n + 1)y = 0 \quad \dots(1)$$

Obviously $x = 0$ is a regular singular point of (1).

Let the series solution of (1) about $x = 0$ be given by

$$y = \sum_{r=0}^{\infty} C_r x^{m+r} \quad \dots(2)$$

$$\text{Diff. } \frac{dy}{dx} = \sum_{r=0}^{\infty} C_r (m+r) x^{m+r-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) x^{m+r-2}$$

Substituting in (1), we get

$$\sum_{r=0}^{\infty} C_r [(m+r)(m+r-1) x^{m+r-1} + (x+n)(m+r) x^{m+r-1} + (n+1) x^{m+r}] = 0$$

$$\text{or } \sum_{r=0}^{\infty} C_r [\{(m+r)(m+r-1) + n(m+r)\} x^{m+r-1} + (m+r+n+1) x^{m+r}] = 0$$

$$\text{or } \sum_{r=0}^{\infty} C_r [(m+r)(m+r+n-1) x^{m+r-1} + (m+r+n+1) x^{m+r}] = 0 \quad \dots(3)$$

Which is an identity so we can equate to zero the coefficients of various powers of x .

Equating to zero the coefficient of lowest power of x i.e., of x^{m-1} , the *indicial equation* of (1) is

$$C_0 m(m+n-1) = 0 \Rightarrow m = 0, -(n-1) \quad \because C_0 \neq 0$$

Which are unequal and their difference $n-1$ is not an integer.

Again equating to zero the coefficient of x^{m+r-1} (lowest power term in x), we get

$$C_r(m+r)(m+r+n-1) + C_{r-1}(m+r-1+n+1) = 0$$

$$\Rightarrow C_r = -\frac{(m+r+n)}{(m+r)(m+r+n-1)} C_{r-1}, \quad r \geq 1 \quad \dots(4)$$

Putting $r = 1, 2, 3, \dots$ in (4), we get

$$C_1 = -\frac{(m+n+1)}{(m+1)(m+n)} C_0, \quad C_2 = -\frac{(m+n+2)}{(m+2)(m+n+1)} C_1 = \frac{(m+n+2)}{(m+1)(m+2)(m+n)} C_0$$

$$C_3 = -\frac{(m+n+3)}{(m+3)(m+n+2)} C_2 = -\frac{(m+n+3)}{(m+1)(m+2)(m+3)(m+n)} C_0$$

and so on.

$\therefore$ from (2), we have,

$$y = C_0 x^m + C_1 x^{m+1} + C_2 x^{m+2} + \dots$$

$$\text{or } y = C_0 \left[1 - \frac{(m+n+1)}{(m+1)(m+n)} x + \frac{(m+n+2)}{(m+1)(m+2)(m+n)} x^2 \right. \\ \left. - \frac{(m+n+3)}{(m+1)(m+2)(m+3)(m+n)} x^3 + \dots \right] \dots(5)$$

Now putting $m=0$ and taking $C_0 = a$, we get

$$y = a \left[1 - \frac{n+1}{1!n} x + \frac{n+2}{2!n} x^2 - \frac{n+3}{3!n} x^3 + \dots \right] = au \text{ (say)}$$

Again putting $m=-(n-1)$ and taking $C_0 = b$, we get

$$y = b \left[1 + \frac{2}{(n-2)} x + \frac{3x^2}{(n-2)(n-3)} - \frac{4x^3}{(n-2)(n-3)(n-4)} + \dots \right] = bv \text{ (say)}$$

Hence, the required general solution of (1) is

$$y = au + bv, \text{ where } a \text{ and } b \text{ are arbitrary constants.}$$

3.9 Case II. Roots of Indicial Equation Unequal Differing by an Integer Making a Coefficient of y Indeterminate (Frobenius Method)

Method : Let $m = m_1$ and $m = m_2$ (say $m_1 > m_2$) be two roots of the indicial equation which are unequal and differing by an integer. If some coefficient in y becomes indeterminate when $m = m_2$, the complete solution is obtained by putting $m = m_2$ in y and this solution will contain two arbitrary constants. If we put $m = m_1$ in y then it will give a numerical multiple of one of the series contained in above solution for $m = m_2$. Thus, this solution is a part of previous solution, so we reject this solution.

Solved Examples

Example 1: Solve in Series $(1 - x^2) y'' + 2xy' + y = 0$

[Meerut 2000; Rajasthan 2001; Guwahati 2007]

Solution: Given $(1 - x^2) y'' + 2xy' + y = 0$... (1)

Let the series solution of (1) be given by

$$y = \sum_{r=0}^{\infty} C_r x^{m+r}, \quad C_0 \neq 0 \quad \dots (2)$$

$$\text{Diff. } y' = \sum_{r=0}^{\infty} C_r (m+r) x^{m+r-1} \text{ and } y'' = \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) x^{m+r-2}$$

Substituting in (1), we get

$$\sum_{r=0}^{\infty} C_r [(1 - x^2)(m+r)(m+r-1) x^{m+r-2} + 2(m+r) x^{m+r} + x^{m+r}] = 0$$

$$\text{or } \sum_{r=0}^{\infty} C_r [(m+r)(m+r-1) x^{m+r-2} - \{(m+r)^2 - 3(m+r) - 1\} x^{m+r}] = 0$$

$$\text{or } \sum_{r=0}^{\infty} C_r [(m+r)(m+r-1) x^{m+r-2} - \{(m+r)(m+r-3) - 1\} x^{m+r}] = 0 \quad \dots (3)$$

Which is an identity, so we can equate to zero the coefficients of various powers of x .

Equating to zero the coefficient of the lowest power of x i.e., of x^{m-2} in (3), the *indicial equation* is

$$C_0 m(m-1) = 0 \Rightarrow m = 0, m = 1 \quad \because C_0 \neq 0$$

Thus the roots of indicial equation are unequal and differing by an integer.

Again, equating to zero the coefficient of next higher power of x i.e., of x^{m-1} in (3), we get

$$C_1 (m+1) m = 0 \quad \dots(4)$$

So for $m=0$, we get $C_1 \cdot 0 = 0 \Rightarrow C_1$ is indeterminate when $m=0$.

Hence in this case C_1 can be taken as arbitrary constant and when $m=1$, $C_1=0$

Now equating to zero the coefficient of x^{m+r-2} , we get

$$\begin{aligned} C_r (m+r) (m+r-1) - \{(m+r-2) (m+r-2-3) - 1\} C_{r-2} &= 0 \\ \Rightarrow C_r &= \frac{\{(m+r-2) (m+r-5) - 1\}}{(m+r-1) (m+r)} C_{r-2}, \text{ for } r \geq 2 \end{aligned} \quad \dots(5)$$

Putting $r=2, 3, 4, \dots$, $C_2 = \frac{m(m-3)-1}{(m+1)(m+2)} C_0$, $C_3 = \frac{(m+1)(m-2)-1}{(m+2)(m+3)} C_1$

$$C_4 = \frac{(m+2)(m-1)-1}{(m+3)(m+4)} C_2 = \frac{\{(m+2)(m-1)-1\} \{m(m-3)-1\}}{(m+1)(m+2)(m+3)(m+4)} C_0$$

$$C_5 = \frac{(m+3)m-1}{(m+4)(m+5)} C_3 = \frac{\{(m+3)m-1\} \{(m+1)(m-2)-1\}}{(m+2)(m+3)(m+4)(m+5)} C_1$$

and so on.

$\therefore$ from (1), we get

$$y = C_0 x^m + C_1 x^{m+1} + C_2 x^{m+2} + C_3 x^{m+3} + C_4 x^{m+4} + C_5 x^{m+5} + \dots$$

$$\begin{aligned} \text{or } y &= C_0 x^m \left[1 + \frac{m(m-3)-1}{(m+1)(m+2)} x^2 + \frac{\{(m+2)(m-1)-1\} \{m(m-3)-1\}}{(m+1)(m+2)(m+3)(m+4)} x^4 + \dots \right] \\ &+ C_1 x^{m+1} \left[x + \frac{(m+1)(m-2)-1}{(m+2)(m+3)} x^3 + \frac{\{m(m+3)-1\} \{(m+1)(m-2)-1\}}{(m+2)(m+3)(m+4)(m+5)} x^5 + \dots \right] \dots(6) \end{aligned}$$

when $m=0$, taking $C_0 = a$ and $C_1 = b$, we get

$$y = a \left[1 - \frac{x^2}{2} + \frac{x^4}{8} - \dots \right] + b \left[x - \frac{x^3}{2} + \frac{x^5}{40} - \dots \right] \quad \dots(7)$$

If we take $m=1$, then from (3), $C_1=0$

$\therefore$ From (6)

$$y = C_0 \left[x - \frac{x^3}{2} + \frac{x^5}{40} - \dots \right]$$

which is a part of the solution (7)

Hence the solution of the given equation is given by (7), where a and b are arbitrary constants.

Example 2: Solve in series : $y'' + x^2 y = 0$

Solution: Given $y'' + x^2 y = 0$... (1)

Let the series solution of (1) be given by

$$y = \sum_{r=0}^{\infty} C_r x^{m+r}, \quad C_0 \neq 0 \quad \dots (2)$$

$$\text{Diff.} \quad y'' = \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) x^{m+r-2}$$

Substituting in (1), we get

$$\sum_{r=0}^{\infty} C_r [(m+r)(m+r-1) x^{m+r-2} + x^{m+r+2}] = 0 \quad \dots (3)$$

which is an identity. So we can equate to zero the coefficients of various powers of x .

Equating to zero, the coefficient of lowest power of x i.e., of x^{m-2} the indicial equation is.

$$C_0 m(m-1) = 0 \Rightarrow m = 0, m = 1 \quad \because C_0 \neq 0$$

Since difference in powers of x in (3) is 4, so we also equate to zero the coefficients of next higher powers of x .

Equating to zero, the coefficient of x^{m-1} , next higher power of x , we get

$$C_1 (m+1) m = 0 \quad \dots (4)$$

$\Rightarrow C_1$ is indeterminate for $m = 0$ as it gives $C_1 \cdot 0 = 0$ i.e., C_1 can have any value

and $C_1 = 0$ for $m = 1$

Equating to zero, the coefficient of x^m (next higher power of x), we get

$$C_2 (m+2)(m+1) = 0 \Rightarrow C_2 = 0 \text{ for } m = 0 \text{ and } m = 1$$

Equating to zero the coefficient of x^{m+1} (next higher power of x), we get

$$C_3 (m+3)(m+2) = 0 \Rightarrow C_3 = 0 \text{ for } m = 0 \text{ and } m = 1$$

Now equating to zero, the coefficient of x^{m+r-2} , we get

$$C_r (m+r)(m+r-1) + C_{r-4} = 0$$

$$\Rightarrow C_r = -\frac{1}{(m+r)(m+r-1)} C_{r-4} \quad r \geq 4 \quad \dots (5)$$

Putting $r = 6, 7, 10, 11, 14, 15, \dots$

$$C_2 = 0 = C_3 = C_6 = C_7 = C_{10} = C_{11} = C_{14} = C_{15} = \dots$$

Again Putting $r = 4, 5, 8, 9, 12, 13, \dots$, we get

$$C_4 = -\frac{1}{(m+3)(m+4)} C_0, C_5 = -\frac{1}{(m+4)(m+5)} C_1$$

$$C_8 = -\frac{1}{(m+7)(m+8)} C_4 = \frac{1}{(m+3)(m+4)(m+7)(m+8)} C_0$$

$$C_9 = -\frac{1}{(m+8)(m+9)} C_5 = \frac{1}{(m+4)(m+5)(m+8)(m+9)} C_1$$

and so on.

$\therefore$ from (2), we get

$$y = C_0 x^m + C_1 x^{m+1} + C_2 x^{m+2} + \dots$$

or

$$y = C_0 x^m \left[1 - \frac{x^4}{(m+3)(m+4)} + \frac{x^8}{(m+3)(m+4)(m+7)(m+8)} - \dots \right] \\ + C_1 x^m \left[x - \frac{x^5}{(m+4)(m+5)} + \frac{x^9}{(m+4)(m+5)(m+8)(m+9)} + \dots \right] \dots (6)$$

When $m=0$, taking $C_0 = a$ and $C_1 = b$, from (6), we get

$$y = a \left[1 - \frac{x^4}{3 \cdot 4} + \frac{x^8}{3 \cdot 4 \cdot 7 \cdot 8} - \dots \right] + b \left[x - \frac{x^5}{4 \cdot 5} + \frac{x^9}{4 \cdot 5 \cdot 8 \cdot 9} - \dots \right] \dots (7)$$

When $m=1$, we get $C_1 = 0$

$$\therefore \text{ from (6) we have } y = C_0 \left[x - \frac{x^5}{4 \cdot 5} + \frac{x^9}{4 \cdot 5 \cdot 8 \cdot 9} - \dots \right]$$

which is a part of solution (7).

Hence required solution of (1) is given by (7), where a and b are arbitrary constant.

Example 3: Find a series solution in ascending powers of x of the Legendre's equation

$$(1 - x^2) y'' - 2xy' + n(n+1)y = 0$$

[Meerut 2012 (Sem. I); Utkal 2003]

Solution: Given, $(1 - x^2) y'' - 2xy' + n(n+1)y = 0$

...(1)

Let the series solution of (1) in ascending power of x , be given by

$$y = \sum_{r=0}^{\infty} C_r \cdot x^{m+r}, C_0 \neq 0 \dots (2)$$

$$\text{Diff. } y' = \sum_{r=0}^{\infty} C_r (m+r) x^{m+r-1} \text{ and } y'' = \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) x^{m+r-2}$$

Substituting in (1), we get

$$\sum_{r=0}^{\infty} C_r [(1-x^2)(m+r)(m+r-1)x^{m+r-2} - 2x(m+r)x^{m+r-1} + n(n+1)x^{m+r}] = 0$$

$$\text{or } \sum_{r=0}^{\infty} C_r [(m+r)(m+r-1)x^{m+r-2} - \{(m+r)^2 - (m+r) + 2(m+r) - n^2 - n\}x^{m+r}] = 0$$

$$\text{or } \sum_{r=0}^{\infty} C_r [(m+r)(m+r-1)x^{m+r-2} - \{(m+r)^2 - n^2 + (m+r-n)\}x^{m+r}] = 0$$

$$\text{or } \sum_{r=0}^{\infty} C_r [(m+r)(m+r-1)x^{m+r-2} - (m+r-n)(m+r+n+1)x^{m+r}] = 0 \dots (3)$$

Which is an identity, so we can equate to zero the coefficients of various powers of x .

Equating to zero the coefficient of lowest powers of x , i.e., of x^{m-2} the *indicial equation*, is

$$C_0 m(m-1) = 0 \Rightarrow m = 0, 1 \quad \because C_0 \neq 0$$

Again equating to zero the coefficient of x^{m+r-2} , we get

$$C_r (m+r)(m+r-1) - (m+r-2-n)(m+r-2+n+1) C_{r-2} = 0$$

$$\text{or } C_r = \frac{(m+r-n-2)(m+r+n-1)}{(m+r)(m+r-1)} C_{r-2}, \quad r \geq 2 \dots (4)$$

Since the difference in suffixes of C_r and C_{r-2} is $r - (r-2) = 2$, so we also equate to zero the coefficient of next higher power to the lowest power of x i.e., of x^{m-1} . Thus, we get

$$C_1 (m+1) \cdot m = 0 \Rightarrow C_1 \text{ is indeterminate for } m=0 \text{ and } C_1 = 0 \text{ for } m=1$$

Now, from (4), putting $r = 2, 4, 6, \dots$, we get

$$C_2 = \frac{(m-n)(m+n+1)}{(m+1)(m+2)} C_0,$$

$$C_4 = \frac{(m-n+2)(m+n+3)}{(m+3)(m+4)} C_2 = \frac{(m-n)(m-n+2)(m+n+1)(m+n+3)}{(m+1)(m+2)(m+3)(m+4)} C_0$$

and so on.

Again putting $r = 3, 5, 7, \dots$, in (4), we get

$$C_3 = \frac{(m-n+1)(m+n+2)}{(m+2)(m+3)} C_1,$$

$$C_5 = \frac{(m-n+3)(m+n+4)}{(m+4)(m+5)} C_3 = \frac{(m-n+1)(m-n+3)(m+n+2)(m+n+4)}{(m+2)(m+3)(m+4)(m+5)} C_1$$

and so on.

$\therefore$ from (2), we get

$$y = C_0 x^m + C_1 x^{m+1} + C_2 x^{m+2} + \dots$$

or

$$\begin{aligned} y = C_0 & \left[x^m + \frac{(m-n)(m+n+1)}{(m+1)(m+2)} x^{m+2} \right. \\ & \left. + \frac{(m-n)(m-n+2)(m+n+1)(m+n+3)}{(m+1)(m+2)(m+3)(m+4)} x^{m+4} + \dots \right] \\ & + C_1 \left[x^{m+1} + \frac{(m-n+1)(m+n+2)}{(m+2)(m+3)} x^{m+3} \right. \\ & \left. + \frac{(m-n+1)(m-n+3)(m+n+2)(m+n+4)}{(m+2)(m+3)(m+4)(m+5)} x^{m+5} + \dots \right] \dots (5) \end{aligned}$$

When $m=0$, taking $C_0 = a$ and $C_1 = b$, we get

$$\begin{aligned} y = a & \left[1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)n(n+1)(n+3)}{4!} x^4 - \dots \right] \\ & + b \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \dots \right] \dots (6) \end{aligned}$$

When $m=1$, $C_1=0$, $\therefore$ from (5), we get

$$y = C_0 \left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \dots \right]$$

which is one of the two series given in (6)

Hence (6) is the required general solution of the given equation, where a and b are arbitrary constant.

Note : In 3.5 we have given method of finding the series solution of a differential equation about an ordinary point $x = x_0$. And given few examples of series solution of differential equations about an ordinary point $x=0$. All those problems may also be solved by Frobenius method given in 3.9.

3.10 Case III. Roots of Indicial Equation Unequal Differing by an Integer, One Root Making a Coefficient of y Infinity (Frobenius Method)

Method : Let $m = m_1$ and m_2 (say $m_1 > m_2$) be two roots of the indicial equation which are unequal and differing by an integer. If some coefficient in y becomes infinity when $m = m_2$ then we modify y by replacing C_0 by $(m - m_2) k$, where $k \neq 0$. Then two independent solutions are obtained by putting $m = m_2$ in the modified form of y and $\frac{\partial y}{\partial m}$. Note that by putting $m = m_1$ in y we get a solution which is numerical multiple of that obtained by putting $m = m_2$ in modified form of y . So we reject this solution.

Solved Examples

Example 1: Solve in series $x^2 y'' + x y' + (x^2 - 4) y = 0$, the Bessel's equation of order 2 near $x = 0$. [Meerut 2001 (BP)]

Solution: Given, $x^2 y'' + x y' + (x^2 - 4) y = 0$... (1)

Obviously $x = 0$ is a regular singular point of (1), (Check)

Let the series solution of (1) about $x = 0$, be given by

$$y = \sum_{r=0}^{\infty} C_r x^{m+r}, \quad C_0 \neq 0 \quad \dots (2)$$

$$\text{Diff.} \quad y' = \sum_{r=0}^{\infty} C_r (m+r) x^{m+r-1} \text{ and } y'' = \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) x^{m+r-2}$$

Substituting in (1), we get

$$\sum_{r=0}^{\infty} C_r [(m+r)(m+r-1) x^{m+r} + (m+r) x^{m+r} + (x^2 - 4) x^{m+r}] = 0$$

$$\text{or} \quad \sum_{r=0}^{\infty} C_r [\{(m+r)^2 - (m+r) + (m+r) - 4\} x^{m+r} + x^{m+r+2}] = 0$$

$$\text{or} \quad \sum_{r=0}^{\infty} C_r [(m+r-2)(m+r+2) x^{m+r} + x^{m+r+2}] = 0 \quad \dots (3)$$

Which is an identity, so we can equate to zero the coefficients of various powers of x .

Equating to zero the coefficient of lowest power of x i.e., of x^m in (3), the *indicial equation* is

$$C_0 (m-2)(m+2) = 0 \Rightarrow m = -2 \text{ and } m = 2 \quad \because C_0 \neq 0$$

which are unequal and their difference is an integer.

Again equating to zero the coefficient of x^{m+r} (lowest power term of x in m and r), we get

$$C_r (m+r-2) (m+r+2) + C_{r-2} = 0$$

$$\Rightarrow C_r = -\frac{1}{(m+r-2)(m+r+2)} C_{r-2}, r \geq 2 \quad \dots(4)$$

Since in (4) the difference of suffixes is $r - (r-2) = 2$, so we equate to zero the coefficient of next higher power of x i.e., of x^{m+1} (next higher to the lowest power term i.e., x^m of x)

$$C_1 (m+1-2) (m+1+2) = 0 \text{ i.e., } C_1 (m-1) (m+3) = 0$$

$$\Rightarrow C_1 = 0 \text{ for both } m = -2 \text{ and } 2.$$

Now putting $r = 3, 5, \dots$, in (4), we get

$$C_1 = 0 = C_3 = C_5 = \dots$$

Again putting $r = 2, 4, 6, \dots$, in (4), we get

$$C_2 = -\frac{1}{m(m+4)} C_0,$$

$$C_4 = -\frac{1}{(m+2)(m+6)} C_2$$

$$= \left(-\frac{1}{(m+2)(m+6)} \right) \cdot \left(-\frac{1}{m(m+4)} C_0 \right) = \frac{1}{m(m+2)(m+4)(m+6)} C_0$$

$$C_6 = -\frac{1}{(m+4)(m+8)} C_4 = -\frac{1}{m(m+2)(m+4)^2(m+6)(m+8)} C_0$$

and so on.

$\therefore$ from (2), we have

$$y = C_0 x^m + C_1 x^{m+1} + C_2 x^{m+2} + C_3 x^{m+3} + C_4 x^{m+4} + \dots$$

or

$$y = C_0 x^m \left[1 - \frac{x^2}{m(m+4)} + \frac{x^4}{m(m+2)(m+4)(m+6)} \right. \\ \left. - \frac{x^6}{m(m+2)(m+4)^2(m+6)(m+8)} + \dots \right] \dots(5)$$

When $m = -2$, coefficients of $x^4, x^6, \dots$ in y becomes infinite.

So to get rid of this difficulty, we modify the solution (5) by replacing C_0 by $[m - (-2)] k$

i.e., $(m+2)k$ in (5). Thus the modified solution is

$$y = kx^m \left[(m+2) - \frac{(m+2)x^2}{m(m+4)} + \frac{x^4}{m(m+4)(m+6)} - \frac{x^6}{m(m+4)^2(m+6)(m+8)} + \dots \right] \dots (6)$$

$\therefore$ when $m = -2$, taking $k = a$ in (6) we get

$$y = ax^{-2} \left[0 - 0 + \frac{x^4}{(-2)(2)(4)} - \frac{x^6}{(-2)(2)^2(4)(6)} + \dots \right]$$

or $y = ax^{-2} \left[-\frac{x^4}{2^2 \cdot 4} + \frac{x^6}{2^3 \cdot 4 \cdot 6} - \frac{x^8}{2^3 \cdot 4^2 \cdot 6 \cdot 8} + \dots \right] = au$ (say) $\dots (7)$

When $m = 2$, from (6), we get

$$\begin{aligned} y &= kx^2 \left[4 - \frac{4x^2}{2 \cdot 6} + \frac{x^4}{2 \cdot 6 \cdot 8} - \frac{x^6}{2 \cdot 6^2 \cdot 8 \cdot 10} + \dots \right] \\ &= 4kx^{-2} \left[x^4 - \frac{x^6}{2 \cdot 6} + \frac{x^8}{4 \cdot 2 \cdot 6 \cdot 8} - \dots \right] \\ &= (4kx^{-2})(-2^2 \cdot 4) \left[-\frac{x^4}{2^2 \cdot 4} + \frac{x^6}{2 \cdot 6 \cdot 2^2 \cdot 4} - \frac{x^8}{4 \cdot 2 \cdot 6 \cdot 8 \cdot 2^2 \cdot 4} + \dots \right] \\ &= -64kx^{-2} \left[-\frac{x^4}{2^2 \cdot 4} + \frac{x^6}{2^3 \cdot 4 \cdot 6} - \frac{x^8}{2^3 \cdot 4^2 \cdot 6 \cdot 8} + \dots \right] = kw \text{ (say)} \dots (8) \end{aligned}$$

From (7) and (8) we have

$$w = -64u \Rightarrow \text{solutions } u \text{ and } w \text{ are not independent.}$$

So we proceed to find one more independent solution of (1).

Now differentiating (6), we get

$$\frac{dy}{dx} = k \left[m(m+2)x^{m-1} - \frac{(m+2)^2 x^{m+1}}{m(m+4)} + \frac{(m+4)x^{m+3}}{m(m+4)(m+6)} \dots \right]$$

and $\frac{d^2y}{dx^2} = k \left[m(m-1)(m+2)x^{m-2} - \frac{(m+1)(m+2)^2}{m(m+4)} x^m + \frac{(m+3)x^{m+2}}{m(m+6)} + \dots \right]$

Substituting these values in the left hand side of equation (1), we get

$$\begin{aligned}
& x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - 4) y \\
&= k x^2 \left[m(m-1)(m+2) x^{m-2} - \frac{(m+1)(m+2)^2}{m(m+4)} x^m + \frac{(m+3)x^{m+2}}{m(m+6)} \dots \right] \\
&\quad + k x \left[m(m+2) x^{m-1} - \frac{(m+2)^2 x^{m+1}}{m(m+4)} + \frac{x^{m+3}}{m(m+6)} \dots \right] \\
&\quad + k (x^2 - 4) x^m \left[(m+2) - \frac{(m+2)x^2}{m(m+4)} + \frac{x^4}{m(m+4)(m+6)} - \dots \right] \\
&= k(m+2)^2(m-2)x^m \text{ (All other terms being cancelled)}
\end{aligned}$$

Differentiating both sides partially w.r.t. m , we get

$$\begin{aligned}
\frac{\partial}{\partial m} \left[x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - 4) y \right] &= k [2(m+2)(m-2) + (m+2)^2] x^m \\
&\quad + k(m+2)^2(m-2)x^m \log x
\end{aligned}$$

Since differential operator is commutative, so this can be written as

$$\left[x^2 \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} + (x^2 - 4) \right] \frac{\partial y}{\partial m} = k [2(m+2)(m-2) + (m+2)^2] x^m + k(m+2)^2(m-2)x^m \log x$$

Putting $m = -2$, we have

$$\left[x^2 \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} + (x^2 - 4) \right] \left(\frac{\partial y}{\partial m} \right)_{m=-2} = 0$$

Hence $\left(\frac{\partial y}{\partial m} \right)_{m=-2}$ is also a solution of given differential equation (1).

Now from (6), differentiating partially w.r.t. m , we get

$$\begin{aligned}
\frac{\partial y}{\partial m} &= k x^m \log x \left[(m+2) - \frac{(m+2)x^2}{m(m+4)} + \frac{x^4}{m(m+4)(m+6)} - \frac{x^6}{m(m+4)^2(m+6)(m+8)} \dots \right] \\
&\quad + k x^m \left[1 - \frac{(m+2)x^2}{m(m+4)} \left(\frac{1}{m+2} - \frac{1}{m} - \frac{1}{m+4} \right) + \frac{x^4}{m(m+4)(m+6)} \left(-\frac{1}{m} - \frac{1}{m+4} - \frac{1}{m+6} \right) \right. \\
&\quad \left. - \frac{x^6}{m(m+4)^2(m+6)(m+8)} \left(-\frac{1}{m} - \frac{2}{m+4} - \frac{1}{m+6} - \frac{1}{m+8} \right) \dots \right]
\end{aligned}$$

(Diff. 2nd, 3rd, ... terms after taking their logarithms) as follows :

$$\text{Let } z_1 = \frac{m+2}{m(m+4)} \Rightarrow \log z_1 = \log(m+2) - \log m - \log(m+4)$$

$$\text{Diff. partially w.r.t } m, \frac{\partial z_1}{\partial m} = z_1 \left(\frac{1}{m+2} - \frac{1}{m} - \frac{1}{m+4} \right) \text{ etc.}$$

$$\begin{aligned} \therefore \left(\frac{\partial y}{\partial m} \right)_{m=-2} &= k x^{-2} \log x \left[0 - 0 + \frac{x^4}{-2 \cdot 2 \cdot 4} - \frac{x^6}{-2 \cdot 2^2 \cdot 4 \cdot 6} + \dots \right] \\ &\quad + k x^{-2} \left[1 - x^2 \left(\frac{1}{-2 \cdot 2} - 0 - 0 \right) + \frac{x^4}{-2 \cdot 2 \cdot 4} \left(\frac{1}{2} - \frac{1}{2} - \frac{1}{4} \right) \right. \\ &\quad \left. - \frac{x^6}{-2 \cdot 2^2 \cdot 4 \cdot 8} \left(\frac{1}{2} - \frac{2}{2} - \frac{1}{4} - \frac{1}{6} \right) + \dots \right] \\ &= x^{-2} k \log x \left[\frac{-x^4}{2^2 \cdot 4} + \frac{x^6}{2^3 \cdot 4 \cdot 6} - \frac{x^8}{2^3 \cdot 4^2 \cdot 6 \cdot 8} \dots \right] \\ &\quad + k x^{-2} \left[1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \dots \right] \\ &= k \left[u \log x + x^{-2} \left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \dots \right) \right] = bv \text{ (say)} \end{aligned}$$

Taking $k = b$

Hence the required solution is

$$y = au + bv, \text{ where } a \text{ and } b \text{ are arbitrary constants.}$$

Example 2: Solve in series $x^2 y'' + xy' + (x^2 - 1)y = 0$, the Bessel's equation of first order near $x = 0$. [Ravishankar 2005; Agra 2006; Delhi Maths (Hons.) 2007]

$$\text{Solution: Given } x^2 y'' + xy' + (x^2 - 1)y = 0 \quad \dots(1)$$

Obviously $x = 0$ is a regular singular point of (1), (Check)

Let the series solution of (1) about $x = 0$, be given by

$$y = \sum_{r=0}^{\infty} C_r x^{m+r}, \quad C_0 \neq 0 \quad \dots(2)$$

$$\text{Diff. } y' = \sum_{r=0}^{\infty} C_r (m+r) x^{m+r-1} \text{ and } y'' = \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) x^{m+r-2}$$

Substituting in (1), we get

$$\sum_{r=0}^{\infty} C_r [(m+r)(m+r-1)x^{m+r} + (m+r)x^{m+r} + (x^2-1)x^{m+r}] = 0$$

or
$$\sum_{r=0}^{\infty} C_r [\{(m+r)^2 - 1\} x^{m+r} + x^{m+r+2}] = 0$$

or
$$\sum_{r=0}^{\infty} C_r [(m+r-1)(m+r+1)x^{m+r} + x^{m+r+2}] = 0 \quad \dots(3)$$

Which is an identity, so we can equate to zero the coefficients of various powers of x .

Equating to zero the coefficient of lowest power of x , i.e., of x^m in (3), the *indicial equation* is

$$C_0 (m-1)(m+1) = 0 \Rightarrow m = -1, m = 1 \quad \because C_0 \neq 0$$

which are unequal and their difference is an integer.

Again equating to zero the coefficient of x^{m+r} (lowest power term of x is m and r), we get

$$C_r (m+r-1)(m+r+1) + C_{r-2} = 0$$

$$\Rightarrow C_r = -\frac{1}{(m+r-1)(m+r+1)} C_{r-2}, \quad r \geq 2 \quad \dots(4)$$

Since in (4) the difference of suffixes of C is $r - (r-2) = 2$,

So we equate to zero the coefficient of next higher power of x i.e., of x^{m+1} (next higher power to the lowest power term i.e., x^m of x)

$$C_1 m(m+2) = 0 \Rightarrow C_1 = 0 \text{ for both } m = -1 \text{ or } 1$$

Putting $r = 3, 5, \dots$ in (4), we get

$$C_1 = 0 = C_3 = C_5 = \dots$$

Again putting $r = 2, 4, 6, \dots$, in (4), we get

$$C_2 = -\frac{1}{(m+1)(m+3)} C_0, C_4 = -\frac{1}{(m+3)(m+5)} C_2 = \frac{1}{(m+1)(m+3)^2(m+5)} C_0$$

and so on.

$\therefore$ From (2), we get

$$y = C_0 x^m + C_1 x^{m+1} + C_2 x^{m+2} + C_3 x^{m+3} + C_4 x^{m+4} + \dots$$

$$\text{or } y = C_0 x^m \left[1 - \frac{x^2}{(m+1)(m+3)} + \frac{x^4}{(m+1)(m+3)^2(m+5)} - \dots \right] \quad \dots(5)$$

When $m = -1$ coefficients of $x^2, x^4, \dots$ in y becomes infinite. So to get rid of this difficulty, we modify the solution (5) by replacing C_0 by $[m - (-1)] k$ i.e., $(m+1)k$ in (5).

Thus the modified solution is

$$y = k x^m \left[(m+1) - \frac{x^2}{(m+3)} + \frac{x^4}{(m+3)^2(m+5)} - \dots \right] \quad \dots(6)$$

$\therefore$ When $m = -1$, taking $k = a$ in (6), we get

$$y = a x^{-1} \left[0 - \frac{x^2}{2} + \frac{x^4}{2^2 \cdot 4} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6} + \dots \right] = au \text{ (say)} \quad \dots(7)$$

When $m = 1$, from (6), we get

$$\begin{aligned} y &= k x \left[2 - \frac{x^2}{4} + \frac{x^4}{4^2 \cdot 6} - \dots \right] \\ &= -2^2 k x^{-1} \left[-\frac{x^2}{2} + \frac{x^4}{2^2 \cdot 4} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6} + \dots \right] = k w \text{ (say)} \end{aligned} \quad \dots(8)$$

From (7) and (8), we have

$$w = -4u \Rightarrow \text{solutions } u \text{ and } w \text{ are not independent solutions of (1).}$$

So we proceed to find one more independent solution of (1).

Now differentiating (6) and substituting in L.H.S. of (1) and on simplification as in last ex. 1, we get

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - 1) y = k x^m (m+1)^2 (m-1),$$

Differentiating both sides partially w.r.t. m , we get

$$\begin{aligned} \frac{\partial}{\partial m} \left(x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - 1) y \right) &= k [2(m+1)(m-1) + (m+1)^2] x^m \\ &\quad + k x^m \log m \cdot (m+1)^2 (m-1) \end{aligned}$$

$$\Rightarrow \left(x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} + (x^2 - 1) \right) \frac{\partial y}{\partial m} = k [2(m^2 - 1) + (m + 1)^2] x^m \\ + k x^m \log m \cdot (m + 1)^2 (m - 1)$$

Putting $m = -1$, we have

$$\left(x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} + (x^2 - 1) \right) \left(\frac{\partial y}{\partial m} \right)_{m=-1} = 0$$

Hence $\left(\frac{\partial y}{\partial m} \right)_{m=-1}$ is also a solution of differential equation (1)

Now from (6), differentiating partially w.r.t. m , we get

$$\frac{\partial y}{\partial m} = k x^m \log x \left[(m + 1) - \frac{x^2}{(m + 3)} + \frac{x^4}{(m + 3)^3 (m + 5)} - \dots \right] \\ + k x^m \left[1 + \frac{x^2}{(m + 3)^2} + \frac{x^4}{(m + 3)^2 (m + 5)} \left(\frac{-2}{m + 3} - \frac{1}{m + 5} \right) - \dots \right]$$

⎧ Taking $z_1 = \frac{1}{(m + 3)^2 (m + 5)}$, and diff. after taking log, we get $\frac{\partial z_1}{\partial m}$ ⎫

$$\Rightarrow \left(\frac{\partial y}{\partial m} \right)_{m=-1} = k x^{-1} \log x \left[0 - \frac{x^2}{2} + \frac{x^4}{2^2 \cdot 4} - \dots \right] \\ + k x^{-1} \left[1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4} \left(-\frac{2}{2} - \frac{1}{4} \right) - \dots \right] \\ = k \left[u \log x + x^{-1} \left(1 + \frac{x^2}{2^2} - \frac{5x^4}{2^2 \cdot 4^2} + \dots \right) \right] = bv \text{ (say) Taking } k = b$$

Hence the required solution is $y = au + bv$,

where a and b are arbitrary constants.

3.1 1 Roots of Indicial Equation Equal [Frobenius Method]

Method : Let $m = \alpha, \alpha$ be two equal roots of the indicial equation. Then two independent solutions say u and v will be obtained by substituting $m = \alpha$ in the series for y and $\frac{\partial y}{\partial m}$. The second solution will always consist of the product of the first solution (or its numerical value) and $\log x$ added to another series.

Solved Examples

Example 1: Find the series solution of $x y'' + (x + 1) y' + 2y = 0$ by Frobenius method.

[Garhwal 2005; Meerut 2010 (Sem I.)]

Solution: Given $x y'' + (x + 1) y' + 2y = 0$

...(1)

Let the series solution of (1) be given by

$$y = \sum_{r=0}^{\infty} C_r x^{m+r}, \quad C_0 \neq 0 \quad \dots(2)$$

$$\text{Diff.} \quad y' = \sum_{r=0}^{\infty} C_r (m+r) x^{m+r-1} \text{ and } y'' = \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) x^{m+r-2}$$

Substituting in (1), we get

$$\sum_{r=0}^{\infty} C_r [(m+r)(m+r-1) x^{m+r-1} + (x+1)(m+r) x^{m+r-1} + 2x^{m+r}] = 0$$

$$\text{or} \quad \sum_{r=0}^{\infty} C_r [\{(m+r)^2 - (m+r) + (m+r)\} x^{m+r-1} + (m+r+2) x^{m+r}] = 0$$

$$\text{or} \quad \sum_{r=0}^{\infty} C_r [(m+r)^2 x^{m+r-1} + (m+r+2) x^{m+r}] = 0 \quad \dots(3)$$

Which is an identity in x . So we can equate to zero the coefficients of various powers of x .

Equating to zero the coefficient of lowest power of x i.e., of x^{m-1} , the *indicial equation* is

$$C_0 m^2 = 0 \Rightarrow m = 0, 0 \text{ (Two equal roots)} \quad \because \quad C_0 \neq 0$$

Now equating to zero the coefficient of x^{m+r-1} , we get

$$(m+r)^2 C_r + (m+r-1+2) C_{r-1} = 0$$

$$\Rightarrow \quad C_r = -\frac{m+r+1}{(m+r)^2} C_{r-1}, \quad r \geq 1. \quad \dots(4)$$

Putting $r = 1, 2, 3, \dots$ in (4), we get

$$C_1 = -\frac{m+2}{(m+1)^2} C_0, \quad C_2 = -\frac{m+3}{(m+2)^2} C_1 = \frac{(m+2)(m+3)}{(m+1)^2(m+2)^2} C_0 = \frac{(m+3)}{(m+1)^2(m+2)} C_0$$

$$C_3 = -\frac{m+4}{(m+3)^2} C_2 = -\frac{(m+3)(m+4)}{(m+1)^2(m+2)(m+3)^2} C_0 = -\frac{(m+4)}{(m+1)^2(m+2)(m+3)} C_0$$

and so on.

∴ from (2),

$$y = C_0 x^m + C_1 x^{m+1} + C_2 x^{m+2} + C_3 x^{m+3} + \dots$$

$$\text{or } y = C_0 x^m \left[1 - \frac{(m+2)}{(m+1)^2} x + \frac{(m+3)}{(m+1)^2 (m+2)} x^2 - \frac{(m+4)}{(m+1)^2 (m+2) (m+3)} x^3 + \dots \right] \quad \dots(5)$$

Putting $m=0$ and taking $C_0 = a$

$$y = a \left[1 - 2x + \frac{3}{2!} x^2 - \frac{4}{3!} x^3 + \dots \right] = a u \text{ (say)} \quad \dots(6)$$

This gives one solution of (1).

To get another independent solution, we proceed as follows :

From (5), we have

$$\begin{aligned} y' &= C_0 \left[m x^{m-1} - \frac{(m+1)(m+2)}{(m+1)^2} x^m + \frac{(m+3)(m+2)}{(m+1)^2 (m+2)} x^{m+1} - \dots \right] \\ &= C_0 \left[m x^{m-1} - \frac{m+2}{m+1} x^m + \frac{m+3}{(m+1)^2} x^{m+1} - \dots \right] \end{aligned}$$

$$\text{and } y'' = C_0 \left[m(m-1) x^{m-2} - \frac{m(m+2)}{m+1} x^{m-1} + \frac{m+3}{m+1} x^m - \dots \right]$$

Putting in the L.H.S. of (1), we get

$$\begin{aligned} x y'' + (x+1) y' + 2y &= C_0 \left[m(m-1) x^{m-1} - \frac{m(m+2)}{m+1} x^m + \frac{m+3}{m+1} x^{m+1} - \dots \right] \\ &\quad + (x+1) C_0 \left[m x^{m-1} - \frac{m+2}{m+1} x^m + \frac{m+3}{(m+1)^2} x^{m+1} - \dots \right] \\ &\quad + 2C_0 \left[x^m - \frac{m+2}{(m+1)^2} x^{m+1} + \dots \right] \\ &= C_0 m^2 x^{m-1} \text{ (Other terms being cancelled)} \end{aligned}$$

Since R.H.S. involves the square of m its partial differential coefficient w.r.t m will vanish when $m=0$.

$$\text{Now } \frac{\partial}{\partial m} [x y'' + (x+1) y' + 2y] = \frac{\partial}{\partial m} (C_0 m^2 x^{m-1}) = C_0 (2m x^{m-1} + m^2 x^{m-1} \log x)$$

$$\Rightarrow \frac{\partial}{\partial m} \left[x \frac{\partial^2}{\partial x^2} + (x+1) \frac{\partial}{\partial x} + 2 \right] y = C_0 (2mx^{m-1} + m^2 x^{m-1} \log x)$$

$$\Rightarrow \left[x \frac{\partial^2}{\partial x^2} + (x+1) \frac{\partial}{\partial m} + 2 \right] \frac{\partial y}{\partial m} = C_0 (2mx^{m-1} + m^2 x^{m-1} \log x)$$

(As differential operator being commutative)

Hence $\frac{\partial y}{\partial m}$ is a solution of (1) if m is put equal to zero after differentiation i.e.,

i.e., $\left(\frac{\partial y}{\partial m} \right)_{m=0}$ is second solution of (1).

Now from (5),

$$\begin{aligned} \frac{\partial y}{\partial m} &= C_0 x^m \log x \left[1 - \frac{m+2}{(m+1)^2} x + \frac{(m+3)}{(m+1)^2 (m+2)} x^2 - \frac{(m+4)}{(m+1)^2 (m+2) (m+3)} x^3 + \dots \right] \\ &+ C_0 x^m \left[0 - x \frac{(m+2)}{(m+1)^2} \left\{ \frac{1}{m+2} - \frac{2}{m+1} \right\} + x^2 \frac{(m+3)}{(m+1)^2 (m+2)} \left\{ \frac{1}{m+3} - \frac{2}{m+1} - \frac{1}{m+2} \right\} \dots \right] \end{aligned}$$

(Taking $z_1 = \frac{m+2}{(m+1)^2}$, $\log z_1 = \log(m+3) - 2 \log(m+1)$ Now diff. find $\frac{\partial z_1}{\partial x}$. etc.)

$$\begin{aligned} \Rightarrow \left(\frac{\partial y}{\partial m} \right)_{m=0} &= C_0 \log x \left[1 - 2x + \frac{3}{2!} x^2 - \frac{4}{3!} x^3 + \dots \right] + C_0 \left[3x - \frac{13}{2 \cdot 2!} x^2 + \dots \right] \\ &= b \left[u \log x + \left\{ 3x - \frac{13}{2 \cdot 2!} x^2 + \dots \right\} \right] \\ &= bv \text{ (say)} \quad \text{Taking } C_0 = b \end{aligned}$$

which is second solution of (1).

Hence the required solution is

$$y = au + bv$$

where a and b are arbitrary constants.

Example 2: Find the series solution of $(x - x^2) y'' + (1 - x) y' - y = 0$ near $x = 0$.

[Delhi Maths (Hons.) 2008]

Solution: Given $(x - x^2) y'' + (1 - x) y' - y = 0$

...(1)

Let the series solution of (1), about $x = 0$ be given by

$$y = \sum_{r=0}^{\infty} C_r x^{m+r}, \quad C_0 \neq 0 \quad \dots(2)$$

Differentiating $y' = \sum_{r=0}^{\infty} C_r (m+r) x^{m+r-1}$ and

$$y'' = \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) x^{m+r-2}$$

Substituting in (1), we get

$$\sum_{r=0}^{\infty} C_r [(x-x^2)(m+r)(m+r-1)x^{m+r-2} + (1-x)(m+r)x^{m+r-1} - x^{m+r}] = 0$$

$$\text{or } \sum_{r=0}^{\infty} C_r [\{(m+r)^2 - (m+r) + (m+r)\} x^{m+r-1} - \{(m+r)^2 - (m+r) + (m+r) + 1\} x^{m+r}] = 0$$

$$\text{or } \sum_{r=0}^{\infty} C_r [(m+r)^2 x^{m+r-1} - \{(m+r)^2 + 1\} x^{m+r}] = 0 \quad \dots(3)$$

Which is an identity, so we can equate to zero the coefficients of various powers of x .

Equating to zero the coefficient of lowest power of x i.e., of x^{m-1} , the *indicial equation* is

$$C_0 m^2 = 0 \Rightarrow m = 0, 0 \text{ (Two equal roots), } \quad \therefore \quad C_0 \neq 0$$

Now equating to zero the coefficient of x^{m+r-1} , we get

$$(m+r)^2 C_r - \{(m+r-1)^2 + 1\} C_{r-1} = 0$$

$$\Rightarrow C_r = \frac{(m+r-1)^2 + 1}{(m+r)^2} C_{r-1}, \quad r \geq 1 \quad \dots(4)$$

Putting $r = 1, 2, 3, \dots$, in (4), we get

$$C_1 = \frac{m^2 + 1}{(m+1)^2} C_0, \quad C_2 = \frac{(m+1)^2 + 1}{(m+2)^2} C_1 = \frac{(m^2 + 1) \{(m+1)^2 + 1\}}{(m+1)^2 (m+2)^2} C_0$$

$$C_3 = \frac{(m+2)^2 + 1}{(m+3)^2} C_2 = \frac{(m^2 + 1) \{(m+1)^2 + 1\} \{(m+2)^2 + 1\}}{(m+1)^2 (m+2)^2 (m+3)^3} C_0$$

and so on.

$$\therefore \quad \text{from (2), } y = C_0 x^m + C_1 x^{m+1} + C_2 x^{m+2} + C_3 x^{m+3} + \dots$$

or

$$y = C_0 x^m \left[1 + \frac{m^2 + 1}{(m+1)^2} x + \frac{(m^2 + 1) \{(m+1)^2 + 1\}}{(m+1)^2 (m+2)^2} x^2 + \frac{(m^2 + 1) \{(m+1)^2 + 1\} \{(m+2)^2 + 1\}}{(m+1)^2 (m+2)^2 (m+3)^2} x^3 + \dots \right] \dots (5)$$

Putting $m=0$ and taking $C_0 = a$, we have

$$y = a \left[1 + x + \frac{2}{4} x^2 + \frac{2.5}{4.9} x^3 + \dots \right] = au \text{ (say)} \quad \dots (6)$$

which is one solution of (1)

Proceeding as in Ex. 1, $\left(\frac{\partial y}{\partial m} \right)_{m=0}$ is second independent solution of (1).

From (5), we have

$$\begin{aligned} \frac{\partial y}{\partial m} = C_0 x^m \log x & \left[1 + \frac{m^2 + 1}{(m+1)^2} x + \frac{(m^2 + 1) \{(m+1)^2 + 1\}}{(m+1)^2 (m+2)^2} x^2 + \dots \right] \\ & + C_0 x^m \left[0 + \frac{m^2 + 1}{(m+1)^2} x \left\{ \frac{2m}{m^2 + 1} - \frac{2}{m+1} \right\} \right] \\ & + \frac{(m^2 + 1) \{(m+1)^2 + 1\}}{(m+1)^2 (m+2)^2} x^2 \cdot \left\{ \frac{2m}{m^2 + 1} + \frac{2(m+1)}{(m+1)^2 + 1} - \frac{2}{m+1} - \frac{2}{m+2} \right\} + \dots \end{aligned}$$

[Taking $z_1 = \frac{m^2 + 1}{(m+1)^2}$, $\log z_1 = \log (m^2 + 1) - 2 \log (m+1)$

Diff. $\frac{\partial z_1}{\partial m} = z_1 \left(\frac{2m}{m^2 + 1} - \frac{2}{m+1} \right)$ etc.]

$$\begin{aligned} \therefore \left(\frac{\partial y}{\partial m} \right)_{m=0} &= C_0 \log x \left[1 + x + \frac{2}{4} x^2 + \frac{2.5}{4.9} x^3 + \dots \right] + C_0 \left(-2x - x^2 - \frac{14}{27} x^3 - \dots \right) \\ &= b \left[u \log x + \left(-2x - x^2 - \frac{14}{27} x^3 - \dots \right) \right] = bv \text{ (say)} \end{aligned}$$

Taking $C_0 = b$

Which is the second solution of (2),

Hence the required solution of (1) is

$$y = au + bv$$

where a and b are arbitrary constants.

Example 3: Solve in series $(x - x^2) y'' + (1 - 5x) y' - 4y = 0$

[Delhi Maths (Hons.) 2005; Agra 2009]

Solution: Given $(x - x^2) y'' + (1 - 5x) y' - 4y = 0$... (1)

Obviously $x = 0$ is a regular singular point of (1) (check)

Let the series solution of (1), be given by

$$y = \sum_{r=0}^{\infty} C_r x^{m+r}, \quad C_0 \neq 0 \quad \dots (2)$$

Differentiating $y' = \sum_{r=0}^{\infty} C_r (m+r) x^{m+r-1}$ and

$$y'' = \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) x^{m+r-2}$$

Substituting in (1), we get

$$\begin{aligned} \sum_{r=0}^{\infty} C_r [(x - x^2)(m+r)(m+r-1) x^{m+r-2} \\ + (1 - 5x)(m+r) x^{m+r-1} - 4x^{m+r}] = 0 \end{aligned}$$

$$\text{or } \sum_{r=0}^{\infty} C_r [(m+r)^2 x^{m+r-1} - \{(m+r)^2 - (m+r) + 5(m+r) + 4\} x^{m+r}] = 0$$

$$\text{or } \sum_{r=0}^{\infty} C_r [(m+r)^2 x^{m+r-1} - \{(m+r)^2 + 4(m+r) + 4\} x^{m+r}] = 0$$

$$\text{or } \sum_{r=0}^{\infty} C_r [(m+r)^2 x^{m+r-1} - (m+r+2)^2 x^{m+r}] = 0 \quad \dots (3)$$

Which is an identity, so we can equate to zero the coefficient of various powers of x .

Equating to zero the coefficient of lowest power of x , i.e., of x^{m-1} , the *indicial equation* is

$$C_0 m^2 = 0 \Rightarrow m = 0, 0 \text{ (equal roots)}, \quad \because C_0 \neq 0$$

Again equating to zero the coefficient of x^{m+r-1} , we get

$$\begin{aligned} C_r (m+r)^2 - (m+r+2)^2 C_{r-1} &= 0 \\ \Rightarrow C_r &= \frac{(m+r+1)^2}{(m+r)^2} C_{r-1}, \quad r \geq 1 \end{aligned} \quad \dots (4)$$

Putting $r = 1, 2, 3, \dots$, we get

$$C_1 = \frac{(m+2)^2}{(m+1)^2} C_0, C_2 = \frac{(m+3)^2}{(m+2)^2} C_1 = \frac{(m+3)^2 (m+2)^2}{(m+2)^2 (m+1)^2} C_0 = \frac{(m+3)^2}{(m+1)^2} C_0$$

$$C_3 = \frac{(m+4)^2}{(m+3)^2} C_2 = \frac{(m+4)^2 (m+3)^2}{(m+3)^2 (m+1)^2} C_0 = \frac{(m+4)^2}{(m+1)^2} C_0$$

and so on.

∴ from (2), we get

$$y = C_0 x^m + C_1 x^{m+1} + C_2 x^{m+2} + \dots$$

$$\text{or } y = C_0 x^m \left[1 + \frac{(m+2)^2}{(m+1)^2} x + \frac{(m+3)^2}{(m+1)^2} x^2 + \frac{(m+4)^2}{(m+1)^2} x^3 + \dots \right] \quad \dots(5)$$

Putting $m=0$ and taking $C_0 = a$, we get

$$y = a [1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \dots] = au \text{ (say)} \quad \dots(6)$$

which is one solution of (1).

Proceeding as in Ex. 1 $\left(\frac{\partial y}{\partial m} \right)_{m=0}$ is second independent solution of (1).

From (5), we have

$$\begin{aligned} \frac{\partial y}{\partial m} = & C_0 x^m \log x \left[1 + \frac{(m+2)^2}{(m+1)^2} x + \frac{(m+3)^2}{(m+1)^2} x^2 + \frac{(m+4)^2}{(m+1)^2} x^3 + \dots \right] \\ & + C_0 x^m \left[0 + \frac{(m+2)^2}{(m+1)^2} x \cdot \left(\frac{2}{m+2} - \frac{2}{m+1} \right) + \frac{(m+3)^2}{(m+1)^2} x^2 \left(\frac{2}{m+3} - \frac{2}{m+1} \right) \right. \\ & \left. + \frac{(m+4)^2}{(m+1)^2} x^3 \cdot \left(\frac{2}{m+4} - \frac{2}{m+1} \right) + \dots \right] \end{aligned}$$

$$\text{[Taking } z_1 = \frac{(m+2)^2}{(m+1)^2}, \log z_1 = 2 \log (m+2) - 2 \log (m+1)$$

$$\therefore \frac{\partial z_1}{\partial x} = z_1 \left(\frac{2}{m+2} - \frac{2}{m+1} \right) \text{ etc.}$$

$$\begin{aligned} \therefore \left(\frac{\partial y}{\partial m} \right)_{m=0} = & C_0 [\log x (1 + 2^2 x + 3^2 x^2 + 4^2 x^3 + \dots) \\ & + \left\{ 2^2 x (1-2) + 3^2 x^2 \cdot \left(\frac{2}{3} - 2 \right) + 4^2 x^3 \left(\frac{2}{4} - 2 \right) \dots \right\}] \end{aligned}$$

$$= b [u \log x - 2 (1.2 x + 2.3 x^2 + 3.4 x^3 + \dots)] = bv \text{ (say)}$$

Taking $C_0 = b$

which is second solution of (1).

Hence the required solution of (1) is $y = au + bv$, where a and b are arbitrary constants.

3.12 Solution of Legendre's Equation (In Decending Powers of x)

Example 1: Find a series solution in decending power of x of the Legendre's equation :

$$(1 - x^2) y'' - 2xy' + n(n+1)y = 0$$

taking $2n$ as non integral.

Solution: Given $(1 - x^2) y'' - 2xy' + n(n+1)y = 0$... (1)

Let the series solution of (1) in decending power of x , be given by

$$y = \sum_{r=0}^{\infty} C_r x^{m-r}, \quad C_0 \neq 0 \quad \dots (2)$$

Differentiating $y' = \sum_{r=0}^{\infty} C_r (m-r) x^{m-r-1}$

and $y'' = \sum_{r=0}^{\infty} C_r (m-r)(m-r-1) x^{m-r-2}$

Substituting in (1), we get

$$\sum_{r=0}^{\infty} C_r [(1 - x^2) (m-r)(m-r-1) x^{m-r-2} - 2x(m-r) x^{m-r-1} + n(n+1) x^{m-r}] = 0$$

or $\sum_{r=0}^{\infty} C_r [(m-r)(m-r-1) x^{m-r-2} - \{(m-r)^2 - (m-r) + 2(m-r) - n(n+1)\} x^{m-r}] = 0$

or $\sum_{r=0}^{\infty} C_r [(m-r)(m-r-1) x^{m-r-2} - \{(m-r)^2 - n^2 + (m-r+n)\} x^{m-r}] = 0$

or $\sum_{r=0}^{\infty} C_r [(m-r)(m-r-1) x^{m-r-2} - (m-r-n)(m-r+n+1) x^{m-r}] = 0 \quad \dots (3)$

Which is an identity, so we can equate to zero the coefficients of various powers of x .

Equating to zero the coefficient of highest powers of x i.e., of x^m , the *indicial equation* is

$$C_0 (m - n) (m + n + 1) = 0 \Rightarrow m = n, -(n + 1), \quad \because C_0 \neq 0$$

Again equating to zero, the coefficient of highest power of x in m and r i.e., of x^{m-r} , we get

$$\begin{aligned} C_{r-2} (m - r + 2) (m - r + 2 - 1) - (m - r - n) (m - r + n + 1) C_r &= 0 \\ \Rightarrow C_r &= -\frac{(m - r + 2) (m - r + 1)}{(n - m + r) (n + m - r + 1)} C_{r-2} \quad r \geq 2 \quad \dots(4) \end{aligned}$$

Since the difference in suffixes of C is $r - (r - 2) = 2$, so we also equate to zero the coefficient of next lower power to the highest power of x i.e., of x^{m-1} , we get

$$-C_1 (m - 1 - n) (m - 1 + n + 1) = 0 \Rightarrow C_1 = 0 \text{ for both } m = n \text{ and } m = -(n + 1)$$

Now putting $r = 3, 5, \dots$ in (4), we get

$$C_3 = -\frac{(m - 1) (m - 2)}{(n - m + 3) (n + m - 2)} C_1 = 0$$

Thus, $C_1 = 0 = C_3 = C_5 = \dots$

Again putting $r = 2, 4, 6, \dots$ in (4), we get

$$\begin{aligned} C_2 &= -\frac{m (m - 1)}{(n - m + 2) (n + m - 1)} C_0 \\ C_4 &= -\frac{(m - 2) (m - 3)}{(n - m + 4) (n + m - 4 + 1)} C_2 \\ &= \frac{m (m - 1) (m - 2) (m - 3)}{(n - m + 2) (n - m + 4) (n + m - 1) (n + m - 3)} C_0 \end{aligned}$$

and so on.

$\therefore$ from (2), we get

$$y = C_0 x^m + C_1 x^{m-1} + C_2 x^{m-2} + C_3 x^{m-3} + \dots$$

$$\begin{aligned} \text{or } y &= C_0 \left[x^m - \frac{m (m - 1)}{(n - m + 2) (n + m - 1)} x^{m-2} \right. \\ &\quad \left. + \frac{m (m - 1) (m - 2) (m - 3)}{(n - m + 2) (n - m + 4) (n + m - 1) (n + m - 3)} x^{m-4} + \dots \right] \dots(5) \end{aligned}$$

Now putting $m = n$ and taking $C_0 = a$, we get

$$y = a \left[x^n - \frac{n(n-1)}{2 \cdot (2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 (2n-1)(2n-3)} x^{n-4} - \dots \right] = au \text{ (say)}$$

Again putting $m = -(n+1)$ and taking $C_0 = b$, we get

$$y = b \left[x^{-n-1} + \frac{(n+1)(n+2)}{2 \cdot (2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4 (2n+3)(2n+5)} x^{-n-5} + \dots \right] = bv \text{ (say)}$$

Hence the required solution is $y = au + bv$, where a and b are arbitrary constants.

3.13 The Particular Integral

If in a linear differential equation R.H.S. is of the form x^m , then the usual method to determine the particular integral corresponding to x^m is followed. For the clear understanding of the method see the following example.

Example 1: Solve the series $(x^2 - x) \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} - 2y = x + \frac{3}{x^2}$, near $x = 0$

Solution: Given $(x^2 - x) \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} - 2y = x + \frac{3}{x^2}$... (1)

Obviously $x = 0$ is a regular singular point (check).

To Find C.F. : i.e., to find the solution of

$$(x^2 - x) \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} - 2y = 0 \quad \dots (2)$$

Let the series solution of (2), near $x = 0$, be given by

$$y = \sum_{r=0}^{\infty} C_r x^{m+r}, \quad C_0 \neq 0 \quad \dots (3)$$

Differentiating $\frac{dy}{dx} = \sum_{r=0}^{\infty} C_r (m+r) x^{m+r-1}$

and $\frac{d^2 y}{dx^2} = \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) x^{m+r-2}$

Substituting in (2), we get

$$\sum_{r=0}^{\infty} C_r [(x^2 - x)(m+r)(m+r-1) x^{m+r-2} + 3(m+r) x^{m+r-1} - 2x^{m+r}] = 0$$

or $\sum_{r=0}^{\infty} C_r [\{(m+r)^2 - (m+r) - 2\} x^{m+r} - \{(m+r)^2 - (m+r) - 3\} x^{m+r-1}] = 0$

$$\text{or } \sum_{r=0}^{\infty} C_r [(m+r-2)(m+r+1)x^{m+r} - (m+r)(m+r-4)x^{m+r-1}] = 0 \quad \dots(4)$$

Which is an identity, so we can equate to zero the coefficients of various powers of x .

Equating to zero the coefficient of lower power of x i.e., of x^{m-1} the indicial equation is

$$C_0(m)(m-4) = 0 \Rightarrow m = 0, 4 \quad \because C_0 \neq 0$$

Again equating to zero the coefficient of x^{m+r-1} , we get

$$\begin{aligned} C_{r-1}(m+r-1-2)(m+r-1+1) - C_r(m+r)(m+r-4) &= 0 \\ \Rightarrow C_r &= \frac{(m+r-3)}{(m+r-4)} C_{r-1}, \quad r \geq 1 \end{aligned} \quad \dots(5)$$

Putting $r = 1, 2, 3, \dots$

$$C_1 = \frac{m-2}{m-3} C_0, C_2 = \frac{(m-1)}{(m-2)} C_1 = \frac{(m-1)(m-2)}{(m-2)(m-3)} C_0 = \frac{m-1}{m-3} C_0$$

$$C_3 = \frac{m}{m-1} C_2 = \frac{m(m-1)}{(m-1)(m-3)} C_0 = \frac{m}{m-3} C_0$$

$$C_4 = \frac{m+1}{m} C_3 = \frac{m+1}{m} \cdot \frac{m}{m-3} C_0 = \frac{m+1}{m-3} C_0 \text{ and so on.}$$

$\therefore$ from (2)

$$y = C_0 x^m + C_1 x^{m+1} + C_2 x^{m+2} + \dots$$

$$\text{or } y = C_0 x^m \left[1 + \frac{m-2}{m-3} x + \frac{m-1}{m-3} x^2 + \frac{m}{m-3} x^3 + \frac{m+1}{m-3} x^4 + \frac{m+2}{m-3} x^5 + \dots \right] \quad \dots(6)$$

when $m = 0$ taking $C_0 = a$

$$y = a \left(1 + \frac{2}{3} x + \frac{1}{3} x^2 - \frac{1}{3} x^4 - \frac{2}{3} x^5 - \frac{3}{3} x^6 - \dots \right) = a u \text{ (say)}$$

when $m = 4$, taking $C_0 = b$

$$y = b x^4 (1 + 2x + 3x^2 + 4x^3 + 5x^4 \dots) = b v \text{ (say)}$$

$\therefore$ solution of (2), i.e., C.F. of (1) is given by C.F. = $au + bv$.

Obviously seeing u and v carefully, we note that

$$u = \left(1 + \frac{2}{3} x + \frac{1}{3} x^2 \right) - \frac{v}{3}$$

$$\therefore \text{C.F.} = \left[a \left(1 + \frac{2}{3} x + \frac{1}{3} x^2 \right) - \frac{av}{3} \right] + bv$$

$$\begin{aligned}
 &= \frac{a}{3} (x^2 + 2x + 3) + (b - a/3)v = A (x^2 + 2x + 3) + Bv \\
 &= A (x^2 + 2x + 3) + B x^4 (1 - x)^{-2}
 \end{aligned}$$

Where $a/3 = A$ and $(b - a/3) = B$ are arbitrary constants.

To Find P.I.

For y given by (6), on simplification, we get

$$(x^2 - x) \frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} - 2y = -m(m-4) C_0 x^{m-1} \quad \dots(7)$$

To find P.I. corresponding to x .

Taking only x on the R.H.S. of (1).

Using (7), we have $-m(m-4) C_0 x^{m-1} = x$

$$\Rightarrow -m(m-4) C_0 = 1 \text{ and } m-1=1 \Rightarrow m=2 \text{ and } C_0 = 1/4$$

For $m=2$ recurrence relation (5) reduces to

$$C_r = \frac{r-1}{r-2} C_{r-1}, \quad r \geq 1$$

Putting $r = 1, 2, 3, \dots$, $C_1 = 0 = C_2 = C_3 = \dots$

$$\therefore \text{ From (2), P.I. corresponding to } x = C_0 x^m = \frac{1}{4} x^2$$

To find P.I. corresponding to $3/x^2$.

Taking $3/x^2$ on the R.H.S. of (1), using (7)

$$-m(m-4) C_0 x^{m-1} = 3/x^2 = 3x^{-2}$$

$$\Rightarrow -m(m-4) C_0 = 3 \text{ and } m-1=-2 \Rightarrow m=-1 \text{ and } C_0 = -3/5$$

for $m=-1$, recurrence relation (5), reduces to

$$C_r = \frac{r-4}{r-5} C_{r-1}, \quad r \geq 1$$

$$\text{Putting } r = 1, 2, 3, \dots, C_1 = \frac{3}{4} C_0, C_2 = \frac{2}{3} \cdot C_1 = \frac{3}{4} \cdot \frac{2}{3} C_0 = \frac{1}{2} C_0$$

$$C_3 = \frac{-1}{-2} C_2 = \frac{1}{2} \cdot \frac{1}{2} C_0 = \frac{1}{4} C_0, C_4 = 0 = C_5 = \dots$$

$\therefore$ from (2), P.I. corresponding to $3/x^2$

$$= C_0 x^m + C_1 x^{m+1} + C_2 x^{m+2} + \dots$$

$$\begin{aligned}
&= x^m (C_0 + C_1 x + C_2 x^2 + C_3 x^3 + 0) \\
&= C_0 x^m \left(1 + \frac{3}{4}x + \frac{1}{2}x^2 + \frac{1}{4}x^3 \right) = -\frac{3}{5}x^{-1} \left(1 + \frac{3}{4}x + \frac{1}{2}x^2 + \frac{1}{4}x^3 \right) \\
&= -\frac{3}{5x} - \frac{3}{20}(x^2 - 2x + 3)
\end{aligned}$$

Hence the complete integral of (1) is

$$y = \text{C.F.} + \text{P.I. of } x + \text{P.I. of } x^2/3$$

or

$$\begin{aligned}
y &= A(x^2 + 2x + 3) + Bx^4(1-x)^{-2} + \frac{1}{4}x^2 - \frac{3}{5x} - \frac{3}{20}(x^2 + 2x + 3) \\
&= \left(A - \frac{3}{20} \right)(x^2 + 2x + 3) + Bx^4(1-x)^{-2} + \frac{x^2}{4} - \frac{3}{5x} \\
&= C(x^2 + 2x + 3) + Bx^4(1-x)^{-2} + \frac{x^2}{4} - \frac{3}{5x}
\end{aligned}$$

$$\text{where } A - \frac{3}{20} = C$$

Hence B and C are arbitrary constants.

3.14 Some Cases where the Frobenius Method Fails

It is not necessary that every equation can have a convergent series solution in ascending powers of x . In this situations in which indicial equation may not have any root or may have one root only, may lead to a divergent series.

For clear understanding see following examples :

Solved Examples

Example 1: Transform the equation $\frac{d^2 y}{dx^2} - y = 0$ by the substitution $x = \frac{1}{z}$ and show that it has no integrals that are regular in descending powers of x .

Solution: Given, $\frac{d^2 y}{dx^2} - y = 0$... (1)

Let $x = \frac{1}{z}$, so that $\frac{dx}{dz} = -\frac{1}{z^2}$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = -z^2 \frac{dy}{dz}$$

$$\begin{aligned}
 \text{and} \quad \frac{d^2 y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dz} \left(-z^2 \frac{dy}{dz} \right) \cdot \frac{dz}{dx} = \left(-z^2 \frac{d^2 y}{dz^2} - 2z \frac{dy}{dz} \right) \cdot (-z^2) \\
 &= z^4 \frac{d^2 y}{dz^2} + 2z^3 \frac{dy}{dz}
 \end{aligned}$$

Substituting in (1) and simplifying, we get

$$z^4 \frac{d^2 y}{dz^2} + 2z^3 \frac{dy}{dz} - y = 0 \quad \dots(2)$$

$$\text{Let} \quad y = \sum_{r=0}^{\infty} C_r z^{m+r}, \quad C_0 \neq 0 \quad \dots(3)$$

be series solution of (2) in ascending powers of z .

$$\text{Diff. (3),} \quad \frac{dy}{dz} = \sum_{r=0}^{\infty} C_r (m+r) z^{m+r-1}$$

$$\text{and} \quad \frac{d^2 y}{dz^2} = \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) z^{m+r-2}$$

Substituting in (2), we get

$$\sum_{r=0}^{\infty} C_r [z^4 (m+r)(m+r-1) z^{m+r-2} + 2z^3 (m+r) z^{m+r-1} - z^{m+r}] = 0$$

$$\text{or} \quad \sum_{r=0}^{\infty} C_r [\{(m+r)^2 - (m+r) + 2(m+r)\} z^{m+r+2} - z^{m+r}] = 0$$

$$\text{or} \quad \sum_{r=0}^{\infty} C_r [(m+r)(m+r+1) z^{m+r+2} - z^{m+r}] = 0 \quad \dots(4)$$

Which is an identity, so we can equate to zero the coefficients of various powers of z ,

Equating to zero the coefficient of lowest power of z i.e., of z^m , we get

$$C_0 = 0 \text{ which has no root, } \therefore C_0 \neq 0$$

$\Rightarrow$ The diff. equation(2) has no regular integrals in ascending powers of z

$\Rightarrow$ Given diff. equation (1) has no regular integrals in decending powers of x

Example 2: Transform Bessel's equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0 \text{ by substitution } x = \frac{1}{z}$$

Hence show that it has no integrals that are regular in decending powers of x .

Solution: Given, $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0$... (1)

Let $x = \frac{1}{z}$, $\therefore \frac{dx}{dz} = -\frac{1}{z^2}$, $\frac{dy}{dx} = -z^2 \frac{dy}{dz}$ and $\frac{d^2 y}{dx^2} = z^4 \frac{d^2 y}{dz^2} + 2z^3 \frac{dy}{dz}$

Substituting in (1) and simplifying, we get

$$z^4 \frac{d^2 y}{dz^2} + z^3 \frac{dy}{dz} + (1 - n^2 z^2) y = 0 \quad \dots (2)$$

Let $y = \sum_{r=0}^{\infty} C_r z^{m+r}$, $C_0 \neq 0$... (3)

be the series solution of (2) in ascending powers of z .

Diff. (3), $\frac{dy}{dz} = \sum_{r=0}^{\infty} C_r (m+r) z^{m+r-1}$ and $\frac{d^2 y}{dz^2} = \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) z^{m+r-2}$

Substituting in (2), we get

$$\sum_{r=0}^{\infty} C_r [(m+r)(m+r-1) z^{m+r+2} + (m+r) z^{m+r+2} + (1 - n^2 z^2) z^{m+r}] = 0$$

or $\sum_{r=0}^{\infty} C_r [\{(m+r)^2 - n^2\} z^{m+r+2} - z^{m+r}] = 0$... (4)

or $\sum_{r=0}^{\infty} C_r [(m+r-n)(m+r+n) z^{m+r+2} - z^{m+r}] = 0$... (5)

Which is an identity, so we can equate to zero the coefficients of various powers of z

Equating to zero the coefficient of lowest power of z , i.e., of z^m , we get

$$C_0 = 0 \text{ which has no root } \because C_0 \neq 0$$

$\Rightarrow$ diff. equation (2) has no regular integrals in ascending powers of z

$\Rightarrow$ given diff. equation (1) has no regular integral in decending powers of x .

Exercise 3.2

Use the method of Frobenius to find the solution near $x=0$ of each of the following differential equations

1. $2x^2 \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + (x-5)y = 0$

2. $2x(1-x)y'' + (1-x)y' + 3y = 0$

3. $2x^2 y'' + x y' - (x+1)y = 0$

4. $2x^2 y'' - x y' + (1-x^2)y = 0$

[Meerut 2001]

5. $2x y'' + (x+1)y' + 3y = 0$

[Delhi Maths (Hons.) 2005]

6. $2x^2 y'' + x y' + (x^2-3)y = 0$

7. $x^2 y'' + x y' + (x^2 - 1/9)y = 0$

8. $2x^2 y'' + x y' + (x^2 - 1)y = 0$

9. $3x y'' - (x-2)y' - 2y = 0$

10. $(1-x^2)y'' - 2x y' + 2y = 0$ (Legendre's equation of order unity)

11. $y'' + y = 0$

12. $y'' - x^2 y' - y = 0$

13. $y'' + x y = 0$

14. $(1-x^2)y'' - x y' + 4y = 0$

15. $y'' + x y' + x^2 y = 0$

16. $(1+x^2)y'' + x y' - y = 0$

17. $(x+x^2+x^3)y'' + 3x^2 y' - 2y = 0$

18. $x(1-x)y'' - 3x y' - y = 0$

19. $x(1-x)y'' - (1+3x)y' - y = 0$

20. $x(x-1)y'' + (3x-1)y' + y = 4x$

21. $x^2 y'' - x(1+x)y' + y = 0$

22. $x y'' + y' + xy = 0$ (Bessel's equation of order zero)

[Rajasthan 2001; Garhwal 2005]

23. $x y'' + y' + x^2 y = 0$

[Kurukshetra 2000, 01]

24. $(1-x^2)y'' - x \frac{dy}{dx} - a^2 y = 0$, where $x=0$, $y=1$ and $\frac{dy}{dx} = a$

25. $x y'' - x y = 1$ where $(y)_0 = 1, (y')_0 = 0$

26. $x y'' + y' + my = 0$, Given $y=1$ when $x=0$

Answers 3.2

1. $y = ax^{5/2} \left(1 - \frac{1}{9}x + \frac{1}{198}x^2 - \frac{1}{7722}x^3 + \dots \right) + bx^{-1} \left(1 + \frac{1}{5}x + \frac{1}{30}x^2 + \frac{1}{90}x^3 + \dots \right)$
2. $y = a \left(1 - 3x + \frac{3x^2}{1.3} + \frac{3x^3}{3.5} + \frac{4x^4}{5.7} + \dots \right) + bx^{1/2} (1 - x)$
3. $y = ax \left(1 + \frac{x}{5} + \frac{x^2}{70} + \dots \right) + bx^{-1/2} \left(1 - x - \frac{x^2}{2} - \dots \right)$
4. $y = ax \left(1 + \frac{x^2}{2.5} + \frac{x^4}{2.4.5.9} + \dots \right) + bx^{1/2} \left(1 + \frac{x^2}{2.3} + \frac{x^4}{2.4.3.7} + \dots \right)$
5. $y = a \left(1 - 3x + 2x^2 - \frac{2x^3}{3} + \dots \right) + bx^{1/2} \left(1 - \frac{7}{6}x + \frac{21}{40}x^2 - \frac{11}{80}x^3 + \dots \right)$
6. $y = ax^{3/2} \left(1 - \frac{1}{18}x^2 + \frac{1}{936}x^4 - \dots \right) + bx^{-1} \left(1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \dots \right)$
7. $y = ax^{1/3} \left(1 - \frac{3x^2}{16} + \frac{9x^4}{896} - \dots \right) + bx^{-1/3} \left(1 - \frac{3x^2}{8} + \frac{9x^4}{320} + \dots \right)$
8. $y = ax \left(1 - \frac{x^2}{14} + \frac{x^4}{616} - \dots \right) + bx^{-1/2} \left(1 - \frac{x^2}{2} + \frac{x^4}{40} - \dots \right)$
9. $y = a \left(1 + x + \frac{3}{10}x^2 + \dots \right) + bx^{1/3} \left(1 + \frac{7x}{12} + \frac{5x^2}{36} + \dots \right)$
10. $y = a \left(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \dots \right) + bx$
11. $y = a \cos x + b \sin x$
12. $y = a \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{20}x^5 + \dots \right) + b \left(x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{12}x^5 + \dots \right)$
13. $y = a \left(1 - \frac{1}{3!}x^3 + \frac{1.4}{6!}x^6 - \frac{1.4.7}{9!}x^9 + \dots \right) + bx \left(1 - \frac{2}{4!}x^3 + \frac{2.5}{7!}x^6 - \frac{2.5.8}{10!}x^9 + \dots \right)$
14. $y = a(1 - 2x^2) + b \left(x - \frac{x^3}{2} - \frac{x^5}{8} + \frac{x^7}{16} + \dots \right)$
15. $y = a \left(1 - \frac{x^4}{12} + \frac{x^6}{90} - \dots \right) + b \left(x - \frac{x^3}{6} - \frac{x^5}{40} - \dots \right)$
16. $y = a \left(1 + \frac{x^2}{2} - \frac{x^4}{8} + \dots \right) + C_1x$
17. $y = au + b \left[u \log x + 1 - x - 5x^2 - x^3 + \frac{11}{3}x^4 - \dots \right]$

where $u = 2x + 2x^2 - x^3 - x^4 + \frac{4}{5}x^5 \dots$

18. $y = au + b[u \log x + (1-x)^{-1}]$, where $u = x(1-x)^{-2}$

19. $y = au + b[u \log x + (-1 + x + 3x^2 + 4x^3 \dots)]$

where $u = 1.2x^2 + 2.3x^3 + 3.4x^4 + \dots$

20. $y = (a + b \log x)(1-x)^{-1} - x^2(1-x)^{-1}$

21. $y = axu + b\left[u \log x + x^2\left(-1 - \frac{3}{4}x \dots\right)\right]$ where $u = 1 + x + \frac{1}{2}x^2 + \frac{1}{2.3}x^3 + \dots$

22. $y = au + b\left[u \log x + \frac{x^2}{2^2} - \frac{x^4}{2^2.4^2}\left(1 + \frac{1}{2}\right) + \frac{x^6}{2^2.4^2.6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) \dots\right]$

where $u = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \dots$

23. $y = au + b\left[u \log x + 2\left\{\frac{x^3}{3^3} - \frac{x^6}{3^5(2!)^2}\left(1 + \frac{1}{2}\right) + \frac{x^9}{3^7(3!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \dots\right\}\right]$

24. $y = 1 + ax + \frac{a^2}{2!}x^2 + \frac{a(a^2 + 1^2)}{3!}x^3 + \frac{a^2(a^2 + 2^2)}{4!}x^4 + \dots$

25. $y = \left(1 + \frac{1}{3}x^3 + \frac{4}{6!}x^6 + \frac{4.7}{9!}x^9 + \dots\right) + \left(\frac{1}{2!}x^2 + \frac{3}{5!}x^5 + \frac{3.6}{8!}x^8 + \frac{3.6.8}{11!}x^{11} + \dots\right)$

26. $y = e^{-mx}$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter (a), (b), (c) or (d).

- For the differential equation $2x^2y'' - xy' + (x-4)y = 0$, the point $x=0$ is ;
 (a) ordinary point (b) regular singular point
 (c) irregular singular point (d) none of these
- The point $x=0$, for the diff. equation $xy'' + y \sin x = 0$ is :
 (a) regular singular point (b) ordinary point
 (c) irregular singular point (d) none of these
- The number of singular points of the diff. equation $y'' + (1-x)y' + (1-x)^2y = 0$ is:
 (a) 1 (b) 2 (c) 3 (d) 0
- Singular points are :
 (a) regular (b) irregular
 (c) regular or irregular (d) none of these

[Agra 2005, 06, 08]

5. For the differential equation $x^2(1-x)y'' + xy' + y = 0$
- (a) $x = 1$ is an ordinary point (b) $x = 1$ is a regular singular point
 (c) $x = 0$ is an irregular singular point (d) $x = 0$ is an ordinary point [GATE 2004]
6. For the equation $x(x-1)y'' + (\sin x)y' + 2x(x-1)y = 0$, consider the following statements
- A : $x = 0$ is a regular singular point
 B : $x = 1$ is a regular singular point
 Then :
- (a) both A and B are true (b) A is false but B is true
 (c) A is true but B is false (d) both A and B are false [GATE 2008]
7. The number of singular points of the differential equation $(x-1)y'' + xy' + (1/x)y = 0$ is :
- (a) 0 (b) 1 (c) 2 (d) 3
8. The regular singular points of the differential equation $(x^2 - 3x)\frac{d^2y}{dx^2} + (x+2)\frac{dy}{dx} + y = 0$ are :
- (a) 0, 1 (b) 0, 3 (c) 1, 2 (d) 2, 3
9. The exponents (Roots of indicial equation) of the diff. equation $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - (x^2 + 5/4)y = 0$ are
- (a) $5/2, -1/2$ (b) $-5/2, 1/2$ (c) $5/2, 1/2$ (d) $-5/2, -1/2$
10. The roots of the indicial equation of the diff. equation $3xy'' + 2y' + x^2y = 0$ are :
- (a) 0, $-1/3$ (b) $1/2, 1/3$ (c) $-1/2, -1/3$ (d) 0, $1/3$

Fill in the Blank

Fill in the blanks “...” so that the following statements are complete and correct.

- For the differential equation $y'' + P(x)y' + Q(x)y = 0$ if both $P(x)$ and $Q(x)$ exists at $x = a$, then $x = a$ is the point of the equation.
- For the diff. equation $(x^2 - 1)y'' + xy' - y = 0$, the point $x = 0$ is an point while the point $x = 1$ is a point.
- For the differential equation $y'' - xy' + 2y = 0$, the point $x = 0$ is point.
- All points are points of the differential equation $y'' + xy' + (x^2 + 2)y = 0$
- The indicial equation of the diff. equation $t^2x'' + t(3 - 2t)x' + (1 - 3t + t^2)x = 0$ is

True / False

Write 'T' for true and 'F' for false statement.

- The point $x = x_0$ is called singular point of the differential equation $y'' + P(x)y' + Q(x)y = 0$ if either or both $P(x)$ and $Q(x)$ are not defined at $x = x_0$.
- The point $x = 0$ is an irregular singular point of $x(x+1)^3 y'' + 2(x+1)^3 y' + 6y = 0$.
- For the differential equation

$$(x^4 - 2x^3 + x^2) \frac{d^2 y}{dx^2} + 2(x-1) \frac{dy}{dx} + x^2 y = 0$$

 $x = 0$ is an irregular singular point and $x = 1$ is a regular singular point.
- The exponents (Roots of the indicial equation) of the differential equation

$$x^2 \frac{d^2 y}{dx^2} + (x^2 - 3x) \frac{dy}{dx} + 3y = 0$$
 are 1 and 3.

Answers

Objective Type Questions

Multiple Choice Questions

1.	(b)	2.	(a)	3.	(d)	4.	(c)	5.	(b)
6.	(d)	7.	(c)	8.	(b)	9.	(a)	10.	(d)

Fill in the Blank

1.	ordinary	2.	ordinary, regular singular point	3.	ordinary
4.	ordinary	5.	$(m+1)^2 = 0$		

True / False

1.	True	2.	False	3.	True	4.	True

○○○



UNIT- 3

CHAPTER 4 : Beta and Gamma Functions

CHAPTER 5 : Gauss Hypergeometric Equation

CHAPTER 6 : Hermite Polynomials



4.1 Introduction

In this chapter we shall discuss two Eulerian integrals i.e., *Beta and Gamma functions*. These integrals occupy a very important place in definite integrals and have got wide application in Mechanics, Physics, Statistics and many other applied sciences.

4.2 Euler's Integrals

Here we define two Eulerian integrals.

1. Beta Function (Eulerian Integral of First Kind) (Def.)

The definite integral

$$\int_0^1 x^{m-1} (1-x)^{n-1} dx, \text{ for } m, n > 0$$

is known as **Beta function** and is denoted by $B(m, n)$ [read as “Beta m, n ”]. It is also called the *Eulerian integral of first kind*.

Thus
$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx, \quad m, n > 0$$

2. Gamma Function (Eulerian Integral of Second Kind) (Def.)

[Meerut 2013 (Sem. I)]

The definite integral

$$\int_0^{\infty} e^{-x} x^{n-1} dx \text{ for } n > 0$$

is known as **Gamma function** and is denoted by $\Gamma(n)$ [read as “Gamma n ”]. It is also called the *Eulerian integral of second kind*.

$$\text{Thus } \Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx \quad \text{for } n > 0$$

4.3 Elementary Properties of Gamma Function

$$\text{I. } \Gamma(1) = 1 \quad \because \quad \Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$\therefore \quad \Gamma(1) = \int_0^{\infty} e^{-x} x^{1-1} dx = \int_0^{\infty} e^{-x} = [-e^{-x}]_0^{\infty} = -e^{-\infty} + e^0 = 0 + 1$$

$$\text{i.e., } \Gamma(1) = 1$$

$$\text{II. } \Gamma(n+1) = n\Gamma(n), \quad n > 0$$

$$\because \quad \Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx, \quad n > 0$$

$$\begin{aligned} \therefore \quad \Gamma(n+1) &= \int_0^{\infty} e^{-x} x^{n+1-1} dx = \int_0^{\infty} e^{-x} x^n dx \\ &= [x^n (-e^{-x})]_0^{\infty} - \int_0^{\infty} x^{n-1} (-e^{-x}) dx, \quad (\text{integrating by parts}) \\ &= -\lim_{n \rightarrow \infty} \frac{x^n}{e^x} + 0 + n \int_0^{\infty} e^{-x} x^{n-1} dx \quad \dots(1) \end{aligned}$$

$$\text{Now } \lim_{x \rightarrow \infty} \frac{x^n}{e^x} = \lim_{x \rightarrow \infty} \frac{x^n}{1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \dots}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{\left[\frac{1}{x^n} + \frac{1}{x^{n-1}} + \frac{1}{2!x^{n-2}} + \dots + \frac{1}{n!} + \frac{x}{(n+1)!} + \dots \right]}$$

Dividing Num. and Den. by x^n .

$$= \frac{1}{0 + \frac{1}{n!} + \infty} = \frac{1}{\infty} = 0$$

$\therefore$ from (1), we get

$$\Gamma(n+1) = n\Gamma(n) \quad \text{for } n > 0$$

III. $\Gamma(n+1) = n!$ if n is a positive integer

If n is a positive integer, then

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) \text{ By Prop. II}$$

$$= n(n-1)(n-2)\dots 1.\Gamma(1) \text{ applying Prop. II again and again.}$$

$$\therefore \Gamma(1) = 1, \therefore \text{ if } n \text{ is a positive integer, then } \Gamma(n+1) = n!$$

IV. $\Gamma(n) = \infty$, if n is zero or a negative integer

$$\text{We have } \Gamma(n+1) = n\Gamma(n)$$

$$\Rightarrow \Gamma(n) = \frac{\Gamma(n+1)}{n} \quad \dots(1)$$

$$\text{Putting } n=0 \text{ in (1), } \Gamma(0) = \frac{\Gamma(1)}{0} = \frac{1}{0} = \infty$$

Putting $n = -1, -2, \dots$ in (1), we get

$$\Gamma(-1) = \frac{\Gamma(0)}{-1} = \frac{\infty}{-1} = \infty, \Gamma(-2) = \frac{\Gamma(-1)}{-2} = \frac{\infty}{-1} = \infty \text{ and so on.}$$

Thus, $\Gamma(n) = \infty$ if n is zero or a negative integer, i.e., $\Gamma(n)$ is not defined for zero and negative integral values of n .

V. $\Gamma(n)$ is defined for $-1 < n < 0$, $-2 < n < -1$, $-3 < n < -2$ etc.

$$\text{We have } \Gamma(n+1) = n\Gamma(n) \Rightarrow \Gamma(n) = \frac{\Gamma(n+1)}{n} \quad \dots(1)$$

For $-1 < n < 0$, $n+1 > 0$, so $\Gamma(n+1)$ is well defined.

$\therefore$ R.H.S. of (1) is well defined for $-1 < n < 0$

i.e., $\Gamma(n)$ is defined for $-1 < n < 0$.

Similarly $\Gamma(n)$ is defined for $-2 < n < -1$, $-3 < n < -2$ etc.

Thus from Prop. IV and V we conclude that $\Gamma(n)$ is defined for all values of n except for $n = 0, -1, -2, -3, \dots$

4.4 To Show that $\Gamma(1/2) = \sqrt{\pi}$

[Agra 2000, 05; Meerut 2007]

$$\text{Proof: We have } \Gamma(n) = \int_0^{\infty} e^{-t} t^{n-1} dt, \quad n > 0 \quad \dots(1)$$

Replacing n by $1/2$, we get

$$\Gamma(1/2) = \int_0^{\infty} e^{-t} t^{-1/2} dt = \int_0^{\infty} e^{-u^2} \cdot \frac{1}{u} \cdot 2u du = 2 \int_0^{\infty} e^{-u^2} du \quad \dots(2)$$

$$\text{Putting } t = u^2, dt = 2u du$$

$\therefore$ By prop. of definite integrals, we have

$$\Gamma(1/2) = 2 \int_0^{\infty} e^{-x^2} dx \quad \dots(3)$$

$$\text{and} \quad \Gamma(1/2) = 2 \int_0^{\infty} e^{-y^2} dy \quad \dots(4)$$

Multiplying the corresponding sides of (3) and (4), we get

$$[\Gamma(1/2)]^2 = 4 \left(\int_0^{\infty} e^{-x^2} dx \right) \left(\int_0^{\infty} e^{-y^2} dy \right) = 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2 + y^2)} dx dy \quad \dots(5)$$

Changing to polars, by substituting $x = r \cos \theta$, $y = r \sin \theta$,

$$\text{so that } x^2 + y^2 = r^2, \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r$$

From (1),

$$\begin{aligned} [\Gamma(1/2)]^2 &= 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2 + y^2)} dx dy = 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r d\theta dr \\ &= 4 \int_{\theta=0}^{\pi/2} \left[\int_{r=0}^{\infty} r e^{-r^2} dr \right] d\theta, dx dy \text{ is replaced by } \frac{\partial(x, y)}{\partial(r, \theta)} d\theta dr = r d\theta dr \\ &= 4 \int_0^{\pi/2} \left(\int_{v=0}^{\infty} e^{-v} \cdot \frac{1}{2} dv \right) d\theta \quad \text{Putting } r^2 = v, 2r dr = dv \\ &= 2 \int_0^{\pi/2} [-e^{-v}]_0^{\infty} d\theta = 2 \int_0^{\pi/2} [-e^{-\infty} + e^0] d\theta \\ &= 2 \int_0^{\pi/2} (0 + 1) d\theta = 2 [\theta]_0^{\pi/2} = \pi \end{aligned}$$

$$\therefore \Gamma(1/2) = \sqrt{\pi}. \text{ Also from (2) } \int_0^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}$$

4.5 Transformation of Gamma Function

$$\text{Form I. } \int_0^{\infty} e^{-k y} y^{n-1} dy = \frac{\Gamma(n)}{k^n}, \quad n > 0, k > 0$$

$$\text{Proof: We have,} \quad \Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$$

$$= \int_0^{\infty} e^{-k y} (k y)^{n-1} \cdot k dy \quad \text{Putting } x = k y, dx = k dy$$

$$= k^n \int_0^{\infty} e^{-k y} y^{n-1} dy$$

$$\Rightarrow \int_0^{\infty} e^{-k y} y^{n-1} dy = \frac{\Gamma(n)}{k^n}$$

Form II. $\Gamma(n) = \int_0^1 [\log(1/y)]^{n-1} dy, \quad n > 0$

Proof: We have, $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$

$$= - \int_1^0 (\log(1/y))^{n-1} dy \quad \text{Putting } x = \log(1/y), y = e^{-x}, dy = -e^{-x} dx$$

$$= \int_0^1 [\log(1/y)]^{n-1} dy$$

Form III. $\Gamma(n) = \frac{1}{n} \int_0^{\infty} e^{-y^{1/n}} dy, \quad n > 0$

Proof: We have, $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$

$$= \frac{1}{n} \int_0^{\infty} e^{-y^{1/n}} dy \quad \text{Putting } x^n = y, x = y^{1/n}, n x^{n-1} dx = dy$$

i.e., $\Gamma(n) = \frac{1}{n} \int_0^{\infty} e^{-y^{1/n}} dy$

Note : 1. From the above result, we have

$$\int_0^{\infty} e^{-y^{1/n}} dy = n \Gamma(n) = \Gamma(n+1)$$

2. Putting $n = 1/2$, $\Gamma(1/2) = 2 \int_0^{\infty} e^{-y^2} dy$ or $\int_0^{\infty} e^{-y^2} dy = \frac{1}{2} \Gamma(1/2) = \frac{\sqrt{\pi}}{2}$

Solved Examples

Example 1: Compute (i) $\Gamma(-1/2)$ [Agra 2006] (ii) $\Gamma(-5/2)$ [Kanpur 2004]

Solution: We have $\Gamma(n+1) = n \Gamma(n) \Rightarrow \Gamma(n) = \frac{\Gamma(n+1)}{n} \dots (1)$

(i) Putting $n = -1/2$ in (1), $\Gamma(-1/2) = \frac{\Gamma(-1/2+1)}{(-1/2)} = \frac{\Gamma(1/2)}{-1/2} = -2\sqrt{\pi}$

(ii) Putting $n = -\frac{5}{2}$ in (1),

$$\begin{aligned}\Gamma(-5/2) &= \frac{\Gamma(-5/2 + 1)}{-5/2} = -\frac{2}{5} \Gamma(-3/2) \\ &= \left(-\frac{2}{5}\right) \cdot \frac{\Gamma(-3/2 + 1)}{(-3/2)} = \left(-\frac{2}{5}\right) \left(-\frac{2}{3}\right) \Gamma(-1/2) \\ &= \left(-\frac{2}{5}\right) \left(-\frac{2}{3}\right) \cdot \frac{\Gamma(-1/2 + 1)}{-1/2} = \left(-\frac{2}{5}\right) \left(-\frac{2}{3}\right) (-2) \Gamma(1/2) = -\frac{8}{15} \sqrt{\pi}\end{aligned}$$

Example 2: Evaluate the followings :

$$(i) \int_0^{\infty} x^4 e^{-x} dx \quad (ii) \int_0^{\infty} x^2 e^{-x^2} dx \quad (iii) \int_0^{\infty} t^{-3/2} (1 - e^{-t}) dt$$

Solution: (i) $\int_0^{\infty} x^4 e^{-x} dx = \int_0^{\infty} x^{5-1} e^{-x} dx = \Gamma(5) = 4! = 24$

$$\begin{aligned}(ii) \int_0^{\infty} x^2 e^{-x^2} dx &= \int_0^{\infty} t \cdot e^{-t} \frac{1}{2} t^{-1/2} dt, \text{ Putting } x^2 = t, x = \sqrt{t}, dx = \frac{1}{2} t^{-1/2} dt \\ &= \frac{1}{2} \int_0^{\infty} t^{1/2} e^{-t} dt = \frac{1}{2} \int_0^{\infty} t^{(3/2)-1} e^{-t} dt = \frac{1}{2} \Gamma(3/2) = \frac{1}{2} \cdot \frac{1}{2} \Gamma(1/2) = \frac{\sqrt{\pi}}{4}\end{aligned}$$

$$\begin{aligned}(iii) \int_0^{\infty} t^{-3/2} (1 - e^{-t}) dt &= [(1 - e^{-t}) \cdot (-2t^{-1/2})]_0^{\infty} - \int_0^{\infty} e^{-t} \cdot (-2t^{-1/2}) dt \\ &= 0 + 2 \int_0^{\infty} t^{(1/2)-1} e^{-t} dt = 2\Gamma(1/2) = 2\sqrt{\pi}\end{aligned}$$

Example 3: Evaluate $\int_0^1 \frac{dx}{\sqrt{(-\log x)}}$

Solution: Putting $-\log x = t, x = e^{-t}, dx = -e^{-t} dt$

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{(-\log x)}} &= \int_{\infty}^0 \frac{-e^{-t} dt}{\sqrt{t}} = \int_0^{\infty} t^{-1/2} e^{-t} dt = \int_0^{\infty} t^{(1/2)-1} e^{-t} dt \\ &= \Gamma(1/2) = \sqrt{\pi}\end{aligned}$$

Example 4: If n is a positive integer and $m > -1$, then prove that

$$\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$$

Solution: Putting $\log x = -t, x = e^{-t}, dx = -e^{-t} dt$

$$\begin{aligned}\int_0^1 x^m (\log x)^n dx &= \int_{-\infty}^0 (e^{-t})^m (-t)^n (-e^{-t}) dt = (-1)^n \int_0^{\infty} e^{-(m+1)t} \cdot t^{(n+1)-1} dt \\ &= (-1)^n \frac{\Gamma(n+1)}{(m+1)^{n+1}} = \frac{(-1)^n n!}{(m+1)^{n+1}}, \quad \because n \text{ is a positive integer}\end{aligned}$$

Example 5: Show that $\int_0^{\infty} x^{2n-1} \cdot e^{-ax^2} dx = \frac{\Gamma(n)}{2a^n}$, $a > 0$, $n > 0$

Solution: Putting $ax^2 = t$, $x = \sqrt{t/a}$, $dx = \frac{1}{\sqrt{a}} \cdot \frac{1}{2} t^{-1/2} dt$

$$\begin{aligned}\int_0^{\infty} x^{2n-1} e^{-ax^2} dx &= \int_0^{\infty} \{\sqrt{t/a}\}^{2n-1} e^{-t} \cdot \frac{1}{2\sqrt{a}} t^{-1/2} dt \\ &= \frac{1}{2a^n} \int_0^{\infty} t^{n-1} e^{-t} dt = \frac{\Gamma(n)}{2a^n}\end{aligned}$$

Example 6: With certain limitations on the values of a , b , m and n prove that

$$\int_0^{\infty} \int_0^{\infty} e^{-(ax^2 + by^2)} x^{2m-1} y^{2n-1} dx dy = \frac{\Gamma(m)\Gamma(n)}{4a^m b^n}$$

Solution: Let $I = \int_0^{\infty} \int_0^{\infty} e^{-(ax^2 + by^2)} x^{2m-1} y^{2n-1} dx dy$

$$= \int_0^{\infty} e^{-ay^2} y^{2n-1} dy \cdot \int_0^{\infty} e^{-ax^2} x^{2m-1} dx = I_1 \cdot I_2 \quad \dots(1)$$

As in ex. 5. $I_1 = \int_0^{\infty} e^{-ax^2} x^{2m-1} dx = \frac{\Gamma(m)}{2a^m}$, taking $m > 0$, $a > 0$

and $I_2 = \int_0^{\infty} e^{-by^2} y^{2n-1} dy = \frac{\Gamma(n)}{2b^n}$, taking $n > 0$, $b > 0$.

Putting in (1), $I = \frac{\Gamma(m)\Gamma(n)}{4a^m b^n}$, for $a, b > 0$ and $m, n > 0$

Example 7: Prove that $\int_0^1 x^{n-1} \left(\log \frac{1}{x}\right)^{m-1} dx = \frac{\Gamma(m)}{n^m}$, $m, n > 0$ [Garhwal 2003]

Solution: Putting $\log(1/x) = t$, $x = e^{-t}$, $dx = -e^{-t} dt$

$$\begin{aligned}\int_0^1 x^{n-1} \left(\log \frac{1}{x}\right)^{m-1} dx &= \int_{\infty}^0 (e^{-t})^{n-1} t^{m-1} \cdot (-e^{-t}) dt \\ &= \int_0^{\infty} t^{m-1} \cdot e^{-nt} dt = \frac{\Gamma(m)}{n^m}, \quad m, n > 0\end{aligned}$$

Example 8: Show that (i) $\int_0^{\infty} \frac{x^c}{c^x} dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}, \quad c > 0$ [Purvanchal 2006]

(ii) $\int_0^{\infty} 3^{-4x^2} dx = \frac{1}{4} \sqrt{\frac{\pi}{(\log 3)}}$

Solution: (i) $\int_0^{\infty} \frac{x^c}{c^x} dx = \int_0^{\infty} \frac{x^c}{e^{x \log c}} dx = \int_0^{\infty} x^c e^{-x \log c} dx$

$$= \int_0^{\infty} x^{(c+1)-1} e^{-(\log c) x} dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}, \quad c > 0$$

(ii) $\int_0^{\infty} 3^{-4x^2} dx = \int_0^{\infty} e^{-4x^2 \log 3} dx$

$$= \frac{1}{4 \sqrt{\log 3}} \int_0^{\infty} e^{-t} \cdot t^{(1/2)-1} dt = \frac{1}{4 \sqrt{\log 3}} \Gamma(1/2)$$

Putting $4x^2 \log 3 = t$

$$x = \frac{1}{2 \sqrt{\log 3}} t^{1/2}$$

$$dx = \frac{1}{4 \sqrt{\log 3}} t^{(1/2)-1} dt$$

$$= \frac{\sqrt{\pi}}{4 \sqrt{\log 3}} = \frac{1}{4} \sqrt{\left(\frac{\pi}{\log 3} \right)}$$

Example 9: Show that $2^n \Gamma(n+1/2) = 1.3.5 \dots (2n-1) \sqrt{\pi}$

where n is a positive integer

Solution: We have

$$\Gamma(n+1/2) = \Gamma[(n-1/2)+1] = (n-1/2) \cdot \Gamma(n-1/2) \quad \because \Gamma(n+1) = n \Gamma(n) \quad \dots (1)$$

$$= (n-1/2) (n-3/2) \Gamma(n-3/2) \quad \text{applying (1) again.}$$

Proceeding similarly, we get

$$\Gamma(n+1/2) = (n-1/2) (n-3/2) (n-5/2) \dots (3/2) (1/2) \Gamma(1/2)$$

$$\therefore 2^n \Gamma(n+1/2) = (2n-1) (2n-3) \dots 5.3.1 \sqrt{\pi}$$

$$= 1.3.5 \dots (2n-3) (2n-1) \sqrt{\pi}$$

Example 10: When n is a positive integer, prove that

$$\Gamma(-n + 1/2) = \frac{(-1)^n 2^n \sqrt{\pi}}{1.3.5 \dots (2n-1)}$$

Solution: We have $\Gamma(n+1) = n \Gamma(n) \Rightarrow \Gamma(n) = \frac{\Gamma(n+1)}{n}$... (1)

$$\begin{aligned} \therefore \Gamma(-n + 1/2) &= \Gamma\left(\frac{-2n+1}{2}\right) = \frac{\Gamma\left(\frac{-2n+1}{2} + 1\right)}{\left(\frac{-2n+1}{2}\right)} && \text{Using (1)} \\ &= \frac{\Gamma\left(\frac{-2n+3}{2}\right)}{\left(\frac{-2n+1}{2}\right)} = \frac{1}{\left(\frac{-2n+1}{2}\right)} \cdot \frac{\Gamma\left(\frac{-2n+3}{2} + 1\right)}{\left(\frac{-2n+3}{2}\right)} && \text{applying (1) again} \\ &= \frac{1}{\left(\frac{-2n+1}{2}\right)} \cdot \frac{1}{\left(\frac{-2n+3}{2}\right)} \cdot \Gamma\left(\frac{-2n+5}{2}\right) \end{aligned}$$

Proceeding similarly, we get

$$\begin{aligned} \Gamma(-n + 1/2) &= \frac{1}{\left(\frac{-2n+1}{2}\right)} \cdot \frac{1}{\left(\frac{-2n+3}{2}\right)} \cdot \frac{1}{\left(\frac{-2n+5}{2}\right)} \dots \frac{1}{\left(-\frac{3}{2}\right)} \Gamma\left(-\frac{3}{2} + 1\right) \\ &= \frac{1}{\left(\frac{-2n+1}{2}\right)} \cdot \frac{1}{\left(\frac{-2n+3}{2}\right)} \dots \frac{1}{\left(-\frac{3}{2}\right)} \Gamma\left(-\frac{1}{2}\right) \\ &= \frac{1}{\left(\frac{-2n+1}{2}\right)} \cdot \frac{1}{\left(\frac{-2n+3}{2}\right)} \dots \frac{1}{\left(-\frac{3}{2}\right)} \cdot \frac{\Gamma\left(-\frac{1}{2} + 1\right)}{\left(-\frac{1}{2}\right)} \\ &= \frac{2^n \sqrt{\pi}}{(-1)^n (2n-1)(2n-3) \dots 5.3.1} = \frac{(-1)^n 2^n \sqrt{\pi}}{1.3.5 \dots (2n-1)} \end{aligned}$$

Example 11: Evaluate the integrals :

$$(i) \int_0^\infty e^{-ax} x^{m-1} \cos bx \, dx \text{ and } (ii) \int_0^\infty e^{-ax} x^{m-1} \sin bx \, dx$$

and deduce that

$$(a) \int_0^\infty x^{m-1} \cos bx \, dx = \frac{\Gamma(m)}{b^m} \cos \frac{m\pi}{2}$$

$$(b) \int_0^\infty x^{m-1} \sin bx \, dx = \frac{\Gamma(m)}{b^m} \sin \frac{m\pi}{2}$$

Solution: $\because e^{ibx} = \cos bx + i \sin bx$

$$\begin{aligned} \therefore \text{Consider } \int_0^\infty e^{ibx} e^{-ax} x^{m-1} dx &= \int_0^\infty e^{-x(a-ib)} x^{m-1} dx \\ &= \frac{\Gamma(m)}{(a-ib)^m} = \frac{\Gamma(m)}{r^m (\cos \theta - i \sin \theta)^m} \end{aligned}$$

Putting $a = r \cos \theta$ and $b = r \sin \theta$

so that $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$

$$\begin{aligned} &= \frac{\Gamma(m)}{r^m} (\cos \theta - i \sin \theta)^{-m} = \frac{\Gamma(m)}{r^m} (\cos m\theta + i \sin m\theta) \\ \Rightarrow \int_0^\infty e^{-ax} x^{m-1} (\cos bx + i \sin bx) dx &= \frac{\Gamma(m)}{(a^2 + b^2)^{m/2}} (\cos m\theta + i \sin m\theta) \end{aligned}$$

Equating real and imaginary parts, we get

$$\int_0^\infty e^{-ax} x^{m-1} \cos bx dx = \frac{\Gamma(m)}{(a^2 + b^2)^{m/2}} \cos m\theta \quad \dots(i)$$

$$\text{and } \int_0^\infty e^{-ax} x^{m-1} \sin bx dx = \frac{\Gamma(m)}{(a^2 + b^2)^{m/2}} \sin m\theta \quad \dots(ii)$$

Deductions : Putting $a=0$, $\theta = \tan^{-1} \infty = \pi/2$

$\therefore$ from (i) and (ii) we have

$$(a) \quad \int_0^\infty x^{m-1} \cos bx dx = \frac{\Gamma(m)}{b^m} \cos(m\pi/2)$$

$$\text{and (b) } \int_0^\infty x^{m-1} \sin bx dx = \frac{\Gamma(m)}{b^m} \sin(m\pi/2)$$

Example 12: Show that

$$\int_0^\infty \cos(bz^{1/n}) dz = \frac{1}{b^{1/n}} \Gamma(n+1) \cos\left(\frac{n\pi}{2}\right)$$

Solution:

$$I = \int_0^\infty \cos(bz^{1/n}) dz = n \int_0^\infty \cos(bx) x^{n-1} dx$$

Putting $z^{1/n} = x$, $z = x^n$, $dz = nx^{n-1} dx$

$$= \text{Real part of } n \int_0^\infty e^{-ibx} x^{n-1} dx$$

$$\begin{aligned}
&= \text{Real part of } n \frac{\Gamma(n)}{(ib)^n} \\
&= \text{Real part of } \frac{n\Gamma(n)}{b^n} \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)^{-n} \\
&= \text{Real part of } \frac{\Gamma(n+1)}{b^n} \left(\cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2} \right) \\
&= \frac{\Gamma(n+1)}{b^n} \cos \left(\frac{n\pi}{2} \right)
\end{aligned}$$

Example 13: Prove that $\int_0^\infty \frac{\sin bz}{z} dz = \frac{\pi}{2}$

Solution: We have $I = \int_0^\infty \int_0^\infty e^{-xz} \sin bz \, dx \, dz$

$$\begin{aligned}
&= \int_0^\infty \left(\frac{e^{-xz}}{-z} \right)_{x=0}^\infty \sin bz \, dz \quad \text{Integrating w.r.t. } x \\
&= \int_0^\infty \frac{\sin bz}{z} dz \quad \dots(1)
\end{aligned}$$

Again, first integrating w.r.t. z we have

$$\begin{aligned}
I &= \int_0^\infty \int_0^\infty e^{-xz} \sin bz \, dx \, dz = \int_0^\infty \left\{ \frac{e^{-xz}}{x^2 + b^2} (-x \sin bz - b \cos bz) \right\}_{z=0}^\infty dx \\
&= \int_0^\infty \frac{b}{x^2 + b^2} dx = \left[\tan^{-1} \left(\frac{x}{b} \right) \right]_0^\infty = \frac{\pi}{2} \quad \dots(2)
\end{aligned}$$

From (1) and (2) we get,

$$\int_0^\infty \frac{\sin bz}{z} dz = \frac{\pi}{2}$$

Exercise 4.1

1. Evaluate the followings :

(i) $\int_0^\infty e^{-4x} x^{5/2} dx$

(ii) $\int_0^\infty \sqrt{x} \cdot e^{-x^3} dx$

(iii) $\int_0^\infty x^2 e^{-x^2} dx$

(iv) $\int_0^\infty x^m e^{-ax^n} dx$

2. Show that $\int_0^{\infty} e^{(2ax - x^2)} dx = \frac{1}{2} \sqrt{\pi} e^{a^2}$

3. Show that

(i) $\int_0^{\infty} x^m e^{-x^n} dx = \frac{1}{n} \Gamma\left(\frac{m+1}{n}\right), (m > -1, n > 0)$

(ii) $\int_0^{\infty} e^{-ax} x^n dx = \frac{1}{a^{n+1}} \Gamma(n+1), (n > -1, a > 0)$

4. Show that $\Gamma\left(\frac{1}{2} + x\right) \Gamma\left(\frac{1}{2} - x\right) = \frac{\pi}{\cos \pi x}$

5. Show that $\Gamma\left(\frac{3}{2} - x\right) \Gamma\left(\frac{3}{2} + x\right) = \left(\frac{1}{4} - x^2\right) \pi \sec \pi x$

provided $-1 < 2x < 1$.

6. Show that

(i) $\int_0^{\pi/2} (\tan x)^n dx = \frac{\pi}{2} \sec\left(\frac{n\pi}{2}\right), (0 < n < 1)$

(ii) $\int_0^{\infty} \cos(bz^{1/n}) dz = \frac{\Gamma(n+1)}{b^n} \cos\left(\frac{n\pi}{2}\right)$

7. Prove that $\int_0^{\pi/2} \sqrt{\tan \theta} d\theta = \frac{1}{2} \Gamma(1/4) \Gamma(3/4) = \pi / \sqrt{2}$

8. Prove that $\int_0^{\pi/2} \sqrt{\sin \theta} d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin \theta}} = \pi$

[Garhwal 2002; Delhi Maths (Hons.) 2005, 08]

9. Evaluate $\int_0^{\infty} x^2 e^{-x^4} dx \times \int_0^{\infty} e^{-x^4} dx$

10. Compute

(i) $\Gamma(-1/2)$

[Agra 2006]

(ii) $\Gamma(-5/2)$ and (iii) $\Gamma(-9/2)$

[Kanpur 2004]

11. Prove that $\int_0^1 x^m (1-x^n)^p dx = \frac{1}{n} B\left(\frac{m+1}{n}, p+1\right)$

12. Prove that $\int_{-\infty}^{\infty} \cos\left(\frac{1}{2} \pi x^2\right) dx = 1$

13. Show that

(i) $\int_0^{\infty} x e^{-\alpha x} \cos \beta x dx = \frac{\alpha^2 - \beta^2}{(\alpha^2 + \beta^2)^2}$

(ii) $\int_0^{\infty} x e^{-\alpha x} \sin \beta x dx = \frac{2\alpha\beta}{(\alpha^2 + \beta^2)^2}$

Answers 4.1

1.	(i) $\frac{15\pi}{1024}$	(ii) $\frac{\sqrt{\pi}}{3}$	(iii) $\frac{\sqrt{\pi}}{4}$	(iv) $\frac{\Gamma\left(\frac{m+1}{n}\right)}{(na^{(m+1)/n})}$
9.	$\frac{\pi\sqrt{2}}{16}$			
10.	(i) $-2\sqrt{\pi}$	(ii) $\frac{-8\sqrt{\pi}}{15}$	(iii) $\frac{-32\sqrt{\pi}}{945}$	

4.6 Symmetric Property of Beta Functions i.e., $B(m, n) = B(n, m)$

We have $B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx = \int_0^1 (1-x)^{m-1} [1-(1-x)]^{n-1} dx$

By Prop. $\int_0^a f(x) dx = \int_0^a f(a-x) dx$ of D.I.

$$= \int_0^1 (1-x)^{m-1} x^{n-1} dx = \int_0^1 x^{n-1} (1-x)^{m-1} dx = B(n, m)$$

Hence, $B(m, n) = B(n, m)$

4.7 Transformation of Beta Function

$$\text{Form I. } B(m, n) = \int_0^\infty \frac{x^{m-1} dx}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx = \frac{1}{2} \int_0^\infty \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$$

[Kanpur 2004; Delhi 2008]

Proof: We have

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx = \int_\infty^0 \left(\frac{1}{1+t}\right)^{m-1} \left(1 - \frac{1}{1+t}\right)^{n-1} \left(-\frac{1}{(1+t)^2}\right) dt$$

$$\text{Putting } x = \frac{1}{1+t}, dx = -\frac{1}{(1+t)^2} dt$$

$$= \int_0^\infty \frac{t^{n-1}}{(1+t)^{m+n}} dt = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx \quad \text{Changing variable } t \text{ to } x.$$

Similarly $B(n, m) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

But $B(m, n) = B(n, m)$

$$B(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

We have, $B(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$. also $B(m, n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$

Adding, $2B(m, n) = \int_0^{\infty} \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

$$\therefore B(m, n) = \frac{1}{2} \int_0^{\infty} \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$$

Hence, $B(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx = \frac{1}{2} \int_0^{\infty} \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

Form II. $B(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

Proof: We have

$$B(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx \quad \dots(1)$$

Now putting $x = \frac{1}{t}$ so that $dx = -\frac{1}{t^2} dt$

$$\int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_{t=1}^0 \frac{(1/t)^{m-1} (-1/t^2) dt}{(1+1/t)^{m+n}} = \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}} dt = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$\therefore$ From (1), we have

$$B(m, n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$$

Form III. $B(m, n) = a^m b^n \int_0^{\infty} \frac{x^{m-1}}{(ax+b)^{m+n}} dx$

[Delhi Maths (Hons.) 2006]

Proof: We have, $B(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

$$= \int_0^{\infty} \frac{(at/b)^{m-1} \cdot (a/b) dt}{(1+at/b)^{m+n}} \quad \text{Putting } x = at/b, dx = (a/b) dt$$

$$= a^m b^n \int_0^{\infty} \frac{t^{m-1}}{(at+b)^{m+n}} dt = a^m b^n \int_0^{\infty} \frac{x^{m-1}}{(ax+b)^{m+n}} dx$$

$$\text{Form IV. } (a-b)^{m+n-1} B(m, n) = \int_b^a (x-b)^{m-1} (a-x)^{n-1} dx$$

$$\text{Proof: We have, } B(m, n) = \int_0^1 y^{m-1} (1-y)^{n-1} dy$$

$$\text{Putting } y = \frac{x-b}{a-b}, dy = \frac{dx}{(a-b)}$$

$$B(m, n) = \int_b^a \left(\frac{x-b}{a-b} \right)^{m-1} \left(1 - \frac{x-b}{a-b} \right)^{n-1} \frac{dx}{(a-b)}$$

$$= \frac{1}{(a-b)^{m+n-1}} \int_b^a (x-b)^{m-1} (a-x)^{n-1} dx$$

$$\Rightarrow (a-b)^{m+n-1} B(m, n) = \int_b^a (x-b)^{m-1} (a-x)^{n-1} dx$$

$$\text{Form V. } B(m, n) = 2a^m b^n \int_0^{\pi/2} \frac{\sin^{2m-1} \theta \cos^{2n-1} \theta d\theta}{(a \sin^2 \theta + b \cos^2 \theta)^{m+n}}$$

$$\text{Proof: We have, } B(m, n) = \int_0^{\infty} \frac{x^{m-1} dx}{(1+x)^{m+n}}$$

$$= \int_0^{\pi/2} \frac{\{(a/b) \tan^2 \theta\}^{m-1} (2a/b) \tan \theta \sec^2 \theta d\theta}{(1 + (a/b) \tan^2 \theta)^{m+n}}$$

$$\text{Putting } x = \frac{a}{b} \tan^2 \theta \text{ and } dx = \frac{2a}{b} \tan \theta \sec^2 \theta d\theta$$

$$= 2a^m b^n \int_0^{\pi/2} \frac{\tan^{2m-1} \theta \sec^2 \theta \cdot \cos^{2(m+n)} \theta d\theta}{(a \sin^2 \theta + b \cos^2 \theta)^{m+n}}$$

$$= 2a^m b^n \int_0^{\pi/2} \frac{\sin^{2m-1} \theta \cos^{2n-1} \theta d\theta}{(a \sin^2 \theta + b \cos^2 \theta)^{m+n}}$$

4.8 Relation between Beta and Gamma Functions

$$B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}, m > 0, n > 0$$

[Delhi Maths (Hons.) 2000, 04; Meerut 2004]

Proof: We have $\int_0^\infty e^{-zx} x^{m-1} dx = \frac{\Gamma(m)}{z^m}$... (1)

$$\Rightarrow \Gamma(m) = \int_0^\infty z^m e^{-zx} x^{m-1} dx$$

Multiplying both sides by $e^{-z} \cdot z^{n-1}$, we get

$$\Gamma(m) e^{-z} z^{n-1} = \int_0^\infty z^{m+n-1} e^{-(x+1)z} x^{m-1} dx$$

Integrating both sides w.r.t. z , from 0 to ∞ , we get

$$\Gamma(m) \int_0^\infty e^{-z} z^{n-1} dz = \int_{z=0}^\infty \int_{x=0}^\infty z^{m+n-1} e^{-(x+1)z} x^{m-1} dx dz$$

$$\Rightarrow \Gamma(m) \Gamma(n) = \int_{x=0}^\infty x^{m-1} \left(\int_{z=0}^\infty z^{m+n-1} e^{-(x+1)z} dz \right) dx$$

Changing the order of integration

$$\begin{aligned} &= \int_{x=0}^\infty x^{m-1} \frac{\Gamma(m+n)}{(x+1)^{m+n}} dx \\ &= \Gamma(m+n) \int_0^\infty \frac{x^{m-1}}{(x+1)^{m+n}} dx = \Gamma(m+n) \cdot B(m, n) \end{aligned}$$

$$\Rightarrow B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

Deduction : (i) $\Gamma(n) \Gamma(1-n) = \frac{\pi}{\sin n\pi}$, $0 < n < 1$ [Delhi Maths (Hons.) 2002, 08; Kanpur 2006]

and (ii) $\Gamma(1/2) = \sqrt{\pi}$

Proof: We have, $B(m, n) = \int_0^\infty \frac{x^{n-1} dx}{(x+1)^{m+n}}$, $m > 0, n > 0$

$$\Rightarrow \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1} dx}{(x+1)^{m+n}}$$

Taking $m+n=1$, we get

$$\Rightarrow \frac{\Gamma(1-n)\Gamma(n)}{\Gamma(1)} = \int_0^\infty \frac{x^{n-1}}{x+1} dx \quad \dots(1)$$

But we know that $\int_0^\infty \frac{x^{n-1}}{x+1} dx = \frac{\pi}{\sin n\pi}$ and $\Gamma(1) = 1$

$\therefore$ From (1), we get

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}, \quad n > 0 \quad \dots(2)$$

(ii) Putting $n = 1/2$ in (2), we get

$$\begin{aligned} \Gamma(1/2) \cdot \Gamma(1/2) &= \frac{\pi}{\sin\left(\frac{\pi}{2}\right)} = \pi \\ \Rightarrow [\Gamma(1/2)]^2 &= \pi \quad \Rightarrow \Gamma(1/2) = \sqrt{\pi} \end{aligned}$$

$$4.9 \quad \int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

Proof: We have, $B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx, \quad m, n > 0$

$$\Rightarrow B(m, n) = \int_0^{\pi/2} (\sin^2 \theta)^{m-1} (1 - \sin^2 \theta)^{n-1} 2 \sin \theta \cos \theta d\theta$$

Putting $x = \sin^2 \theta, dx = 2 \sin \theta \cos \theta d\theta$

$$\Rightarrow \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta \quad \dots(1)$$

Taking $2m-1 = p$ and $2n-1 = q$, so that $m = \frac{p+1}{2}, n = \frac{q+1}{2}$

and $m > 0 \Rightarrow p > -1, \quad n > 0 \Rightarrow q > -1$ in (1)

we get

$$\int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}, \quad p, q > -1$$

Particular Cases (i)
$$\int_0^{\pi/2} \sin^p \theta \, d\theta = \int_0^{\pi/2} \sin^p \theta \cos^0 \theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{0+1}{2}\right)}{2 \Gamma\left(\frac{p+0+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right) \cdot \sqrt{\pi}}{2 \Gamma\left(\frac{p+2}{2}\right)}$$

(ii) Similarly,
$$\int_0^{\pi/2} \cos^q \theta \, d\theta = \int_0^{\pi/2} \sin^0 \theta \cos^q \theta \, d\theta = \frac{\sqrt{\pi} \Gamma\left(\frac{q+1}{2}\right)}{2 \Gamma\left(\frac{q+2}{2}\right)}$$

4.10 Legendre Duplication Formula

$\Gamma(m) \cdot \Gamma(m + 1/2) = \frac{\sqrt{\pi}}{2^{2m-1}} \cdot \Gamma(2m)$ where m is an integer.

[Agra 2000, 01, 02, 03, 06, 08, Delhi Maths (Hons.) 2005, 07]

Proof: We have,
$$B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}, \quad m, n > 0$$

Putting $n = m$, we get
$$B(m, m) = \frac{[\Gamma(m)]^2}{\Gamma(2m)} \quad \dots(1)$$

Also, we have
$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$$

$\therefore$
$$B(m, m) = \int_0^1 x^{m-1} (1-x)^{m-1} \, dx$$

or
$$B(m, m) = \int_0^{\pi/2} (\sin^2 \theta)^{m-1} (1 - \sin^2 \theta)^{m-1} \cdot 2 \sin \theta \cos \theta \, d\theta$$

Putting $x = \sin^2 \theta, dx = 2 \sin \theta \cos \theta \, d\theta$

$$= 2 \int_0^{\pi/2} (\sin \theta \cos \theta)^{2m-1} \, d\theta = 2 \int_0^{\pi/2} \left(\frac{1}{2} \sin 2\theta\right)^{2m-1} \, d\theta$$

$$= \frac{1}{2^{2m-2}} \cdot \frac{1}{2} \int_0^{\pi} \sin^{2m-1} t \, dt \quad \text{Putting } 2\theta = t, 2d\theta = dt$$

$$= \frac{1}{2^{2m-2}} \cdot \frac{1}{2} \cdot 2 \int_0^{\pi/2} \sin^{2m-1} t \, dt \quad \text{By prop. of D.I.}$$

$$\therefore \sin^{2m-1}(\pi - t) = \sin^{2m-1} t$$

$$= \frac{1}{2^{2m-2}} \cdot \frac{\Gamma(m) \Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{2m-1+0+2}{2}\right)} = \frac{\Gamma(m) \cdot \sqrt{\pi}}{2^{2m-1} \cdot \Gamma(m+1/2)} \quad \dots(2)$$

From (1) and (2)

$$\frac{[\Gamma(m)]^2}{\Gamma(2m)} = \frac{\Gamma(m) \cdot \sqrt{\pi}}{2^{2m-1} \Gamma(m+1/2)}$$

$$\text{or } \Gamma(m) \Gamma(m+1/2) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma(2m)$$

4.1.1 To Prove that

$$\Gamma\left(\frac{1}{n}\right) \cdot \Gamma\left(\frac{2}{n}\right) \cdot \Gamma\left(\frac{3}{n}\right) \dots \Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}$$

where n is a positive integer

$$\textbf{Proof:} \text{ Let } A = \Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{2}{n}\right) \Gamma\left(\frac{3}{n}\right) \dots \Gamma\left(\frac{n-2}{n}\right) \Gamma\left(\frac{n-1}{n}\right) \quad \dots(1)$$

Writing the expression in the reversed order, we have

$$A = \Gamma\left(\frac{n-1}{n}\right) \Gamma\left(\frac{n-2}{n}\right) \dots \Gamma\left(\frac{2}{n}\right) \Gamma\left(\frac{1}{n}\right)$$

$$\text{or } A = \Gamma\left(1 - \frac{1}{n}\right) \Gamma\left(1 - \frac{2}{n}\right) \dots \Gamma\left(1 - \frac{n-2}{n}\right) \Gamma\left(1 - \frac{n-1}{n}\right) \quad \dots(2)$$

Multiplying (1) and (2), we get

$$\begin{aligned} A^2 &= \left\{ \Gamma\left(\frac{1}{n}\right) \cdot \Gamma\left(1 - \frac{1}{n}\right) \right\} \left\{ \Gamma\left(\frac{2}{n}\right) \cdot \Gamma\left(1 - \frac{2}{n}\right) \right\} \dots \left\{ \Gamma\left(\frac{n-1}{n}\right) \cdot \Gamma\left(1 - \frac{n-1}{n}\right) \right\} \\ &= \frac{\pi}{\sin \frac{\pi}{n}} \cdot \frac{\pi}{\sin \frac{2\pi}{n}} \dots \frac{\pi}{\sin \left(\frac{n-1}{n}\right) \pi} \end{aligned}$$

$$\text{or } A^2 = \frac{\pi^{n-1}}{\sin \frac{\pi}{n} \cdot \sin \frac{2\pi}{n} \dots \sin \left(\frac{n-1}{n}\right) \pi} \quad \dots(3)$$

To calculate the value of $\sin \frac{\pi}{n} \cdot \sin \frac{2\pi}{n} \dots \sin \left(\frac{n-1}{n}\right) \pi$, we factorize $1 - x^{2n}$.

Since the roots of the equation $1 - x^{2n} = 0$, are given by

$$x = (1)^{1/2n} = (\cos 2r\pi + i \sin 2r\pi)^{1/2n} = \cos \frac{r\pi}{n} + i \sin \frac{r\pi}{n}$$

where $r = 0, 1, 2, \dots, (m-1)$

If the roots are given by $\alpha_r = \cos \frac{r\pi}{n} + i \sin \frac{r\pi}{n}$

$$r = 0, 1, 2, \dots, (m-1)$$

then $\alpha_0 = \cos 0 + i \sin 0 = 1, \alpha_1 = \cos \frac{\pi}{n} + i \sin \frac{\pi}{n}, \dots$

$$\alpha_{n-1} = \cos \left(\frac{(n-1)\pi}{n} \right) + i \sin \left(\frac{(n-1)\pi}{n} \right), \alpha_n = \cos \pi + i \sin \pi = -1$$

$$\begin{aligned} \alpha_{n+1} &= \cos \left(\frac{n+1}{n} \pi \right) + i \sin \left(\frac{n+1}{n} \pi \right) = \cos \left(2\pi - \frac{n-1}{n} \pi \right) + i \sin \left(2\pi - \frac{n-1}{n} \pi \right) \\ &= \cos \left(\frac{n-1}{n} \pi \right) - i \sin \left(\frac{n-1}{n} \pi \right) \dots \end{aligned}$$

$$\begin{aligned} \alpha_{2n-1} &= \cos \left(\frac{2n-1}{n} \pi \right) + i \sin \left(\frac{2n-1}{n} \pi \right) = \cos \left(2\pi - \frac{\pi}{n} \right) + i \sin \left(2\pi - \frac{\pi}{n} \right) \\ &= \cos \frac{\pi}{n} - i \sin \frac{\pi}{n} \end{aligned}$$

$$\therefore 1 - x^{2n} = -(x - \alpha_0)(x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_{n-1})(x - \alpha_n)(x - \alpha_{n+1}) \dots (x - \alpha_{2n-1})$$

$$\begin{aligned} &= -\{(x - \alpha_0) \cdot (x - \alpha_n)\} \{(x - \alpha_1)(x - \alpha_{2n-1})\} \{(x - \alpha_2)(x - \alpha_{2n-2})\} \\ &\quad \dots \{(x - \alpha_{n-1})(x - \alpha_{n+1})\} \end{aligned}$$

$$= -\{(x-1)(x+1)\} \left\{ \left(x - \cos \frac{\pi}{n} - i \sin \frac{\pi}{n} \right) \left(x - \cos \frac{\pi}{n} + i \sin \frac{\pi}{n} \right) \right\}$$

$$\dots \left\{ x - \cos \left(\frac{n-1}{n} \pi \right) - i \sin \left(\frac{n-1}{n} \pi \right) \right\} \left\{ x - \cos \left(\frac{n-1}{n} \pi \right) + i \sin \left(\frac{n-1}{n} \pi \right) \right\}$$

$$= (1 - x^2) \left\{ \left(x - \cos \frac{\pi}{n} \right)^2 - i^2 \sin^2 \frac{\pi}{n} \right\} \dots \left\{ \left(x - \cos \left(\frac{n-1}{n} \pi \right) \right)^2 - i^2 \sin^2 \left(\frac{n-1}{n} \pi \right) \right\}$$

$$= (1 - x^2) \left(x^2 - 2x \cos \frac{\pi}{n} + 1 \right) \left(x^2 - 2x \cos \frac{2\pi}{n} + 1 \right) \dots \left(x^2 - 2x \cos \left(\frac{n-1}{n} \pi \right) + 1 \right)$$

$$\therefore \frac{1-x^{2n}}{1-x^2} = \left(x^2 - 2x \cos \frac{\pi}{n} + 1\right) \left(x^2 - 2x \cos \frac{2\pi}{n} + 1\right) \dots \left(x^2 - 2x \cos \left(\frac{n-1}{n} \pi\right) + 1\right)$$

Taking limit of both sides as $x \rightarrow 1$ and $x \rightarrow -1$ successively, we get

$$n = \left(2 - 2 \cos \frac{\pi}{n}\right) \left(2 - 2 \cos \frac{2\pi}{n}\right) \dots \left(2 - 2 \cos \left(\frac{n-1}{n} \pi\right)\right) \quad \dots(4)$$

$$\text{and} \quad n = \left(2 + 2 \cos \frac{\pi}{n}\right) \left(2 + 2 \cos \frac{2\pi}{n}\right) \dots \left(2 + 2 \cos \left(\frac{n-1}{n} \pi\right)\right) \quad \dots(5)$$

Multiplying (4) and (5), we get

$$\begin{aligned} n^2 &= 2^{2(n-1)} \left(1 - \cos^2 \frac{\pi}{n}\right) \left(1 - \cos^2 \frac{2\pi}{n}\right) \dots \left(1 - \cos^2 \left(\frac{n-1}{n} \pi\right)\right) \\ &= 2^{2(n-1)} \sin^2 \frac{\pi}{n} \cdot \sin^2 \frac{2\pi}{n} \dots \sin^2 \left(\frac{n-1}{n} \pi\right) \end{aligned}$$

$$\Rightarrow \sin \frac{\pi}{n} \cdot \sin \frac{2\pi}{n} \dots \sin \left(\frac{n-1}{n} \pi\right) = \frac{n}{2^{n-1}}$$

Putting in (3), we get

$$A^2 = \frac{\pi^{n-1}}{n/2^{n-1}} = \frac{(2\pi)^{n-1}}{n} \Rightarrow A = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}$$

Note : The value of $\sin \frac{\pi}{n} \cdot \sin \frac{2\pi}{n} \dots \sin \left(\frac{n-1}{n} \pi\right)$ can also be found by using the identity.

$$\frac{\sin n\theta}{\sin \theta} = 2^{n-1} \sin \left(\theta + \frac{\pi}{n}\right) \sin \left(\theta + \frac{2\pi}{n}\right) \dots \sin \left(\theta + \frac{n-1}{n} \pi\right)$$

Taking limit as $\theta \rightarrow 0$, we get

$$2^{n-1} \sin \frac{\pi}{n} \cdot \sin \frac{2\pi}{n} \dots \sin \left(\frac{n-1}{n} \pi\right) = \lim_{\theta \rightarrow 0} \frac{\sin n\theta}{\sin \theta}$$

$$= \lim_{\theta \rightarrow 0} \left(n \cdot \frac{\sin n\theta}{n\theta} \cdot \frac{\theta}{\sin \theta} \right) = n \cdot 1 \cdot 1 = n$$

$$\Rightarrow \sin \frac{\pi}{n} \cdot \sin \frac{2\pi}{n} \dots \sin \left(\frac{n-1}{n} \pi\right) = \frac{n}{2^{n-1}}$$

Solved Examples

Example 1: Evaluate

$$(i) \int_0^a x^4 \sqrt{(a^2 - x^2)} dx$$

$$(ii) \int_0^2 x (8 - x^3)^{1/3} dx$$

$$(iii) \int_0^a \frac{dx}{(a^n - x^n)^{1/n}}$$

$$(iv) \int_0^1 \frac{x^{m-1} (1-x)^{n-1}}{(a+x)^{m+n}} dx$$

Solution: (i) $\int_0^a x^4 \sqrt{(a^2 - x^2)} dx = \int_0^1 a^4 t^2 \cdot \sqrt{(a^2 - a^2 t^2)} \cdot \frac{a}{2} t^{(1/2)-1} dt$

Putting $x^2 = a^2 t$, $x = a\sqrt{t}$, $dx = \frac{a}{2} t^{(1/2)-1} dt$

$$= \frac{a^6}{2} \int_0^1 t^{(5/2)-1} \cdot (1-t)^{(3/2)-1} dt = \frac{a^6}{2} B\left(\frac{5}{2}, \frac{3}{2}\right)$$

$$= \frac{a^6}{2} \cdot \frac{\Gamma(5/2) \Gamma(3/2)}{\Gamma(5/2 + 3/2)} = \frac{a^6}{2} \cdot \frac{\left(\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}\right) \left(\frac{1}{2} \cdot \sqrt{\pi}\right)}{3 \cdot 2 \cdot 1} = \frac{a^6 \pi}{32}$$

Aliter : Putting $x = a \sin \theta$, $dx = a \cos \theta d\theta$

$$\int_0^a x^4 \sqrt{(a^2 - x^2)} dx = a^6 \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta d\theta = \frac{a^6 \Gamma(5/2) \Gamma(3/2)}{2\Gamma\left(\frac{4+2+2}{2}\right)} = \frac{a^6 \pi}{32}$$

$$(ii) \int_0^2 x (8 - x^3)^{1/3} dx = \int_0^1 2t^{1/3} (8 - 8t)^{1/3} (2/3) t^{(1/3)-1} dt$$

Putting $x^3 = 8t$, $x = 2t^{1/3}$, $dx = (2/3) t^{(1/3)-1} dt$

$$= \frac{8}{3} \int_0^1 t^{(2/3)-1} (1-t)^{(4/3)-1} dt = \frac{8}{3} B(2/3, 4/3)$$

$$= \frac{8}{3} \frac{\Gamma(2/3) \Gamma(4/3)}{\Gamma(2/3 + 4/3)} = \frac{8}{3} \frac{\Gamma\left(\frac{2}{3}\right) \frac{1}{3} \Gamma\left(\frac{1}{3}\right)}{\Gamma(2)} = \frac{8}{9} \frac{\Gamma\left(1 - \frac{1}{3}\right) \Gamma\left(\frac{1}{3}\right)}{1}$$

$$= \frac{8}{9} \cdot \frac{\pi}{\sin\left(\frac{\pi}{3}\right)} = \frac{8\pi}{9} \cdot \frac{1}{\sqrt{3}/2} = \frac{16\pi}{9\sqrt{3}}$$

$$(iii) \int_0^a \frac{dx}{(a^n - x^n)^{1/n}} = \int_0^1 (a^n - a^n t)^{-1/n} dx$$

Putting $x^n = a^n t$, $x = at^{1/n}$, $dx = \frac{a}{n} t^{(1/n)-1} dt$

$$\begin{aligned}
 &= \int_0^1 (a^n - a^n t)^{-1/n} \frac{a}{n} t^{(1/n)-1} dt = \frac{1}{n} \int_0^1 t^{(1/n)-1} (1-t)^{[1-(1/n)]-1} dt \\
 &= \frac{1}{n} \cdot B[1/n, 1-1/n] = \frac{1}{n} \cdot \frac{\Gamma(1/n) \Gamma(1-1/n)}{\Gamma\left(\frac{1}{n} + 1 - \frac{1}{n}\right)} = \frac{1}{n} \cdot \frac{\pi}{\sin(\pi/n)}
 \end{aligned}$$

(iv) Putting $\frac{x(1+a)}{a+x} = y$, $x = \frac{ay}{1+a-y}$

$$1-x = 1 - \frac{ay}{1+a-y} = \frac{(1+a)(1-y)}{1+a-y}, \quad a+x = a + \frac{ay}{1+a-y} = \frac{a(1+a)}{1+a-y}$$

$$dx = \frac{a(1+a)}{(1+a-y)^2} dy$$

$$\begin{aligned}
 &\int_0^1 \frac{x^{m-1} \cdot (1-x)^{n-1}}{(a+x)^{m+n}} dx \\
 &= \int_0^1 \left(\frac{ay}{1+a-y} \right)^{m-1} \frac{(1+a)^{n-1} (1-y)^{n-1}}{(1+a-y)^{n-1}} \frac{(1+a-y)^{m+n}}{a^{m+n} (1+a)^{m+n}} \frac{a(1+a)dy}{(1+a-y)^2} \\
 &= \frac{1}{a^n (1+a)^m} \int_0^1 y^{m-1} (1-y)^{n-1} dy = \frac{B(m, n)}{a^n (1+a)^m} = \frac{\Gamma(m) \Gamma(n)}{a^n (1+a)^m \Gamma(m+n)}
 \end{aligned}$$

Example 2: Evaluate (i) $\int_0^2 \frac{x^2 dx}{\sqrt{(2-x)}}$ (ii) $\int_0^\infty \frac{x^4 (1+x^5)}{(1+x)^{15}} dx$

Solution: (i) $\int_0^2 \frac{x^2 dx}{\sqrt{(2-x)}} = \int_0^1 \frac{4t^2 \cdot 2dt}{\sqrt{(2-2t)}}$ Putting $x = 2t$, $dx = 2dt$

$$= 4\sqrt{2} \int_0^1 t^2 (1-t)^{-1/2} dt = 4\sqrt{2} \int_0^1 t^{3-1} (1-t)^{(1/2)-1} dt$$

$$= 4\sqrt{2} B(3, 1/2) = 4\sqrt{2} \cdot \frac{\Gamma(3) \Gamma(1/2)}{\Gamma(3+1/2)} = 4\sqrt{2} \cdot \frac{2 \cdot 1 \cdot \sqrt{\pi}}{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}} = \frac{64\sqrt{2}}{15}$$

(ii) $\int_0^\infty \frac{x^4 (1+x^5)}{(1+x)^{15}} dx = \int_0^\infty \frac{x^4 dx}{(1+x)^{15}} + \int_0^\infty \frac{x^9 dx}{(1+x)^{15}}$

$$= \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}} dx + \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+5}} dx = B(5, 10) + B(10, 5)$$

$$= 2 \frac{\Gamma(5) \Gamma(10)}{\Gamma(5+10)} = \frac{2 \cdot 4! \cdot 9!}{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9!} = \frac{1}{5005}$$

Example 3: Show that (i) $\int_0^1 \frac{x^{m-1} (1-x)^{n-1} dx}{\{b + (a-b)x\}^{m+n}} = \frac{1}{a^m b^n} B(m, n)$

(ii) $\int_0^1 \frac{x^{m-1} (1-x)^{n-1} dx}{(b+cx)^{m+n}} = \frac{1}{(b+c)^m b^n} B(m, n)$

Solution: (i) We have, $B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad \dots(1)$

Putting $\frac{b}{y} - \frac{a}{x} = a-b$, so that $x = \frac{ay}{b + (a-b)y}$

$$dx = \frac{ab dy}{\{b + (a-b)y\}^2}, \quad 1-x = 1 - \frac{ay}{b + (a-b)y} = \frac{b(1-y)}{b + (a-b)y}$$

when $x=0$, $y=0$ and when $x=1$, $y=1$

$$\begin{aligned} \therefore \text{ from (1), } B(m, n) &= \int_0^1 \left[\frac{ay}{b + (a-b)y} \right]^{m-1} \left[\frac{b(1-y)}{b + (a-b)y} \right]^{n-1} \cdot \frac{ab dy}{[b + (a-b)y]^2} \\ &= a^m b^n \int_0^1 \frac{y^{m-1} (1-y)^{n-1} dy}{[b + (a-b)y]^{m+n}} = a^m b^n \int_0^1 \frac{x^{m-1} (1-x)^{n-1} dx}{\{b + (a-b)x\}^{m+n}} \end{aligned}$$

By prop. of D.I., changing independent variable from y to x .

$$\therefore \int_0^1 \frac{x^{m-1} (1-x)^{n-1} dx}{\{b + (a-b)x\}^{m+n}} = \frac{B(m, n)}{a^m b^n}$$

(ii) Putting $a-b=c$, $a=b+c$ in part (i) we get

$$\int_0^1 \frac{x^{m-1} (1-x)^{n-1} dx}{(b+cx)^{m+n}} = \frac{B(m, n)}{(b+c)^m b^n}$$

Example 4: Show that $\int_{-1}^1 (1+x)^{p-1} (1-x)^{q-1} dx = 2^{p+q-1} \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

Solution: Putting $1+x=(1-x)t$ i.e., $t = \frac{1+x}{1-x}$, $x = \frac{t-1}{t+1}$, $1-x = \frac{2}{t+1}$

$$dx = \frac{2}{(t+1)^2} dt, \text{ when } x=-1, t=0; \text{ when } x=1, t=\infty$$

$$\int_{-1}^1 (1+x)^{p-1} (1-x)^{q-1} dx = \int_{t=0}^{\infty} \{(1-x)t\}^{p-1} (1-x)^{q-1} \cdot \frac{2}{(1+t)^2} dt$$

$$\begin{aligned}
 &= 2 \int_0^{\infty} \frac{t^{p-1}}{(1+t)^2} (1-x)^{p+q-2} dt = 2 \int_0^{\infty} \frac{t^{p-1}}{(1+t)^2} \cdot \left(\frac{2}{t+1}\right)^{p+q-2} dt \\
 &= 2^{p+q-1} \int_0^{\infty} \frac{t^{p-1}}{(1+t)^{p+q}} dt = 2^{p+q-1} B(p, q) = 2^{p+q-1} \cdot \frac{\Gamma(p) \Gamma(q)}{\Gamma(p+q)}
 \end{aligned}$$

Example 5: Show that $B(m+1, n) + B(m, n+1) = B(m, n)$, $m > 0$, $n > 0$

[Agra 2006, 07]

Solution: $B(m+1, n) + B(m, n+1)$

$$\begin{aligned}
 &= \frac{\Gamma(m+1) \Gamma(n)}{\Gamma(m+1+n)} + \frac{\Gamma(m) \Gamma(n+1)}{\Gamma(m+n+1)} = \frac{(m) \Gamma(m) \Gamma(n) + \Gamma(m) (n) \Gamma(n)}{\Gamma(m+n+1)} \\
 &= \frac{(m+n) \Gamma(m) \Gamma(n)}{(m+n) \Gamma(m+n)} = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} = B(m, n)
 \end{aligned}$$

Example 6: Show that $\int_0^{\infty} \frac{x dx}{1+x^6} = \frac{1}{6} B(1/3, 2/3) = \frac{\pi}{3\sqrt{3}}$ [Delhi Maths (Hons.) 2007, 08]

Solution: $\int_0^{\infty} \frac{x dx}{1+x^6} = \frac{1}{6} \int_0^{\infty} \frac{t^{1/6} \cdot t^{-5/6} dt}{1+t}$ Putting $x^6 = t$, $x = t^{1/6}$, $dx = \frac{1}{6} t^{-5/6} dt$

$$\begin{aligned}
 &= \frac{1}{6} \int_0^{\infty} \frac{t^{(1/3)-1}}{(1+t)^{(1/3)+(2/3)}} dt = \frac{1}{6} B(1/3, 2/3) = \frac{1}{6} \frac{\Gamma(1/3) \Gamma(2/3)}{\Gamma(1/3+2/3)} \\
 &= \frac{1}{6} \frac{\Gamma(1/3) \Gamma(1-1/3)}{\Gamma(1)} = \frac{1}{6} \frac{\pi}{\sin(\pi/3)} = \frac{\pi}{6(\sqrt{3}/2)} = \frac{\pi}{3\sqrt{3}}
 \end{aligned}$$

Example 7: Prove that $\int_0^1 \frac{x^2 dx}{(1-x^4)^{1/2}} \times \int_0^1 \frac{dx}{(1+x^4)^{1/2}} = \frac{\pi}{4\sqrt{2}}$

Solution: $\int_0^1 \frac{x^2 dx}{(1-x^4)^{1/2}} = \int_0^{\pi/2} \frac{\sin \theta}{(\cos^2 \theta)^{1/2}} \cdot \frac{\cos \theta d\theta}{2\sqrt{\sin \theta}}$

Putting $x^2 = \sin \theta$, $dx = \frac{\cos \theta d\theta}{2x} = \frac{\cos \theta d\theta}{2\sqrt{\sin \theta}}$

$$\begin{aligned}
 &= \frac{1}{2} \int_0^{\pi/2} \sqrt{\sin \theta} d\theta = \frac{1}{2} \int_0^{\pi/2} \sin^{1/2} \theta \cdot \cos^0 \theta d\theta \\
 &= \frac{1}{2} \frac{\Gamma(3/4) \Gamma(1/2)}{2\Gamma(5/4)} = \frac{1}{4} \cdot \frac{\Gamma(3/4) \cdot \sqrt{\pi}}{(1/4) \Gamma(1/4)} = \frac{\sqrt{\pi} \Gamma(3/4)}{\Gamma(1/4)}
 \end{aligned}$$

Also $\int_0^1 \frac{dx}{(1+x^4)^{1/2}} = \int_0^{\pi/4} \frac{1}{\sec \theta} \frac{\sec^2 \theta d\theta}{2\sqrt{\tan \theta}}$

$$\text{Putting } x^2 = \tan \theta, x = \sqrt{\tan \theta}, dx = \frac{\sec^2 \theta d\theta}{2\sqrt{\tan \theta}}$$

$$= \frac{1}{2} \int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin \theta \cos \theta}} = \frac{\sqrt{2}}{2} \int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin 2\theta}} = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} (\sin t)^{-1/2} dt$$

$$\text{Putting } 2\theta = t, d\theta = (1/2) dt$$

$$= \frac{1}{2\sqrt{2}} \int_0^{\pi/2} (\sin t)^{-1/2} \cdot \cos^0 t dt = \frac{1}{2\sqrt{2}} \frac{\Gamma(1/4) \Gamma(1/2)}{2\Gamma(3/4)} = \frac{\sqrt{\pi}}{4\sqrt{2}} \frac{\Gamma(1/4)}{\Gamma(3/4)}$$

$$\therefore \int_0^1 \frac{x^2 dx}{(1-x^4)^{1/2}} \times \int_0^1 \frac{dx}{(1+x^4)^{1/2}} = \sqrt{\pi} \frac{\Gamma(3/4)}{\Gamma(1/4)} \times \frac{\sqrt{\pi}}{4\sqrt{2}} \frac{\Gamma(1/4)}{\Gamma(3/4)} = \frac{\pi}{4\sqrt{2}}$$

Example 8: By means of Beta function, show that

$$\int_t^z \frac{dx}{(z-x)^{1-\alpha} (x-t)^\alpha} = \frac{\pi}{\sin \pi\alpha}, \quad 0 < \alpha < 1$$

Solution: Putting $x-t = (z-t)y$ i.e., $x = t + (z-t)y$, $dx = (z-t)dy$,

when $x = t$, $y = 0$ when $x = z$, $y = 1$ (Note that z and t are constants)

$$\begin{aligned} \int_t^z \frac{dx}{(z-x)^{1-\alpha} (x-t)^\alpha} &= \int_0^1 \frac{(z-t)dy}{\{z-t-(z-t)y\}^{1-\alpha} \{(z-t)y\}^\alpha} \\ &= \int_0^1 \frac{(z-t)dy}{(z-t)^{1-\alpha} (1-y)^{1-\alpha} \cdot (z-t)^\alpha \cdot y^\alpha} = \int_0^1 y^{-\alpha} (1-y)^{\alpha-1} dy \\ &= \int_0^1 y^{(1-\alpha)-1} \cdot (1-y)^{\alpha-1} dy = B[(1-\alpha), \alpha] = \frac{\Gamma(1-\alpha) \Gamma(\alpha)}{\Gamma(1-\alpha+\alpha)} \\ &= \frac{\pi}{\sin \pi\alpha} \end{aligned}$$

Example 9: When n is positive integer, show that

$$\Gamma(n+1/2) = \frac{(2n)!}{2^{2n} \cdot n!} \sqrt{\pi} \quad [\text{Delhi Maths (Hons.) 2003}]$$

Solution : From Legendre Duplication formula, we have

$$\begin{aligned} \Gamma(n) \Gamma(n+1/2) &= \frac{\sqrt{\pi}}{2^{2n-1}} \Gamma(2n) \\ \Rightarrow \Gamma(n+1/2) &= \frac{\sqrt{\pi}}{2^{2n-1}} \cdot \frac{\Gamma(2n)}{\Gamma(n)} = \frac{\sqrt{\pi}}{2^{2n-1}} \frac{(2n-1)!}{(n-1)!} \quad \because n \text{ is positive integer} \\ &= \frac{\sqrt{\pi}}{2^{2n-1}} \cdot \frac{2n \cdot (2n-1)!}{2n \cdot (n-1)!} = \frac{\sqrt{\pi}}{2^{2n}} \cdot \frac{(2n)!}{(n)!} \end{aligned}$$

Example 10: Prove that

$$\Gamma(n) \Gamma\left(\frac{1-n}{2}\right) = \frac{\sqrt{\pi} \cdot \Gamma\left(\frac{1}{2}n\right)}{2^{1-n} \cos\left(\frac{1}{2}n\pi\right)}, \quad 0 < n < 1$$

Solution: We know that

$$\Gamma(m) \Gamma(1-m) = \frac{\pi}{\sin(m\pi)} \quad \dots(1)$$

$$\text{and} \quad \Gamma(m) \Gamma(m+1/2) = \frac{\sqrt{\pi} \Gamma(2m)}{2^{2m-1}} \quad \dots(2)$$

Now putting $m = (n+1)/2$ in (1), we get

$$\Gamma\left(\frac{n+1}{2}\right) \Gamma\left(1 - \frac{n+1}{2}\right) = \frac{\pi}{\sin\left((n+1)\frac{\pi}{2}\right)}$$

$$\text{or} \quad \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{1-n}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi}{2} + \frac{n\pi}{2}\right)} = \frac{\pi}{\cos\left(\frac{n\pi}{2}\right)} \quad \dots(3)$$

Also putting $m = n/2$ in (2), we get

$$\Gamma(n/2) \Gamma\left(\frac{n}{2} + \frac{1}{2}\right) = \frac{\sqrt{\pi} \cdot \Gamma(n)}{2^{n-1}}$$

$$\text{or} \quad \frac{\Gamma[(n+1)/2]}{\Gamma(n)} = \frac{\sqrt{\pi}}{2^{n-1} \Gamma(n/2)} \quad \dots(4)$$

Dividing (3) by (4), we get

$$\Gamma(n) \Gamma\left(\frac{1-n}{2}\right) = \frac{\pi}{\cos\left(\frac{n\pi}{2}\right)} \frac{2^{n-1} \Gamma(n/2)}{\sqrt{\pi}} = \frac{\sqrt{\pi} \cdot \Gamma(n/2)}{2^{1-n} \cos\left(\frac{1}{2}n\pi\right)}$$

Example 11: Show that $\Gamma(1) \Gamma(2) \dots \Gamma(9) = \frac{(2\pi)^{9/2}}{\sqrt{10}}$

Solution: From 4.11, we have

$$\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{2}{n}\right) \dots \Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{(n-1)/2}}{\sqrt{n}}$$

$$\therefore \Gamma(1) \Gamma(2) \dots \Gamma(9)$$

$$= \Gamma\left(\frac{1}{10}\right) \Gamma\left(\frac{2}{10}\right) \dots \Gamma\left(\frac{10-1}{10}\right) = \frac{(2 \cdot \pi)^{(10-1)/2}}{\sqrt{10}} = \frac{(2\pi)^{9/2}}{\sqrt{10}}$$

Exercise 4.2

1. Evaluate : (i) $\int_0^1 \frac{dx}{\sqrt{(1-x^4)}}$ [Delhi Maths (Hons.) 2003; Meerut 2007]
 (ii) $\int_0^1 \frac{dx}{\sqrt{(1+x^4)}}$ (iii) $\int_0^1 (1-x^n)^{1/n} dx$
 (iv) $\int_0^1 \frac{dx}{\sqrt{(1-x^n)}}$ [Meerut 2004; Purvanchal 2006]
2. Show that $\int_0^a \frac{dx}{(a^n - x^n)^{1/n}} = \frac{\pi}{n \sin(\pi/n)}$
3. Prove that (i) $\int_0^1 x^{m-1} (1-x^2)^{n-1} dx = \frac{1}{2} B(m/2, n)$
 (ii) $\int_0^\infty \frac{x dx}{1+x^6} = \frac{1}{6} B(1/3, 2/3) = \frac{\pi}{3\sqrt{3}}$ [Delhi Maths (Hons.) 2007, 08]
4. Show that $\int_b^a (x-b)^{m-1} (a-x)^{n-1} dx = (a-b)^{m+n-1} B(m, n)$
 [Hint : Put $x = \frac{t-b}{a-b}$ in $B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$]
5. Show that $\int_0^1 \frac{x^{m-1} (1-x)^{n-1} dx}{\{a + (b-a)x\}^{m+n}} = \frac{1}{a^n b^m} B(m, n)$
 [Hint : See Ex. 3 (i)]
6. Find the value of $\Gamma\left(\frac{1}{9}\right) \Gamma\left(\frac{2}{9}\right) \dots \Gamma\left(\frac{8}{9}\right)$
7. Show that $3^{1/2} [\Gamma(1/3)]^2 = 2^{1/3} \cdot \sqrt{\pi} \cdot \Gamma(1/6)$
 [Hint : Take $n = 1/3$ in Legendre Duplication Formula given in 4.10]
8. If $p > 0, q > 0, m+1 > 0, n+1 > 0$, prove that

$$\int_0^p x^m (p^q - x^q)^n dx = \frac{p^{nq+m+1}}{q} B\left(n+1, \frac{m+1}{q}\right)$$
 [Hint : Put $x^q = p^q \cdot t$]
9. Prove that $l \cdot B(l, m+1) = m \cdot B(l+1, m)$
10. Prove that $B(m, m) \cdot B(m+1/2, m+1/2) = \frac{\pi \cdot m^{-1}}{2^{4m-1}}$
11. Prove that $B(l, m) B(l+m, n) = B(m, n) B(m+n, l)$
 $= B(n, l) \cdot B(n+l, m)$
12. Prove that $B(l, m) B(l+m, n) B(l+m+n, p) = \frac{\Gamma(l) \Gamma(m) \Gamma(n) \Gamma(p)}{\Gamma(l+m+n+p)}$

Answers 4.2

1.	(i) $\frac{[\Gamma(1/4)]^2}{6\sqrt{2\pi}}$ (ii) $\frac{\Gamma(1/2)\Gamma(1/4)}{4\sqrt{2}\Gamma(3/4)}$ (iii) $\frac{[\Gamma(1/n)]^2}{2n\Gamma(2/n)}$ (iv) $\frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma(1/n)}{\Gamma(1/n + 1/2)}$
6.	$\frac{16\pi^4}{3}$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter (a), (b), (c) and (d).

- The value of $\int_0^\infty e^{-x} x^{-(1/2)} dx$ is
 (a) $\sqrt{\pi}$ (b) $\sqrt{\pi}/2$ (c) $2\sqrt{\pi}$ (d) $3\sqrt{\pi}$
- The value of $\int_0^\infty e^{-4x} x^{5/2} dx$ is
 (a) $\frac{15\sqrt{\pi}}{512}$ (b) $\frac{15\pi}{512}$ (c) $\frac{15\sqrt{\pi}}{1024}$ (d) $\frac{15\pi}{1024}$
- $\Gamma(n)\Gamma(1-n)$ where $0 < n < 1$ is equal to
 (a) π (b) $\sin n\pi$ (c) $\pi \sin n\pi$ (d) $\pi/\sin n\pi$
- The value of $\int_0^\infty e^{-x^2} dx$ is equal to
 (a) $\sqrt{\pi}$ (b) $\sqrt{\pi}/2$ (c) $\pi/2$ (d) $\pi/4$
- $\int_0^\infty x^{m-1} e^{-2x} dx$ is equal to
 (a) $\Gamma(2)/2^m$ (b) $\Gamma(m)/m^2$ (c) $\Gamma(m)$ (d) $m^2 \Gamma(m)$
- $\int_0^1 x^4 (1-x)^5 dx$ is equal to
 (a) $\frac{1}{126}$ (b) $\frac{1}{252}$ (c) $\frac{1}{462}$ (d) $\frac{1}{231}$
- The value of $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$ is equal to

- (a) $B(m, n)$ (b) $2B(m, n)$ (c) $\frac{1}{2}B(m, n)$ (d) $\gamma(m+n)$

8. The value of $\int_0^{\pi/2} \sin^n \theta \, d\theta$ is equal to

- (a) $B\left(\frac{1}{2}, \frac{n+1}{2}\right)$ (b) $\frac{1}{2}B\left(\frac{1}{2}, \frac{n+1}{2}\right)$
 (c) $B\left(\frac{n+1}{2}, \frac{1}{2}\right)$ (d) $2B\left(\frac{1}{2}, \frac{n+1}{2}\right)$

9. $\int_0^1 x^{m-1} \left(\log \frac{1}{x}\right)^{n-1} dx$ is equal to

- (a) $\frac{\Gamma(n)}{m^n}$ (b) $\frac{\Gamma(m)}{n^m}$ (c) $\frac{\Gamma(m)}{m^n}$ (d) $\frac{\Gamma(n)}{n^m}$

10. $\int_0^1 \left(\log \frac{1}{x}\right)^{m-1} dx$ is equal to

- (a) $\Gamma(m+1)$ (b) $\Gamma(m)$ (c) $\Gamma(m/2)$ (d) $\Gamma(m+1)$

Fill in the Blank

Fill in the blank's "....." so that the following statements are complete and correct.

- The value of $\int_0^\infty e^{-2x} \cdot x^6 \, dx$ is
- The value of $\int_0^1 \frac{35x^2 \, dx}{32\sqrt{1-x}}$ is
- The value of $\Gamma(1/4) \cdot \Gamma(3/4)$ is
- $\int_0^1 x^4 (1-x)^2 \, dx$ is equal to
- $\Gamma(-3/2)$ is equal to
- $\int_0^\infty \frac{dt}{\sqrt{t}(1+t)} = \dots\dots\dots$

True / False

Write 'T' for true and 'F' for false statement.

- $\Gamma(n-1) = n\Gamma(n)$
- $\int_0^\infty \sqrt{x} e^{-x^3} \, dx = \sqrt{\pi}/3$

$$3. \int_0^{\infty} e^{-x^2} \cdot x^2 dx = \sqrt{\pi} / 3$$

$$4. \int_0^{\infty} \frac{x^8 (1-x^6)}{(1+x)^{2n}} dx = 0$$

$$5. \int_0^1 \left(\frac{1}{x} - 1 \right)^{1/4} dx = \frac{\pi}{2\sqrt{2}}$$

$$6. \int_0^{\infty} e^{-ax} x^n dx = \frac{\Gamma(n)}{a^n}, \quad n > -1, \quad a > 0$$

Answers

Objective Type Questions

Multiple Choice Questions

1.	(a)	2.	(c)	3.	(d)
4.	(b)	5.	(b)	6.	(c)
7.	(a)	8.	(b)	9.	(a)
10.	(b)				

Fill in the Blank

1.	$45/8$	2.	1	3.	$\pi\sqrt{2}$
4.	$1/105$	5.	$4\sqrt{\pi}/3$	6.	π

True / False

1.	False	2.	True	3.	False
4.	True	5.	True	6.	False

○○○

5.1 The Pochhammer Symbol (Def.)

If n is a positive integer, then the abbreviation

$$(\alpha)_n = \alpha (\alpha + 1) (\alpha + 2) \dots (\alpha + n - 1)$$

with $(\alpha)_0 = 1$

is called the **Pochhammer symbol**.

Another Def. We have

$$\begin{aligned} (\alpha)_n &= \alpha (\alpha + 1) (\alpha + 2) \dots (\alpha + n - 1) \\ &= \frac{1.2.3 \dots (\alpha - 1) . \alpha (\alpha + 1) (\alpha + 2) \dots (\alpha + n - 1)}{1.2.3 \dots (\alpha - 1)} \\ &= \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} \end{aligned}$$

Thus, **Pochhammer symbol** in the form of gamma function can also be defined as

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)}, n = 0, 1, 2, 3, \dots$$

Note : The Pochhammer Symbol $(\alpha)_n$ is sometimes replaced by Appel's symbol (α, n) . But throughout this chapter we shall use Pochhammer symbol.

5.2 Identities Satisfied by Pochhammer Symbol $(\alpha)_n$

1. $(1)_n = n!$
2. $(\alpha + n)(\alpha)_n = \alpha(\alpha + 1)_n = (\alpha)_{n+1}$
3. $\binom{-\alpha}{n} = \frac{(-1)^n}{n!} (\alpha)_n$
4. $(\alpha)_{n+1} = \alpha(\alpha + 1)_n$
5. $(\alpha)_{n+k} = (\alpha)_k (\alpha + k)_n = (\alpha)_n (\alpha + n)_k$ (Addition formula)
6. $(\alpha)_{k-n} = (-1)^n (\alpha)_k / (1 - \alpha - k)_n$
7. $(\alpha)_{-n} = (-1)^n / (1 - \alpha)_n, n = 1, 2, 3, \dots$
8. $(-k)_n = \begin{cases} \frac{(-1)^n k!}{(k-n)!}, & 0 \leq n \leq k \\ 0, & n > k \end{cases}$

Proof: 1. $(1)_n = 1(1+1)(1+2) \dots (1+n-1) = 1 \cdot 2 \cdot 3 \dots n = n!$

$$2. \quad (\alpha + n)(\alpha)_n = (\alpha)_n \cdot (\alpha + n) = \alpha(\alpha + 1)(\alpha + 2) \dots (\alpha + n - 1) \cdot (\alpha + n) \\ = \alpha \cdot \{(\alpha + 1)(\alpha + 2) \dots (\alpha + 1 + n - 1)\} = \alpha(\alpha + 1)_n$$

$$\text{Also } (\alpha + n)(\alpha)_n = \alpha(\alpha + 1)(\alpha + 2) \dots (\alpha + n - 1) \cdot (\alpha + n + 1 - 1) = (\alpha)_{n+1}$$

$$\text{Hence } (\alpha + n)(\alpha)_n = \alpha(\alpha + 1)_n = (\alpha)_{n+1}$$

$$3. \quad \binom{-\alpha}{n} = \frac{(-\alpha)(-\alpha-1)(-\alpha-2) \dots (-\alpha-n+1)}{n!} \\ = \frac{(-1)^n \alpha(\alpha+1)(\alpha+2) \dots (\alpha+n-1)}{n!} = \frac{(-1)^n}{n!} (\alpha)_n$$

$$4. \quad (\alpha)_{n+1} = \alpha(\alpha+1)(\alpha+2) \dots (\alpha+n+1-1) \\ = \alpha \cdot (\alpha+1)(\alpha+1+1) \dots (\alpha+1+n-1) = \alpha \cdot (\alpha+1)_n$$

$$5. \quad (\alpha)_{n+k} = \alpha(\alpha+1)(\alpha+2) \dots (\alpha+k-1)(\alpha+k) \dots (\alpha+n+k-1) \\ = \{\alpha(\alpha+1)(\alpha+2) \dots (\alpha+k-1)\} \{(\alpha+k)(\alpha+k+1) \dots (\alpha+k+n-1)\} \\ = (\alpha)_k \cdot (\alpha+k)_n \quad \text{if } k < n.$$

Also,

$$(\alpha)_{n+k} = \{\alpha(\alpha+1)(\alpha+2) \dots (\alpha+n-1)\} \{(\alpha+n)(\alpha+n+1) \dots (\alpha+n+k-1)\} \\ = (\alpha)_n \cdot (\alpha+n)_k \quad \text{if } k > n$$

$$\text{Hence } (\alpha)_{n+k} = (\alpha)_k (\alpha+k)_n = (\alpha)_n (\alpha+n)_k$$

$$\begin{aligned}
 6. \quad (\alpha)_{k-n} &= \alpha (\alpha+1) (\alpha+2) \dots (\alpha+k-n-1) \\
 &= \frac{\alpha (\alpha+1) (\alpha+2) \dots (\alpha+k-n-1) (\alpha+k-n) (\alpha+k-n+1) \dots (\alpha+k-1)}{(\alpha+k-n) (\alpha+k-n+1) \dots (\alpha+k-1)} \\
 &= \frac{(\alpha)_k}{(-1)^n (1-\alpha-k) (1-\alpha-k+1) \dots (1-\alpha-k+n-1)} \\
 &= \frac{(-1)^n (\alpha)_k}{(1-\alpha-k)_n}
 \end{aligned}$$

7. In (6) we have proved that

$$(\alpha)_{k-n} = \frac{(-1)^n (\alpha)_k}{(1-\alpha-k)_n}$$

$$\text{Putting } k=0, \text{ we get } (\alpha)_{-n} = \frac{(-1)^n}{(1-\alpha)_n}, \quad \because (\alpha)_0 = 1$$

$$\begin{aligned}
 8. \quad (-k)_n &= (-k) (-k+1) (-k+2) \dots (-k+n-1) \\
 &= \frac{(-1)^n \cdot k (k-1) (k-2) \dots (k-(n-1)) \cdot (k-n) \dots 2 \cdot 1}{(k-n) (k-n-1) \dots 2 \cdot 1} \quad \text{if } 0 \leq n \leq k \\
 &= \frac{(-1)^n k!}{(k-n)!} \quad \text{if } 0 \leq n \leq k \\
 &= 0 \quad \text{if } n > k, \quad \because (-p)! = \infty, p > 0
 \end{aligned}$$

5.3 Hypergeometric Series

The series defined by

$$1 + \frac{\alpha\beta}{\gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)} \frac{x^2}{2!} + \dots = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \frac{x^r}{r!} \quad \dots(1)$$

is called the **hypergeometric series**.

Note : 1. Throughout this chapter α, β, γ are assumed to be real.

2. Above series is given the name hypergeometric series from the fact that, for $\alpha=1, \gamma=\beta$, this series reduces to the geometric series

$$1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n$$

The above series (1) converges absolutely if $|x| < 1$ and it is divergent for all $|x| > 1$. if $|x| = 1$ then the series converges absolutely if $\gamma - \alpha - \beta > 0$ i.e., $\gamma > \alpha + \beta$

5.4 Hypergeometric Function

The function defined by the hypergeometric series $\sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \frac{x^r}{r!}$, $|x| < 1$ is called the **hypergeometric function**. It is commonly denoted by the symbol ${}_2F_1(\alpha, \beta; \gamma; x)$ or simply $F(\alpha, \beta, \gamma, x)$

$$\text{Thus } {}_2F_1(\alpha, \beta; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \frac{x^r}{r!} \quad \dots(1)$$

where 2 and 1 refer to the number of numerator and denominator parameters respectively, in its series representation. The semicolons separate the numerator parameter α and β (which are themselves separated by a comma), the denominator parameter γ and the argument x .

Note : 1. If $\gamma = 0$ or a negative integer, then the series (1) does not exist and so the function ${}_2F_1(\alpha, \beta; \gamma; x)$ is not defined.

2. If either α or β (or both) is zero or a negative integer, then the series is finite and thus converges for all x .

$\therefore$ If $\alpha = -m$ then

$$(\alpha)_n = (-m)_n = 0 \text{ when } n > m$$

5.5 Different Forms of Hypergeometric Function

The hypergeometric function $F(\alpha, \beta, \gamma, x)$ can be put in the following forms as well

$$\begin{aligned} F(\alpha, \beta, \gamma, x) &= (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta; \gamma; x) \\ &= (1-x)^{-\alpha} F\left(\alpha, \gamma-\beta; \gamma; \frac{x}{x-1}\right) \\ &= (1-x)^{-\beta} F\left(\beta, \gamma-\alpha; \gamma; \frac{x}{x-1}\right) \end{aligned}$$

5.6 Confluent Hypergeometric Function (Kummer Function)

The function denoted by ${}_1F_1(\alpha; \beta; x)$ and defined by

$${}_1F_1(\alpha; \beta; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\beta)_r} \frac{x^r}{r!}$$

is called the **Confluent Hypergeometric Function** or **Kummer Function**

The Confluent Hypergeometric Function ${}_1F_1(\alpha; \beta; x)$ is also denoted by $M(\alpha, \beta, x)$

5.7 General Hypergeometric Function

The general hypergeometric function denoted by ${}_mF_n(\alpha_1, \alpha_2, \dots, \alpha_m; \beta_1, \beta_2, \dots, \beta_n; x)$ is defined by

$${}_mF_n(\alpha_1, \alpha_2, \dots, \alpha_m; \beta_1, \beta_2, \dots, \beta_n; x) = \sum_{r=0}^{\infty} \frac{(\alpha_1)_r (\alpha_2)_r \dots (\alpha_m)_r}{(\beta_1)_r (\beta_2)_r \dots (\beta_n)_r} \frac{x^r}{r!}$$

Another notation often used for general hypergeometric function is

$${}_mF_n \left[\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_m \\ \beta_1, \beta_2, \dots, \beta_n \end{matrix} ; x \right]$$

- Note :** 1. Confluent hypergeometric function (5.6) is special case of general hypergeometric function when $m = n = 1$.
2. Hypergeometric function (5.4) is special case of general hypergeometric function when $m = 2$ and $n = 1$.

5.8 Symmetric Property of Hypergeometric Function

[Kanpur 2002, 03]

Hypergeometric function does not change if the parameters α and β are interchanged, keeping γ fixed i.e., $F(\alpha, \beta; \gamma; x) = F(\beta, \alpha; \gamma; x)$

Proof: By the definition of hypergeometric function, we have

$$\begin{aligned} F(\alpha, \beta; \gamma; x) &= \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \frac{x^r}{r!} \\ &= \sum_{r=0}^{\infty} \frac{(\beta)_r (\alpha)_r}{(\gamma)_r} \frac{x^r}{r!} = F(\beta, \alpha; \gamma; x) \end{aligned}$$

5.9 Particular Cases of Hypergeometric Series

Here we give some well known series which can be denoted by hypergeometric function.

- $e^x = {}_1F_1(\alpha; \alpha; x)$
- $(1-x)^{-1} = {}_2F_1(\alpha, 1; \alpha; x) = {}_2F_1(1, \alpha; \alpha; x), |x| < 1$
- $(1-x)^{-\alpha} = {}_2F_1(\alpha, \beta; \beta; x) = {}_2F_1(\beta, \alpha; \beta; x)$

[Kanpur 2004]

$$4. \quad (1+x)^n = {}_2F_1(-n, 1; 1; -x) = {}_2F_1(1, -n; 1; -x)$$

[Meerut 2011 (Sem. I)]

$$5. \quad \log_e(1+x) = x {}_2F_1(1, 1; 2; -x)$$

$$6. \quad \log_e(1-x) = -x {}_2F_1(1, 1; 2; x)$$

[Kanpur 2000]

$$7. \quad \log\left(\frac{1+x}{1-x}\right) = 2x {}_2F_1(1/2, 1; 3/2; x^2) = 2x {}_2F_1(1, 1/2; 3/2; x^2)$$

$$8. \quad \sin^{-1} x = x {}_2F_1(1/2, 1/2; 3/2; x^2)$$

$$9. \quad \tan^{-1} x = x {}_2F_1(1/2, 1; 3/2; -x^2) = x {}_2F_1(1, 1/2; 3/2; -x^2)$$

Proof: 1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$= 1 + \frac{\alpha}{\alpha} \cdot \frac{x}{1!} + \frac{\alpha(\alpha+1)}{\alpha(\alpha+1)} \cdot \frac{x^2}{2!} + \dots = {}_1F_1(\alpha; \alpha; x)$$

$$2. \quad (1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$= 1 + \frac{\alpha \cdot 1}{\alpha} \cdot \frac{x}{1!} + \frac{\alpha(\alpha+1) \cdot 1(1+1)}{\alpha(\alpha+1)} \cdot \frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2) \cdot 1(1+1)(1+2)}{\alpha(\alpha+1)(\alpha+2)} \cdot \frac{x^3}{3!} + \dots$$

$$= {}_2F_1(\alpha, 1; \alpha; x) = {}_2F_1(1, \alpha; \alpha; x)$$

(By symmetric property)

$$3. \quad (1-x)^{-\alpha} = 1 + (-\alpha)(-x) + \frac{(-\alpha)(-\alpha-1)}{2!}(-x)^2 + \frac{(-\alpha)(-\alpha-1)(-\alpha-2)}{3!}(-x)^3 + \dots$$

$$= 1 + \alpha x + \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{3!}x^3 + \dots$$

$$= 1 + \frac{\alpha \cdot \beta}{\beta} x + \frac{\alpha(\alpha+1) \cdot \beta(\beta+1)}{\beta(\beta+1)} \cdot \frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2) \cdot \beta(\beta+1)(\beta+2)}{\beta(\beta+1)(\beta+2)} \cdot \frac{x^3}{3!} + \dots$$

$$= {}_2F_1(\alpha, \beta; \beta; x) = {}_2F_1(\beta, \alpha; \beta; x)$$

(By symmetric property)

$$4. \quad (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

$$= 1 + \frac{(-n) \cdot 1}{1} \cdot \frac{(-x)}{1!} + \frac{(-n)(-n+1) \cdot 1(1+1)}{1(1+1)} \cdot \frac{(-x)^2}{2!}$$

$$+ \frac{(-n)(-n+1)(-n+2) \cdot 1(1+1)(1+2)}{1(1+1)(1+2)} \cdot \frac{(-x)^3}{3!} + \dots$$

$$= {}_2F_1(-n, 1; 1; -x) = {}_2F_1(1, -n; 1; -x)$$

(By symmetric property)

$$5. \quad \log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$= x \left[1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \dots \right]$$

$$= x \left[1 + \frac{1 \cdot 1}{2} \cdot \frac{(-x)}{1!} + \frac{1(1+1) \cdot 1(1+1)}{2(2+1)} \cdot \frac{(-x)^2}{2!} + \frac{1(1+1)(1+2) \cdot 1(1+1)(1+2)}{2(2+1)(2+2)} \cdot \frac{(-x)^3}{3!} + \dots \right]$$

$$= x {}_2F_1(1, 1; 2; -x)$$

$$6. \quad \log_e (1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots = -x \left[1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \dots \right]$$

$$= -x \left[1 + \frac{1 \cdot 1}{2} \cdot \frac{x}{1!} + \frac{1(1+1) \cdot 1(1+1)}{2(2+1)} \cdot \frac{x^2}{2!} + \frac{1(1+1)(1+2) \cdot 1(1+1)(1+2)}{2(2+1)(2+2)} \cdot \frac{x^3}{3!} + \dots \right]$$

$$= -x {}_2F_1(1, 1; 2; x)$$

$$7. \quad \log \left(\frac{1+x}{1-x} \right) = 2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \right)$$

$$= 2x \left(1 + \frac{x^2}{3} + \frac{x^4}{5} + \dots \right)$$

$$= 2x \left[1 + \frac{(1/2) \cdot 1}{3/2} \cdot \frac{x^2}{1!} + \frac{(1/2)(1/2+1) \cdot 1(1+1)}{(3/2)(3/2+1)} \cdot \frac{(x^2)^2}{2!} + \dots \right]$$

$$= 2x {}_2F_1(1/2, 1; 3/2; x^2) = 2x {}_2F_1(1, 1/2; 3/2; x^2) \quad (\text{By symmetric property})$$

$$8. \quad \sin^{-1} x = x + 1^2 \cdot \frac{x^3}{3!} + 1^2 \cdot 3^2 \cdot \frac{x^5}{5!} + 1^2 \cdot 3^2 \cdot 5^2 \cdot \frac{x^7}{7!} + \dots$$

$$= x \left[1 + 1^2 \cdot \frac{x^2}{3!} + 1^2 \cdot 3^2 \cdot \frac{x^4}{5!} + 1^2 \cdot 3^2 \cdot 5^2 \cdot \frac{x^6}{7!} + \dots \right]$$

$$= x \left[1 + \frac{1/2 \cdot 1/2}{3/2} \cdot \frac{x^2}{1!} + \frac{(1/2)(1/2+1)(1/2)(1/2+1)}{(3/2)(3/2+1)} \cdot \frac{(x^2)^2}{2!} + \dots \right]$$

$$= x {}_2F_1(1/2, 1/2; 3/2; x^2)$$

Aliter We have ${}_2F_1(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha \cdot \beta}{\gamma} \frac{x}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)} \frac{x^2}{2!} + \dots$

Replacing α, β, γ, x by $1/2, 1/2, 3/2, x^2$ respectively, we get

$$\begin{aligned} & x \cdot {}_2F_1(1/2, 1/2; 3/2; x^2) \\ &= x \left[1 + \frac{1/2 \cdot 1/2}{3/2} \cdot \frac{x^2}{1!} + \frac{(1/2) \cdot (1/2 + 1) \cdot 1/2 (1/2 + 1)}{(3/2)(3/2 + 1)} \cdot \frac{(x^2)^2}{2!} + \dots \right] \\ &= x + 1^2 \cdot \frac{x^2}{3!} + 1^2 \cdot 3^2 \cdot \frac{x^5}{5!} + 1^2 \cdot 3^2 \cdot 5^2 \cdot \frac{x^7}{7!} + \dots \end{aligned}$$

$$9. \quad \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$\begin{aligned} &= x \left[1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \dots \right] \\ &= x \left[1 + \frac{1/2 \cdot 1}{3/2} \cdot \frac{(-x^2)}{1!} + \frac{(1/2)(1/2 + 1) \cdot 1(1 + 1)}{(3/2)(3/2 + 1)} \cdot \frac{(-x^2)^2}{2!} + \dots \right] \\ &= x {}_2F_1(1/2, 1; 3/2; -x^2) = x {}_2F_1(1, 1/2; 3/2; -x^2) \end{aligned}$$

(By symmetric property)

Aliter : We have, ${}_2F_1(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha \cdot \beta}{\gamma} \cdot \frac{x}{1!} + \frac{\alpha(\alpha + 1) \cdot \beta(\beta + 1)}{\gamma(\gamma + 1)} \cdot \frac{x^2}{2!} + \dots$

Replacing α, β, γ, x by $1/2, 1, 3/2, -x^2$, we get

$$\begin{aligned} & x \cdot {}_2F_1(1/2, 1; 3/2; -x^2) \\ &= x \left[1 + \frac{1/2 \cdot 1}{3/2} \cdot \frac{(-x^2)}{1!} + \frac{(1/2)(1/2 + 1) \cdot 1(1 + 1)}{(3/2)(3/2 + 1)} \cdot \frac{(-x^2)^2}{2!} + \dots \right] \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \end{aligned}$$

5.10 Gauss's Hypergeometric Equation or Gauss's Equation or Hypergeometric Equation

The differential equation

$$x(1-x) \frac{d^2 y}{dx^2} + \{\gamma - (\alpha + \beta + 1)x\} \frac{dy}{dx} - \alpha\beta y = 0$$

is called **Gauss's hypergeometric equation** or **Gauss's equation** or **hypergeometric equation**.

5.1.1 Solution of the Hypergeometric Equation

[Meerut 2010 (Sem. I); Kanpur 2006]

Hypergeometric equation is

$$x(1-x) \frac{d^2 y}{dx^2} + \{\gamma - (\alpha + \beta + 1)x\} \frac{dy}{dx} - \alpha\beta y = 0 \quad \dots(1)$$

Dividing by $x(1-x)$, we get $\frac{d^2 y}{dx^2} + \frac{\gamma - (\alpha + \beta + 1)x}{x(1-x)} \frac{dy}{dx} - \frac{\alpha\beta}{x(1-x)} y = 0$

Comparing with $\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = 0$, we get

$$P(x) = \frac{\gamma - (\alpha + \beta + 1)x}{x(1-x)} \text{ and } Q(x) = -\frac{\alpha\beta}{x(1-x)}$$

Since $P(x)$ and $Q(x)$ are not finite at $x=0$ and $x=1$, and $xP(x)$ and $x^2Q(x)$ are both finite and so analytic at $x=0$ also $(x-1)P(x)$ and $(x-1)^2Q(x)$ are also both finite and so analytic at $x=1$.

Thus the points $x=0$ and $x=1$ are both regular singular points of (1).

To find series solution of (1) near $x=0$

[Kanpur 2006]

$$\text{Let } y = \sum_{r=0}^{\infty} C_r x^{m+r}, \quad C_0 \neq 0 \quad \dots(2)$$

be series solution of (1) near $x=0$

Diff. (2), we get

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} C_r (m+r) x^{m+r-1} \text{ and } \frac{d^2 y}{dx^2} = \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) x^{m+r-2}$$

Substituting in (1), we get

$$\sum_{r=0}^{\infty} C_r [(x-x^2)(m+r)(m+r-1)x^{m+r-2} + \{\gamma - (\alpha + \beta + 1)x\}(m+r)x^{m+r-1} - \alpha\beta x^{m+r}] = 0$$

$$\begin{aligned} \text{or } \sum_{r=0}^{\infty} C_r \{[(m+r)(m+r-1) + \gamma(m+r)]x^{m+r-1} \\ - \{(m+r)(m+r-1) + (\alpha + \beta + 1)(m+r) + \alpha\beta\}x^{m+r}\} = 0 \end{aligned}$$

$$\text{or } \sum_{r=0}^{\infty} C_r [(m+r)(m+r+\gamma-1)x^{m+r-1} - \{(m+r)^2 + (\alpha+\beta)(m+r) + \alpha\beta\}x^{m+r}] = 0$$

$$\text{or } \sum_{r=0}^{\infty} C_r [(m+r)(m+r+\gamma-1)x^{m+r-1} - (m+r+\alpha)(m+r+\beta)x^{m+r}] = 0 \quad \dots(3)$$

which is an identity, so we can equate to zero the coefficients of various powers of x ,

Equating to zero the coefficient of lowest power of x i.e., of x^{m-1} , the *indicial equation* is :

$$C_0 m(m+\gamma-1) = 0 \Rightarrow m=0 \text{ or } m=1-\gamma$$

which are unequal and not differing by an integer as γ is not an integer.

Also equating to zero the coefficient of x^{m+r-1} , we get

$$\begin{aligned} C_r(m+r)(m+r+\gamma-1) - (m+r-1+\alpha)(m+r-1+\beta)C_{r-1} &= 0 \\ \Rightarrow C_r &= \frac{(m+r-1+\alpha)(m+r-1+\beta)}{(m+r)(m+r+\gamma-1)} C_{r-1}, \quad \text{for } r \geq 1 \end{aligned} \quad \dots(4)$$

Case I. When $m=0$ Putting $m=0$, in (4), we get

$$C_r = \frac{(\alpha+r-1)(\beta+r-1)}{r(\gamma+r-1)} C_{r-1}, \quad \text{for } r \geq 1$$

Putting $r=1, 2, 3, \dots$,

$$C_1 = \frac{\alpha \cdot \beta}{1 \cdot \gamma} C_0, \quad C_2 = \frac{(\alpha+1)(\beta+1)}{2(\gamma+1)} C_1 = \frac{\alpha(\alpha+1)\beta(\beta+1)}{2! \gamma(\gamma+1)} C_0$$

$$C_3 = \frac{(\alpha+2)(\beta+2)}{3(\gamma+2)} C_2 = \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{3! \gamma(\gamma+1)(\gamma+2)} C_0 \text{ and so on.}$$

Putting in (2), we get

$$y = C_0 \left[1 + \frac{\alpha \cdot \beta}{\gamma} \frac{x}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)} \frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)} \frac{x^3}{3!} + \dots \right]$$

$$\text{or } y = C_0 {}_2F_1(\alpha, \beta; \gamma; x)$$

$$\text{Taking } C_0 = 1, \text{ we see that } y = {}_2F_1(\alpha, \beta; \gamma; x) \quad \dots(5)$$

called hypergeometric function is one solution of (1).

Case II. When $m = 1 - \gamma$

Putting $m = 1 - \gamma$ in (4), we get

$$C_r = \frac{(1 - \gamma + r - 1 + \alpha)(1 - \gamma + r - 1 + \beta)}{(1 - \gamma + r)(1 - \gamma + r + \gamma - 1)} C_{r-1}$$

$$\Rightarrow C_r = \frac{(\alpha' + r - 1)(\beta' + r - 1)}{r(\gamma' + r - 1)} C_{r-1}, \quad r \geq 1 \quad \dots(6)$$

where $\alpha' = 1 - \gamma + \alpha$, $\beta' = 1 - \gamma + \beta$ and $\gamma' = 2 - \gamma$

Putting $r = 1, 2, 3, \dots$ in (6), we get

$$C_1 = \frac{\alpha' \beta'}{1 \cdot \gamma'} C_0, \quad C_2 = \frac{(\alpha' + 1)(\beta' + 1)}{2(\gamma' + 1)} C_1 = \frac{\alpha'(\alpha' + 1) \cdot \beta'(\beta' + 1)}{2! \gamma'(\gamma' + 1)} C_0$$

$$C_3 = \frac{(\alpha' + 2)(\beta' + 2)}{3(\gamma' + 2)} C_2 = \frac{\alpha'(\alpha' + 1)(\alpha' + 2) \cdot \beta'(\beta' + 1)(\beta' + 2)}{3! \gamma'(\gamma' + 1)(\gamma' + 2)} C_0, \text{ and so on.}$$

Putting in (2), we get

$$y = C_0 x^{1-\gamma} \left[1 + \frac{\alpha' \cdot \beta'}{\gamma'} \cdot \frac{x}{1!} + \frac{\alpha'(\alpha' + 1) \beta'(\beta' + 1)}{\gamma'(\gamma' + 1)} \cdot \frac{x^2}{2!} + \frac{\alpha'(\alpha' + 1)(\alpha' + 2) \cdot \beta'(\beta' + 1)(\beta' + 2)}{\gamma'(\gamma' + 1)(\gamma' + 2)} \cdot \frac{x^3}{3!} + \dots \right]$$

$$\text{or } y = C_0 x^{1-\gamma} {}_2F_1(\alpha', \beta'; \gamma'; x)$$

$$\text{or } y = C_0 x^{1-\gamma} {}_2F_1(1 - \gamma + \alpha, 1 - \gamma + \beta; 2 - \gamma; x)$$

$$\text{Taking } C_0 = 1, y = x^{1-\gamma} {}_2F_1(1 - \gamma + \alpha, 1 - \gamma + \beta; 2 - \gamma; x) \quad \dots(7)$$

Which is another independent solution of (1).

Hence the general solution of (1) near $x = 0$ is

$$y = a {}_2F_1(\alpha, \beta; \gamma; x) + b x^{1-\gamma} {}_2F_1(1 - \gamma + \alpha, 1 - \gamma + \beta, 2 - \gamma, x) \quad \dots(8)$$

where a and b are arbitrary constants.

To find series solution of (1) near $x = 1$.

To find series solution of diff. equation (1) near $x = 1$ i.e., in powers of $x - 1$, we change the independent variable from x to z , by the substitution

$1 - x = z$, so that $\frac{dz}{dx} = -1$, $\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = -\frac{dy}{dz}$

and
$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(-\frac{dy}{dz} \right) = -\frac{d^2 y}{dz^2} \cdot \frac{dz}{dx} = \frac{d^2 y}{dz^2}$$

$\therefore$ Diff., equation (1), reduces to

$$(1-z)z \cdot \frac{d^2 y}{dz^2} + \{\gamma - (\alpha + \beta + 1)(1-z)\} \left(-\frac{dy}{dz} \right) - \alpha\beta y = 0$$

or
$$z(1-z) \frac{d^2 y}{dz^2} + \{(\alpha + \beta - \gamma + 1) - (\alpha + \beta + 1)z\} \frac{dy}{dz} - \alpha\beta y = 0 \quad \dots(9)$$

which is identical with (1) with γ replaced by $\alpha + \beta - \gamma + 1$ and

Hence from (8), solution of (9) is

$$y = a \cdot {}_2F_1(\alpha, \beta; \alpha + \beta - \gamma + 1; z) + b \cdot z^{1 - (\alpha + \beta - \gamma + 1)} \cdot {}_2F_1(\gamma - \beta, \gamma - \alpha, \gamma - \alpha - \beta + 1; z)$$

$\Rightarrow$ Solution of (1) near $x = 1$ is

$$y = a \cdot {}_2F_1(\alpha, \beta; \alpha + \beta - \gamma + 1; 1-x) + b(1-x)^{\gamma - \alpha - \beta} \cdot {}_2F_1(\gamma - \alpha, \gamma - \beta; \gamma - \alpha - \beta + 1; 1-x) \quad \dots(10)$$

5.12 Derivatives of Hypergeometric Functions

We have,
$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \cdot \frac{x^n}{n!}$$

Differentiating both sides, w.r.t. x

$$\begin{aligned} \frac{d}{dx} {}_2F_1(\alpha, \beta; \gamma; x) &= \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \cdot \frac{n x^{n-1}}{n!} \\ &= \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \cdot \frac{x^{n-1}}{(n-1)!} \\ &= \sum_{m=0}^{\infty} \frac{(\alpha)_{m+1} (\beta)_{m+1}}{(\gamma)_{m+1}} \cdot \frac{x^m}{m!}, \quad \text{Taking } n-1 = m \end{aligned}$$

Since $(\alpha)_{m+1} = \alpha(\alpha+1)(\alpha+2) \dots (\alpha+m+1-1)$

$$= \alpha \cdot (\alpha+1)(\alpha+2) \dots (\alpha+1+m-1)$$

$$= \alpha(\alpha+1)_m$$

$\dots(1)$

$$\therefore \frac{d}{dx} {}_2F_1(\alpha, \beta; \gamma; x) = \frac{\alpha\beta}{\gamma} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m (\beta+1)_m}{(\gamma+1)_m} \cdot \frac{x^m}{m!} \quad \dots(2)$$

$$\Rightarrow \frac{d}{dx} {}_2F_1(\alpha, \beta; \gamma; x) = \frac{(\alpha)_1 \cdot (\beta)_1}{(\gamma)_1} {}_2F_1(\alpha+1, \beta+1; \gamma+1; x)$$

$$\therefore (\alpha)_1 = \alpha \text{ etc.}$$

[Kanpur 2005; Kurukshetra 2006; Garhwal 2005]

Differentiating again

$$\begin{aligned} \frac{d^2}{dx^2} {}_2F_1(\alpha, \beta; \gamma; x) &= \frac{\alpha\beta}{\gamma} \sum_{m=1}^{\infty} \frac{(\alpha+1)_m (\beta+1)_m}{(\gamma+1)_m} \cdot \frac{mx^{m-1}}{m!} \\ &= \frac{\alpha\beta}{\gamma} \sum_{m=1}^{\infty} \frac{(\alpha+1)_m (\beta+1)_m}{(\gamma+1)_m} \cdot \frac{x^{m-1}}{(m-1)!} \\ &= \frac{\alpha\beta}{\gamma} \sum_{n=0}^{\infty} \frac{(\alpha+1)_{n+1} (\beta+1)_{n+1}}{(\gamma+1)_{n+1}} \cdot \frac{x^n}{n!} \quad \text{Taking } m-1=n \\ &= \frac{\alpha\beta}{\gamma} \sum_{n=0}^{\infty} \frac{(\alpha+1)(\alpha+2)_n (\beta+1)(\beta+2)_n}{(\gamma+1)(\gamma+2)_n} \cdot \frac{x^n}{n!} \\ &\quad [\because \text{From (1), } (\alpha+1)_{n+1} = (\alpha+1)(\alpha+2)_n] \\ &= \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)} \sum_{n=0}^{\infty} \frac{(\alpha+2)_n (\beta+2)_n}{(\gamma+2)_n} \cdot \frac{x^n}{n!} \quad \dots(3) \end{aligned}$$

$$\Rightarrow \frac{d^2}{dx^2} {}_2F_1(\alpha, \beta; \gamma; x) = \frac{(\alpha)_2 \cdot (\beta)_2}{(\gamma)_2} {}_2F_1(\alpha+2, \beta+2; \gamma+2; x)$$

$$\therefore (\alpha)_2 = \alpha(\alpha+2-1) = \alpha(\alpha+1) \text{ etc.}$$

5.13 n-th Derivative of Hypergeometric Function

$$\text{To Prove } \frac{d^n}{dx^n} {}_2F_1(\alpha, \beta; \gamma; x) = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} {}_2F_1(\alpha+n, \beta+n; \gamma+n; x) \quad \dots(1)$$

where n is a positive integer.

[Meerut 2010 (Sem. I)]

Proof: In 5.12 we have proved that

$$\frac{d}{dx} {}_2F_1(\alpha, \beta; \gamma; x) = \frac{(\alpha)_1 (\beta)_1}{(\gamma)_1} {}_2F_1(\alpha+1, \beta+1; \gamma+1; x) \quad \dots(2)$$

i.e., (1) is true for $n=1$ Now we assume that (1) is true for a positive integer $m < n$, then

$$\frac{d^m}{dx^m} {}_2F_1(\alpha, \beta; \gamma; x) = \frac{(\alpha)_m (\beta)_m}{(\gamma)_m} {}_2F_1(\alpha + m, \beta + m; \gamma + m; x)$$

Differentiating both sides, w.r.t. x , we get

$$\begin{aligned} \frac{d^{m+1}}{dx^{m+1}} {}_2F_1(\alpha, \beta; \gamma; x) &= \frac{(\alpha)_m (\beta)_m}{(\gamma)_m} \cdot \frac{d}{dx} {}_2F_1(\alpha + m, \beta + m; \gamma + m; x) \\ &= \frac{(\alpha)_m (\beta)_m}{(\gamma)_m} \cdot \frac{(\alpha + m)(\beta + m)}{(\gamma + m)} {}_2F_1(\alpha + m + 1, \beta + m + 1; \gamma + m + 1; x) \end{aligned}$$

(using (1) for $\alpha + m, \beta + m, \gamma + m$ in place of α, β, γ respectively and with $(\alpha)_1 = \alpha$)

$$= \frac{(\alpha)_{m+1} (\beta)_{m+1}}{(\gamma)_{m+1}} {}_2F_1(\alpha + m + 1, \beta + m + 1; \gamma + m + 1; x) \quad \dots(3)$$

$$\therefore (\alpha)_m \cdot (\alpha + m) = (\alpha)_{m+1}$$

from (3) it follows that (1) is true for $m + 1$.

i.e., assuming that (1) is true for positive integer $m < n$, (1) is also true for $m + 1$.

But from (2), it is obvious that (1) is true for $n = 1$, hence by mathematical induction (1) is true for all positive integral value of n .

5.14 Derivatives of Hypergeometric Function at $x = 0$

1. $\frac{d}{dx} {}_2F_1(\alpha, \beta; \gamma; 0) = 1$
2. $\frac{d^2}{dx^2} {}_2F_1(\alpha, \beta; \gamma; 0) = \frac{\alpha \cdot \beta}{\gamma} = \frac{(\alpha)_1 (\beta)_1}{(\gamma)_1}$
3. $\frac{d^n}{dx^n} {}_2F_1(\alpha, \beta; \gamma; 0) = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n}$

Proof: We have,

$$\begin{aligned} {}_2F_1(\alpha, \beta; \gamma; x) &= \sum_{n=0}^{\infty} \frac{(\alpha)_n \cdot (\beta)_n}{(\gamma)_n} \cdot \frac{x^n}{n!} \\ &= \frac{(\alpha)_0 \cdot (\beta)_0}{(\gamma)_0} \cdot \frac{1}{0!} + \frac{(\alpha)_1 \cdot (\beta)_1}{(\gamma)_1} \cdot \frac{x}{1!} + \frac{(\alpha)_2 \cdot (\beta)_2}{(\gamma)_2} \cdot \frac{x^2}{2!} + \dots + \frac{(\alpha)_n \cdot (\beta)_n}{(\gamma)_n} \cdot \frac{x^n}{n!} + \dots \end{aligned}$$

$$\begin{aligned} \therefore \frac{d}{dx} {}_2F_1(\alpha, \beta; \gamma; x) &= 0 + \frac{(\alpha)_1 (\beta)_1}{(\gamma)_1} \cdot 1 + \frac{(\alpha)_2 (\beta)_2}{(\gamma)_2} \cdot \frac{2x}{2!} + \frac{(\alpha)_3 (\beta)_3}{(\gamma)_3} \cdot \frac{3x^2}{3!} + \dots \\ &\quad + \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \cdot \frac{nx^{n-1}}{n!} + \dots \\ \frac{d^2}{dx^2} {}_2F_1(\alpha, \beta; \gamma; x) &= 0 + \frac{(\alpha)_2 (\beta)_2}{(\gamma)_2} \cdot 1 + \frac{(\alpha)_3 (\beta)_3}{(\gamma)_3} \cdot \frac{3 \cdot 2x}{3!} + \dots \\ &\quad + \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \cdot \frac{n(n-1)x^{n-2}}{n!} + \dots \end{aligned}$$

Proceeding similarly

$$\frac{d^n}{dx^n} {}_2F_1(\alpha, \beta; \gamma; x) = 0 + \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \cdot \frac{n!}{n!} + \frac{(\alpha)_{n+1} (\beta)_{n+1}}{(\gamma)_{n+1}} \cdot \frac{(n+1)n(n-1)\dots 2}{n!} x + \dots$$

Putting $x=0$ we get

$$\begin{aligned} {}_2F_1(\alpha, \beta; \gamma; x) &= \frac{(\alpha)_0 (\beta)_0}{(\gamma)_0} = 1 \\ \frac{d}{dx} {}_2F_1(\alpha, \beta; \gamma; 0) &= \frac{(\alpha)_1 (\beta)_1}{(\gamma)_1} \quad \text{or} \quad \frac{\alpha\beta}{\gamma} \\ \frac{d^2}{dx^2} {}_2F_1(\alpha, \beta; \gamma; 0) &= \frac{(\alpha)_2 (\beta)_2}{(\gamma)_2} \quad \text{or} \quad \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)} \\ \frac{d^n}{dx^n} {}_2F_1(\alpha, \beta; \gamma; 0) &= \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \end{aligned}$$

5.15 Integral Formula for Hypergeometric Function

If $|x| < 1$ and $\gamma > \beta > 0$, then

$${}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-xt)^{-\alpha} dt$$

$$\text{or } {}_2F_1(\alpha, \beta; \gamma; x) = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-xt)^{-\alpha} dt$$

[Rajasthan 2000, 09; Garhwal 2004; Kanpur 2004, 07; Purvanchal 2007]

Proof: We have the hypergeometric function

$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \cdot \frac{x^n}{n!} \quad \dots(1)$$

where $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)$

$$= \frac{1.2 \dots (\alpha-1) \alpha (\alpha+1) \dots (\alpha+n-1)}{1.2 \dots (\alpha-1)} = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

$$\begin{aligned} \therefore \frac{(\beta)_n}{(\gamma)_n} &= \frac{\Gamma(\beta+n)}{\Gamma(\beta)} \cdot \frac{\Gamma(\gamma)}{\Gamma(\gamma+n)} = \frac{\Gamma(\gamma)}{\Gamma(\beta)} \frac{\Gamma(\beta+n)}{\Gamma(\gamma+n)} \cdot \frac{\Gamma(\gamma-\beta)}{\Gamma(\gamma-\beta)} \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)} \cdot \frac{\Gamma(\beta+n)}{\Gamma(\beta+n+\gamma-\beta)} \cdot \frac{1}{\Gamma(\gamma-\beta)} \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta) \Gamma(\gamma-\beta)} \cdot B(\beta+n, \gamma-\beta) \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta) \Gamma(\gamma-\beta)} \cdot \int_0^1 t^{\beta+n-1} \cdot (1-t)^{\gamma-\beta-1} dt \end{aligned}$$

$\therefore$ From (1) ${}_2F_1(\alpha, \beta; \gamma; x)$

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{\Gamma(\gamma)}{\Gamma(\beta) \Gamma(\gamma-\beta)} \int_0^1 t^{\beta+n-1} (1-t)^{\gamma-\beta-1} \cdot (\alpha)_n \cdot \frac{x^n}{n!} dt \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(\gamma)}{\Gamma(\beta) \Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (\alpha)_n \cdot \frac{(tx)^n}{n!} dt \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta) \Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} \left\{ \sum_{n=0}^{\infty} (\alpha)_n \cdot \frac{(xt)^n}{n!} \right\} dt \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta) \Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} \cdot \left\{ (\alpha)_0 \cdot 1 + (\alpha)_1 \cdot \frac{xt}{1!} + (\alpha)_2 \cdot \frac{(xt)^2}{2!} + \dots \right\} dt \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta) \Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} \cdot \left\{ 1 + \alpha \cdot xt + \alpha(\alpha+1) \cdot \frac{(xt)^2}{2!} + \dots \right\} dt \end{aligned}$$

$$\text{or } {}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta) \Gamma(\gamma-\beta)} \cdot \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-xt)^{-\alpha} dt$$

which is the required integral formula.

Which can also be written as

$${}_2F_1(\alpha, \beta; \gamma; x) = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-xt)^{-\alpha} dt$$

Integral formula for hypergeometric function is valid for $|x| < 1, \gamma > \beta > 0$.

5.16 Kummer's Theorem (For Hypergeometric Function)

$${}_2F_1(\alpha, \beta; \beta - \alpha + 1; -1) = \frac{\Gamma(\beta - \alpha + 1) \Gamma(\beta/2 + 1)}{\Gamma(\beta + 1) \Gamma(\beta/2 - \alpha + 1)} \quad [\text{Purvanchal 2005}]$$

Proof: The integral formula for hypergeometric function is

$${}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta) \Gamma(\gamma - \beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-xt)^{-\alpha} dt$$

Putting $x = -1$ and $\gamma = \beta - \alpha + 1$, we get

$${}_2F_1(\alpha, \beta; \beta - \alpha + 1; -1) = \frac{\Gamma(\beta - \alpha + 1)}{\Gamma(\beta) \Gamma(1 - \alpha)} \int_0^1 t^{\beta-1} (1-t)^{-\alpha} (1+t)^{-\alpha} dt$$

$$= \frac{\Gamma(\beta - \alpha + 1)}{\Gamma(\beta) \Gamma(1 - \alpha)} \int_0^1 t^{\beta-1} (1-t^2)^{-\alpha} dt$$

$$= \frac{\Gamma(\beta - \alpha + 1)}{\Gamma(\beta) \Gamma(1 - \alpha)} \int_0^1 z^{(\beta-1)/2} (1-z)^{-\alpha} \cdot \frac{1}{2} z^{-1/2} dz$$

Putting $t^2 = z$, $t = z^{1/2}$ and $dt = \frac{1}{2} z^{-1/2} dz$

$$= \frac{1}{2} \frac{\Gamma(\beta - \alpha + 1)}{\Gamma(\beta) \Gamma(1 - \alpha)} \int_0^1 z^{\beta/2-1} (1-z)^{(1-\alpha)-1} dz$$

$$= \frac{1}{2} \frac{\Gamma(\beta - \alpha + 1)}{\Gamma(\beta) \Gamma(1 - \alpha)} \cdot B(\beta/2, 1 - \alpha)$$

$$= \frac{1}{2} \frac{\Gamma(\beta - \alpha + 1)}{\Gamma(\beta) \Gamma(1 - \alpha)} \cdot \frac{\Gamma(\beta/2) \cdot \Gamma(1 - \alpha)}{\Gamma(\beta/2 + 1 - \alpha)}$$

$$= \frac{\Gamma(\beta - \alpha + 1) \cdot (\beta/2) \Gamma(\beta/2)}{\beta \Gamma(\beta) \Gamma(\beta/2 - \alpha + 1)}$$

$$\text{or } {}_2F_1(\alpha, \beta; \beta - \alpha + 1; -1) = \frac{\Gamma(\beta - \alpha + 1) \cdot \Gamma(\beta/2 + 1)}{\Gamma(\beta + 1) \Gamma(\beta/2 - \alpha + 1)}$$

which is the required Kummer's theorem (or formula).

5.17 Gauss's Theorem

$${}_2F_1(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma) \cdot \Gamma(\gamma - \beta - \alpha)}{\Gamma(\gamma - \alpha) \cdot \Gamma(\gamma - \beta)}$$

[Kanpur 2005, 06, 07; Meerut 2001, 03, 12 (Sem. I); Rajasthan 2000, 07, 08]

Proof: The integral formula for hypergeometric function is

$${}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-xt)^{-\alpha} dt$$

Putting $x=1$, we get

$$\begin{aligned} {}_2F_1(\alpha, \beta; \gamma; 1) &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-t)^{-\alpha} dt \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{(\gamma-\beta-\alpha)-1} dt \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} B(\beta, \gamma-\beta-\alpha) \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \cdot \frac{\Gamma(\beta) \cdot \Gamma(\gamma-\beta-\alpha)}{\Gamma(\beta+\gamma-\beta-\alpha)} \end{aligned}$$

$$\text{or } {}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma) \cdot \Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}$$

which is the required Gauss's theorem.

5.18 Vandermonde's Theorem

$${}_2F_1(-n, \beta; \gamma; 1) = \frac{(\gamma-\beta)_n}{(\gamma)_n}$$

[Kanpur 2000, 01; Meerut 2001, 03]

Proof: The integral formula for hypergeometric function is

$${}_2F_1(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-xt)^{-\alpha} dt$$

Putting $x=1$ and $\alpha=-n$, we have

$$\begin{aligned} {}_2F_1(-n, \beta; \gamma; 1) &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-t)^n dt \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{(\gamma+n-\beta)-1} dt \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} B(\beta, \gamma+n-\beta) \\ &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \cdot \frac{\Gamma(\beta) \cdot \Gamma(\gamma+n-\beta)}{\Gamma(\beta+\gamma+n-\beta)} = \frac{\Gamma(\gamma)\Gamma(\gamma-\beta+n)}{\Gamma(\gamma-\beta)\Gamma(\gamma+n)} \end{aligned}$$

$$\begin{aligned}
&= \frac{\Gamma(\gamma) (\gamma - \beta + n - 1) (\gamma - \beta + n - 2) \dots (\gamma - \beta) \Gamma(\gamma - \beta)}{\Gamma(\gamma - \beta) \cdot (\gamma + n - 1) (\gamma + n - 2) \dots \gamma \cdot \Gamma(\gamma)} \\
&= \frac{(\gamma - \beta) (\gamma - \beta + 1) \dots (\gamma - \beta + n - 2) (\gamma - \beta + n - 1)}{\gamma (\gamma + 1) \dots (\gamma + n - 2) (\gamma + n - 1)}
\end{aligned}$$

$$\text{or } {}_2F_1(-n, \beta; \gamma; 1) = \frac{(\gamma - \beta)_n}{(\gamma)_n}$$

which is the required Vandermonde's theorem.

5.19 Confluent Hypergeometric Equation (Or Kummer's Equation) and its Solution

The hypergeometric differential equation is

$$x(1-x) \frac{d^2 y}{dx^2} \{ \gamma - (\alpha + \beta + 1)x \} \frac{dy}{dx} - \alpha \beta y = 0 \quad \dots(1)$$

Replacing x by x/β in (1), we get

$$\begin{aligned}
&\frac{x}{\beta} \left(1 - \frac{x}{\beta} \right) \beta^2 \frac{d^2 y}{dx^2} + \left\{ \gamma - (\alpha + \beta + 1) \frac{x}{\beta} \right\} \beta \frac{dy}{dx} - \alpha \beta y = 0 \\
\Rightarrow &x \left(1 - \frac{x}{\beta} \right) \frac{d^2 y}{dx^2} + \left\{ \gamma - \left(1 + \frac{1 + \alpha}{\beta} \right) x \right\} \frac{dy}{dx} - \alpha y = 0 \quad \dots(2)
\end{aligned}$$

Since ${}_2F_1(\alpha, \beta; \gamma; x)$ is the solution of (1), $\therefore {}_2F_1(\alpha, \beta; \gamma; x/\beta)$ is the solution of (2)

When $\beta \rightarrow \infty$, the equation (2) reduces to

$$x \frac{d^2 y}{dx^2} + (\gamma - x) \frac{dy}{dx} - \alpha y = 0 \quad \dots(3)$$

whose solution is given by

$$\begin{aligned}
\lim_{\beta \rightarrow \infty} {}_2F_1(\alpha, \beta; \gamma; x/\beta) &= \lim_{\beta \rightarrow \infty} \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \cdot \frac{1}{r!} \left(\frac{x}{\beta} \right)^r \\
&= \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\gamma)_r} \cdot \frac{x^r}{r!} = {}_1F_1(\alpha; \gamma; x)
\end{aligned}$$

$$\text{Since } \lim_{\beta \rightarrow \infty} \frac{(\beta)_r}{\beta^r} = \lim_{\beta \rightarrow \infty} \frac{\beta(\beta+1)(\beta+2) \dots (\beta+r-1)}{\beta^r}$$

$$= \lim_{\beta \rightarrow \infty} 1 \cdot \left(1 + \frac{1}{\beta}\right) \left(1 + \frac{2}{\beta}\right) \dots \left(1 + \frac{r-1}{\beta}\right) = 1$$

i.e., solution of equation (3) is ${}_1F_1(\alpha; \gamma; x)$

This function ${}_1F_1(\alpha; \gamma; x)$ is called **confluent hypergeometric function** which is the solution of the equation (3) called **Confluent hypergeometric equation or Kummer's equation**.

5.20 Series Solution of Confluent Hypergeometric Differential Equation near $x = 0$ when γ is not an Integer

The confluent hypergeometric differential equation is

$$x \frac{d^2 y}{dx^2} + (\gamma - x) \frac{dy}{dx} - \alpha y = 0 \quad \dots(1)$$

Comparing with $\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)y = 0$, we get

$$P(x) = \frac{\gamma - x}{x} \text{ and } Q(x) = -\frac{\alpha}{x}$$

None of which is finite i.e., analytic at $x = 0$ and $xP(x) = \gamma - x$ and $x^2Q(x) = -\alpha x$, which are both finite i.e., analytic at $x = 0$.

So $x = 0$ is a regular singular point of (1).

Let the solution of (1), near the point $x = 0$, be given by the infinite series

$$y = \sum_{r=0}^{\infty} C_r x^{m+r}, \quad C_0 \neq 0 \quad \dots(2)$$

$$\text{Diff.} \quad \frac{dy}{dx} = \sum_{r=0}^{\infty} C_r (m+r) x^{m+r-1}$$

$$\text{and} \quad \frac{d^2 y}{dx^2} = \sum_{r=0}^{\infty} C_r (m+r)(m+r-1) x^{m+r-2}$$

Substituting in (1), we get

$$\sum_{r=0}^{\infty} C_r [(m+r)(m+r-1) x^{m+r-1} + (\gamma - x)(m+r) x^{m+r-1} - \alpha x^{m+r}] = 0$$

$$\text{or} \quad \sum_{r=0}^{\infty} C_r [(m+r)(m+r-1+\gamma) x^{m+r-1} - (m+r+\alpha) x^{m+r}] = 0 \quad \dots(3)$$

Which is an identity, so we can equate to zero the coefficients of various powers of x .

Equating to zero the coefficient of lowest power of x i.e., of x^{m-1} , the *indicial equation* is

$$C_0 m(m-1+\gamma) = 0 \Rightarrow m=0 \text{ or } 1-\gamma \quad \because C_0 = 0$$

which are unequal and not differing by an integer.

Now equating to zero the coefficient of x^{m+r-1} , we get

$$\begin{aligned} & C_r(m+r)(m+r-1+\gamma) - (m+r-1+\alpha)C_{r-1} = 0 \\ \Rightarrow & C_r = \frac{m+\alpha+r-1}{(m+r)(m+r-1+\gamma)} C_{r-1}, \quad \text{for } r \geq 1 \end{aligned} \quad \dots(4)$$

Case I : When $m = 0$:

$$\text{In this case } C_r = \frac{\alpha+r-1}{r(r+\gamma-1)} C_{r-1}, \quad r \geq 1$$

Putting $r = 1, 2, 3, \dots$,

$$C_1 = \frac{\alpha}{\gamma} C_0, C_2 = \frac{\alpha+1}{2 \cdot (\gamma+1)} C_1 = \frac{\alpha(\alpha+1)}{1 \cdot 2 \gamma(\gamma+1)} C_0$$

$$C_3 = \frac{\alpha+2}{3(\gamma+2)} C_2 = \frac{\alpha(\alpha+1)(\alpha+2)}{1 \cdot 2 \cdot 3 \gamma(\gamma+1)(\gamma+2)} C_0 \text{ and so on.}$$

Putting in (2), we get

$$\begin{aligned} y &= \sum_{r=0}^{\infty} C_r x^r, \text{ when } m=0 \\ &= C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots \end{aligned}$$

$$\text{or } y = C_0 \left[1 + \frac{\alpha}{\gamma} x + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)} \cdot \frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)} \cdot \frac{x^3}{3!} + \dots \right]$$

Taking $C_0 = 1$

$$y = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\gamma)_r} \cdot \frac{x^r}{r!} = {}_1F_1(\alpha; \gamma; x)$$

which is a solution of (1) and is called the **confluent hypergeometric function of first kind**.

Case II. When $m = 1 - \gamma$: In this case, from (4)

$$C_r = \frac{1-\gamma+\alpha+r-1}{(1-\gamma+r)(1-\gamma+r-1+\gamma)} C_0 = \frac{(\alpha-\gamma+1)+r-1}{r \{(2-\gamma)+r-1\}} C_0$$

$$\Rightarrow C_r = \frac{\alpha' + r - 1}{r(\gamma' + r - 1)} C_0 \quad \text{for } r \geq 1$$

where $\alpha' = \alpha - \gamma + 1$ and $\gamma' = 2 - \gamma$

Putting $r = 1, 2, 3, \dots$, we get

$$C_1 = \frac{\alpha'}{1 \cdot \gamma'} C_0, C_2 = \frac{\alpha' + 1}{2(\gamma' + 1)} C_1 = \frac{\alpha'(\alpha' + 1)}{1 \cdot 2 \gamma'(\gamma' + 1)} C_0$$

$$C_3 = \frac{\alpha' + 2}{3 \cdot (\gamma' + 2)} C_2 = \frac{\alpha'(\alpha' + 1)(\alpha' + 2)}{1 \cdot 2 \cdot 3 \gamma'(\gamma' + 1)(\gamma' + 2)} C_0 \text{ and so on.}$$

$\therefore$ from (2), we have

$$\begin{aligned} y &= \sum_{r=0}^{\infty} C_r x^{1-\gamma+r} = C_0 x^{1-\gamma} + C_1 x^{2-\gamma} + C_2 x^{3-\gamma} + \dots \\ &= C_0 x^{1-\gamma} \left[1 + \frac{\alpha'}{\gamma'} \frac{x}{1!} + \frac{\alpha'(\alpha' + 1)}{\gamma'(\gamma' + 1)} \cdot \frac{x^2}{2!} + \frac{\alpha'(\alpha' + 1)(\alpha' + 2)}{\gamma'(\gamma' + 1)(\gamma' + 2)} \frac{x^3}{3!} + \dots \right] \end{aligned}$$

Taking $C_0 = 1$

$$y = x^{1-\gamma} \sum_{r=0}^{\infty} \frac{(\alpha')_r}{(\gamma')_r} \cdot \frac{x^r}{r!} = x^{1-\gamma} {}_1F_1(\alpha'; \gamma'; x)$$

$$\text{i.e., } y = x^{1-\gamma} {}_1F_1(\alpha - \gamma + 1; 2 - \gamma; x)$$

which is other independent solution of (1) and is called the **confluent hypergeometric function of second kind**.

Hence the general solution of confluent hypergeometric equation (1) is

$$y = A {}_1F_1(\alpha; \gamma; x) + B x^{1-\gamma} {}_1F_1(\alpha - \gamma + 1; 2 - \gamma; x)$$

where $\gamma > 0$ is not integral and A, B are arbitrary constants.

5.21 Derivatives of Confluent Hypergeometric Function

The confluent hypergeometric function is given by

$${}_1F_1(\alpha; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \cdot \frac{x^n}{n!}$$

Differentiating both sides w.r.t. x , we get

$$\begin{aligned}
\frac{d}{dx} {}_1F_1(\alpha; \gamma; x) &= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \cdot \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \cdot \frac{x^{n-1}}{(n-1)!} \\
&= \sum_{m=0}^{\infty} \frac{(\alpha)_{m+1}}{(\gamma)_{m+1}} \cdot \frac{x^m}{m!} \quad \text{Putting } n-1=m \\
&= \frac{\alpha}{\gamma} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{(\gamma+1)_m} \cdot \frac{x^m}{m!} \quad [\because (\alpha)_{m+1} = \alpha(\alpha+1)_m] \\
\therefore &= \frac{\alpha}{\gamma} {}_1F_1(\alpha+1; \gamma+1; x)
\end{aligned}$$

$$i.e., \frac{d}{dx} {}_1F_1(\alpha; \gamma; x) = \frac{\alpha}{\gamma} {}_1F_1(\alpha+1; \gamma+1; x) \quad \dots(1)$$

Differentiating (1) again, w.r.t x we get

$$\begin{aligned}
\frac{d^2}{dx^2} {}_1F_1(\alpha; \gamma; x) &= \frac{\alpha}{\gamma} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m}{(\gamma+1)_m} \cdot \frac{m x^{m-1}}{m!} \\
&= \frac{\alpha}{\gamma} \sum_{m=1}^{\infty} \frac{(\alpha+1)_m}{(\gamma+1)_m} \cdot \frac{x^{m-1}}{(m-1)!} \\
&= \frac{\alpha}{\gamma} \sum_{n=0}^{\infty} \frac{(\alpha+1)_{n+1}}{(\gamma+1)_{n+1}} \cdot \frac{x^n}{n!} \quad \text{Taking } m-1=n \\
&= \frac{\alpha}{\gamma} \sum_{n=0}^{\infty} \frac{(\alpha+1)(\alpha+2)_n}{(\gamma+1)(\gamma+2)_n} \cdot \frac{x^n}{n!} \quad [\because (\alpha+1)_{n+1} = (\alpha+1)(\alpha+2)_n] \\
&= \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)} \sum_{n=0}^{\infty} \frac{(\alpha+2)_n}{(\gamma+2)_n} \cdot \frac{x^n}{n!} = \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)} {}_1F_1(\alpha+2; \gamma+2; x)
\end{aligned}$$

$$i.e., \frac{d^2}{dx^2} {}_1F_1(\alpha; \gamma; x) = \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)} {}_1F_1(\alpha+2; \gamma+2; x) = \frac{(\alpha)_2}{(\gamma)_2} {}_1F_1(\alpha+2; \gamma+2; x) \quad \dots(2)$$

Proceeding similarly, we can assume that

$$\frac{d^p}{dx^p} {}_1F_1(\alpha; \gamma; x) = \frac{(\alpha)_p}{(\gamma)_p} {}_1F_1(\alpha+p; \gamma+p; x) \quad \dots(3)$$

where p is a positive integer.

Differentiating again, w.r.t. ' x ', we get

$$\begin{aligned}
 \frac{d^{p+1}}{dx^{p+1}} {}_1F_1(\alpha; \gamma; x) &= \frac{(\alpha)_p}{(\gamma)_p} \cdot \frac{d}{dx} {}_1F_1(\alpha + p; \gamma + p; x) \\
 &= \frac{(\alpha)_p}{(\gamma)_p} \cdot \frac{\alpha + p}{\gamma + p} {}_1F_1(\alpha + p + 1; \gamma + p + 1; x) \\
 &= \frac{(\alpha)_{p+1}}{(\gamma)_{p+1}} {}_1F_1(\alpha + p + 1; \gamma + p + 1; x)
 \end{aligned}$$

$$\therefore (\alpha)_p \cdot (\alpha + p) = (\alpha)_{p+1}$$

i.e., if (3) is true for positive integer p , then it is also true for positive integer $p + 1$. But from (1) it is true for $p = 1$. Hence by mathematical induction, (3) is the for every positive integer n .

$$\frac{d^n}{dx^n} {}_1F_1(\alpha; \gamma; x) = \frac{(\alpha)_n}{(\gamma)_n} \cdot {}_1F_1(\alpha + n; \gamma + n; x)$$

5.22 Integral Formula for Confluent Hypergeometric Function ${}_1F_1(\alpha; \gamma; x)$

$${}_1F_1(\alpha; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 (1-t)^{\gamma-\alpha-1} \cdot t^{\alpha-1} e^{xt} dt$$

$$\text{or } {}_1F_1(\alpha; \gamma; x) = \frac{1}{B(\alpha, \gamma-\alpha)} \int_0^1 (1-t)^{\gamma-\alpha-1} \cdot t^{\alpha-1} e^{xt} dt$$

[Rajasthan 2003; Purvanchal 2006]

Proof: We have, the confluent hypergeometric function

$${}_1F_1(\alpha; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \cdot \frac{x^n}{n!} \quad \dots(1)$$

$$\text{Since } (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \quad (\text{See 5.15})$$

$$\begin{aligned}
 \therefore \frac{(\alpha)_n}{(\gamma)_n} &= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\gamma)}{\Gamma(\gamma+n)} = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \cdot \frac{\Gamma(\gamma-\alpha)\Gamma(\alpha+n)}{\Gamma(\gamma+n)} \\
 &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \cdot \frac{\Gamma(\gamma-\alpha)\Gamma(\alpha+n)}{\Gamma(\gamma-\alpha+\alpha+n)} = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \cdot B(\alpha+n, \gamma-\alpha) \\
 &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 (1-t)^{\gamma-\alpha-1} t^{\alpha+n-1} dt
 \end{aligned}$$

$\therefore$ from (1), we have

$$\begin{aligned}
{}_1F_1(\alpha; \gamma; x) &= \sum_{n=0}^{\infty} \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\gamma-\alpha)} \int_0^1 (1-t)^{\gamma-\alpha-1} t^{\alpha+n-1} \cdot \frac{x^n}{n!} dt \\
&= \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\gamma-\alpha)} \int_0^1 (1-t)^{\gamma-\alpha-1} t^{\alpha-1} \left(\sum_{n=0}^{\infty} \frac{(xt)^n}{n!} \right) dt \\
&= \frac{\Gamma(\gamma)}{\Gamma(\alpha) \Gamma(\gamma-\alpha)} \int_0^1 (1-t)^{\gamma-\alpha-1} t^{\alpha-1} \cdot e^{xt} dt
\end{aligned}$$

or ${}_1F_1(\alpha; \gamma; x) = \frac{1}{B(\alpha, \gamma-\alpha)} \cdot \int_0^1 (1-t)^{\gamma-\alpha-1} t^{\alpha-1} e^{xt} dt$

5.23 Kummer's Theorem (for Confluent Hypergeometric Function)

$${}_1F_1(\alpha; \beta; x) = e^x {}_1F_1(\beta - \alpha; \beta; -x) \quad [\text{Kanpur 2000, 01; Rajasthan 2002, 03}]$$

Proof: Integral formula for confluent hypergeometric function is :

$${}_1F_1(\alpha; \beta; x) = \frac{\Gamma(\beta)}{\Gamma(\alpha) \Gamma(\beta-\alpha)} \int_0^1 (1-t)^{\beta-\alpha-1} t^{\alpha-1} \cdot e^{xt} dt \quad \dots(1)$$

Replacing α by $\beta - \alpha$ and x by $-x$, we get

$$\begin{aligned}
{}_1F_1(\beta - \alpha; \beta; x) &= \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha) \Gamma[\beta - (\beta - \alpha)]} \int_0^1 (1-t)^{\beta - (\beta - \alpha) - 1} t^{\beta - \alpha - 1} e^{-xt} dt \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha) \Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} t^{\beta - \alpha - 1} e^{-xt} dt \\
&= \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha) \Gamma(\alpha)} \int_1^0 u^{\alpha-1} (1-u)^{\beta - \alpha - 1} e^{-x(1-u)} (-1) du
\end{aligned}$$

Putting $1-t=u$, $dt=-du$

$$\begin{aligned}
&= e^{-x} \frac{\Gamma(\beta)}{\Gamma(\alpha) \Gamma(\beta - \alpha)} \int_0^1 (1-u)^{\beta - \alpha - 1} u^{\alpha-1} e^{xu} du \\
&= e^{-x} \frac{\Gamma(\beta)}{\Gamma(\alpha) \Gamma(\beta - \alpha)} \int_0^1 (1-t)^{\beta - \alpha - 1} t^{\alpha-1} e^{xt} dt
\end{aligned}$$

By Prop of D.I.

$$= e^{-x} F(\alpha; \beta; x) \text{ from (1)}$$

$$\Rightarrow {}_1F_1(\alpha; \beta; x) = e^x {}_1F_1(\beta - \alpha; \beta; -x)$$

5.24 Whittaker's Confluent Hypergeometric Function

The Confluent Hypergeometric equation is

$$x \frac{d^2 y}{dx^2} + (\gamma - x) \frac{dy}{dx} - \alpha y = 0 \quad \dots(1)$$

whose general solution is

$$y = Ay_1(x) + By_2(x) \quad \dots(2)$$

where A and B are arbitrary constants and

$$y_1(x) = {}_1F_1(\alpha; \gamma; x) \quad \dots(3)$$

and

$$y_2(x) = x^{1-\gamma} {}_1F_1(\alpha - \gamma + 1; 2 - \gamma; x) \quad \dots(4)$$

Putting

$$y(x) = x^{-\gamma/2} e^{x/2} W(x) \quad \dots(5)$$

so that,

$$\begin{aligned} \frac{dy}{dx} &= \left\{ -\left(\frac{\gamma}{2}\right) x^{-\gamma/2-1} e^{x/2} + \frac{1}{2} x^{-\gamma/2} e^{x/2} \right\} W(x) + x^{-\gamma/2} e^{x/2} \frac{dW}{dx} \\ \frac{d^2 y}{dx^2} &= \left\{ -(\gamma/2) \cdot x^{-\gamma/2-1} e^{x/2} + \frac{1}{2} x^{-\gamma/2} e^{x/2} \right\} \frac{dW}{dx} \\ &\quad + \left\{ \left(\frac{\gamma}{2}\right) \left(\frac{\gamma}{2} + 1\right) e^{-(\gamma/2)-2} \cdot e^{x/2} - \left(\frac{\gamma}{2}\right) x^{-(\gamma/2)-1} \cdot \frac{1}{2} e^{x/2} \right. \\ &\quad \left. - \frac{\gamma}{4} x^{-(\gamma/2)-1} e^{x/2} + \frac{1}{4} x^{-\gamma/2} e^{x/2} \right\} W(x) \\ &\quad + x^{-\gamma/2} e^{x/2} \frac{d^2 W}{dx^2} + \left\{ \left(-\frac{\gamma}{2}\right) x^{-(\gamma/2)-1} e^{x/2} + \frac{1}{2} x^{-\gamma/2} e^{x/2} \right\} \frac{dW}{dx} \\ &= \left\{ -\gamma x^{-\gamma/2-1} e^{x/2} + x^{-\gamma/2} e^{x/2} \right\} \frac{dW}{dx} \\ &\quad + \left\{ \left(\gamma \frac{(\gamma+2)}{4}\right) x^{-(\gamma/2)-2} e^{x/2} - \left(\frac{\gamma}{2}\right) x^{-(\gamma/2)-1} e^{x/2} + \left(\frac{1}{4}\right) e^{-\gamma/2} e^{x/2} \right\} \\ &\quad W(x) + x^{-\gamma/2} e^{x/2} \frac{d^2 W}{dx^2} \end{aligned}$$

in (1), it reduces to

$$\begin{aligned} x \cdot x^{-\gamma/2} e^{x/2} \left[\frac{d^2 W}{dx^2} + (-\gamma x^{-1} + 1) \frac{dW}{dx} + \left\{ \left(\gamma \frac{(\gamma+2)}{4}\right) x^{-2} - \left(\frac{\gamma}{2}\right) x^{-1} + \frac{1}{4} \right\} W(x) \right] \\ + (\gamma - x) \cdot x^{-\gamma/2} e^{x/2} \left[\frac{dW}{dx} + \left\{ \left(-\frac{\gamma}{2}\right) x^{-1} + \frac{1}{2} \right\} W(x) \right] - \alpha x^{-\gamma/2} e^{x/2} W(x) = 0 \end{aligned}$$

$$\begin{aligned}
\text{or } x \frac{d^2 W}{dx^2} + \left[\left(\frac{\gamma(\gamma+2)}{4} \right) x^{-1} - \frac{\gamma}{2} + \frac{x}{4} + (\gamma-x) \left\{ \left(-\frac{\gamma}{2} \right) x^{-1} + \frac{1}{2} \right\} - \alpha \right] W(x) &= 0 \\
\text{or } \frac{d^2 W}{dx^2} + \left[-\frac{1}{4} + \frac{\gamma/2 - \alpha}{x} - \frac{(-\gamma^2/4 + \gamma/2)}{x^2} \right] W(x) &= 0 \\
\text{or } \frac{d^2 W}{dx^2} + \left[-\frac{1}{4} + \frac{\gamma/2 - \alpha}{x} + \frac{1/4 - (\gamma/2 - 1/2)^2}{x^2} \right] W(x) &= 0 \\
\text{or } \frac{d^2 W}{dx^2} + \left[-\frac{1}{4} + \frac{k}{x} + \frac{1/4 - m^2}{x^2} \right] W(x) &= 0 \quad \dots(6)
\end{aligned}$$

where $k = \gamma/2 - \alpha$ and $m = \gamma/2 - 1/2$

The solutions $W(x)$ of (6) are called the Whittaker's Confluent Hypergeometric Functions

If $2m$ is neither 1 nor an integer then the solution of equation (6) is given by (2) where $\gamma = 2m + 1$ and $\alpha = \gamma/2 - k = 1/2 + m - k$

Thus the solutions of (6) i.e., Whittaker's Confluent hypergeometric functions are given by

$$\begin{aligned}
M_{k,m}(x) &= x^{\gamma/2} e^{-x/2} y_1(x) \\
&= x^{m+1/2} \cdot e^{-x/2} {}_1F_1(1/2 + m - k; 2m + 1; x)
\end{aligned}$$

$$\begin{aligned}
\text{and } M_{k,-m}(x) &= x^{\gamma/2} e^{-x/2} y_2(x) \\
&= x^{1/2-m} \cdot e^{-x/2} {}_1F_1(1/2 - m - k; 1 - 2m; x)
\end{aligned}$$

5.25 Contiguous Hypergeometric Functions

According to Gauss, the function $F(\alpha'; \beta'; \gamma'; x)$ is said to be contiguous to function $F(\alpha, \beta; \gamma; x)$ if it is obtained by increasing or decreasing one and only one of the parameters α, β, γ by unity.

Thus there exist six hypergeometric functions contiguous to $F(\alpha, \beta; \gamma; x)$ and are denoted as given below

$$\begin{aligned}
F(\alpha + 1, \beta; \gamma; x) &= F_{\alpha+}, & F(\alpha - 1, \beta; \gamma; x) &= F_{\alpha-} \\
F(\alpha, \beta + 1; \gamma; x) &= F_{\beta+}, & F(\alpha, \beta - 1; \gamma; x) &= F_{\beta-} \\
F(\alpha, \beta; \gamma + 1; x) &= F_{\gamma+}, & F(\alpha, \beta; \gamma - 1; x) &= F_{\gamma-}
\end{aligned}$$

5.26 Theorem : Contiguity Relationship

There exists a linear relation with polynomial coefficients between a hypergeometric function $F(\alpha, \beta; \gamma; x)$ and its any two contiguous hypergeometric functions

$$\text{i.e.,} \quad (\alpha - \beta) F(\alpha, \beta; \gamma; x) = \alpha \cdot F_{\alpha+} - \beta \cdot F_{\beta+} \text{ (etc.)}$$

Which holds for $|x| < 1$.

Proof: We have,

$$\begin{aligned} \alpha \cdot F_{\alpha+} - \beta \cdot F_{\beta+} &= \alpha \cdot F(\alpha + 1, \beta; \gamma; x) - \beta \cdot F(\alpha, \beta + 1; \gamma; x) \\ &= \alpha \cdot \frac{(\alpha + 1)_r (\beta)_r}{(\gamma)_r} \cdot \frac{x^r}{r!} - \beta \cdot \frac{(\alpha)_r (\beta + 1)_r}{(\gamma)_r} \cdot \frac{x^r}{r!} \\ &= \left[[\alpha (\alpha + 1)]_r (\beta)_r - (\alpha)_r \beta (\beta + 1)_r \right] \cdot \frac{x^r}{(\gamma)_r r!} \\ &= \left[[(\alpha)_r \cdot (\alpha + r) \cdot (\beta)_r - (\alpha)_r (\beta)_r (\beta + r)] \right] \frac{x^r}{(\gamma)_r r!} \end{aligned}$$

$$\begin{aligned} \therefore \quad \alpha (\alpha + 1)_r &= \alpha (\alpha + 1) (\alpha + 2) \dots (\alpha + 1 + r - 1) \\ &= \{\alpha (\alpha + 1) (\alpha + 2) \dots (\alpha + r - 1)\} (\alpha + r) \\ &= (\alpha)_r \cdot (\alpha + r) \end{aligned}$$

Similarly, $\beta (\beta + 1)_r = (\beta)_r (\beta + r)$

$$\begin{aligned} \therefore \quad \alpha \cdot F_{\alpha+} - \beta \cdot F_{\beta+} &= [(\alpha + r) - (\beta + r)] \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \cdot \frac{x^r}{r!} \\ &= (\alpha - \beta) \frac{(\alpha)_r \cdot (\beta)_r}{(\gamma)_r} \cdot \frac{x^r}{r!} = (\alpha - \beta) F(\alpha, \beta; \gamma; x) \end{aligned}$$

Solved Examples

Example 1: Show that $(1 + n)_n = 4^n (1/2)_n$

Solution: $(1 + n)_n = (1 + n) (1 + n + 1) (1 + n + 2) \dots (1 + n + n - 1)$

$$\begin{aligned} &= (n + 1) (n + 2) (n + 3) \dots (2n) \\ &= \frac{n!}{n!} \cdot (n + 1) (n + 2) \dots (2n) = \frac{2n!}{n!} \\ &= \frac{2n (2n - 1) (2n - 2) (2n - 3) \dots 4 \cdot 3 \cdot 2 \cdot 1}{n!} \end{aligned}$$

$$\begin{aligned}
&= \frac{\{2n(2n-2)(2n-4)\dots 4 \cdot 2\}}{n!} \cdot \{(2n-1)(2n-3)\dots 3 \cdot 1\} \\
&= \frac{2^n n(n-1)(n-2)\dots 2 \cdot 1}{n!} \cdot 2^n \left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) \dots \left(\frac{3}{2}\right) \left(\frac{1}{2}\right) \\
&= 2^{2n} \cdot \frac{n!}{n!} \cdot \left(\frac{1}{2}\right) \left(\frac{1}{2} + 1\right) \left(\frac{1}{2} + 2\right) \dots \left(\frac{1}{2} + n - 1\right) \\
&= (2^2)^n \cdot \left(\frac{1}{2}\right)_n = 4^n \left(\frac{1}{2}\right)_n.
\end{aligned}$$

Example 2: Show that $1 - nx {}_2F_1(1-n, 1; 2; x) = (1-x)^n$

where n is any natural number.

Solution: We have

$${}_2F_1(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha\beta}{\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)} \frac{x^2}{2!} + \dots$$

$$\therefore 1 - nx {}_2F_1(1-n, 1; 2; x)$$

$$\begin{aligned}
&= 1 - nx \left[1 + \frac{(1-n) \cdot 1}{2} \cdot \frac{x}{1!} + \frac{(1-n)(2-n) \cdot 1(1+1)}{2(2+1)} \cdot \frac{x^2}{2!} + \dots \right. \\
&\quad \left. + \frac{(1-n)(2-n)\dots(n-1-n) \cdot 1(1+1)\dots(1+n-2)}{2(2+1)\dots(2+n-2)} \cdot \frac{x^{n-1}}{(n-1)!} \right. \\
&\quad \left. + \frac{(1-n)(2-n)\dots(n-n) \cdot 1(1+1)\dots(1+n-1)}{2(2+1)\dots(2+n-1)} \cdot \frac{x^n}{n!} + \dots \right] \\
&= 1 - nx \left[1 - \frac{(n-1)}{2!}x + \frac{(n-1)(n-2)}{3!}x^2 + \dots \right. \\
&\quad \left. + (-1)^{n-1} \frac{(n-1)(n-2)\dots[n-(n-1)]}{n \cdot (n-1)!} x^{n-1} + 0 \right] \\
&= 1 + n(-x) + \frac{n(n-1)}{2!}(-x)^2 + \frac{n(n-1)(n-2)}{3!}(-x)^3 + \dots \\
&\quad + \frac{n(n-1)(n-2)\dots 1}{n!}(-x)^n
\end{aligned}$$

$$= 1 + {}^nC_1(-x) + {}^nC_2(-x)^2 + {}^nC_3(-x)^3 + \dots + {}^nC_n(-x)^n = (1-x)^n$$

Example 3: Find the third derivative of ${}_2F_1(2, 1; 4; x)$ w.r.t. x

Solution: We have $\frac{d^n}{dx^n} {}_2F_1(\alpha, \beta; \gamma; x) = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} {}_2F_1(\alpha+n, \beta+n; \gamma+n; x)$

$$\begin{aligned}
 \therefore \frac{d^3}{dx^3} {}_2F_1(2, 1; 4; x) &= \frac{(2)_3 \cdot (1)_3}{(4)_3} {}_2F_1(2+3, 1+3; 4+3; x) \\
 &= \frac{2(2+1)(2+2) \cdot 1(1+1)(1+2)}{4(4+1)(4+2)} {}_2F_1(5, 4; 7; x) \\
 &= \frac{6}{5} \cdot {}_2F_1(5, 4; 7; x)
 \end{aligned}$$

Example 4: Show that $\lim_{a, b \rightarrow \infty} {}_2F_1(a, b; 1/2; x^2/4ab) = \cosh x$

Solution: ${}_2F_1(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha\beta}{\gamma} \frac{x}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)} \frac{x^2}{2!} + \dots$

$$\begin{aligned}
 \therefore {}_2F_1(a, b; 1/2; x^2/4ab) &= 1 + \frac{ab}{(1/2)} \frac{(x^2/4ab)}{1!} + \frac{a(a+1)b(b+1)}{(1/2)(1/2+1)} \cdot \frac{(x^2/4ab)^2}{2!} + \dots \\
 &= 1 + \frac{x^2}{2!} + \left(1 + \frac{1}{a}\right) \left(1 + \frac{1}{b}\right) \cdot \frac{x^4}{24} + \dots
 \end{aligned}$$

$$\therefore \lim_{a, b \rightarrow \infty} {}_2F_1(a, b; 1/2; x^2/4ab) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \cosh x$$

Example 5: If $|z| < 1$ and $\left| \frac{z}{1-z} \right| < 1$,

prove that ${}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1((a, c-b; c; -z/(1-z)))$

[Kanpur 2000, 01, 03; Rajasthan 2001; Meerut 2003]

Solution: From integral formula for the hypergeometric series, we have

$${}_2F_1(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt \quad \dots(1)$$

$$= \frac{1}{B(b, c-b)} \int_0^1 (1-v)^{b-1} v^{c-b-1} \{1-z(1-v)\}^{-a} dv$$

Putting $1-t=v$, $dt=-dv$

$$= \frac{1}{B(b, c-b)} \int_0^1 (1-v)^{b-1} v^{c-b-1} \cdot (1-z)^{-a} \left(1 - \frac{z}{-1+z} v\right)^{-a} dv$$

$$= \frac{(1-z)^{-a}}{B(c-b, b)} \int_0^1 v^{(c-b)-1} (1-v)^{c-(c-b)-1} \left(1 - \frac{z}{-1+z} v\right)^{-a} dv$$

$$\begin{aligned}
 &= (1-z)^{-a} {}_2F_1 \left(a, c-b; c; \frac{z}{-1+z} \right), \text{ [By (1)]} \\
 &= (1-z)^{-a} {}_2F_1 \left(a, c-b; c; \frac{-z}{1-z} \right)
 \end{aligned}$$

Example 6: Prove that ${}_1F_1(\alpha; \gamma; x) = e^x \cdot {}_1F_1(\gamma - \alpha; \gamma; -x)$

Solution: From Integral formula for the confluent hypergeometric series, we have

$${}_1F_1(\alpha; \gamma; x) = \frac{1}{B(\alpha, \gamma - \alpha)} \int_0^1 (1-t)^{\gamma - \alpha - 1} t^{\alpha - 1} e^{xt} dt \quad \dots(1)$$

Replacing x by $-x$ and α by $\gamma - \alpha$, we have

$$\begin{aligned}
 {}_1F_1(\gamma - \alpha; \gamma; -x) &= \frac{1}{B(\alpha, \gamma - \alpha)} \int_0^1 (1-t)^{\gamma - (\gamma - \alpha) - 1} t^{(\gamma - \alpha) - 1} e^{-xt} dt \\
 &= \frac{1}{B(\alpha, \gamma - \alpha)} \int_0^1 (1-t)^{\alpha - 1} t^{\gamma - \alpha - 1} e^{-xt} dt \\
 &= \frac{1}{B(\alpha, \gamma - \alpha)} \int_0^1 v^{\alpha - 1} (1-v)^{\gamma - \alpha - 1} e^{-x(1-v)} dv \\
 &\quad \text{Putting } 1-t=v, dt=-dv \\
 &= \frac{e^{-x}}{B(\alpha, \gamma - \alpha)} \int_0^1 (1-v)^{\gamma - \alpha - 1} v^{\alpha - 1} e^{xv} dv \\
 &= e^{-x} {}_1F_1(\alpha; \gamma; x) \text{ By (1)}
 \end{aligned}$$

$$\Rightarrow {}_1F_1(\alpha; \gamma; x) = e^x {}_1F_1(\gamma - \alpha; \gamma; -x)$$

Example 7: Show that $\int_0^{\pi/2} (1-x^2 \sin^2 \theta)^{1/2} d\theta = \frac{\pi}{2} {}_2F_1(-1/2, 1/2; 1; x^2), |x| < 1$

Solution: Putting $\sin^2 \theta = t, 2 \sin \theta \cos \theta d\theta = dt$

$$\begin{aligned}
 \int_0^{\pi/2} (1-x^2 \sin^2 \theta)^{1/2} d\theta &= \int_0^1 (1-x^2 t)^{1/2} \cdot \frac{1}{2 \sin \theta \cos \theta} dt \\
 &= \frac{1}{2} \int_0^1 (1-x^2 t)^{1/2} \cdot \frac{1}{\sqrt{t} \cdot \sqrt{1-t}} dt = \frac{1}{2} \int_0^1 t^{1/2-1} (1-t)^{1/2-1} \cdot (1-x^2 t)^{1/2} dt
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \int_0^1 t^{1/2-1} (1-t)^{1-1/2-1} (1-x^2t)^{-1/2} dt \\
 &= \frac{1}{2} B(1/2, 1-1/2) {}_2F_1(-1/2, 1/2; 1; x^2)
 \end{aligned}$$

By integral formula for hypergeometric function

$$\begin{aligned}
 {}_2F_1(a, b; c; z) &= \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt \\
 &= \frac{1}{2} \cdot \frac{\Gamma(1/2) \Gamma(1/2)}{\Gamma(1/2+1/2)} \cdot {}_2F_1(-1/2, 1/2; 1; x^2) \\
 &= \frac{\pi}{2} {}_2F_1(-1/2, 1/2; 1; x^2)
 \end{aligned}$$

Example 8: Show that

$$P_n(\cos \theta) = \cos^n \theta {}_2F_1\left(-\frac{n}{2}, -\frac{n-1}{2}; 1; -\tan^2 \theta\right)$$

Solution: From Laplace's first integral, we have

$$P_n(x) = \frac{1}{\pi} \int_0^\pi [x \pm \sqrt{(x^2-1)} \cos \phi]^n d\phi$$

Putting $x = \cos \theta$ and taking positive sign on R.H.S. we get

$$\begin{aligned}
 P_n(\cos \theta) &= \frac{1}{\pi} \int_0^\pi [\cos \theta + \sqrt{(\cos^2 \theta - 1)} \cos \phi]^n d\phi \\
 &= \frac{1}{\pi} \int_0^\pi [\cos \theta + i \sin \theta \cos \phi]^n d\phi \\
 &= \frac{1}{\pi} \cos^n \theta \int_0^\pi (1 + i \tan \theta \cos \phi)^n d\phi \\
 &= \frac{1}{\pi} \cos^n \theta \int_0^\pi \left[1 + n \cdot i \tan \theta \cos \phi + \frac{n(n-1)}{2!} i^2 \tan^2 \theta \cos^2 \phi \right. \\
 &\quad \left. + \frac{n(n-1)(n-2)}{3!} i^3 \tan^3 \theta \cos^3 \phi + \frac{n(n-1)(n-2)(n-3)}{4!} i^4 \tan^4 \theta \cos^4 \phi + \dots \right] d\phi \\
 &= \frac{1}{\pi} \cos^n \theta \left[\int_0^\pi d\phi + ni \tan \theta \int_0^\pi \cos \phi d\phi - \frac{n(n-1)}{2!} \tan^2 \theta \int_0^\pi \cos^2 \phi d\phi \right. \\
 &\quad \left. - i \frac{n(n-1)(n-2)}{3!} \tan^3 \theta \int_0^\pi \cos^3 \phi d\phi \right. \\
 &\quad \left. + \frac{n(n-1)(n-2)(n-3)}{4!} \tan^4 \theta \int_0^\pi \cos^4 \phi d\phi + \dots \right]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\pi} \cos^n \theta \left[\pi + 0 - \frac{n(n-1)}{2!} \tan^2 \theta \cdot 2 \int_0^{\pi/2} \cos^2 \phi \, d\phi - 0 \right. \\
&\quad \left. + \frac{n(n-1)(n-2)(n-3)}{4!} \tan^4 \theta \cdot 2 \int_0^{\pi/2} \cos^4 \phi \, d\phi + \dots \right] \\
&\quad \text{(Applying prop. of D.I.)} \\
&= \frac{1}{\pi} \cos^n \theta \left[\pi - \frac{n(n-1)}{2!} \tan^2 \theta \cdot 2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} + \frac{n(n-1)(n-2)(n-3)}{4!} \cdot 2 \cdot \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{\pi}{2} \dots \right] \\
&= \cos^n \theta \left[1 + \frac{\left(-\frac{n}{2}\right) \left(-\frac{n-1}{2}\right)}{1} \frac{(-\tan^2 \theta)}{1!} \right. \\
&\quad \left. + \frac{\left(-\frac{n}{2}\right) \left(-\frac{n}{2} + 1\right) \left(-\frac{n-1}{2}\right) \left(-\frac{n-1}{2} + 1\right)}{1(1+1)} \frac{(-\tan^2 \theta)^2}{2!} + \dots \right] \\
&= \cos^n \theta {}_2F_1 \left(-\frac{n}{2}, -\frac{n-1}{2}; 1; -\tan^2 \theta \right)
\end{aligned}$$

Example 9: Prove that

$$P_n(x) = {}_2F_1(-n, n+1; 1; (1-x)/2)$$

[Kanpur 2003, 04, 06; Punjab 2005; Purvanchal 2005, 06]

Solution: From Rodrigue's Formula, we have

$$\begin{aligned}
P_n(x) &= \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n = \frac{1}{n!} \frac{d^n}{dx^n} \left[(x-1)^n \left\{ \frac{1}{2} (x+1)^n \right\} \right] \\
&= \frac{1}{n!} \frac{d^n}{dx^n} \left[(x-1)^n \left\{ 1 - \frac{1-x}{2} \right\}^n \right] \\
&= \frac{1}{n!} (-1)^n \frac{d^n}{dx^n} \left[(1-x)^n \left\{ 1 - n \left(\frac{1-x}{2} \right) + \frac{n(n-1)}{2!} \left(\frac{1-x}{2} \right)^2 - \frac{n(n-1)(n-2)}{3!} \left(\frac{1-x}{2} \right)^3 + \dots \right\} \right] \\
&= \frac{1}{n!} (-1)^n \frac{d^n}{dx^n} \left[(1-x)^n - \frac{n}{2} (1-x)^{n+1} + \frac{n(n-1)}{2! \cdot 2^2} (1-x)^{n+2} \right. \\
&\quad \left. - \frac{n(n-1)(n-2)}{3! \cdot 2^3} (1-x)^{n+3} + \dots \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{n!} (-1)^n \left[(-1)^n n! - \frac{n}{2} (-1)^n \frac{(n+1)!}{1!} (1-x)^1 + \frac{n(n-1)}{2! \cdot 2^2} (-1)^n \frac{(n+2)!}{2!} (1-x)^2 \right. \\
&\quad \left. - \frac{n(n-1)(n-2)}{3! \cdot 2^3} (-1)^n \frac{(n-3)!}{3!} (1-x)^3 + \dots \right] \\
&\because D^n [ax+b]^m = a^n \frac{m!}{(m-n)!} (ax+b)^{m-n}, n \leq m
\end{aligned}$$

$$\begin{aligned}
&= 1 + \frac{(-n)(n+1)}{1.1!} \left(\frac{1-x}{2} \right) + \frac{(-n)(-n+1)(n+1)}{1(1+1)} \frac{(n+1+1)}{2!} \left(\frac{1-x}{2} \right)^2 + \dots \\
&= {}_2F_1 \left(-n, n+1; 1; \frac{1-x}{2} \right)
\end{aligned}$$

Example 10: Prove that $(a-c+1)F = {}_aF_{a+} - (c-1)F_{c-}$

Solution: We have, $(a-c+1) \cdot \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} F(a, b; c; z)$

$$\begin{aligned}
&= (a-c+1) \cdot \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \{ (a+n) - (c-1+n) \} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \cdot \frac{z^n}{n!} \\
&\quad [\because \Gamma(a) \cdot (a)_n = \{1.2 \dots (a-1)\} \cdot \{a(a+1) \dots (a+n-1)\} = \Gamma(a+n), \text{ etc.}] \\
&= \sum_{n=0}^{\infty} (a+n) \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \cdot \frac{z^n}{n!} - \sum_{n=0}^{\infty} \frac{(c+n-1)\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \cdot \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(a+n+1)\Gamma(b+n)}{\Gamma(c+n)} \cdot \frac{z^n}{n!} - \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n-1)} \cdot \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\Gamma(a+1) \cdot (a+1)_n \cdot \Gamma(b) (b)_n}{\Gamma(c) (c)_n} \cdot \frac{z^n}{n!} - \sum_{n=0}^{\infty} \frac{\Gamma(a) (a)_n \cdot \Gamma(b) (b)_n}{\Gamma(c-1) (c-1)_n} \cdot \frac{z^n}{n!} \\
&\because \Gamma(a) \cdot (a)_n = \Gamma(a+n) \\
&= \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(c)} \cdot a \sum_{n=0}^{\infty} \frac{(a+1)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!} - \frac{(c-1) \cdot \Gamma(a) \Gamma(b)}{(c-1) \Gamma(c-1)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c-1)_n} \cdot \frac{z^n}{n!} \\
&= \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} [a {}_2F_1(a+1, b; c; z) - (c-1) {}_2F_1(a, b; c-1; z)]
\end{aligned}$$

$$\Rightarrow (a-c-1)F = {}_aF_{a+} - (c-1)F_{c-}$$

Exercise

1. Prove that $(a-n)_n = (-1)^n (1-a)_n$

2. Show that $(1-x)^{-1} = {}_2F_1(1, 1; 1; x), |x| < 1$

3. Show that $\lim_{a \rightarrow \infty} {}_2F_1(1, a; 1, x/a) = e^x$

4. Show that

(i) $(1-x)^{-2} = {}_2F_1(\alpha, \beta; \beta; x)$

(ii) $e^x = {}_1F_1(\alpha; \alpha; x)$

5. Prove Ramanujam's formula

$${}_1F_1(\alpha; \beta; z) \cdot {}_1F_1(\alpha; \beta; -z) = {}_2F\left[\begin{matrix} \alpha, \beta - \alpha; \\ \beta, \beta/2, \beta/2 + 1/2 \end{matrix} \middle| z^2/4\right]$$

6. Show that the function $w = {}_0F_1(\rho; z)$ satisfies the equation $z \frac{d^2 w}{dz^2} + \rho \frac{dw}{dz} - w = 0$

Hence show that

$${}_1F_1(\alpha; 2\alpha; 2z) = e^z {}_0F_1(\alpha, \alpha + 1/2; z^2/4)$$

Provided that 2α is not a negative integer or zero.

7. If $P_n(x)$ be Legendre's polynomial of degree n , prove that

$$P_n(x) = \left(\frac{x-1}{2}\right)^n {}_2F_1\left(-n, -n; 1, \frac{x+1}{x-1}\right)$$

8. Show that ${}_2F_1(\alpha, 1-\alpha; \gamma; 1/2) = \frac{\Gamma(\gamma/2) \Gamma(\gamma/2 + 1/2)}{\Gamma(\alpha/2 + 1/2) \Gamma(1/2 - \alpha/2 + \gamma/2)}$

9. Show that ${}_2F_1(\alpha, \beta; (\alpha + \beta + 1)/2; 1/2) = \frac{\Gamma(1/2) \Gamma[(1 + \alpha + \beta)/2]}{\Gamma[(1 + \alpha)/2] \Gamma[(1 + \beta)/2]}$

10. Prove that

(i) $P_n(\cos \theta) = (-1)^n {}_2F_1(n+1, -n; 1; \cos^2 \theta/2)$

(ii) $P_n(\cos \theta) = {}_2F_1(-n, n+1; 1; \sin^2 \theta/2)$

[Hint : Put $x = \cos \theta$ and $x = -\cos \theta$ successively in Ex. 9]

11. Find the 3rd derivative of ${}_2F_1(2, 5; 6; x)$

12. Find the 4th derivative of ${}_2F_1(2, -3; 4; x)$

13. Prove the following relations :

- (i) $F(\alpha + 1; \gamma; x) - F(\alpha; \gamma; x) = (x/\gamma) \cdot F(\alpha + 1; \gamma + 1; x)$
- (ii) $\beta F(\alpha - 1; \beta; x) + x F(\alpha; \beta + 1; x) = \beta F(\alpha; \beta; x)$
- (iii) $\alpha F(\alpha + 1; \beta; x) - (\beta - 1) F(\alpha; \beta - 1; x) = (\alpha - \beta + 1) F(\alpha; \beta; x)$

14. Show that

$${}_2F_1(a-1, b-1; c; x) - {}_2F_1(a, b-1; c; x) = (1-b)(x/2) \cdot {}_2F_1(a, b; c+1; x)$$

15. Show that

$$\int_0^{\pi/2} \frac{d\theta}{(1-x^2 \sin^2 \theta)^{1/2}} = \frac{\pi}{2} \cdot {}_2F_1(1/2, 1/2, 1; x^2), |x| < 1$$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter (a), (b), (c) and (d).

1. $(-n)_n$ is equal to :

- (a) $(-1)^n$
- (b) $n!$
- (c) $(-1)^n n!$
- (d) none of these

2. $(a+1)_n / (a)_{n+1}$ is equal to

- (a) a
- (b) $1/a$
- (c) $(a)_n$
- (d) 1

3. If $\int_0^{\pi/2} (1-k^2 \sin^2 \phi)^{-1/2} d\phi = A \cdot {}_2F_1(1/2, 1/2; 1; k^2)$ then $A =$

where $|k| < 1$

- (a) $\pi/2$
- (b) $2/\pi$
- (c) π
- (d) 2π

4. Fourth derivative of ${}_2F_1(2, -2; 5; x)$ w.r.t x is

- (a) 1
- (b) 0
- (c) ${}_2F_1(1, -1; 2; x)$
- (d) ${}_2F_1(6, 4; 7; x)$

5. If ${}_2F_1(\alpha, \beta; \gamma; 1/2) = A \cdot {}_2F_1(\alpha, \gamma - \beta; \gamma; -1)$ then $A =$

- (a) 2^{-x}
- (b) 2^α
- (c) 2^β
- (d) 2^x

Fill in the Blank

Fill in the blanks "...", so that the following statements are complete and correct.

- 1. $2^{2n} (3/2)_n n! = \dots\dots\dots$
- 2. $(a-1)_n + n(a)_{n-1} = \dots\dots\dots$
- 3. $2^{2n} \cdot (1/2)_n \cdot n! = \dots\dots\dots$

4. ${}_2F_1(a, b; b; x) = \dots\dots\dots$
5. $\frac{d^3}{dx^3} {}_2F_1(a, b; c; x) = \dots\dots\dots$

True / False

Write 'T' for true and 'F' for false statement.

1. $(a+1)_n - n(a+1)_{n-1} = (a)_n$
2. $\binom{n+a-1}{n} = (a)_n$
3. ${}_2F_1(0, b; c; x) = 1$
4. $\frac{d^3}{dx^3} {}_2F_1(a, b; c; 0) = \frac{3!(a)_3(b)_3}{(c)_3}$
5. $\frac{d^n}{dx^n} {}_2F_1(a, b; c; 0) = \frac{(a)_n(b)_n}{(c)_n}$

Answers

Exercise

11.	$240 \cdot {}_2F_1(5, 8; 9; x)$	12.	0

Objective Type Questions

Multiple Choice Questions

1.	(c)	2.	(b)	3.	(a)
4.	(b)	5.	(b)		

Fill in the Blank

1.	$(2n + 1)!$	2.	$(a)_n$	3.	$(2n)!$
4.	$(1 - x)^{-a}$	5.	ab / c		

True / False

1.	True	2.	False	3.	True
4.	False	5.	True		

Unit - 3

Chapter

6

Hermite Polynomials

6.1 Hermite Differential Equation

[Meerut 2013 (Sem. I)]

The differential equation

$$\frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2\lambda y = 0$$

where λ is a constant, is called the **Hermite differential equation**.

6.2 Solution of Hermite Differential Equation

[Meerut 2013 (Sem. I)]

The Hermite equation is

$$\frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2\lambda y = 0 \quad \dots(1)$$

We solve (1) in series by using method of Frobenius.

$$\text{Let } y = \sum_{r=0}^{\infty} C_r x^{m+r}, \quad C_0 \neq 0 \quad \dots(2)$$

be the series solution of (1)

$$\text{Diff. (2), } \frac{dy}{dx} = \sum_{r=0}^{\infty} C_r (m+r) x^{m+r-1} \text{ and } \frac{d^2 y}{dx^2} = \sum_{r=0}^{\infty} C_r (m+r) (m+r-1) x^{m+r-2}$$

Substituting in (1), we get

$$\sum_{r=0}^{\infty} C_r [(m+r)(m+r-1)x^{m+r-2} - 2(m+r)x^{m+r} + 2\lambda x^{m+r}] = 0$$

$$\text{or } \sum_{r=0}^{\infty} C_r [(m+r)(m+r-1)x^{m+r-2} - 2(m+r-\lambda)x^{m+r}] = 0 \quad \dots(3)$$

Which is an identity, so we can equate to zero the coefficients of various powers of x .

Equating to zero the coefficient of lowest power of x i.e., of x^{m-2} , the indicial equation is

$$C_0 m(m-1) = 0 \Rightarrow m = 0, 1 \quad \because C_0 \neq 0$$

Thus the roots of indicial equation are distinct integers.

Now equating to zero the coefficient of next higher power of x i.e., of x^{m-1} , we get

$$C_1(m+1) \cdot m = 0 \Rightarrow C_1 \text{ is indeterminate when } m = 0 \text{ and } C_1 = 0 \text{ when } m = 1$$

Now equating to zero the coefficient of x^{m+r-2} , we get

$$\begin{aligned} C_r(m+r)(m+r-1) - 2(m+r-2-\lambda)C_{r-2} &= 0 \\ \Rightarrow C_r &= \frac{2(m+r-2-\lambda)}{(m+r-1)(m+r)} C_{r-2}, \quad r \geq 2 \end{aligned} \quad \dots(4)$$

Case I : When $m = 0$ from (4),

$$C_r = \frac{-2(\lambda-r+2)}{(r-1)r} C_{r-2}, \quad r \geq 2 \quad \dots(5)$$

Putting $r = 2, 4, 6, \dots$ we get

$$C_2 = \frac{-2\lambda}{1.2} C_0 = \frac{-2\lambda}{2!} C_0, C_4 = \frac{-2(\lambda-2)}{3.4} C_2 = \frac{(-2)^2 \lambda(\lambda-2)}{4!} C_0$$

$$C_6 = \frac{-2(\lambda-4)}{5.6} C_4 = \frac{(-2)^3 \lambda(\lambda-2)(\lambda-4)}{6!} C_0$$

and so on

$$\therefore C_{2m} = \frac{(-2)^m \lambda(\lambda-2)(\lambda-4) \dots (\lambda-2m+2)}{(2m)!} C_0$$

Again putting $r = 1, 3, 5, \dots$, in (5), we get

$$C_3 = \frac{-2(\lambda-1)}{2.3} C_1 = \frac{-2(\lambda-1)}{3!} C_1, C_5 = \frac{(-2)(\lambda-3)}{4.5} C_3 = \frac{(-2)^2 (\lambda-1)(\lambda-3)}{5!} C_1$$

and so on.

$$\therefore C_{2m+1} = \frac{(-2)^m (\lambda-1)(\lambda-3)\dots(\lambda-2m+1)}{(2m+1)!} C_1$$

$$\therefore \text{ From (2) } y = \sum_{r=0}^{\infty} C_r x^r \quad \because m=0$$

$$\text{or } y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots$$

$$\begin{aligned} \text{or } y = C_0 & \left[1 - \frac{2\lambda}{2!} x^2 + \frac{(-2)^2 \lambda (\lambda-2)}{4!} x^4 + \dots \right. \\ & \left. + \frac{(-2)^m \lambda (\lambda-2)(\lambda-4)\dots(\lambda-2m+2)}{(2m)!} x^{2m} + \dots \right] \\ & + C_1 \left[x - \frac{2(\lambda-1)}{3!} x^3 + \frac{(-2)^2 (\lambda-1)(\lambda-3)}{5!} x^5 + \dots \right. \\ & \left. + \frac{(-2)^m (\lambda-1)(\lambda-3)\dots(\lambda-2m+1)}{(2m+1)!} x^{2m+1} + \dots \right] \dots (6) \end{aligned}$$

Case II : When $m=1$. In this case $C_1=0$

$\therefore$ from (4), we get

$$C_r = -\frac{2(\lambda-r+1)}{r(r+1)} C_{r-2}, \quad r \geq 2 \quad \dots (7)$$

Putting $r=3, 5, \dots$

$$C_3 = \frac{-2(\lambda-2)}{3 \cdot 4} C_1 = 0 = C_5 = C_7 = \dots$$

Again putting $r=2, 4, 6, \dots$ in (7)

$$C_2 = \frac{-2(\lambda-1)}{2 \cdot 3} C_0 = \frac{-2(\lambda-1)}{3!} C_0,$$

$$C_4 = \frac{-2(\lambda-3)}{4 \cdot 5} C_2 = \frac{(-2)^2 (\lambda-1)(\lambda-3)}{5!} C_0$$

and so on.

$$\therefore C_{2m} = \frac{(-2)^m (\lambda-1)(\lambda-3)\dots(\lambda-2m+1)}{(2m)!} C_0$$

$$\therefore \text{ from (2) } y = \sum_{r=0}^{\infty} C_r x^{r+1} \quad \because m=1$$

$$\text{or } y = C_0 x + C_1 x^2 + C_2 x^3 + \dots$$

$$\begin{aligned} \text{or } y = C_0 \left[x - \frac{-2(\lambda-1)}{3!} x^3 + \frac{(-2)^2(\lambda-1)(\lambda-3)}{5!} x^5 + \dots \right. \\ \left. + \frac{(-2)^m(\lambda-1)(\lambda-3)\dots(\lambda-2m+1)}{(2m+1)!} x^{2m+1} + \dots \right] \dots (8) \end{aligned}$$

Which is the second bracket in solution (6) i.e., solution (8) is a part of solution (6).

Hence the general solution of Hermite equation (1) is given by (6)

$$\text{i.e., by } y = C_0 u + C_1 v = y_1 + y_2$$

where C_0 and C_1 are arbitrary constants.

y_1 and y_2 are given by

$$\begin{aligned} y_1 = C_0 u = C_0 \left[1 - \frac{2\lambda}{2!} x^2 + (-2)^2 \frac{\lambda(\lambda-2)}{4!} x^4 + \dots \right. \\ \left. + \frac{(-2)^m \lambda(\lambda-2)\dots(\lambda-2m+2)}{(2m)!} x^{2m} + \dots \right] \dots (9) \end{aligned}$$

$$\begin{aligned} \text{and } y_2 = C_1 v = C_1 \left[x - \frac{2(\lambda-1)}{3!} x^3 + \frac{(-2)^2(\lambda-1)(\lambda-3)}{5!} x^5 + \dots \right. \\ \left. + \frac{(-2)^m(\lambda-1)(\lambda-3)\dots(\lambda-2m+1)}{(2m+1)!} x^{2m+1} + \dots \right] \dots (10) \end{aligned}$$

Obviously y_1 and y_2 are two independent solutions of (1), since u/v is not merely a constant.

6.3 Hermite Polynomial

If $\lambda = n$ is an even integer and $a_0 = (-1)^{n/2} \frac{n!}{(n/2)!}$, then in (9) of 6.2, the coefficient of x^n

$$\begin{aligned} &= (-1)^{n/2} \frac{n!}{(n/2)!} \cdot \frac{(-2)^{n/2} n(n-2)(n-4)\dots(n-n+2)}{n!} \\ &= 2^n \frac{(n/2)(n/2-1)(n/2-2)\dots 1}{(n/2)!} = 2^n \end{aligned}$$

Coefficient of x^{n-2}

$$\begin{aligned}
 &= \frac{(-1)^{n/2} n!}{(n/2)!} \cdot \frac{(-2)^{(n-2)/2} n(n-2)(n-4) \dots (n-n+2+2)}{(n-2)!} \\
 &= -2^{n-2} \cdot n(n-1) \cdot \frac{(n/2)(n/2-1)(n/2-2) \dots 2}{(n/2)!} = \frac{-n(n-1)}{1!} 2^{n-2}
 \end{aligned}$$

and so on.

Obviously the coefficients of $x^{n+2}, x^{n+4}, \dots$ in (9) are zero.

$\therefore$ (9) yields

$$\begin{aligned}
 y = y_1 = 2^n x^n - \frac{n(n-1)}{1!} 2^{n-2} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!} 2^{n-4} x^{n-4} \dots \\
 + (-1)^{n/2} \frac{n!}{(n/2)!}
 \end{aligned}$$

Which is a polynomial of degree n .

This polynomial is known as the **Hermite Polynomial** and is denoted by $H_n(x)$. Thus

$$\begin{aligned}
 H_n(x) = (2x)^n - \frac{n(n-1)}{1!} (2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!} (2x)^{n-4} \\
 + \dots + (-1)^{n/2} \frac{n!}{(n/2)!}
 \end{aligned}$$

or
$$H_n(x) = \sum_{r=0}^{(n/2)} \frac{(-1)^r n!}{r! (n-2r)!} \cdot (2x)^{n-2r}$$

where
$$(n/2) = \begin{cases} n/2, & \text{when } n \text{ is even} \\ (n-1)/2 & \text{when } n \text{ is odd.} \end{cases}$$

6.4 Generating Function for Hermite Polynomial $H_n(x)$

To Prove
$$e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x).$$

[Garhwal 2000; Rajasthan 2001]

Proof: We have

$$\begin{aligned}
 e^{2tx - t^2} &= e^{2tx} \cdot e^{-t^2} \\
 &= \sum_{r=0}^{\infty} \frac{(2tx)^r}{r!} \cdot \sum_{s=0}^{\infty} \frac{(-t^2)^s}{s!} = \sum_{r,s=0}^{\infty} (-1)^s \cdot \frac{(2x)^r}{r! s!} \cdot t^{r+2s}
 \end{aligned}$$

Coefficient of t^n (for fixed value of s)

$$= (-1)^s \frac{(2x)^{n-2s}}{(n-2s)!s!}$$

Obtained by putting $r + 2s = n$ i.e., $r = n - 2s$

The total value of coefficient of t^n is obtained by summing over all allowed values of s , and since $r = n - 2s$

$$\therefore r \geq 0 \Rightarrow n - 2s \geq 0 \text{ or } s \leq n/2$$

Thus if n is even s goes from 0 to $n/2$ and if n is odd, s goes from 0 to $(n-1)/2$.

$$\therefore \text{Coefficient of } t^n = \sum_{s=0}^{(n/2)} (-1)^r \frac{(2x)^{n-2s}}{(n-2s)!s!} = \frac{H_n(x)}{n!}$$

Hence
$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

or
$$e^{x^2-(t-x)^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x).$$

6.5 Other Forms for the Hermite Polynomials

$$(I) \quad H_n(x) = 2^n \left\{ \exp. \left(-\frac{1}{4} \frac{d^2}{dx^2} \right) \right\} x^n.$$

[Kanpur 2000]

Proof: We have

$$\frac{1}{2} \frac{d}{dx} e^{2tx} = t e^{2tx}$$

$$\frac{d}{dx} \left(\frac{1}{2} \frac{d}{dx} e^{2tx} \right) = 2t^2 e^{2tx}$$

$$\therefore \frac{1}{2} \frac{d}{dx} \left(\frac{1}{2} \frac{d}{dx} e^{2tx} \right) = t^2 e^{2tx} \quad \text{or} \quad \left(\frac{1}{2} \frac{d}{dx} \right)^2 e^{2tx} = t^2 e^{2tx}.$$

Hence
$$\left(\frac{1}{2} \frac{d}{dx} \right)^n e^{2tx} = t^n e^{2tx} \quad \dots(1)$$

Thus
$$\left\{ \exp. \left(-\frac{1}{4} \frac{d^2}{dx^2} \right) \right\} e^{2tx} = \left[\sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{4} \frac{d^2}{dx^2} \right)^n \right] e^{2tx}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{1}{2} \frac{d}{dx} \right)^{2n} e^{2tx} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n} e^{2tx} \quad [\text{From (1)}] \\
&= e^{2tx} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n} = e^{2tx} \sum_{n=0}^{\infty} \frac{1}{n!} (-t^2)^n \\
&= e^{2tx} \cdot e^{-t^2} = e^{-t^2 + 2tx}
\end{aligned}$$

$$\text{or } \left\{ \exp. \left(-\frac{1}{4} \frac{d^2}{dx^2} \right) \right\} \sum_{n=0}^{\infty} \frac{1}{n!} (2tx)^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \quad [\text{from 6.4}]$$

Equating the coefficient of t^n from the two sides, we have

$$\begin{aligned}
\left\{ \exp. \left(-\frac{1}{4} \frac{d^2}{dx^2} \right) \right\} \frac{1}{n!} 2^n x^n &= \frac{1}{n!} H_n(x) \\
H_n(x) &= 2^n \left\{ \exp. \left(-\frac{1}{4} \frac{d^2}{dx^2} \right) \right\} x^n
\end{aligned}$$

(II) The Rodrigues Formula for $H_n(x)$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}) \quad [\text{Garhwal 2000; Kanpur 2005, 07}]$$

Proof: We have

$$\begin{aligned}
f(x, t) = e^{x^2 - (t-x)^2} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \\
&= \frac{H_0(x)}{0!} t_0 + \frac{H_1(x)}{1!} t + \frac{H_2(x)}{2!} t^2 + \dots + \frac{H_n(x)}{n!} t^n + \frac{H_{n+1}(x)}{(n+1)!} t^{n+1} + \dots
\end{aligned}$$

Differentiating both sides, partially w.r.t. t , n time and then putting $t=0$, we have

$$\frac{H_n(x)}{n!} n! = \left[\frac{\partial^n}{\partial t^n} e^{-(t-x)^2} \right]_{t=0} \cdot e^{x^2}$$

Now let $t-x=u$ i.e., at $t=0$, $x=-u$

$$\therefore \frac{\partial}{\partial t} \equiv \frac{\partial}{\partial u} \text{ and at } t=0, \frac{\partial}{\partial u} = -\frac{\partial}{\partial x}$$

$$\text{or } \left[\frac{\partial^n}{\partial t^n} e^{-(t-x)^2} \right]_{t=0} = \frac{\partial^n}{\partial u^n} (e^{-u^2}) = (-1)^n \frac{\partial^n}{\partial x^n} (e^{-x^2}) = (-1)^n \frac{d^n}{dx^n} (e^{-x^2})$$

$$\therefore H_n(x) = (-1)^n \cdot e^{x^2} \cdot \frac{d^n}{dx^n} (e^{-x^2})$$

6.6 To find First Few Hermite Polynomials

We have $H_n(x) = (-1)^n e^{x^2} \cdot \frac{d^n}{dx^n} (e^{-x^2})$

Putting $n = 0, 1, 2, 3, \dots$, we have

$$H_0(x) = e^{x^2} e^{-x^2} = 1$$

$$H_1(x) = (-1) e^{x^2} \frac{d}{dx} (e^{-x^2}) = 2x$$

$$\begin{aligned} H_2(x) &= (-1)^2 e^{x^2} \frac{d^2}{dx^2} (e^{-x^2}) = e^{x^2} \frac{d}{dx} (-2x e^{-x^2}) \\ &= e^{x^2} (4x^2 e^{-x^2} - 2e^{-x^2}) = e^{x^2} \{ (4x^2 - 2) e^{-x^2} \} = 4x^2 - 2 \end{aligned}$$

$$\begin{aligned} H_3(x) &= (-1)^3 e^{x^2} \frac{d^3}{dx^3} (e^{-x^2}) = -e^{x^2} \frac{d}{dx} \left\{ \frac{d^2}{dx^2} (e^{-x^2}) \right\} \\ &= -e^{x^2} \frac{d}{dx} \{ (4x^2 - 2) e^{-x^2} \} = -e^{x^2} \{ -2x (4x^2 - 2) e^{-x^2} + 8x e^{-x^2} \} \end{aligned}$$

[Meerut 2012 (Sem. I)]

$$= -e^{x^2} \{ (-8x^3 + 12x) e^{-x^2} \} = 8x^3 - 12x$$

Similarly $H_4(x) = 16x^4 - 48x^2 + 12$

$$H_5(x) = 32x^5 - 160x^3 + 120x \text{ etc.}$$

6.7 Orthogonal Properties of Hermite Polynomials

To prove that

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = \begin{cases} 0 & \text{if } m \neq n \\ \sqrt{\pi} 2^n n! & \text{if } m = n \end{cases} = \sqrt{\pi} 2^n n! \cdot \delta_{m,n}$$

Where $\delta_{m,n}$ is Kronecker Delta defined by $\delta_{m,n} = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$

[Rajasthan 2001; Kanpur 2007; Meerut 2010 (Sem. I), 11 (Sem. I)]

Proof: We have $e^{-t^2 + 2tx} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$

and $e^{-s^2 + 2sx} = \sum_{m=0}^{\infty} H_m(x) \frac{s^m}{m!}$

$$\therefore e^{-t^2 + 2tx} \cdot e^{-s^2 + 2sx} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \cdot \sum_{m=0}^{\infty} H_m(x) \frac{s^m}{m!}$$

$$\therefore \frac{1}{n!m!} H_n(x) H_m(x) = \text{Coeff. of } t^n s^m \text{ in the expansion of } e^{-t^2 + 2tx} \cdot e^{-s^2 + 2sx}$$

$$\therefore \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = n!m! \text{ times the coefficient of } t^n s^m \text{ in the expansion of}$$

$$\int_{-\infty}^{\infty} e^{-x^2} e^{-t^2 + 2tx} \cdot e^{-s^2 + 2sx} dx \quad \dots (1)$$

Now $\int_{-\infty}^{\infty} e^{-x^2} e^{-t^2 + 2tx} \cdot e^{-s^2 + 2sx} dx$

$$= e^{-t^2 - s^2} \int_{-\infty}^{\infty} e^{-x^2 + 2tx + 2sx} dx$$

$$= e^{-t^2 - s^2} \int_{-\infty}^{\infty} e^{-[x - (t+s)]^2 + (t+s)^2} dx$$

$$= e^{2ts} \int_{-\infty}^{\infty} e^{-[x - (t+s)]^2} dx$$

$$= e^{2ts} \int_{-\infty}^{\infty} e^{-u^2} du \quad \text{Putting } x - (t+s) = u$$

$$= e^{2ts} \cdot \sqrt{\pi}. \quad \text{Since } \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$$

$$= \sqrt{\pi} \left[1 + \frac{2ts}{1!} + \frac{(2ts)^2}{2!} + \dots + \frac{(2ts)^n}{n!} + \dots \right]$$

$\therefore$ Coefficient of $t^n s^m$ in the expansion of

$$\int_{-\infty}^{\infty} e^{-x^2} \cdot e^{-t^2 + 2tx} \cdot e^{-s^2 + 2sx} dx$$

$$\text{is } 0, \text{ if } m \neq n$$

$$\text{and } \frac{2^n \sqrt{\pi}}{n!}, \text{ if } m = n$$

∴ from (1), we have

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = n! m! \cdot 0 = 0 \text{ if } m \neq n$$

and $\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_n(x) dx = n! \cdot n! \cdot \frac{2^n \sqrt{\pi}}{n!}, \text{ if } m = n$

i.e., $\int_{-\infty}^{\infty} e^{-x^2} [H_n(x)]^2 dx = \sqrt{\pi} 2^n \cdot n!$ [Kanpur 2007]

Hence, $\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = \begin{cases} 0 & \text{if } m \neq n \\ \sqrt{\pi} 2^n (n)! & \text{if } m = n \end{cases} = \sqrt{\pi} 2^n n! \cdot \delta_{m,n}$
[Meerut 2011 (Sem. I)]

6.8 Recurrence Formulae for Hermite Polynomials

(I) $H'_n(x) = 2n H_{n-1}(x) \quad n \geq 1$ [Meerut 2001]

Proof: We have

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = e^{-t^2 + 2tx}$$

Differentiating both sides w.r.t 'x', we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{H'_n(x)}{n!} t^n &= e^{-t^2 + 2tx} \cdot 2t \\ &= 2t \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \\ &= 2 \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^{n+1} \\ &= 2 \sum_{n=1}^{\infty} \frac{H_{n-1}(x)}{(n-1)!} t^n \end{aligned}$$

Equating the coefficient of t^n , on both the sides, we have

$$\frac{H'_n(x)}{n!} = 2 \frac{H_{n-1}(x)}{(n-1)!}$$

i.e., $H'_n(x) = 2n H_{n-1}(x)$

$$(II) \quad 2x H_n(x) = 2n H_{n-1}(x) + H_{n+1}(x)$$

[Meerut 2010 (Sem. I)]

Proof: We have $\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = e^{-t^2 + 2tx}$

Differentiating both sides w.r.t. 't' we have

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} n t^{n-1} = e^{-t^2 + 2tx} \cdot (-2t + 2x)$$

$$\text{or} \quad \sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)!} t^{n-1} = -2te^{-t^2 + 2tx} + 2xe^{-t^2 + 2tx}$$

(Since term of L.H.S. corresponding to $n=0$ is zero)

$$\text{or} \quad \sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)!} t^{n-1} = -2t \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n + 2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

$$\text{or} \quad 2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = 2 \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^{n+1} + \sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)!} t^{n-1}$$

$$\text{or} \quad 2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = 2 \sum_{n=1}^{\infty} \frac{H_{n-1}(x)}{(n-1)!} t^n + \sum_{n=0}^{\infty} \frac{H_{n+1}(x)}{n!} t^n$$

Equating the coefficients of t^n , on the two sides, we have

$$2x \frac{H_n(x)}{n!} = 2 \frac{H_{n-1}(x)}{(n-1)!} + \frac{H_{n+1}(x)}{n!}$$

$$\text{or} \quad 2x H_n(x) = 2n H_{n-1}(x) + H_{n+1}(x)$$

Note : Equating the coefficients of t , we have $2x H_0(x) = H_1(x)$.

$$(III) \quad H'_n(x) = 2x H_n(x) - H_{n+1}(x)$$

Proof: Writing recurrence formula I and II, we have

$$H'_n(x) = 2n H_{n-1}(x) \quad \dots(1)$$

$$\text{and} \quad 2x H_n(x) = 2n H_{n-1}(x) + H_{n+1}(x) \quad \dots(2)$$

Subtracting (2) from (1), we have

$$\text{i.e.,} \quad H'_n(x) - 2x H_n(x) = -H_{n+1}(x)$$

$$\text{or} \quad H'_n(x) = 2x H_n(x) - H_{n+1}(x)$$

$$(IV) \quad H''_n(x) - 2x H'_n(x) + 2n H_n(x) = 0$$

Proof: Hermite's differential equation is

$$\frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2ny = 0 \quad \dots(1)$$

$\therefore H_n(x)$ is the solution of (1), therefore, we have

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$$

$$(V) \quad xH_n'(x) = nH_{n-1}'(x) + nH_n(x)$$

Proof: We have, $e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$... (1)

Differentiating (1), w.r.t. 'x', we get

$$2te^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n'(x) \quad \dots(2)$$

Again differentiating (1), w.r.t. 't', we get

$$2(x-t)e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{nt^{n-1}}{n!} H_n(x)$$

or $2(x-t)e^{2tx - t^2} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} H_n(x)$... (3)

Multiplying (2) by $(x-t)$ and (3) by t and subtracting, we get

$$(x-t) \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n'(x) = t \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} H_n(x)$$

or $x \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n'(x) - \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} H_n'(x) = \sum_{n=1}^{\infty} \frac{t^n}{(n-1)!} H_n(x)$

or $x \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n'(x) - \sum_{n=1}^{\infty} \frac{t^n}{(n-1)!} H_{n-1}'(x) = \sum_{n=1}^{\infty} \frac{t^n}{(n-1)!} H_n(x)$

(Replacing n by $n-1$)

Equating coefficients of t^n from both sides, we get

$$x \cdot \frac{1}{n!} H_n'(x) - \frac{1}{(n-1)!} H_{n-1}'(x) = \frac{1}{(n-1)!} H_n(x)$$

or $xH_n'(x) = nH_{n-1}'(x) + nH_n(x)$

Solved Examples

Example 1: Prove that $H''_n(x) = 4n(n-1)H_{n-2}(x)$

Solution: From recurrence formula I, we have

$$H'_n(x) = 2nH_{n-1}(x) \quad \dots(1)$$

$$\text{Differentiating, } H''_n(x) = 2nH'_{n-1}(x) \quad \dots(2)$$

Replacing n by $(n-1)$ in (1), we get

$$H'_{n-1}(x) = 2(n-1)H_{n-2}(x) \quad \dots(3)$$

Substituting from (3) and (2), we get

$$H''_n(x) = 2^2 n(n-1)H_{n-2}(x) = 4n(n-1)H_{n-2}(x)$$

which can be written as, $H''_n = 4n(n-1)H_{n-2}$.

Example 2: Prove that, if $m < n$

$$\frac{d^m}{dx^m} \{H_n(x)\} = \frac{2^m n!}{(n-m)!} H_{n-m}(x) \quad [\text{Garhwal 2004, 05}]$$

$$\text{Solution: We have, } \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = e^{-t^2 + 2tx} \quad \dots(1)$$

Differentiating both sides w.r.t. 'x' m times, we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{d^m}{dx^m} \{H_n(x)\} &= \frac{d^m}{dx^m} (e^{-t^2 + 2tx}) = (2t)^m \cdot e^{-t^2 + 2tx} \\ &= (2t)^m \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) \quad \text{From (1)} \\ &= 2^m \sum_{n=0}^{\infty} \frac{t^{n+m}}{n!} H_n(x) = 2^m \sum_{s=m}^{\infty} \frac{t^s}{(s-m)!} H_{s-m}(x) \end{aligned}$$

Putting $m+n=s$

Equating coefficients of t^n , from the two sides, we get

$$\frac{1}{n!} \frac{d^m}{dx^m} \{H_n(x)\} = 2^m \cdot \frac{1}{(n-m)!} H_{n-m}(x)$$

$$\text{or } \frac{d^m}{dx^m} \{H_n(x)\} = \frac{2^m n!}{(n-m)!} H_{n-m}(x)$$

Aliter : From recurrence formula I, we have

$$H'_n(x) = 2nH_{n-1}(x) \quad \dots(2)$$

Differentiating both sides w.r.t 'x'

$$H''_n(x) = 2nH'_{n-1}(x) = 2n \cdot 2(n-1) H_{n-2}(x)$$

Substituting for $H'_{n-1}(x)$ from (1), obtained by replacing n by $n-1$.

or
$$H''_n(x) = 2^2 n(n-1) H_{n-2}(x)$$

Differentiating successively $m-2$ times more and substituting in each step from (2), we get

$$\begin{aligned} \frac{d^m}{dx^m} \{H_n(x)\} &= 2^m n(n-1)(n-2) \dots (n-m+1) H_{n-m}(x) \\ &= \frac{2^m n!}{(n-m)!} H_{n-m}(x) \end{aligned}$$

Example 3: Express polynomial $P(x) = x^4 + 2x^3 + 2x^2 - x - 3$ in terms of Hermite's polynomials.

Solution: From 6.6, we have

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2,$$

$$H_3(x) = 8x^3 - 12x, \quad H_4(x) = 16x^4 - 48x^2 + 12,$$

From
$$H_4(x) = 16x^4 - 48x^2 + 12, \quad x^4 = \frac{1}{16} H_4(x) + 3x^2 - \frac{3}{4}$$

From
$$H_3(x) = 8x^3 - 12x, \quad x^3 = \frac{1}{8} H_3(x) + \frac{3}{2}x$$

From
$$H_2(x) = 4x^2 - 2, \quad x^2 = \frac{1}{4} H_2(x) + \frac{1}{2}$$

From
$$H_1(x) = 2x, \quad x = \frac{1}{2} H_1(x)$$

$\therefore$
$$P(x) = x^4 + 2x^3 + 2x^2 - x - 3$$

$$= \left(\frac{1}{16} H_4(x) + 3x^2 - \frac{3}{4} \right) + 2x^3 + 2x^2 - x - 3$$

$$= \frac{1}{16} H_4(x) + 2x^3 + 5x^2 - x - \frac{15}{4}$$

$$\begin{aligned}
&= \frac{1}{16} H_4(x) + 2 \left[\frac{1}{8} H_3(x) + \frac{3}{2} x \right] + 5x^2 - x - \frac{15}{4} \\
&= \frac{1}{16} H_4(x) + \frac{1}{4} H_3(x) + 5x^2 + 2x - \frac{15}{4} \\
&= \frac{1}{16} H_4(x) + \frac{1}{4} H_3(x) + 5 \left[\frac{1}{4} H_2(x) + \frac{1}{2} \right] + 2x - \frac{15}{4} \\
&= \frac{1}{16} H_4(x) + \frac{1}{4} H_3(x) + \frac{5}{4} H_2(x) + 2x - \frac{5}{4} \\
&= \frac{1}{16} H_4(x) + \frac{1}{4} H_3(x) + \frac{5}{4} H_2(x) + H_1(x) - \frac{5}{4} H_0(x)
\end{aligned}$$

Example 4: Prove that $H_n(-x) = (-1)^n H_n(x)$.

[Kanpur 2004]

Hence prove that $H_n(x)$ is even or odd function of x according as n is even or odd respectively.

Solution: We have, $e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$... (1)

Replacing x by $-x$ in (1), we get

$$e^{-2tx - t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(-x)$$

$$\text{or } \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(-x) = e^{2x(-t) - (-t)^2} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} H_n(x) \quad [\text{from (1)}]$$

Equating coefficients of t^n , from both sides, we get

$$\frac{1}{n!} H_n(-x) = \frac{(-1)^n}{n!} H_n(x) \Rightarrow H_n(-x) = (-1)^n H_n(x) \quad \dots (2)$$

Second part : If n is even, $(-1)^n = 1$

$$\therefore (2) \Rightarrow H_n(-x) = H_n(x)$$

$\Rightarrow H_n(x)$ is even function of x when n is even

If n is odd, then $(-1)^n = -1$

$$\therefore (2) \Rightarrow H_n(-x) = -H_n(x)$$

$\Rightarrow H_n(x)$ is odd function of x when n is odd.

Example 5: Prove that (i) $H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}$ and (ii) $H_{2n+1}(0) = 0$ [Kanpur 2006]

Solution: We have

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = e^{-t^2} + 2tx \text{ from 6.4}$$

Putting $x = 0$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(0) = e^{-t^2} = \left[1 - t^2 + \frac{(t^2)^2}{2!} + \dots + (-1)^n \frac{(t^2)^n}{n!} + \dots \right] \quad \dots(1)$$

(i) Equating the coefficients of t^{2n} , on the two sides, we have

$$\frac{1}{(2n)!} H_{2n}(0) = (-1)^n \frac{1}{n!}$$

$$\therefore H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}$$

(ii) Again equating the coefficients of t^{2n+1} , on the two sides of (1), we have

$$\frac{1}{(2n+1)!} H_{2n+1}(0) = 0$$

(Since R.H.S. of (1) do not involve, odd powers of t).

$$\therefore H_{2n+1}(0) = 0.$$

Example 6: Evaluate :

$$\int_{-\infty}^{\infty} x e^{-x^2} H_n(x) H_m(x) dx.$$

Solution: From Recurrence formula II, we have

$$x H_n(x) = n H_{n-1}(x) + \frac{1}{2} H_{n+1}(x)$$

$$\begin{aligned} \therefore \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx &= \int_{-\infty}^{\infty} e^{-x^2} \{n H_{n-1}(x) + \frac{1}{2} H_{n+1}(x)\} H_m(x) dx \\ &= n \int_{-\infty}^{\infty} e^{-x^2} H_{n-1}(x) H_m(x) dx + \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} H_{n+1}(x) H_m(x) dx \\ &= n \sqrt{\pi} 2^{n-1} (n-1)! \delta_{n-1, m} + \frac{1}{2} \sqrt{\pi} 2^{n+1} (n+1)! \delta_{n+1, m} \end{aligned}$$

$$= \sqrt{\pi} 2^{n-1} n! \delta_{n-1, m} + \sqrt{\pi} 2^n (n+1)! \delta_{n+1, m}$$

where δ is Kronecker delta.

Example 7: Prove that $\int_0^x e^{-y^2} H_n(y) dy = H_{n-1}(0) - e^{-x^2} H_{n-1}(x)$.

[Rajasthan 2000]

Solution: From Rodrigue's formula, we have

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} (e^{-y^2})$$

$$\text{or} \quad e^{-y^2} H_n(y) = (-1)^n \frac{d^n}{dy^n} (e^{-y^2}) \quad \dots(1)$$

Integrating both sides w.r.t. 'y' from 0 to x

$$\begin{aligned} \int_0^x e^{-y^2} H_n(y) dy &= (-1)^n \left[\frac{d^{n-1}}{dy^{n-1}} (e^{-y^2}) \right]_0^x \\ &= (-1)^1 \left[(-1)^{n-1} \frac{d^{n-1}}{dy^{n-1}} (e^{-y^2}) \right]_0^x \\ &= (-1) \left[e^{-y^2} H_{n-1}(y) \right]_0^x \quad \text{From (1) replacing } n \text{ by } n-1 \\ &= -e^{-x^2} H_{n-1}(x) + H_{n-1}(0). \end{aligned}$$

Example 8: Prove that $P_n(x) = \frac{2}{\sqrt{\pi} n!} \int_0^\infty t^n e^{-t^2} H_n(xt) dt$.

Solution: We have $H_n(x) = \sum_{r=0}^{(n/2)} (-1)^r \frac{n!}{r!(n-2r)!} (2x)^{n-2r}$

$$\therefore H_n(xt) = \sum_{r=0}^{(n/2)} (-1)^r \frac{n!}{r!(n-2r)!} (2xt)^{n-2r}$$

$$\begin{aligned} \therefore \frac{2}{\sqrt{\pi} n!} \int_0^\infty t^n e^{-t^2} H_n(xt) dt &= \frac{2}{\sqrt{\pi} n!} \int_0^\infty t^n e^{-t^2} \sum_{r=0}^{(n/2)} (-1)^r \frac{n!}{r!(n-2r)!} (2xt)^{n-2r} dt \\ &= \sum_{r=0}^{(n/2)} \frac{2^{n-2r+1} (-1)^r x^{n-2r}}{\sqrt{\pi} r!(n-2r)!} \int_0^\infty e^{-t^2} t^{2n-2r} dt \end{aligned}$$

$$= \sum_{r=0}^{(n/2)} \frac{2^{n-2r+1} (-1)^r x^{n-2r}}{\sqrt{\pi} r! (n-2r)!} \int_0^\infty e^{-t^2} t^{2(n-r+1/2)-1} dt$$

$$= \sum_{r=0}^{(n/2)} \frac{2^{n-2r+1} (-1)^r x^{n-2r}}{\sqrt{\pi} r! (n-2r)!} \cdot \frac{1}{2} \Gamma\left(n-r+\frac{1}{2}\right)$$

$$\text{Since } 2 \int_0^\infty e^{-t^2} \cdot t^{(2n-1)} dt = \Gamma(n)$$

$$= \sum_{r=0}^{(n/2)} \frac{2^{n-2r} (-1)^r x^{n-2r} \cdot [2(n-r)]!}{\sqrt{\pi} r! (n-2r)! 2^{2(n-r)} (n-r)!} \sqrt{\pi}$$

$$\left(\text{Since } \Gamma\left(x + \frac{1}{2}\right) = \frac{(2x)!}{2^{2x} x!} \sqrt{\pi} \right)$$

$$= \sum_{r=0}^{(n/2)} (-1)^r \frac{(2n-2r)!}{2^n(r)!(n-2r)!(n-r)!} x^{n-2r} = P_n(x)$$

Hence

$$P_n(x) = \frac{2}{\sqrt{\pi} n!} \int_0^\infty t^n e^{-t^2} H_n(xt) dt$$

Example 9: Show that :

$$\sum_{k=0}^n \frac{H_k(x) H_k(z)}{2^k k!} = \frac{H_{n+1}(y) H_n(x) - H_{n+1}(x) H_n(y)}{2^{n+1} n! (y-x)}$$

Solution: From Recurrence formula II, we have

$$x H_n(x) = n H_{n-1}(x) + \frac{1}{2} H_{n+1}(x) \quad \dots(1)$$

$$\therefore y H_n(y) = n H_{n-1}(y) + \frac{1}{2} H_{n+1}(y) \quad \dots(2)$$

Multiplying (2) by $H_n(x)$ and (1) by $H_n(y)$ and then subtracting, we have

$$\begin{aligned} (y-x) H_n(x) H_n(y) &= \frac{1}{2} [H_{n+1}(y) H_n(x) - H_{n+1}(x) H_n(y)] \\ &\quad - n [H_{n-1}(x) H_n(y) - H_{n-1}(y) H_n(x)] \quad \dots(3) \end{aligned}$$

Putting $n = 0, 1, 2, 3, \dots, (n-1), n$ in (3) respectively, we have

$$(y-x) H_0(x) H_0(y) = \frac{1}{2} [H_1(y) H_0(x) - H_1(x) H_0(y)] - 0 \quad \dots(A_0)$$

$$\int_{-\infty}^{\infty} \Psi_m(x) \Psi'_n(x) dx = \begin{cases} 0 & , \text{ if } m \neq n \pm 1 \\ 2^{n-1} n! \sqrt{\pi} & , \text{ if } m = n - 1 \\ -2^n (n+1)! \sqrt{\pi} & , \text{ if } m = n + 1 \end{cases}$$

Solution: Given, $\psi_n(x) = e^{-x^2/2} H_n(x)$, $\therefore \psi_m(x) = e^{-x^2/2} H_m(x)$

$$\text{Differentiating } \psi'_n(x) = -x e^{-x^2/2} H_n(x) + e^{-x^2/2} H'_n(x) \quad \dots(1)$$

Now from Recc. formula I and II, we have

$$H'_n(x) = 2n H_{n-1}(x)$$

$$\text{and } x H_n(x) = n H_{n-1}(x) + \frac{1}{2} H_{n+1}(x)$$

Substituting in (1), we get

$$\begin{aligned} \psi'_n(x) &= -[n H_{n-1}(x) + \frac{1}{2} H_{n+1}(x)] e^{-x^2/2} + e^{-x^2/2} [2n H_{n-1}(x)] \\ &= n e^{-x^2/2} H_{n-1}(x) - \frac{1}{2} e^{-x^2/2} H_{n+1}(x) \end{aligned}$$

$$\therefore \int_{-\infty}^{\infty} \psi_m(x) \psi'_n(x) dx = \int_{-\infty}^{\infty} e^{-x^2/2} H_m(x).$$

$$\left[n e^{-x^2/2} H_{n-1}(x) - \frac{1}{2} e^{-x^2/2} H_{n+1}(x) \right] dx$$

$$= n \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_{n-1}(x) dx - \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_{n+1}(x) dx$$

$$= n \cdot 2^{n-1} (n-1)! \sqrt{\pi} \delta_{n-1, m} - \frac{1}{2} \cdot 2^{n+1} (n+1)! \sqrt{\pi} \delta_{n+1, m}$$

(Using orthogonal prop. 6.7)

$$= 2^{n-1} n! \sqrt{\pi} \delta_{n-1, m} - 2^n (n+1)! \sqrt{\pi} \delta_{n+1, m}$$

$$= 0 - 0 = 0, \quad \text{if } m \neq n-1 \text{ and } m \neq n+1$$

$$\text{or } = 2^{n-1} n! \sqrt{\pi} \cdot 1 - 0, \quad \text{if } m = n-1 \text{ i.e., } m \neq n+1$$

$$\text{or } = 0 - 2^n (n+1)! \sqrt{\pi} \cdot 1, \quad \text{if } m = n+1 \text{ i.e., } m \neq n-1$$

$$\text{Hence, } \int_{-\infty}^{\infty} \psi_m(x) \psi'_n(x) dx = \begin{cases} 0 & , \text{ if } m \neq n \pm 1 \\ 2^{n-1} n! \sqrt{\pi} & , \text{ if } m = n-1 \\ -2^n (n+1)! \sqrt{\pi} & , \text{ if } m = n+1 \end{cases}$$

Example 11: Prove that $x^n = \sum_{r=0}^{(n/2)} \frac{n!}{2^n r! (n-2r)!} \frac{H_{n-2r}(x)}{2^n}$

[Kanpur 2004]

Solution: We have

$$\begin{aligned}
 e^{-t^2 + 2xt} &= \sum_{m=0}^{\infty} \frac{H_m(x)}{m!} t^m \\
 \Rightarrow e^{2xt} &= e^{t^2} \cdot \sum_{m=0}^{\infty} \frac{H_m(x)}{m!} t^m \\
 \Rightarrow \sum_{n=0}^{\infty} \frac{(2xt)^n}{n!} &= \sum_{r=0}^{\infty} \frac{(t^2)^r}{r!} \cdot \sum_{m=0}^{\infty} \frac{H_m(x)}{m!} t^m \\
 \Rightarrow \sum_{n=0}^{\infty} \frac{(2xt)^n}{n!} &= \sum_{r,m=0}^{\infty} \frac{H_m(x)}{r! m!} t^{2r+m} \quad \dots(1)
 \end{aligned}$$

On R.H.S. coeff. of t^n , (for fixed value of r)

$$= \frac{H_{n-2r}}{r! (n-2r)!} \quad [\text{Obtained by putting } 2r + m = n \text{ i.e., } m = n - 2r]$$

Total value of coeff. of t^n is obtained by summing R.H.S. over all allowed values of m and since $m = n - 2r \therefore m \geq 0 \Rightarrow n - 2r \geq 0 \Rightarrow r \leq n/2$

$\therefore$ If n is even r goes from 0 to $n/2$ and if n is odd r goes from 0 to $(n-1)/2$

$$\therefore \text{Coeff. of } t^n \text{ on R.H.S.} = \sum_{r=0}^{(n/2)} \frac{H_{n-2r}}{r! (n-2r)!}$$

$$\text{And coeff. of } t^n \text{ on L.H.S.} = \frac{2^n x^n}{n!}$$

$\therefore$ Equating the coefficient of t^n on both sides of (1), we get

$$\begin{aligned}
 \frac{2^n x^n}{n!} &= \sum_{r=0}^{(n/2)} \frac{H_{n-2r}(x)}{r! (n-2r)!} \\
 \Rightarrow x^n &= \sum_{r=0}^{(n/2)} \frac{n!}{2^n \cdot r! (n-2r)!} H_{n-2r}(x)
 \end{aligned}$$

Exercise

1. For what value of n , $H_n(0) = 0$?
2. Show that $H_1(x) = 2x H_0(x)$
3. Prove that $H_5(x) = 32x^5 - 160x^3 + 120x$
4. Show that $H_n^n(x) = 2^n n! H_0(x)$
5. Express the following polynomials in terms of Hermite's polynomials :

(i) $P_1(x) = x^3 + 2x^2 + 3x + 2$

(ii) $P_2(x) = 4x^4 + 2x^3 + 4x^2 + x - 3$

6. Prove that
$$\int_{-\infty}^{\infty} x e^{-x^2} H_m(x) \cdot H_n(x) dx = \begin{cases} 0 & , \text{ if } m \neq n \pm 1 \\ \sqrt{\pi} 2^{n-1} \sqrt{n} & , \text{ if } m = n - 1 \\ \sqrt{\pi} 2^n (n+1)! & , \text{ if } m = n + 1 \end{cases}$$

(Hint : Substitute the values of $\delta_{n-1, m}$ and $\delta_{n+1, m}$ in Ex. 6)

7. Show that
$$\int_0^x H_n(y) dy = \frac{1}{2(n+1)} [H_{n+1}(x) - H_{n+1}(0)]$$

[Hint : From Recc. formula I, $H_{n-1}(y) = \frac{1}{2n} H'_n(y)$

Replacing n by $n+1$, $H_n(y) = \frac{1}{2(n+1)} H'_{n+1}(y)$

Now integrate from 0 to x]

8. Show that
$$\int_{-\infty}^{\infty} x^2 e^{-x^2} \{H_n(x)\}^2 dx = \sqrt{\pi} 2^n n! (n+1/2)$$

[Hint : From Recc. formula II, $x H_n(x) = n H_{n-1}(x) + \frac{1}{2} H_{n+1}(x)$

Substitute in the integral and apply orthogonal property.]

9. If $\psi_n(x) = e^{-x^2/2} H_n(x)$, where $H_n(x)$ is a Hermite polynomial of degree n , then prove that

$$\int_{-\infty}^{\infty} \psi_m(x) \psi_n(x) dx = 2^n n! \sqrt{\pi} \delta_{m, n}$$

10. Show that for $n = 0, 1, 2, \dots$

$$(i) \quad H_{2n}(x) = \frac{2^{n+1}(-1)^n}{\sqrt{\pi}} e^{x^2} \int_0^\infty e^{-t^2} \cdot t^{2n} \cos 2xt \, dt$$

$$(ii) \quad H_{2n+1}(x) = \frac{2^{2n+2}(-1)^n}{\sqrt{\pi}} e^{x^2} \int_0^\infty e^{-t^2} t^{2n+1} \sin 2xt \, dt$$

Objective Type Questions

Multiple Choice Questions

Indicate correct answer for each question by writing the corresponding letter (a), (b), (c) or (d)

1. The Hermite differential equation is given by :

- (a) $\frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2\lambda y = 0$ (b) $\frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} + 2\lambda y = 0$
 (c) $\frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} - 2\lambda y = 0$ (d) $\frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} + \lambda y = 0$

2. The Hermite polynomial $H_n(x)$ is given by

- (a) $H_n(x) = \sum_{r=0}^{n/2} \frac{(-1)^r n!}{r! (n-2r)!} (2x)^{n-2r}$
 (b) $H_n(x) = \sum_{r=0}^{(n/2)} \frac{(-1)^r n!}{r! (n-2r)!} (2x)^{n-2r}$
 (c) $H_n(x) = \sum_{r=0}^{n/2} \frac{(-1)^r r!}{n! (n-2r)!} (2x)^{n-2r}$
 (b) $H_n(x) = \sum_{r=0}^{(n/2)} \frac{(-1)^r r!}{n! (n-2r)!} (2x)^{n-2r}$

3. Generating function for Hermite polynomial $H_n(x)$ is

- (a) $e^{2tx+t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$ (b) $e^{2tx-x^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$
 (c) $e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$ (d) $e^{tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$

4. The Rodrigues formula for $H_n(x)$ is :

- (a) $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2})$ (b) $H_n(x) = e^{x^2} \frac{d^n}{dx^n} (e^{-x^2})$
 (c) $H_n(x) = (-1)^n e^x \frac{d^n}{dx^n} (e^{-x})$ (d) $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x})$

5. For what value of n , $H_n(x) = 0$?

- (a) $n = \text{even integers}$ (b) $n = \text{odd integers}$
 (c) $n = \text{any integer}$ (d) None of these

6. If $m \neq n$, then $\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx$ is equal to

- (a) $\sqrt{\pi} 2^n$ (b) 2^n
 (c) 0 (d) $2^n \cdot n!$

7. If $m = n$, then $\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx$ is equal to

- (a) 0 (b) $\sqrt{\pi} \cdot 2^n$
 (c) $2^n \cdot n!$ (d) $\sqrt{\pi} \cdot 2^n n!$

8. Which one of these is correct?

- (a) $2xH_n(x) = 2nH_{n-1}(x) - H_{n+1}(x)$
 (b) $2xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$
 (c) $2xH_n(x) = 2nH_{n-1}(x) - xH_{n+1}(x)$
 (d) $xH_n(x) = 2nH_{n-1}(x) + H_{n+1}(x)$

9. The value of $H_n^n(x)$ is equal to

- (a) 0 (b) $n!$
 (c) $2^n n!$ (d) 2^n

10. If $P(x) = x^3 + 2x^2 + 3x + 2$, then $P(x) =$

- (a) $H_3 + 4H_2 + 18H_1 + 24H_0$
 (b) $(H_3 + 4H_2 + 18H_1 + 24H_0) / 8$
 (c) $H_3 - 4H_2 + 18H_1 - 24H_0$
 (d) $(H_3 - 4H_2 + 18H_1 - 24H_0) / 8$
-

Fill in the Blank(s)

Fill in the blanks so that the following statements are complete and correct.

1. $H_1(x) = \dots\dots\dots H_0(x)$
 2. $H_n(-x) = \dots\dots\dots H_n(x)$
 3. $H'_n(x) = 2n \dots\dots\dots$
 4. $H'_n(x) = \dots\dots\dots n(n-1)(n-2) H_{n-3}(x)$
 5. $H_3(x) = 8x^3 - \dots\dots\dots x$
 6. $H_4(x) = 16x^4 - 48x^2 + \dots\dots\dots$
 7. $H_{2n+1}(0) = \dots\dots\dots$
 8. $\frac{d^m}{dx^m} \{H_n(x)\} = \frac{2^m \cdot n!}{(n-m)!} \dots\dots\dots$ if $m < n$.
 9. $\text{In } H_n(x) = \sum_{r=0}^{(n/2)} \frac{(-1)^r n!}{n! \cdot (n-2r)!} \cdot (2x)^{n-2r}, (n/2) = \dots\dots\dots$
 10. $H_{2n}(0) = (-1)^n \cdot (2n)! / \dots\dots\dots$
-

Answers

Exercise

1.	for odd integers
5.	(i) $\frac{1}{8}(H_3 + 4H_2 + 18H_1 + 24H_0)$ (ii) $\frac{1}{4}(H_4 + H_3 + 16H_2 + 8H_1 + 8H_0)$

Objective Type Questions

Multiple Choice Questions

1.	(a)	2.	(b)	3.	(c)
4.	(a)	5.	(b)	6.	(c)
7.	(d)	8.	(b)	9.	(c)
10.	(b)				

Fill in the Blank(s)

1.	$2x$	2.	$(-1)^n$
3.	$H_{n-1}(x)$	4.	2^3
5.	12	6.	12
7.	0	8.	H_{n-m}
9.	$n/2$ if n is even, $(n-1)/2$ if n is odd.	10.	$n!$

○○○



UNIT- 4

CHAPTER 7 : Partial Differential Equations of the First Order (Origin of First Order Partial Differential Equations and Classification)

CHAPTER 8 : Non-Linear Partial Differential Equations of First Order (Charpit's and Jacobi's Methods)

CHAPTER 9 : Partial Differential Equations of the Second Order with Variable Coefficients (Origin and Classification)

CHAPTER 10: Linear Partial Differential Equations with Constant Coefficients



Partial Differential
Equations of the First Order(Origin of First Order Partial Differential
Equations and Classification)

7.1 Introduction

If a differential equation contains one or more partial derivatives, it is called a **Partial Differential Equation**. [Delhi Maths (Hons.) 2001]

The partial derivatives occurs only when there are two or more independent variables. So in partial differential equations there are at least two independent variables.

If we consider the case of two independent variables only, we shall usually take them to be x and y and take z to be the dependent variable.

Thus $z = f(x, y)$

In the study of partial differential equations we shall use the following notations.

1. If there are two independent variables x and y and z is the dependent variable, then the following notations are used :

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s \text{ and } \frac{\partial^2 z}{\partial y^2} = t$$

2. If there are n independent variables, we take them to be $x_1, x_2, \dots, x_n$ and z the dependent variable, then the following notations are used :

$$\frac{\partial z}{\partial x_1} = p_1, \frac{\partial z}{\partial x_2} = p_2, \dots, \frac{\partial z}{\partial x_n} = p_n$$

3. If u is the dependent variable and x, y the independent variables, then the partial derivatives are also denoted by making use of suffixes, as follows :

$$\frac{\partial u}{\partial x} = u_x, \frac{\partial u}{\partial y} = u_y, \frac{\partial^2 u}{\partial x^2} = u_{xx}, \frac{\partial^2 u}{\partial x \partial y} = u_{xy}, \frac{\partial^2 u}{\partial y^2} = u_{yy}$$

7.2 Order and Degree of a Partial Differential Equation

[Delhi Maths (Hons.) 2001]

1. **Order of a Partial Differential Equation** : The order of the highest partial derivative appearing in the equation is called the **order** of the partial differential equation. Thus, if a partial differential equation contains n -th and lower order derivatives, it is said to be of order n .

For example, the order of differential equation

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = kz \quad \text{is } 1$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 2z \quad \text{is } 2$$

$$\frac{\partial^2 z}{\partial x^2} + x \frac{\partial z}{\partial y} + z = 0 \quad \text{is } 2$$

2. **Degree of a Partial Differential Equation** : The **degree** of a partial differential equation is the greatest exponent (*i.e.*, power) of the highest order partial derivative involved in the equation, when the equation has been made rational and integral as far as the powers of the derivatives are concerned.

For example, the partial differential equation :

$$\left(\frac{\partial z}{\partial x}\right)^2 + 2 \frac{\partial z}{\partial y} - 2z = 0 \text{ is of order 1 and degree 2.}$$

$$\frac{\partial^2 z}{\partial x^2} + x \left(\frac{\partial z}{\partial y}\right)^2 + xy = 0 \text{ is of order 2 and degree 1}$$

$$x \frac{\partial^2 z}{\partial x^2} + y \left(\frac{\partial z}{\partial y}\right)^{1/3} + kz = 0 \text{ is of order 2 and degree 3}$$

as in this case after making the powers of derivatives integral, it reduces to

$$\left(x \frac{\partial^2 z}{\partial x^2} + kz\right)^3 = -y^3 \frac{\partial z}{\partial y}$$

or
$$x^3 \left(\frac{\partial^2 z}{\partial x^2}\right)^3 + 3kx^2z \left(\frac{\partial^2 z}{\partial x^2}\right)^2 + 3k^2xz^2 \frac{\partial^2 z}{\partial x^2} + y^3 \frac{\partial z}{\partial y} + k^3z^3 = 0$$

7.3 Classification of First Order Partial Differential Equations into Linear, Semi-linear, Quasi-linear and Non-linear

A first order partial differential equation in two independent variables x and y and z as dependent variable in its most general form is given by

$$F(x, y, z, p, q) = 0$$

where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$

1. **Linear Partial Differential Equation :** *A first order partial differential equation $F(x, y, z, p, q) = 0$ is known as linear equation if it is linear in p, q and z i.e., if it is of the first degree in the dependent variable and its partial derivatives.*

Thus in a linear partial differential equation the powers or the products of the dependent variable and its partial derivatives must not occur i.e., a linear partial differential equation is of the form $f(x, y)p + g(x, y)q + h(x, y)z = \phi(x, y)$

where $f(x, y), g(x, y), h(x, y), \phi(x, y)$ are functions of x and y only and do not contain any term of z .

For example : $(x^2 + y)p + (x^3 + 3y^2)q + xyz = x^2y^3$

$$p + 2q = 6z + \tan(y - 3x)$$

and $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 6u + \frac{x^2y}{z}$

are all linear partial differential equations of first order.

2. **Semi-linear Partial Differential Equation :** *A first order partial differential equation $F(x, y, z, p, q) = 0$ is known as semi linear equation, if it is linear in p and q and the coefficients of p and q are functions of x and y only i.e., if the given equation is of the form*

$$f(x, y)p + g(x, y)q = \phi(x, y, z)$$

where $f(x, y)$ and $g(x, y)$ are functions of x and y only and do not contain any term of z and $\phi(x, y, z)$ is a function of x, y and z .

For example : $(x + y)p + (x^2y^3)q = (x^2y + xy^3 + z^2)$

$$\sin(x + y)p + \cos(x + y)q = \sin x + e^y + \log z$$

are semi-linear partial differential equations of first order.

3. **Quasi-linear Partial Differential Equation :** *A first order partial differential equation $F(x, y, z, p, q) = 0$ is known as quasi-linear equation, if it is linear in p and q i.e., if the given equation is of the form :*

$$f(x, y, z) p + g(x, y, z) q = \phi(x, y, z)$$

where $f(x, y, z)$, $g(x, y, z)$ and $\phi(x, y, z)$ are functions of x , y and z .

For example : $(x^2 + 2yz) p + (y^2 + 2zx) q = x^2 + xy + z^2$

$$2zp + 5yq = 3x^2 + \log y + e^z$$

are quasi linear partial differential equations of first order.

4. **Non-linear Partial Differential Equation :** A first order partial differential equation $F(x, y, z, p, q) = 0$ which does not come under the above three types i.e., which is neither linear, nor semi-linear or quasi-linear is known as a non-linear equation.

For example : $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$

$$x^2 p^2 + y^2 q^2 = z^2, \quad p^2 + q^2 = z$$

are all non-linear partial differential equations of first order.

7.4 Origin (Derivation) of First Order Partial Differential Equation

Here we shall discuss the origin (derivation) of first order partial differential equation by the elimination of arbitrary constants or arbitrary functions.

7.4.1 Origin (Derivation) of First Order Partial Differential Equation by the Elimination of Arbitrary Constants

Consider an equation

$$F(x, y, z, a, b) = 0 \quad \dots(1)$$

where a and b are two arbitrary constants.

Taking z as a function of two independent variables x and y , differentiating (1) w.r.t., x and y partially, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p = 0 \quad \dots(2)$$

$$\text{and} \quad \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0 \quad \text{or} \quad \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q = 0 \quad \dots(3)$$

Eliminating arbitrary constants a and b between (1), (2) and (3) we shall obtain a partial differential equation of first order of the form

$$F(x, y, z, p, q) = 0.$$

Note : If in the function f , there are more arbitrary constants than the number of independent variables, the above procedure of elimination of arbitrary constants will give rise to a partial differential equation of higher order than the first. For clear understanding of this see chapter 9.

Solved Examples

Example 1: Form a partial differential equation by eliminating a and b from

$$z = a x e^y + \frac{1}{2} a^2 e^{2y} + b. \quad [\text{Meerut 2006}]$$

Solution: Given $z = a x e^y + \frac{1}{2} a^2 e^{2y} + b \quad \dots(1)$

Differentiating (1) partially w.r.t. x and y , we get

$$\frac{\partial z}{\partial x} = a e^y \quad \dots(2)$$

and $\frac{\partial z}{\partial y} = a x e^y + a^2 e^{2y} = x (a e^y) + (a e^y)^2 \quad \dots(3)$

Substituting the value of $a e^y$ from (2) in (3), we get

$$\frac{\partial z}{\partial y} = x \frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x} \right)^2 \quad \text{or} \quad q = x p + p^2$$

which is the required partial differential equation.

Example 2: Form a partial differential equation by eliminating a and b from

$$z = (x^2 + a)(y^2 + b)$$

Solution: Given $z = (x^2 + a)(y^2 + b) \quad \dots(1)$

Differentiating (1) partially w.r.t. x and y , we get

$$\frac{\partial z}{\partial x} = 2x(y^2 + b) \quad \text{or} \quad y^2 + b = \frac{1}{2x} \cdot \frac{\partial z}{\partial x} \quad \dots(2)$$

and $\frac{\partial z}{\partial y} = 2y(x^2 + a) \quad \text{or} \quad x^2 + a = \frac{1}{2y} \cdot \frac{\partial z}{\partial y} \quad \dots(3)$

Substituting the values of $y^2 + b$ and $x^2 + a$ from (2) and (3) in (1), we get

$$z = \frac{1}{4xy} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \quad \text{or} \quad \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = 4xyz \quad \text{or} \quad pq = 4xyz$$

which is the required partial differential equation.

Example 3: Form a potential differential equation by eliminating arbitrary constants a and b from

$$2z = (ax + y)^2 + b$$

Solution: Given $2z = (ax + y)^2 + b$... (1)

Differentiating (1) partially w.r.t. x and y , we get

$$2p = 2(ax + y)a \quad \dots (2)$$

and $2q = 2(ax + y)$... (3)

Dividing (2) by (3), we get $\frac{p}{q} = a$

Putting in (3), the required partial differential equation is

$$q = \left(\frac{px}{q} + y \right) \text{ or } px + qy = q^2$$

Example 4: Eliminate a and b from the equation $ax^2 + by^2 + z^2 = 1$ to obtain a first order differential equation. [Meerut 2010 (Sem. I)]

Solution: Given $ax^2 + by^2 + z^2 = 1$... (1)

Differentiating (1) partially w.r.t. x and y , we get

$$2ax + 2zp = 0 \quad \dots (2)$$

and $2by + 2zq = 0$... (3)

From (2) and (3), we get $a = -\frac{zp}{x}$ and $b = -\frac{zq}{y}$

Putting in (1), the required equation is

$$\left(-\frac{zp}{x} \right) x^2 + \left(-\frac{zq}{y} \right) y^2 + z^2 = 1 \quad \text{i.e., } z^2 - (px + qy)z = 1$$

Example 5: Form the partial differential equation by eliminating the arbitrary constants a and b from $\log(az - 1) = x + ay + b$. [I.A.S. 2002]

Solution: Given $\log(az - 1) = x + ay + b$... (1)

Differentiating (1) partially w.r.t x and y , we get

$$\frac{1}{az - 1} \cdot ap = 1 \quad \dots (2)$$

and $\frac{1}{az - 1} \cdot aq = a$... (3)

Dividing (3) by (2) $a = \frac{q}{p}$

Putting in (3), the required equation is

$$q = z \cdot \frac{q}{p} - 1 \text{ or } pq = zq - p \text{ i.e., } (1 + q)p = zq$$

Example 6: Find partial differential equation by eliminating constants a, b, c from

$$z = a(x + y) + b(x - y) + abt + c$$

Solution: Given $z = a(x + y) + b(x - y) + abt + c$... (1)

Here we have three constants a, b and c and three independent variables x, y and t . So differentiating (1) partially w.r.t., x, y and t , we get

$$\frac{\partial z}{\partial x} = a + b \quad \dots (2)$$

$$\frac{\partial z}{\partial y} = a - b \quad \dots (3)$$

$$\text{and} \quad \frac{\partial z}{\partial t} = ab \quad \dots (4)$$

$$\text{From (2) and (3), } a = \frac{1}{2} \left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \right) \text{ and } b = \frac{1}{2} \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right)$$

Putting in (4), the required equation is

$$\frac{\partial z}{\partial t} = \frac{1}{4} \left[\left(\frac{\partial z}{\partial x} \right)^2 - \left(\frac{\partial z}{\partial y} \right)^2 \right] \text{ i.e., } \left(\frac{\partial z}{\partial x} \right)^2 - \left(\frac{\partial z}{\partial y} \right)^2 = 4 \frac{\partial z}{\partial t}$$

Example 7: Show that the differential equation of all cones which have their vertices at the origin is $px + qy = z$. Verify that $yz + zx + xy = 0$ is a surface satisfying the above equation.

Solution: From coordinate geometry of three dimensions, the equation of cone with the vertex at the origin is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \dots (1)$$

where a, b, c, f, g, h are constants.

Differentiating (1) partially w.r.t. x and y , we get

$$2ax + 2czp + 2fyp + 2g(z \cdot 1 + xp) + 2hy = 0$$

$$\text{or} \quad (ax + gz + hy) + (cz + fy + gx)p = 0 \quad \dots (2)$$

and $2by + 2czq + 2f(1.z + yq) + 2gxq + 2hx = 0$

or $(by + fz + hx) + (cz + fy + gx)q = 0 \quad \dots(3)$

Multiplying (2) and (3) by x and y respectively and adding, we get

$$(ax + gz + hy)x + (by + fz + hx)y + (cz + fy + gx)(px + qy) = 0$$

or $ax^2 + by^2 + gzx + 2hxy + fy + yz + (cz + fy + gx)(px + qy) = 0 \quad \dots(4)$

Now subtracting (1) from (4), we get

$$(-cz^2 - gzx - fy + yz) + (cz + fy + gx)(px + qy) = 0$$

Substituting from (1)

$$(cz + fy + gx)(px + qy - z) = 0$$

$\Rightarrow px + qy - z = 0 \quad \dots(5)$

is the required differential equation.

Again, differentiating $yz + zx + xy = 0$

partially w.r.t. x and y , we get

$$yp + (z + px) + y = 0 \text{ and } (1.z + yq) + xq + x = 0$$

$\therefore p = -\frac{y+z}{x+y} \quad \text{and} \quad q = -\frac{z+x}{x+y}$

Putting in (5),

$$px + qy - z = -\left(\frac{y+z}{x+y}\right)x - \left(\frac{z+x}{x+y}\right)y - z = -\frac{2(xy + yz + zx)}{x+y}$$

$\Rightarrow yz + zx + xy = 0$ is a surface satisfying the differential equation $px + qy = z$.

Example 8: Find the partial differential equation of all spheres of radius c having centres in the xy plane.

Solution: Let $(h, k, 0)$ be the centre in xy plane of the sphere of radius c . Then the equation of the sphere is :

$$(x-h)^2 + (y-k)^2 + z^2 = c^2 \quad \dots(1)$$

Here h and k are the arbitrary constants.

Differentiating (1) partially w.r.t. x and y , we get

$$2(x-h) + 2z \frac{\partial z}{\partial x} = 0 \quad \text{or} \quad x-h = -z \frac{\partial z}{\partial x} \quad \dots(2)$$

$$\text{and} \quad 2(y-k) + 2z \frac{\partial z}{\partial y} = 0 \quad \text{or} \quad y-k = -z \frac{\partial z}{\partial y} \quad \dots(3)$$

Substituting the values of $x-h$ and $y-k$ from (2) and (3) in (1), we get

$$z^2 \left[\left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 + 1 \right] = c^2 \quad \text{or} \quad z^2 (p^2 + q^2 + 1) = c^2.$$

which is the required partial differential equation.

Example 9: Formulate a partial differential equation by eliminating arbitrary constants a and b from the equation $(x+a)^2 + (y+b)^2 + z^2 = 1$. Examine whether the partial differential equation is linear or non-linear find its order and degree. [Delhi (Hons.) 2008]

Solution: Proceed as in Ex. 8. Here $c=1$

$\therefore$ The required differential equation is

$$z^2 (p^2 + q^2 + 1) = 1$$

Which is a non-linear differential equation of order one and degree two.

Example 10: Find the differential equation of the set of all right circular cones whose axis coincide with z -axis.

Solution: The equation of a right circular cone whose axis coincide with z -axis, having semi-vertical angle α and vertex at $(0, 0, c)$ is given by

$$x^2 + y^2 = (z-c)^2 \tan^2 \alpha \quad \dots(1)$$

Here c and α are arbitrary constants.

Differentiating (1) partially w.r.t. x and y , we get

$$2x = 2(z-c) \frac{\partial z}{\partial x} \tan^2 \alpha \quad \text{or} \quad x = (z-c) \frac{\partial z}{\partial x} \tan^2 \alpha \quad \dots(2)$$

$$\text{and} \quad 2y = 2(z-c) \frac{\partial z}{\partial y} \tan^2 \alpha \quad \text{or} \quad y = (z-c) \frac{\partial z}{\partial y} \tan^2 \alpha \quad \dots(3)$$

Multiplying (2) by y and (3) by x , we get

$$xy = y(z-c) \frac{\partial z}{\partial x} \tan^2 \alpha \quad \text{and} \quad xy = x(z-c) \frac{\partial z}{\partial y} \tan^2 \alpha$$

$$\therefore \quad y(z-c) \frac{\partial z}{\partial x} \tan^2 \alpha = x(z-c) \frac{\partial z}{\partial y} \tan^2 \alpha$$

or $y \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial y} = 0$ or $yp - xq = 0$

which is the required differential equation.

7.4.2 Origin (Derivation) of First Order Partial Differential Equation by the Elimination of Arbitrary Function

Let u and v be two functions of x , y and z which are connected by the relation.

$$f(u, v) = 0 \quad \dots(1)$$

Differentiating (1) partially w.r.t. x and y , we get

$$\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial x} \right) = 0$$

or
$$\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p \right) = 0 \quad \dots(2)$$

and
$$\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial y} \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial y} \right) = 0$$

or
$$\frac{\partial f}{\partial u} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q \right) + \frac{\partial f}{\partial v} \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q \right) = 0 \quad \dots(3)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from (2) and (3), we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p & \frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p \\ \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q \end{vmatrix} = 0$$

or
$$\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p \right) \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q \right) - \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q \right) \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p \right) = 0$$

or
$$\left(\frac{\partial u}{\partial y} \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \frac{\partial v}{\partial y} \right) p + \left(\frac{\partial u}{\partial z} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial z} \right) q = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}$$

or
$$Pp + Qq = R$$

where
$$P = \frac{\partial u}{\partial y} \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \frac{\partial v}{\partial y} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} = \frac{\partial(u, v)}{\partial(y, z)}$$

$$Q = \frac{\partial u}{\partial z} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial z} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} = \frac{\partial(u, v)}{\partial(z, x)}$$

$$\text{and } R = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u, v)}{\partial(x, y)}$$

Note : If the given relation between x , y and z contains two arbitrary functions, then in general their elimination will give rise to a partial differential equation of order higher than one. For clear understanding of this see chapter 9.

Solved Examples

Example 1: Form a partial differential equation by eliminating the function f from

$$z = y^2 + 2f(1/x + \log y) \quad [\text{I.A.S. 2007}]$$

Solution: Given $z = y^2 + 2f(1/x + \log y) \quad \dots(1)$

Differentiating (1) partially w.r.t. x and y , we get

$$\frac{\partial z}{\partial x} = 2f'(1/x + \log y) \cdot (-1/x^2)$$

or $-x^2 \frac{\partial z}{\partial x} = 2f'(1/x + \log y) \quad \dots(2)$

and $\frac{\partial z}{\partial y} = 2y + 2f'(1/x + \log y) \cdot (1/y)$

or $y \frac{\partial z}{\partial y} - 2y^2 = 2f'(1/x + \log y) \quad \dots(3)$

From (2) and (3), we get

$$-x^2 \frac{\partial z}{\partial x} = y \frac{\partial z}{\partial y} - 2y^2$$

or $x^2 \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2y^2 \quad \text{or} \quad x^2 p + yq = 2y^2$

which is the required partial differential equation.

Example 2: Form a partial differential equation by eliminating the function ϕ from

$$z = (x - y) \phi(x^2 + y^2).$$

Solution: Given $z = (x - y) \phi(x^2 + y^2)$... (1)

Differentiating (1) partially w.r.t. x and y , we get

$$\frac{\partial z}{\partial x} = \phi(x^2 + y^2) + (x - y) \phi'(x^2 + y^2) \cdot 2x \quad \dots (2)$$

$$\text{and} \quad \frac{\partial z}{\partial y} = -\phi(x^2 + y^2) + (x - y) \phi'(x^2 + y^2) \cdot 2y \quad \dots (3)$$

Now to eliminate $\phi(x^2 + y^2)$ and $\phi'(x^2 + y^2)$, between (1), (2) and (3), we proceed as follows :

Multiplying (2) by y and (3) by x , and subtracting, we get

$$\begin{aligned} y \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial y} &= (y + x) \phi(x^2 + y^2) \\ &= (y + x) z / (x - y) \text{ substituting from (1)} \end{aligned}$$

$$\text{or} \quad (x - y) \left(y \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial y} \right) = (x + y) z$$

$$\text{or} \quad (x - y) (y p - x q) = (x + y) z$$

which is the required partial differential equation.

Example 3: Form a partial differential equation by eliminating the arbitrary function ϕ from $\phi(x + y + z, x^2 + y^2 - z^2) = 0$ What is the order of this partial differential equation ?
[Vikram 2001; Jiwaji 2003]

Solution: Given $\phi(x + y + z, x^2 + y^2 - z^2) = 0$... (1)

$$\text{Let} \quad x + y + z = u \text{ and } x^2 + y^2 - z^2 = v \quad \dots (2)$$

$$\text{Thus, (1) becomes, } \phi(u, v) = 0 \quad \dots (3)$$

Differentiating u and v given in (2), w.r.t. x and y , we get

$$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 1, \frac{\partial u}{\partial z} = 1, \frac{\partial v}{\partial x} = 2x, \frac{\partial v}{\partial y} = 2y, \frac{\partial v}{\partial z} = -2z$$

Differentiating (3), w.r.t. x , we get

$$\frac{\partial \phi}{\partial u} \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} \right) + \frac{\partial \phi}{\partial v} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial x} \right) = 0$$

$$\text{or} \quad \frac{\partial \phi}{\partial u} (1 + p) + \frac{\partial \phi}{\partial v} (2x - 2z p) = 0 \quad \dots(4)$$

Again differentiating (3), w.r.t. y , we get

$$\frac{\partial \phi}{\partial u} \left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial y} \right) + \frac{\partial \phi}{\partial v} \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \frac{\partial z}{\partial y} \right) = 0$$

$$\text{or} \quad \frac{\partial \phi}{\partial u} (1 + q) + \frac{\partial \phi}{\partial v} (2y - 2z q) = 0 \quad \dots(5)$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ between (4) and (5), we get

$$\begin{vmatrix} 1+p & 2(x-zp) \\ 1+q & 2(y-zq) \end{vmatrix} = 0$$

$$\text{or} \quad 2(1+p)(y-zq) - 2(1+q)(x-zp) = 0$$

$$\text{or} \quad (y+z)p - (x+z)q = x-y$$

which is the required partial differential equation whose order is one.

Example 4: Formulate the partial differential equation by eliminating the function from

$$z = f(xy/z)$$

[Kurukshetra 2004]

$$\textbf{Solution:} \text{ Given } z = f(xy/z) \quad \dots(1)$$

Differentiating (1) partially w.r.t x and y , we get

$$\frac{\partial z}{\partial x} = f' \left(\frac{xy}{z} \right) \left(\frac{y}{z} - \frac{xy}{z^2} \frac{\partial z}{\partial x} \right) \quad \dots(2)$$

$$\text{and} \quad \frac{\partial z}{\partial y} = f' \left(\frac{xy}{z} \right) \left(\frac{x}{z} - \frac{xy}{z^2} \frac{\partial z}{\partial y} \right) \quad \dots(3)$$

Dividing (2) by (3), we get

$$\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = \frac{\frac{y}{z} - \frac{xy}{z^2} \frac{\partial z}{\partial x}}{\frac{x}{z} - \frac{xy}{z^2} \frac{\partial z}{\partial y}}$$

$$\text{or} \quad x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = 0 \quad \text{or} \quad px - qy = 0$$

which is the required differential equation.

Example 5: Form a partial differential equation by eliminating the arbitrary function f from the equation

$$x + y + z = f(x^2 + y^2 + z^2)$$

Solution: Given, $x + y + z = f(x^2 + y^2 + z^2)$... (1)

Differentiating (1) partially w.r.t. x and y , we get

$$1 + 0 + p = f'(x^2 + y^2 + z^2) \cdot 2(x + z p) \quad \dots (2)$$

and $0 + 1 + q = f'(x^2 + y^2 + z^2) \cdot 2(y + z q) \quad \dots (3)$

Dividing (2) by (3), we get

$$\frac{p+1}{q+1} = \frac{x+z p}{y+z q}$$

or $(p+1)(y+z q) = (q+1)(x+z p)$ or $(y-z)p - (x-z)q = x-y$

which is the required differential equation.

Example 6: Form the partial differential equation by eliminating the arbitrary function f from

$$z = x y f(x^2 + y^2 + z^2)$$

Solution: Given $z = x y f(x^2 + y^2 + z^2)$... (1)

Differentiating (1) partially w.r.t. x and y , we get

$$p = y f(x^2 + y^2 + z^2) + x y f'(x^2 + y^2 + z^2) \cdot (2x + 2z p) \quad \dots (2)$$

and $q = x f(x^2 + y^2 + z^2) + x y f'(x^2 + y^2 + z^2) \cdot (2y + 2z q) \quad \dots (3)$

Required differential equation is obtained by eliminating the function f and f' from (1), (2) and (3). From (2) and (3), we get

$$p - y f(x^2 + y^2 + z^2) = 2x y (x + z p) f'(x^2 + y^2 + z^2)$$

and $q - x f(x^2 + y^2 + z^2) = 2x y (y + z q) f'(x^2 + y^2 + z^2)$

Dividing,
$$\frac{p - y f(x^2 + y^2 + z^2)}{q - x f(x^2 + y^2 + z^2)} = \frac{x + z p}{y + z q}$$

or $\frac{p - z/x}{q - z/y} = \frac{x + z p}{y + z q}$ Substituting $f(x^2 + y^2 + z^2) = z/x \cdot y$ from (1)

$$y(p x - z)(y + z q) = x(q y - z)(x + z p)$$

or $p x y (y + z q) - q x y (x + z p) - y z (y + z q) + x z (x + z p) = 0$

or $p x y^2 - q x^2 y - y^2 z - y z^2 q + x^2 z + x z^2 p = 0$

or $x(y^2 + z^2)p - y(x^2 + z^2)q = z(y^2 - x^2)$

which is the required differential equation.

Exercise 7.1

Eliminating the arbitrary constants from the following equations, form the partial differential equations.

1. $a x^3 + b y^3 = z^3$

2. $a z + b = a^2 x + y$

3. $z = a x + b y + a^2 + b^2$

4. $z = a x + a^2 y^2 + b$

5. $z = (x + a)(y + b)$

6. $z = (x - a)^2 + (y - b)^2$

7. $z = a x + (1 - a) y + b$

8. Find the differential equation of the family of spheres of radius 4 with centre on the $x y$ plane.

9. Find the differential equation of all spheres whose centres lie on z -axis.

[Hint : equation of spheres is $x^2 + y^2 + (z - c)^2 = a^2$]

Form the partial differential equations by eliminating the arbitrary functions from the following equations.

10. $z = x^n f(y/x)$

11. $z = e^{my} \phi(x - y)$

12. $l x + m y + n z = \phi(x^2 + y^2 + z^2)$

13. $z = x + y + f(x y)$

14. $z = e^y f(x + y)$

15. $z = e^{ax + by} \cdot f(ax - by)$

16. $z = x y + f(x^2 + y^2)$

17. $x y z = f(x + y + z)$

18. $f(x^2 + y^2 + z^2, z^2 - 2x y) = 0$

19. $f(x + y + z, x^2 + y^2 + z^2) = 0$

Answers 7.1

1.	$px + qy = z$	2.	$pq = 1$
3.	$px + qy + p^2 + q^2 = z$	4.	$pq = 1$
5.	$z = pq$	6.	$4z = p^2 + q^2$
7.	$p + q = 1$	8.	$z^2 (p^2 + q^2 + 1) = 16$
9.	$py - qx = 0$	10.	$px + qy = xz$
11.	$p + q = mz$	12.	$(l + np)y + (lq - mp)z = (m + nq)x$
13.	$px - qy = x - y$	14.	$q = z + p$
15.	$bp + aq = 2abz$	16.	$py - qx = y^2 - x^2$
17.	$x(y - z)p + y(z - x)q = z(x - y)$	18.	$(p - q)z = y - x$
19.	$(y - z)p - (x - z)q = x - y$		

7.5 Some Definitions

1. **Solution of a Differential Equation** : A **solution** or **integral** of a differential equation is a relation between the variables (dependent and independent) by means of which and the derivatives obtained from it, the differential equation is satisfied.

2. **Complete Integral (C.I.)** : Let $f(x, y, z, p, q) = 0$... (1)

be a partial differential equation containing two independent variables x and y and a dependent variable z .

If a relation $F(x, y, z, a, b) = 0$... (2)

containing as many arbitrary constants a and b as there are the independent variables, is derived from (1) by using some methods, then the relation (2) is called the **complete integral** of the partial differential equation (1).

3. **Particular Integral (P.I.)** : A **particular integral** of the partial differential equation $f(x, y, z, p, q) = 0$ is obtained by giving particular values to the arbitrary constants a and b in its complete integral $F(x, y, z, a, b) = 0$

4. **General Integral (G.I.)** : Let $F(x, y, z, a, b) = 0$... (1)

be the complete integral of the differential equation

$$f(x, y, z, p, q) = 0 \quad \dots (2)$$

If one of the constants in (1) is a function of the other, say $b = \phi(a)$, then (1), becomes

$$F[x, y, z, a, \phi(a)] = 0 \quad \dots(3)$$

Then the envelope of the family of surface (3), obtained by eliminating a between (3) and $\frac{\partial F}{\partial a} = 0$, is called the **general integral** of the differential equation (2).

5. **Singular Integral (S.I.)** : Let $F(x, y, z, a, b) = 0$...(1)

be the complete integral of the differential equation

$$f(x, y, z, p, q) = 0 \quad \dots(2)$$

Then the envelope of the family of surface (1), obtained by eliminating a and b between (1), $\frac{\partial F}{\partial a} = 0$ and $\frac{\partial F}{\partial b} = 0$ is called the **singular integral** of the differential equation (2).

Since other relations may appear in the process of getting the singular integral, so it is necessary to test that the equation of singular integral satisfies the given differential equation.

7.6 Lagrange's Linear Partial Differential Equation

The partial differential equation of the form $Pp + Qq = R$ where P, Q, R are functions of x, y, z is the standard form of linear partial differential equation of order one and is called **Lagrange's Linear equation**.

7.7 Lagrange's Solution of the Lagrange's Linear Equation. (Lagrange's Method of Solving the Linear Partial Differential Equation of Order One Namely $Pp + Qq = R$)

[Delhi Maths (Hons.) 2000, 04, 08; Meerut 2003; Poona 2003]

Theorem: *The general solution of the linear partial differential equation*

$$Pp + Qq = R$$

where P, Q, R are functions of x, y, z is $f(u, v) = 0$

where $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$, (c_1 and c_2 are two arbitrary constants)

form a solution of the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

[Meerut 2008 (BP), 10 (Sem. I)]

Proof: In 7.4.2 we have shown that, eliminating function f , from the relation

$$f(u, v) = 0 \quad \dots(1)$$

where u and v are two functions of x, y and z , we obtain a linear partial differential equation

$$P p + Q q = R \quad \dots(2)$$

where

$$P = \frac{\partial u}{\partial y} \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \frac{\partial v}{\partial y}, \quad Q = \frac{\partial u}{\partial z} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial z}, \quad R = \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}$$

Hence, relation (1) is the solution (general solution) of the partial differential equation (2). Thus in order to find the solution of partial differential equation (2), we have to find two functions u and v , which when substituted in (1), will give the general solution of (2).

Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2 \quad \dots(3)$

be two functions where c_1 and c_2 are arbitrary constants. Differentiating functions u and v given in (3), we get

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0$$

Solving these, we get

$$\frac{dx}{\frac{\partial u}{\partial y} \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x}}$$

i.e., $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \dots(4)$

Thus, (4) is the differential equation whose solutions are $u = c_1$ and $v = c_2$

Hence, $f(u, v) = 0$ is the general solution of the differential equation (2), where $u = c_1$ and $v = c_2$ are solutions of (4).

The equations (4) are called **Lagrange's auxiliary equations** or **Lagrange's subsidiary equations**.

7.8 Working Method

To solve the partial differential equation

$$P p + Q q = R \quad \dots(1)$$

By Lagrange's method we proceed as follows.

Step 1 : Write the Lagrange's auxiliary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \dots(2)$$

Step 2 : Find two independent solutions $u = c_1$ and $v = c_2$ of the auxiliary equations (2).

Step 3 : The general solution of partial differential equations (1) is obtained by substituting the values of u and v obtained in step 2, in any one of the following equivalent forms.

$$f(u, v) = 0, v = f(u) \quad \text{or} \quad u = f(v)$$

7.9 The Linear Partial Differential Equation with n Independent Variables

Let the linear partial differential equation with n independent variables $x_1, x_2, \dots, x_n$ be

$$P_1 p_1 + P_2 p_2 + \dots + P_n p_n = R \quad \dots(1)$$

where $p_i = \frac{\partial z}{\partial x_i}, i = 1, 2, \dots, n$

and $P_1, P_2, \dots, P_n$ and R are functions of $x_1, x_2, \dots, x_n$ and z .

The general solution of (1) is given by

$$f(u_1, u_2, \dots, u_n) = 0$$

where $u_1 = c_1, u_2 = c_2, \dots, u_n = c_n$ ($c_1, c_2, \dots, c_n$ are n arbitrary constants)

are n independent integrals of the subsidiary equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \dots = \frac{dx_n}{P_n} = \frac{dz}{R}$$

Note : This method is the generalization of Lagrange's method for two independent variables discussed in 7.7 and 7.8

Solved Examples

Example 1: Solve $x^2 p + y^2 q = z^2$.

[Jabalpur 2000; Ravishankar 2001]

Solution: Lagrange's subsidiary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$$

From 1st and 2nd fractions, we have

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Integrating, $-\frac{1}{x} = -\frac{1}{y} - c_1$ i.e., $u \equiv \frac{1}{x} - \frac{1}{y} = c_1$

Similarly, from 1st and 3rd fractions, we have $v \equiv \frac{1}{x} - \frac{1}{z} = c_2$

Hence, the required general solution is

$$f(u, v) = 0 \text{ i.e., } f\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{x} - \frac{1}{z}\right) = 0$$

Example 2: Solve $yzp + zxq = xy$.

[Ravishankar 2004]

Solution: Lagrange's subsidiary equations are

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

From first two fractions, we have $x dx - y dy = 0$

Integrating, $u \equiv x^2 - y^2 = c_1$

From first and third fractions, we have, $x dx - z dz = 0$

Integrating $v \equiv x^2 - z^2 = c_2$

Hence the required general solution is

$$f(u, v) = 0 \text{ i.e., } f(x^2 - y^2, x^2 - z^2) = 0$$

Example 3: Solve $p \tan x + q \tan y = \tan z$.

[Madras 2005; Kanpur 2007]

Solution: Lagrange's subsidiary equations are

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

From first two fractions, we have $\cot x dx - \cot y dy = 0$

Integrating, $\log \sin x - \log \sin y = \log c_1$ or $u \equiv \sin x / \sin y = c_1$

From first and third fractions, we have $\cot x dx - \cot z dz = 0$

Integrating, $\log \sin x - \log \sin z = \log c_2$ or $v \equiv \sin x / \sin z = c_2$

Hence, the required general solution is

$$f(u, v) = 0 \text{ i.e., } f(\sin x / \sin y, \sin x / \sin z) = 0$$

Note : The general solution of the problem can also be taken as $\sin x / \sin y = f(\sin x / \sin z)$.

Example 4: Solve $p + 3q = 5z + \tan(y - 3x)$.

[Indore 2002; Ravishankar 2003; Meerut 2003; Agra 2006]

Solution: Lagrange's subsidiary equation are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)}$$

From 1st and 2nd fractions, $dy - 3dx = 0$

Integrating, $u \equiv y - 3x = c_1$... (1)

From 1st and 3rd fractions,

$$dx = \frac{dz}{5z + \tan c_1}, \text{ using (1)}$$

Integrating, $5x = \log(5z + \tan c_1) - \log c_2$

or $\log c_2 / (5z + \tan c_1) = -5x$

or $c_2 = e^{-5x} (5z + \tan c_1)$

or $v \equiv e^{-5x} \{5z + \tan(y - 3x)\} = c_2$

Hence, the required general solution is

$$f(u, v) = 0 \text{ i.e., } f[y - 3x, e^{-5x} \{5z + \tan(y - 3x)\}] = 0.$$

Example 5: Solve $xz p + yz q = xy$.

[Jiwaji 2000; Vikram 2000; Punjab 2005; Agra 2007]

Solution: Lagrange's subsidiary equation are

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{x y}$$

From 1st and 2nd fractions, $\frac{dx}{x} = \frac{dy}{y}$

Integrating $\log x = \log y + \log c_1 \quad \therefore \quad u \equiv x / y = c_1$... (1)

From 2nd and 3rd fractions,

$$\frac{dy}{z} = \frac{dz}{x} \text{ or } x dy = z dz$$

or $c_1 y dy = z dz$, using (1)

Integrating, $c_1 y^2 - z^2 = c_2$ i.e., $v \equiv x y - z^2 = c_2$. Putting $c_1 = x / y$

Hence, the required general solution is

$$f(u, v) = 0 \text{ i.e., } f(x/y, x y - z^2) = 0$$

Example 6: Solve $py + qx = xyz^2 (x^2 - y^2)$.

Solution: Lagrange's subsidiary equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2 (x^2 - y^2)}$$

From 1st and 2nd fractions, $x dx = y dy$.

Integrating, $u \equiv x^2 - y^2 = c_1$... (1)

From 2nd and 3rd fractions, $y dy = \frac{dz}{z^2 c_1}$ using (1)

or $c_1 y dy = dz / z^2$. Integrating $c_1 y^2 = -2 / z + c_2$

or $v \equiv y^2 (x^2 - y^2) + 2 / z = c_2$. Putting $c_1 = x^2 - y^2$

Hence the required general solution is

$$v = f(u) \text{ i.e., } y^2 (x^2 - y^2) + 2 / z = f(x^2 - y^2)$$

Example 7: Solve $(x^2 + 2y^2) p - xyq = xz$

[Kurukshetra 2005]

Solution: The subsidiary equations of the given equation are

$$\frac{dx}{x^2 + 2y^2} = \frac{dy}{-x y} = \frac{dz}{x z}$$

Taking 2nd and 3rd fractions, we get

$$\frac{dy}{y} + \frac{dz}{z} = 0 \text{ Integrating } \log y + \log z = \log c_1 \Rightarrow u \equiv yz = c_1$$

Again taking 1st and 2nd fractions, we get

$$\frac{dy}{dx} = -\frac{xy}{x^2 + 2y^2}$$

which is homogenous. $\therefore$ Putting $y = vx, \frac{dy}{dx} = v + x \frac{dv}{dx}$.

We get

$$\begin{aligned} v + x \frac{dv}{dx} &= -\frac{v}{1+2v^2} \Rightarrow x \frac{dv}{dx} = -v \left(1 + \frac{1}{1+2v^2} \right) = -\frac{v(2+2v^2)}{1+2v^2} \\ \Rightarrow 2 \frac{dx}{x} + \frac{1+2v^2}{v(1+v^2)} &= 0 \Rightarrow 2 \frac{dx}{x} + \frac{1}{v} \cdot \frac{(1+v^2)+v^2}{1+v^2} dv = 0 \\ \Rightarrow 4 \frac{dx}{x} + \left(\frac{2}{v} + \frac{2v}{1+v^2} \right) dv &= 0 \end{aligned}$$

Integrating, $4 \log x + 2 \log v + \log(1+v^2) = \log c_2$

$$\Rightarrow \log x^4 v^2 (1+v^2) = \log c_2 \Rightarrow x^4 v^2 + x^4 v^4 = c_2$$

$$\Rightarrow v \equiv x^2 y^2 + y^4 = c_2$$

Hence the required general solution is

$$f(u, v) = 0 \text{ i.e., } f(yz, x^2 y^2 + y^4) = 0$$

Example 8: Solve $xp + zq + y = 0$.

[M.D.U. Rohtak 2004]

Solution: The given differential equation can be written as

$$xp + zq = -y$$

$\therefore$ The subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{z} = \frac{dz}{-y} \quad \dots(1)$$

From 2nd and 3rd fractions, we get

$$y dy + z dz = 0$$

$$\text{Integrating } y^2 + z^2 = c_1 \text{ i.e., } u \equiv y^2 + z^2 = c_1 \quad \dots(2)$$

Again taking 1st and 2nd fractions, we have

$$\frac{dx}{x} = \frac{dy}{z} \Rightarrow \frac{dx}{x} = \frac{dy}{\sqrt{c_1 - y^2}}$$

$$\text{Integrating } \log x = \sin^{-1} \left(\frac{y}{\sqrt{c_1}} \right) + c_2 \Rightarrow \log x = \sin^{-1} \left(\frac{y}{\sqrt{y^2 + z^2}} \right) + c_2$$

$$\Rightarrow \quad v \equiv \log x - \tan^{-1} \left(\frac{y}{z} \right) = c_2 \quad \because \quad \sin^{-1} \left(\frac{y}{\sqrt{(y^2 + z^2)}} \right) = \tan^{-1} \frac{y}{z}$$

Hence the required general solution is

$$f(u, v) = 0 \text{ i.e., } f[y^2 + z^2, \log x - \tan^{-1}(y/z)] = 0$$

Example 9: Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$.

[Delhi Maths (Hons.) 2005; M.D.U. Rohtak 2005; Meerut 2006; Guwahati 2007; Agra 2008]

Solution: Lagrange's subsidiary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

$$\therefore \text{ each fraction } = \frac{dx - dy}{(x - y)(x + y + z)} = \frac{dy - dz}{(y - z)(x + y + z)} = \frac{dz - dx}{(z - x)(x + y + z)}$$

$$\text{Taking 1st and 2nd fractions, } \frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}$$

$$\text{Integrating, } \log(x - y) = \log(y - z) + \log c_1 \therefore u \equiv \frac{x - y}{y - z} = c_1$$

$$\text{Again taking 1st and 3rd fractions, } \frac{dy - dz}{y - z} = \frac{dz - dx}{z - x}$$

$$\text{Integrating, } \log(y - z) = \log(z - x) + \log c_2$$

$$\therefore \quad v \equiv \frac{y - z}{z - x} = c_2$$

Hence, the required general solution is

$$f(u, v) = 0 \text{ i.e., } f\left(\frac{x - y}{y - z}, \frac{y - z}{z - x}\right) = 0$$

Example 10: Solve

$$\begin{aligned} (2x^2 + y^2 + z^2 - 2yz - zx - xy)p + (x^2 + 2y^2 + z^2 - yz - 2zx - xy)q \\ = x^2 + y^2 + 2z^2 - yz - zx - 2xy \end{aligned}$$

Solution: Lagrange's subsidiary equations are

$$\frac{dx}{2x^2 + y^2 + z^2 - 2yz - zx - xy} = \frac{dy}{x^2 + 2y^2 + z^2 - yz - 2zx - xy}$$

$$= \frac{dz}{x^2 + y^2 + 2z^2 - yz - zx - 2xy}$$

$$\therefore \text{ each fraction} = \frac{dx - dy}{x^2 - y^2 - yz + zx} = \frac{dy - dz}{y^2 - z^2 - zx + xy} = \frac{dz - dx}{z^2 - x^2 - xy + yz}$$

$$\text{or } \frac{dx - dy}{(x - y)(x + y + z)} = \frac{dy - dz}{(y - z)(x + y + z)} = \frac{dz - dx}{(z - x)(x + y + z)}$$

$$\text{Taking 1st and 2nd fractions, } \frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}$$

$$\text{Integrating, } \log(x - y) = \log(y - z) + \log c_1 \quad \therefore u \equiv \frac{x - y}{y - z} = c_1$$

$$\text{Again taking 2nd and 3rd fractions, } \frac{dy - dz}{y - z} = \frac{dz - dx}{z - x}$$

$$\text{Integrating, } \log(y - z) = \log(z - x) + \log c_2 \quad \therefore v \equiv \frac{y - z}{z - x} = c_2$$

Hence, the required general solution is

$$f(u, v) = 0 \text{ i.e., } f\left(\frac{x - y}{y - z}, \frac{y - z}{z - x}\right) = 0$$

Example 11: Solve $p \cos(x + y) + q \sin(x + y) = z$.

[Meerut 2007; Delhi Maths (Hons.) 2007]

Solution: Lagrange's subsidiary equations are

$$\frac{dx}{\cos(x + y)} = \frac{dy}{\sin(x + y)} = \frac{dz}{z}$$

$$\therefore \frac{dx}{\cos(x + y)} = \frac{dy}{\sin(x + y)} = \frac{dz}{z} = \frac{dx + dy}{\cos(x + y) + \sin(x + y)} = \frac{dx - dy}{\cos(x + y) - \sin(x + y)}$$

Taking 3rd and 4th fractions,

$$\frac{dz}{z} = \frac{(dx + dy)}{\sqrt{2} \sin\left(x + y + \frac{\pi}{4}\right)} \quad \text{or} \quad \sqrt{2} \frac{dz}{z} = \frac{(dx + dy)}{\sin\left(x + y + \frac{\pi}{4}\right)}$$

$$\text{or} \quad \sqrt{2} \frac{dz}{z} = \operatorname{cosec} t dt, \quad \text{putting } x + y + \frac{\pi}{4} = t$$

$$\text{Integrating, } \sqrt{2} \log z = \log \tan\left(\frac{t}{2}\right) + \log c_1$$

or
$$\log \frac{z^{\sqrt{2}}}{\tan \left(\frac{x}{2} + \frac{y}{2} + \frac{\pi}{8} \right)} = \log c_1$$

$\therefore u \equiv z^{\sqrt{2}} \cot \left(\frac{x}{2} + \frac{y}{2} + \frac{\pi}{8} \right) = c_1$

Again taking 4th and 5th fractions

$$dx - dy = \frac{\cos(x+y) - \sin(x+y)}{\cos(x+y) + \sin(x+y)} (dx + dy)$$

Integrating $x - y = \log \{ \cos(x+y) + \sin(x+y) \} - \log c_2$

or $e^{x-y} = \{ \cos(x+y) + \sin(x+y) \} / c_2$

$\therefore v \equiv e^{y-x} \{ \cos(x+y) + \sin(x+y) \} = c_2$

Hence the required general solution is

$$f(u, v) = 0 \text{ i.e., } f \left[z^{\sqrt{2}} \cot \left(\frac{x}{2} + \frac{y}{2} + \frac{\pi}{8} \right), e^{y-x} \{ \cos(x+y) + \sin(x+y) \} \right] = 0$$

Example 12: Solve $p x(x+y) - q y(x+y) + (x-y)(2x+2y+z) = 0$

[Delhi Maths (Hons.) 2006]

Solution: The given differential equation can be written as

$$x(x+y)p - y(x+y)q = -(x-y)(2x+2y+z)$$

$\therefore$ The Lagrange's subsidiary equations are

$$\frac{dx}{x(x+y)} = \frac{dy}{-y(x+y)} = \frac{dz}{-(x-y)(2x+2y+z)} \quad \dots(1)$$

Taking 1st and 2nd fractions, we get

$$\frac{dx}{x} + \frac{dy}{y} = 0$$

Integrating $\log x + \log y = \log c_1 \Rightarrow \log xy = \log c_1 \Rightarrow u \equiv xy = c_1$.

Again each fraction of (1) is equal to

$$\begin{aligned} \frac{dx + dy}{(x-y)(x+y)} &= \frac{dx + dy + dz}{x(x+y) - y(x+y) - (x-y)(2x+2y+z)} \\ &= \frac{dx + dy + dz}{(x-y)(x+y) - (x-y)(2x+2y+z)} = \frac{dx + dy + dz}{-(x-y)(x+y+z)} \end{aligned}$$

$$\Rightarrow \frac{dx + dy}{x + y} = \frac{dx + dy + dz}{-(x + y + z)}$$

Integrating $\log(x + y) + \log(x + y + z) = \log c_2$

$$\Rightarrow \log(x + y)(x + y + z) = \log c_2 \Rightarrow v \equiv (x + y)(x + y + z) = c_2$$

Hence the required general solution is

$$f[xy, (x + y)(x + y + z)] = 0$$

Example 13: Solve $2x(y + z^2)p + y(2y + z^2)q = z^3$. [Delhi Maths (Hons.) 2007]

Solution: The subsidiary equations of the given equation are

$$\frac{dx}{2x(y + z^2)} = \frac{dy}{y(2y + z^2)} = \frac{dz}{z^3}$$

Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, each fraction

$$= \frac{\left(\frac{1}{x}\right)dx - \left(\frac{1}{y}\right)dy - \left(\frac{1}{z}\right)dz}{2(y + z^2) - (2y + z^2) - z^2} = \frac{\left(\frac{1}{x}\right)dx - \left(\frac{1}{y}\right)dy - \left(\frac{1}{z}\right)dz}{0}$$

$$\therefore \frac{1}{x}dx - \frac{1}{y}dy - \frac{1}{z}dz = 0$$

Integrating, $\log x - \log y - \log z = \log c_1 \Rightarrow \log(x/yz) = \log c_1 \Rightarrow u \equiv x/yz = c_1$

Again taking 2nd and 3rd fractions, we get

$$\frac{dy}{dz} = \frac{y(2y + z^2)}{z^3} = \frac{2y^2}{z^3} + \frac{y}{z}$$

$$\text{or } \frac{1}{y^2} \frac{dy}{dz} - \frac{1}{z} \frac{1}{y} = \frac{2}{z^3} \quad (\text{Dividing by } y^2)$$

Putting $-\frac{1}{y} = t$, $\frac{1}{y^2} \frac{dy}{dz} = \frac{dt}{dz}$, it reduces to

$$\frac{dt}{dz} + \frac{1}{z}t = \frac{2}{z^3} \quad \dots(1)$$

Which is linear differential equation, whose

$$\text{I.F.} = e^{\int (1/z) dz} = e^{\log z} = z$$

$\therefore$ Solution of (1) is

$$\text{t.z.} = c_2 + \int \frac{2}{z^3} \cdot z \, dz = c_2 + 2 \int \frac{1}{z^2} \, dz = c_2 - \frac{2}{z}$$

or $-\frac{z}{y} + \frac{2}{z} = c_2$ i.e., $v \equiv \frac{2}{z} - \frac{z}{y} = c_2$

Hence, the required general solution is

$$f(u, v) = 0 \text{ i.e., } f\left(\frac{x}{yz}, \frac{2}{z} - \frac{z}{y}\right) = 0$$

Example 14: Solve $(x + 2z)p + (4zx - y)q = 2x^2 + y$

Solution: Lagrange's subsidiary equations are

$$\frac{dx}{x + 2z} = \frac{dy}{4zx - y} = \frac{dz}{2x^2 + y} \quad \dots(1)$$

Using $y, x - 2z$ as multipliers, each fraction in (1)

$$= \frac{y \, dx + x \, dy - 2z \, dz}{y(x + 2z) + x(4zx - y) - 2z(2x^2 + y)} = \frac{y \, dx + x \, dy - 2z \, dz}{0}$$

$$\therefore y \, dx + x \, dy - 2z \, dz = 0 \quad \text{or} \quad d(xy) - 2z \, dz = 0$$

Integrating, $u \equiv xy - z^2 = c_1$

Again using $2x, -1, -1$ as multipliers, each fraction in (1)

$$= \frac{2x \, dx - dy - dz}{2x(x + 2z) - (4zx - y) - (2x^2 + y)} = \frac{2x \, dx - dy - dz}{0}$$

$$\therefore 2x \, dx - dy - dz = 0. \text{ Integrating } v \equiv x^2 - y - z = c_2$$

Hence, the required general solution is

$$f(u, v) = 0 \quad \text{i.e., } f(xy - z^2, x^2 - y - z) = 0$$

Example 15: Solve $z - xp - yq = a\sqrt{x^2 + y^2 + z^2}$

Solution: The given equation can be written as

$$xp - yq = z - a\sqrt{x^2 + y^2 + z^2}$$

$\therefore$ Lagrange's subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a\sqrt{x^2 + y^2 + z^2}}$$

Taking 1st and 2nd fractions, $\frac{dx}{x} = \frac{dy}{y}$

Integrating, $\log x = \log y + \log c_2$ or $x = y c_1$

$\therefore u \equiv x / y = c_1$

Now using x, y, z as multipliers

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a\sqrt{(x^2 + y^2 + z^2)}} = \frac{x dx + y dy + z dz}{x^2 + y^2 + z^2 - az\sqrt{(x^2 + y^2 + z^2)}}$$

Putting $x^2 + y^2 + z^2 = u^2$, so that $x dx + y dy + z dz = u du$

$$\frac{dx}{x} = \frac{dz}{z - au} = \frac{u du}{u^2 - azu} = \frac{du}{u - az} = \frac{dz + du}{(1 - a)(z + u)}$$

Taking $\frac{dx}{x} = \frac{dz + du}{(1 - a)(z + u)}$, $(1 - a) \frac{dx}{x} = \frac{dz + du}{z + u}$

Integrating, $(1 - a) \log x = \log(z + u) + \log c_2$

or $x^{1-a} = (z + u) c_2$ i.e., $v \equiv \frac{x^{1-a}}{(z + u)} = c_2$

Hence, the required general solution is

$$v = f(u) \text{ i.e., } \frac{x^{1-a}}{\{z + \sqrt{(x^2 + y^2 + z^2)}\}} = f(x/y)$$

or $x^{1-a} = \{z + \sqrt{(x^2 + y^2 + z^2)}\} f(x/y)$

Example 16: Solve, $(y^3 x - 2x^4) p + (2y^4 - x^3 y) q = 9z(x^3 - y^3)$. [Jabalpur 2004]

Solution: Lagrange's subsidiary equations are

$$\frac{dx}{y^3 x - 2x^4} = \frac{dy}{2y^4 - x^3 y} = \frac{dz}{9z(x^3 - y^3)}$$

Taking 1st and 2nd fractions, $(2y^4 - x^3 y) dx = (y^3 x - 2x^4) dy$

or $\left(\frac{2y}{x^3} - \frac{1}{y^2}\right) dx = \left(\frac{1}{x^2} - \frac{2x}{y^3}\right) dy$, Dividing by $x^3 y^3$

or $\left(\frac{1}{x^2} dy - \frac{2y}{x^3} dx\right) + \left(\frac{1}{y^2} dx - \frac{2x}{y^3} dy\right) = 0$ or $d\left(\frac{y}{x^2}\right) + d\left(\frac{x}{y^2}\right) = 0$

Integrating, $u \equiv \frac{y}{x^2} + \frac{x}{y^2} = c_1$

Now using $1/x, 1/y, 1/3z$ as multipliers.

$$\text{Each fraction} = \frac{\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{3z} dz}{(y^3 - 2x^3) + (2y^3 - x^3) + 3(x^3 - y^3)} = \frac{\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{3z} dz}{0}$$

$$\therefore \frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{3} \frac{dz}{z} = 0. \text{ Integrating } \log x + \log y + \frac{1}{3} \log z = \log c_2$$

$$\therefore v \equiv xyz^{1/3} = c_2$$

Hence, the required general solution is

$$f(u, v) = 0 \text{ i.e., } f(y/x^2 + x/y^2, xyz^{1/3}) = 0$$

Example 17: Solve $(mz - ny)p + (nx - lz)q = ly - mx$

[Bhopal 2004; Madras 2005; Kanpur 2005, 06]

Solution: Lagrange's subsidiary equations for the given equation are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Choosing x, y, z as multipliers, each fraction

$$= \frac{x dx + y dy + z dz}{x(mz - ny) + y(nx - lz) + z(ly - mx)} = \frac{x dx + y dy + z dz}{0}$$

$$\therefore x dx + y dy + z dz = 0$$

Integrating, we get $u \equiv x^2 + y^2 + z^2 = c_1$

Again choosing l, m, n as multipliers, each fraction

$$= \frac{l dx + m dy + n dz}{0}$$

$$\therefore l dx + m dy + n dz = 0$$

Integrating, we get $v \equiv lx + my + nz = c_2$

Hence, the required general solution is

$$f(u, v) = 0 \text{ i.e., } f(x^2 + y^2 + z^2, lx + my + nz) = 0$$

Example 18: Solve $(y + zx)p - (x + yz)q = x^2 - y^2$ [Gate 2001; Meerut 2010 (Sem. I)]

Solution: Lagrange's subsidiary equations for the given equation are

$$\frac{dx}{y + zx} \equiv -\frac{dy}{(x + yz)} = \frac{dz}{x^2 - y^2}$$

Using $y, x, 1$ as multipliers, each fraction

$$= \frac{y dx + x dy + 1 \cdot dz}{y(y + zx) - x(x + yz) + 1 \cdot (x^2 - y^2)} = \frac{y dx + x dy + dz}{0}$$

$$\therefore y dx + x dy + dz = 0 \quad \text{or} \quad d(xy) + dz = 0$$

Integrating, $u \equiv xy + z = c_1$

Again using $x, y, -z$ as multipliers, each fraction

$$= \frac{x dx + y dy - z dz}{x(y + zx) - y(x + yz) - z(x^2 - y^2)} = \frac{x dx + y dy - z dz}{0}$$

$$\therefore x dx + y dy - z dz = 0$$

Integrating, $v \equiv x^2 + y^2 - z^2 = c_2$

Hence, the required general solution is

$$f(u, v) = 0 \quad \text{i.e., } f(xy + z, x^2 + y^2 - z^2) = 0$$

Example 19: Solve $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$

[I.A.S. 2004; Agra 2005; Delhi Maths (Hons.) 2006; Purvanchal 2007]

Solution: Lagrange's subsidiary equations for the given equation are

$$\frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{z(x^2 - y^2)}$$

Choosing $1/x, 1/y, 1/z$ as multipliers, each fraction

$$= \frac{\left(\frac{1}{x}\right) dx + \left(\frac{1}{y}\right) dy + \left(\frac{1}{z}\right) dz}{(y^2 + z) - (x^2 + z) + (x^2 - y^2)} = \frac{\left(\frac{1}{x}\right) dx + \left(\frac{1}{y}\right) dy + \left(\frac{1}{z}\right) dz}{0}$$

$$\therefore \frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz = 0, \text{ Integrating } \log x + \log y + \log z = \log c_1$$

$$\Rightarrow \log xyz = \log c_1 \Rightarrow u \equiv xyz = c_1$$

Again choosing $x, y, -1$ as multipliers, each fraction

$$= \frac{x dx + y dy - 1 dz}{x^2 (y^2 + z) - y^2 (x^2 + z) - z (x^2 - y^2)} = \frac{x dx + y dy - dz}{0}$$

$$\therefore x dx + y dy - dz = 0, \text{ Integrating } v \equiv x^2 + y^2 - 2z = c_2$$

Hence, the required general solution is

$$f(u, v) = 0 \text{ i.e., } f(xyz, x^2 + y^2 - 2z) = 0$$

Example 20: Solve $x(x^2 + 3y^2)p - y(3x^2 + y^2)q = 2z(y^2 - x^2)$

[Delhi Maths (Hons.) 2000]

Solution: Lagrange's subsidiary equations for the given equation are

$$\frac{dx}{x(x^2 + 3y^2)} = \frac{dy}{-y(3x^2 + y^2)} = \frac{dz}{2z(y^2 - x^2)}$$

Choosing $1/x, 1/y, -1/z$ as multipliers, each fraction

$$= \frac{\left(\frac{1}{x}\right)dx + \left(\frac{1}{y}\right)dy - \left(\frac{1}{z}\right)dz}{0} \quad \therefore \frac{1}{x}dx + \frac{1}{y}dy - \frac{1}{z}dz = 0$$

$$\text{Integrating, } \log x + \log y - \log z = \log c_1 \Rightarrow \log \left(\frac{xy}{z}\right) = \log c_1$$

$$\Rightarrow u \equiv \frac{xy}{z} = c_1$$

Now, from the first two fractions, we have

$$\frac{dy}{dx} = \frac{-y(3x^2 + y^2)}{x(x^2 + 3y^2)}$$

which is homogeneous, so putting $y = vx$, $\frac{dy}{dx} = v + x \frac{dv}{dx}$, we get

$$v + x \frac{dv}{dx} = -\frac{v(3 + v^2)}{1 + 3v^2}$$

$$\Rightarrow x \frac{dv}{dx} = -\frac{v(3 + v^2)}{1 + 3v^2} - v = -\frac{4(1 + v^2)v}{1 + 3v^2}$$

$$\Rightarrow 4 \frac{dx}{x} + \frac{1 + 3v^2}{(1 + v^2)v} dv = 0 \Rightarrow 4 \frac{dx}{x} + \frac{1 + v^2 + 2v^2}{(1 + v^2)v} dv = 0$$

$$\Rightarrow 4 \frac{dx}{x} + \left(\frac{1}{v} + \frac{2v}{1+v^2} \right) dv = 0$$

Integrating, $4 \log x + \log v + \log (1+v^2) = \log c'_2$

$$\Rightarrow \log x^4 v (1+v^2) = \log c'_2$$

$$\Rightarrow x^4 \left(\frac{y}{x} \right) (1+y^2/x^2) = c'_2 \Rightarrow xy(x^2+y^2) = c'_2 \Rightarrow c_1 z (x^2+y^2) = c'_2$$

$$\Rightarrow v \equiv z (x^2+y^2) = c'_2 / c_1 = c_2, \text{ where } c_2 = c'_2 / c_1$$

Hence, the required general solution is

$$f(u, v) = 0 \text{ i.e., } f\left(\frac{xy}{z}, z(x^2+y^2)\right) = 0$$

Example 21: Solve $px(z-2y^2) = (z-xy)(z-y^2-2x^3)$ [I.A.S. 2006; Delhi 2008]

Solution: The given equation can be written as

$$x(z-2y^2)p + y(z-y^2-2x^3)q = z(z-y^2-2x^3)$$

Whose, Lagrange's subsidiary equation are

$$\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)} \quad \dots(1)$$

Taking 2nd and 3rd fractions, we get

$$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \frac{dz}{z} - \frac{dy}{y} = 0 \text{ Integrating } \log z - \log y = \log c_1$$

$$\Rightarrow \frac{z}{y} = c_1 \Rightarrow u \equiv \frac{z}{y} = c_1 \quad \dots(2)$$

Now taking 0, $-2y$, 1 as multipliers, each fraction

$$= \frac{-2y dy + dz}{-2y^2(z-y^2-2x^3) + z(z-y^2-2x^3)} = \frac{dz - 2y dy}{(z-2y^2)(z-y^2-2x^3)}$$

Taking this fraction and 1st fraction in (1), we get

$$\frac{dx}{x(z-2y^2)} = \frac{dz - 2y dy}{(z-2y^2)(z-y^2-2x^3)}$$

$$\Rightarrow \frac{dx}{x} = \frac{dz - 2y dy}{z - y^2 - 2x^3} = \frac{dt}{t - 2x^3}, \text{ taking } z - y^2 = t, dz - 2y dy = dt$$

$$\Rightarrow \frac{dt}{dx} = \frac{t - 2x^3}{x} \Rightarrow \frac{dt}{dx} - \frac{1}{x} t = -2x^2 \quad \dots(3)$$

Which is a L.D. equation in t .

$$\therefore \text{I.F.} = e^{\int (-1/x) dx} = e^{-\log x} = e^{\log (x^{-1})} = 1/x$$

$\therefore$ solution of (3) is

$$t \cdot (1/x) = c_2 + \int (-2x^2) \cdot (1/x) dx = c_2 - \int 2x dx = c_2 - x^2$$

$$\text{or} \quad \frac{z - y^2}{x} + x^2 = c_2 \quad \text{or} \quad v \equiv \frac{z - y^2 + x^3}{x}$$

Hence, the required general solution is

$$f(u, v) = 0 \quad \text{i.e.,} \quad f\left(\frac{z}{y}, \frac{(z - y^2 + x^3)}{x}\right) = 0$$

Aliter : To find second function v .

Taking 1st and 2nd fractions in (1), we get

$$\frac{dx}{x(z - 2y^2)} = \frac{dy}{y(z - y^2 - 2x^3)} \Rightarrow \frac{dx}{x(c_1 y - 2y^2)} = \frac{dy}{y(c_1 y - y^2 - 2x^3)}$$

$\because z = c_1 y$ from (2)

$$\Rightarrow (c_1 y - y^2 - 2x^3) dx + x(2y - c_1) dy = 0 \quad \dots(4)$$

Comparing (4) with $M dx + N dy = 0$ we get

$$M = c_1 y - y^2 - 2x^3 \text{ and } N = 2xy - c_1 x \therefore \frac{\partial M}{\partial y} = c_1 - 2y \text{ and } \frac{\partial N}{\partial x} = 2y - c_1$$

$$\therefore \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = \frac{1}{2xy - c_1 x} (2c_1 - 4y) = -\frac{2}{x}, \text{ which is a function of } x \text{ alone.}$$

$$\therefore \text{I.F. of (4)} = e^{\int (-2/x) dx} = e^{-2 \log x} = e^{\log x^{-2}} = 1/x^2$$

Multiplying (4) by $1/x^2$, we get

$$\left(\frac{c_1 y - y^2}{x^2} - 2x \right) dx + \frac{2y - c_1}{x} dy = 0 \quad \dots(5)$$

Which is exact, as for this equation $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ Solution of (5) is

$$\int M dx + \int (\text{Terms of } N, \text{ free from } x) dy = c'_2$$

treating y as constant

$$\Rightarrow \int [(c_1 y - y^2) x^{-2} - 2x] dx + 0 = c'_2$$

treating y as constant

$$\Rightarrow (c_1 y - y^2) (-x^{-1}) - x^2 = c'_2 \Rightarrow v \equiv z - y^2 + x^3 = c_2$$

Example 22: Find the surface whose tangent planes cut off an intercept of constant length k from the axis of z .

Solution: Equation of tangent plane at the point (x, y, z) is

$$(X - x)p + (Y - y)q = Z - z \quad \dots(1)$$

If this plane cut off intercept of constant length k from the axis of z , then it will pass through the point $(0, 0, k)$.

$\therefore$ Putting $X = 0, Y = 0, Z = k$ in (1), we get

$$xp + yq = z - k \quad \dots(2)$$

which is Lagrange's linear equation

$\therefore$ Lagrange's subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - k}$$

Taking 1st and 2nd fractions, $\frac{dx}{x} = \frac{dy}{y}$

$$\text{Integrating, } \log x = \log y - \log c_1 \quad \therefore u \equiv \frac{y}{x} = c_1$$

$$\text{Again taking 1st and 3rd fractions, } \frac{dx}{x} = \frac{dz}{z - k}$$

$$\text{Integrating, } \log x = \log (z - k) - \log c_2 \quad \therefore v \equiv (z - k) / x = c_2$$

Hence, the required general solution is

$$f[y/x, (z - k)/x] = 0$$

Example 23: Solve $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = xyz$.

Solution: Lagrange's subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{du}{xyz}$$

Taking 1st and 2nd fractions $\frac{dx}{x} = \frac{dy}{y}$

Integrating, $\log x = \log y + \log c_1$ or $x/y = c_1 \therefore u \equiv x/y = c_1$

Similarly taking 2nd and 3rd fractions, we get, $y/z = c_2 \therefore v \equiv y/z = c_2$

Again we have

$$\frac{yz dx + zx dy + xy dz}{3xyz} = \frac{du}{xyz} \quad \text{or} \quad yz dx + zx dy + xy dz = 3du$$

Integrating, $xyz - 3u = c_3 \therefore w \equiv xyz - 3u = c_3$

Hence, the required general solution is

$$f(u, v, w) = 0, \text{ i.e., } f\left(\frac{x}{y}, \frac{y}{z}, xyz - 3u\right) = 0$$

Example 24: Solve $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} + t \frac{\partial z}{\partial t} = az + \frac{xy}{t}$.

Solution: Lagrange's subsidiary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{t} = \frac{dz}{az + xy/t}$$

Taking 1st and 2nd fractions, $\frac{dx}{x} = \frac{dy}{y}$

Integrating $\log x = \log y - \log c_1 \therefore u \equiv y/x = c_1$

Similarly, taking 1st and 3rd fractions, we have

$$v \equiv t/x = c_2$$

Again taking 1st and 4th members, we have

$$\frac{dx}{x} = \frac{dz}{az + xy/t}$$

$$\text{or} \quad \frac{dz}{dx} - \frac{a}{x} z = \frac{y}{t} \quad \text{or} \quad \frac{dz}{dx} - \frac{a}{x} z = \frac{c_1}{c_2}$$

which is linear differential equation.

$$\text{I.F.} = e^{-\int \frac{a}{x} dx} = e^{-a \log x} = \frac{1}{x^a}$$

$$\therefore z \cdot \frac{1}{x^a} = \frac{c_1}{c_2} \int x^{-a} dx + c_3 \quad \text{or } zx^{-a} = \frac{y}{t} \cdot \frac{x^{1-a}}{1-a} + c_3$$

$$\therefore w \equiv z \cdot x^{-a} - \frac{y}{t} \cdot \frac{x^{1-a}}{1-a} = c_3$$

Hence, the required general solution is

$$f(u, v, w) = 0 \quad \text{i.e.,} \quad f\left(\frac{y}{x}, \frac{t}{x}, zx^{-a} - \frac{y}{t} \cdot \frac{x^{1-a}}{1-a}\right) = 0.$$

Example 25: Solve $(y + z + t) \frac{\partial t}{\partial x} + (z + x + t) \frac{\partial t}{\partial y} + (x + y + t) \frac{\partial t}{\partial z} = x + y + z$.

[Kanpur 2004; Ravishankar 2004]

Solution: Lagrange's subsidiary equations are

$$\frac{dx}{y + z + t} = \frac{dy}{y + x + t} = \frac{dz}{x + y + t} = \frac{dt}{x + y + z}$$

$$\therefore \frac{dx - dy}{-(x - y)} = \frac{dy - dz}{-(y - z)} = \frac{dz - dt}{-(z - t)} = \frac{dx + dy + dz + dt}{3(x + y + z + t)}$$

Taking 1st and 2nd, fractions $\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}$

Integrating $\log(x - y) = \log(y - z) + \log c_1 \therefore u \equiv \frac{x - y}{y - z} = c_1$

Similarly taking 2nd and 3rd we have $v \equiv \frac{z - t}{y - z} = c_2$

Again taking 3rd and 4th, we have

$$\frac{dz - dt}{z - t} + \frac{dx + dy + dz + dt}{3(x + y + z + t)} = 0$$

Integrating $\log(z - t) + \frac{1}{3} \log(x + y + z + t) = \log c_3$

$$\therefore w \equiv (z - t)(x + y + z + t)^{1/3} = c_3$$

Hence, the required general solution is

$$f(u, v, w) = 0 \quad \text{i.e.,} \quad f\left(\frac{x - y}{y - z}, \frac{z - t}{y - z}, (z - t)(x + y + z + t)^{1/3}\right) = 0$$

Example 26: Solve $(x_3 - x_2) p_1 + x_2 p_2 - x_3 p_3 + x_2^2 - (x_2 x_1 + x_2 x_3) = 0$.

Solution: The given equation can be written as :

$$(x_3 - x_2) p_1 + x_2 p_2 - x_3 p_3 = x_2 x_1 + x_2 x_3 - x_2^2.$$

$\therefore$ Lagrange's subsidiary equations are

$$\frac{dx_1}{x_3 - x_2} = \frac{dx_2}{x_2} = \frac{dx_3}{-x_3} = \frac{dz}{x_2 x_1 + x_2 x_3 - x_2^2}$$

Taking 2nd and 3rd fractions $\frac{dx_2}{x_2} + \frac{dx_3}{x_3} = 0$

Integrating $\log x_2 + \log x_3 = \log c_1$ or $x_2 x_3 = c_1$, $\therefore u \equiv x_2 x_3 = c_1$

Again, each fraction $= \frac{dx_1 + dx_2 + dx_3}{0}$

$\therefore dx_1 + dx_2 + dx_3 = 0$. Integrating $x_1 + x_2 + x_3 = c_2$

$\therefore v \equiv x_1 + x_2 + x_3 = c_2$

Again using $x_2, x_1, 0, -1$ as multipliers, each fraction

$$= \frac{x_2 dx_1 + x_1 dx_2 - dz}{x_2 (x_3 - x_2) + x_1 x_2 - (x_2 x_1 + x_2 x_3 - x_2^2)} = \frac{x_2 dx_1 + x_1 dx_2 - dz}{0}$$

$\therefore x_2 dx_1 + x_1 dx_2 - dz = 0$ or $d(x_1 x_2) - dz = 0$

Integrating, $w \equiv x_1 x_2 - z = c_3$

Hence, the required general solution is

$$f(u, v, w) = 0 \text{ i.e., } f(x_2 x_3, x_1 + x_2 + x_3, x_1 x_2 - z) = 0$$

Example 27: If u is a function of x, y and z which satisfy

$$(y - z) \frac{\partial u}{\partial x} + (z - x) \frac{\partial u}{\partial y} + (x - y) \frac{\partial u}{\partial z} = 0,$$

show that u contains x, y, z only in combination of $x + y + z$ and $x^2 + y^2 + z^2$.

[Nagpur 2002, 05]

Solution: The subsidiary equations of the given equation are

$$\frac{dx}{y - z} = \frac{dy}{z - x} = \frac{dz}{x - y} = \frac{du}{0} \quad \dots(1)$$

Each fraction of (1) = $\frac{dx + dy + dz}{0} = \frac{x dx + y dy + z dz}{0} = \frac{du}{0}$

$\therefore dx + dy + dz = 0, x dx + y dy + z dz = 0$ and $du = 0$

Integrating $u_1 \equiv x + y + z = c_1, u_2 \equiv x^2 + y^2 + z^2 = c_2$ and $u_3 = u \equiv c_3$

Hence, the required general solution is

$$u_3 = f(u_1, u_2) \text{ i.e., } u = f(x + y + z, x^2 + y^2 + z^2).$$

Exercise 7.2

Solve the following partial differential equations.

1. $p + q = \frac{z}{a}$

2. $p - q = \frac{z}{(x + y)}$

3. $\frac{y^2 z}{x} p + zxq = y^2$

[Indore 2004]

4. $(y + z)p + (z + x)q = x + y$

[Jabalpur 2000; Jiwaji 2002; Ravishankar 2002, 05; Purvanchal 2007]

5. $pz - qz = z^2 + (x + y)^2$

6. $xyp + y^2 q = zxy - 2x^2$

[Garhwal 2005]

7. $p + 3q = z + \cot(y - 3x)$

[M.D.U. Rohtak 2006]

8. $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

[Delhi Maths 2007]

9. $x^2(y - z)p + (z - x)y^2 q = z^2(x - y)$

[Hint : Use $1/x^2, 1/y^2, 1/z^2$ and $1/x, 1/y, 1/z$ multipliers]

10. $\left(\frac{y - z}{yz}\right)p + \left(\frac{z - x}{zx}\right)q = \frac{x - y}{xy}$

[Vikram 2003; I.A.S. 2005]

11. $(y^2 + z^2 - x^2)p - 2xyq + 2zx = 0$

12. $z(xp - yq) = y^2 - x^2$

13. $(1 + y)p + (1 + x)q = z$

14. $px(z - 2y^2) = (z - yq)(z - y^2 - 2x^2)$

[Delhi Maths (Hons.) 2002]

[Hint : Proceed as in Ex. 21]

15. $x^2 p + y^2 q = (x + y)z$

16. $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$

17. $(x + 2z)p + (4xz - y)q = 2x^2 + y$

[Meerut 2005]

[Hint : Use $2x, -1, -1$ and $y, x, -2z$ multipliers]

18. $(x - y)p + (x + y)q = 2xz$

19. $p + q = x + y + z$

[Bilaspur 2000, 02; Ravishankar 2002]

20. $x_2x_3p_1 + x_3x_1p_2 + x_1x_2p_3 + x_1x_2x_3 = 0$

21. $p_2 + p_3 = 1 + p_1$

22. $p_1 + x_1p_2 + x_1x_2p_3 = x_1x_2x_3\sqrt{z}$

Answers 7.2

1.	$z = e^{y/a} f(x - y)$ or $f(x - y, ze^{-y/a})$	2.	$(x + y) \log z = x + f(x + y)$
3.	$f(x^3 - y^3, x^2 - z^2) = 0$	4.	$f\left[\frac{x - y}{y - z}, (x - y)^2(x + y + z)\right] = 0$
5.	$f[x + y, \log(x^2 + y^2 + z^2 + 2xy) - 2x] = 0$	6.	$f[x/y, x - \log(z - 2x^2/y^2)] = 0$
7.	$f[y - 3x, x - \log z + \cot(y - 3x)] = 0$	8.	$f(x^2 + y^2 + z^2, xyz) = 0$
9.	$f(xyz, 1/x + 1/y + 1/z) = 0$	10.	$f(x + y + z, xyz) = 0$
11.	$x^2 + y^2 + z^2 = z f(y/z)$	12.	$f(xy, x^2 + y^2 + z^2) = 0$
13.	$f[(1 + x)^2 - (1 + y)^2, (2 + x + y)/z] = 0$	14.	$f[y/z, (z - y^2 + 2x^2)/x] = 0$
15.	$f\left[\frac{(x - y)}{xy}, \frac{xy}{z}\right] = 0$	16.	$f(x^2 + y^2 + z^2, y^2 - 2yz - z^2) = 0$
17.	$f(x^2 - y - z, xy - z^2) = 0$	18.	$(x^2 + y^2)e^{-2 \tan^{-1}(y/x)} = f(x + y - \log z)$
19.	$e^{-y}(x + y + z + 2) = f(x - y)$	20.	$f(x_1^2 + 2z, x_1^2 - x_2^2, x_1^2 - x_3^2) = 0$
21.	$f(x_1 + z, x_1 + x_2, x_1 + x_3) = 0$	22.	$(4\sqrt{z} - x_3^2, 2x_3 - x_2^2, 2x_2 - x_1^2) = 0$

7.10 Integral Surfaces Passing Through a Given Curve

Here we present method to find the integral surface of the partial differential equation

$$Pp + Qq = R$$

which passes through a given curve.

Working method : Let

$$Pp + Qq = R \quad \dots(1)$$

be a given differential equation

$$\text{Let} \quad u(x, y, z) = c_1 \quad \text{and} \quad v(x, y, z) = c_2 \quad \dots(2)$$

be two independent solutions of the Lagrange's subsidiary equations.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Suppose we wish to obtain the integral surface passing through the curve which is determined by the two equations.

$$\phi(x, y, z) = 0 \quad \text{and} \quad \psi(x, y, z) = 0 \quad \dots(3)$$

For this we eliminate x, y, z between equations given in (2) and (3) and get a relation between c_1 and c_2 . Then required integral surface is obtained by replacing c_1 by $u(x, y, z)$ and c_2 by $v(x, y, z)$ from (2), in this relation between c_1 and c_2 .

Solved Examples

Example 1: Find the equation of the surface satisfying.

$$4yzp + q + 2y = 0 \text{ and passing through } y^2 + z^2 = 1 \text{ and } x + z = 2.$$

Solution: The given equation can be written as

$$4yzp + q = -2y \quad \dots(1)$$

$\therefore$ Lagrange's subsidiary equations are

$$\frac{dx}{4yz} = \frac{dy}{1} = \frac{dz}{-2y}$$

From 1st and 3rd, fractions, $dx + 2z dz = 0$

$$\text{Integrating,} \quad u \equiv x + z^2 = c_1 \quad \dots(2)$$

Again from 2nd and 3rd fractions, $2ydy + dz = 0$

Integrating, $v \equiv y^2 + z = c_2$... (3)

The surface satisfying (1), are required to pass through $y^2 + z^2 = 1$... (4)

and $x + z = 2$... (5)

To eliminate x, y, z between (2), (3), (4) and (5), we add (2) and (3), and get

$$y^2 + z^2 + x + z = c_1 + c_2$$

$$\Rightarrow 1 + 2 = c_1 + c_2 \Rightarrow c_1 + c_2 = 3$$
 ... (6)

Substituting the values of c_1 and c_2 from (2) and (3) is (6) the required surface is

$$x + z^2 + y^2 + z = 3 \quad \text{i.e.,} \quad y^2 + z^2 + x + z - 3 = 0$$

Example 2: Find the equation of the integral surface of the differential equation

$$2y(z-3)p + (2x-z)q = y(2x-3),$$

which pass through the circle $z = 0, x^2 + y^2 = 2x$. [Meerut 2007]

Solution: The given differential equation is

$$2y(z-3)p + (2x-z)q = y(2x-3)$$
 ... (1)

$\therefore$ Lagrange's subsidiary equations are

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)}$$

Taking 1st and 3rd fractions, $(2x-3)dx = 2(z-3)dz$

Integrating, $u \equiv x^2 - 3x - z^2 + 6z = c_1$... (2)

Now using $1/2, y, -1$ as multipliers, each fraction

$$= \frac{(1/2)dx + ydy - dz}{y(z-3) + y(2x-z) - y(2x-3)} = \frac{(1/2)dx + ydy - dz}{0}$$

$\therefore \frac{1}{2}dx + ydy - dz = 0$ or $dx + 2ydy - 2dz = 0$

Integrating $v \equiv x + y^2 - 2z = c_2$... (3)

The integral surface of (1) given by (2) and (3), is required to pass through the circle

$$x^2 + y^2 = 2x, z = 0 \quad \dots(4)$$

For which we shall eliminate x, y, z between (2), (3) and (4).

$\therefore$ Putting $z = 0$ in (2) and (3), we get

$$x^2 - 3x = c_1 \text{ and } x + y^2 = c_2$$

$$\text{or } c_1 + c_2 = 0, \text{ using (4)} \quad \dots(5)$$

Substituting the values of c_1 and c_2 from (2) and (3) in (5), the required integral surface is

$$(x^2 - 3x - z^2 + 6z) + (x + y^2 - 2z) = 0$$

$$\text{or } x^2 + y^2 - z^2 - 2x + 4z = 0$$

Example 3: Find the integral surface of the equation

$$(x - y) y^2 p + (y - x) x^2 q = (x^2 + y^2) z, \text{ through the curve } xz = a^3, y = 0.$$

Solution: The given equation is

$$(x - y) y^2 p + (y - x) x^2 q = (x^2 + y^2) z \quad \dots(1)$$

The Lagrange's subsidiary equations are

$$\frac{dx}{(x - y) y^2} = \frac{dy}{(y - x) x^2} = \frac{dz}{(x^2 + y^2) z}$$

Taking 1st and 2nd fractions, $x^2 dx + y^2 dy = 0$

$$\text{Integrating } u \equiv x^3 + y^3 = c_1 \quad \dots(2)$$

$$\text{Again, } \frac{dx - dy}{(x - y)(x^2 + y^2)} = \frac{dz}{(x^2 + y^2)z} \quad \text{or} \quad \frac{dx - dy}{(x - y)} = \frac{dz}{z}$$

Integrating $\log(x - y) - \log z = \log c_2$

$$\therefore v \equiv (x - y) / z = c_2 \quad \dots(3)$$

The integral surface of (1) given by (2), (3) is required to pass through the curve

$$xz = a^3, y = 0 \quad \dots(4)$$

For which we shall eliminate x, y, z between (2), (3) and (4).

Putting $y = 0$ from (4) in (2) and (3), we get

$$x^3 = c_1 \quad \dots(5)$$

and $x/z = c_2$ or $x(x/a^3) = c_2$ using (4)

$$\text{or} \quad x^2 = a^3 c_2 \quad \dots(6)$$

$$\text{From (5) and (6) } (x^3)^2 = (x^2)^3 \Rightarrow c_1^2 = (a^3 c_2)^3 \Rightarrow a^9 c_2^3 \quad \dots(7)$$

Substituting the values of c_1 and c_2 from (2) and (3) in (7) the required integral surface is

$$(x^3 + y^3)^2 = a^9 [(x - y)/z]^3 \text{ i.e., } z^3 (x^3 + y^3)^2 = a^9 (x - y)^3.$$

Example 4: Find the integral surface of $x^2 p + y^2 q + z^2 = 0$, which passes through the hyperbola $xy = x + y, z = 1$.

Solution: The given partial differential equation can be written as :

$$x^2 p + y^2 q = -z^2 \quad \dots(1)$$

which is Lagrange's equation. So the Lagrange's subsidiary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{-z^2}$$

Taking the 1st and 2nd fractions, $\frac{dx}{x^2} = \frac{dy}{y^2}$

$$\text{Integrating, } -1/x = -1/y + c_1 \quad \therefore u \equiv 1/y - 1/x = c_1 \quad \dots(2)$$

Taking 1st and 3rd fractions, $\frac{dx}{x^2} + \frac{dz}{z^2} = 0$

$$\text{Integrating, } -1/x - 1/z + c_2 = 0 \quad \therefore v \equiv 1/x + 1/z = c_2 \quad \dots(3)$$

The integral surface of (1) given by (2) and (3) is required to pass through the curve

$$xy = x + y, z = 1 \quad \dots(4)$$

For which we eliminate x, y, z between (2), (3) and (4).

Putting $z = 1$ from (4) in (3), we get

$$1/x = c_2 - 1. \quad \therefore \text{from (2)} \quad 1/y = 1/x + c_1 = c_1 + c_2 - 1$$

$$\text{From (4),} \quad 1 = 1/y + 1/x = (c_1 + c_2 - 1) + c_2 - 1$$

$$\text{or} \quad c_1 + 2c_2 = 3 \quad \dots(5)$$

Substituting the values of c_1 and c_2 from (2) and (3) in (5), the required integral surface is

$$(1/y - 1/x) + 2(1/x + 1/z) = 3 \text{ i.e., } xz + yz + 2xy = 3xyz.$$

Example 5: Find the integral surface of the linear first order partial differential equation $yp + xq = z - 1$ which passes through the curve $z = x^2 + y^2 + 1$, $y = 2x$.

Solution: Given differential equation is

$$yp + xq = z - 1 \quad \dots(1)$$

Whose subsidiary equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z-1}$$

Taking 1st and 2nd fractions, we get

$$y \, dy - x \, dx = 0. \text{ Integrating } y^2 - x^2 = c_1 \quad \dots(2)$$

Again taking 1st and 3rd fractions, we get

$$\frac{dx}{\sqrt{c_1 + x^2}} = \frac{dz}{z-1}, \text{ using } y^2 - x^2 = c_1 \text{ from (2)}$$

Integrating, $\log(z-1) - \log(x + \sqrt{c_1 + x^2}) = \log c_2$

$$\Rightarrow \frac{(z-1)}{(x + \sqrt{c_1 + x^2})} = c_2 \Rightarrow \frac{(z-1)}{(x+y)} = c_2 \quad \dots(3)$$

The integral surface of (1) given by (2) and (3) is required to pass through the curve

$$z = x^2 + y^2 + 1, \quad y = 2x \quad \dots(4)$$

For which we eliminate x, y, z between (2), (3) and (4). Parametric form of curve (4) is

$$x = t, \quad y = 2t, \quad z = t^2 + 4t^2 + 1 = 5t^2 + 1 \quad \dots(5)$$

Substituting from (5) in (2) and (3), we get

$$4t^2 - t^2 = 3t^2 = c_1 \quad \text{and} \quad \frac{5t^2}{3t} = \frac{5t}{3} = c_2$$

Eliminating t , we get

$$3 \cdot (3c_2/5)^2 = c_1 \quad \text{or} \quad 27c_2^2 = 25c_1$$

Substituting the values of c_1 and c_2 from (2) and (3) the required integral surface is

$$27 \frac{(z-1)^2}{(x+y)^2} = 25 (y^2 - x^2)$$

i.e., $27 (z-1)^2 = 25 (y-x)(x+y)^3.$

Exercise 7.3

- Find the integral surface of the partial differential equation $(x-y)p + (y-x-z)q = z$, through the circle $z=1, x^2 + y^2 = 1$. [Nagpur 2002]
- Find the integral surface of the partial differential equation $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ which contains the straight line $x+y=0, z=1$. [Meerut 2008 (BP); Delhi 2008]
- Solve $xp + yq = z$. Find a solution representing a surface meeting the parabola $y^2 = 4x, z=1$.
- Find the general integral of the partial differential equation $(2xy-1)p + (z-2x^2)q = 2(x-yz)$ and also the particular integral which passes through the line $x=1, y=0$. [I.A.S. 2008; Meerut 2011 (Sem. I)]
- Find the equation of the integral surface of the differential equation $(x^2 - yz)p + (y^2 - zx)q = x^2 - xy$ which passes through the line $x=1, y=0$.

Answers 7.3

1.	$z^4 (x+y+z)^2 + (y-x-z)^2 - 2z^4 (x+y+z) + 2z^2 (y-x-z) = 0$
2.	$2xyz + x^2 + y^2 - 2z + 2 = 0$
3.	$f(x/2, y/2) = 0, y^2 = 4xz$
4.	$f(xz + y, x^2 + y^2 + z) = 0; x^2 + y^2 + z - xz - y = 1$
5.	$(x-y)(xy + yz + zx) + y - z = 0$

7.1.1 Surface Orthogonal to a Given System of Surface

- I. The surfaces represented by $Pp + Qq = R$ are orthogonal to the surfaces represented by $Pdx + Qdy + Rdz = 0$ (if integrable).

Proof: The d.r.'s of the normal at the point (x, y, z) to the surface given by

$$Pp + Qq = R \quad \dots(1)$$

are $p, q, -1$

And the d.r.'s of the normal at the point (x, y, z) to the surface represented by

$$Pdx + Qdy + Rdz = 0 \quad \dots(2)$$

are P, Q, R

Since $p.P + q.Q + (-1).R = Pp + Qq - R = 0$, by (1)

so the two normals are perpendicular to each other.

Hence the two surfaces represented by the differential equations (1) and (2) are orthogonal to each other.

- II. The partial differential equation of the surfaces orthogonal to the surfaces represented by $f(x, y, z) = c$ is given by $\frac{\partial f}{\partial x} p + \frac{\partial f}{\partial y} q = \frac{\partial f}{\partial z}$.

Proof: The d.r.'s of the normal at the point (x, y, z) to the surfaces represented by

$$f(x, y, z) = c \quad \dots(1)$$

are $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

Let $z = \phi(x, y) \quad \dots(2)$

represent the surfaces which are orthogonal to the surfaces represented by (1).

The d.r.'s of the normal at the point (x, y, z) to the surface represented by (2) are

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1 \text{ i.e., } p, q, -1$$

Since the surfaces represented by (1) and (2) are orthogonal

$$\therefore \frac{\partial f}{\partial x} . p + \frac{\partial f}{\partial y} . q + \frac{\partial f}{\partial z} (-1) = 0$$

or
$$\frac{\partial f}{\partial x} p + \frac{\partial f}{\partial y} q = \frac{\partial f}{\partial z} \quad \dots(3)$$

which is the partial differential equation of the surfaces, orthogonal to the surfaces given by (1).

Working Method

To find the equation of the surfaces orthogonal to the surfaces given by $f(x, y, z) = c$

find
$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$$

Then form the differential equation of the orthogonal surfaces *i.e.*,

$$\frac{\partial f}{\partial x} p + \frac{\partial f}{\partial y} q = \frac{\partial f}{\partial z}$$

Solving this differential equation, find the required equation of the orthogonal surfaces.

Solved Examples

Example 1: Find the family of surfaces orthogonal to the family of surfaces given by the equation

$$(y + z)p + (z + x)q = x + y$$

Solution: Given $(y + z)p + (z + x)q = x + y \quad \dots(1)$

Comparing with $Pp + Qq = R$, we get

$$P = y + z, Q = z + x, R = x + y$$

$\therefore$ The differential equation of the family of surfaces orthogonal to the family of surfaces given by (1) is

$$P dx + Q dy + R dz = 0$$

or
$$(y + z) dx + (z + x) dy + (x + y) dz = 0$$

or
$$(x dy + y dx) + (y dz + z dy) + (z dx + x dz) = 0$$

or
$$d(xy) + d(yz) + d(zx) = 0$$

Integrating
$$xy + yz + zx = c$$

Which is the required family of orthogonal surfaces.

Example 2: Find the family orthogonal to $\phi\{z(x+y)^2, x^2 - y^2\} = 0$

Solution: Given, $\phi\{z(x+y)^2, x^2 - y^2\} = 0$

$$\text{or} \quad \phi(u, v) = 0 \quad \dots(1)$$

$$\text{where} \quad u = z(x+y)^2, v = x^2 - y^2$$

Differentiating (1) partially w.r.t. x and y , we get

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial x} = 0$$

$$\text{i.e.,} \quad \frac{\partial \phi}{\partial u} \cdot \{p(x+y)^2 + 2z(x+y)\} + \frac{\partial \phi}{\partial v} (2x - 0) = 0 \quad \dots(2)$$

$$\text{and} \quad \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial y} = 0$$

$$\text{i.e.,} \quad \frac{\partial \phi}{\partial u} \{q(x+y)^2 + 2z(x+y)\} + \frac{\partial \phi}{\partial v} (0 - 2y) = 0 \quad \dots(3)$$

Eliminating $\frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v}$ from (2) and (3), we get

$$\begin{vmatrix} p(x+y)^2 + 2z(x+y) & 2x \\ q(x+y)^2 + 2z(x+y) & -2y \end{vmatrix} = 0$$

$$\text{or} \quad \{-y(x+y)^2 p - 2yz(x+y)\} - \{x(x+y)^2 q + 2zx(x+y)\} = 0$$

$$\text{or} \quad y(x+y)^2 p + x(x+y)^2 q = -2z(x+y)(y+x)$$

$$\text{or} \quad yp + xq = -2z \quad \dots(4)$$

Which is the partial differential equation of the given surfaces.

Comparing (4) with $Pp + Qq = R$, we get $P = y, Q = x$ and $R = -2z$

$\therefore$ The differential equation of the family of surfaces orthogonal to the family of surfaces given by (4) i.e., (1), is

$$Pdx + Qdy + Rdz = 0$$

$$\text{or} \quad ydx + xdy - 2zdz = 0 \quad \text{or} \quad d(xy) - 2zdz = 0$$

$$\text{Integrating, we get} \quad xy - z^2 = c$$

which is the required surface.

Example 3: Find the surface which cut orthogonally the cones of the family

$$x^2 + y^2 + z^2 = Cxy$$

Solution: Given system of surfaces is

$$f(x, y, z) = \frac{x^2 + y^2 + z^2}{xy} = \frac{x}{y} + \frac{y}{x} + \frac{z^2}{xy} = C \quad \dots(1)$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{1}{y} - \frac{y}{x^2} - \frac{z^2}{x^2 y} = \frac{x^2 - y^2 - z^2}{x^2 y}, \quad \frac{\partial f}{\partial y} = \frac{-x}{y^2} + \frac{1}{x} - \frac{z^2}{xy^2} = \frac{-x^2 + y^2 - z^2}{xy^2}$$

and $\frac{\partial f}{\partial z} = \frac{2z}{xy}$

$\therefore$ The differential equation of orthogonal surface is

$$\frac{\partial f}{\partial x} p + \frac{\partial f}{\partial y} q = \frac{\partial f}{\partial z}$$

or $\left(\frac{x^2 - y^2 - z^2}{x^2 y} \right) p + \left(\frac{-x^2 + y^2 - z^2}{xy^2} \right) q = \frac{2z}{xy}$

or $y(x^2 - y^2 - z^2)p + x(-x^2 + y^2 - z^2)q = 2xyz$

$\therefore$ The Lagrange's subsidiary equations are

$$\frac{dx}{y(x^2 - y^2 - z^2)} = \frac{dy}{x(-x^2 + y^2 - z^2)} = \frac{dz}{2xyz}$$

Taking x, y, z as multipliers, each fraction

$$= \frac{x dx + y dy + z dz}{xy(x^2 - y^2 - z^2) + xy(-x^2 + y^2 - z^2) + 2xyz} = \frac{x dx + y dy + z dz}{0}$$

$\therefore x dx + y dy + z dz = 0$. Integrating, $u \equiv x^2 + y^2 + z^2 = c_1 \quad \dots(2)$

Now taking 2nd and 3rd fractions, we get

$$\begin{aligned} \frac{dy}{-x^2 + y^2 - z^2} &= \frac{dz}{2yz} \Rightarrow \frac{dy}{2y^2 - c_1} = \frac{dz}{2yz} \quad \because x^2 + z^2 = c_1 - y^2 \text{ from (2)} \\ &\Rightarrow \frac{4y dy}{2y^2 - c_1} = \frac{2dz}{z} \end{aligned}$$

Integrating, $\log(2y^2 - c_1) = 2 \log z + \log c_2$

$$\Rightarrow \log [(2y^2 - c_1)/z^2] = \log c'_2$$

$$\Rightarrow (2y^2 - c_1)/z^2 = c'_2 \quad \Rightarrow \{2y^2 - (x^2 + y^2 + z^2)\}/z^2 = c'_2$$

$$\Rightarrow \frac{y^2 - x^2}{z^2} = 1 + c'_2 = c_2, \text{ where } 1 + c'_2 = c_2$$

$$\Rightarrow v \equiv \frac{(y^2 - x^2)}{z^2} = c_2$$

Hence, the general equation of the surface which are orthogonal to the given family of surface (1) is of the form,

$$\frac{(y^2 - x^2)}{z^2} = \phi(x^2 + y^2 + z^2)$$

where ϕ is an arbitrary constant.

Example 4: Find the surface which intersects the surfaces of the system $z(x + y) = C(3z + 1)$ orthogonally and which passes through the circle $x^2 + y^2 = 1, z = 1$. [Meerut 2013 (Sem. I)]

Solution: Given system of surfaces is

$$f(x, y, z) = \frac{z(x + y)}{3z + 1} = C \quad \dots(1)$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{z}{3z + 1}, \quad \frac{\partial f}{\partial y} = \frac{z}{3z + 1} \text{ and } \frac{\partial f}{\partial z} = \frac{x + y}{(3z + 1)^2}$$

$\therefore$ The differential equation of orthogonal surfaces is

$$\frac{\partial f}{\partial x} p + \frac{\partial f}{\partial y} q = \frac{\partial f}{\partial z}$$

$$\text{or } \frac{z}{3z + 1} p + \frac{z}{3z + 1} q = \frac{x + y}{(3z + 1)^2}$$

$$\text{or } z(3z + 1)p + z(3z + 1)q = x + y$$

$\therefore$ The Lagrange's subsidiary equations are

$$\frac{dx}{z(3z + 1)} = \frac{dy}{z(3z + 1)} = \frac{dz}{x + y} \quad \dots(2)$$

Taking 1st and 2nd fractions, we get

$$dx - dy = 0 \text{ Integrating } u \equiv x - y = C_1 \quad \dots(3)$$

Again taking $x, y, -z(3z+1)$ as multipliers, each fraction

$$= \frac{x dx + y dy - z(3z+1) dz}{(x+y)z(3z+1) - (x+y) \cdot z(3z+1)} = \frac{x dx + y dy - z(3z+1) dz}{0}$$

$$\Rightarrow x dx + y dy - (3z^2 + z) dz = 0$$

$$\text{Integrating } v \equiv x^2 + y^2 - 2z^3 - z^2 = C_2 \quad \dots(4)$$

$\therefore$ the equation of the surfaces which are orthogonal to the given family of surfaces (1) is of the form,

$$x^2 + y^2 - 2z^3 - z^2 = \phi(x - y) \quad \dots(5)$$

If surfaces (5) passes through the circle

$$x^2 + y^2 = 1, z = 1, \text{ then from (5) } \phi(x - y) = 1 - 2 - 1 = -2$$

Hence, the required surfaces are

$$x^2 + y^2 - 2z^3 - z^2 = -2 \text{ i.e., } x^2 + y^2 - 2z^3 - z^2 + 2 = 0$$

Exercise 7.4

- Find the equation of the system of surfaces which cut orthogonally the system of surfaces given by $x(x^2 + y^2 + z^2) = Cy^2$. [I.A.S. 2001]
- Find the surface which is orthogonal to the one parameter family $z(x^2 + y^2) = Cxy$ and which passes through the hyperbola $x^2 - y^2 = a^2, z = 0$.

Answers 7.4

1.	$x^2 + y^2 + z^2 = z \phi\{(2x^2 + y^2)/z^2\}$
2.	$x^2 - y^2 = a^2 e^{2z^2/(x^2 + y^2)}$

7.12 Compatible System of First Order Equations

[Delhi Maths (Hons.) 2007]

Definition : Two first order partial differential equations

$$f(x, y, z, p, q) = 0 \quad \dots(1)$$

$$g(x, y, z, p, q) = 0 \quad \dots(2)$$

are said to be **compatible** if every solution of one equation is also a solution of the other.

To find the condition for the pair of equations (1) and (2) to be compatible.

[Delhi Maths (Hons.) 2000, 02, 08]

Let
$$J = \frac{\partial(f, g)}{\partial(p, q)} \neq 0 \quad \dots(3)$$

Then the equations (1) and (2) can be solved to obtain the explicit expressions for p and q given by

$$p = \phi(x, y, z) \quad \text{and} \quad q = \psi(x, y, z) \quad \dots(4)$$

Then the condition that the pair of equations (1) and (2) should be compatible reduces to the condition that the system of equations (4) should be completely integrable *i.e.*, the equation

$$dz = p dx + q dy \quad \text{i.e.,} \quad \phi dx + \psi dy - dz = 0 \quad \dots(5)$$

should be integrable.

But we know that the single differential equation $Pdx + Qdy + Rdz = 0$ will be completely integrable, if and only if

$$P \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) + Q \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) + R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = 0$$

$\therefore$ The single differential equation (5) is completely integrable, if and only if

$$\phi \left(\frac{\partial \psi}{\partial z} - 0 \right) + \psi \left(0 - \frac{\partial \phi}{\partial z} \right) + (-1) \left(\frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \right) = 0$$

i.e., if and only if
$$\frac{\partial \psi}{\partial x} + \phi \frac{\partial \psi}{\partial z} = \frac{\partial \phi}{\partial y} + \psi \frac{\partial \phi}{\partial z} \quad \dots(6)$$

Substituting from (4) in (1) and differentiating w.r.t. x and z respectively, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial p} \frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial q} \frac{\partial \psi}{\partial x} = 0 \quad \dots(7)$$

and
$$\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p} \frac{\partial \phi}{\partial z} + \frac{\partial f}{\partial q} \frac{\partial \psi}{\partial z} = 0 \quad \dots(8)$$

Multiplying (8) by ϕ and adding to (7), we get

$$\frac{\partial f}{\partial x} + \phi \frac{\partial f}{\partial z} + \left(\frac{\partial \phi}{\partial x} + \phi \frac{\partial \phi}{\partial z} \right) \frac{\partial f}{\partial p} + \left(\frac{\partial \psi}{\partial x} + \phi \frac{\partial \psi}{\partial z} \right) \frac{\partial f}{\partial q} = 0 \quad \dots(9)$$

In a similar manner substituting from (4) in (2) and differentiating w.r.t. x and z and proceeding as above, we get

$$\frac{\partial g}{\partial x} + \phi \frac{\partial g}{\partial z} + \left(\frac{\partial \phi}{\partial x} + \phi \frac{\partial \phi}{\partial z} \right) \frac{\partial g}{\partial p} + \left(\frac{\partial \psi}{\partial x} + \phi \frac{\partial \psi}{\partial z} \right) \frac{\partial g}{\partial q} = 0 \quad \dots(10)$$

Multiplying (9) by $\frac{\partial g}{\partial p}$ and (10) by $\frac{\partial f}{\partial p}$ and subtracting, we get

$$\left(\frac{\partial \psi}{\partial x} + \phi \frac{\partial \psi}{\partial z} \right) \left(\frac{\partial f}{\partial p} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} \right) = \left(\frac{\partial f}{\partial x} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} \right) + \phi \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial z} \right)$$

$$\text{or} \quad \frac{\partial \psi}{\partial x} + \phi \frac{\partial \psi}{\partial z} = \frac{1}{J} \left[\frac{\partial(f, g)}{\partial(x, p)} + \phi \frac{\partial(f, g)}{\partial(z, p)} \right] \quad \dots(11)$$

Again substituting from (4) in (1) and (2), and differentiating w.r.t. y and z and proceeding as above, we get

$$\frac{\partial \phi}{\partial y} + \psi \frac{\partial \phi}{\partial z} = -\frac{1}{J} \left[\frac{\partial(f, g)}{\partial(y, p)} + \psi \frac{\partial(f, g)}{\partial(z, q)} \right] \quad \dots(12)$$

Substituting from (11) and (12) in (6) and replacing ϕ by p and ψ by q , the condition that the two equation (1) and (2) are compatible is that

$$[f, g] = \frac{\partial(f, g)}{\partial(x, p)} + p \frac{\partial(f, g)}{\partial(z, p)} + \frac{\partial(f, g)}{\partial(y, q)} + q \frac{\partial(f, g)}{\partial(z, q)} = 0 \quad \dots(13)$$

7.12.1 Particular Case

Theorem: The first order partial differential equations, $p = P(x, y)$ and $q = Q(x, y)$ are compatible, if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Proof: If z is a function of two independent variables x and y , then we have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = p dx + q dy$$

$\therefore$ the given partial differential equations

$p = P(x, y)$ and $q = Q(x, y)$ are compatible if and only if the single differential equation.

$dz = P dx + Q dy$ is integrable.

But we know that if P and Q are functions of two variables x and y , then $Pdx + Qdy$ is an exact differential of some function $\phi(x, y)$.

i.e., $Pdx + Qdy = d\phi(x, y)$, if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$\therefore$ the single differential equation $dz = Pdx + Qdy$ is integrable iff $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

Hence, the difference equations $p = P(x, y)$ and $q = Q(x, y)$ are compatible, iff

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Solved Examples

Example 1: Show that the differential equations

$$p = 4x + 3y + 1, q = 3x + 2y + 1$$

are compatible and find their solution.

Solution: We know that the differential equations

$$p = P(x, y) \text{ and } q = Q(x, y) \text{ are compatible iff } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Here $P(x, y) = 4x + 3y + 1$ and $Q(x, y) = 3x + 2y + 1$

$$\therefore \frac{\partial P}{\partial y} = 3 = \frac{\partial Q}{\partial x}$$

Hence, the given differential equations are compatible.

$$\begin{aligned} \text{Now } dz &= p dx + q dy = Pdx + Qdy = (4x + 3y + 1) dx + (3x + 2y + 1) dy \\ &= (4x + 1) dx + 3(y dx + x dy) + (2y + 1) dy \end{aligned}$$

Integrating, $z = 2x^2 + x + 3xy + y^2 + y + c$. Where c is an arbitrary constant.

Which is the solution of the given differential equations.

Example 2: Show that the differential equations

$$\frac{\partial z}{\partial x} = (x + y)^2, \frac{\partial z}{\partial y} = x^2 + 2xy - y^2$$

are compatible and solve them.

Solution: Given differential equations are

$$p = \frac{\partial z}{\partial x} = (x + y)^2 \text{ and } q = \frac{\partial z}{\partial y} = x^2 + 2xy - y^2$$

or $p = P(x, y)$ and $q = Q(x, y)$

where $P(x, y) = (x + y)^2$ and $Q(x, y) = x^2 + 2xy - y^2$

and $\frac{\partial P}{\partial y} = 2(x + y) = \frac{\partial Q}{\partial x}$

$\therefore$ the given differential equations are compatible.

Now $dz = p dx + q dy = P dx + Q dy$

$$\begin{aligned} \text{or } dz &= (x + y)^2 dx + (x^2 + 2xy - y^2) dy \\ &= (x + y)^2 dx + \{(x + y)^2 - 2y^2\} dy \\ &= (x + y)^2 (dx + dy) - 2y^2 dy \end{aligned} \quad \dots(1)$$

Integrating both sides, the common solution of the given differential equations is

$$z = \frac{1}{3} (x + y)^3 - \frac{2}{3} y^3 + c$$

where c is an arbitrary constant.

Equation (1), can also be integrated as an exact differential equation as, here $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$$\begin{aligned} z &= \int P dx + \int (\text{Terms in } Q \text{ which do not contain } x) dy + c \\ &\quad \text{Treading } y \text{ const.} \\ &= \int (x + y)^2 dx + \int (-y^2) dy + c \\ &\quad \text{Treading } y \text{ const.} \\ &= \int (x^2 + 2xy + y^2) dx - \frac{1}{3} y^3 + c \\ &\quad \text{Treading } y \text{ const.} \\ &= \frac{1}{3} x^3 + x^2 y + x y^2 - \frac{1}{3} y^3 + c = \frac{1}{3} (x + y)^3 - \frac{2}{3} y^3 + c. \end{aligned}$$

Example 3: Show that the equations $xp = yq$, $z(xp + yq) = 2xy$ are compatible and solve them. [Delhi Maths (Hons.) 2005, 07]

Solution: Let $f = xp - yq = 0 \quad \dots(1)$

and $g = z(xp + yq) - 2xy = 0 \quad \dots(2)$

$$\begin{aligned}
 \text{Now } \frac{\partial(f, g)}{\partial(x, p)} &= \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p & x \\ zp - 2y & zx \end{vmatrix} = zpx - x(zp - 2y) = 2xy \\
 \frac{\partial(f, g)}{\partial(z, p)} &= \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & x \\ xp + yq & zx \end{vmatrix} = 0 - x(xp + yq) = -x(xp + yq) \\
 \frac{\partial(f, g)}{\partial(y, q)} &= \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} -q & -y \\ zq - 2x & zy \end{vmatrix} = -qzy + y(zq - 2x) = -2xy \\
 \frac{\partial(f, g)}{\partial(z, q)} &= \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -y \\ xp + yq & zy \end{vmatrix} = 0 + y(xp + yq) = y(xp + yq)
 \end{aligned}$$

$$\begin{aligned}
 \therefore [f, g] &= \frac{\partial(f, g)}{\partial(x, p)} + p \frac{\partial(f, g)}{\partial(z, p)} + \frac{\partial(f, g)}{\partial(y, q)} + q \frac{\partial(f, g)}{\partial(z, q)} \\
 &= 2xy - px(xp + yq) - 2xy + yq(xp + yq) \\
 &= (-px + qy)(xp + yq) = 0 \quad \because px = qy. \text{ From (1)}
 \end{aligned}$$

$\therefore$ the given equations are compatible.

Solving (1) and (2), we get $p = \frac{y}{z}$ and $q = \frac{x}{z}$

Now $dz = p dx + q dy$ gives $dz = (y/z) dx + (x/z) dy$

or $z dz = y dx + x dy = d(xy)$

Integrating, the solution of the given equations is

$$z^2 = 2xy + c$$

where c is an arbitrary constant.

Example 4: Show that the equations

$$xp - yq = x, \quad x^2 p + q = xz$$

are compatible and find their solution.

[Delhi Maths (Hons.) 2007]

Solution: Let $f = xp - yq - x = 0$

...(1)

and $g = x^2 p + q - xz = 0 \quad \dots(2)$

Now
$$\frac{\partial(f, g)}{\partial(x, p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p-1 & x \\ 2px-z & x^2 \end{vmatrix}$$

$$= x^2(p-1) - x(2px-z) = -x^2 - px^2 + zx,$$

$$\frac{\partial(f, g)}{\partial(z, p)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & x \\ -x & x^2 \end{vmatrix} = 0 - (-x^2) = x^2,$$

$$\frac{\partial(f, g)}{\partial(y, q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} -q & -y \\ 0 & 1 \end{vmatrix} = -q$$

and
$$\frac{\partial(f, g)}{\partial(z, q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -y \\ -x & 1 \end{vmatrix} = -xy$$

$$\begin{aligned} \therefore [f, g] &= \frac{\partial(f, g)}{\partial(x, p)} + p \frac{\partial(f, g)}{\partial(z, p)} + \frac{\partial(f, g)}{\partial(y, q)} + q \frac{\partial(f, g)}{\partial(z, q)} \\ &= (-x^2 - px^2 + zx) + p \cdot x^2 - q + q(-xy) \\ &= -x^2 + zx - q - qxy \\ &= -x^2 + (x^2 p + q) - q - qxy \quad \because xz = x^2 p + q, \text{ from (2)} \\ &= -x^2 + x(px - qy) \\ &= -x^2 + x \cdot x = 0 \quad \because x = xp - yq, \text{ from (1)} \end{aligned}$$

Hence, the given equations are compatible.

Solving (1) and (2), we get $p = \frac{1+yz}{1+xy}$ and $q = \frac{x(z-x)}{1+xy}$

Now $dz = p dx + q dy$ gives $dz = \frac{(1+yz) dx + x(z-x) dy}{1+xy}$

or $(1+xy) dz = (1+yz) dx + x(z-x) dy$

$$\text{or} \quad (1 + xy) dz - z (y dx + x dy) - dx + x^2 dy = 0$$

$$\text{or} \quad (1 + xy) (dz - dx) - (z - x) (y dx + x dy) + (1 + xy) dx$$

$$1 - x (y dx + x dy) - dx + x^2 dy = 0$$

$$\text{or} \quad (1 + xy) (dz - dx) - (z - x) (y dx + x dy) = 0$$

$$\text{or} \quad \frac{(1 + xy) d(z - x) - (z - x) d(1 + xy)}{(1 + xy)^2} = 0$$

$$\text{or} \quad d \left(\frac{z - x}{1 + xy} \right) = 0$$

Integrating, the required solution of the given equations is

$$\frac{z - x}{1 + xy} = c, \quad \text{or} \quad z = x + c(1 + xy)$$

where c is an arbitrary constant.

Example 5: Show that the equation $z = px + qy$ is compatible with any equation $f(x, y, z, p, q) = 0$ which is homogeneous in x, y, z . [Delhi Maths (Hons.) 2001, 06]

$$\text{Solution:} \quad \text{Given,} \quad f(x, y, z, p, q) = 0 \quad \dots(1)$$

$$\text{Let} \quad g(x, y, z, p, q) = px + qy - z = 0 \quad \dots(2)$$

Since (1) is homogeneous in x, y, z , so we can write

$$f(x, y, z, p, q) = z^n F(x/z, y/z, p, q) = z^n \phi(u, v, p, q)$$

$$\text{where} \quad u = x/z \text{ and } v = y/z \quad \dots(3)$$

$\therefore$ (1) can be written as

$$F(x, y, z, p, q) = \phi(u, v, p, q) = 0 \quad \dots(4)$$

$$\text{Now} \quad \frac{\partial(g, F)}{\partial(x, p)} = \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial p} \end{vmatrix} = \frac{\partial g}{\partial x} \cdot \frac{\partial F}{\partial p} - \frac{\partial g}{\partial p} \cdot \frac{\partial F}{\partial x} = p \frac{\partial \phi}{\partial p} - x \frac{\partial \phi}{\partial x}$$

$$= p \frac{\partial \phi}{\partial p} - x \frac{\partial \phi}{\partial u} \cdot \frac{\partial u}{\partial x} = p \frac{\partial \phi}{\partial p} - \frac{x}{z} \frac{\partial \phi}{\partial u} = p \frac{\partial \phi}{\partial p} - u \frac{\partial \phi}{\partial u}$$

$$\frac{\partial(g, F)}{\partial(z, p)} = \begin{vmatrix} \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p} \\ \frac{\partial F}{\partial z} & \frac{\partial F}{\partial p} \end{vmatrix} = \frac{\partial g}{\partial z} \cdot \frac{\partial F}{\partial p} - \frac{\partial g}{\partial p} \cdot \frac{\partial F}{\partial z} = -1 \frac{\partial \phi}{\partial p} - x \frac{\partial \phi}{\partial z}$$

$$\begin{aligned}\frac{\partial(g, F)}{\partial(y, q)} &= \begin{vmatrix} \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \\ \frac{\partial F}{\partial y} & \frac{\partial F}{\partial q} \end{vmatrix} = \frac{\partial g}{\partial y} \cdot \frac{\partial F}{\partial q} - \frac{\partial g}{\partial q} \cdot \frac{\partial F}{\partial y} = q \frac{\partial \phi}{\partial q} - y \frac{\partial \phi}{\partial y} \\ &= q \frac{\partial \phi}{\partial q} - y \frac{\partial \phi}{\partial v} \cdot \frac{\partial v}{\partial y} = q \frac{\partial \phi}{\partial q} - y \frac{\partial \phi}{\partial v} \frac{1}{z} = q \frac{\partial \phi}{\partial q} - v \frac{\partial \phi}{\partial v}\end{aligned}$$

$$\frac{\partial(g, F)}{\partial(z, q)} = \begin{vmatrix} \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \\ \frac{\partial F}{\partial z} & \frac{\partial F}{\partial q} \end{vmatrix} = \frac{\partial g}{\partial z} \cdot \frac{\partial F}{\partial q} - \frac{\partial g}{\partial q} \cdot \frac{\partial F}{\partial z} = -1 \frac{\partial \phi}{\partial q} - y \frac{\partial \phi}{\partial z}$$

$$\begin{aligned}\therefore [g, F] &= \frac{\partial(g, F)}{\partial(x, p)} + p \frac{\partial(g, F)}{\partial(z, p)} + \frac{\partial(g, F)}{\partial(y, q)} + q \frac{\partial(g, F)}{\partial(z, q)} \\ &= p \frac{\partial \phi}{\partial p} - u \frac{\partial \phi}{\partial u} + p \left(-\frac{\partial \phi}{\partial p} - x \frac{\partial \phi}{\partial z} \right) + q \frac{\partial \phi}{\partial q} - v \frac{\partial \phi}{\partial v} + q \left(-\frac{\partial \phi}{\partial q} - y \frac{\partial \phi}{\partial z} \right) \\ &= -u \frac{\partial \phi}{\partial u} - (px + qv) \frac{\partial \phi}{\partial z} - v \frac{\partial \phi}{\partial v} \\ &= -u \frac{\partial \phi}{\partial u} - z \cdot \left(\frac{\partial \phi}{\partial u} \frac{\partial u}{\partial z} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial z} \right) - v \frac{\partial \phi}{\partial v} \\ &= -u \frac{\partial \phi}{\partial u} - z \left[\frac{\partial \phi}{\partial u} \left(\frac{-x}{z^2} \right) + \frac{\partial \phi}{\partial v} \left(\frac{-y}{z^2} \right) \right] - v \frac{\partial \phi}{\partial v} \\ &= -u \frac{\partial \phi}{\partial u} + u \frac{\partial \phi}{\partial u} + v \frac{\partial \phi}{\partial v} - v \frac{\partial \phi}{\partial v} = 0\end{aligned}$$

Hence, the given equations are compatible.

Exercise 7.5

Show that the following pairs of equations are compatible and find their common solution.

1. $p = x^2 - ay, q = y^2 - ax.$

2. $p = 6x + 3y, q = 3x - 4y.$

3. $p = x - \frac{y}{x^2 + y^2}, q = y + \frac{x}{x^2 + y^2}.$

4. $p = y(1 + 1/x) + \cos x, q = x + \log x - x \sin y.$

5. $2p - 4q + 4x - 5y + 1 = 0, 5p + 6q + 3x + 4y + 2 = 0$

6. $z = px + qy, 2xy(p^2 + q^2) = z(y p + x q).$

[Delhi Maths (Hons.) 2006]

7. $p = 1 + 4xy + 2y^2, q = 1 + 4xy + 2x^2$

8. $p = x^4 - 2xy^2 + y^4, q = -(2x^2y - 4xy^3 + \sin y)$

9. Show that the equations $f(x, y, p, q) = 0$ and $g(x, y, p, q) = 0$ are compatible if

$$\frac{\partial(f, g)}{\partial(x, p)} + \frac{\partial(f, g)}{\partial(y, q)} = 0.$$

Answers 7.5

1.	$z = \frac{(x^3 + y^3)}{3} - axy + c$
2.	$z = 3(x^2 + xy) - 2y^2 + c$
3.	$z = \frac{x^2}{2} - \tan^{-1}\left(\frac{x}{y}\right) + \frac{y^2}{2} + c$
4.	$z = y(x + \log x) + x \cos y + c$
5.	$z = -\frac{9}{16}x^2 + \frac{7}{16}xy - \frac{7}{16}x + \frac{1}{32}y - \frac{33}{64}y^2 + c$
6.	$x^2 + y^2 = cz^2 \quad \text{or} \quad z^2 = cxy$
7.	$z = x + 2x^2y + 2xy^2 + y + c$
8.	$z = \frac{1}{5}x^5 - x^2y^2 + xy^4 + \cos y + c$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter (a), (b), (c) or (d).

- The partial differential equation $2zp + 7yq = 3z^4 + \log x - e^{2y}$ is :
 (a) linear (b) semi-linear (c) quasi-linear (d) non-linear
- The partial differential equation $x^3 yp + y^2 q = (x + y) z^3 + \log xy$ is :
 (a) linear (b) semi-linear (c) quasi-linear (d) non-linear
- The equation $(x^2 + y^2) \frac{\partial z}{\partial x} + (x + y) \frac{\partial z}{\partial y} + (3x - 4y) z = x^2 + y^2$ is :
 (a) linear (b) quasi-linear (c) semi-linear (d) non-linear [Agra 2008]
- The equation $(2x + 3y) p + 4x q - 8pq = x + y$ is :
 (a) linear (b) non-linear
 (c) quasi-linear (d) semi-linear [Agra 2005, 06]
- The equation $\frac{\partial^2 z}{\partial x^2} - 2 \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2 + \frac{\partial^2 z}{\partial y^2} = 0$ is of order
 (a) one (b) two
 (c) three (d) none of these [Agra 2005, 06]
- The equation $p \tan y + q \tan x = \sec^2 z$ is of order
 (a) 1 (b) 2
 (c) 0 (d) none of these [Agra 2005, 08]
- The partial differential equation obtained by a and b from $az + b = a^2 x + y$ is
 (a) $\frac{\partial z}{\partial x} = 2 \frac{\partial z}{\partial y}$ (b) $\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = 1$
 (c) $\left(\frac{\partial z}{\partial x} \right)^2 = \frac{\partial z}{\partial y}$ (d) none of these
- The partial differential equation obtained by eliminating the arbitrary constant ϕ from $z = \phi(x + y)$ is :
 (a) $z = pq$ (b) $z = p + q$
 (c) $pq = 1$ (d) $p = q$

Fill in the Blank

Fill in the blanks “....” so that the following statement are complete and correct.

1. A partial differential equation $F(x, y, z, p, q) = 0$ in which z is dependent variable and is function of independent variables x and y is if in this equation z and its partial derivatives p and q all occur in first degree.
2. The partial differential equation $p + 3q = 5z + 6xy$ is
3. The partial differential equation $p^2x + q^2y = 3z$ is
4. The partial differential equation $(z^2 + 2yz + x^2)p + x^3q = x(y + z)$ is
5. The partial differential equation $\sin x \cdot \frac{\partial z}{\partial x} + (x^2 + y^2) \frac{\partial z}{\partial y} = x^3 + y^3 + z^3$ is
6. The partial differential equation $(2x + 3y)p - xq + pq = x^2 + y^2$ is
7. The partial differential equation of the form

$$f(x, y, z) \frac{\partial z}{\partial x} + g(x, y, z) \frac{\partial z}{\partial y} = h(x, y, z)$$
 is called [Meerut 2003]
8. The order of the differential equation $3p + 4yq = 3z$ is
9. The quasi-linear partial differential equation $(\sin x)p + (\cos y)zq = zx$ is
10. The equation $p = ax + hy + q, q = hx + by + f$ are

True / False

Write ‘T’ for true and ‘F’ for false statement.

1. The subsidiary equations of Lagrange’s equation $Pp + Qq = R$ are $\frac{dx}{Q} = \frac{dy}{P} = \frac{dz}{R}$
2. The partial differential equation $\frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = 3x$ is non-linear. [Meerut 2003]
3. The partial differential equation obtained by eliminating constants a and b from $z = ax^2 - by^2$ is $2z = px - qy$.
4. The equation $(x + 2y - z)p - (3x + 2y)q - 2z = x - y$ is not quasi linear.
5. The equations $p = y^2e^{xy^2} + 4x^3, q = 2xye^{xy^2} - 3y^2$ are compatible.

Answers

Objective Type Questions

Multiple Choice Questions

1.	(c)	2.	(b)	3.	(a)
4.	(b)	5.	(b)	6.	(a)
7.	(b)	8.	(d)		

Fill in the Blank

1.	linear	2.	linear
3.	non-linear	4.	quasi-linear
5.	semi-linear	6.	non-linear
7.	quasi-linear	8.	1
9.	Lagrange's equation	10.	compatible

True / False

1.	False	2.	True	3.	True
4.	False	5.	True		

○○○

Non-Linear Partial Differential Equations of First Order (Charpit's and Jacobi's Methods)

8.1 Introduction

In the previous chapter we have discussed linear partial differential equations of first order. Here in this chapter we shall discuss methods for the solution of non-linear partial differential equations of first order.

In general a non-linear partial differential equation of order one is written as

$$f(x, y, z, p, q) = 0$$

where f is not linear in p and q .

8.2 Solution of Partial Differential Equations of First Order and any Degree in Some Standard Forms

There is a general method (Charpit's Method) for solving a linear or non-linear partial differential equation of first order (partial differential equation of first order and any degree). Before giving this method, we shall give methods for the solution of partial differential equations in some standard forms.

While solving a non-linear partial differential equation, the general and singular solutions should also be given with the complete integral of the problem. To find the complete integral only is considered to be the incomplete solution of the differential equation.

8.3 Standard Form I : Equation Involving Only p and q and no x , y and z

Let the equation of this form be

$$f(p, q) = 0 \quad \dots(1)$$

Complete Integral : The complete integral of (1) is given by

$$z = ax + by + c \quad \dots(2)$$

where a and b are connected by the relation

$$f(a, b) = 0 \quad \dots(3)$$

Since from (2), $p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$, which when substituted in (3), gives (1). Thus, (2) satisfy (1) while a and b satisfy (3).

From (3) we can find b in terms of a . Let $b = \phi(a)$.

Substituting in (2), the **complete integral** of differential equation (1) is given by

$$z = ax + \phi(a) \cdot y + c \quad \dots(4)$$

which contains two arbitrary constants a and c which is equal to the number of independent variables x and y .

General Integral : Taking $c = \psi(a)$, where ψ is an arbitrary function, in the complete integral (4), we get

$$z = ax + y\phi(a) + \psi(a) \quad \dots(5)$$

Now, differentiating (5) partially w.r.t. a , we get

$$0 = x + y\phi'(a) + \psi'(a) \quad \dots(6)$$

The general integral of (1) is obtained by eliminating a between (5) and (6).

Singular Integral : The singular integral of equation (1) is obtained by eliminating a and c between the complete integral (4) and the equations obtained by differentiating (4) partially w.r.t. a and c successively.

i.e., between (4) and the equations

$$0 = x + y\phi'(a) \text{ and } 0 = 1.$$

But the last equation $0 = 1$ is inadmissible.

So we conclude that the equations of standard form I given by (1) have no singular integral (or singular solution).

Note : Sometimes, the change of variable transforms the equations to standard form I.

Solved Examples

Example 1: Solve $pq = p + q$.

Solution: The given equation is

$$pq = p + q \quad \dots(1)$$

which is of the form $f(p, q) = 0$

$\therefore$ its solution can be taken as $z = ax + by + c$... (2)

Putting $p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$, in (1), we get

$$ab = a + b \Rightarrow b = a / (a - 1)$$

Putting in (2), the complete integral of (1) is

$$z = ax + \{a / (a - 1)\} y + c \quad \dots(3)$$

which contains two arbitrary constants a and c .

General Integral : Taking $c = \psi(a)$ in complete integral (3), we get

$$z = ax + \{a / (a - 1)\} y + \psi(a) \quad \dots(4)$$

Differentiating (4) partially w.r.t. a , we get

$$0 = x - \{1 / (a - 1)^2\} y + \psi'(a) \quad \dots(5)$$

The general integral of (1) is obtained by eliminating a between (4) and (5).

Singular Integral : The singular integral of (1) is obtained by eliminating a and c between the complete integral

$$z = ax + \{a / (a - 1)\} y + c$$

and its partial derivatives w.r.t. a and c

$$\text{i.e.,} \quad 0 = x - \{1 / (a - 1)^2\} y \quad \text{and} \quad 0 = 1.$$

$\therefore$ $0 = 1$ is inadmissible, so the given equation has no singular integral.

Example 2: Solve $p^2 + q^2 = m^2$.

Solution: The given equation is

$$p^2 + q^2 = m^2 \quad \dots(1)$$

which is of the form, $f(p, q) = 0$

$\therefore$ its solution can be taken as $z = ax + by + c$... (2)

Putting $p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$, in (1), we get

$$a^2 + b^2 = m^2 \Rightarrow b = \sqrt{m^2 - a^2}$$

Putting in (2), the complete integral of (1) is

$$z = ax + \sqrt{m^2 - a^2} y + c \quad \dots(3)$$

which contains two arbitrary constants a and c .

General Integral : Taking $c = \psi(a)$, in complete integral (3), we get

$$z = ax + \sqrt{m^2 - a^2} y + \psi(a) \quad \dots(4)$$

Differentiating (4) partially, w.r.t. a , we get

$$0 = x - a y / \sqrt{m^2 - a^2} + \psi'(a) \quad \dots(5)$$

The general integral of (1) is obtained by eliminating a between (4) and (5).

The problem have no singular integral can be proved as in last example 1.

Example 3: Solve $x^2 p^2 + y^2 q^2 = z^2$.

[Jabalpur 2000, 03; Meerut 2001; I.A.S. 2003; Sagar 2004; Rohilkhand 2004; Gulberg 2005]

Solution: The given equation can be written as

$$\left(\frac{x}{z} \frac{\partial z}{\partial x} \right)^2 + \left(\frac{y}{z} \frac{\partial z}{\partial y} \right)^2 = 1 \quad \dots(1)$$

Putting $\frac{1}{x} dx = dX$, $\frac{1}{y} dy = dY$ and $\frac{1}{z} dz = dZ$

so that $\log x = X$, $\log y = Y$, $\log z = Z$

i.e., $x = e^X$, $y = e^Y$ and $z = e^Z$, equation (1) reduces to

$$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1 \quad \dots(2)$$

or $P^2 + Q^2 = 1$, where $\frac{\partial Z}{\partial X} = P$ and $\frac{\partial Z}{\partial Y} = Q$

which of the standard form $f(P, Q) = 0$

$\therefore$ its solution can be taken as $Z = aX + bY + c$... (3)

Putting $P = \frac{\partial Z}{\partial X} = a$, $Q = \frac{\partial Z}{\partial Y} = b$ in (2), we get

$$a^2 + b^2 = 1 \Rightarrow b = \sqrt{1 - a^2}.$$

Putting in (3), the complete integral of (2) is

$$Z = aX + \sqrt{1 - a^2} Y + c$$

or $\log z = a \log x + \sqrt{1 - a^2} \log y + c$

Taking $a = \cos \alpha$ and $c = \log c'$

$$\begin{aligned} \log z &= \cos \alpha \log x + \sin \alpha \log y + \log c' \\ &= \log (c' x^{\cos \alpha} y^{\sin \alpha}) \end{aligned}$$

or $z = c' x^{\cos \alpha} y^{\sin \alpha}$... (4)

Which contains two arbitrary constants α and c' and is the complete integral of (1).

General Integral : Taking $c' = \psi(\alpha)$, in the complete integral (4), we get

$$z = \psi(\alpha) x^{\cos \alpha} y^{\sin \alpha} \quad \dots(5)$$

Differentiating (5) partially w.r.t. α , we get

$$\begin{aligned} 0 &= \psi'(\alpha) x^{\cos \alpha} y^{\sin \alpha} + \psi(\alpha) x^{\cos \alpha} \log_e x \cdot (-\sin \alpha) y^{\sin \alpha} \\ &\quad + \psi(\alpha) x^{\cos \alpha} y^{\sin \alpha} \log_e y \cdot (\cos \alpha) \quad \dots(6) \end{aligned}$$

The general integral of (1) is obtained by eliminating α between (5) and (6).

Singular Integral : Differentiating the complete integral (4), partially, w.r.t. α and c' , we get

$$0 = c' x^{\cos \alpha} \log_e x \cdot (-\sin \alpha) \cdot y^{\sin \alpha} + c' x^{\cos \alpha} y^{\sin \alpha} \log_e y \cdot \cos \alpha \quad \dots(7)$$

and $0 = x^{\cos \alpha} y^{\sin \alpha}$... (8)

Eliminating α and c' between (4), (7) and (8), the singular integral is $z = 0$, which satisfy the given equation (1).

Example 4: Find the complete integral of $x^2 p^2 + y^2 q^2 = z$. [Delhi Maths (Hons.) 2004]

Solution: The given equation can be written as

$$\left(\frac{x}{\sqrt{z}} \frac{\partial z}{\partial x} \right)^2 + \left(\frac{y}{\sqrt{z}} \frac{\partial z}{\partial y} \right)^2 = 1 \quad \dots (1)$$

Putting $\frac{dx}{x} = dX$, $\frac{dy}{y} = dY$ and $\frac{dz}{\sqrt{z}} = dZ$

So that $\log x = X$, $\log y = Y$ and $2\sqrt{z} = Z$

i.e., $x = e^X$, $y = e^Y$ and $z = Z^2 / 4$

(1) reduces to $\left(\frac{\partial Z}{\partial X} \right)^2 + \left(\frac{\partial Z}{\partial Y} \right)^2 = 1$... (2)

or $P^2 + Q^2 = 1$, where $P = \frac{\partial Z}{\partial X}$ and $Q = \frac{\partial Z}{\partial Y}$

which is of the standard form $f(P, Q) = 0$

$\therefore$ its solution can be taken as $Z = aX + bY + c$... (3)

$\therefore$ $P = \frac{\partial Z}{\partial X} = a$, $Q = \frac{\partial Z}{\partial Y} = b$, Putting in (2),

$$a^2 + b^2 = 1 \Rightarrow b = \sqrt{1 - a^2}$$

Putting in (2), the required complete integral is

$$Z = aX + \sqrt{1 - a^2} Y + c \quad \text{or} \quad 2\sqrt{z} = a \log x + \sqrt{1 - a^2} \log y + c$$

or $2\sqrt{z} = \log (x^a \cdot y^{\sqrt{1 - a^2}} / c')$, where $c = -\log c'$

or $x^a y^{\sqrt{1 - a^2}} = c' e^{2\sqrt{z}}$.

Example 5: Find the complete integral, general integral and singular integral of $pq = 4xy$. Show that the equation is satisfied by $z = 2xy + c$, c being an arbitrary constant. [Delhi Maths (Hons.) 2007]

Solution: The given equation can be written as

$$\left(\frac{1}{2x} \frac{\partial z}{\partial x}\right) \left(\frac{1}{2y} \frac{\partial z}{\partial y}\right) = 1 \quad \dots(1)$$

Taking $2x \, dx = dX$, $2y \, dy = dY$, so that $x^2 = X$, $y^2 = Y$

$$\text{the equation (1) reduces to } \frac{\partial z}{\partial X} \cdot \frac{\partial z}{\partial Y} = 1 \quad \dots(2)$$

or $PQ = 1$ where $P = \frac{\partial z}{\partial X}$, $Q = \frac{\partial z}{\partial Y}$, which is of standard form $f(P, Q) = 0$

$\therefore$ its solution can be taken as $z = aX + bY + c$... (3)

$$\Rightarrow P = \frac{\partial z}{\partial X} = a \text{ and } Q = \frac{\partial z}{\partial Y} = b \quad \therefore \text{ from (2) } ab = 1 \Rightarrow b = 1/a$$

$$\therefore \text{ from (3), } z = ax^2 + (1/a)y^2 + c \quad \dots(4)$$

is complete integral of the given equation as it contains two arbitrary constants a and c .

To find General Integral : Putting $c = \phi(a)$ in (4),

$$z = ax^2 + (1/a)y^2 + \phi(a) \quad \dots(5)$$

Diff. (5) partially w.r.t. a ,

$$0 = x^2 - (1/a^2)y^2 + \phi'(a) \quad \dots(6)$$

The general integral is obtained by eliminating a between (5) and (6).

To find Singular Integral : Differentiating (4) partially w.r.t. a and c , we get

$$0 = x^2 - (1/a^2)y^2 \text{ and } 0 = 1$$

Since $0 = 1$ is absurd, so the given equation has no singular integral.

To show that (1) is satisfied by $z = 2xy + c$... (7)

$$\text{From (7), } p = \frac{\partial z}{\partial x} = 2y \text{ and } q = \frac{\partial z}{\partial y} = 2x$$

which satisfy (1).

Hence $z = 2x + y + c$ satisfy (1).

Example 6: Find the complete integral of

$$p^m \sec^{2m} x + z^l q^n \operatorname{cosec}^{2n} y = z^{lm/(m-n)}.$$

Solution: The given equation can be written as

$$\left(\frac{z^{-l/(m-n)}}{\cos^2 x} \frac{\partial z}{\partial x} \right)^m + \left(\frac{z^{-l/(m-n)}}{\sin^2 y} \frac{\partial z}{\partial y} \right)^n = 1 \quad \dots(1)$$

Putting, $\cos^2 x \, dx = dX$, $\sin^2 y \, dy = dY$ and $z^{-l/(m-n)} dz = dZ$

so that, $X = \frac{1}{2} \left(x + \frac{1}{2} \sin 2x \right)$, $Y = \frac{1}{2} \left(y - \frac{1}{2} \sin 2y \right)$ and $Z = \frac{m-n}{m-n-l} z^{(m-n-l)/(m-n)}$

equation (1) reduces to

$$\left(\frac{\partial Z}{\partial X} \right)^m + \left(\frac{\partial Z}{\partial Y} \right)^n = 1 \quad \text{or} \quad P^m + Q^n = 1 \quad \dots(2)$$

Where $\frac{\partial Z}{\partial X} = P$ and $\frac{\partial Z}{\partial Y} = Q$

Equation (1) is of the standard form $f(P, Q) = 0$

$\therefore$ its solution can be taken as $Z = aX + bY + c$... (3)

$\therefore$ $P = \frac{\partial Z}{\partial X} = a$, $Q = \frac{\partial Z}{\partial Y} = b$, putting in (2), we get

$$a^m + b^n = 1 \Rightarrow b = (1 - a^m)^{1/n}$$

Putting in (2), the complete integral of (2) is

$$Z = aX + (1 - a^m)^{1/n} Y + c$$

or
$$\frac{m-n}{m-n-l} z^{(m-n-l)/(m-n)} = \frac{1}{2} a \left(x + \frac{1}{2} \sin 2x \right) + \frac{1}{2} (1 - a^m)^{1/n} \left(y - \frac{1}{2} \sin 2y \right) + c \quad \dots(4)$$

Which contains two arbitrary constants a and c and is the complete integral of (1).

Example 7: Find the complete integral of $pq = x^m y^n z^{2l}$.

[I.A.S. 2000; Delhi Maths (Hons.) 2007]

Solution: The given equation can be written as

$$\frac{z^{-l}}{x^m} \frac{\partial z}{\partial x} \cdot \frac{z^{-l}}{y^n} \frac{\partial z}{\partial y} = 1 \quad \dots(1)$$

Putting $x^m dx = dX$, $y^n dy = dY$ and $z^{-l} dz = dZ$

so that, $X = \frac{x^{m+1}}{m+1}$, $Y = \frac{y^{n+1}}{n+1}$ and $Z = \frac{z^{-l+1}}{(-l+1)}$, equation (1) reduces to

$$\frac{\partial Z}{\partial X} \cdot \frac{\partial Z}{\partial Y} = 1 \quad \text{or} \quad PQ = 1 \quad \dots(2)$$

Where $\frac{\partial Z}{\partial X} = P$ and $\frac{\partial Z}{\partial Y} = Q$

Equation (2) is of the standard form $f(P, Q) = 0$,

$\therefore$ its solution can be taken as $Z = aX + bY + c$ $\dots(3)$

$\therefore$ $P = \frac{\partial Z}{\partial X} = a$, $Q = \frac{\partial Z}{\partial Y} = b$, putting in (2), we get

$$ab = 1 \Rightarrow b = \frac{1}{a}$$

Putting in (2), the complete integral of (2) is

$$Z = aX + \left(\frac{1}{a}\right)Y + c$$

or
$$\frac{z^{-l+1}}{(-l+1)} = a \frac{x^{m+1}}{(m+1)} + \frac{y^{n+1}}{\{a(n+1)\}} + c$$

which contains two arbitrary constants a and c and is the complete integral of (1).

Example 8: Find the complete integral of

$$z^2 p^2 y + 6zpxy + 2zqx^2 + 4x^2 y = 0.$$

Solution: Dividing by $x^2 y$, the given equation can be written as

$$\left(\frac{z}{x} \frac{\partial z}{\partial x}\right)^2 + 6 \left(\frac{z}{x} \frac{\partial z}{\partial x}\right) + 2 \left(\frac{z}{y} \frac{\partial z}{\partial y}\right) + 4 = 0 \quad \dots(1)$$

Putting $x dx = dX$, $y dy = dY$ and $z dz = dZ$, so that

$$X = \frac{x^2}{2}, Y = \frac{y^2}{2} \text{ and } Z = \frac{z^2}{2}, \text{ equation (1) reduces to}$$

$$\left(\frac{\partial Z}{\partial X}\right)^2 + 6 \frac{\partial Z}{\partial X} + 2 \frac{\partial Z}{\partial Y} + 4 = 0$$

or
$$P^2 + 6P + 2Q + 4 = 0 \quad \dots(2)$$

where $\frac{\partial Z}{\partial X} = P$ and $\frac{\partial Z}{\partial Y} = Q$

Equation (2) is of the form, $f(P, Q) = 0$

∴ its solution can be taken as $Z = aX + bY + c$... (3)

∴ $P = \frac{\partial Z}{\partial X} = a$ and $Q = \frac{\partial Z}{\partial Y} = b$, putting in (2), we get

$$a^2 + 6a + 2b + 4 = 0 \Rightarrow b = -(a^2 + 6a + 4) / 2$$

Putting in (3), the complete integral of (2) is

$$Z = aX - \{(a^2 + 6a + 4) / 2\} Y + c$$

or $z^2 / 2 = a x^2 / 2 - \{(a^2 + 6a + 4) / 2\} y^2 / 2 + c$

or $z^2 = a x^2 - \{(a^2 + 6a + 4) / 2\} y^2 + c'$, where $c' = 2c$

Which contains two arbitrary constants a and c' and is the complete integral of (1).

Example 9: Solve $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$.

[Kanpur 2006]

Solution: To reduce the given equation to standard form I , let

$$x + y = X^2 \text{ and } x - y = Y^2$$

∴ $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y} \frac{\partial Y}{\partial x} = \frac{1}{2X} \frac{\partial z}{\partial X} + \frac{1}{2Y} \frac{\partial z}{\partial Y}$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y} \frac{\partial Y}{\partial y} = \frac{1}{2X} \frac{\partial z}{\partial X} - \frac{1}{2Y} \frac{\partial z}{\partial Y}$

so that $p + q = \frac{1}{X} \frac{\partial z}{\partial X}$ and $p - q = \frac{1}{Y} \frac{\partial z}{\partial Y}$

Putting in the given equation, it reduces to

$$\left(\frac{\partial z}{\partial X} \right)^2 + \left(\frac{\partial z}{\partial Y} \right)^2 = 1 \quad \text{or} \quad P^2 + Q^2 = 1 \quad \dots (1)$$

where $\frac{\partial z}{\partial X} = P$ and $\frac{\partial z}{\partial Y} = Q$

Equation (1) is of the form $f(P, Q) = 0$

∴ its solution can be taken as $z = aX + bY + c$... (2)

∴ $P = \frac{\partial z}{\partial X} = a$ and $Q = \frac{\partial z}{\partial Y} = b$, putting in (2), we get

$$a^2 + b^2 = 1 \Rightarrow b = \sqrt{1 - a^2}$$

Putting in (2), the complete integral of (2) is

$$z = aX + \sqrt{(1-a^2)} Y + c$$

or

$$z = a\sqrt{(x+y)} + \sqrt{(1-a^2)}\sqrt{(x-y)} + c$$

which contains two arbitrary constants a and c and is the complete integral of the given equation.

Example 10: Find the complete integral of

$$(y-x)(qy-px) = (p-q)^2.$$

[Kanpur 2001, 04]

Solution: To reduce the given equation to standard form I, let

$$x+y=X \quad \text{and} \quad x-y=Y$$

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y} \frac{\partial Y}{\partial x} = \frac{\partial z}{\partial X} + y \frac{\partial z}{\partial Y}$$

and

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y} \frac{\partial Y}{\partial y} = \frac{\partial z}{\partial X} + x \frac{\partial z}{\partial Y}$$

so that

$$p-q = (y-x) \frac{\partial z}{\partial Y} \quad \text{and} \quad qy-px = (y-x) \frac{\partial z}{\partial X}$$

Putting in the given equation, it reduces to

$$(y-x)^2 \frac{\partial z}{\partial X} = (y-x)^2 \left(\frac{\partial z}{\partial Y} \right)^2$$

$$\text{or} \quad \frac{\partial z}{\partial X} = \left(\frac{\partial z}{\partial Y} \right)^2 \quad \text{or} \quad P = Q^2 \quad \dots(1)$$

$$\text{where} \quad \frac{\partial z}{\partial X} = P \quad \text{and} \quad \frac{\partial z}{\partial Y} = Q$$

Equation (1) is of the form $f(P, Q) = 0$

$$\therefore \text{ its solution can be taken as } z = aX + bY + c \quad \dots(2)$$

$$\therefore P = \frac{\partial z}{\partial X} = a \quad \text{and} \quad Q = \frac{\partial z}{\partial Y} = b$$

Putting in (1), we get $a = b^2$

Putting in (2), the complete integral of (1) is

$$z = b^2 X + bY + c \quad \text{or} \quad z = b^2(x+y) + bxy + c$$

which contains two arbitrary constants b and c and is the complete integral of the given equation.

Example 11: Solve $(x^2 + y^2)(p^2 + q^2) = 1$

[Vikram 2000; Indore 2004; Agra 2008]

Solution: To reduce the given equation to the standard form I, let $x = r \cos \theta$, $y = r \sin \theta$,

so that $x^2 + y^2 = r^2$, $\theta = \tan^{-1}(y/x)$. Differentiating $x^2 + y^2 = r^2$, w.r.t. x and y partially,

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos \theta, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin \theta$$

Differentiating $\theta = \tan^{-1}(y/x)$, w.r.t. x and y partially,

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + (y/x)^2} \left(-\frac{y}{x^2} \right) = \frac{-y}{x^2 + y^2} = -\frac{r \sin \theta}{r^2} = -\frac{1}{r} \sin \theta$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + (y/x)^2} \left(\frac{1}{x} \right) = \frac{x}{x^2 + y^2} = \frac{r \cos \theta}{r^2} = \frac{1}{r} \cos \theta$$

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} = \frac{\partial z}{\partial r} \cdot \cos \theta - \frac{\sin \theta}{r} \frac{\partial z}{\partial \theta}$$

$$\text{and } q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta} \cdot \frac{\partial \theta}{\partial y} = \frac{\partial z}{\partial r} \cdot \sin \theta + \frac{\cos \theta}{r} \frac{\partial z}{\partial \theta}$$

Putting in the given equation, it reduces to

$$r^2 \left\{ \left(\frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta} \right)^2 \right\} = 1 \quad \text{or} \quad \left(r \frac{\partial z}{\partial r} \right)^2 + \left(\frac{\partial z}{\partial \theta} \right)^2 = 1 \quad \dots(1)$$

Again putting $(1/r)dr = dR$, so that $R = \log r$, (1) reduces to

$$\left(\frac{\partial z}{\partial R} \right)^2 + \left(\frac{\partial z}{\partial \theta} \right)^2 = 1 \quad \text{or} \quad P^2 + Q^2 = 1 \quad \dots(2)$$

where $P = \frac{\partial z}{\partial R}$ and $Q = \frac{\partial z}{\partial \theta}$

Equation (2) is of the form $f(P, Q) = 0$

$\therefore$ its solution can be taken as $z = aR + b\theta + c$...(3)

$$\therefore P = \frac{\partial z}{\partial R} = a, Q = \frac{\partial z}{\partial \theta} = b$$

Putting in (2), we get $a^2 + b^2 = 1 \Rightarrow b = \sqrt{1 - a^2}$

Putting in (3), the complete integral of (2) is

$$z = aR + \sqrt{(1-a^2)} \theta + c$$

or
$$z = a \log r + \sqrt{(1-a^2)} \tan^{-1}(y/x) + c$$

or
$$z = \frac{1}{2} a \log (x^2 + y^2) + \sqrt{(1-a^2)} \tan^{-1}(y/x) + c$$

Which contains two arbitrary constants a and c and is the complete integral of the given equation.

Exercise 8.1

Find the complete integrals of the followings :

1. $pq = \lambda$ where λ is constant. 2. $q = 3p^2$

3. $p = e^q$

4. $p^2 + q^2 = 1$

[Meerut 2011 (Sem. I)]

[Hint. : Put $m = 1$ in Ex.2]

5. $p^2 - q^2 = \lambda$

6. $p^2 + p = q^2$

7. $p^2 + q^2 = npq$

8. $q = pq + p^2$

9. $p + q = pq$, [Gulberg 2005]

10. $pq = x^m y^n z^l$

11. $z^2 = pq \cdot xy$

[Punjab-2005]

[Hint : equation can be written as

$$\left(\frac{x}{z} \frac{\partial z}{\partial x}\right) \left(\frac{y}{z} \frac{\partial z}{\partial y}\right) = 1, \text{ take } \frac{1}{x} dx = dX, \frac{1}{y} dy = dY \text{ and } \frac{1}{z} dz = dZ].$$

12. $z y^2 p = x(y^2 + z^2 q^2)$

[Hint: Equation can be written as

$$\frac{z}{x} \frac{\partial z}{\partial x} = 1 + \left(\frac{z}{y} \frac{\partial z}{\partial y}\right)^2 \text{ take } x dx = dX, y dy = dY, z dz = dZ].$$

13. $p^2 + q^2 = z$

14. $p^2 x + q^2 y = z$

[Hint: Equation can be written as $\left(\frac{\sqrt{x}}{\sqrt{z}} \frac{\partial z}{\partial x}\right)^2 + \left(\frac{\sqrt{y}}{\sqrt{z}} \frac{\partial z}{\partial y}\right)^2 = 1$

take $\frac{1}{\sqrt{x}} dx = dX, \frac{1}{\sqrt{y}} dy = dY, \frac{1}{\sqrt{z}} dz = dZ].$

15. $(1-x^2)yp^2 + x^2q = 0$

[Hint: equation can be written as $\left(\frac{1}{x}\sqrt{1-x^2}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{1}{y}\frac{\partial z}{\partial y}\right)^2 = 0$

Take $\frac{x}{\sqrt{1-x^2}}dx = dX$, $ydy = dY$ i.e., $X = -\sqrt{1-x^2}$, $Y = \frac{y^2}{2}$.

Answers 8.1

1.	$z = ax + \frac{y\lambda}{a} + b$ or $a^2x + y\lambda - az = c$	2.	$z = ax + 3a^2y + c$
3.	$z = ax + y \log a + c$	4.	$z = ax + \sqrt{(1-a^2)}y + c$
5.	$z = ax + \sqrt{(a^2 - \lambda)}y + c$	6.	$z = ax + y\sqrt{(a^2 + a)} + c$
7.	$z = ax + (a/2)[n \pm \sqrt{(n^2 - 4)}]y + c$	8.	$z = ax + \{a^2/(1-a)\}y + c$
9.	$z = ax + \{a/(a-1)\}y + c$		
10.	$\frac{z}{1-l/2} = \frac{a}{m+1}x^{m+1} + \frac{1}{a(n+1)}y^{n+1} + c$		
11.	$z = x^a y^{1/a} + c$	12.	$z^2 = ax^2 + y^2\sqrt{(a-1)} + c$
13.	$2\sqrt{z} = ax + \sqrt{(1-a^2)}y + c$	14.	$\sqrt{z} = a\sqrt{x} + \sqrt{y}\sqrt{(1-a^2)} + c$
15.	$z = -a\sqrt{(1-x^2)} - (a^2y^2)/2 + c$		

8.4 Standard Form II : Equations Involving Only p, q & z

[Nagpur 2003; Delhi Maths (Hons.) 2006]

Let the equation of this form be

$$f(p, q, z) = 0 \quad \dots(1)$$

Complete Integral : Let $z = f(X)$ where $X = x + ay$, a is an arbitrary constant be a solution of (1), then

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial x} = \frac{dz}{dX}$$

and

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial y} = a \frac{dz}{dX}$$

$\therefore$ Equation (1) reduces to the form

$$f\left(\frac{dz}{dX}, a \frac{dz}{dX}, z\right) = 0 \quad \dots(2)$$

which is an ordinary differential equation of first order.

Solving (2), we shall get z as a function of X . Then replacing X by $x + ay$, we get the complete integral of the given equation.

Working Method : To solve the differential equation of the form $f(p, q, z) = 0$, replace p and q by $\frac{dz}{dX}$ and $a \frac{dz}{dX}$ respectively, in the given equation, where $X = x + ay$. This will reduce the equation to an ordinary differential equation. Solve this equation by usual method and in the last replace X by $x + ay$ to get the complete integral of the equation.

General Integral : If $F = 0$ is the complete integral involving two constants a and b then replacing b by $\phi(a)$, the general integral is obtained by eliminating a between $F = 0$ and $\frac{\partial F}{\partial a} = 0$.

Singular Integral : If $F = 0$ is the complete integral involving two arbitrary constants a and b then the singular integral is obtained by eliminating a and b between

$$F = 0, \quad \frac{\partial F}{\partial a} = 0 \quad \text{and} \quad \frac{\partial F}{\partial b} = 0.$$

Solved Examples

Example 1: Solve $z = p^2 - q^2$.

Solution: The given partial differential equation is of the form $f(p, q, z) = 0$

$\therefore$ Let $z = f(X)$; where $X = x + ay$, a is an arbitrary constant.

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial x} = \frac{dz}{dX} \quad \text{and} \quad q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial y} = a \frac{dz}{dX}$$

Putting in the given equation, it reduces to

$$z = (1 - a^2) \left(\frac{dz}{dX} \right)^2 \quad \text{or} \quad \sqrt{z} = \sqrt{(1 - a^2)} \frac{dz}{dX}$$

$\therefore \sqrt{(1-a^2)} \cdot z^{-1/2} dz = dX$, separating the variables.

Integrating, $\sqrt{(1-a^2)} \cdot 2\sqrt{z} = X + b$, where b is an arbitrary constant

$$\text{or} \quad 4(1-a^2)z = (x + ay + b)^2 \quad \dots(1)$$

which contains two arbitrary constants a and b and is the *complete integral* of the given differential equation.

General Integral : Taking $b = \phi(a)$, in (1), we get

$$4(1-a^2)z = \{x + ay + \phi(a)\}^2 \quad \dots(2)$$

Differentiating (2), partially w.r.t. a , we get

$$-8az = 2\{x + ay + \phi(a)\} \cdot \{y + \phi'(a)\} \quad \dots(3)$$

The general integral, can be obtained by eliminating a between (2) and (3).

Singular Integral : Differentiating the complete integral (1), partially w.r.t. a and b , we get

$$-8az = 2(x + ay + b) \cdot a \quad \text{or} \quad -4z = x + ay + b \quad \dots(4)$$

$$\text{and} \quad 0 = 2(x + ay + b) \quad \dots(5)$$

Eliminating a and b between (1), (4) and (5), the singular integral of given equation is $z = 0$, as it satisfy the given differential equation, ($\because z = 0 \Rightarrow p = 0, q = 0$).

Example 2: Solve $p^3 + q^3 = 27z$.

[Ravishankar 2005]

Solution: The given differential equation is of the form $f(p, q, z) = 0$

$\therefore$ Let $z = f(X)$, where $X = x + ay$, a is an arbitrary constant,

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{dz}{dX} \quad \text{and} \quad q = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial y} = a \frac{dz}{dX}$$

Putting in the given equation, it reduces to

$$(1 + a^3) \left(\frac{dz}{dX} \right)^3 = 27z \quad \text{or} \quad (1 + a^3)^{1/3} \frac{dz}{dX} = 3z^{1/3}$$

$$\text{or} \quad \frac{1}{3} (1 + a^3)^{1/3} z^{-1/3} dz = dX, \text{ separating the variables.}$$

Integrating $\frac{1}{3}(1+a^3)^{1/3} \frac{3}{2} z^{2/3} = X + b$, where b is an arbitrary constant.

$$\text{or} \quad (1+a^3) z^2 = 8(x+ay+b)^3 \quad \dots(1)$$

which is the *complete integral* of the given equation.

General Integral : Taking $b = \phi(a)$, in (1), we get

$$(1+a^3) z^2 = 8\{x+ay+\phi(a)\}^3 \quad \dots(2)$$

Differentiating (2), partially, w.r.t. a we get

$$a^2 z^2 = 8\{x+ay+\phi(a)\}^2 \{y+\phi'(a)\} \quad \dots(3)$$

The general integral, can be obtained by eliminating a between (2) and (3).

Singular Integral : Differentiating the complete integral (1), partially w.r.t. a and b , we get

$$a^2 z^2 = 8(x+ay+b)^2 y \quad \dots(4)$$

$$\text{and} \quad 0 = 24(x+ay+b)^2 \quad \dots(5)$$

Eliminating a and b between (1), (4), (5) the singular integral of given equation is $z=0$, as it satisfy the given differential equation ($\because z=0 \Rightarrow p=0, q=0$).

Example 3: Find the complete integral of $p(1+q^2) = q(z-a)$.

[Bilaspur 2002; Jiwaji 2003; Vikram 2004; Ravishankar 2005]

Solution: The given differential equation is of the form $f(p, q, z) = 0$.

$\therefore$ Let $z = f(X)$, where $X = x + ay$, a is an arbitrary constant.

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial x} = \frac{dz}{dX} \text{ and } q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial y} = a \frac{dz}{dX}$$

Putting in the given equation, it reduces to

$$\frac{dz}{dX} \cdot \left[1 + a^2 \left(\frac{dz}{dX} \right)^2 \right] = a \frac{dz}{dX} (z-a) \quad \text{or} \quad a^2 \left(\frac{dz}{dX} \right)^2 = a(z-a) - 1$$

$$\text{or} \quad \frac{a dz}{\sqrt{a(z-a)-1}} = dX$$

Integrating, $2\sqrt{a(z-a)-1} = X + b$, b is an arbitrary constant.

or $4a(z-a) = 4 + (x+ay+b)^2$

which is the required complete integral.

Example 4: Find complete integral of

$$z^2(p^2z^2 + q^2) = 1.$$

Find also the singular integral if it exists.

[Delhi Maths (Hons.) 2005; Meerut 2007]

Solution: The given differential equation is of the form $f(p, q, z) = 0$

∴ Let $z = f(X)$, where $X = x + ay$, a is an arbitrary constant

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial x} = \frac{dz}{dX}, \quad q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial y} = a \frac{dz}{dX}$$

Putting in the given equation it reduces to

$$z^2(z^2 + a^2) \left(\frac{dz}{dX} \right)^2 = 1 \quad \text{or} \quad z \sqrt{(z^2 + a^2)} dz = dX$$

Integrating, $\frac{1}{3} (z^2 + a^2)^{3/2} = X + b$, b is an arbitrary constant.

$$\text{or} \quad (z^2 + a^2)^3 = 9(x + ay + b)^2 \quad \dots(1)$$

which is the complete integral of the given equation.

Singular Integral : Differential (1) partially w.r.t. a and b , we get

$$6a(z^2 + a^2)^2 = 18y(x + ay + b) \quad \dots(2)$$

$$\text{and} \quad 0 = 18(x + ay + b) \quad \dots(3)$$

$$\text{Now from (2) and (3), we get } a = 0 \quad \dots(4)$$

From (1), (3) and (4), we get $z = 0$

∴ $z = 0$ may be the singular integral.

But when $z = 0$, $p = 0$, $q = 0$, which does not satisfy given differential equation. Hence, $z = 0$ is not the singular solution.

Hence, the problem does not have singular solution.

Example 5: Find a complete integral of $pz = 1 + q^2$.

Solution: The given differential equation is of the form $f(p, q, z) = 0$

∴ Let $z = f(X)$, where $X = x + ay$, a is an arbitrary constant

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial x} = \frac{dz}{dX} \text{ and } q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial y} = a \frac{dz}{dX}$$

Putting in the given equation, it reduces to

$$z \frac{dz}{dX} = 1 + a^2 \left(\frac{dz}{dX} \right)^2 \text{ or } a^2 \left(\frac{dz}{dX} \right)^2 - z \frac{dz}{dX} + 1 = 0$$

$$\therefore \frac{dz}{dX} = \frac{z \pm \sqrt{z^2 - 4a^2}}{2a^2} \text{ or } dX = \frac{2a^2 dz}{z \pm \sqrt{z^2 - 4a^2}}$$

$$\text{or } dX = \frac{2a^2 \{z \mp \sqrt{z^2 - 4a^2}\} dz}{\{z \mp \sqrt{z^2 - 4a^2}\} \cdot \{z \pm \sqrt{z^2 - 4a^2}\}} = \frac{2a^2}{4a^2} \{z \pm \sqrt{z^2 - 4a^2}\} dz$$

$$\text{or } 4dX = 2 \{z \pm \sqrt{z^2 - 4a^2}\} dz$$

$$\text{Integrating } 4X + b = z^2 \pm 2 \left[\frac{1}{2} z \sqrt{z^2 - 4a^2} - \frac{4a^2}{2} \log \{z + \sqrt{z^2 - 4a^2}\} \right]$$

$$\text{or } 4(x + ay) + b = z^2 \pm [z \sqrt{z^2 - 4a^2} - 4a^2 \log \{z + \sqrt{z^2 - 4a^2}\}]$$

where a and b are arbitrary constants

which is the complete integral.

Example 6: Find a complete integral of $p^2 = z^2 (1 - pq)$.

[Meerut 2001]

Solution: The given differential equation is of the form $f(p, q, z) = 0$

∴ Let $z = f(X)$, where $X = x + ay$, a is an arbitrary constant.

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial x} = \frac{dz}{dX} \text{ and } q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial y} = a \frac{dz}{dX}$$

Putting in the given equation, it reduces to

$$\left(\frac{dz}{dX} \right)^2 = z^2 \left\{ 1 - a \left(\frac{dz}{dX} \right)^2 \right\} \text{ or } (1 + az^2) \left(\frac{dz}{dX} \right)^2 = z^2$$

$$\begin{aligned}
 \text{or } dX &= \frac{\sqrt{1+az^2}}{z} dz = \frac{1+az^2}{z\sqrt{1+az^2}} dz \\
 &= \left[\frac{dz}{z\sqrt{1+az^2}} + \frac{az dz}{\sqrt{1+az^2}} \right] \\
 &= -\frac{dt}{\sqrt{t^2+a}} + dv \quad \text{Putting } z=1/t, dz=-(1/t^2) dt \text{ in first integral}
 \end{aligned}$$

and $1+az^2=v^2$, $az dz=v dv$ in second integral

Integrating, $X+b=-\log\{t+\sqrt{t^2+a}\}+v$

$$\text{or } x+ay+b=-\log\left\{\frac{1+\sqrt{1+az^2}}{z}\right\}+\sqrt{1+az^2}$$

$$\text{or } x+ay+b=-\log\{1+\sqrt{1+az^2}\}+\log z+\sqrt{1+az^2}.$$

which is the required complete integral.

Example 7: Find a complete integral of the equation $p^2 z^2 + q^2 = 1$.

[Meerut 2008]

Solution: The given differential equation is of the form $f(p, q, z) = 0$

$\therefore$ Let $z = f(X)$, where $X = x + ay$, a is an arbitrary constant.

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial x} = \frac{dz}{dX} \text{ and } q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial y} = a \frac{dz}{dX}$$

Putting in the given equation, it reduces to

$$(z^2 + a^2) \left(\frac{dz}{dX} \right)^2 = 1$$

$$\text{or } \sqrt{z^2 + a^2} dz = dX$$

$$\text{Integrating } \frac{1}{2} z \sqrt{z^2 + a^2} + \frac{1}{2} a^2 \log\{z + \sqrt{z^2 + a^2}\} = X + b$$

$$\text{or } z \sqrt{z^2 + a^2} + a^2 \log\{z + \sqrt{z^2 + a^2}\} = 2(x + ay + b).$$

Note : If we take $X = ax + y$, then $p = a \frac{dz}{dX}$ and $q = \frac{dz}{dX}$

In this case the complete integral is

$$az \sqrt{(a^2 z^2 + 1)} + \log \{az + \sqrt{(a^2 z^2 + 1)}\} = 2a (ax + y + b).$$

Example 8: Find a complete integral and singular integral of $z^2 (p^2 + q^2 + 1) = c^2$.

[Jabalpur 2004]

Solution: The given differential equation is of the form $f(p, q, z) = 0$

$\therefore$ Let $z = f(X)$, where $X = x + ay$, a is an arbitrary constant.

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial x} = \frac{dz}{dX} \text{ and } q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X} \frac{\partial X}{\partial y} = a \frac{dz}{dX}$$

Putting in the given equation, it reduces to

$$z^2 \left[(1 + a^2) \left(\frac{dz}{dX} \right)^2 + 1 \right] = c^2 \quad \text{or} \quad z^2 (1 + a^2) \left(\frac{dz}{dX} \right)^2 = c^2 - z^2$$

$$\text{or} \quad \pm \frac{1}{2} \sqrt{(1 + a^2)} \frac{(-2z)}{\sqrt{(c^2 - z^2)}} dz = dX$$

$$\text{Integrating, } \pm \sqrt{(1 + a^2)} \sqrt{(c^2 - z^2)} = X + b$$

$$\text{or} \quad (1 + a^2) (c^2 - z^2) = (x + ay + b)^2 \quad \dots(1)$$

which is the complete integral of the given equation.

Singular Integral : Differentiating (1) partially w.r.t. a and b , we have

$$2a (c^2 - z^2) = 2 (x + ay + b) \cdot y \quad \text{and} \quad 0 = 2 (x + ay + b)$$

$$\Rightarrow \quad x + ay + b = 0 \quad \text{and} \quad a = 0$$

Putting in (1), we get $z^2 = c^2$

$$\text{From } z^2 = c^2, \text{ we get } p = 0 \quad \text{and} \quad q = 0$$

These values satisfy the given differential equation.

$\therefore z^2 = c^2$ is the singular integral of the equation.

8.4.1 Partial Differential Equations Reducible to Standard Form II, $f(p, q, z) = 0$

Example 9: Find a complete integral of $pq = x^m y^n z^l$.

Solution: The given equation can be written as

$$\left(\frac{1}{x^m} \frac{\partial z}{\partial x} \right) \left(\frac{1}{y^n} \frac{\partial z}{\partial y} \right) = z^l \quad \dots(1)$$

Putting $x^m dx = dX$ and $y^n dy = dY$,

so that $\frac{x^{m+1}}{m+1} = X$ and $\frac{y^{n+1}}{n+1} = Y$, (1) reduces to

$$\frac{\partial z}{\partial X} \cdot \frac{\partial z}{\partial Y} = z^l \quad \text{or} \quad PQ = z^l \quad \dots(2)$$

where $\frac{\partial z}{\partial X} = P$ and $\frac{\partial z}{\partial Y} = Q$

Equation (2) is of the form $f(P, Q, z) = 0$

$\therefore$ Let $z = f(u)$, where $u = X + aY$, a is an arbitrary constant.

$$\therefore P = \frac{\partial z}{\partial X} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial X} = \frac{dz}{du} \quad \text{and} \quad Q = \frac{\partial z}{\partial Y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial Y} = a \frac{dz}{du}$$

Putting in (2), it reduces to

$$a \left(\frac{dz}{du} \right)^2 = z^l \quad \text{or} \quad z^{-l/2} dz = \frac{du}{\sqrt{a}}$$

Integrating, $\frac{z^{-l/2+1}}{-l/2+1} = \frac{u}{\sqrt{a}} + b$

$$\text{or} \quad \frac{z^{-l/2+1}}{-l/2+1} = \frac{1}{\sqrt{a}} \left(\frac{x^{m+1}}{m+1} + a \frac{y^{n+1}}{n+1} \right) + b$$

which is the complete integral of the given equation, a and b are arbitrary constant.

Example 10: Find a complete integral of $q^2 y^2 = z(z - p x)$.

Solution: The given equation can be written as

$$\left(y \frac{\partial z}{\partial y}\right)^2 = z^2 - z \left(x \frac{\partial z}{\partial x}\right) \quad \dots(1)$$

Putting $\frac{1}{x} dx = dX$ and $\frac{1}{y} dy = dY$, so that $X = \log x$ and $Y = \log y$, (1) reduces to

$$\left(\frac{\partial z}{\partial Y}\right)^2 = z^2 - z \left(\frac{\partial z}{\partial X}\right) \quad \text{or} \quad Q^2 = z^2 - zP \quad \dots(2)$$

where $P = \frac{\partial z}{\partial X}$ and $Q = \frac{\partial z}{\partial Y}$

Equation (2) is of the form $f(P, Q, z)$

$\therefore$ Let $z = f(u)$ where $u = X + aY$

$$\therefore P = \frac{\partial z}{\partial X} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial X} = \frac{dz}{du}, \quad Q = \frac{\partial z}{\partial Y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial Y} = a \frac{dz}{du}$$

Putting in (2), it reduces to

$$a^2 \left(\frac{dz}{du}\right)^2 = z^2 - z \cdot \frac{dz}{du} \quad \text{or} \quad a^2 \left(\frac{dz}{du}\right)^2 + z \frac{dz}{du} - z^2 = 0$$

or
$$\frac{dz}{du} = \frac{-z \pm \sqrt{(z^2 + 4a^2 z^2)}}{2a^2} = \frac{z}{2a^2} [-1 \pm \sqrt{(1 + 4a^2)}] = z k$$

where $k = [-1 \pm \sqrt{(1 + 4a^2)}] / 2a^2$

or
$$\frac{1}{k} \frac{dz}{z} = du$$

Integrating, $\frac{1}{k} \log z = u + \log b = X + aY + \log b = \log x + a \log y + \log b$

or
$$\log z^{1/k} = \log (x b y^a) \quad \text{or} \quad z^{1/k} = b x y^a$$

or
$$z = (b x y^a)^k, \text{ where } k = [-1 \pm \sqrt{(1 + 4a^2)}] / 2a^2.$$

which is the complete integral of the given problem.

Exercise 8.2

Find the complete integrals of the following :

1. $p^2 = zq$

[Sagar 2004]

2. $p^3 = qz$

[Delhi (Maths) 2007]

3. $z = pq$

4. $zpq = p + q$

[Delhi (Maths) 2007]

5. $9(p^2z + q^2) = 4$

[Delhi Maths (Hons.) 2006]

6. $p(z + p) + q = 0$

7. $zpq = p + q$

8. $p^3 + q^3 - 3pqz = 0$

9. $q^2 = z^2p^2(1 - p^2)$

10. $z^2(p^2 + q^2 + 1) = 1$

11. $4(1 + z^3) = 9z^4pq$

12. $p(1 + q) = qz$

[Gulberg 2005]

13. $p^2x^2 = z(z - qy)$

[Hint : Given equation can be written as

$$\left(x \frac{\partial z}{\partial x}\right)^2 = z \left[z - y \frac{\partial z}{\partial y}\right] \text{ Take } \frac{1}{x} dx = dX \text{ and } \frac{1}{y} dy = dY.$$

Answers 8.2

1.	$z = b e^{ax + a^2 y}$	2.	$4z = (x + a y + b)^2$
3.	$z = (x + a y + b)^2 / 4a$	4.	$x + a y + b = 4az / (1 - a)$
5.	$(z + a^2)^3 = (x + a y + b)^2$	6.	$z + a = b e^{-(x + a y)}$
7.	$4az = (1 - a)(x + a y + b)$	8.	$(1 + a^3) \log z = 3a(x + a y) + b$
9.	$(x + a y + b)^2 = z^2 - a^2$	10.	$(1 + a^2)(1 - z^2) = (x + a y + b)^2$
11.	$a(1 + z^3) = (x + a y + b)^2$	12.	$az = 1 + b e^{x + a y}$
13.	$x b y^a = z^{1/k}, k = [-1 \pm \sqrt{(a^2 + 4)}] / 2a^2$		

8.5 Standard Form III :

Equations of the Form $f_1(x, p) = f_2(y, q)$

In partial differential equations of standard form III, z does not appear and the terms containing p and x can be separated from the terms containing q and y i.e., the given equation can be written as $f_1(x, p) = f_2(y, q)$.

To solve the equations of this form each side is taken equal to an arbitrary constant.

$$\text{i.e., let } f_1(x, p) = a = f_2(y, q) \quad \dots(1)$$

where a is an arbitrary constant.

Solving (1) for p and q , we get

$$p = F_1(x, a) \text{ and } q = F_2(y, a)$$

Now using well known formula, $dz = p dx + q dy$, we get

$$dz = F_1(x, a) dx + F_2(y, a) dy$$

Integrating,

$$z = \int F_1(x, a) dx + \int F_2(y, a) dy + b \quad \dots(2)$$

which contains two arbitrary constants a and b and is the **complete integral** of the given equation.

General Integral : The general equation can be obtained as in other cases.

Singular Integral : Differentiating (2); partially w.r.t. b , we get $0=1$ which is inadmissible. So there is no singular integral in this case.

Solved Examples

Example 1: Solve $py + qx + pq = 0$.

[Kurukshetra 2004]

Solution: The given equation can be written as

$$p(y + q) = -qx \quad \text{or} \quad \frac{p}{x} = -\frac{q}{(y + q)} \quad \dots(1)$$

which is of the form $f_1(x, p) = f_2(y, q)$

Equating each side equal to an arbitrary constant a , we have, $\frac{p}{x} = a = -\frac{q}{(y + q)}$

$$\Rightarrow \quad p = ax \quad \text{and} \quad q = -\frac{ay}{(1+a)}$$

$$\therefore \quad \text{from,} \quad dz = p dx + q dy, \text{ we get } dz = ax dx - \left\{ \frac{a}{(1+a)} \right\} y dy$$

$$\text{Integrating,} \quad z = \frac{1}{2} ax^2 - \left\{ \frac{a}{(1+a)} \right\} \frac{1}{2} y^2 + b \quad \dots(2)$$

which is the required complete integral containing two arbitrary constants a and b .

General Integral : Taking $b = \phi(a)$ in (2), we get

$$z = \frac{1}{2} ax^2 - \frac{1}{2} \left\{ \frac{a}{(1+a)} \right\} y^2 + \phi(a) \quad \dots(3)$$

Differentiating (3) partially w.r.t. a , we get

$$z = \frac{1}{2} x^2 - \frac{1}{2} \left\{ \frac{1}{(1+a)^2} \right\} y^2 + \phi'(a) \quad \dots(4)$$

The general integral of the given equation can be obtained by eliminating a between (3) and (4).

Singular Integral : Differentiating (2) partially w.r.t. a and b , we get

$$0 = \frac{1}{2} x^2 - \frac{1}{2} \left\{ \frac{1}{(1+a)^2} \right\} y^2 \quad \dots(5) \quad \text{and} \quad 0 = 1 \quad \dots(6)$$

Since (6) is inadmissible, hence the given equation has no singular integral.

Example 2: Find a complete integral of $x(1+y)p = y(1+x)q$.

Solution: The given equation can be written as

$$\frac{x p}{1+x} = \frac{y q}{1+y} \quad \dots(1)$$

which is of the form $f_1(x, p) = f_2(y, q)$

$\therefore$ Equating each side to an arbitrary constant a , we have

$$\frac{x p}{1+x} = a = \frac{y q}{1+y} \Rightarrow p = \frac{1+x}{x} a, q = \frac{1+y}{y} a$$

$$\therefore \quad \text{from } dz = p dx + q dy, \text{ we get } dz = \left(\frac{1}{x} + 1 \right) a dx + \left(\frac{1}{y} + 1 \right) a dy$$

Integrating, $z = a (\log x + x) + a (\log y + y) + b$

or $z = a (\log xy + x + y) + b$

Which is the required complete integral containing two arbitrary constants a and b .

Example 3: Solve $q = x y p^2$.

Solution: The given equation can be written as

$$x p^2 = q / y \quad \dots(1)$$

which is of the form $f_1(x, p) = f_2(y, q)$.

$\therefore$ Let $x p^2 = a = \frac{q}{y} \Rightarrow p = \sqrt{(a/x)}, q = a y, a$ is an arbitrary constant.

$\therefore$ from $dz = p dx + q dy$, we get $dz = \sqrt{a} x^{-1/2} dx + a y dy$.

$$\text{Integrating} \quad z = 2 \sqrt{(ax)} + \frac{1}{2} ay^2 + b \quad \dots(2)$$

which is the complete integral of the given equation.

General Integral : Taking $b = \phi(a)$, from (2), we get

$$z = 2 \sqrt{(ax)} + \frac{1}{2} ay^2 + \phi(a) \quad \dots(3)$$

Differentiating (3) partially w.r.t. a , we get

$$0 = \sqrt{x} / \sqrt{a} + \frac{1}{2} y^2 + \phi'(a) \quad \dots(4)$$

General integral of the given equation can be obtained by eliminating a between (3) and (4).

Singular Integral : There is no singular integral of the problem.

(Proceed as in the last example 3)

Example 4: Find a complete integral of $q = p x + p^2$.

[Bilaspur 2004]

Solution: The given equation is of the form $f_1(x, p) = f_2(y, q)$.

$\therefore$ Let $p x + p^2 = a = q$, where a is an arbitrary constant.

$$\Rightarrow q = a \text{ and } p^2 + p x - a = 0$$

$$\Rightarrow q = a \text{ and } p = [-x \pm \sqrt{(x^2 + 4a)}] / 2.$$

$$\therefore \text{ from } dz = p dx + q dy, \text{ we get } dz = \frac{1}{2} [-x \pm \sqrt{(x^2 + 4a)}] dx + a dy$$

$$\text{Integrating, } z = -\frac{1}{4}x^2 \pm \frac{1}{2} \left[\frac{1}{2} x \sqrt{(x^2 + 4a)} + 2a \cdot \log \left\{ x + \sqrt{(x^2 + 4a)} \right\} \right] + ay + b.$$

which is the complete integral of the given equation.

Example 5: Find a complete integral of $yp = 2yx + \log q$.

[Ravishankar 2005]

Solution: The given equation can be written as

$$p - 2x = (\log q) / y, \text{ which is of the form } f_1(x, p) = f_2(y, q)$$

$$\therefore \text{ Let } p - 2x = a = (\log q) / y, \text{ where } a \text{ is an arbitrary constant.}$$

$$\Rightarrow p = 2x + a \text{ and } \log q = a y \text{ i.e., } q = e^{ay}$$

$$\therefore \text{ from } dz = p dx + q dy, \text{ we get } dz = (2x + a) dx + e^{ay} dy$$

$$\text{Integrating } z = x^2 + ax + \frac{e^{ay}}{a} + b \quad \dots(1)$$

which is the complete integral of the given equation.

Example 6: Find a complete integral of $p^2 q (x^2 + y^2) = p^2 + q$.

Solution: The given equation can be written as

$$x^2 + y^2 = 1/q + 1/p^2 \text{ or } 1/p^2 - x^2 = y^2 - 1/q$$

$$\text{which is of the form } f_1(x, p) = f_2(y, q)$$

$$\therefore \text{ Let } 1/p^2 - x^2 = a = y^2 - 1/q, \text{ where } a \text{ is an arbitrary constant.}$$

$$\Rightarrow p = \frac{1}{\sqrt{(x^2 + a)}} \text{ and } q = \frac{1}{(y^2 - a)}$$

$$\therefore \text{ from } dz = p dx + q dy, \text{ we get } dz = \frac{dx}{\sqrt{(x^2 + a)}} + \frac{dy}{y^2 - a}$$

$$\text{Integrating, } z = \log \{x + \sqrt{(x^2 + a)}\} + \frac{1}{2\sqrt{a}} \log \left(\frac{y - \sqrt{a}}{y + \sqrt{a}} \right) + b$$

which is the complete integral of the given problem.

8.5.1 Partial Differential Equations Reducible to the Form

$$f_1(x, p) = f_2(y, q)$$

Example 7: Find a complete integral of $z(p^2 - q^2) = x - y$. [Indore 2002; Bilaspur 2003]

Solution: The given equation can be written as

$$\left(\sqrt{z} \frac{\partial z}{\partial x}\right)^2 - \left(\sqrt{z} \frac{\partial z}{\partial y}\right)^2 = x - y \quad \dots(1)$$

Putting $\sqrt{z} dz = dZ$, so that $(2/3) z^{3/2} = Z$, equation (1) reduces to

$$\left(\frac{\partial Z}{\partial x}\right)^2 - \left(\frac{\partial Z}{\partial y}\right)^2 = x - y$$

or $P^2 - Q^2 = x - y$, where $\frac{\partial Z}{\partial x} = P$, $\frac{\partial Z}{\partial y} = Q$

or $P^2 - x = Q^2 - y$, which is of the form $f_1(x, P) = f_2(y, Q)$

$\therefore$ Let $P^2 - x = a = Q^2 - y$, where a is an arbitrary constant.

$$\Rightarrow P = \sqrt{(x+a)} \text{ and } Q = \sqrt{(a+y)}$$

$\therefore$ from $dZ = Pdx + Qdy$, we get $dZ = \sqrt{(x+a)} dx + \sqrt{(a+y)} dy$

$$\text{Integrating } Z = (2/3) [(x+a)^{3/2} + (a+y)^{3/2}] + b$$

$$\text{or } (2/3) z^{3/2} = (2/3) [(x+a)^{3/2} + (a+y)^{3/2}] + b$$

$$\text{or } z^{3/2} = (x+a)^{3/2} + (a+y)^{3/2} + c, \text{ where } c = 3b/2.$$

which is the complete integral of the given problem.

Example 8: Find a complete integral of $z^2(p^2 + q^2) = x^2 + y^2$.

[Meerut 2003; Jabalpur 2004; Agra 2006]

Solution: The given equation can be written as :

$$\left(z \frac{\partial z}{\partial x}\right)^2 + \left(z \frac{\partial z}{\partial y}\right)^2 = x^2 + y^2 \quad \dots(1)$$

Putting $z dz = dZ$, so that $Z = \frac{1}{2} z^2$, (1) reduces to

$$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = x^2 + y^2$$

or $P^2 + Q^2 = x^2 + y^2$, where $P = \frac{\partial Z}{\partial x}$, $Q = \frac{\partial Z}{\partial y}$

or $P^2 - x^2 = y^2 - Q^2 \quad \dots(2)$

which is of the form $f_1(x, P) = f_2(y, Q)$

$\therefore$ Let $P^2 - x^2 = a = y^2 - Q^2$, where a is an arbitrary constant.

$\Rightarrow \quad P = \sqrt{(x^2 + a)}$ and $Q = \sqrt{(y^2 - a)}$

$\therefore$ from $dZ = Pdx + Qdy$, we get $dZ = \sqrt{(x^2 + a)} dx + \sqrt{(y^2 - a)} dy$

Integrating, $Z = \frac{1}{2} x \sqrt{(x^2 + a)} + \frac{a}{2} \log \{x + \sqrt{(x^2 + a)}\}$
 $+ \frac{1}{2} y \sqrt{(y^2 - a)} - \frac{a}{2} \log \{y + \sqrt{(y^2 - a)}\} + b$

or $z^2 = x \sqrt{(x^2 + a)} + y \sqrt{(y^2 - a)} + a [\log \{x + \sqrt{(x^2 + a)}\}$
 $- \log \{y + \sqrt{(y^2 - a)}\}] + c$

$\therefore \quad Z = \frac{1}{2} z^2$, where $c = 2b$

which is the complete integral of the given problem.

Exercise 8.3

Find the complete integrals of the followings :

1. $p e^y = q e^x$ 2. $p^2 + q^2 = x + y$ [Meerut 2007]

3. $p^2 + q^2 = x^2 + y^2$

4. $p q = x y$

5. $p^2 y (1 + x^2) = q x^2$

6. $\sqrt{p} + \sqrt{q} = 2x$

7. $p - 3x^2 = q^2 - y$

8. $p^2 q^2 + x^2 y^2 = x^2 q^2 (x^2 + y^2)$

[Delhi Maths (Hons.) 2002; Agra 2005]

[Hint : Equation can be written as $p^2 / x^2 - x^2 = y^2 (1 - 1/q^2)$]

9. $y z p^2 = q$

10. $z (x p - y q) = y^2 - x^2$

11. $p^2 + q^2 = (x^2 + y^2) z$

12. $z^2 (p^2 + q^2) = x^2 + e^2 y$

[Delhi Maths (Hons.) 2005]

Answers 8.3

1.	$z = a(e^x + e^y) + b$	2.	$z = \frac{2}{3}[(x+a)^{3/2} + (y-a)^{3/2}] + b$
3.	$2z = x\sqrt{x^2 + a^2} + a^2 \sinh^{-1}(x/a) + y\sqrt{y^2 - a^2} - a^2 \cosh^{-1}(y/a) + b$		
4.	$2az = a^2x^2 + y^2 + 2ab$	5.	$z = a\sqrt{1+x^2} + a^2y^2/2 + b$
6.	$z = \frac{1}{6}(a+2x)^3 + a^2y + b$	7.	$z = x^3 + ax \pm \frac{2}{3}(y+a)^{3/2} + b$
8.	$z = \frac{1}{3}(x^2 + a^2)^{3/2} + (y^2 - a^2)^{1/2} + b$	9.	$z^2 = 2ax + a^2y + b$
10.	$z^2 = 2a \log(xy) - x^2 - y^2 + b$		
11.	$4\sqrt{z} = x\sqrt{x^2 + a^2} + a^2 \sinh^{-1} \frac{x}{a} + y\sqrt{y^2 - a^2} - a^2 \cosh^{-1} \frac{y}{a} + b$		
12.	$z^2 = x\sqrt{x^2 + a^2} + a \sinh^{-1}(x/\sqrt{a}) + 2\sqrt{e^{2y} - a} - \sqrt{a} \tan^{-1}\{(e^{2y} - a)/a\}^{1/2} + b$		

8.6 Standard Form IV :

Equations of the Form $z = px + qy + f(p, q)$

Clairaut Equation : A first order partial differential equation is said to be of **Clairaut form**, if it can be written in the form [Nagpur 2002]

$$z = px + qy + f(p, q) \quad \dots(1)$$

The complete integral of the equation in this form is given by

$$z = ax + by + f(a, b) \quad \dots(2)$$

It can be verified that (2) satisfy (1)

From (2), we have, $p = \frac{\partial z}{\partial x} = a$, $q = \frac{\partial z}{\partial y} = b$

Putting in (2), we get $z = px + qy + f(p, q)$

which is the given equation (1).

Working Method : The complete integral of the partial differential in standard form (1) is obtained by replacing p by a and q by b in the equation, where a and b are arbitrary constants.

General Integral : General integral is obtained as in other cases.

Singular Integral : Differentiating the complete integral (2) partially w.r.t. a and b , we get

$$0 = x + \frac{\partial f}{\partial a} \quad \dots(3) \quad \text{and} \quad 0 = y + \frac{\partial f}{\partial b} \quad \dots(4)$$

Singular integral is obtained by eliminating a and b between (2), (3) and (4).

Solved Examples

Example 1: Solve $z = p x + q y - 2 \sqrt{pq}$.

Solution: The given differential equation is of clairaut form.

$\therefore$ Replacing p by a and q by b , the complete integral of the equation is

$$z = a x + b y - 2 \sqrt{ab} \quad \dots(1)$$

where a and b are arbitrary constants.

General Integral : Replacing b by $\phi(a)$ in (1), we get

$$z = a x + y \phi(a) - 2 \sqrt{a \phi(a)} \quad \dots(2)$$

Differentiating (2) partially w.r.t. a , we get

$$0 = x + y \phi'(a) - [a \phi'(a) + \phi(a)] / \sqrt{a \phi(a)} \quad \dots(3)$$

The *general integral* is obtained by eliminating a between (2) and (3).

Singular Integral : Differentiating (1) partially w.r.t. a and b , we get

$$0 = x - b / \sqrt{ab} \quad \dots(4) \quad \text{and} \quad 0 = y - a / \sqrt{ab} \quad \dots(5)$$

From (4), $x = \sqrt{b/a}$ and from (5) $y = \sqrt{a/b}$

$\therefore$ Eliminating a and b , we get

$x y = 1$, which is the *singular integral*.

Example 2: Solve $z = p x + q y + \log(pq)$. [Indore 2004; Kurukshetra 2006; Meerut 2012]

Solution: The given differential equation is of clairaut form.

∴ Its complete integral is, $z = ax + by + \log(ab)$... (1)

where a and b are arbitrary constants.

General Integral : Replacing b by $\phi(a)$ in (1), we get

$$z = ax + y\phi(a) + \log[a\phi(a)]$$

or $z = ax + y\phi(a) + \log a + \log \phi(a)$... (2)

Differentiating (2) partially w.r.t. a , we get

$$0 = x + y\phi'(a) + \frac{1}{a} + \frac{\phi'(a)}{\phi(a)} \quad \dots (3)$$

The general integral is obtained by eliminating a between (2) and (3).

Singular Integral : Differentiating (1) partially w.r.t. a and b , we get

$$0 = x + 1/a \quad \dots (4) \quad \text{and} \quad 0 = y + 1/b \quad \dots (5)$$

From (4) and (5), $a = -1/x, b = -1/y$

Putting in (1), the singular integral of given problem is given by

$$z = -1 - 1 + \log(1/xy) \text{ or } z = -2 - \log(xy)$$

which satisfy the given equation.

Example 3: Solve $z = px + qy + c\sqrt{1 + p^2 + q^2}$.

Solution: The given differential equation is of Clairaut form.

∴ Its complete integral is

$$z = ax + by + c\sqrt{1 + a^2 + b^2} \quad \dots (1)$$

which a and b are arbitrary constants.

General Integral : Replacing, $b = \phi(a)$ in (1), we get

$$z = ax + y\phi(a) + c\sqrt{1 + a^2 + \phi^2(a)} \quad \dots (2)$$

Differentiating (2) partially w.r.t. a , we get

$$0 = x + y\phi'(a) + c\{a + \phi(a)\phi'(a)\} / \sqrt{1 + a^2 + \phi^2(a)} \quad \dots (3)$$

General integral is obtained by eliminating a between (2) and (3).

Singular Integral : Differentiating (1) partially w.r.t. a and b , we get

$$0 = x + c a / \sqrt{(1 + a^2 + b^2)} \dots(4) \text{ and } 0 = y + c b / \sqrt{(1 + a^2 + b^2)} \dots(5)$$

To obtain the singular integral by eliminating a and b between (1), (4) and (5). We proceed as follows.

$$\text{From (4) and (5). } x^2 + y^2 = c^2 (a^2 + b^2) / (1 + a^2 + b^2)$$

$$\text{or } x^2 + y^2 - c^2 = -c^2 / (1 + a^2 + b^2)$$

$$\text{or } 1 + a^2 + b^2 = c^2 / (c^2 - x^2 - y^2).$$

$$\therefore \text{ from (4) and (5) } a = \frac{-x \sqrt{(1 + a^2 + b^2)}}{c} = \frac{-x}{\sqrt{(c^2 - x^2 - y^2)}}$$

$$\text{and } b = \frac{-y \sqrt{(1 + a^2 + b^2)}}{c} = \frac{-y}{\sqrt{(c^2 - x^2 - y^2)}}$$

Putting in (1), the singular integral of the problem is

$$z = \frac{-(x^2 + y^2)}{\sqrt{(c^2 - x^2 - y^2)}} + \frac{c^2}{\sqrt{(c^2 - x^2 - y^2)}} = \sqrt{(c^2 - x^2 - y^2)}$$

$$\text{or } x^2 + y^2 + z^2 = c^2$$

which satisfy the given equation.

Example 4: Prove that the complete integral of $z = p x + q y - 2 p - 3 q$ represents all possible planes through the point $(2, 3, 0)$. Also find the envelope of all planes represented by the complete integral. [M.D.U. Rohtak 2006]

Solution: The given equation

$$z = p x + q y - 2 p - 3 q \dots(1)$$

is of Clairaut form. So its Complete integral is

$$z = a x + b y - 2 a - 3 b \dots(2)$$

where a and b are arbitrary constants.

Since (2) is a linear equation in x, y, z so it represents a plane. Also the point $(2, 3, 0)$ satisfy (2). Hence the complete integral of the given equation represents all possible planes through the point $(2, 3, 0)$.

To Find the Envelope : Differentiating (2) partially w.r.t. a and b , we get

$$0 = x - 2 \quad \text{and} \quad 0 = y - 3$$

$$\Rightarrow \quad x = 2 \quad \text{and} \quad y = 3$$

Putting in (1), the equation of *singular integral i.e., the envelope* of all planes represented by the complete integral is $z = 0$.

which satisfy the given equation.

8.6.1 Partial Differential Equations Reducible to Clairaut Form

Example 5: Find a complete and the singular integral of

$$4xyz = pq + 2px^2y + 2qxy^2. \quad [\text{Meerut 2011 (B.P.)}]$$

Solution: The given equation can be written as

$$z = \left(\frac{1}{2x} \frac{\partial z}{\partial x} \right) \left(\frac{1}{2y} \frac{\partial z}{\partial y} \right) + \left(\frac{1}{2x} \frac{\partial z}{\partial x} \right) x^2 + \left(\frac{1}{2y} \frac{\partial z}{\partial y} \right) y^2 \quad \dots(1)$$

Putting $2x \, dx = dX$ and $2y \, dy = dY$, so that $x^2 = X$ and $y^2 = Y$, (1) reduced to

$$z = \left(\frac{\partial z}{\partial X} \right) \left(\frac{\partial z}{\partial Y} \right) + \left(\frac{\partial z}{\partial X} \right) X + \left(\frac{\partial z}{\partial Y} \right) Y$$

$$\text{or} \quad z = PX + QY + PQ \quad \dots(2)$$

$$\text{where} \quad \frac{\partial z}{\partial X} = P \quad \text{and} \quad \frac{\partial z}{\partial Y} = Q$$

Equation (2) is of Clairaut's form, so its complete integral is

$$z = aX + bY + ab, \text{ where } a \text{ and } b \text{ are arbitrary constants}$$

$$\text{or} \quad z = ax^2 + by^2 + ab \quad \dots(3)$$

which is the *complete integral* of the given equation.

Singular Integral : Differentiating (2) partially w.r.t. a and b we get

$$0 = x^2 + b \text{ and } 0 = y^2 + a$$

$$\Rightarrow \quad a = -y^2, b = -x^2$$

Putting in (3), the singular integral is

$$z = -y^2x^2 - x^2y^2 + x^2y^2 \quad \text{or} \quad z = -x^2y^2.$$

Exercise 8.4

Solve the following partial differential equations :

1. $z = px + qy + p^2 \cdot q^2$

2. $z = px + qy - p^2q$

3. $z = px + qy + pq$

[Bangalore 2005]

4. $z = px + qy + p^2 - q^2$

[Purvanchal 2007]

5. $pqz = p^2(xq + p^2) + q^2(yq + q^2)$

[Meerut 2007 (B.P.); Delhi Maths 2008]

[Hint : Equation can be written as $pqz = p^2qx + p^2q^2y + (p^4 + q^4)$

or $z = px + qy + (p^4 + q^4) / pq$

6. $(p + q)(z - px - qy) = 1$

7. $z = px + qy$

8. Prove that the complete integral of the equation $(px + qy - z)^2 = 1 + p^2 + q^2$ is $ax + by + cz = (a^2 + b^2 + c^2)^{1/2}$

9. Show that the complete integral of the equation $z = px + qy + \sqrt{(p^2 + q^2 + 1)}$ represents all planes at unit distance from the origin.

10. Prove that the complete integral of $z = px + qy + pq / (pq - p - q)$ represents all planes such that the algebraic sum of the intercepts on three coordinate axes is unity.

11. Find the complete integral of $z = px + qy - \sin(pq)$

[Gate 2003]

12. Find the complete integral, envelope and singular integral of the partial differential equation $z - px - qy - p^2 - q^2 = 0$

[Meerut 2009 (B.P.)]

Answers 8.4

1.	$z = ax + by + a^2b^2$	S.I. $z = -(3/4)(4x^2y^2)^{1/3}$
2.	$z = ax + by - a^2b$	S.I. $z = x\sqrt{y}$
3.	$z = ax + by + ab$	S.I. $z = -xy$
4.	$z = ax + by + a^2 - b^2$	S.I. $z = -(x^2 - y^2)/4$
5.	$abz = a^2(xb + a^2) + b^2(ya + b^2)$	
6.	$z = ax + by + 1/(a + b)$	7. $z = ax + by$ S.I. $z = 0$
11.	$z = ax + by - \sin(ab)$	
12.	$z = ax + by + a^2 + b^2$	Env. S.I. $z = -(x^2 + y^2)/4$.

8.7 Charpit's Method : General Method of Solution of Non-linear Partial Differential Equation of Order One with Two Independent Variables

[Delhi Maths (Hons.) 2000,05, 06; Rohilkhand 2001, 04; Agra 2003; Nagpur 2002, 04, 06, 08; Meerut 2003, 05, 11 (B.P.)]

Charpit's method is the general method for solving partial differential equations of order one but of any degree with two independent variables.

Let the given equation be

$$f(x, y, z, p, q) = 0 \quad \dots(1)$$

The fundamental idea in Charpit's method is to find a second partial differential equation

$$F(x, y, z, p, q, a) = 0 \quad \dots(2)$$

Which contains an arbitrary constant a , such that (1) and (2) can be solved to give

$$p = p(x, y, z, a) \text{ and } q = q(x, y, z, a)$$

$$\text{and when substituted in } dz = p dx + q dy \quad \dots(3)$$

it becomes integrable.

The integration of (3) will give the complete integral of equation (1).

To obtain relation (2), differentiating (1) and (2) w.r.t. x and y , we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot p + \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial x} = 0 \quad \dots(4)$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot p + \frac{\partial F}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial F}{\partial q} \cdot \frac{\partial q}{\partial x} = 0 \quad \dots(5)$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot q + \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial y} = 0 \quad \dots(6)$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \cdot q + \frac{\partial F}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial F}{\partial q} \cdot \frac{\partial q}{\partial y} = 0 \quad \dots(7)$$

Eliminating $\frac{\partial p}{\partial x}$ from (4) and (5), we have

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial x} \right) \frac{\partial F}{\partial p} - \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial q} \cdot \frac{\partial q}{\partial x} \right) \cdot \frac{\partial f}{\partial p} = 0$$

or
$$\left(\frac{\partial f}{\partial x} \cdot \frac{\partial F}{\partial p} - \frac{\partial F}{\partial x} \cdot \frac{\partial f}{\partial p} \right) + \left(\frac{\partial f}{\partial z} \cdot \frac{\partial F}{\partial p} - \frac{\partial F}{\partial z} \cdot \frac{\partial f}{\partial p} \right) p + \left(\frac{\partial f}{\partial q} \cdot \frac{\partial F}{\partial p} - \frac{\partial F}{\partial q} \cdot \frac{\partial f}{\partial p} \right) \frac{\partial q}{\partial x} = 0 \quad \dots(8)$$

Similarly, eliminating, $\frac{\partial q}{\partial y}$ from (6) and (7), we have

$$\left(\frac{\partial f}{\partial y} \cdot \frac{\partial F}{\partial q} - \frac{\partial F}{\partial y} \cdot \frac{\partial f}{\partial q} \right) + \left(\frac{\partial f}{\partial z} \cdot \frac{\partial F}{\partial q} - \frac{\partial F}{\partial z} \cdot \frac{\partial f}{\partial q} \right) q + \left(\frac{\partial f}{\partial p} \cdot \frac{\partial F}{\partial q} - \frac{\partial F}{\partial p} \cdot \frac{\partial f}{\partial q} \right) \frac{\partial p}{\partial y} = 0 \quad \dots(9)$$

Since
$$\frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial p}{\partial y}$$

Adding (8) and (9), we have

$$\begin{aligned} \left(\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} \right) \frac{\partial F}{\partial p} + \left(\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z} \right) \frac{\partial F}{\partial q} + \left(-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q} \right) \frac{\partial F}{\partial z} \\ + \left(-\frac{\partial f}{\partial p} \right) \frac{\partial F}{\partial x} + \left(-\frac{\partial f}{\partial q} \right) \frac{\partial F}{\partial y} = 0 \quad \dots(10) \end{aligned}$$

which is linear equation of first order with x, y, z, p, q as independent variables and F as dependent variable.

The auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{\partial F}{0} \quad \dots(11)$$

The equations given by (11) are known as **Charpit's auxiliary equations**.

Any integral of (11) will satisfy (10). Now we find a solution of (11) in which either p or q occur for $F \equiv 0$. Then from $f \equiv 0$ given by (1) and $F \equiv 0$, find the values of p and q and substitute in $dz = p dx + q dy$. Integrating this we obtain the complete integral of (1).

Note : The Charpit's method is complicated enough and is applied only when the four standard methods given in 8.3, 8.4, 8.5, 8.6 fail to solve the equation or when it is required to solve the equation by Charpit's method.

Working method : To find the complete integral by *Charpit's method*, proceed as follows :

Step I : Write the given differential equation in the form

$$f(x, y, z, p, q) = 0 \quad \dots(1)$$

Step II : Write the Charpit's auxiliary equations

$$\frac{\frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}}}{\frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}}} = \frac{\frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}}}{\frac{dx}{-\frac{\partial f}{\partial p}}} = \frac{dy}{-\frac{\partial f}{\partial q}} \quad \dots(2)$$

Step III : Select two fractions in (2), which on integration give a simplest relation involving at least one of p and q .

Let this relation be

$$F(x, y, z, p, q, a) = 0 \quad \dots(3)$$

where a is an arbitrary constant.

Step IV : Solve (1) and (3), to get the values of p and q in terms of x, y and z .

Step V : Substitute the values of p and q obtained in step IV in

$$dz = p dx + q dy \quad \dots(4)$$

Step VI : Integrate (4) to get the complete integral of given equation (1).

The general and singular integrals can be obtained in the usual manner.

Solved Examples

Example 1: Apply Charpit's method to find complete integral of

$$z = p x + q y + p^2 + q^2. \quad [\text{Ravishankar 2000, 04; Kurukshetra 2005; I.A.S. 2005; Agra 2010}]$$

Solution: Here $f \equiv z - p x - q y - p^2 - q^2 = 0$

$\therefore$ Charpit's auxiliary equations are

$$\frac{\frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}}}{\frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}}} = \frac{\frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}}}{\frac{dx}{-\frac{\partial f}{\partial p}}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

or
$$\frac{dp}{-p + p} = \frac{dq}{-q + q} = \frac{dz}{-p(-x - 2p) - q(-y - 2q)} = \frac{dx}{-(-x - 2p)} = \frac{dy}{-(-y - 2q)}$$

$\therefore dp = 0$ and $dq = 0$. Integrating $p = a, q = b$

Putting in (1), the complete integral is

$$z = ax + by + a^2 + b^2.$$

Note : The given equation is of standard form IV.

Example 2: Solve $p = (qy + z)^2$.

[Agra 2001; Kurukshetra 2005; Delhi Maths 2006, 08; Meerut 2008]

Solution: Here $f \equiv p - (qy + z)^2 = 0$... (1)

$\therefore$ Charpit's auxiliary equations are

$$\frac{dp}{2p(qy + z)} = \frac{dq}{2q(qy + z)} = \frac{dz}{(-p)(-1) - q \cdot 2(qy + z)y} = \frac{dx}{-(-1)} = \frac{dy}{-2y(qy + z)}$$

Taking the 1st and 5th fractions, we have

$$\frac{dp}{p} + \frac{dy}{y} = 0. \text{ Integrating, } \log p + \log y = \log a \text{ or } p = a/y$$

$\therefore$ From (1), $a/y = (qy + z)^2$

or $\sqrt{\left(\frac{a}{y}\right)} = qy + z \quad \therefore \quad q = \frac{\sqrt{a}}{y^{3/2}} - \frac{z}{y}$

Substituting these values in $dz = p dx + q dy$, we have

$$dz = \frac{a}{y} dx + \left(\frac{\sqrt{a}}{y^{3/2}} - \frac{z}{y} \right) dy \text{ or } y dz + z dy = a dx + \frac{\sqrt{a}}{\sqrt{y}} dy$$

Integrating, $yz = ax + 2\sqrt{ay} + b$... (2)

which is the complete integral.

General Integral : Taking $b = \phi(a)$ in (2), we have

$$yz = ax + 2\sqrt{ay} + \phi(a) \quad \dots (3)$$

Differentiating (3) partially w.r.t. a , we get

$$0 = x + \sqrt{(y/a)} + \phi'(a) \quad \dots (4)$$

The general integral is obtained by eliminating a between (3) and (4),

Singular Integral : Differentiating (2) partially w.r.t. a and b , we get

$$0 = x + \sqrt{(y/a)} \quad \dots (5) \quad \text{and} \quad 0 = 1 \quad \dots (6)$$

Since (6) is inadmissible, hence there is no singular integral.

Example 3: Find the complete integral of

$$pxy + pq + qy = yz. \quad [\text{Meerut 2001, 02, 12; Garhwal 2001; Kanpur 2003, 05}]$$

Solution: Here $f \equiv pxy + pq + qy - yz = 0$... (1)

$\therefore$ Charpit's auxiliary equations are

$$\frac{dp}{py + p(-y)} = \frac{dq}{(px + q) + q(-y)} = \frac{dz}{-p(xy + q) - q(p + y)} = \frac{dx}{-(xy + q)} = \frac{dy}{-(p + y)}$$

Taking 1st and 2nd fractions, $dp = 0 \Rightarrow p = a$

Putting in (1), we get $axy + aq + qy = yz$

$$\therefore q = \frac{y(z - ax)}{a + y}$$

Substituting the values of p and q in $dz = p dx + q dy$, we get

$$dz = a dx + \frac{y(z - ax)}{a + y} dy \quad \text{or} \quad \frac{dz - a dx}{z - ax} = \frac{y}{a + y} dy$$

$$\text{or} \quad \frac{dz - a dx}{z - ax} = \left(1 - \frac{a}{a + y}\right) dy$$

Integrating, $\log(z - ax) = y - a \log(a + y) + \log b$

$$\text{or} \quad z - ax = b e^y (a + y)^{-a}$$

which is the complete integral.

Example 4: Find a complete integral of $2zx - px^2 - 2qxy + pq = 0$.

[Rajasthan 2000, 03, 05; Delhi Maths 2001, 05; Kanpur 2001, 03;
M.D.U. Rohtak 2004; Bhopal 2004; Meerut 2005, 07 (B.P.); I.A.S. 2007, 08]

Solution: Here $f \equiv 2zx - px^2 - 2qxy + pq = 0$... (1)

$\therefore$ Charpit's auxiliary equations are

$$\frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dz}{px^2 - pq + 2xyq - pq} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dF}{0}$$

$$\therefore dq = 0 \quad \text{or} \quad q = a$$

Putting in (1), we have $2zx - px^2 - 2axy + pa = 0$

$$\text{or} \quad p(x^2 - a) = 2x(z - ay)$$

$$\therefore p = \frac{2x(z - ay)}{x^2 - a}$$

Substituting these values in $dz = p dx + q dy$, we have

$$dz = \frac{2x(z - ay)}{x^2 - a} dx + a dy \quad \text{or} \quad \frac{dz - a dy}{z - ay} = \frac{2x}{x^2 - a} dx$$

Integrating $\log(z - ay) = \log(x^2 - a) + \log b$

$$\therefore z - ay = b(x^2 - a) \quad \text{or} \quad z = ay + b(x^2 - a)$$

which is the complete integral.

Example 5: Find a complete integral of $(p^2 + q^2)y = qz$.

[Agra 2001; Rohilkhand 2001; Delhi Maths (Hons.) 2003, 05;
Garhwal 2005, Kurukshetra 2004; Guwahati 2007; Meerut 2005, 10]

Solution: Here $f \equiv (p^2 + q^2)y - qz = 0$...(1)

$\therefore$ The charpit's auxiliary equations are

$$\frac{dp}{-pq} = \frac{dq}{(p^2 + q^2) - q^2} = \frac{dz}{-2p^2y - 2q^2y + qz} = \frac{dx}{-2py} = \frac{dy}{-2qy + z}$$

Taking the first two fractions, we have

$$\frac{dp}{-pq} = \frac{dq}{p^2} \quad \text{or} \quad p dp + q dq = 0$$

Integrating $p^2 + q^2 = a^2$ (say) ...(2)

Putting in (1), $q = \frac{a^2 y}{z}$

$$\therefore \text{From (2), } p = \sqrt{(a^2 - q^2)} = \sqrt{\left\{a^2 - \frac{a^4 y^2}{z^2}\right\}} = \frac{a}{z} \sqrt{(z^2 - a^2 y^2)}$$

$\therefore$ From $dz = p dx + q dy$, we have

$$dz = \frac{a}{z} \sqrt{(z^2 - a^2 y^2)} dx + \frac{a^2 y}{z} dy$$

or $\frac{z dz - a^2 y dy}{\sqrt{(z^2 - a^2 y^2)}} = a dx$ or $dt = a dx$ Putting $z^2 - a^2 y^2 = t^2$

Integrating, we have $t = \sqrt{(z^2 - a^2 y^2)} = ax + b$

$$\text{or} \quad z^2 = a^2 y^2 + (ax + b)^2. \quad \dots(3)$$

which is the complete integral.

Note : Differentiating (3) partially w.r.t. a and b and then eliminating a, b between (3) and these two relations obtained by differentiation, the singular integral of the problem is $y = 0$.

Example 6: Find the complete integral of $(p^2 + q^2)x = pz$.

[Ravishankar 2001; Delhi Maths (Hons.) 2001, 04, 05;
Agra 2003; I.A.S. 2004; Rajasthan 2005; Meerut 2007, 11 (B.P.)]

And deduce the surface solution which passes through the curve $x = 0, z^2 = 4y$.

[Meerut 2007]

Solution: Here $f \equiv (p^2 + q^2)x - pz = 0$ $\dots(1)$

$\therefore$ The Charpit's auxiliary equations are

$$\frac{dp}{(p^2 + q^2) + p(-p)} = \frac{dq}{q(-p)} = \frac{dz}{-p(2px - z) - q(2qx)} = \frac{dx}{-(2px - z)} = \frac{dy}{-2qx}$$

Taking the first two fractions, we get

$$\frac{dp}{q^2} = \frac{dq}{-pq} \quad \text{or} \quad p dp + q dq = 0$$

Integrating $p^2 + q^2 = a^2$ (say) $\dots(2)$

Putting in (1), $p = a^2 x / z \therefore$ from (2), $q = a \sqrt{(z^2 - a^2 x^2)} / z$

$\therefore$ Putting in $dz = p dx + q dy$, we get

$$dz = \frac{a^2 x}{z} dx + \frac{a \sqrt{(z^2 - a^2 x^2)}}{z} dy$$

or $\frac{z dz - a^2 x dx}{\sqrt{(z^2 - a^2 x^2)}} = a dy$

or $\frac{1}{2} t^{-1/2} dt = a dy$ Putting $z^2 - a^2 x^2 = t$

Integrating, $\sqrt{t} = ay + b$ or $t = z^2 - a^2 x^2 = (ay + b)^2$

$$\text{or} \quad z^2 = a^2 x^2 + (a y + b)^2 \quad \dots(3)$$

which is the complete integral.

2nd Part : Since the curve (3) passes through the curve $x = 0, z^2 = 4y$, so putting in (3), we get

$$4y = (a y + b)^2 \quad \text{or} \quad a^2 y^2 + (2ab - 4)y + b^2 = 0 \quad \dots(4)$$

For the equation (4), to have real roots, we must have

$$(2ab - 4)^2 - 4a^2 b^2 = 0 \Rightarrow ab = 1 \Rightarrow b = 1/a.$$

Putting in (3), one parameter family is

$$z^2 = a^2 x^2 + (ay + 1/a)^2$$

$$\text{or} \quad (x^2 + y^2) a^4 + (2y - z^2) a^2 + 1 = 0 \quad \dots(5)$$

Now differentiating (5) partially w.r.t. 'a' we get

$$(x^2 + y^2) \cdot 4a^3 + (2y - z^2) 2a = 0$$

$$\Rightarrow \quad a^2 = \frac{z^2 - 2y}{2(x^2 + y^2)}$$

Putting in (5), the required surface is

$$(x^2 + y^2) \frac{(z^2 - 2y)^2}{4(x^2 + y^2)^2} + (2y - z^2) \frac{(z^2 - 2y)}{2(x^2 + y^2)} + 1 = 0$$

$$\text{or} \quad (z^2 - 2y)^2 = 4(x^2 + y^2).$$

Example 7: Find a complete integral of the equation.

$$p^2 x + q^2 y = z \quad [\text{Punjab 2001; Kurukshetra 2001; Agra 2004; Gujarat 2005; I.A.S. 2004, 06}]$$

$$\text{Solution:} \quad \text{Here} \quad f \equiv p^2 x + q^2 y - z = 0 \quad \dots(1)$$

$\therefore$ The Charpit's auxiliary equations are

$$\frac{dp}{-p + p^2} = \frac{dq}{-q + q^2} = \frac{dz}{-2(p^2 x + q^2 y)} = \frac{dx}{-2px} = \frac{dy}{-2qy}$$

from which, we have

$$\frac{p^2 dx + 2px dp}{p^2 x} = \frac{q^2 dy + 2qy dq}{q^2 y}$$

$$\text{Integrating, } \log p^2 x = \log q^2 y + \log a \quad \therefore p^2 x = a q^2 y \quad \dots(2)$$

where a is a constant.

From (1) and (2), we have $a q^2 y + q^2 y = z$

$$\therefore q = \left\{ \frac{z}{(1+a)y} \right\}^{1/2} \quad \therefore \text{From (1), } p = \left\{ \frac{az}{(1+a)x} \right\}^{1/2}$$

Putting in $dz = p dx + q dy$, we have

$$dz = \left\{ \frac{az}{(1+a)x} \right\}^{1/2} dx + \left\{ \frac{z}{(1+a)y} \right\}^{1/2} dy$$

$$\text{or } \sqrt{(1+a)} \frac{dz}{\sqrt{z}} = \sqrt{a} \frac{dx}{\sqrt{x}} + \frac{dy}{\sqrt{y}}$$

$$\text{Integrating } \sqrt{\{(1+a)z\}} = \sqrt{(ax)} + \sqrt{y} + b$$

which is the complete integral.

Example 8: Find the complete integral of the equation

$$2(z + xp + yq) = yp^2.$$

[Delhi Maths 2007, 08]

$$\text{Solution: Here } f \equiv 2(z + xp + yq) - yp^2 = 0 \quad \dots(1)$$

$\therefore$ The Charpit's auxiliary equations are

$$\frac{dp}{2p+2p} = \frac{dq}{2q-p^2+2q} = \frac{dz}{-p(2x-2yp)-2qy} = \frac{dx}{-2x+2yp} = \frac{dy}{-2y}$$

Taking 1st and last fractions, we have

$$\frac{dp}{p} + 2 \frac{dy}{y} = 0. \text{ Integrating } \log p + 2 \log y = \log a \text{ i.e., } p y^2 = a \quad \dots(2)$$

From (2), $p = a/y^2$ putting in (1), we get

$$2z + 2x \cdot \frac{a}{y^2} + 2yp = y \cdot \frac{a^2}{y^4} \quad \therefore q = -\frac{z}{y} - \frac{ax}{y^3} + \frac{a^3}{2y^4}$$

$\therefore$ Putting in $dz = p dx + q dy$, we get

$$dz = \frac{a}{y^2} dx + \left(-\frac{z}{y} - \frac{ax}{y^3} + \frac{a^2}{2y^4} \right) dy$$

or $y dz + z dy = a \left(\frac{y dx - x dy}{y^2} \right) + \frac{a^2}{2y^3} dy$

Integrating, $yz = \frac{ax}{y} - \frac{a^2}{4y^2} + b \quad \therefore z = \frac{ax}{y^2} - \frac{a^2}{4y^3} + \frac{b}{y}$

which is the complete integral.

Example 9: Find the complete integral of the equation

$$px^5 - 4q^3x^2 + 6x^2z - 2 = 0. \quad [\text{I.A.S. 2002, 04, 08; Meerut 2005 (B.P.), 12}]$$

Solution: Here $f \equiv px^5 - 4q^3x^2 + 6x^2z - 2 = 0 \quad \dots(1)$

$\therefore$ The Charpit's auxiliary equations are

$$\frac{dp}{(5px^4 - 8q^3x + 12xz) + p \cdot 6x^2} = \frac{dq}{q \cdot 6x^2} = \frac{dz}{-p \cdot x^5 + q \cdot 12q^2x^2} = \frac{dx}{-x^5} = \frac{dy}{12q^2x^2}$$

From 2nd and 5th fractions, we get

$$2q dy = dy \quad \Rightarrow \quad q^2 = y + a \quad \dots(2)$$

From (2), $q = \sqrt{(y + a)}$, putting in (1),

$$p = [4x^2 (y + a)^{3/2} - 6x^2z + 2] / x^5$$

$\therefore$ From $dz = p dx + q dy$, we get

$$dz = [4x^2 (y + a)^{3/2} - 6x^2z + 2] / x^5 dx + (y + a)^{1/2} dy$$

or $dz + (6/x^3) z dx = (4/x^3) (y + a)^{3/2} dx + (y + a)^{1/2} dy + (2/x^5) dx \quad \dots(3)$

Multiplying by e^{-3/x^2} , we get

$$e^{-3/x^2} dz + \frac{6ze^{-3/x^2}}{x^3} dx = \frac{4}{x^3} e^{-3/x^2} (y + a)^{3/2} dx + e^{-3/x^2} (y + a)^{1/2} dy + \frac{2}{x^5} e^{-3/x^2} dx - \frac{1}{9} t e^t dt$$

or $d(z e^{-3/x^2}) = d \left\{ \frac{2}{3} e^{-3/x^2} (y + a)^{3/2} \right\} - \frac{1}{9} t e^t dt$

Putting $-3/x^2 = t$, so that $(6/x^3) dx = dt$

$$\begin{aligned} \text{Integrating, } z e^{-3/x^2} &= \frac{2}{3} e^{-3/x^2} (y+a)^{3/2} - \frac{1}{9} (t-1) e^t + b \\ &= \frac{2}{3} e^{-3/x^2} (y+a)^{3/2} - \frac{1}{9} \left(-\frac{3}{x^2} - 1 \right) e^{-3/x^2} + b \end{aligned}$$

or
$$z = \frac{2}{3} (y+a)^{3/2} + \frac{1}{3} x^{-2} + \frac{1}{9} + b e^{3/x^2}$$

which is the complete integral.

Note : 1. L.H.S. of equation (3) is $dz + (6z/x^3) dx = Mdz + Ndx$ (say)

Here $\frac{1}{M} \left(\frac{\partial N}{\partial z} - \frac{\partial M}{\partial x} \right) = \frac{6}{x^3} = f(x) \therefore \text{I.F.} = e^{\int (6/x^3) dx} = e^{-3/x^2}$

Thus, to make L.H.S. the differential of a function both sides are multiplied by e^{-3/x^2} .

2. If we take 2nd and 4th fractions in Charpit's equations then we get

$$q = a e^{3/x^2} \therefore \text{from (1) } p = (4x^2 a^3 e^{9/x^2} - 6x^2 z + 2)/5$$

When substituted in $dz = p dx + q dy$, then the complete integral of the given equation is

$$z = -\frac{a^3}{3} e^{9/x^2} + \frac{1}{3x^2} + \frac{1}{9} + (ay + b) e^{3/x^2}$$

Example 10: Find the complete integral of the equation

$$x p + 3 y q = 2 (z - x^2 q^2).$$

[Meerut 2007 (B.P.)]

Solution: Here $f \equiv x p + 3 y q - 2z + 2x^2 q^2 = 0$... (1)

$\therefore$ The Charpit's equations are

$$\frac{dp}{p + 4x q^2 + p(-2)} = \frac{dq}{3q + q(-2)} = \frac{dz}{-p x - q(3y + 4x^2 q)} = \frac{dx}{-x} = \frac{dy}{-(3y + 4x^2 q)}$$

Taking 2nd and 4th fractions, we get

$$\frac{dq}{q} + \frac{dx}{x} = 0 \text{ Integrating } \log q + \log x = \log a \quad \therefore q = a/x.$$

Putting $q = a/x$ in (1), $p = (-3a y + 2z x - 2a^2 x)/x^2$

Putting in $dz = p dx + q dy$, we get

$$dz = \frac{-3a y + 2z x - 2a^2 x}{x^2} dx + \frac{a}{x} dy$$

or $x^2 dz - 2z x dx = -3a y dx + a x dy - 2a^2 x dx$

or $\frac{x^2 dz - 2z x dx}{x^4} = -\frac{3a y}{x^4} dx + \frac{a}{x^3} dy - \frac{2a^2}{x^3} dx$

or $d\left(\frac{z}{x^2}\right) = a d\left(y \cdot \frac{1}{x^3}\right) - \frac{2a^2}{x^3} dx$

Integrating $\frac{z}{x^2} = \frac{a y}{x^3} + \frac{a^2}{x^2} + b$

or $z = a\left(\frac{y}{x} + a\right) + b x^2$

which is the complete integral.

Example 11: Solve $p x + q y = p q$.

[Vikram 2000, 03, 04; Kanpur 2004; Jiwaji 2004;

Gulberg 2005; I.A.S. 2005; Kurukshetra 2006; Agra 2009; Rajasthan 2001, 10; Meerut 2013 (Sem. I)]

Solution: Here $f \equiv p x + q y - p q = 0$.

...(1)

$\therefore$ The Charpit's auxiliary equations are

$$\frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-p(x-q) - q(y-p)} = \frac{dx}{-(x-q)} = \frac{dy}{-(y-p)}$$

Taking the first two members, we have

$$\frac{dp}{p} = \frac{dq}{q}$$

Integrating, $\log p = \log q + \log a$ or $p = aq$

...(2)

Putting $p = aq$ in (1), we have

$$aqx + qy = aq^2 \quad \therefore \quad q = \frac{y + ax}{a}$$

from (2) $p = aq = y + ax$

Putting in $dz = p dx + q dy$, we have

$$dz = (y + ax) dx + \frac{(y + ax)}{a} dy$$

or $a dz = (y + ax)(dy + a dx)$

$$\text{Integrating} \quad az = \frac{1}{2}(y + ax)^2 + b \quad \dots(3)$$

which is the *complete integral*.

General Integral : Writing $b = \phi(a)$ in (3), we get

$$az = \frac{1}{2}(y + ax)^2 + \phi(a) \quad \dots(4)$$

Differentiating (4) partially w.r.t. a , we have

$$z = x(y + ax) + \phi'(a) \quad \dots(5)$$

General integral is obtained by eliminating a from (4) and (5).

Singular Integral : Differentiating the complete integral partially w.r.t. a and b , we have

$$z = x(y + ax) \quad \text{and} \quad 0 = 1$$

The last relation $0 = 1$ is inadmissible.

Hence there is no singular integral.

Note : (i) Taking the 2nd and 4th fractions, we get $q^2 - 2xq + a = 0$

$\therefore q = \frac{1}{2}[2x + 2\sqrt{(x^2 - a)}] = x + \sqrt{(x^2 - a)}$. Taking +ve sign only. Proceeding as usual another complete integral of the problem is

$$z = x y + y \sqrt{(x^2 - a)} + b.$$

(ii) Taking the 1st and 5th fractions, we get $p^2 - 2yp + a = 0$.

$\therefore p = y + \sqrt{(y^2 - a)}$. Proceeding as usual the third complete integral is

$$z = x y + x \sqrt{(y^2 - a)} + b.$$

Example 12: Find the complete integral of the equation $z^2 = pq \cdot xy$.

[Delhi Maths (Hons.) 2004; Jabalpur 2004; Meerut 2006]

Solution: Here $f \equiv z^2 - pq \cdot xy = 0$...(1)

$\therefore$ The Charpit's auxiliary equations are

$$\frac{dp}{-pq \cdot y + 2pz} = \frac{dq}{-pq \cdot x + 2qz} = \frac{dz}{2pq \cdot xy} = \frac{dx}{q \cdot xy} = \frac{dy}{p \cdot xy} \quad \dots(2)$$

Now each fraction in (2) $= \frac{x \cdot dp + p \cdot dx}{2p \cdot z \cdot x} = \frac{y \cdot dq + q \cdot dy}{2q \cdot z \cdot y}$

$$\Rightarrow \frac{x dp + p dx}{p x} = \frac{y dq + q dy}{q y}$$

Integrating $\log p x = \log (q y) + \log a^2$

or $p x = a^2 q y$ or $p = a^2 q y / x$... (3)

Putting in (1), $q = \frac{z}{(a y)}$ $\therefore$ from (3) $p = (a z) / x$

$\therefore$ from $dz = p dx + q dy$, we get

$$dz = (a z / x) dx + (z / a y) dy$$

or $\frac{dz}{z} = \frac{a}{x} dx + \frac{1}{a y} dy$

Integrating $\log z = a \log x + \frac{1}{a} \log y + \log b$ or $z = b x^a y^{1/a}$.

Example 13: Find the complete integral of $z^2 (p^2 z^2 + q^2) = 1$. [Meerut 2007, 08]

Solution: Here $f \equiv z^2 (p^2 z^2 + q^2) - 1 = 0$... (1)

$\therefore$ The Charpit's equations are

$$\frac{dp}{p(4p^2 z^3 + 2q^2 z)} = \frac{dq}{q(4p^2 z^3 + 2q^2 z)} = \frac{dz}{-p(2pz^4) - q(2qz^2)} = \frac{dx}{-2pz^4} = \frac{dy}{-2qz^2}$$

Taking the 1st and 2nd fractions, we have

$$\frac{dp}{p} = \frac{dq}{q} \Rightarrow \log p = \log q + \log a \Rightarrow p = aq$$

Putting in (1), $z^2 (a^2 z^2 q^2 + q^2) = 1$

$\therefore q^2 = \frac{1}{z^2 (a^2 z^2 + 1)}$ or $q = \frac{1}{z \sqrt{(a^2 z^2 + 1)}}$ $\therefore p = \frac{a}{z \sqrt{(a^2 z^2 + 1)}}$

Substituting in $dz = p dx + q dy$, we have

$$dz = \frac{a}{z \sqrt{(a^2 z^2 + 1)}} dx + \frac{1}{z \sqrt{(a^2 z^2 + 1)}} dy$$

or $z \sqrt{(a^2 z^2 + 1)} dz = a dx + dy$ or $\frac{t^2}{a^2} dt = a dx + dy$, Putting $a^2 z^2 + 1 = t^2$

Integrating, $\frac{1}{3a^2} t^3 = \frac{1}{3a^2} (a^2 z^2 + 1)^{3/2} = ax + y + b$

or $(a^2 z^2 + 1)^3 = 9a^4 (ax + y + b)^2$

which is the complete integral.

Note : The given equation is of the standard form II.

Example 14: Find a complete integral of the equation

$$p^2 + q^2 - 2px - 2qy + 1 = 0.$$

[Meerut 2003]

Solution: Here $f \equiv p^2 + q^2 - 2px - 2qy + 1 = 0$... (1)

Charpit's auxiliary equations are

$$\frac{dp}{-2p} = \frac{dq}{-2q} = \frac{dz}{-p(2p-2x) - q(2q-2y)} = \frac{dx}{-(2p-2x)} = \frac{dy}{-(2q-2y)}$$

Taking the first two members, we have

$$\frac{dp}{p} = \frac{dq}{q} \quad \therefore p = aq$$

Putting in (1), $a^2 q^2 + q^2 - 2aqx + 2qy + 1 = 0$

$$(a^2 + 1)q^2 - 2(ax + y)q + 1 = 0$$

$$\therefore q = \frac{2(ax + y) \pm \sqrt{4(ax + y)^2 - 4(a^2 + 1)}}{2(a^2 + 1)}$$

$$\text{or } q = \frac{ax + y + \sqrt{\{(ax + y)^2 - (a^2 + 1)\}}}{a^2 + 1} \quad (\text{Taking positive sign})$$

$$\text{and } p = aq = \frac{a[(ax + y) + \sqrt{\{(ax + y)^2 - (a^2 + 1)\}}]}{a^2 + 1}$$

Substituting in $dz = p dx + q dy$, we have

$$dz = \frac{(ax + y) + \sqrt{\{(ax + y)^2 - (a^2 + 1)\}}}{a^2 + 1} (a dx + dy)$$

Putting $ax + y = t$ so that $a dx + dy = dt$,

we have, $(a^2 + 1) dz = [t \sqrt{t^2 - (a^2 + 1)}] dt$

Integrating,

$$(a^2 + 1) z = \frac{t^2}{2} + \frac{t}{2} \sqrt{t^2 - (a^2 + 1)} - \frac{a^2 + 1}{2} \log [t + \sqrt{t^2 - (a^2 + 1)}] + b$$

where $t = ax + y$

which is the complete integral.

Example 15: Solve $px + qy = z\sqrt{1 + pq}$.

[Meerut 2001, 02, 11 (Sem. I)]

Solution: Here $f \equiv px + qy - z(1 + pq)^{1/2} = 0$... (1)

The Charpit's auxiliary equations are

$$\frac{dp}{p - p(1 + pq)^{1/2}} = \frac{dq}{q - q(1 + pq)^{1/2}} = \dots$$

Taking the first two members, we have

$$\frac{dp}{p} = \frac{dq}{q}$$

Integrating, $\log p = \log q + \log a$. $\therefore p = aq$

Putting in (1), we have $aqx + qy = z(1 + aq^2)^{1/2}$

$$\text{or } q^2(ax + y)^2 = z^2(1 + aq^2) \quad \text{or } q^2 = \frac{z^2}{(ax + y)^2 - az^2}$$

$$\text{or } q = \frac{z}{\sqrt{\{(ax + y)^2 - az^2\}}} \quad \therefore p = \frac{az}{\sqrt{\{(ax + y)^2 - az^2\}}}$$

Substituting, these values in $dz = p dx + q dy$, we have

$$dz = \frac{(az dx + z dy)}{\sqrt{\{(ax + y)^2 - az^2\}}} \quad \text{or } \frac{dz}{z} = \frac{a dx + dy}{\sqrt{\{(ax + y)^2 - az^2\}}}$$

Putting $ax + y = u\sqrt{a}$

$$\text{we get } \frac{dz}{z} = \frac{\sqrt{a} du}{\sqrt{au^2 - az^2}} \quad \text{or } \frac{du}{dz} = \frac{1}{z} \sqrt{u^2 - z^2}$$

Again putting $u = vz$, we get

$$\frac{du}{dz} = v + z \frac{dv}{dz} = \frac{1}{z} \sqrt{(v^2 z^2 - z^2)}$$

$$\text{or} \quad v + z \frac{dv}{dz} = \sqrt{(v^2 - 1)} \quad \text{or} \quad z \frac{dv}{dz} = \sqrt{(v^2 - 1)} - v$$

$$\text{or} \quad \frac{dz}{z} = \frac{dv}{\sqrt{(v^2 - 1)} - v} \quad \text{or} \quad \frac{dz}{z} = -\{\sqrt{(v^2 - 1)} + v\} dv$$

$$\text{Integrating,} \quad \log z = -\left[\frac{v}{2} \sqrt{(v^2 - 1)} - \frac{1}{2} \log \{v + \sqrt{(v^2 - 1)}\}\right] - \frac{v^2}{2} + b$$

$$\text{or} \quad \log z + \frac{v^2}{2} + \frac{v}{2} \sqrt{(v^2 - 1)} - \frac{1}{2} \log \{v + \sqrt{(v^2 - 1)}\} = b$$

which is the complete integral, where $v = \frac{u}{z} = \frac{ax + y}{z\sqrt{a}}$.

Example 16: Find the complete integral of the equation

$$p^2 + q^2 - 2px - 2qy + 2xy = 0. \quad [\text{Meerut 2003, 06 (B.P.); Kurukshetra 2006}]$$

$$\textbf{Solution:} \quad \text{Here} \quad f \equiv p^2 + q^2 - 2px - 2qy + 2xy = 0 \quad \dots(1)$$

$\therefore$ The Charpit's equations are

$$\frac{dp}{-2p + 2y} = \frac{dq}{-2q + 2x} = \frac{dz}{-p(2p - 2x) - q(2q - 2y)} = \frac{dx}{-(2p - 2x)} = \frac{dy}{-(2q - 2y)}$$

$$\Rightarrow \quad \frac{dp}{-p + y} = \frac{dq}{-q + x} = \frac{dx}{x - p} = \frac{dy}{y - q}$$

$$\Rightarrow \quad \frac{dp + dq}{x + y - p - q} = \frac{dx + dy}{x + y - p - q}$$

$$\Rightarrow \quad dp + dq = dx + dy \quad \Rightarrow \quad p + q = x + y + a \quad \dots(2)$$

Equation (1) can be written as

$$(p - x)^2 + (q - y)^2 = (x - y)^2 \quad \dots(3)$$

Putting the value of $(q - y)$ from (2) in (3), we have

$$(p - x)^2 + [a - (p - x)]^2 = (x - y)^2$$

$$\text{or} \quad 2(p - x)^2 - 2a(p - x) + \{a^2 - (x - y)^2\} = 0$$

$$\begin{aligned}\therefore p - x &= \frac{2a + \sqrt{[4a^2 - 4 \cdot 2 \{a^2 - (x - y)^2\}]}}{4} \text{ (Taking +ve sign only)} \\ &= \frac{1}{2} [a + \sqrt{2(x - y)^2 - a^2}] \text{ or } p = x + \frac{1}{2} [a + \sqrt{2(x - y)^2 - a^2}]\end{aligned}$$

$$\therefore \text{ From (2), } q - y = a - \frac{1}{2} [a + \sqrt{2(x - y)^2 - a^2}]$$

$$\text{or } q = y + \frac{1}{2} [a - \sqrt{2(x - y)^2 - a^2}]$$

Substituting these values in $dz = p dx + q dy$, we have

$$\begin{aligned}dz &= x dx + y dy + \frac{a}{2} (dx + dy) + \frac{1}{2} \sqrt{2(x - y)^2 - a^2} (dx - dy) \\ &= x dx + y dy + \frac{a}{2} (dx + dy) + \frac{1}{\sqrt{2}} \sqrt{[(x - y)^2 - (1/2) a^2]} (dx - dy)\end{aligned}$$

Integrating, we have

$$\begin{aligned}z &= \frac{x^2}{2} + \frac{y^2}{2} + \frac{a}{2} (x + y) \\ &\quad + \frac{1}{\sqrt{2}} \left[\frac{x - y}{2} \sqrt{\left\{ (x - y)^2 - \frac{a^2}{2} \right\}} - \frac{a^2}{4} \log \left\{ (x - y) + \sqrt{\left((x - y)^2 - \frac{a^2}{2} \right)} \right\} \right] + b'\end{aligned}$$

$$\text{or } 2z = x^2 + y^2 + ax + ay$$

$$+ \frac{1}{\sqrt{2}} \left[(x - y) \sqrt{\left\{ (x - y)^2 - \frac{a^2}{2} \right\}} - \frac{a^2}{4} \log \left\{ (x - y) + \sqrt{\left((x - y)^2 - \frac{a^2}{2} \right)} \right\} \right] + b$$

where $2b' = b$

which is the complete integral.

Example 17: Find the complete integral of the equation

$$(p^2 + q^2)^n (qx - py) = 1.$$

$$\text{Solution: Here } f \equiv (p^2 + q^2)^n (qx - py) - 1 = 0 \quad \dots(1)$$

$\therefore$ The Charpit's equations are

$$\frac{dp}{q(p^2 + q^2)^n} = \frac{dq}{-p(p^2 + q^2)^n} = \dots$$

$$\Rightarrow 2p dp + 2q dq = 0 \Rightarrow p^2 + q^2 = 1/a^2 \text{ (say)} \quad \dots(2)$$

From (1) and (2), we have $(qx - py)/a^{2n} = 1$

$$\Rightarrow q = (a^{2n} + py)/x$$

Putting in (2), $p^2 + (a^{2n} + py)^2/x^2 = 1/a^2$

or $(x^2 + y^2)p^2 + 2a^{2n}yp + a^{4n} - x^2/a^2 = 0$

$$\therefore p = \frac{-2a^{2n}y \pm \sqrt{4a^{4n}y^2 - 4(x^2 + y^2)(a^{4n} - x^2/a^2)}}{2(x^2 + y^2)}$$

or $p = \frac{-a^{2n}y + x\sqrt{\{-a^{4n} + (x^2 + y^2)/a^2\}}}{(x^2 + y^2)}$ (Taking +ve sign only)

$$\therefore q = \frac{a^{2n}x + y\sqrt{\{-a^{4n} + (x^2 + y^2)/a^2\}}}{(x^2 + y^2)}$$

Putting in $dz = p dx + q dy$, we get

$$\begin{aligned} dz &= a^{2n} \left(\frac{x dy - y dx}{x^2 + y^2} \right) + \sqrt{\left(\frac{x^2 + y^2}{a^2} - a^{4n} \right)} \left(\frac{x dx + y dy}{x^2 + y^2} \right) \\ &= a^{2n} d \left(\tan^{-1} \frac{y}{x} \right) + \frac{1}{a} \sqrt{(t^2 - a^{4n+2})} \cdot \frac{dt}{t} \end{aligned}$$

Putting $x^2 + y^2 = t^2$, $x dx + y dy = t dt$

Integrating, $z = a^{2n} \tan^{-1} \left(\frac{y}{x} \right) + \frac{1}{a} \int \frac{\sqrt{(t^2 - a^{4n+2})}}{t} dt + b$

or $z = a^{2n} \tan^{-1} \left(\frac{y}{x} \right) + \frac{1}{a} \int \left\{ \frac{t}{\sqrt{(t^2 - a^{4n+2})}} - \frac{a^{4n+2}}{t \sqrt{(t^2 - a^{4n+2})}} \right\} dt + b$

or $z = a^{2n} \tan^{-1} \left(\frac{y}{x} \right) + \frac{1}{a} \int \left[\sqrt{(t^2 - a^{4n+2})} - a^{2n+1} \sec^{-1} \left(\frac{t}{a^{2n+1}} \right) \right] \frac{dt}{t} + b,$

which is the complete integral, where $t = \sqrt{(x^2 + y^2)}$.

Example 18: Find the complete integral of the equation

$$(x^2 - y^2) pq - xy (p^2 - q^2) - 1 = 0$$

Solution: Here $f \equiv (x^2 - y^2) pq - xy (p^2 - q^2) - 1 = 0$... (1)

The Charpit's auxiliary equations are

$$\begin{aligned} \frac{dp}{2pqx - y(p^2 - q^2)} &= \frac{dq}{-2y pq - x(p^2 - q^2)} = \frac{dx}{-(x^2 - y^2)q + 2pxy} \\ &= \frac{dy}{-(x^2 - y^2)p - 2qxy} = \dots \end{aligned}$$

Using x, y, p, q as multipliers,

$$\text{Each fraction} = \frac{x dp + y dq + p dx + q dy}{0}$$

$$\therefore (x dp + p dx) + (q dy + y dq) = 0$$

$$\text{Integrating} \quad p x + q y = a, \quad \therefore p = \frac{a - q y}{x}$$

$\therefore$ From (1), we have

$$(x^2 - y^2) \left(\frac{a - q y}{x} \right) q - xy \left[\frac{(a - q y)^2}{x^2} - q^2 \right] - 1 = 0$$

$$\text{or} \quad \frac{a - q y}{x} \{ (x^2 - y^2) q - (a - q y) y \} + xy q^2 - 1 = 0$$

$$\text{or} \quad \frac{a - q y}{x} (x^2 q - a y) + xy q^2 - 1 = 0$$

$$\text{or} \quad (a - q y) (x^2 q - a y) + x^2 y q^2 - x = 0$$

$$\text{or} \quad a x^2 q - a^2 y - x^2 y q^2 + a y^2 q + x^2 y q^2 - x = 0$$

$$\text{or} \quad qa (x^2 + y^2) = a^2 y + x \quad \text{or} \quad q = \frac{a^2 y + x}{a (x^2 + y^2)}$$

$$\therefore p = \frac{1}{x} \left[a - \frac{(a^2 y + x) y}{a (x^2 + y^2)} \right] = \frac{a^2 x - y}{a (x^2 + y^2)}$$

Substituting these values in $dz = p dx + q dy$, we have

$$dz = \frac{(a^2 x - y) dx + (a^2 y + x) dy}{a(x^2 + y^2)}$$

or
$$dz = a \frac{(x dx + y dy)}{(x^2 + y^2)} + \frac{x dy - y dx}{a(x^2 + y^2)}$$

Integrating, we have

$$z = \frac{a}{2} \log(x^2 + y^2) + \frac{1}{a} \tan^{-1} \frac{y}{x} + b$$

which is the complete integral.

Example 19: Find the complete integral of the equation

$$2(pq + yp + qx) + x^2 + y^2 = 0.$$

[Meerut 2009]

Solution: Here $f \equiv 2(pq + yp + qx) + x^2 + y^2 = 0$... (1)

∴ The Charpit's auxiliary equations are

$$\frac{dp}{2q + 2x} = \frac{dq}{2p + 2y} = \frac{dz}{-p(2q + 2y) - q(2p + 2x)} = \frac{dx}{-(2q + 2y)} = \frac{dy}{-(2p + 2x)} \quad \dots (2)$$

∴ each fraction in (2) = $\frac{dp + dq + dx + dy}{0}$

$$\Rightarrow dp + dq + dx + dy = 0 \Rightarrow p + q + x + y = a$$

$$\Rightarrow (p + x) + (q + y) = a \quad \dots (3)$$

From (1), we have $2(p + x)(q + y) + (x - y)^2 = 0$

$$\text{or } 2(p + x)\{a - (p + x)\} + (x - y)^2 = 0$$

Substituting $q + y = a - (p + x)$ (from 3)

$$\text{or } 2(p + x)^2 - 2a(p + x) - (x - y)^2 = 0$$

$$\therefore p + x = \frac{2a \pm \sqrt{4a^2 + 8(x - y)^2}}{4} = \frac{1}{2} [a + \sqrt{a^2 + 2(x - y)^2}]$$

[Taking + sign only]

$$\text{or } p = -x + \frac{1}{2} [a + \sqrt{a^2 + 2(x - y)^2}]$$

$$\therefore q = -y + a - \frac{1}{2} [a + \sqrt{a^2 + 2(x - y)^2}] = -y + \frac{1}{2} [a - \sqrt{a^2 + 2(x - y)^2}]$$

∴ Putting in $dz = p dx + q dy$, we get

$$dz = -(x dx + y dy) + \frac{1}{2} a (dx + dy) + \frac{1}{2} \sqrt{a^2 + 2(x-y)^2} \cdot (dx - dy)$$

or
$$dz = -(x dx + y dy) + \frac{1}{2} a (dx + dy) + \frac{\sqrt{2}}{2} \sqrt{\left(\frac{a^2}{2} + t^2\right)} dt$$

Putting $x - y = t$, $dx - dy = dt$

Integrating
$$z = -\frac{1}{2} (x^2 + y^2) + \frac{1}{2} a (x + y) + \frac{\sqrt{2}}{2} \left[\frac{t}{2} \sqrt{\left(\frac{a^2}{2} + t^2\right)} + \frac{a^2/2}{2} \log \left\{ t + \sqrt{\left(\frac{a^2}{2} + t^2\right)} \right\} \right] + b$$

or
$$z = -\frac{1}{2} (x^2 + y^2) + \frac{1}{2} a (x + y) + \frac{\sqrt{2}}{4} \left[(x - y) \sqrt{\frac{a^2}{2} + (x - y)^2} + \frac{1}{2} a^2 \log \left\{ (x - y) \sqrt{\frac{a^2}{2} + (x - y)^2} \right\} \right] + b$$

which is the complete integral.

Example 20: Find the complete integral of $xp - yq = xqf(z - px - qy)$.

Solution: Here, $F \equiv xp - yq - xqf(z - px - qy) = 0$... (1)

∴ The Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial F}{\partial x} + p \frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q \frac{\partial F}{\partial z}} = \frac{dz}{-p \frac{\partial F}{\partial p} - q \frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

or
$$\frac{dp}{p - qf + xqp f' - pxq f'} = \frac{dq}{-q + xq^2 f' - xq^2 f'} = \dots$$

or
$$\frac{dp}{p - qf} = \frac{dq}{-q} = \dots \quad \dots (2)$$

∴ each fraction
$$= \frac{x dp + y dq}{px - yq - xqf} = \frac{x dp + y dq}{0}, \text{ by (1)}$$

∴
$$x dp + y dq = 0$$

$$(x dp + p dx) + (y dq + q dy) = (p dx + q dy)$$

$$\text{or } d(p x) + d(q y) = dz$$

$$\text{or } d(z - p x - q y) = 0 \Rightarrow z - p x - q y = a \quad \dots(3)$$

$$\therefore \text{ from (1), we get } xp - yq - xq f(a) = 0 \quad \dots(4)$$

$$\text{Adding (3) and (4), we get } q = \frac{z - a}{2y + x f(a)}$$

$$\text{Putting in (3), } p = \frac{(z - a) \{y + x f(a)\}}{x \{2y + x f(a)\}}$$

$$\text{Substituting in } dz = p dx + q dy, \text{ we get}$$

$$dz = \frac{(z - a) \{y dx + x f(a) dx + x dy\}}{x \{2y + x f(a)\}}$$

$$\text{or } 2 \frac{dz}{z - a} = \frac{2(x dy + y dx) + 2x f(a) dx}{2x y + x^2 f(a)}$$

$$\text{Integrating, } 2 \log(z - a) = \log \{2x y + x^2 f(a)\} + \log b$$

$$\text{or } (z - a)^2 = b \{2x y + x^2 f(a)\}$$

which is the complete integral.

Example 21: Find the complete integral by Charpit's method of

$$p^2 + q^2 - 2pq \tanh 2y = \operatorname{sech}^2 2y. \quad [\text{I.A.S. 2004}]$$

$$\textbf{Solution:} \text{ Let } f \equiv p^2 + q^2 - 2pq \tanh 2y - \operatorname{sech}^2 2y = 0 \quad \dots(1)$$

The Charpit's auxiliary equations are

$$\begin{aligned} \frac{dp}{0} &= \frac{dq}{-4pq \operatorname{sech}^2 2y + 4 \operatorname{sech}^2 2y \tanh 2y} = \frac{dx}{-(2p - 2q \tanh 2y)} \\ &= \frac{dy}{-(2q - 2p \tanh 2y)} \end{aligned}$$

$$\Rightarrow dp = 0 \Rightarrow p = a \text{ (constant)}$$

Putting in (1), we get

$$q^2 - (2a \tanh 2y) q + (a^2 - \operatorname{sech}^2 2y) = 0$$

$$\begin{aligned}
 \Rightarrow \quad q &= \frac{2a \tanh 2y \pm \sqrt{[4a^2 \tanh^2 2y - 4(a^2 - \operatorname{sech}^2 2y)]}}{2} \\
 &= a \tanh 2y \pm \sqrt{[\operatorname{sech}^2 2y - a^2 (1 - \tanh^2 2y)]} \\
 &= a \tanh 2y \pm \sqrt{(1 - a^2)} \operatorname{sech} 2y
 \end{aligned}$$

Substituting in $dz = p dx + q dy$, we have

$$dz = a dx + [a \tanh 2y \pm \sqrt{(1 - a^2)} \operatorname{sech} 2y] dy$$

Integrating,
$$z = ax + \frac{1}{2} a \log \cosh 2y \pm \sqrt{(1 - a^2)} \int \frac{2dy}{e^{2y} + e^{-2y}} + b$$

or
$$z = ax + b + \frac{1}{2} a \log \cosh 2y \pm \sqrt{(1 - a^2)} \int \frac{2e^{2y} dy}{(e^{2y})^2 + 1}$$

or
$$z = ax + b + \frac{1}{2} a \log \cosh 2y \pm \sqrt{(1 - a^2)} \tan^{-1} (e^{2y})$$

which is the required complete integral.

Exercise 8.5

Applying Charpit's method to find the complete integrals of the following equations

- $z = pq$ [Sagar 2004; Rewa 2004; Ravishankar 2004; Agra 2008]
- $q = 3y^2$
- $q = 3p^2$ [Agra 2006]
- $q = (z + px)^2$ [Indore 2004; Ravishankar 2005]
- $f(p, q) = 0$
- $yz p^2 - q = 0$
- $p^2 - y^2 q = y^2 - x^2$ [M.D.U. Rohtak 2006]
- $p(1 + q^2) - q(z - c) = 0$
- $px + qy + pq = 0$
- $2z + p^2 + qy + 2y^2 = 0$ [Bilaspur 2004; M.D.U. Rohtak 2005]
- $zpq = p + q$ [Meerut 2006; Delhi Maths 2007]
- $(p + q)(px + qy) = 1$ [Delhi Maths (Hons.) 2007; Purvanchal 2007; Gurunanak 2010]

13. $16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$

14. $2x(z^2q^2 + 1) = pz$

15. $q = px + p^2$

[Sagar 2003; Vikram 2006; Agra 2010; Kanpur 2010]

16. $q = -px + p^2$

[Indore 2010; Vikram 2010]

17. $2(y + qz) = q(xp + yq)$

18. $z = \frac{1}{2}(p^2 + q^2) + (p - x)(q - y)$

[I.A.S. 2002]

19. Write short note on the Charpit's method.

[Meerut 2005, 05 (B.P.), 11 (B.P.)]

Answers 8.5

1.	$2\sqrt{z} = \sqrt{ax} + \frac{y}{\sqrt{a}} + b$	2.	$z = ax + y^3 + b$
3.	$z = ax + 3a^2y + b$	4.	$xz = 2\sqrt{ax} + ay + b$
5.	$z = ax + f(a)y + b$	6.	$z^2(a^2 - y^2) = (x + b)^2$
7.	$z = \frac{1}{2}[x\sqrt{(a^2 - x^2)} + a^2 \sin^{-1}(x/a)] - a^2/y - y + b$		
8.	$2\sqrt{a(z - c - a)} = ax + y + b, a, b \text{ are arbitrary constants.}$		
9.	$az = -(1/2)(y + ax)^2 + b$	10.	$y^2[(x - a)^2 + y^2 + 2z] = b$
11.	$z^2 = 2(a + 1)(x + y/a) + b$	12.	$z\sqrt{(1 + a)} = 2\sqrt{(x + ay)} + b$
13.	$\sqrt{\{(16a^2 + 9)(1 - z^2)\}} + 2(ax + y) = b$	14.	$z^2 = 2(a^2 + 1)x^2 + 2ay + b$
15.	$z = -\frac{1}{4}x^2 \pm \frac{1}{2}\left[\frac{1}{2}x\sqrt{x^2 + 4a} + 2a \log\{x + \sqrt{x^2 + 4a}\}\right] + ay + b$		
16.	$z = \frac{1}{4}x^2 \pm \frac{1}{2}\left[\frac{1}{2}x\sqrt{x^2 + 4a} + 2a \log\{x + \sqrt{x^2 + 4a}\}\right] + ay + b$		
17.	$(2z - ax^2) \pm \sqrt{\{(2z - ax^2)^2 + 8y^2\}} = by^2$		
18.	$z = \frac{1}{2}[(x + a)^2 + (y + b)^2] + ab$		

8.8 Jacobi's Methods

[Meerut 2005, 06, 07 (B.P.), 11 (Sem. I)]

This method is used for the solution of non-linear partial differential equation of order one with three or more independent variables.

Let us consider the partial differential equation with three independent variables x_1, x_2, x_3 , as

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = 0 \quad \dots(1)$$

z is the dependent variable not occurring in the equation except in its partial derivatives p_1, p_2, p_3 w.r.t. x_1, x_2, x_3 respectively.

The idea of Jacobi's method is to find two additional equations

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1 \quad \dots(2)$$

$$\text{and} \quad F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2 \quad \dots(3)$$

where a_1, a_2 are arbitrary constants such that (1), (2), (3) may be solved for p_1, p_2, p_3 as functions of x_1, x_2, x_3 which when substituted in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 \quad \dots(4)$$

it becomes integrable, for which conditions are

$$\frac{\partial p_2}{\partial x_1} = \frac{\partial^2 z}{\partial x_1 \partial x_2} = \frac{\partial p_1}{\partial x_2}, \quad \frac{\partial p_3}{\partial x_2} = \frac{\partial p_2}{\partial x_3}, \quad \frac{\partial p_1}{\partial x_3} = \frac{\partial p_3}{\partial x_1}$$

Now differentiating (1) and (2), partially w.r.t. x_1 , we have

$$\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial p_1} \cdot \frac{\partial p_1}{\partial x_1} + \frac{\partial f}{\partial p_2} \cdot \frac{\partial p_2}{\partial x_1} + \frac{\partial f}{\partial p_3} \cdot \frac{\partial p_3}{\partial x_1} = 0 \quad \dots(5)$$

$$\frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial p_1} \cdot \frac{\partial p_1}{\partial x_1} + \frac{\partial F_1}{\partial p_2} \cdot \frac{\partial p_2}{\partial x_1} + \frac{\partial F_1}{\partial p_3} \cdot \frac{\partial p_3}{\partial x_1} = 0 \quad \dots(6)$$

Eliminating $\frac{\partial p_1}{\partial x_1}$ from (5) and (6), we have

$$\left(\frac{\partial f}{\partial x_1} \frac{\partial F_1}{\partial p_1} - \frac{\partial f}{\partial p_1} \frac{\partial F_1}{\partial x_1} \right) + \left(\frac{\partial f}{\partial p_2} \frac{\partial F_1}{\partial p_1} - \frac{\partial f}{\partial p_1} \frac{\partial F_1}{\partial p_2} \right) \frac{\partial p_2}{\partial x_1} + \left(\frac{\partial f}{\partial p_3} \frac{\partial F_1}{\partial p_1} - \frac{\partial f}{\partial p_1} \frac{\partial F_1}{\partial p_3} \right) \frac{\partial p_3}{\partial x_1} = 0 \quad \dots(7)$$

Similarly, we have

$$\left(\frac{\partial f}{\partial x_2} \frac{\partial F_1}{\partial p_2} - \frac{\partial f}{\partial p_2} \frac{\partial F_1}{\partial x_2} \right) + \left(\frac{\partial f}{\partial p_1} \frac{\partial F_1}{\partial p_2} - \frac{\partial f}{\partial p_2} \frac{\partial F_1}{\partial p_1} \right) \frac{\partial p_2}{\partial x_2} + \left(\frac{\partial f}{\partial p_3} \frac{\partial F_1}{\partial p_2} - \frac{\partial f}{\partial p_2} \frac{\partial F_1}{\partial p_3} \right) \frac{\partial p_3}{\partial x_2} = 0 \quad \dots(8)$$

$$\text{and } \left(\frac{\partial f}{\partial x_3} \frac{\partial F_1}{\partial p_3} - \frac{\partial f}{\partial p_3} \frac{\partial F_1}{\partial x_3} \right) + \left(\frac{\partial f}{\partial p_1} \frac{\partial F_1}{\partial p_3} - \frac{\partial f}{\partial p_3} \frac{\partial F_1}{\partial p_1} \right) \frac{\partial p_1}{\partial x_3} + \left(\frac{\partial f}{\partial p_2} \frac{\partial F_1}{\partial p_3} - \frac{\partial f}{\partial p_3} \frac{\partial F_1}{\partial p_2} \right) \frac{\partial p_2}{\partial x_3} = 0 \quad \dots(9)$$

Adding (7), (8) and using $\frac{\partial p_2}{\partial x_1} = \frac{\partial p_1}{\partial x_2}$, $\frac{\partial p_3}{\partial x_2} = \frac{\partial p_2}{\partial x_3}$ and $\frac{\partial p_1}{\partial x_3} = \frac{\partial p_3}{\partial x_1}$, we have

$$\left(\frac{\partial f}{\partial x_1} \frac{\partial F_1}{\partial p_1} - \frac{\partial f}{\partial p_1} \frac{\partial F_1}{\partial x_1} \right) + \left(\frac{\partial f}{\partial x_2} \frac{\partial F_1}{\partial p_2} - \frac{\partial f}{\partial p_2} \frac{\partial F_1}{\partial x_2} \right) + \left(\frac{\partial f}{\partial x_3} \frac{\partial F_1}{\partial p_3} - \frac{\partial f}{\partial p_3} \frac{\partial F_1}{\partial x_3} \right) = 0$$

i.e., $(f, F_1) = \sum_{r=1}^3 \left(\frac{\partial f}{\partial x_r} \frac{\partial F_1}{\partial p_r} - \frac{\partial f}{\partial p_r} \frac{\partial F_1}{\partial x_r} \right) = 0 \quad \dots(10)$

Similarly, we have

$$(f, F_2) = \sum_{r=1}^3 \left(\frac{\partial f}{\partial x_r} \frac{\partial F_2}{\partial p_r} - \frac{\partial f}{\partial p_r} \frac{\partial F_2}{\partial x_r} \right) = 0 \quad \dots(11)$$

$$(F_1, F_2) = \sum_{r=1}^3 \left(\frac{\partial F_1}{\partial x_r} \frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r} \frac{\partial F_2}{\partial x_r} \right) = 0 \quad \dots(12)$$

Since (10) is a linear equation of the first order with $x_1, x_2, x_3, p_1, p_2, p_3$ as independent variables and F_1 as dependent variable.

$\therefore$ The auxiliary equations are

$$\frac{dp_1}{\frac{\partial f}{\partial x_1}} = \frac{dx_1}{-\frac{\partial f}{\partial p_1}} = \frac{dp_2}{\frac{\partial f}{\partial x_2}} = \frac{dx_2}{-\frac{\partial f}{\partial p_2}} = \frac{dp_3}{\frac{\partial f}{\partial x_3}} = \frac{dx_3}{-\frac{\partial f}{\partial p_3}} \quad \dots(13)$$

which are known as **Jacobi's auxiliary equations** and any integral of (13) will satisfy (10). [Meerut 2011 (Sem. I)]

Now we try to find two independent integrals.

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1 \text{ and } F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2$$

with the help of (13). If these relations satisfy (12) then these relations are the additional relations (2) and (3).

Solving (1), (2) and (3) we find the values of p_1, p_2, p_3 in terms of x_1, x_2, x_3 . Substituting these values in (4), it becomes integrable. Its integral will give the *complete integral* of the equation (1).

Working Method : To find the complete integral by Jacobi's method, proceed as follows:

For partial differential equation with three independent variables x_1, x_2, x_3 .

Step I : Write the given differential equation in the form

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = 0 \quad \dots(1)$$

Step II : Write the Jacobi's auxiliary equations

$$\frac{\frac{dp_1}{\partial f}}{\frac{dx_1}{\partial p_1}} = \frac{\frac{dp_2}{\partial f}}{\frac{dx_2}{\partial p_2}} = \frac{\frac{dp_3}{\partial f}}{\frac{dx_3}{\partial p_3}} = \frac{dp_3}{\partial f} = \frac{dx_3}{-\partial f}$$

Solving these equations obtain two additional equations

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1 \quad \dots(2)$$

$$\text{and} \quad F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2 \quad \dots(3)$$

where a_1, a_2 are arbitrary constants.

Try to find the simplest relations (2) and (3) so that the further solution may be as simple as possible.

Step III : Verify that the relations (2) and (3) satisfy the condition.

$$(F_1, F_2) = \sum_{r=1}^3 \left(\frac{\partial F_1}{\partial x_r} \frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r} \frac{\partial F_2}{\partial x_r} \right) = 0 \quad \dots(4)$$

Step IV : If (4) is satisfied, then solve (1), (2) and (3) to get the values of p_1, p_2, p_3 in terms of x_1, x_2, x_3 .

Step V : Substitute the values of p_1, p_2, p_3 in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 \quad \dots(5)$$

Step VI : Integrate (5) to get the complete integral of the equation (1).

For partial differential equation with four independent variables x_1, x_2, x_3, x_4 .

Step I : Write the given differential equation in the form

$$f(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = 0 \quad \dots(1)$$

Step II : Write the Jacobi's auxiliary equations, which are as follows :

$$\frac{\frac{dp_1}{\partial f}}{\frac{dx_1}{\partial p_1}} = \frac{\frac{dp_2}{\partial f}}{\frac{dx_2}{\partial p_2}} = \frac{\frac{dp_3}{\partial f}}{\frac{dx_3}{\partial p_3}} = \frac{\frac{dp_4}{\partial f}}{\frac{dx_4}{\partial p_4}} = \frac{dp_4}{\partial f} = \frac{dx_4}{-\partial f} \quad \dots(2)$$

Solving these equations obtain three additional equations (simplest possible)

$$F_1(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_1 \quad \dots(3)$$

$$F_2(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_2 \quad \dots(4)$$

$$F_3(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_3 \quad \dots(5)$$

where a_1, a_2, a_3 are arbitrary constants.

Step III : Verify that the relations (3), (4) and (5) satisfy conditions.

$$(F_1, F_2) = \sum_{i=1}^4 \left(\frac{\partial F_1}{\partial x_i} \frac{\partial F_2}{\partial p_i} - \frac{\partial F_1}{\partial p_i} \frac{\partial F_2}{\partial x_i} \right) = 0 \quad \dots(6)$$

$$(F_2, F_3) = \sum_{i=1}^4 \left(\frac{\partial F_2}{\partial x_i} \frac{\partial F_3}{\partial p_i} - \frac{\partial F_2}{\partial p_i} \frac{\partial F_3}{\partial x_i} \right) = 0 \quad \dots(7)$$

and $(F_3, F_1) = \sum_{i=1}^4 \left(\frac{\partial F_3}{\partial x_i} \frac{\partial F_1}{\partial p_i} - \frac{\partial F_3}{\partial p_i} \frac{\partial F_1}{\partial x_i} \right) = 0 \quad \dots(8)$

Step IV : If (6), (7) and (8) are satisfied, then solve (1), (3), (4) and (5) to get the values of p_1, p_2, p_3 and p_4 in terms of x_1, x_2, x_3 and x_4 .

Step V : Substitute the values of p_1, p_2, p_3 and p_4 in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 + p_4 dx_4 \quad \dots(9)$$

Step VI : Integrate (9) to get the complete integral of the equation (1).

Solved Examples

Example 1: Apply Jacobi's method to find the complete integral of

$$2p_1x_1x_3 + 3p_2x_3^2 + p_2^2p_3 = 0 \quad [\text{Meerut 2013 (Sem. I)}]$$

Solution: Here $f \equiv 2p_1x_1x_3 + 3p_2x_3^2 + p_2^2p_3 = 0 \quad \dots(1)$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{\frac{dp_1}{\partial f}}{\frac{\partial x_1}{\partial p_1}} = \frac{\frac{dx_1}{-\partial f}}{\frac{\partial p_1}{\partial x_2}} = \frac{\frac{dp_2}{\partial f}}{\frac{\partial x_2}{\partial p_2}} = \frac{\frac{dx_2}{-\partial f}}{\frac{\partial p_2}{\partial x_3}} = \frac{\frac{dp_3}{\partial f}}{\frac{\partial x_3}{\partial p_3}} = \frac{\frac{dx_3}{-\partial f}}{\frac{\partial p_3}{\partial p_3}}$$

or $\frac{dp_1}{2p_1x_3} = \frac{dx_1}{-2x_1x_3} = \frac{dp_2}{0} = \frac{dx_2}{-3x_3^2 - 2p_2p_3} = \frac{dp_3}{2p_1x_1 + 6p_2x_3} = \frac{dx_3}{-p_2^2}$

Taking 1st and 2nd fractions,

$$\frac{dp_1}{p_1} + \frac{dx_1}{x_1} = 0 \Rightarrow \log p_1 + \log x_1 = \log a_1 \Rightarrow p_1 x_1 = a_1 \quad a_1 \text{ is arbitrary constant.}$$

$$\therefore F_1 = p_1 x_1 = a_1 \quad \dots(2)$$

Taking 1st and 3rd fractions

$$dp_2 = 0 \Rightarrow p_2 = a_2 \quad \therefore F_2 = p_2 = a_2 \quad \dots(3)$$

where a_2 is arbitrary constant.

We have

$$\begin{aligned} (F_1, F_2) &= \sum_{r=1}^3 \left(\frac{\partial F_1}{\partial x_r} \frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r} \frac{\partial F_2}{\partial x_r} \right) \\ &= \left(\frac{\partial F_1}{\partial x_1} \frac{\partial F_2}{\partial p_1} - \frac{\partial F_1}{\partial p_1} \frac{\partial F_2}{\partial x_1} \right) + \left(\frac{\partial F_1}{\partial x_2} \frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2} \frac{\partial F_2}{\partial x_2} \right) + \left(\frac{\partial F_1}{\partial x_3} \frac{\partial F_2}{\partial p_3} - \frac{\partial F_1}{\partial p_3} \frac{\partial F_2}{\partial x_3} \right) \\ &= (p_1 \cdot 0 - x_1 \cdot 0) + (0 \cdot 1 - 0 \cdot 0) + (0 \cdot 0 - 0 \cdot 0) = 0 \end{aligned}$$

i.e., $(F_1, F_2) = 0$ is satisfied by $F_1 = a_1$ and $F_2 = a_2$ given by (2) and (3).

From (2) $p_1 = \frac{a_1}{x_1}$, from (3) $p_2 = a_2$, $\therefore$ from (1) $p_3 = -(2a_1 x_3 + 3a_2 x_3^2) / a_2^2$

Putting in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we get

$$dz = (a_1 / x_1) dx_1 + a_2 dx_2 - \{(2a_1 x_3 + 3a_2 x_3^2) / a_2^2\} dx_3$$

Integrating $z = a_1 \log x_1 + a_2 x_2 - (a_1 x_3^2 + a_2 x_3^3) / a_2^2 + a_3$

which is the complete integral.

Example 2: Apply Jacobi's method to find the complete integral of

$$p_1 x_1 + p_2 x_2 = p_3^2.$$

[Meerut 2007]

Solution: Here $f \equiv p_1 x_1 + p_2 x_2 - p_3^2 = 0$...(1)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{p_1} = \frac{dx_1}{-x_1} = \frac{dp_2}{p_2} = \frac{dx_2}{-x_2} = \frac{dp_3}{0} = \frac{dx_3}{-2p_3}$$

Taking 1st and 2nd fractions.

$$\frac{dp_1}{p_1} + \frac{dx_1}{x_1} = 0 \Rightarrow \log p_1 + \log x_1 = \log a_1 \Rightarrow F_1 = p_1 x_1 = a_1 \quad \dots(2)$$

Taking 1st and 5th fractions.

$$dp_3 = 0 \Rightarrow p_3 = a_2 \Rightarrow F_2 = p_3 = a_2 \quad \dots(3)$$

Obviously for F_1, F_2 given by (2) and (3)

$$(F_1, F_2) = \sum_{r=1}^3 \left(\frac{\partial F_1}{\partial x_r} \frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r} \frac{\partial F_2}{\partial x_r} \right) = 0 \quad (\text{verify yourself})$$

From (2) $p_1 = a_1 / x_1$, from (3), $p_3 = a_2$, $\therefore$ from (1) $p_2 = (a_2^2 - a_1) / x_2$

Putting in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we get

$$dz = \frac{a_1}{x_1} dx_1 + \frac{a_2^2 - a_1}{x_2} dx_2 + a_2 dx_3$$

Integrating, $z = a_1 \log x_1 + (a_2^2 - a_1) \log x_2 + a_2 x_3 + a_3$

which is the complete integral.

Example 3: Find the complete integral of

$$x_3^2 p_1^2 p_2^2 p_3^2 + p_1^2 p_2^2 - p_3^2 = 0. \quad [\text{Meerut 2006 (B.P.)}; \text{Delhi Maths (Hons.) 2006}]$$

$$\textbf{Solution:} \text{ Here } f \equiv x_3^2 p_1^2 p_2^2 p_3^2 + p_1^2 p_2^2 - p_3^2 = 0 \quad \dots(1)$$

$\therefore$ Jacobi's auxiliary equations are

$$\begin{aligned} \frac{dp_1}{0} &= \frac{dx_1}{-2(x_3^2 p_1^2 p_2^2 p_3^2 + p_1^2 p_2^2)} = \frac{dp_2}{0} = \frac{dx_2}{-2(x_3^2 p_1^2 p_2^2 p_3^2 + p_1^2 p_2^2)} \\ &= \frac{dp_3}{2x_3 p_1^2 p_2^2 p_3^2} = \frac{dx_3}{-2(x_3^2 p_1^2 p_2^2 p_3^2 - p_3)} \end{aligned}$$

Here we have, $dp_1 = 0$ and $dp_2 = 0 \Rightarrow p_1 = a_1, p_2 = a_2$

$$\therefore F_1 = p_1 = a_1 \quad \dots(2) \quad \text{and} \quad F_2 = p_2 = a_2 \quad \dots(3)$$

a_1, a_2 are arbitrary constants.

Obviously these functions F_1 and F_2 satisfy

$$(F_1, F_2) = \sum_{r=1}^3 \left(\frac{\partial F_1}{\partial x_r} \frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r} \frac{\partial F_2}{\partial x_r} \right) = 0 \quad (\text{Verify yourself})$$

Putting the values of p_1 and p_2 in (1),

$$p_3 = \pm a_1 a_2 / \sqrt{(1 - x_3^2 a_1^2 a_2^2)}$$

$\therefore$ Substituting in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we get

$$dz = a_1 dx_1 + a_2 dx_2 \pm \frac{a_1 a_2}{\sqrt{(1 - a_1^2 a_2^2 x_3^2)}} dx_3$$

Integrating, $z = a_1 x_1 + a_2 x_2 \pm \sin^{-1}(a_1 a_2 x_3) + a_3$

which is the complete integral.

Example 4: Find the complete integral of

$$p_3 x_3 (p_1 + p_2) + x_1 + x_2 = 0. \quad [\text{Delhi Maths (Hons.) 2000, 05, 08, 10}]$$

Solution: Here $f \equiv p_3 x_3 (p_1 + p_2) + x_1 + x_2 = 0$...(1)

$\therefore$ The Jacobi's auxiliary equations are

$$\frac{dp_1}{1} = \frac{dx_1}{-p_3 x_3} = \frac{dp_2}{1} = \frac{dx_2}{-p_3 x_3} = \frac{dp_3}{p_3 (p_1 + p_2)} = \frac{dx_3}{-x_3 (p_1 + p_2)}$$

Taking 1st and 3rd fractions,

$$dp_1 - dp_2 = 0 \quad \Rightarrow \quad F_1 = p_1 - p_2 = a_1 \quad \dots(2)$$

Taking 5th and 6th fractions,

$$\frac{dp_3}{p_3} + \frac{dx_3}{x_3} = 0 \quad \Rightarrow \quad \log p_3 + \log x_3 = \log a_2 \quad \Rightarrow \quad F_2 = p_3 x_3 = a_2 \quad \dots(3)$$

Obviously F_1 and F_2 given by (2) and (3) satisfy

$$(F_1, F_2) = \sum_{r=1}^3 \left(\frac{\partial F_1}{\partial x_r} \frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r} \frac{\partial F_2}{\partial x_r} \right) = 0 \quad (\text{Verify yourself})$$

From (2) $p_1 = p_2 + a_1$, from (3), $p_3 = a_2 / x_3$ putting in (1), we get

$$a_2 (2p_2 + a_1) + x_1 + x_2 = 0 \quad \therefore \quad p_2 = -\frac{a_1}{2} - \frac{x_1 + x_2}{2a_2}$$

$$\therefore \quad p_1 = \left(-\frac{a_1}{2} - \frac{x_1 + x_2}{2a_2} \right) + a_1 = \frac{a_1}{2} - \frac{x_1 + x_2}{2a_2}$$

Substituting in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we get

$$dz = \left(\frac{a_1}{2} - \frac{x_1 + x_2}{2a_2} \right) dx_1 + \left(\frac{-a_1}{2} - \frac{x_1 + x_2}{2a_2} \right) dx_2 + \frac{a_2}{x_3} dx_3$$

or
$$dz = \frac{a_1}{2} (dx_1 - dx_2) - \frac{1}{2a_2} (x_1 + x_2) (dx_1 + dx_2) + \frac{a_2}{x_3} dx_3$$

Integrating,
$$z = \frac{a_1}{2} (x_1 - x_2) - \frac{1}{4a_2} (x_1 + x_2)^2 + a_2 \log x_3 + a_3$$

which is the complete integral.

Example 5: Apply Jacobi's method to find the complete integral of

$$(p_1 + x_1)^2 + (p_2 + x_2)^2 + (p_3 + x_3)^2 = 3(x_1 + x_2 + x_3).$$

Solution: Here $f \equiv (p_1 + x_1)^2 + (p_2 + x_2)^2 + (p_3 + x_3)^2 - 3(x_1 + x_2 + x_3) = 0$... (1)

∴ The Jacobi's auxiliary equations are

$$\begin{aligned} \frac{dp_1}{2(p_1 + x_1) - 3} &= \frac{dx_1}{-2(p_1 + x_1)} = \frac{dp_2}{2(p_2 + x_2) - 3} = \frac{dx_2}{-2(p_2 + x_2)} \\ &= \frac{dp_3}{2(p_3 + x_3) - 3} = \frac{dx_3}{-2(p_3 + x_3)} \end{aligned} \quad \dots (2)$$

∴ Each fraction in (2) $= \frac{dp_1 + dx_1}{-3} = \frac{dp_2 + dx_2}{-3} = \frac{dp_3 + dx_3}{-3}$

∴ $dp_1 + dx_1 = dp_2 + dx_2$ and $dp_1 + dx_1 = dp_3 + dx_3$

Integrating, $p_1 + x_1 = p_2 + x_2 + a_1$ and $p_1 + x_1 = p_3 + x_3 + a_2$

∴ Let $F_1 = p_1 - p_2 + x_1 - x_2 = a_1$... (3)

and $F_2 = p_1 - p_3 + x_1 - x_3 = a_2$... (4)

Obviously F_1 and F_2 given by (3) and (4) satisfy

$$(F_1, F_2) = \sum_{r=1}^3 \left(\frac{\partial F_1}{\partial x_r} \frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r} \frac{\partial F_2}{\partial x_r} \right) = 0 \text{ (Verify yourself)}$$

From (3), $p_2 = p_1 + x_1 - x_2 - a_1$, from (4) $p_3 = p_1 + x_1 - x_3 - a_2$

Putting (1), we get

$$(p_1 + x_1)^2 + (p_1 + x_1 - a_1)^2 + (p_1 + x_1 - a_2)^2 - 3(x_1 + x_2 + x_3) = 0$$

or $3(p_1 + x_1)^2 - 2(a_1 + a_2)(p_1 + x_1) + \{a_1^2 + a_2^2 - 3(x_1 + x_2 + x_3)\} = 0$

$$\therefore p_1 + x_1 = \frac{2(a_1 + a_2) \pm \sqrt{[4(a_1 + a_2)^2 - 12\{a_1^2 + a_2^2 - 3(x_1 + x_2 + x_3)\}]}}{6}$$

$$\text{or } p_1 = -x_1 + \frac{1}{3}[(a_1 + a_2) \pm \sqrt{\{9(x_1 + x_2 + x_3) - 2(a_1^2 + a_2^2 - a_1 a_2)\}}]$$

$$\begin{aligned} \therefore p_2 &= p_1 + x_1 - x_2 - a_1 \\ &= -x_2 + \frac{1}{3}[(-2a_1 + a_2) \pm \sqrt{\{9(x_1 + x_2 + x_3) - 2(a_1^2 + a_2^2 - a_1 a_2)\}}] \end{aligned}$$

$$\begin{aligned} p_3 &= p_1 + x_1 - x_3 - a_2 \\ &= -x_3 + \frac{1}{3}[(a_1 - 2a_2) \pm \sqrt{\{9(x_1 + x_2 + x_3) - 2(a_1^2 + a_2^2 - a_1 a_2)\}}] \end{aligned}$$

Substituting in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we get

$$\begin{aligned} dz &= -(x_1 dx_1 + x_2 dx_2 + x_3 dx_3) + \frac{1}{3}\{(a_1 + a_2) dx_1 + (-2a_1 + a_2) dx_2 + (a_1 - 2a_2) dx_3\} \\ &\quad \pm \frac{1}{3}\sqrt{\{9(x_1 + x_2 + x_3) - 2(a_1^2 + a_2^2 - a_1 a_2)\}}(dx_1 + dx_2 + dx_3) \end{aligned}$$

Integrating, we get

$$\begin{aligned} z &= -\frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + \frac{1}{3}\{(a_1 + a_2)x_1 + (-2a_1 + a_2)x_2 + (a_1 - 2a_2)x_3\} \\ &\quad \pm (2/81)\{9(x_1 + x_2 + x_3) - 2(a_1^2 + a_2^2 - a_1 a_2)\}^{3/2} + a_3 \end{aligned}$$

a_1, a_2, a_3 are arbitrary constants.

which is the complete integral.

Example 6: Find the complete integral of

$$p_1 p_2 p_3 = z^3 x_1 x_2 x_3.$$

Solution: The given equation is not in the standard form as it involve the dependent variable z . So first we reduce the equation to the standard form. The given equation can be written as

$$\left(\frac{1}{z} \frac{\partial z}{\partial x_1}\right) \left(\frac{1}{z} \frac{\partial z}{\partial x_2}\right) \left(\frac{1}{z} \frac{\partial z}{\partial x_3}\right) = x_1 x_2 x_3 \quad \dots(1)$$

Putting $\frac{1}{z} dz = dZ$, so that $\log z = Z$, (1) reduces to

$$\frac{\partial Z}{\partial x_1} \cdot \frac{\partial Z}{\partial x_2} \cdot \frac{\partial Z}{\partial x_3} = x_1 x_2 x_3 \quad \text{or} \quad P_1 P_2 P_3 = x_1 x_2 x_3$$

where $P_1 = \frac{\partial Z}{\partial x_1}$, $P_2 = \frac{\partial Z}{\partial x_2}$ and $P_3 = \frac{\partial Z}{\partial x_3}$

∴ Let $f \equiv P_1 P_2 P_3 - x_1 x_2 x_3 = 0$... (2)

∴ The Jacobi's auxiliary equations are

$$\frac{dP_1}{-x_2 x_3} = \frac{dx_1}{-P_2 P_3} = \frac{dP_2}{-x_1 x_3} = \frac{dx_2}{-P_1 P_3} = \frac{dP_3}{-x_1 x_2} = \frac{dx_3}{-P_1 P_2}$$

Taking the 1st and 2nd fractions, we get

$$\frac{x_1 dP_1}{-x_1 x_2 x_3} = \frac{P_1 dx_1}{-P_1 P_2 P_3} \Rightarrow x_1 dP_1 = P_1 dx_1 \text{ by (2)}$$

⇒ $\frac{dP_1}{P_1} = \frac{dx_1}{x_1}$, Integrating $\log P_1 = \log x_1 + \log a_1$ or $P_1 = a_1 x_1$

∴ Let $F_1 = P_1 - a_1 x_1 = 0$... (3)

Similarly, taking 3rd and 4th fractions, we get another additional equation as

$$F_2 = P_2 - a_2 x_2 = 0 \quad \dots (4)$$

From (3), $P_1 = a_1 x_1$, from (4) $P_2 = a_2 x_2$, Putting in (2), $P_3 = x_3 / (a_1 a_2)$

Substituting in $dZ = P_1 dx_1 + P_2 dx_2 + P_3 dx_3$, we get

$$dZ = a_1 x_1 dx_1 + a_2 x_2 dx_2 + (1 / a_1 a_2) x_3 dx_3$$

Integrating $2Z = a_1 x_1^2 + a_2 x_2^2 + x_3^2 / (a_1 a_2) + a_3$

or $2 \log z = a_1 x_1^2 + a_2 x_2^2 + x_3^2 / (a_1 a_2) + a_3$

which is the complete integral.

Example 7: Find the complete integral of

$$(x_2 + x_3)(p_2 + p_3)^2 + z p_1 = 0.$$

Solution: The given equation is not in the standard form as it involve the dependent variable z . So first we reduce the equation to the standard form. The given equation can be written as

$$(x_2 + x_3) \left(\frac{1}{z} \frac{\partial z}{\partial x_2} + \frac{1}{z} \frac{\partial z}{\partial x_3} \right)^2 + \frac{1}{z} \frac{\partial z}{\partial x_1} = 0 \quad \dots (1)$$

Putting $\frac{1}{z} dz = dZ$, so that $\log z = Z$, equation (1) reduces to

$$(x_2 + x_3) \left(\frac{\partial Z}{\partial x_2} + \frac{\partial Z}{\partial x_3} \right)^2 + \frac{\partial Z}{\partial x_1} = 0$$

or $(x_2 + x_3) (P_2 + P_3)^2 + P_1 = 0$

Where $P_1 = \frac{\partial Z}{\partial x_1}$, $P_2 = \frac{\partial Z}{\partial x_2}$ and $P_3 = \frac{\partial Z}{\partial x_3}$

$\therefore$ Let $f \equiv (x_2 + x_3) (P_2 + P_3)^2 + P_1 = 0$... (2)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dP_1}{0} = \frac{dx_1}{-1} = \frac{dP_2}{(P_2 + P_3)^2} = \frac{dx_2}{-2(x_2 + x_3)(P_2 + P_3)} = \frac{dP_3}{(P_2 + P_3)^2} = \frac{dx_3}{-2(x_2 + x_3)(P_2 + P_3)}$$

Taking 1st and 2nd fractions, $dP_1 = 0 \Rightarrow P_1 = -a_1$

$\therefore$ Let $F_1 = P_1 = -a_1$... (3)

Taking 3rd and 5th fractions, $dP_2 - dP_3 = 0 \Rightarrow P_2 - P_3 = a_2$

$\therefore$ Let $F_2 = P_2 - P_3 = a_2$... (4)

Obviously F_1 and F_2 given by (3) and (4) satisfy

$$(F_1, F_2) = \sum_{r=1}^3 \left(\frac{\partial F_1}{\partial x_r} \frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r} \frac{\partial F_2}{\partial x_r} \right) = 0$$

From (3), $P_1 = -a_1$, from (4) $P_2 = P_3 + a_2$, Putting in (2), we get

$$(x_2 + x_3) (2P_3 + a_2)^2 - a_1 = 0 \Rightarrow P_3 = -\frac{a_2}{2} \pm \sqrt{\left(\frac{a_1}{x_2 + x_3} \right)}$$

$\therefore$ $P_2 = P_3 + a_2 = -\frac{a_2}{2} \pm \sqrt{\left(\frac{a_1}{x_2 + x_3} \right)}$

Substituting in $dZ = P_1 dx_1 + P_2 dx_2 + P_3 dx_3$, we get

$$dZ = -a_1 dx_1 + \left[\frac{a_2}{2} \pm \sqrt{\left(\frac{a_1}{x_2 + x_3} \right)} \right] dx_2 + \left[-\frac{a_2}{2} \pm \sqrt{\left(\frac{a_1}{x_2 + x_3} \right)} \right] dx_3$$

or $dZ = -a_1 dx_1 + \frac{a_2}{2} (dx_2 - dx_3) \pm \sqrt{\left(\frac{a_1}{x_2 + x_3} \right)} (dx_2 + dx_3)$

Integrating $Z = -a_1 x_1 + \frac{1}{2} a_2 (x_2 - x_3) \pm 2 \sqrt{a_1 (x_2 + x_3)} + a_3$

or $\log z = -a_1 x_1 + (a_2 / 2) (x_2 - x_3) \pm 2 \sqrt{a_1 (x_2 + x_3)} + a_3$

which is the complete integral.

Example 8: Find the complete integral of

$$p_1 p_2 p_3 + p_4^3 x_1 x_2 x_3 x_4^3 = 0.$$

Solution: Here the given differential equation have four independent variables x_1, x_2, x_3 and x_4 .

$\therefore$ Let $f \equiv p_1 p_2 p_3 + p_4^3 x_1 x_2 x_3 x_4^3 = 0$... (1)

The Jacobi's auxiliary equations are

$$\begin{aligned} \frac{dp_1}{p_4^3 x_2 x_3 x_4^3} &= \frac{dx_1}{-p_2 p_3} = \frac{dp_2}{p_4^3 x_1 x_3 x_4^3} = \frac{dx_2}{-p_1 p_3} = \frac{dp_3}{p_4^3 x_1 x_2 x_4^3} = \frac{dx_3}{-p_1 p_2} \\ &= \frac{dp_4}{3 p_4^3 x_1 x_2 x_3 x_4^3} = \frac{dx_4}{-3 p_4^2 x_1 x_2 x_3 x_4^3} \end{aligned} \quad \dots (2)$$

Taking 1st and 2nd fractions, we have

$$\begin{aligned} \frac{x_1 dp_1}{p_4^3 x_1 x_2 x_3 x_4^3} &= \frac{p_1 dx_1}{-p_1 p_2 p_3} \Rightarrow \frac{x_1 dp_1}{-p_1 p_2 p_3} = \frac{p_1 dx_1}{-p_1 p_2 p_3}, \text{ By (1)} \\ \Rightarrow \frac{dp_1}{p_1} - \frac{dx_1}{x_1} &= 0 \Rightarrow \log p_1 - \log x_1 = \log a_1 \Rightarrow p_1 = a_1 x_1 \end{aligned}$$

$\therefore$ Let $F_1 = p_1 - a_1 x_1 = 0$... (3)

Similarly, taking 2nd and 3rd, 4th and 5th fraction, we obtain the following two more additional equations

$$F_2 = p_2 - a_2 x_2 = 0 \quad \dots (4) \quad \text{and} \quad F_3 = p_3 - a_3 x_3 = 0 \quad \dots (5)$$

where a_1, a_2, a_3 are arbitrary constants.

Putting the values of p_1, p_2, p_3 from (3), (4) and (5) in (1), we have

$$p_4 = \frac{(a_1 a_2 a_3)^{1/3}}{x_4}$$

Substituting the value of p_1, p_2, p_3, p_4 in

$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 + p_4 dx_4$, we get

$$dz = a_1 x_1 dx_1 + a_2 x_2 dx_2 + a_3 x_3 dx_3 - (a_1 a_2 a_3)^{1/3} (1/x_4) dx_4$$

Integrating,
$$z = \frac{1}{2} (a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2) - (a_1 a_2 a_3)^{1/3} \cdot \log x_4 + a_4.$$

which is the complete integral.

Exercise 8.6

Apply Jacobi's method to find the complete integral of the following :

- $p_1^3 + p_2^3 + p_3 - 1 = 0$ [Meerut 2006]
- $2x_1 x_3 z p_1 p_3 + x_2 p_2 = 0$
- $p_1^2 + p_2 p_3 - z(p_2 + p_3) = 0$
- Write short note on the Jacobi's method. [Meerut 2005 (B.P.)]

Answers 8.6

1.	$z = a_1 x_1 + a_2 x_2 + (1 - a_1^3 - a_2^2) x_3 + a_3.$
2.	$z^2 = a_1 \log x_1 - a_1 a_2 \log x_2 + a_2 \log x_3 + a_3.$
3.	$(1 + a_1 a_2) \log z = (a_1 + a_2) (x_1 + a_2 x_2 + a_2 x_3 + a_3).$

8.9 Jacobi's Method of Solving a Non-linear First Order Partial Differential Equation in Two Independent Variables

[Himachal 2003, 05]

Let a non linear first order partial differential equation in two independent variables x and y be given by $f(x, y, z, p, q) = 0$... (1)

The solution of this equation (1) is of the form $u(x, y, z) = 0$... (2)

From (2) it is obvious that u can be treated as the dependent variable which is a function of three independent variables x, y, z .

Differentiating (2) partially w.r.t. x and y respectively, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \quad p_1 + p_3 \cdot p = 0 \quad \text{and} \quad p_2 + p_3 \cdot q = 0$$

Taking $\frac{\partial u}{\partial x} = p_1, \frac{\partial u}{\partial y} = p_2, \frac{\partial u}{\partial z} = p_3$, where $\frac{\partial z}{\partial x} = p$ and $\frac{\partial z}{\partial y} = q$

$$\Rightarrow \quad p = -\frac{p_1}{p_3} \quad \text{and} \quad q = -\frac{p_2}{p_3}$$

Using these results, and taking $x = x_1, y = y_1, z = z_1$, (1) reduces to

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = 0 \quad \dots(3)$$

which is a partial differential equation with three independent variables x_1, x_2, x_3 and can be solved by Jacobi's method discussed in 8.8. Replacing x_1, x_2, x_3 by original variables x, y, z respectively in the above complete integral, so obtained, we get the solution of given equation (1) which contains the original variables x, y, z and a new dependent variable u . Also this solution contains three arbitrary constants, while we need only two arbitrary constants in the complete solution of equation (1). Hence the required solution $u=0$ of (1) is obtained by giving some definite value to one of the arbitrary constants.

Solved Examples

Example 1: Solve $p^2 x + q^2 y = z$ by Jacobi's method.

[Himachal 2003, 05]

Solution: Given $p^2 x + q^2 y = z \quad \dots(1)$

The solution of (1) is of the form $u(x, y, z) = 0 \quad \dots(2)$

Differentiating (2) partially, w.r.t. x and y and taking $\frac{\partial u}{\partial x} = x_1, \frac{\partial u}{\partial y} = x_2, \frac{\partial u}{\partial z} = x_3$, we get

$$p = -\frac{p_1}{p_3} \quad \text{and} \quad q = -\frac{p_2}{p_3}$$

Substituting in (1) and taking $x = x_1, y = y_1$ and $z = z_1$, it reduces to

$$\left(-\frac{p_1}{p_3}\right)^2 x_1 + \left(-\frac{p_2}{p_3}\right)^2 x_2 = z$$

$$\Rightarrow x_1 p_1^2 + x_2 p_2^2 - x_3 p_3^2 = 0 \quad \dots(3)$$

$$\therefore \text{ Let } f(x_1, x_2, x_3, p_1, p_2, p_3) \equiv x_1 p_1^2 + x_2 p_2^2 - x_3 p_3^2 = 0 \quad \dots(4)$$

$\therefore$ The Jacobi's auxiliary equations are

$$\frac{dp_1}{\frac{\partial f}{\partial x_1}} = \frac{dx_1}{-\frac{\partial f}{\partial p_1}} = \frac{dp_2}{\frac{\partial f}{\partial x_2}} = \frac{dx_2}{-\frac{\partial f}{\partial p_2}} = \frac{dp_3}{\frac{\partial f}{\partial x_3}} = \frac{dx_3}{-\frac{\partial f}{\partial p_3}}$$

$$\text{or } \frac{dp_1}{p_1^2} = \frac{dx_1}{-2x_1 p_1} = \frac{dp_2}{p_2^2} = \frac{dx_2}{-2x_2 p_2} = \frac{dp_3}{-p_3^2} = \frac{dx_3}{-2x_3 p_3}$$

Taking 1st and 2nd fractions, we get $\frac{2dp_1}{p_1} + \frac{dx_1}{x_1} = 0$

$$\text{Integrating } 2 \log p_1 + \log x_1 = \log a_1 \Rightarrow p_1^2 x_1 = a_1 \Rightarrow p_1 = \sqrt{(a_1 / x_1)}$$

Similarly taking 3rd and 4th fractions, we get

$$p_2 = \sqrt{(a_2 / x_2)}$$

Substituting the values of p_1 and p_2 in (3), we get

$$x_3 p_3^2 = a_1 + a_2 \Rightarrow p_3 = \sqrt{\{(a_1 + a_2) / x_3\}}$$

Putting in $du = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we get

$$du = \sqrt{(a_1 / x_1)} dx_1 + \sqrt{(a_2 / x_2)} dx_2 + \sqrt{\{(a_1 + a_2) / x_3\}} dx_3$$

$$\text{Integrating, } u = \sqrt{(a_1 x_1)} + 2 \sqrt{(a_2 x_2)} + 2 \sqrt{\{(a_1 + a_2) x_3\}} + a_3$$

which contains three arbitrary constants a_1 , a_2 and a_3 .

Taking $a_2 = 1$ (particular value), and replacing x_1, x_2, x_3 by original variables x, y, z respectively the solution $u = 0$ of (1) is given by

$$2 \sqrt{a_1} \sqrt{x} + 2 \sqrt{y} + 2 \sqrt{(a_1 + 1)} \sqrt{z} + a_3 = 0$$

which contains two arbitrary constants and so is the required complete integral of (1).

Exercise 8.7

Solve by Jacobi's Method.

1. $p^2 + q^2 = \lambda^2$

[Delhi Maths (Hons.) 2007]

2. $xpq + yq^2 = 1$

[Nagpur 2005]

Objective Type Questions

Multiple Choice Questions

Indicate correct answer for each questions by writing the corresponding letter (a), (b), (c) or (d).

- The complete integral of the differential equation $pq = p - q$ is :
 - $z = ax + \{a / (a + 1)\}y + c$
 - $z = ax + \{a / (a - 1)\}y + c$
 - $z = ax + (a / a - 1)y$
 - None of these
- The complete integral of $p(1 + q) = qz$ is :
 - $az = be^{x+ay}$
 - $az = 1 + be^{x+ay}$
 - $z = 1 + be^{x+ay}$
 - None of these
- The complete integral of $\sqrt{p} - \sqrt{q} = 2x$ is :
 - $z = (a + 2x)^3 + a^2y + b$
 - $z = (a + 2x)^3 - a^2y + b$
 - $z = (a + 2x)^3 / 6 + a^2y + b$
 - None of these
- The singular integral of the differential equation on $z = px + qy - 2\sqrt{pq}$ is :
 - $xy = 1$
 - $x / y = 1$
 - $x^2 + y^2 = 1$
 - None of these
- The complete integral of the differential equation $(p + q)(z - px - qy) = 1$ is :
 - $(z + ax + (y)(a + b)) = 1$
 - $z = ax + by + 1 / (a + b)$
 - $(z + ax + by)(a - b) + 1 = 0$
 - None of these
- For the solution of the equatuon of the form $f(p, q, z) = 0$, we take $z = f(X)$ where $X = x + ay$ as its complete integral. Then p and q are given by :
 - $p = \frac{\partial z}{\partial X}, q = \frac{\partial z}{\partial X}$
 - $p = a \frac{\partial z}{\partial X}, q = \frac{\partial z}{\partial X}$
 - $p = \frac{\partial z}{\partial X}, q = a \frac{\partial z}{\partial X}$
 - $p = a \frac{\partial z}{\partial X}, q = a^2 \frac{\partial z}{\partial y}$

7. To solve a differential equation of the form $f_1(x, p) = f_2(y, q)$, we take
- (a) $f_1(x, p) = a = f_2(y, q)$ (b) $f_1(x, p) \cdot f_2(y, q) = a$
 (c) $f_1(x, p) / f_2(y, q) = a$ (d) None of these
8. A partial differential equation is said to be of Clairaut form, if it can be written as
- (a) $z = py + qx + f(x, y)$ (b) $z = px + qy + f(p, q)$
 (c) $z = px + qy + f(px, qy)$ (d) None of these
9. The complete integral of the equation $z = px + qy + \log(pq)$ is :
- (a) $z = ax + by + \log(a^2 + b^2)$ (b) $z = ax + by + \log(ab)$
 (c) $z = ax - by + \log(ab)$ (d) None of these
10. The complete integral of the equation $4xyz = pq + 2px^2y + 2qxy^2$ is :
- (a) $z = ax^2 + by^2 + ab$ (b) $z = ax^2 - by^2 + ab$
 (c) $z = ax + by + a^2b^2$ (d) $z = ax + by^2 - ab$

Fill in the Blank

Fill in the blank's "....." so that the following statements are complete and correct :

- The complete integral of $p^2 - q^2 = z$ is $2z^{1/2} = ax + \dots + c$.
- The complete integral of $3pq = p + q$ is $4az = (1-a)(x + \dots)$.
- The complete integral of $pe^y = qe^x$ is $z = a(e^x + \dots) + b$.
- A first order partial differential equation is said to be of if it can be written as $z = px + qy + f(p, q)$.
- The complete integral of the equation $p = (qy + z)^2$ is $yz = ax + \dots + b$.
- The complete integral of the equation $q = 3y^2$ is
- The complete integral of the equation $pqz = p^2(xq + p^2) + q^2(yq + q^2)$ is

True/False

Write 'T' for true and 'F' for false statement.

- The Charpit's auxiliary equations of the differential equation $f(x, y, z, p, q) = 0$ are given by

$$\frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

2. The Charpit's auxiliary equations for the equation $2zx - px^2 - 2qxy + pq = 0$ are

$$\frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dz}{px^2 - pq + 2xyq} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p}$$

3. The Jacobi's auxiliary equations for the non-linear partial differential equation $f(x_1, x_2, x_3, p_1, p_2, p_3) = 0$, are given by

$$\frac{\frac{dp_1}{\partial f}}{\frac{\partial x_1}{\partial p_1}} = \frac{\frac{dx_1}{\partial f}}{\frac{\partial p_1}{\partial x_2}} = \frac{\frac{dp_2}{\partial f}}{\frac{\partial x_2}{\partial p_2}} = \frac{\frac{dx_2}{\partial f}}{\frac{\partial p_2}{\partial x_3}} = \frac{\frac{dp_3}{\partial f}}{\frac{\partial x_3}{\partial p_3}} = \frac{\frac{dx_3}{\partial f}}{\frac{\partial p_3}{\partial p_3}}$$

4. The Jacobi's auxiliary equations for the equation $p_1x_1 + p_2x_2 + p_3^2 = 0$ are :

$$\frac{dp_1}{p_1} = \frac{dx_1}{-x_1} = \frac{dp_2}{p_2} = \frac{dx_2}{-x_2} = \frac{dp_3}{2p_3} = \frac{dx_3}{-2p_3}$$

Answers

Exercise 8.7

1.	$a_1x + x_2 + [\sqrt{(a_1^2 + 1)} / \lambda] x_3 + a_3 = 0$	2.	$2a_1 \sqrt{(x_1 + x_2)} + a_1x_3 + a_3 = 0$

Objective Type Questions

Multiple Choice Questions

1.	(a)	2.	(b)	3.	(c)	4.	(a)
5.	(b)	6.	(c)	7.	(a)	8.	(b)
9.	(b)	10.	(a)				

Fill in the Blank

1.	$\sqrt{(a^2 - 1)} \cdot y$	2.	$ay + b$
3.	e^y	4.	Clairaut form
5.	$2\sqrt{(ay)}$	6.	$z = ax + y^3 + b$
7.	$abz = a^2(xb + a^2) + b^2(ya + b^2)$		

True/False

1.	True	2.	False
3.	True	4.	False

Unit - 4

Chapter

9

Partial Differential Equations of the Second Order with Variable Coefficients (Origin and Classification)

9.1 Introduction

In this chapter we shall discuss partial differential equations of second order with variable coefficients. A partial differential equation is said to be of the second order when it involves, at least one of the partial derivative of second order *i.e.*, r , s and t but no partial derivative of third or higher order. The first order partial derivatives p and q may also be present in these equations.

The second order partial differential equation is given by

$$f(x, y, z, p, q, r, s, t) = 0.$$

The general linear partial differential equation of second order in two independent variables x and y , with z as dependent variable and variable coefficients is given by

$$Rr + Ss + Tt + Pp + Qq + Zz = F$$

Where R, S, T, P, Q, Z and F are functions of functions x and y only and not all R, S, T are zero together.

Recall that : $r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial p}{\partial x}$, $s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$, $t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial q}{\partial y}$.

9.2 Origin (Derivation) of Second Order Partial Differential Equation

In 7.4 of chapter 7 we have discussed the origin (derivation) of partial differential equation by eliminating the arbitrary constants from the given relation which contains the arbitrary constants and by eliminating the function if the given relation consists of a function of dependent and independent variables.

If $f(x, y, z, a, b) = 0$ is a function of a dependent variable z and two independent variables x and y and contains at the most two arbitrary constants, then by eliminating a and b from this relation, we obtained first order partial differential equation. However if $f(x, y, z, a, b, c) = 0$ is a function of a dependent variable z and two independent variables x and y and contains three arbitrary constants, then by eliminating a, b and c from this relation we shall obtain second order partial differential equation. Similarly if the relation between dependent variable z and independent variables x and y contains only one function, then by eliminating this function from this relation, we obtained first order partial differential equation. However if the relation between dependent variable z and independent variables x and y contains two arbitrary functions, then by eliminating these two functions from this relation we shall obtain second order partial differential equation.

1. Origin (Derivation) of Second Order Partial Differential Equation by the Elimination of Arbitrary Constants :

Consider an equation $f(x, y, z, a, b, c) = 0$... (1)

where z is a function of two independent variables x and y , a, b and c arbitrary constants.

Differentiating (1) partially w.r.t x and y , we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p = 0 \quad \dots (2)$$

$$\text{and} \quad \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial y} = 0 \quad \text{or} \quad \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q = 0 \quad \dots (3)$$

It is not necessary that a, b and c may be eliminated between (1), (2) and (3) to give a partial differential equation. In that case differentiate (2) partially w.r.t x or differentiate (3) partially w.r.t y or differentiate either (2) partially w.r.t 'y' or (3) partially w.r.t 'x'. Thus we get the following equations.

$$\text{Differential (2) partially w.r.t 'x',} \quad \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial z} p + \frac{\partial f}{\partial z} \cdot \frac{\partial p}{\partial x} = 0 \quad \dots (4)$$

$$\text{Differential (3) partially w.r.t 'y',} \quad \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial y \partial z} q + \frac{\partial f}{\partial z} \cdot \frac{\partial q}{\partial y} = 0 \quad \dots (5)$$

Differential (2) partially w.r.t 'y'

$$\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial y \partial z} p + \frac{\partial f}{\partial z} \cdot \frac{\partial p}{\partial y} = 0 \quad \dots(6)$$

Now eliminating a, b and c between (1), (2), (3) and one or more (if needed) of (4), (5), and (6), we shall get a second order partial differential equation. For clear understanding of the method see the following examples.

Solved Examples

Example 1: Find a partial differential equation by eliminating a, b, c from

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1. \quad [\text{Vikram 2002, 04; Jabalpur 2000, 03; Bhopal 2004}]$$

Solution: Given
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \dots(1)$$

Differentiating (1) partially w.r.t x and y successively we get

$$\frac{2x}{a^2} + \frac{2z}{c^2} \frac{\partial z}{\partial x} = 0 \quad \text{or} \quad c^2 x + a^2 z p = 0 \quad \dots(2)$$

and
$$\frac{2y}{b^2} + \frac{2z}{c^2} \frac{\partial z}{\partial y} = 0 \quad \text{or} \quad c^2 y + b^2 z q = 0 \quad \dots(3)$$

Here it is not possible to eliminate a, b, c between (1), (2) and (3).

So again differentiating (2) partially w.r.t x' , we get

$$c^2 + a^2 \left(\frac{\partial z}{\partial x} p + z \frac{\partial p}{\partial x} \right) = 0 \quad \text{or} \quad c^2 + a^2 (p^2 + z r) = 0 \quad \dots(4)$$

$$\therefore \frac{\partial p}{\partial x} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x^2} = r$$

Now from (2) $c^2 = -\frac{a^2 z}{x} p$. Substituting in (4), we get

$$-\frac{a^2 z}{x} p + a^2 (p^2 + z r) = 0 \quad \text{or} \quad x z r + x p^2 - z p = 0 \quad \dots(5)$$

Again differentiating (3) partially w.r.t. 'y', we get

$$c^2 + b^2 \left(\frac{\partial z}{\partial y} q + z \frac{\partial q}{\partial y} \right) = 0 \quad \text{or} \quad c^2 + b^2 (q^2 + z t) = 0 \quad \dots(6)$$

$$\therefore \frac{\partial q}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y^2} = t$$

Now from (3), $c^2 = -\frac{b^2 z}{y} q$. Substituting in (6), we get

$$-\frac{b^2 z}{y} q + b^2 (q^2 + z t) = 0 \quad \text{or} \quad y z t + y q^2 - z q = 0 \quad \dots(7)$$

Again differentiating (2) partially w.r.t 'y' or (3) w.r.t 'x', we get

$$a^2 \left(\frac{\partial z}{\partial y} p + z \frac{\partial p}{\partial y} \right) = 0 \quad \text{or} \quad p q + z s = 0 \quad \dots(8)$$

$$\therefore \frac{\partial p}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = s.$$

The equations (5), (7) and (8) are the required second order partial differential equations.

Note : We may get more than one second order partial differential equation by eliminating a , b and c from a relation containing a , b , c . As in the above example we got three second order p.d. equations.

Example 2: Obtain the partial differential equation by eliminating arbitrary constants from

$$z = A e^{-lt} \cos mx \sin ny \text{ where } l^2 = m^2 + n^2.$$

Solution: Given, $z = A e^{-lt} \cos mx \sin ny$...(1)

Differentiating (1) partially w.r.t the independent variables t , x and y , we get

$$\frac{\partial z}{\partial t} = -A l e^{-lt} \cos mx \sin ny \quad \dots(2)$$

$$\frac{\partial z}{\partial x} = -A m e^{-lt} \sin mx \sin ny \quad \dots(3)$$

and $\frac{\partial z}{\partial y} = A n e^{-lt} \cos mx \cos ny$...(4)

Differentiating (2), (3) and (4) partially w.r.t t , x and y respectively, we get

$$\frac{\partial^2 z}{\partial t^2} = A l^2 e^{-lt} \cos mx \sin ny \quad \dots(5)$$

$$\frac{\partial^2 z}{\partial x^2} = -A m^2 e^{-lt} \cos mx \sin ny \quad \dots(6)$$

and
$$\frac{\partial^2 z}{\partial y^2} = -A n^2 e^{-lt} \cos mx \sin ny \quad \dots(7)$$

$\therefore$
$$(5) + (6) + (7)$$

$$\frac{\partial^2 z}{\partial t^2} + \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = A (l^2 - m^2 - n^2) e^{-lt} \cos mx \sin ny$$

$\Rightarrow$
$$\frac{\partial^2 z}{\partial t^2} + \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \quad (\because l^2 = m^2 + n^2)$$

Which is the required second order partial differential equation.

Note : From the above example we conclude that a second order partial differential equation can be obtained when the number of arbitrary constants is equal to the number of independent variables.

2. Origin (Derivation) of Second Order Partial Differential Equation by the Elimination of Arbitrary Functions :

If the given relation between the dependent variable z and two independent variables x and y contains two arbitrary functions then the elimination of these two functions from the given relation will give rise to a second order partial differential equation in general.

The procedure will be clear from the following examples.

Solved Examples

Example 1: Form a partial differential equation by eliminating the arbitrary functions f and ϕ from

$$z = y f(x) + x \phi(y).$$

Solution: Given, $z = y f(x) + x \phi(y)$... (1)

Differentiating (1) partially w.r.t x and y , we get

$$\frac{\partial z}{\partial x} = y f'(x) + \phi(y) \quad \dots(2)$$

and
$$\frac{\partial z}{\partial y} = f(x) + x \phi'(y) \quad \dots(3)$$

Again differentiating (2) partially w.r.t 'y', we get

$$\frac{\partial^2 z}{\partial y \partial x} = f'(x) + \phi'(y) \quad \dots(4)$$

From (2) and (3), $f'(x) = \frac{1}{y} \left[\frac{\partial z}{\partial x} - \phi(y) \right]$ and $\phi'(y) = \frac{1}{x} \left[\frac{\partial z}{\partial y} - f(x) \right]$

Substituting in (4), we get

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= \frac{1}{y} \left[\frac{\partial z}{\partial x} - \phi(y) \right] + \frac{1}{x} \left[\frac{\partial z}{\partial y} - f(x) \right] = \frac{1}{y} \frac{\partial z}{\partial x} + \frac{1}{x} \frac{\partial z}{\partial y} - \frac{1}{xy} [x \phi(y) + y f(x)] \\ \Rightarrow xy \frac{\partial^2 z}{\partial x \partial y} &= x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} - z, \quad \text{substituting from (1)} \end{aligned}$$

$$\Rightarrow x y s = x p + y q - z.$$

which is the required partial differential equation of order 2.

Example 2: Form a partial differential equation by eliminating arbitrary functions f and g from $z = f(x^2 - y) + g(x^2 + y)$.

Solution: Given, $z = f(x^2 - y) + g(x^2 + y)$... (1)

Differentiating (1) partially w.r.t 'x' and 'y', we get

$$\frac{\partial z}{\partial x} = 2x f'(x^2 - y) + 2x g'(x^2 + y) \quad \dots (2)$$

$$\text{and} \quad \frac{\partial z}{\partial y} = -f'(x^2 - y) + g'(x^2 + y) \quad \dots (3)$$

Differentiating (2) partially w.r.t 'x' and (3) w.r.t 'y', we get

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= 4x^2 f''(x^2 - y) + 2f'(x^2 - y) + 4x^2 g''(x^2 + y) + 2g'(x^2 + y) \\ &= 4x^2 [f''(x^2 - y) + g''(x^2 + y)] + 2[f'(x^2 - y) + g'(x^2 + y)] \quad \dots (4) \end{aligned}$$

$$\text{and} \quad \frac{\partial^2 z}{\partial y^2} = f''(x^2 - y) + g''(x^2 + y) \quad \dots (5)$$

$$\text{from (2),} \quad f'(x^2 - y) + g'(x^2 + y) = \frac{1}{2x} \frac{\partial z}{\partial x} \quad \dots (6)$$

Substituting from (5) and (6) in (4), we get

$$\frac{\partial^2 z}{\partial x^2} = 4x^2 \frac{\partial^2 z}{\partial y^2} + 2 \cdot \frac{1}{2x} \frac{\partial z}{\partial x}$$

$$\text{or} \quad x \frac{\partial^2 z}{\partial x^2} = 4x^3 \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} \quad \text{or} \quad x r = 4x^3 t + p$$

which is the required differential equation of order 2.

3. The Origin (Derivation) of Second Order Partial Differential Equation of Special Type :

[Meerut 2007]

Let
$$z = f(u) + g(v) + w \quad \dots(1)$$

where f and g are functions of u and v respectively and u , v and w are prescribed functions of x and y .

Differentiating both sides of (1) partially w.r.t. x and y respectively, we get

$$p = f'(u) \cdot u_x + g'(v) \cdot v_x + w_x \quad \dots(2)$$

and
$$q = f'(u) \cdot u_y + g'(v) \cdot v_y + w_y \quad \dots(3)$$

Again differentiating (2) partially w.r.t. x and y and differentiating (3) partially w.r.t. y , we get

$$r = \frac{\partial p}{\partial x} = f''(u) \cdot u_x^2 + g''(v) \cdot v_x^2 + f'(u) \cdot u_{xx} + g'(v) \cdot v_{xx} + w_{xx} \quad \dots(4)$$

$$s = \frac{\partial p}{\partial y} = f''(u) \cdot u_y u_x + g''(v) \cdot v_y v_x + f'(u) \cdot u_{xy} + g'(v) \cdot v_{xy} + w_{xy} \quad \dots(5)$$

and
$$t = \frac{\partial q}{\partial y} = f''(u) \cdot u_y^2 + g''(v) \cdot v_y^2 + f'(u) \cdot u_{yy} + g'(v) \cdot v_{yy} + w_{yy} \quad \dots(6)$$

Equations (2)-(6) can be written as

$$(p - w_x) - f'(u) \cdot u_x - g'(v) \cdot v_x = 0 \quad \dots(7)$$

$$(q - w_y) - f'(u) \cdot u_y - g'(v) \cdot v_y = 0 \quad \dots(8)$$

$$(r - w_{xx}) - f''(u) \cdot u_{xx} - g''(v) \cdot v_{xx} - f''(u) \cdot u_x^2 - g''(v) \cdot v_x^2 = 0 \quad \dots(9)$$

$$(s - w_{xy}) - f''(u) \cdot u_{xy} - g''(v) \cdot v_{xy} - f''(u) \cdot u_x u_y - g''(v) \cdot v_x v_y = 0 \quad \dots(10)$$

$$(t - w_{yy}) - f''(u) \cdot u_{yy} - g''(v) \cdot v_{yy} - f''(u) \cdot u_y^2 - g''(v) \cdot v_y^2 = 0 \quad \dots(11)$$

Eliminating, $f'(u)$, $g'(v)$, $f''(u)$ and $g''(v)$ from equation (7)-(11), we get

$$\begin{vmatrix} p - w_x & u_x & v_x & 0 & 0 \\ q - w_y & u_y & v_y & 0 & 0 \\ r - w_{xx} & u_{xx} & v_{xx} & u_x^2 & v_x^2 \\ s - w_{xy} & u_{xy} & v_{xy} & u_x u_y & v_x v_y \\ t - w_{yy} & u_{yy} & v_{yy} & u_y^2 & v_y^2 \end{vmatrix} = 0$$

Expanding with respect to the first column, we get a differential equation of the form

$$Rr + Ss + Tt + Pp + Qq = W \quad \dots(12)$$

where R, S, T, P, Q and W are known functions of x and y .

Equation (12) is a **linear partial differential equation of second order**, and is obtained by eliminating the functions f and g . This equation (12) is of special type in which the dependent variable z does not occur.

The relation (1) is a solution of the second order linear partial differential equation (12).

Solved Examples

Example 1: Verify that the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \frac{2z}{x^2}$$

is satisfied by, $z = \frac{1}{x} \phi(y-x) + \phi'(y-x)$ where ϕ is an arbitrary function.

Solution: Given, $z = \frac{1}{x} \phi(y-x) + \phi'(y-x)$... (1)

Differentiating (1) twice w.r.t. x and y partially, we get

$$\frac{\partial z}{\partial x} = -\frac{1}{x} \phi'(y-x) - \frac{1}{x^2} \phi(y-x) - \phi''(y-x), \quad \frac{\partial z}{\partial y} = \frac{1}{x} \phi'(y-x) + \phi''(y-x)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{x} \phi''(y-x) + \frac{2}{x^2} \phi'(y-x) + \frac{2}{x^3} \phi(y-x) + \phi'''(y-x)$$

and
$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{x} \phi''(y-x) + \phi'''(y-x)$$

$$\therefore \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \frac{2}{x^2} \left[\phi'(y-x) + \frac{1}{x} \phi(y-x) \right] = \frac{2z}{x^2}$$

... (2)

Hence (1) satisfy equation (2).

Example 2: Show that if f and g are arbitrary functions of a single variable, then $u = f(x-vt+i\alpha y) + g(x-vt-i\alpha y)$ is a solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

provided that $\alpha^2 = 1 - v^2/c^2$.

Solution: Given, $u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y)$... (1)

Differentiating partially w.r.t x, y, z respectively.

$$\begin{aligned}\therefore \frac{\partial u}{\partial x} &= f'(x - vt + i\alpha y) + g'(x - vt - i\alpha y) \\ \frac{\partial u}{\partial y} &= i\alpha f'(x - vt + i\alpha y) - i\alpha g'(x - vt - i\alpha y) \\ \frac{\partial u}{\partial t} &= -v f'(x - vt + i\alpha y) - v g'(x - vt - i\alpha y)\end{aligned}$$

Differentiating again partially w.r.t. x, y, z respectively.

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y) \\ \frac{\partial^2 u}{\partial y^2} &= i^2 \alpha^2 f''(x - vt + i\alpha y) + i^2 \alpha^2 g''(x - vt - i\alpha y)\end{aligned}$$

and $\frac{\partial^2 u}{\partial t^2} = v^2 f''(x - vt + i\alpha y) + v^2 g''(x - vt - i\alpha y)$

$$\begin{aligned}\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= (1 - \alpha^2) [f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y)] \\ &= \frac{(1 - \alpha^2)}{v^2} \frac{\partial^2 u}{\partial t^2}\end{aligned}$$

or $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$... (2)

provided $\alpha^2 = 1 - v^2/c^2$.

Hence (1) is the solution of equation (2).

Exercise 9.1

Eliminating the arbitrary constants from the following equations, form the partial differential equations.

1. $z = A e^{-l^2 t} \cos mx$

2. $z = a x + b y + c x y$

3. $z = A e^{pt} \sin px$

4. $z = a x + b y^2 + c$

5. $a x^2 + b y^2 + c z^2 = 1$

6. $z = a x^2 + b x y + c y^2$

7. $z^2 = a x^3 + b y^3 + a b$

Form the partial differential equations by eliminating the arbitrary functions from the following equations.

8. $z = f(xy) + g(x/y)$

9. $z = f(x + iy) + g(x - iy)$ [Bilaspur 2004]

10. $z = f(x - at) + F(x + at)$

[Jabalpur 2002]

11. $z = f(x \cos \alpha + y \sin \alpha - at) + \phi(x \cos \alpha + y \sin \alpha + at)$

12. Show that $z = f(x + ay) + g(x - ay)$, satisfy the partial differential equation $t = az$.

Answers 9.1

1.	$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial t}$	2.	$px + qy - xys = 0$ or $r = 0$ or $t = 0$
3.	$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$	4.	$q = ty$
5.	$zy \frac{\partial^2 z}{\partial y^2} + y \left(\frac{\partial z}{\partial y} \right)^2 - z \frac{\partial z}{\partial y} = 0$	6.	$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 2z$.
7.	$9x^2 y^2 z = 6x^3 y^2 \frac{\partial z}{\partial x} + 6x^2 y^3 \frac{\partial z}{\partial y} + 4z \cdot \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}$		
8.	$x^2 r - y^2 t + xp - yq = 0$	9.	$r + t = 0$
10.	$a^2 \frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}$	11.	$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{a^2} \frac{\partial^2 z}{\partial t^2}$

9.3 Special Types of Second Order Partial Differential Equations

Recall that the general linear partial differential equation of second order in two independent variables x and y with z as dependent variable is given by

$$Rr + Ss + Tt + Pp + Qq + Zz = F$$

where R, S, T, P, Q, Z and F are functions of x and y only and not all R, S, T are zero.

1. Type I : Under this type we consider equations of the following forms :

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{F}{R} = f_1(x, y)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{F}{S} = f_2(x, y), \quad t = \frac{\partial^2 z}{\partial y^2} = \frac{F}{T} = f_3(x, y)$$

Equations of this type are solved by direct integration of both sides partially w.r.t. x or y as possible. When integration partially w.r.t. x is done then y is treated as constant and the constant of integration is taken as some function of y and when integration partially w.r.t. y is done then x is treated as constant and the constant of integration is taken as some function of x .

Solved Examples

Example 1: Solve $ar = x y$.

[Vikram 2003; Meerut 2001]

Solution: Given, $a \frac{\partial^2 z}{\partial x^2} = x y$

Integrating both sides partially w.r.t. ' x ', treating y constant, we get

$$a \frac{\partial z}{\partial x} = \frac{1}{2} x^2 y + f(y)$$

Again integrating both sides partially w.r.t. ' x ', treating y constant, we get

$$az = (x^3/6)y + x f(y) + \phi(y)$$

where $f(y)$ and $\phi(y)$ are arbitrary functions of y .

Example 2: Solve $xy_s = 1$.

[Agra 2007; Vikram 2005; Rewa 2004]

Solution: Given, $s = \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{x y}$

Integrating both sides partially w.r.t. ' x ' treating y constant, we get

$$\frac{\partial z}{\partial y} = \frac{1}{y} \log x + f(y)$$

Again integrating both sides partially w.r.t. ' y ', treating x constant, we get

$$z = \log y \cdot \log x + \int f(y) dy + \psi(x)$$

or $z = \log y \cdot \log x + \phi(y) + \psi(x)$

where $\phi(y)$ and $\psi(x)$ are arbitrary functions.

Example 3: Solve $t = \sin(xy)$.

Solution: Given, $t = \frac{\partial^2 z}{\partial y^2} = \sin(xy)$

Integrating partially w.r.t. 'y', $\frac{\partial z}{\partial y} = -\frac{1}{x} \cos(xy) + f(x)$

Again integrating partially w.r.t. 'y', $z = -\frac{1}{x^2} \sin(xy) + yf(x) + \phi(x)$

where $f(x)$ and $\phi(x)$ are arbitrary functions.

Example 4: Solve $\log s = x + y$.

Solution: The given equation can be written as

$$s = \frac{\partial^2 z}{\partial x \partial y} = e^{x+y}.$$

Integrating partially w.r.t. 'x', $\frac{\partial z}{\partial y} = e^{x+y} + f(y)$

Again integrating partially w.r.t. 'y', $z = e^{x+y} + \int f(y) dy + \psi(x)$

or $z = e^{x+y} + \phi(y) + \psi(x)$

where $\phi(y)$ and $\psi(x)$ are arbitrary functions.

Example 5: Solve $xy^2 s = 1 - 2x^2 y$.

Solution: The given equation can be written as

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{1-2x^2 y}{xy^2} = \frac{1}{xy^2} - 2 \frac{x}{y}$$

Integrating partially w.r.t. 'x', (treating y constant), we get

$$\frac{\partial z}{\partial y} = \frac{1}{y^2} \log x - \frac{x^2}{y} + f(y)$$

Again integrating partially w.r.t. y, we get

$$z = -\frac{1}{y} \log x - x^2 \log y + \int f(y) dy + \psi(x)$$

or $z = -\frac{1}{y} \log x - x^2 \log y + \phi(y) + \psi(x)$

where $\phi(y)$ and $\psi(x)$ are arbitrary functions.

Example 6: Solve $xr = (n-1)p$.

Solution: The given equation can be written as

$$\frac{\frac{\partial p}{\partial x}}{p} = \frac{n-1}{x}$$

Integrating partially w.r.t. x , we get

$$\log p = (n-1) \log x + \log f(y) \quad \text{or} \quad p = \frac{\partial z}{\partial x} = x^{n-1} f(y)$$

Again integrating w.r.t. ' x ', we get $z = \frac{x^n}{n} f(y) + \psi(y)$

which can also be written as

$$z = x^n \phi(y) + \psi(y), \text{ where } \phi(y) = \frac{1}{n} f(y)$$

where $\phi(y)$ and $\psi(y)$ are arbitrary functions.

2. Type II : Under this type we consider equations of the following forms

$$Rr + Pp = F \quad \text{i.e.} \quad \frac{\partial p}{\partial x} + \frac{P}{R} p = \frac{F}{R}, \quad \text{Here } S=0=T=Q=Z, R \neq 0, P \neq 0$$

$$Ss + Pp = F \quad \text{i.e.} \quad \frac{\partial p}{\partial y} + \frac{P}{S} p = \frac{F}{S}, \quad \text{Here } R=0=T=Q=Z, S \neq 0, P \neq 0$$

$$Ss + Qq = F \quad \text{i.e.} \quad \frac{\partial q}{\partial x} + \frac{Q}{S} q = \frac{F}{S}, \quad \text{Here } R=0=T=P=Z, S \neq 0, Q \neq 0$$

$$\text{and} \quad Tt + Qq = F \quad \text{i.e.} \quad \frac{\partial q}{\partial y} + \frac{Q}{T} q = \frac{F}{T}, \quad \text{Here } R=0=S=P=Z, T \neq 0, Q \neq 0$$

These equations are linear equations of the form

$$\frac{dy}{dx} + Py = Q \quad \text{whose I.F.} = e^{\int P dx}$$

$$\text{and the solution is} \quad y e^{\int P dx} = \int Q e^{\int P dx} dx + C$$

Solved Examples

Example 1: Solve $xr + p = 9x^2 y^3$.

Solution: The given equation can be written as

$$\frac{\partial p}{\partial x} + \frac{1}{x} p = 9x y^3$$

which is linear in p , whose I.F. = $e^{\int \frac{1}{x} dx} = e^{\log x} = x$

$$\therefore p \cdot x = \int (9xy^3) \cdot x dx + f(y) = 3x^3 y^3 + \phi(y) \quad (\text{Treating } y \text{ constant})$$

$$\text{or } p = \frac{\partial z}{\partial x} = 3x^2 y^3 + \frac{1}{x} \phi(y)$$

Integrating partially w.r.t. 'x', we get

$$z = x^3 y^3 + \phi(y) \cdot \log x + \psi(y)$$

where ϕ and ψ are arbitrary functions of y .

Aliter : The given equation can also be written as

$$\frac{\partial}{\partial x} (x p + 1) = 9x^2 y^3$$

Integrating partially w.r.t. 'x', (treating y constant) we get

$$x p + 1 = 3x^3 y^3 + f(y)$$

$$\text{or } p = \frac{\partial z}{\partial x} = 3x^2 y^3 + \frac{1}{x} \{f(y) - 1\}$$

Again integrating partially w.r.t. 'x', (treating y constant), we get

$$z = x^3 y^3 + \{f(y) - 1\} \log x + \psi(y)$$

$$\text{or } z = x^3 y^3 + \phi(y) \log x + \psi(y), \text{ where } \phi(y) = f(y) - 1$$

which is the required solution.

Example 2: Solve $ys + p = \cos(x + y) - y \sin(x + y)$.

Solution: The given equation can be written as

$$\frac{\partial p}{\partial y} + \frac{1}{y} p = \frac{1}{y} [\cos(x + y) - y \sin(x + y)]$$

which is linear in p , whose I.F. = $e^{\int \frac{1}{y} dy} = e^{\log y} = y$.

$$\begin{aligned}\therefore y \cdot p &= \int y \cdot \frac{1}{y} [\cos(x+y) - y \sin(x+y)] dy + f(x) \\ &= \int \cos(x+y) dy - \int y \sin(x+y) dy + f(x) \\ &= \sin(x+y) - [-y \cos(x+y) + \sin(x+y)] + f(x) \\ &= y \cos(x+y) + f(x)\end{aligned}$$

or
$$p = \frac{\partial z}{\partial x} = \cos(x+y) + \frac{1}{y} f(x)$$

Integrating partially w.r.t. 'x', (treating y constant), we get

$$z = \sin(x+y) + \frac{1}{y} \int f(x) dx + \phi(y)$$

or
$$yz = y \sin(x+y) + F(x) + \psi(y), \text{ where } F(x) = \int f(x) dx, \psi(y) = y \phi(y)$$

when $F(x)$ and $\psi(y)$ are arbitrary functions.

Aliter : The given equation can also be written as

$$\frac{\partial}{\partial x} (yq + z) = \cos(x+y) - y \sin(x+y)$$

Integrating partially w.r.t. 'x', $yq + z = \sin(x+y) + y \cos(x+y) + f(y)$

or
$$\frac{\partial}{\partial y} (yz) = \frac{\partial}{\partial y} \{y \sin(x+y)\} + f(y)$$

Integrating partially w.r.t. 'y',

$$yz = y \sin(x+y) + \int f(y) dy + F(x)$$

or
$$yz = y \sin(x+y) + \psi(y) + F(x)$$

where $F(x)$ and $\psi(y)$ are arbitrary functions.

Example 3: Solve $xy^2 - qy = x^2$.

Solution: The given equation can be written as

$$\frac{\partial q}{\partial x} - \frac{1}{x} q = \frac{x}{y}$$

which is linear in q , whose I.F. = $e^{\int -\frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$

$$\therefore q \frac{1}{x} = \int \frac{x}{y} \frac{1}{x} dx + f(y) = \frac{1}{y} \int dx + f(y) = \frac{x}{y} + f(y)$$

$$\text{or } q = \frac{\partial z}{\partial y} = \frac{x^2}{y} + x f(y)$$

Integrating partially w.r.t. 'y', (treating x constant), we get

$$z = x^2 \log y + x \int f(y) dy + \psi(x)$$

$$\text{or } z = x^2 \log y + x \phi(y) + \psi(x), \quad \text{where } \phi(y) = \int f(y) dy$$

where $\phi(y)$ and $\psi(x)$ are arbitrary functions.

Example 4: Solve $2yq + y^2 t = 1$.

Solution: The given equation can be written as

$$\frac{\partial q}{\partial y} + \frac{2}{y} q = \frac{1}{y^2}$$

which is linear in q , whose I.F. $= e^{\int \frac{2}{y} dy} = e^{2 \log y} = y^2$

$$\therefore q \cdot y^2 = \int \frac{1}{y^2} \cdot y^2 dy + f(x) = y + f(x)$$

$$\text{or } q = \frac{\partial z}{\partial y} = \frac{1}{y} + \frac{1}{y^2} f(x)$$

Integrating partially w.r.t. 'y', (treating x constant), we get

$$z = \log y - \frac{1}{y} f(x) + \phi(x)$$

where $f(x)$ and $\phi(x)$ are arbitrary functions.

Aliter : The given differential equation can also be written as

$$\frac{\partial}{\partial y} (y^2 q) = 1$$

Integrating partially w.r.t. 'y', $y^2 q = y + f(x)$

$$\text{or } q = \frac{\partial z}{\partial y} = \frac{1}{y} + \frac{1}{y^2} f(x)$$

Again integrating partially w.r.t. 'y', we get

$$z = \log y - \frac{1}{y} f(x) + \phi(x).$$

Example 5: Solve $s x + q = 4x + 2y + 2$.

Solution: The given equation can be written as

$$\frac{\partial q}{\partial x} + \frac{1}{x} q = \frac{4x + 2y + 2}{x}$$

which is linear in q , whose I.F. = $e^{\int \frac{1}{x} dx} = e^{\log x} = x$.

$$\begin{aligned} \therefore q \cdot x &= \int x \cdot \{(4x + 2y + 2)/x\} dx + f(y) \\ &= \int (4x + 2y + 2) dx + f(y) = 2x^2 + 2(y+1)x + f(y) \end{aligned}$$

$$\text{or } q = \frac{\partial z}{\partial y} = 2x + 2(y+1) + \frac{1}{x} f(y)$$

Integrating partially w.r.t. y , we get

$$z = 2x \cdot y + y^2 + 2y + \frac{1}{x} \phi(y) + \psi(x), \quad \text{where } \int f(y) dy = \phi(y).$$

$$\text{or } xz = 2x^2 y + x y^2 + 2xy + \phi(y) + F(x), \quad \text{where } F(x) = x \psi(x).$$

where $\phi(y)$ and $F(x)$ are arbitrary functions.

Aliter : Hint : The given differential equation can also be written as

$$\frac{\partial}{\partial x} (xq) = 4x + 2y + 2$$

Now proceed as usual.

Example 6: Solve $t - xq = -\sin y - x \cos y$.

Solution: The given equation can be written as

$$\frac{\partial q}{\partial y} - xq = -\sin y - x \cos y$$

which is linear in q (treating x constant), whose I.F. = $e^{\int -x dy} = e^{-x y}$

$$\begin{aligned} \therefore q \cdot e^{-x y} &= \int (-\sin y - x \cos y) e^{-x y} dy + f(x) \\ &= -\int e^{-x y} \sin y dy - x \int e^{-x y} \cos y dy + f(x) \end{aligned}$$

$$\begin{aligned}
 &= -\left[-e^{-x}y \cos y + \int (-x)e^{-x}y \cos y dy\right] - x \int e^{-x}y \cos y dy + f(x) \\
 &= e^{-x}y \cos y + f(x)
 \end{aligned}$$

or $q = \frac{\partial z}{\partial y} = \cos y + e^x y f(x)$

Integrating partially w.r.t. y , we get

$$z = \sin y + \frac{1}{x} e^x y f(x) + \phi(x)$$

where $f(x)$ and $\phi(x)$ are arbitrary functions.

Example 7: Solve $xs + q - xp - z = (1 - y)(1 + \log x)$.

Solution: The given equation can be written as

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{1}{x} \frac{\partial z}{\partial y} - \frac{\partial z}{\partial x} - \frac{z}{x} = \frac{1-y}{x} (1 + \log x)$$

or $\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} + \frac{z}{x} \right) - \left(\frac{\partial z}{\partial x} + \frac{z}{x} \right) = \frac{1-y}{x} (1 + \log x)$

Taking $\frac{\partial z}{\partial x} + \frac{z}{x} = u$, we get

$$\frac{\partial u}{\partial y} - u = (1-y) \frac{1 + \log x}{x}$$

which is linear in u (treating x constant), whose I.F. = $e^{\int -dy} = e^{-y}$

$$\begin{aligned}
 \therefore u \cdot e^{-y} &= \int (1-y)e^{-y} \left(\frac{1 + \log x}{x} \right) dy + f(x) \\
 &= \left(\frac{1 + \log x}{x} \right) \int (1-y)e^{-y} dy + f(x) \\
 &= \left(\frac{1 + \log x}{x} \right) \left[-(1-y)e^{-y} + \int (-1)e^{-y} dy \right] + f(x) \\
 &= \left(\frac{1 + \log x}{x} \right) [-(1-y)e^{-y} + e^{-y}] + f(x)
 \end{aligned}$$

or $u = \left(\frac{1 + \log x}{x} \right) \cdot y + e^y f(x)$

or $\frac{\partial z}{\partial x} + \frac{z}{x} = \left(\frac{1 + \log x}{x} \right) y + e^y f(x)$

which is linear in z (treating y constant), whose I. F. = $e^{\int \frac{1}{x} dx} = e^{\log x} = x$

$$\therefore xz = \int x \left\{ \left(\frac{1 + \log x}{x} \right) y + e^y f(x) \right\} dx + \psi(y)$$

$$\text{or } xz = y \int (1 + \log x) dx + e^y \int f(x) dx + \psi(y)$$

$$\text{or } xz = y \left[(1 + \log x) \cdot x - \int \frac{1}{x} \cdot x dx \right] + e^y \phi(x) + \psi(y), \text{ where } \phi(x) = \int f(x) dx$$

$$\text{or } xz = y[(1 + \log x)x - x] + e^y \phi(x) + \psi(y)$$

$$\text{or } xz = x y \log x + e^y \phi(x) + \psi(y)$$

where $\phi(x)$ and $\psi(y)$ are arbitrary constants.

3. Type III : Under this type we consider equations of the forms

$$Rr + Ss + Pp = F \quad \text{i.e.,} \quad R \frac{\partial p}{\partial x} + S \frac{\partial p}{\partial y} = F - Pp,$$

$$\text{Here} \quad T = 0 = Q = Z, \quad R, S, P \neq 0$$

$$\text{and} \quad Ss + Tt + Qq = F \quad \text{i.e.,} \quad S \frac{\partial q}{\partial x} + T \frac{\partial q}{\partial y} = F - Qq,$$

$$\text{Here} \quad R = 0 = P = Z, \quad S, T, Q \neq 0.$$

These equations are linear partial differential equations of order one with p or q as dependent variable and x, y as independent variables. Such equations are solved for p or q (as the case may be) by Lagrange's method see 7.7 page 245.

Solved Examples

Example 1: Solve $xy r + x^2 s - y p = x^3 e^y$.

Solution: The given equation can be written as

$$xy \frac{\partial p}{\partial x} + x^2 \frac{\partial p}{\partial y} = x^3 e^y + y p \quad \dots(1)$$

which is Lagrange's equation in p and the Lagrange's auxiliary equations are

$$\frac{dx}{xy} = \frac{dy}{x^2} = \frac{dp}{x^3 e^y + y p}$$

$$\text{Taking 1st and 2nd fractions, we get } x dx - y dy = 0, \text{ Integrating, } x^2 - y^2 = c_1 \quad \dots(2)$$

Taking 2nd and 3rd fractions, we get

$$\frac{dp}{dy} = \frac{x^3 e^y + y p}{x^2} = x e^y + y p / x^2$$

$$\text{or} \quad \frac{dp}{dy} - \frac{y}{y^2 + c_1} p = \sqrt{(y^2 + c_1)} e^y \quad \because x = \sqrt{(y^2 + c_1)}, \text{ from (2)}$$

which is linear in p , whose

$$\text{I.F.} = e^{\int -\frac{y}{y^2 + c_1} dy} = e^{-\frac{1}{2} \log(y^2 + c_1)} = (y^2 + c_1)^{-1/2}$$

$$\therefore p \cdot (y^2 + c_1)^{-1/2} = \int \sqrt{(y^2 + c_1)} e^y \cdot (y^2 + c_1)^{-1/2} dy + c_2$$

$$\text{or} \quad \frac{p}{\sqrt{y^2 + c_1}} = \int e^y dy + c_2 = e^y + c_2$$

$$\text{or} \quad \frac{p}{x} = e^y + c_2 \quad \dots(3)$$

From (2) and (3), the solution of (1) is

$$\frac{p}{x} = e^y + f(x^2 - y^2)$$

$$\text{or} \quad p = \frac{\partial z}{\partial x} = x e^y + x f(x^2 - y^2)$$

Integrating partially w.r.t. 'x', (treating y constant) we get

$$z = e^y \int x dx + \int x f(x^2 - y^2) dx + \psi(y)$$

$$\text{or} \quad z = \frac{1}{2} x^2 e^y + \frac{1}{2} \int 2x \cdot f(x^2 - y^2) dx + \psi(y)$$

$$\text{or} \quad z = \frac{1}{2} x^2 e^y + \frac{1}{2} \phi(x^2 - y^2) + \psi(y)$$

$$\text{or} \quad 2z = x^2 e^y + \phi(x^2 - y^2) + F(y) \quad \text{where } 2\psi(y) = F(y)$$

where ϕ and F are arbitrary functions.

Example 2: Solve $r + \frac{y}{x} s = 15x y^2$.

Solution: The given equation can be written as

$$\frac{\partial p}{\partial x} + \frac{y}{x} \frac{\partial p}{\partial y} = 15xy^2 \quad \dots(1)$$

which is Lagrange's equation in p and the Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{y/x} = \frac{dp}{15xy^2}$$

Taking 1st and 2nd fractions, we get

$$\frac{dy}{y} = \frac{dx}{x}, \text{ Integrating } \log y - \log x = \log c_1 \quad \text{i.e., } \frac{y}{x} = c_1 \quad \dots(2)$$

Taking 1st and 3rd fractions, $dp = 15x y^2 dx = 15c_1^2 x^3 dx$, using (2)

$$\text{Integrating, } p = \frac{15}{4} c_1^2 x^4 + c_2 \Rightarrow p - \frac{15}{4} \left(\frac{y}{x}\right)^2 x^4 = c_2$$

$$\text{or } p - \frac{15}{4} x^2 y^2 = c_2 \quad \dots(3)$$

From (2) and (3), the solution of (1) is

$$p - \frac{15}{4} x^2 y^2 = f(y/x)$$

$$\text{or } p = \frac{\partial z}{\partial x} = \frac{15}{4} x^2 y^2 + f(y/x)$$

Integrating partially w.r.t. 'x', (treating y constant), we get

$$\begin{aligned} z &= \frac{5}{4} x^3 y^2 + \int f(y/x) dx + \psi(y) \\ &= \frac{5}{4} x^3 y^2 + y \int \frac{1}{(-y^2/x^2)} f(y/x) d(y/x) + \psi(y) \\ &= \frac{5}{4} x^3 y^2 + y \int \left(-\frac{1}{t^2}\right) f(t) dt + \psi(y) \quad \text{Putting } y/x = t \end{aligned}$$

$$\text{or } z = \frac{5}{4} x^3 y^2 + y \phi(y/x) + \psi(y), \quad \text{where } \int \left(-\frac{1}{t^2}\right) f(t) dt = \phi(t) = \phi(y/x)$$

which is the required solution of (1), where $\phi\left(\frac{y}{x}\right)$ and $\psi(y)$ are arbitrary functions.

Example 3: Solve $xr + ys + p = 10xy^3$.

Solution: The given equation can be written as

$$x \frac{\partial p}{\partial x} + y \frac{\partial p}{\partial y} = -p + 10xy^3 \quad \dots(1)$$

which is Lagrange's equation in p and the Lagrange's auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dp}{-p + 10xy^3}$$

Taking 1st and 2nd fractions, we get $\frac{dx}{x} = \frac{dy}{y}$

Integrating, $\log x - \log y = \log c_1 \Rightarrow x/y = c_1$... (2)

Taking from 2nd and 3rd fractions, we get

$$\frac{dp}{dy} + \frac{1}{y}p = 10x y^2 = 10c_1 y^3$$

using (2)

which is linear in p , whose I.F. = $e^{\int \frac{1}{y} dy} = e^{\log y} = y$

$$\therefore y \cdot p = \int y \cdot 10c_1 y^3 dy + c_2 = 2c_1 y^5 + c_2 = 2x y^4 + c_2$$

$$\text{or } p y - 2x y^4 = c_2$$

... (3)

From (2) and (3), the solution of (1) is $p y - 2x y^4 = f(x/y)$

$$\text{or } p = \frac{\partial z}{\partial x} = 2xy^3 + \frac{1}{y} f(x/y)$$

Integrating partially w.r.t. x (treating y constant), we get

$$\begin{aligned} z &= x^2 y^3 + \int \left(\frac{1}{y} \right) f\left(\frac{x}{y} \right) dx + \psi(y) \\ &= x^2 y^3 + \int f(t) dt + \psi(y), \quad \text{Putting } \frac{x}{y} = t, \frac{1}{y} dx = dt \end{aligned}$$

$$\text{or } z = x^2 y^3 + \phi(x/y) + \psi(y), \quad \text{where } \int f(t) dt = \phi(t) = \phi(x/y)$$

which is required solution where $\phi(x/y)$ and $\psi(y)$ are arbitrary functions.

Example 4: Solve $s y - 2x r - 2p = 6x y$.

Solution: The given equation can be written as

$$2x \frac{\partial p}{\partial x} - y \frac{\partial p}{\partial y} = -(6x y + 2p)$$

... (1)

which is Lagrange's equation in p , and the Lagrange's auxiliary equations are

$$\frac{dx}{2x} = \frac{dy}{-y} = \frac{dp}{-(6xy + 2p)}$$

Taking 1st and 2nd fractions, we get

$$\frac{dx}{x} + \frac{2}{y} dy = 0. \text{ Integrating } \log x + 2 \log y = \log c_1 \Rightarrow x y^2 = c_1 \quad \dots(2)$$

Taking, 2nd and 3rd fractions, we get $\frac{dp}{dy} - \frac{2}{y} p = 6x = \frac{6c_1}{y^2}$

which is linear in p , whose I.F. = $e^{\int -\frac{2}{y} dy} = e^{-2 \log y} = \frac{1}{y^2}$

$$\begin{aligned} \therefore p \cdot \left(\frac{1}{y^2} \right) &= \int \left(\frac{6c_1}{y^2} \right) \cdot \left(\frac{1}{y^2} \right) dy + c_2 = 6c_1 \int \frac{1}{y^4} dy + c_2 \\ &= 6c_1 \left(-\frac{1}{3y^3} \right) + c_2 = -2 \frac{xy^2}{y^3} + c_2 \end{aligned}$$

$$\text{or } \frac{p}{y^2} + \frac{2x}{y} = c_2 \quad \dots(3)$$

From (2) and (3), the solution of (1) is

$$\frac{p}{y^2} + \frac{2x}{y} = f(xy^2)$$

$$\text{or } p = \frac{\partial z}{\partial x} = -2xy + y^2 f(xy^2)$$

Integrating w.r.t. 'x', (treating y constant), we get

$$z = -x^2 y + \int y^2 f(xy^2) dx + \psi(y) = -x^2 y + \int f(t) dt + \psi(y)$$

$$\text{Putting } xy^2 = t, y^2 dx = dt$$

$$\text{or } z = -x^2 y + \phi(xy^2) + \psi(y), \text{ where } \int f(t) dt = \phi(t) = \phi(xy^2)$$

which is the required solution of (1) where

$\phi(xy^2)$, $\psi(y)$ are arbitrary functions.

Example 5: Solve $yt + xs + q = 8yx^2 + 9y^2$.

Solution: The given equation can be written as

$$x \frac{\partial q}{\partial x} + y \frac{\partial q}{\partial y} = -q + 8yx^2 + 9y^2 \quad \dots(1)$$

which is Lagrange's equation in q and the Lagrange's auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dq}{-q + 8yx^2 + 9y^2}$$

Taking 1st and 2nd fractions, we have $\frac{dy}{y} - \frac{dx}{x} = 0$

Integrating, $\log y - \log x = \log c_1 \Rightarrow y/x = c_1 \quad \dots(2)$

Taking 2nd and 3rd fractions, we get

$$\frac{dq}{dy} + \frac{1}{y}q = 8x^2 + 9y = \frac{8y^2}{c_1^2} + 9y \quad \text{using (2)}$$

or
$$\frac{dq}{dy} + \frac{1}{y}q = \frac{8}{c_1^2}y^2 + 9y$$

which is linear in q , whose I.F. = $e^{\int \frac{1}{y} dy} = e^{\log y} = y$

$$\therefore y \cdot q = \int \left\{ \left(\frac{8}{c_1^2} \right) y^2 + 9y \right\} y dy + c_2 = \left(\frac{8}{c_1^2} \right) (y^4/4) + 3y^3 + c_2$$

$$\text{or} \quad yq - 2x^2y^2 - 3y^3 = c_2 \quad \dots(3)$$

From (2) and (3), solution of (1) is

$$yq - 2x^2y^2 - 3y^3 = f(y/x)$$

$$\text{or} \quad q = \frac{\partial z}{\partial y} = 2x^2y + 3y^2 + (1/y)f(y/x)$$

Integrating w.r.t. 'y', (treating x constant), we get

$$z = x^2y^2 + y^3 + \int \left(\frac{1}{y} \right) f(y/x) dy + \psi(x) \quad \text{(Treating } x \text{ constant)}$$

$$\text{or} \quad z = x^2y^2 + y^3 + \int (1/t)f(t)dt + \psi(x) \quad \text{Putting } y/x = t, (1/x)dy = dt$$

$$\text{or} \quad z = x^2y^2 + y^3 + \phi(y/x) + \psi(x), \quad \text{where } \int (1/t)f(t)dt = \phi(t) = \phi(y/x)$$

which is the required solution of (1), where $\phi(y/x)$ and $\psi(x)$ are arbitrary functions.

Example 6: Solve $p + r + s = 1$.

[Meerut 2005, 06 (B.P.); Kanpur 2000, 04]

Solution: The given equation can be written as

$$\frac{\partial p}{\partial x} + \frac{\partial p}{\partial y} = 1 - p \quad \dots(1)$$

which is Lagrange's equation in p , and the Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dp}{1-p}$$

$$\text{Taking 1st and 2nd fractions, we get } dx - dy = 0 \Rightarrow x - y = c_1 \quad \dots(2)$$

Taking 2nd and 3rd fractions, we get

$$dy = \frac{dp}{1-p}$$

$$\text{Integrating } y = -\log(1-p) + \log c_2 \Rightarrow 1-p = c_2 e^{-y} \quad \dots(3)$$

From (2) and (3), the solution of (1) is

$$1-p = e^{-y} f(x-y)$$

$$\text{or } p = \frac{\partial z}{\partial x} = 1 - e^{-y} f(x-y)$$

Integrating partially w.r.t. 'x', (treating y constant), we get

$$z = x - e^{-y} \int f(x-y) dx + \psi(y)$$

$$\text{or } z = x + e^{-y} \phi(x-y) + \psi(y)$$

which is the required solution of (1), where $\phi(x-y)$ and $\psi(y)$ are arbitrary functions.

Aliter : The given equation can also be written as

$$\frac{\partial z}{\partial x} + \frac{\partial p}{\partial x} + \frac{\partial q}{\partial x} = 1$$

Integrating partially w.r.t. 'x', (treating y constant), we get

$$z + p + q = x + f(y)$$

$$\text{or } p + q = x + f(y) - z$$

which is Lagrange's equation and Lagrange's subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{x + f(y) - z}$$

Taking 1st and 2nd fractions, we get $dx - dy = 0 \Rightarrow x - y = c_1$

Taking 2nd and 3rd fractions, we get

$$\frac{dz}{dy} + z = x + f(y) = c_1 + y + f(y)$$

which is linear, whose I.F. = $e^{\int 1 \cdot dy} = e^y$

$$\begin{aligned} \therefore z \cdot e^y &= \int (c_1 + y + f(y)) e^y dy + c_2 \\ &= c_1 \int e^y dy + \int \{y + f(y)\} e^y dy + c_2 = c_1 e^y + F(y) + \phi(x - y) \\ &= (x - y) e^y + F(y) + \phi(x - y), \quad \text{where } F(y) = \int \{y + f(y)\} e^y dy \\ &= x e^y + \{F(y) - y e^y\} + \phi(x - y) \end{aligned}$$

$$\therefore z = x + \{e^{-y} F(y) - y\} + e^{-y} \phi(x - y)$$

$$\text{or } z = x + \psi(y) + e^{-y} \phi(x - y), \quad \text{where } \psi(y) = \{e^{-y} \cdot F(y) - y\}.$$

4. Type IV : Under this type, we consider equations of the forms

$$Rr + Pp + Zz = F \quad \text{i.e.,} \quad R \frac{\partial^2 z}{\partial x^2} + p \frac{\partial z}{\partial x} + Zz = F \quad \dots(1)$$

$$\text{and } Tt + Qq + Zz = F \quad \text{i.e.,} \quad T \frac{\partial^2 z}{\partial y^2} + Q \frac{\partial z}{\partial y} + Zz = F \quad \dots(2)$$

These equations are ordinary linear differential equations of order two with independent variable x in (1) and y in (2).

Solved Examples

Example 1: Solve $r - p - (1/y)(1/y - 1)z = x^2 y - x^2 y^2 + 2x y^3 - 2y^3$.

Solution: Taking $D \equiv \frac{\partial}{\partial x}$, the given differential equation can be written as

$$[D^2 - D - (1/y)(1/y - 1)]z = x^2 y - x^2 y^2 + 2x y^3 - 2y^3 \quad \dots(1)$$

$$\text{or } [D - 1/y][D + (1/y - 1)]z = x^2 y - x^2 y^2 + 2x y^3 - 2y^3 \quad \dots(2)$$

whose $C.F. = e^{x/y} \phi_1(y) + e^{x(1-1/y)} \phi_2(y)$

To find the P.I. of (1), let

$$z = F_1 x^2 + F_2 x + F_3 \quad \dots(3)$$

where F_1, F_2, F_3 are functions of y or constants.

$$\therefore Dz = \frac{\partial z}{\partial x} = 2F_1 x + F_2$$

$$\text{and} \quad D^2 z = \frac{\partial^2 z}{\partial x^2} = 2F_1$$

Putting in (1), we get

$$2F_1 - (2F_1 x + F_2) - (1/y)(1/y - 1)(F_1 x^2 + F_2 x + F_3) = x^2 y - x^2 y^2 + 2x y^3 - 2y^3$$

Equating coefficients of like powers of x , we get

$$-(1/y)[1/y - 1]F_1 = y - y^2 \quad \dots(4)$$

$$-2F_1 - (1/y)[1/y - 1]F_2 = 2y^3 \quad \dots(5)$$

$$\text{and} \quad 2F_1 - F_2 - (1/y)[1/y - 1]F_3 = -2y^3 \quad \dots(6)$$

From (4), $F_1 = -y^3$, $\therefore$ from (5) $F_2 = 0$ and from (6) $F_3 = 0$

$\therefore$ from (3), P.I. $= -y^3 x^2$.

Hence the required solution of (1) is

$$z = C.F. + P.I.$$

or $z = e^{x/y} \phi_1(y) + e^{x(1-1/y)} \phi_2(y) - x^2 y^3$.

Example 2: Solve $t - 2xq + x^2 z = (x - 2) e^{3x+2y}$.

Solution: Taking $D' \equiv \frac{\partial}{\partial y}$, the given equation can be written as

$$(D'^2 - 2x D' + x^2)z = (x - 2) e^{3x+2y}$$

$$\text{or} \quad (D' - x)^2 z = (x - 2) e^{3x+2y}$$

$$\therefore \quad C.F. = e^{xy} \{f_1(x) + x f_2(x)\}$$

and

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D' - x)^2} (x - 2) e^{3x+2y} = (x - 2) \cdot \frac{1}{(D' - x)^2} \cdot e^{3x+2y} \\ &= (x - 2) \cdot \frac{1}{(2 - x)^2} e^{3x+2y} = \frac{e^{3x+2y}}{(x - 2)} \end{aligned}$$

∴ The required solution is

$$z = \text{C.F.} + \text{P.I.} = e^{xy} \{f_1(x) + x f_2(x)\} + \frac{e^{3x+2y}}{x-2}.$$

9.4 Solutions of Equations under Given Conditions

Here, we first find the solution of the given differential equation containing some arbitrary functions of x and y . Then the values of these functions are determined by using the given conditions. By substituting the values of these functions in the solution already obtained, the solution of the equation satisfying the given conditions is obtained.

Solved Examples

Example 1: Find a surface satisfying $t = 6x^3y$, containing the two lines, $y = 0 = z$, $y = 1 = z$. [Kanpur 2001; Sagar 2004]

Solution: The given differential equation can be written as

$$\frac{\partial q}{\partial y} = 6x^3y.$$

Integrating partially w.r.t. ' y ', we get $q = 3x^3y^2 + f(x)$ or $\frac{\partial z}{\partial y} = 3x^3y^2 + f(x)$

Again integrating partially, w.r.t. ' y ',

$$z = x^3y^3 + yf(x) + \phi(y) \quad \dots(1)$$

If (1) contains the two lines $y = 0 = z$ and $y = 1 = z$, then putting in (1), we get

$$0 = \phi(y) \quad \dots(2)$$

and $1 = x^3 + f(x) + \phi(y) \quad \dots(3)$

From (2) and (3), we get $\phi(y) = 0$ and $f(x) = 1 - x^3$

Putting in (1), the required solution is

$$z = x^3y^3 + y(1 - x^3).$$

Example 2: Find the surface passing through the parabolas

$$z = 0, y^2 = 4ax \text{ and } z = 1, y^2 = -4ax$$

and satisfying the equation $xr + 2p = 0$.

[Kanpur 2000, IAS 2006]

Solution: The given differential equation can be written as

$$x \frac{\partial p}{\partial x} + 2p = 0 \quad \text{or} \quad x^2 \frac{\partial p}{\partial x} + 2x p = 0 \quad \text{or} \quad \frac{\partial}{\partial x}(x^2 p) = 0$$

Integrating partially w.r.t. 'x', we get

$$x^2 p = f(y) \quad \text{or} \quad p = \frac{\partial z}{\partial x} = \frac{1}{x^2} f(y)$$

Again integrating partially w.r.t. 'x', we get

$$z = -\frac{1}{x} f(y) + \phi(y) \quad \dots(1)$$

If (1) passes through the parabola $z = 0, y^2 = 4ax$, then putting $z = 0$ and $x = y^2/4a$ in (1), we get

$$0 = -\frac{4a}{y^2} f(y) + \phi(y) \quad \dots(2)$$

If (1) also passes through the parabola $z = 1, y^2 = -4ax$, then putting $z = 1$ and $x = -y^2/4a$ in (1), we get

$$1 = \frac{4a}{y^2} f(y) + \phi(y) \quad \dots(3)$$

Solving (2) and (3), we get $\phi(y) = 1/2$ and $f(y) = y^2/8a$

Putting in (1), the required surface is given by

$$z = -\frac{1}{x} \cdot \frac{y^2}{8a} + \frac{1}{2} \quad \text{or} \quad 8axz = 4ax - y^2.$$

Example 3: Show that a surface of revolution satisfying the differential equation $r = 12x^2 + 4y^2$ and touching the plane $z = 0$ is $z = (x^2 + y^2)^2$. [Agra 2000, 02]

Solution: The given differential equation can be written as

$$r = \frac{\partial^2 z}{\partial x^2} = 12x^2 + 4y^2$$

Integrating partially w.r.t. 'x', we get

$$\frac{\partial z}{\partial x} = p = 4x^3 + 4xy^2 + f(y) \quad \dots(1)$$

Again integrating partially w.r.t. 'x', we get

$$z = x^4 + 2x^2 y^2 + x f(y) + \phi(y) \quad \dots(2)$$

If the surface (2) touches the plane $z = 0$

$$\text{then for } z = 0, \quad \frac{\partial z}{\partial x} = p = 0 \quad \therefore \quad 4x^3 + 4x y^2 + f(y) = 0$$

$$\text{or } f(y) = -(4x^3 + 4x y^2) \quad \dots(3)$$

Since L.H.S. of (3) is a function of y alone while the R.H.S is not a function of y alone. So the equation (3) is satisfied only when both sides are equal to zero.

$$\therefore \quad f(y) = 0 \quad \text{and} \quad 4x^3 + 4x y^2 = 0 \quad \text{i.e., } y^2 = -x^2$$

Putting $f(y) = 0$, $x^2 = -y^2$, and $z = 0$ in (2), we get

$$0 = y^4 - 2y^4 + 0 + \phi(y) \Rightarrow \phi(y) = y^4.$$

Hence from (2), the required surface is given by

$$z = x^4 + 2x^2 y^2 + 0 + y^4 \quad \text{i.e. } z = (x^2 + y^2)^2.$$

Example 4: Find a surface satisfying $r + s = 0$ and touching the elliptic paraboloid $z = 4x^2 + y^2$ along its section by the plane $y = 2x + 1$. [Meerut 2000 (P)]

Solution: The given differential equation can be written as

$$r + s = 0 \quad \dots(1)$$

$$\frac{\partial p}{\partial x} + \frac{\partial q}{\partial x} = 0$$

Integrating partially w.r.t. 'x', we get

$$p + q = f(y)$$

This equation is Lagrange's equation. The Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{f(y)}$$

Taking the 1st and 2nd fractions, we get

$$dy = dx \Rightarrow y - x = c_1 \quad \dots(2)$$

Again taking 2nd and 3rd fractions, we get

$$dz = f(y) dy, \text{ Integrating}$$

$$z = \phi(y) + c_2, \text{ where } \phi(y) = \int f(y) dy \quad \dots(3)$$

From (2) and (3), the solution of equation (1) is

$$z = \phi(y) + \psi(y - x) \quad \dots(4)$$

For (4),
$$p = \frac{\partial z}{\partial x} = -\psi'(y - x)$$

and
$$q = \frac{\partial z}{\partial y} = \phi'(y) + \psi'(y - x)$$

Also for the elliptic paraboloid

$$z = 4x^2 + y^2 \quad \dots(5)$$

$$p = \frac{\partial z}{\partial x} = 8x \quad \text{and} \quad q = \frac{\partial z}{\partial y} = 2y$$

If the surface (4) touches the elliptic paraboloid $z = 4x^2 + y^2$ i.e., (5) along its section by the plane $y = 2x + 1$, then the values of p and q for (4) and (5), for any point on $y = 2x + 1$ should be equal.

$$\therefore -\psi'(y - x) = 8x \quad \dots(6)$$

$$\phi'(y) + \psi'(y - x) = 2y \quad \dots(7)$$

when $y = 2x + 1 \quad \dots(8)$

From (6) and (8), we get

$$\psi'(y - x) = -8(y - x - 1)$$

$$\therefore \psi(y - x) = -8 \left[\frac{1}{2} (y - x)^2 - (y - x) \right] + c_1$$

or
$$\psi(y - x) = -4(y - x)^2 + 8(y - x) + c_1 \quad \dots(9)$$

Again from (6) and (7), we get

$$\phi'(y) = 8x + 2y = \frac{8(y - 1)}{2} + 2y, \quad \text{using (8)}$$

or $\phi'(y) = 6y - 4$

Integrating w.r.t. 'y', we get

$$\phi(y) = 3y^2 - 4y + c_2 \quad \dots(10)$$

Substituting from (9) and (10) in (4), we get

$$z = 3y^2 - 4y - 4(y-x)^2 + 8(y-x) + c, \text{ where } c = c_1 + c_2.$$

or $z = -4x^2 - y^2 + 8xy - 8x + 4y + c \quad \dots(11)$

Now equating the values of z from (5) and (11) when $y = 2x + 1$, we get

$$4x^2 + (2x + 1)^2 = -4x^2 - (2x + 1)^2 + 8x(2x + 1) - 8x + 4(2x + 1) + c$$

or $8x^2 + 4x + 1 = 8x^2 + 4x + 3 + c$

or $c = -2$

Putting $c = -2$ in (11), the required surface is

$$z + 4x^2 + y^2 - 8xy + 8x - 4y + 2 = 0.$$

Example 5: Show that a surface satisfying $r = 6x + 2$ and touching $z = x^3 + y^3$ along its section by the plane $x + y + 1 = 0$ is $z = x^3 + y^3 + (x + y + 1)^2$. [Kurukshetra 2004]

Solution: The given equation can be written as

$$\frac{\partial p}{\partial x} = 6x + 2$$

Integrating partially w.r.t. 'x', we get

$$p = 3x^2 + 2x + f(y)$$

or $\frac{dz}{dx} = 3x^2 + 2x + f(y)$

Again integrating partially w.r.t. 'x', we get

$$z = x^3 + x^2 + x f(y) + \phi(y) \quad \dots(1)$$

For (1), $p = \frac{\partial z}{\partial x} = 3x^2 + 2x + f(y)$

and $q = \frac{\partial z}{\partial y} = x f'(y) + \phi'(y)$

Given $z = x^3 + y^3$... (2)

For (2), $p = \frac{\partial z}{\partial x} = 3x^2$ and $q = \frac{\partial z}{\partial y} = 3y^2$

If (1) touches (2) along its section by the plane $x + y + 1 = 0$, then the values of p and q for (1) and (2) for any point on the plane $x + y + 1 = 0$ will be equal. So we have

$$3x^2 + 2x + f(y) = 3x^2 \quad \dots(3)$$

$$x f'(y) + \phi'(y) = 3y^2 \quad \dots(4)$$

and $x + y + 1 = 0$... (5)

From (3) and (5), we have, $f(y) = -2x = 2(y + 1) \quad \therefore f'(y) = 2$

Putting in (4) $\phi'(y) = 3y^2 - 2x = 3y^2 + 2(y + 1)$

Integrating w.r.t. 'y', $\phi(y) = y^3 + y^2 + 2y + c$

$\therefore$ From (1), we have

$$z = x^3 + x^2 + 2x(y + 1) + y^3 + y^2 + 2y + c \quad \dots(6)$$

Now equating the values of z from (2) and (6),

when $x + y + 1 = 0$ i.e., when $y = -(x + 1)$, we get

$$x^3 - (x + 1)^3 = x^3 + x^2 + 2x(-x) - (x + 1)^3 + (x + 1)^2 - 2(x + 1) + c$$

$$\Rightarrow 0 = -1 + c \Rightarrow c = 1$$

Putting in (6), the required solution is

$$z = x^3 + x^2 + 2x(y + 1) + y^3 + y^2 + 2y + 1$$

or $z = x^3 + y^3 + (x + y + 1)^2.$

Exercise 9.2

Solve the following partial differential equations.

Type I

1. $r = 2y^2.$

2. $t = 2x^2 \cos(xy).$

3. $xr = p.$

[Agra 2007]

4. $s = x/y + a.$

5. $x^2s = \sin y.$

6. $2yq + y^2t = 0.$

7. $r = x^2e^y.$

[Indore 2004]

8. $s = x^2 - y^2.$

9. $xr + 2p = 0.$

Type II

10. $t - xq = x^2.$

11. $yt - q = xy.$

12. $xr + 2p = 0.$

13. $ys - p = xy^2 \cos(xy).$

14. $yt + 2q = (9y + 6)e^{2x+3y}.$

Type III

15. $t + s + q = 0.$

16. $xs + yt + q = 10x^3y.$

17. $s + r = x \cos(x + y).$

18. $2xr - ys + 2p = xy^2.$

19. $s - t = x/y^2.$ [Ravishankar 2005]

Type IV

20. $t - q - (1/x)(1/x - 1)z = xy^2 - x^2y^2 + 2x^3y - 2x^3.$

[Hint : Equation can be written as,

$$(D' - 1/x)\{D' + (1/x - 1)\} = xy^2 - x^2y^2 + 2x^3y - 2x^3, \text{ where } D' \equiv \partial/\partial y.$$

Take P.I. = $f_1y^2 + f_2y + f_3$, f_1, f_2, f_3 functions of x or constant.]

21. Show that a surface passing through the circle $z = 0, x^2 + y^2 = 1$ and satisfying the differential equation $s = 8xy$ is $z = (x^2 + y^2)^2 - 1$.

22. Find the surface satisfying $t = 6x^2y$ containing two lines $y = 0 = z$ and $y = 2 = z$.

23. Find the surface passing through the circle $x^2 + y^2 = a^2, z = 0$ and satisfying the differential equation $s = 8xy$.

Answers 9.2

1.	$z = x^2 y^2 + x f(y) + \phi(y)$	2.	$z = -2 \cos(xy) + y f(x) + \phi(x)$
3.	$z = (1/2)x^2 f(y) + \phi(y)$	4.	$z = (1/2)x^2 \log y + axy + \phi(y) + f(x)$
5.	$z = (1/x) \cos y + f(y) + \phi(x)$	6.	$z = -(1/y) f(x) + \phi(x)$
7.	$z = x^4 e^y / 12 + x f(y) + \phi(y)$	8.	$z = (x^3 y - xy^3) / 3 + f(y) + \phi(x)$
9.	$z = -(1/x) f(y) + \phi(y)$	10.	$z = -xy + e^{xy} f(x) + \phi(x)$
11.	$z = (1/2)xy^2 \log y - (1/4)xy^2 + (1/2)y^2 f(x) + \phi(x)$		
12.	$z = (1/x) f(y) + \phi(y)$	13.	$z = -\cos(xy) + y f(x) + \phi(y)$
14.	$z = e^{2x+3y} - (1/y) f(x) + \phi(x)$	15.	$z e^x = f(x) + \phi(x-y)$
16.	$z x = x^3 y^2 + f(y/x) + \phi(x)$		
17.	$z = -(1/x) \cos(x+y) + (1/4) \sin(x+y) + f(y) + \phi(y-x)$		
18.	$z = (1/4)x^2 y^2 + f(xy^2) + \phi(y)$	19.	$z = x + (x+y) \log y + \phi(x) + \psi(y)$
20.	$z = -x^3 y^2 + e^{y/x} f(x) + e^{y-x} y/x \phi(x)$	21.	$z = (x^2 + y^2)^2 - 1$
22.	$z = x^2 y^3 + y(1-4x^2)$	23.	$z = (x^2 + y^2)^2 - a^2$

9.5 Classification of Linear Partial Differential Equations of Second Order in n-Independent Variables

A linear partial differential equation of second order in n independent variables $x_1, x_2, \dots, x_n$ can be written as

$$\sum_{i=1}^n \sum_{j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} + c_i u = 0 \quad \dots(1)$$

where a_{ij} , b_i and c_i are constants or functions of $x_1, x_2, \dots, x_n$.

Now let $\delta_i \equiv \frac{\partial}{\partial x_i}$, $\delta_i \delta_j \equiv \frac{\partial^2}{\partial x_i \partial x_j}$

where $i = 1, 2, 3, \dots, n$ and $j = 1, 2, 3, \dots, n$.

Then consider the operator

$$\phi = \sum_{i=1}^n \sum_{j=1}^n a_{ij} \delta_i \delta_j \quad \dots(2)$$

for all non-zero real values of $\delta_i \delta_j$ positive or negative.

At any point $(x_1, x_2, \dots, x_n)$, the differential equation (1) is said to be :

1. **Elliptic** : If ϕ is positive for all real values of δ_i and δ_j and it reduces to zero only when all δ_i 's and δ_j 's are zero.
2. **Hyperbolic** : If ϕ can be both positive or negative and
3. **Parabolic** : If the determinant Δ vanishes, where

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & \dots & a_{nn} \end{vmatrix}$$

Note : 1. If all a_{ij} are constants, the differential equation will have the same nature (Elliptic, hyperbolic or parabolic) throughout the region.

2. If a_{ij} are functions of $x_1, x_2, \dots, x_n$, the same differential equation can be elliptic, hyperbolic or parabolic at different points of the region.

9.5.1 Working Method

1. Find the operator $\phi = \sum_{i=1}^n \sum_{j=1}^n a_{ij} \cdot \delta_i \delta_j$

where $\delta_i \equiv \frac{\partial}{\partial x_i}$, and $\delta_i \delta_j \equiv \frac{\partial^2}{\partial x_i \partial x_j}$

For all values of $\delta_i \delta_j$, positive or negative.

Obviously ϕ consists of the operator of all second order partial derivatives in the given equation.

2. Find the values of all a_{ij} $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n$.
3. Find the value of the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & \dots & a_{nn} \end{vmatrix}$$

4. Result

- (i) **Parabolic** : If $\Delta = 0$, then the given equation is parabolic.
- (ii) **Elliptic** : If $\Delta \neq 0$ and ϕ is positive for all non-zero real values of δ_i 's and δ_j 's and it reduces to zero only when all δ_i 's and δ_j 's are zero.
- (iii) **Hyperbolic** : If $\Delta \neq 0$ and ϕ can be both positive or negative.

9.6 Classification of Linear Partial Differential Equation of Second Order in Two Independent Variables

A linear partial differential equation of second order in two independent variables x and y can be written as

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \quad \dots(1)$$

where A is positive.

Here the operator $\phi = A\delta_1^2 + B\delta_1\delta_2 + C\delta_2^2$ where $\delta_1 \equiv \frac{\partial}{\partial x}$, $\delta_2 \equiv \frac{\partial}{\partial y}$ and $\delta_1\delta_2 \equiv \frac{\partial^2}{\partial x \partial y}$

At any point (x, y) , the differential equation (1) is said to be

1. **Elliptic** : If $B^2 - 4AC < 0$
2. **Hyperbolic** : If $B^2 - 4AC > 0$ and
3. **Parabolic** : If $B^2 - 4AC = 0$

Note : 1. If all A, B, C are constants, the differential equation (1) will have the same nature (Elliptic, Hyperbolic or Parabolic) throughout the region.

2. If A, B, C are functions of x, y , the same equation (1) can be elliptic, hyperbolic or parabolic at different points of the region.

It will be elliptic in the region where $B^2 - 4AC < 0$

It will be hyperbolic in the region where $B^2 - 4AC > 0$

It will be parabolic in the region where $B^2 - 4AC = 0$

Solved Examples

Example 1: Classify the following operators.

$$(i) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}$$

$$(ii) \quad \frac{\partial^2 u}{\partial x^2} - 4 \frac{\partial^2 u}{\partial x \partial y} + 2 \frac{\partial^2 u}{\partial y^2}$$

$$(iii) \quad \frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + 4 \frac{\partial^2 u}{\partial y^2}$$

[Meerut 2007 (B.P.)]

and (iv) $2 \frac{\partial^2 u}{\partial r^2} + 4 \frac{\partial^2 u}{\partial x \partial r} + 3 \frac{\partial^2 u}{\partial x^2}$.

[Meerut 2006]

Solution: (i) Here $A = 1$, $B = 1$ and $C = 1$.

$$\therefore B^2 - 4AC = 1 - 4 = -3 < 0 \Rightarrow \text{operator is elliptic.}$$

(ii) Here $A = 1$, $B = -4$ and $C = 2$.

$$\therefore B^2 - 4AC = 16 - 8 = 8 > 0 \Rightarrow \text{Operator is hyperbolic.}$$

(iii) Here $A = 1$, $B = 4$ and $C = 4$.

$$\therefore B^2 - 4AC = 16 - 16 = 0 \Rightarrow \text{Operator is parabolic.}$$

(iv) Here $A = 2$, $B = 4$ and $C = 3$.

$$\therefore B^2 - 4AC = -8 < 0 \Rightarrow \text{operator is elliptic.}$$

Example 2: Find where the following operators are hyperbolic, parabolic and elliptic.

$$(i) \quad \frac{\partial^2 u}{\partial x^2} + t \frac{\partial^2 u}{\partial x \partial t} + x \frac{\partial^2 u}{\partial t^2}$$

$$(ii) \quad x^2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u$$

and (iii) $t \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial^2 u}{\partial x \partial t} + x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$.

[Meerut 2005 (B.P.)]

Solution: (i) Here $A = 1$, $B = t$ and $C = x$ $\therefore B^2 - 4AC = t^2 - 4x$.

$\therefore$ The operator, is hyperbolic, if $t^2 - 4x > 0$ $i.e., \text{ if } t^2 > 4x$

is parabolic, if $t^2 - 4x = 0$ $i.e., \text{ if } t^2 = 4x$

is elliptic if $t^2 - 4x < 0$ $i.e., \text{ if } t^2 < 4x$

(ii) Here $A = -1$, $B = 0$ and $C = x^2$

$$\therefore B^2 - 4AC = 4x^2$$

$\therefore$ The operator, is hyperbolic, if $4x^2 > 0$ *i.e.*, if $|x| > 0$

is parabolic, if $4x^2 = 0$ *i.e.*, if $x = 0$

Since $4x^2$ cannot be negative $\therefore$ the operator cannot be elliptic.

(iii) Here $A = x$, $B = 2$ and $C = t$ $\therefore B^2 - 4AC = 4 - 4tx$.

$\therefore$ The operator, is hyperbolic, if $4 - 4tx > 0$ *i.e.*, if $tx < 1$

is parabolic, if $4 - 4tx = 0$ *i.e.*, if $tx = 1$

is elliptic, if $4 - 4tx < 0$ *i.e.*, if $tx > 0$.

Example 3: In each of the following cases determine whether the equation is hyperbolic, parabolic or elliptic.

(i) $8 \frac{\partial^2 u}{\partial x^2} - 2 \frac{\partial^2 u}{\partial x \partial y} - 3 \frac{\partial^2 u}{\partial y^2} = 0$

(ii) $\frac{\partial^2 u}{\partial x^2} + 6 \frac{\partial^2 u}{\partial x \partial y} + 9 \frac{\partial^2 u}{\partial y^2} = 0$

(iii) $\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + 5 \frac{\partial^2 u}{\partial y^2} = 0.$

[Meerut 2011 (Sem. I)]

Solution: (i) Here $A = 8$, $B = -2$ and $C = -3$

$\therefore B^2 - 4AC = 4 + 96 = 100 > 0 \Rightarrow$ Equation is *hyperbolic*

(ii) Here $A = 1$, $B = 6$ and $C = 9$

$\therefore B^2 - 4AC = 36 - 36 = 0 \Rightarrow$ Equation is *parabolic*.

(iii) Here $A = 1$, $B = 2$ and $C = 5$

$\therefore B^2 - 4AC = 2 - 20 = -18 < 0 \Rightarrow$ Equation is *elliptic*.

Example 4: Classify the following equations.

(i) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ (Laplace's Equation)

[Meerut 2007, 08, 11 (B.P.), 12 (Sem. I)]

(ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$ (Wave Equation)

[Meerut 2007, 09 (B.P.), 10, 11 (Sem. I), 11, (B.P.)]

and (iii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}$ (Heat Equation)

[Meerut 2005, 07]

Solution: (i) Here, the operator $\phi = \delta_1^2 + \delta_2^2 + \delta_3^2$,

and $a_{11} = 1 = a_{22} = a_{33}, a_{12} = 0 = a_{13} = a_{21} = a_{23} = a_{31} = a_{32}$

Now $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$

Since ϕ is positive for all non-zero real values of $\delta_1, \delta_2, \delta_3$ and reduces to zero only when $\delta_1 = 0, \delta_2 = 0, \delta_3 = 0$

Hence, the given equation is *elliptic*.

(ii) The given equation is of second order with four independent variables x, y, z and t .

Here, the operator $\phi = \delta_1^2 + \delta_2^2 + \delta_3^2 - \frac{1}{c^2} \delta_4^2$,

and $a_{11} = 1 = a_{22} = a_{33}, a_{44} = -\frac{1}{c^2}$. All $a_{ij} = 0$ if $i \neq j, i, j = 1, 2, 3, 4$.

Now $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{c^2} \end{vmatrix} = -\frac{1}{c^2} \neq 0$

Since the operator ϕ can be both positive or negative, hence the given equation is *hyperbolic*.

(iii) The given equation is of second order with four independent variables x, y, z and t .

Here the operator $\phi = \delta_1^2 + \delta_2^2 + \delta_3^2$

and $a_{11} = 1 = a_{22} = a_{33}, a_{44} = 0$. All $a_{ij} = 0$ if $i \neq j, i, j = 1, 2, 3, 4$.

Now $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$

Since $\Delta = 0$, hence the given equation is *parabolic*.

Example 5: Classify the following equation

$$u_{xx} + u_{yy} + u_{zz} + 2u_{yz} = 0.$$

Solution: Given equation is partial differential equation of second order with three independent variables x, y and z .

Here the operator $\phi = \delta_1^2 + \delta_2^2 + \delta_3^2 + 2\delta_2\delta_3$

where $\delta_1 \equiv \frac{\partial}{\partial x}$, $\delta_2 \equiv \frac{\partial}{\partial y}$, $\delta_3 \equiv \frac{\partial}{\partial z}$

and $a_{11} = 1 = a_{22} = a_{33}$, $a_{12} = 0 = a_{13}$, $a_{21} = a_{31}$ and $a_{23} = 1 = a_{32}$

Now
$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

Hence the given equation is parabolic.

Example 6: Classify the equation

$$(1-x^2)\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + (1-y^2)\frac{\partial^2 z}{\partial y^2} + 2x\frac{\partial z}{\partial x} + 6x^2y\frac{\partial z}{\partial y} - 6z = 0.$$

Solution: Here the operator

$$\phi = A\delta_1^2 + B\delta_1\delta_2 + C\delta_2^2,$$

where $\delta_1 \equiv \frac{\partial}{\partial x}$, $\delta_2 \equiv \frac{\partial}{\partial y}$, $A = 1-x^2$, $B = -2xy$ and $C = 1-y^2$,

$$\therefore B^2 - 4AC = 4x^2y^2 - 4(1-x^2)(1-y^2) = 4(-1+x^2+y^2)$$

As A , B and C are functions of x and y , the differential equation is hyperbolic in the region, where $B^2 - 4AC > 0$ i.e., where $x^2 + y^2 > 1$, parabolic in the region, where $B^2 - 4AC = 0$ i.e., where $x^2 + y^2 = 1$ and elliptic in the region, where $B^2 - 4AC < 0$ i.e., where $x^2 + y^2 < 1$.

Hence, the given differential equation is respectively hyperbolic, parabolic and elliptic in the region out side the circle, on the circle and inside the circle, whose equation is $x^2 + y^2 = 1$.

Exercise 9.3

Classify the following operators.

1. $u_{xx} + 2u_{xy} + 3u_{yy}$.

2. $4u_{xx} - 4u_{xy} + u_{yy}$.

[Meerut 2008 (B.P.)]

3. $x^2 u_{tt} - 2u_{xx} + 100u$.

4. $t u_{xx} - x u_{xt} + u_{tt} + 2u_t + 5u_x$.

5. $\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + 5 \frac{\partial^2 u}{\partial y^2}$.

[Meerut 2011 (Sem. I)]

6. $\frac{\partial^2 u}{\partial t^2} + 4 \frac{\partial^2 u}{\partial x \partial t} + 4 \frac{\partial^2 u}{\partial x^2}$.

[Meerut 2007 (B.P.)]

Classify the following equations.

7. $u_{xx} + u_{yy} = u_z$.

8. $u_{xx} + u_{yy} = 0$.

9. $3u_{xx} + 2u_{yy} - u_x = 0$.

10. $u_{xx} - 7u_{xy} + 6u_{yy} = 0$.

11. $u_{xx} + u_{yy} - u_{zz} = 0$.

12. $u_{xx} + 2u_{yy} + u_{zz} = 2u_{xy} + 2u_{yz}$.

13. Determine all the points (x, y) at which the equation

$$\frac{\partial^2 u}{\partial x^2} + x \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y^2} - xy \frac{\partial u}{\partial x} = 0 \text{ is}$$

(i) hyperbolic

(ii) parabolic

(iii) elliptic.

[Meerut 2012]

14. Show that the equation

$$(1 - x^2) \frac{\partial^2 u}{\partial x^2} + 2y \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

is hyperbolic for all (x, y) outside the region bounded by the circle $x^2 + y^2 = 1$, parabolic on the boundary of this region and elliptic for all (x, y) inside this region.

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter (a), (b), (c) or (d).

- The partial differential equation obtained by eliminating the arbitrary constants from $z = Ae^{pt} \cos px$ is :
 - $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$
 - $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial t^2} = 0$
 - $\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial t} = 0$
 - $\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial t^2} = 0$
 - The partial differential equation obtained by eliminating the arbitrary constants from $z = ax^2 - bxy + cy^2$ is :
 - $2z = x^2r + 2xys + y^2t$
 - $2z = x^2r - 2xys + y^2t$
 - $z = x^2r + 2xys + y^2t$
 - None of these
 - The partial differential equation obtained by eliminating the arbitrary functions from $x = f(z) + \phi(y)$ is :
 - $ps + qr = 0$
 - $ps - qr = 0$
 - $pr - qs = 0$
 - $pr + qs = 0$
 - The partial differential equation obtained by eliminating the arbitrary functions from $z = f(x) + e^y \phi(x)$ is
 - $q + s = 0$
 - $t + q = 0$
 - $t - q = 0$
 - None of these
 - The solution of the equation $t = \cos(xy)$ is
 - $zx^2 = -\cos(xy) + x^2\{yf(x) + \phi(x)\}$
 - $z = -\cos(xy) + yf(x) + \phi(x)$
 - $3x^2 = \cos(xy) + x^2\{yf(x) + \phi(x)\}$
-

- (d) $z = \cos(xy) + yf(x) + \phi(x)$
6. The solution of the equation $2yq + y^2t = 1$ is :
- (a) $z = 1 + \log y + f(x) + \phi(x)$
 (b) $-[1 + \log y + f(x)] + \phi(x)$
 (c) $z = -[1 + \log y + f(x)] / y + \phi(x)$
 (d) None of these
7. The solution of the equation $r = 2y^2 \cos(xy)$ is :
- (a) $z = 2 \cos(xy) + xf(y) + \phi(y)$
 (b) $z = -2 \cos(xy) + xf(y) + \phi(y)$
 (c) $z = -2 \sin(xy) + xf(y) + \phi(y)$
 (d) None of these
8. The solution of the equation $r - yp = y^2$ is
- (a) $z = xy + f(y)e^{xy} + \phi(y)$
 (b) $z = xy - f(y)e^{xy} + \phi(y)$
 (c) $z = -xy + f(y)e^{xy} + \phi(y)$
 (d) None of these
9. The equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2}$ is
- (a) Elliptic (b) Parabolic
 (c) Hyperbolic (d) None of these
10. The equation $t \frac{\partial^2 u}{\partial x^2} - x \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial u}{\partial t} + 5 \frac{\partial u}{\partial x}$ is elliptic if
- (a) $x^2 < 4t$ (b) $x^2 = 4t$
 (c) $x^2 > 4t$ (d) None of these

Fill in the Blank

Fill in the blank's, so that the following statements are complete and correct.

1. The partial equation obtained by eliminating the arbitrary constants from $z = ax^2 + by^2 + c$, where $a - b = l$ is
2. The partial differential equation obtained by eliminate ϕ and ψ from $z = \phi(y + ax) + \psi(y - ax)$ is $a^2 \frac{\partial^2 z}{\partial y^2}$

3. The partial differential equation obtained by eliminating ϕ and ψ from $z = \phi(x + at) + x\psi(x + at)$ is $a^2 \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = \dots\dots\dots$
 4. The solution of the differential equation $2yq + y^2t = 0$ is $\dots\dots\dots$.
 5. The equation $2 \frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}$ is $\dots\dots\dots$.
 6. The equation $\frac{\partial^2 u}{\partial x^2} + x \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y^2} + xy \frac{\partial u}{\partial x} = 0$ is elliptic if $\dots\dots\dots$.
-

Answers

Exercise 9.3

1.	Elliptic	2.	Parabolic
3.	Hyperbolic if $x \neq 0$ and parabolic if $x = 0$ Operator cannot be elliptic.		
4.	Hyperbolic if $x^2 > 4t$, parabolic if $x^2 = 4t$ and elliptic if $x^2 < 4t$.		
5.	Elliptic	6.	Elliptic
7.	Parabolic	8.	Elliptic
9.	Elliptic	10.	Hyperbolic
11.	Hyperbolic	12.	Hyperbolic
13.	Hyperbolic at all (x, y) where $y < x^2/4$, parabolic at all (x, y) where $y = x^2/4$ and elliptic at all (x, y) where $y > x^2/4$.		

Objective Type Questions

Multiple Choice Questions

1.	(a)	2.	(a)	3.	(b)	4.	(c)
5.	(a)	6.	(c)	7.	(b)	8.	(c)
9.	(c)	10.	(a)				

Fill in the Blank

1.	$r - t = 2l$	2.	$\frac{\partial^2 z}{\partial x^2}$
3.	$2a \frac{\partial^2 z}{\partial x \partial t}$	4.	$z = -(1/y) f(x) + \phi(x)$
5.	hyperbolic	6.	$x^2 < 4y$

Linear Partial Differential Equations with Constant Coefficients

10.1 Introduction

In this chapter we shall discuss *linear partial differential equations* with constant coefficients. Recall that a partial differential equation in which the dependent variable and its derivatives appear only in the first degree and are not multiplied together is called a *linear partial differential equation*. If the coefficients of dependent variable and all its derivatives in a linear partial differential equation are constants, then it is called a *linear partial differential equation with constant coefficients*.

The general form of a linear partial differential equation is

$$\begin{aligned} & \left(A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + A_2 \frac{\partial^n z}{\partial x^{n-2} \partial^2 y^2} + \dots + A_n \frac{\partial^n z}{\partial y^n} \right) \\ & + \left(B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + B_1 \frac{\partial^{n-1} z}{\partial x^{n-2} \partial y} + \dots + B_{n-1} \frac{\partial^{n-1} z}{\partial y^{n-1}} \right) + \dots + \left(M_0 \frac{\partial z}{\partial x} + M_1 \frac{\partial z}{\partial y} \right) + N_0 z \\ & = \phi(x, y) \quad \dots(1) \end{aligned}$$

where the coefficients $A_0, A_1, \dots, A_n, B_0, B_1, \dots, B_{n-1}, \dots, M_0, M_1, N_0$ are constants or functions of x and y .

If all these coefficients are constants, then equation (1) is called a *linear partial differential equation with constant coefficients*.

Denoting $\frac{\partial}{\partial x}$ by D and $\frac{\partial}{\partial y}$ by D' , equation (1) can be written as

$$\begin{aligned} & [(A_0 D^n + A_1 D^{n-1} D' + \dots + A_n D'^n) + (B_0 D^{n-1} + B_1 D^{n-2} D' + \dots + B_{n-1} D'^{n-1}) \\ & + \dots + (M_0 D + M_1 D') + N_0] z = \phi(x, y) \end{aligned} \quad \dots(2)$$

$$\text{or} \quad f(D, D') z = \phi(x, y) \quad \dots(3)$$

10.2 Homogeneous and Non-homogeneous Linear Partial Differential Equations with Constant Coefficients

If all derivatives appearing in a linear partial differential equation with constants coefficients given by (1) or (2) or (3) are of the same order, then it is called a *homogeneous linear partial differential equation with constant coefficients*.

Thus a linear homogeneous partial differential equation with constant coefficients is given by

$$(A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \dots + A_n D'^n) z = \phi(x, y)$$

where $A_0, A_1, \dots, A_n$ are all constants.

or briefly, $f(D, D') z = \phi(x, y)$

If all the derivatives in a linear Partial differential equation with constant coefficients are not of the same order then it is called a *non-homogeneous linear partial differential equation with constant coefficients*.

10.3 Solution of a Homogeneous Linear Partial Differential Equation with Constant Coefficients

A homogeneous linear partial differential equation is given by

$$(A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \dots + A_n D'^n) z = \phi(x, y)$$

$$\text{i.e.,} \quad f(D, D') z = \phi(x, y) \quad \dots(1)$$

The complete solution of (1) will consist of the sum of the **complementary function** (abbreviated as **C.F.**) and the **Particular integral** (abbreviated as **P.I.**)

i.e., the solution of (1) is given by

$$z = \text{C.F.} + \text{P.I.}$$

10.4 Methods of Finding the Complementary Function (C.F.) of the Homogeneous Linear Partial Differential Equation with Constant Coefficients

Let a homogeneous linear partial differential equation with constant coefficients be given by

$$f(D, D')z = \phi(x, y) \quad \dots(1)$$

The *complementary function* (C.F.) of (1) is the general solution of

$$f(D, D')z = 0 \quad \dots(2)$$

$$\text{i.e., of} \quad f(D, D')z = (D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \dots + A_n D^n)z = 0 \quad \dots(3)$$

$$\text{or} \quad (D - m_1 D')(D - m_2 D') \dots (D - m_n D')z = 0 \quad \dots(4)$$

(Taking $A_0 = 1$. If $A_0 \neq 1$, then dividing by A_0 , the coefficient of D^n becomes 1)

Obviously the solution of any one of the equations

$$(D - m_1 D')z = 0, (D - m_2 D')z = 0, \dots, (D - m_n D')z = 0$$

is also a solution of (4).

$$\text{To find solution of } (D - m_r D')z = 0 \quad \dots(5)$$

$$\begin{aligned} (D - m_r D')z = 0 & \Rightarrow Dz - m_r D'z = 0 \\ \Rightarrow \frac{\partial z}{\partial x} - m_r \frac{\partial z}{\partial y} = 0 & \Rightarrow p - m_r q = 0 \end{aligned}$$

which is m_r Lagrange's form.

$\therefore$ Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-m_r} = \frac{dz}{0}$$

Taking 1st and 2nd fractions, $dy + m_r dx = 0 \Rightarrow y + m_r x = c_1$

Taking 1st and 3rd fractions, $dz = 0 \Rightarrow z = c_2$

$\therefore$ The general solution of (5) is given by

$$z = F(y + m_r x)$$

where F is an arbitrary function.

Thus a solution of (2), (3) or (4) is of the form $z = F(y + m_r x)$

Now $Dz = \frac{\partial}{\partial x} F(y + mx) = m F'(y + mx), D'z = \frac{\partial}{\partial y} F(y + mx) = F'(y + mx)$

$$D^2 z = \frac{\partial^2}{\partial x^2} F(y + mx) = m^2 F''(y + mx), D'z = \frac{\partial^2}{\partial y^2} F(y + mx) = F''(y + mx)$$

$$\dots\dots\dots$$

$$D^n z = \frac{\partial^n}{\partial x^n} F(y + mx) = m^n F^n(y + mx), D'^n z = \frac{\partial^n}{\partial y^n} F(y + mx) = F^n(y + mx)$$

Thus, $D^r D'^{(n-r)} z = \frac{\partial^n z}{\partial x^r \partial y^{n-r}} = m^r F^n(y + mx)$

Substituting in (3), we get

$$(m^n + A_1 m^{n-1} + A_2 m^{n-2} + \dots + A_n) F^n(y + mx) = 0$$

which is satisfied if

$$m^n + A_1 m^{n-1} + \dots + A_n = 0 \quad \dots(6)$$

This equation is known as the **Auxiliary equation** (Abbreviated as **A.E.**) which can be obtained by putting $D = m$ and $D' = 1$ in (3).

Let $m_1, m_2, \dots, m_n$ be the roots of this equation (A.E.). There are two cases of these roots.

Case I: Distinct Roots : When $m_1, m_2, \dots, m_n$ are distinct.

Then part of C.F. corresponding to $m = m_r$, is

$$z = f_r(y + m_r x), \text{ for } r = 1, 2, \dots, n.$$

$\therefore$ C.F. of (1) is $z = f_1(y + m_1 x) + f_2(y + m_2 x) + \dots + f_n(y + m_n x)$

where $f_1, f_2, \dots, f_n$ are arbitrary functions.

Case II: Repeated Roots : Let m be a repeated root of (6).

$\therefore$ Consider $(D - mD')^2 z = 0$

or $(D - mD') v = 0 \quad \dots(7)$

Taking $(D - mD') z = v \quad \dots(8)$

$\therefore$ proceeding as above, $v = \phi_1(y + mx)$

$\therefore$ from (8), we have,

$$\frac{\partial z}{\partial x} - m \frac{\partial z}{\partial y} = \phi_1(y + mx)$$

which in Lagrange's form, so its auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi_1(y + mx)}$$

Taking 1st and 2nd, fractions, $dy + m dx = 0 \Rightarrow y + mx = C$

Taking 1st and 3rd fractions, we get

$$dz = \phi_1(y + mx) dx = \phi_1(C) dx$$

$$\Rightarrow z = x \phi_1(C) + C' \Rightarrow z - x \phi_1(y + mx) = C'$$

$$\therefore z - x \phi_1(y + mx) = \phi_2(y + mx)$$

or $z = \phi_2(y + mx) + x \phi_1(y + mx)$ where ϕ_1 and ϕ_2 are arbitrary functions,

is a part of C.F. corresponding to repeated root m . In general if m is a root of A.E. (6) repeated r times, then corresponding to this root

$$f_1(y + mx) + x f_2(y + mx) + x^2 f_3(y + mx) + \dots + x^{r-1} f_r(y + mx) \text{ is a part of C.F.}$$

The C.F. of (1) is the sum of all parts of C.F.'s. corresponding to all roots of A.E.

10.5 Working Method of Finding C.F. of a Homogeneous Linear Partial Differential Equation with Constant Coefficients

Let the given partial differential equation be

$$f(D, D')z = \phi(x, y) \quad \dots(1)$$

The complementary function (C.F.) of (1) can be obtained by any of the following two methods.

Method 1 : Find the A.E. by replacing D by m and D' by 1 in $f(D, D')$ and equating the result to zero $f(D, D')$ is the coefficient of z in (1).

Thus A.E. is $f(m, 1) = 0$.

Solve this A.E. for m and write the parts of C.F.'s for all the roots according to the following rules. The C.F. of equation (1) will be the sum of all the parts of C.F.'s obtained above.

S.N.	Roots	Part of C.F.
1.	For $m = m_1, m_2, \dots, m_n$ (distinct roots).	$f_1(y + m_1x) + f_2(y + m_2x) + \dots$ $+ f_n(y + m_nx)$
2.	For $m = b_1/a_1, b_2/a_2, \dots, b_n/a_n$ (distinct roots)	$f_1(a_1y + b_1x) + f_2(a_2y + b_2x) + \dots$ $+ f_n(a_ny + b_nx)$
3.	For $m = -b_1/a_1, -b_2/a_2, \dots, -b_n/a_n$ (distinct roots)	$f_1(a_1y - b_1x) + f_2(a_2y - b_2x) + \dots$ $+ f_n(a_ny - b_nx)$
4.	For $m = m'$ (repeated r – times)	$f_1(y + m'x) + x f_2(y + m'x) + x^2 f_3(y + m'x)$ $+ \dots + x^{r-1} f_r(y + m'x)$
5.	For $m = b/a$ (repeated r – times)	$f_1(ay + bx) + x f_2(ay + bx) + x^2 f_3(ay + bx)$ $+ \dots + x^{r-1} f_r(ay + bx)$
6.	For $m = -b/a$ (repeated i – times)	$f_1(ay - bx) + x f_2(ay - bx) + x^2 f_3(ay - bx)$ $+ \dots + x^{r-1} f_r(ay - bx)$
7.	For a factor D' (non – repeated) on the L.H.S. of (1)	$f_1(x)$
8.	For a repeated factor D'^r on the L.H.S of (1)	$f_1(x) + y f_2(x) + y^2 f_3(x) + \dots + y^{r-1} f_r(x)$

Method 2 : Factorise $f(D, D')$ into linear factors of the form $(aD - bD')$. Then write the parts of C.F. for each factor according to the following rules. The required C.F. of (1) is obtained by adding all these parts of C.F's for all factors.

S.N.	Factors	Parts of C.F.
1.	For non – repeated factor $(a_1D - b_1D')$	$f_1(a_1y + b_1x)$
2.	For non – repeated factor $(a_1D + b_1D')$	$f_1(a_1y - b_1x)$
3.	For repeated factor $(aD - bD')^r$	$f_1(ay + bx) + x f_2(ay + bx) + x^2 f_3(ay + bx)$ $+ \dots + x^{r-1} f_r(ay + bx)$
4.	For repeated factor $(aD + bD')^r$	$f_1(ay - bx) + x f_2(ay - bx) + x^2 f_3(ay - bx)$ $+ \dots + x^{r-1} f_r(ay - bx)$
5.	For non – repeated factor D	$f_1(y)$
6.	For repeated factor D^r	$f_1(y) + x f_2(y) + x^2 f_3(y) + \dots + x^{r-1} f_r(y)$
7.	For non – repeated factor D'	$f_1(x)$
8.	For repeated factor D'^r	$f_1(x) + y f_2(x) + y^2 f_3(x) + \dots + y^{r-1} f_r(x)$

Solved Examples

Example 1: Solve $2r + 5s + 2t = 0$.

Solution: The given equation can be written as

$$(2D^2 + 5DD' + 2D'^2)z = 0$$

A.E. is $2m^2 + 5m + 2 = 0$

or $(2m + 1)(m + 2) = 0 \quad \therefore m = -\frac{1}{2}, -2$

$\therefore z = f\left(y - \frac{x}{2}\right) + \psi(y - 2x).$

or $z = \phi(2y - x) + \psi(y - 2x).$

Example 2: Solve $r = a^2t$.

Solution: The given equation can be written as

$$(D^2 - a^2 D'^2)z = 0.$$

A.E. is $m^2 - a^2 = 0.$

$\therefore (m - a)(m + a) = 0 \quad \text{or} \quad m = -a, a.$

Hence, the solution is

$$y = f_1(y + ax) + f_2(y - ax).$$

Example 3: Solve $\frac{\partial^3 z}{\partial x^3} - 3 \frac{\partial^3 z}{\partial x^2 \partial y} + 2 \frac{\partial^3 z}{\partial x \partial y^2} = 0.$

[Meerut 2012 (Sem. I)]

Solution: The given equation can be written as

$$(D^3 - 3D^2 D' + 2DD'^2)z = 0.$$

A.E. is $m^3 - 3m^2 + 2m = 0$

or $m(m - 1)(m - 2) = 0 \quad \therefore m = 0, 1, 2.$

Hence, the solution is

$$z = f_1(y) + f_2(y + x) + f_3(y + 2x).$$

Example 4: Solve $(D^3 - 6D^2 D' + 11DD'^2 - 6D'^3)z = 0.$

[Agra 2005]

Solution: Here A.E. is $m^3 - 6m^2 + 11m - 6 = 0.$

$$\therefore (m-1)(m-2)(m-3) = 0 \quad \text{or} \quad m = 1, 2, 3.$$

Hence the solution is

$$z = f_1(y+x) + f_2(y+2x) + f_3(y+3x).$$

Example 5: Solve $25r - 40s + 16t = 0$.

[Jabalpur 2002; Sagar 2004]

Solution: The given equation can be written as

$$(25D^2 - 40DD' + 16D'^2)z = 0$$

A.E. is $25m^2 - 40m + 16 = 0$

or $(5m-4)^2 = 0 \quad \therefore m = \frac{4}{5}, \frac{4}{5}.$

Hence the solution is

$$z = f_1\left(y + \frac{4}{5}x\right) + x f_2\left(y + \frac{4}{5}x\right)$$

or $z = \phi_1(5y+4x) + x \phi_2(5y+4x).$

Example 6: Solve $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0$.

Solution: The given equation can be written as

$$(D^4 - D'^4)z = 0$$

A.E. is $m^4 - 1 = 0$

or $(m-1)(m+1)(m^2+1) = 0.$

$\therefore m = 1, -1, \pm i.$

Hence the solution is

$$z = f_1(y+x) + f_2(y-x) + f_3(y+ix) + f_4(y-ix).$$

Example 7: Solve $(D^3 - 4D^2 D' + 4DD'^2)z = 0$.

[Bhopal 2000, 03]

Solution: Here A.E. is $m^3 - 4m^2 + 4m = 0$

or $m(m-2)^2 = 0 \quad \therefore m = 0, 2, 2.$

Hence the solution is

$$z = \phi(y) + f_1(y+2x) + x f_2(y+2x).$$

Example 8: Solve $\frac{\partial^3 z}{\partial x^3} - 7 \frac{\partial^3 z}{\partial x \partial y^2} + 6 \frac{\partial^3 z}{\partial y^3} = 0$.

Solution: The given equation can be written as

$$(D^3 - 7DD'^2 + 6D'^3)z = 0$$

$$\therefore \text{A.E. is} \quad m^3 - 7m + 6 = 0$$

$$\text{or} \quad (m-1)(m-2)(m+3) = 0 \quad \therefore \quad m = 1, 2, -3.$$

Hence the solution

$$z = f_1(y+x) + f_2(y+2x) + f_3(y-3x).$$

Example 9: Solve $(D^4 - 2D^3D' + 2DD'^3 - D'^4)z = 0$.

$$\text{Solution: A.E. is} \quad m^4 - 2m^3 + 2m - 1 = 0$$

$$\text{or} \quad (m+1)(m-1)^3 = 0 \quad \therefore \quad m = -1, 1, 1, 1.$$

Hence the solution is

$$z = f_1(y-x) + f_2(y+x) + x f_3(y+x) + x^2 f_4(y+x).$$

Exercise 10.1

Solve the following partial differential equations

1. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0.$

2. $\frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2} = 0.$

3. $(D^2 - 3aDD' + 2a^2D'^2)z = 0.$

4. $(4D^2 + 12DD' + 9D'^2)z = 0.$

5. $(2D^2 - 3DD' - 2D'^2)z = 0.$

6. $r + t + 2s = 0.$

7. $r - 4s + 4t = 0.$

8. $(D^3 - 7DD'^2 + 6D'^3)z = 0.$

9. $(D^3 - 2D^2D' + DD'^2)z = 0.$

10. $(D^3 - 3aDD' + 2a^2D'^2)z = 0.$

[Kanpur 2007]

11. $(D^3 - 3D^2D' + 3DD'^2 - D'^3)z = 0.$

12. $(D^4 + D'^4 - 2D^2D'^2)z = 0.$

13. $(D^4 + D'^4)z = 0.$

14. $(D^3D'^2 + D^2D'^3)z = 0.$

Answers 10.1

1.	$z = f_1(y+x) + f_2(y-x).$	2.	$z = f_1(y+ax) + f_2(y-ax).$
3.	$z = f_1(y+ax) + f_2(y+2ax).$	4.	$z = f_1(2y-3x) + x f_2(2y-3x).$
5.	$z = f_1(2y-x) + f_2(y+2x).$	6.	$z = f_1(y-x) + x f_2(y-x).$
7.	$z = f_1(y+2x) + x f_2(y+2x).$	8.	$z = f_1(y+x) + f_2(y+2x) + f_3(y-3x).$
9.	$z = f_1(y) + f_2(y+x) + x f_3(y+x).$	10.	$z = f_1(y+ax) + f_2(y+2ax).$
11.	$z = f_1(y+x) + x f_2(y+x) + x^2 f_2(y+x).$		
12.	$z = f_1(y+x) + x f_2(y+x) + f_3(y-x) + x f_4(y-x).$		
13.	$z = f_1(y+\alpha x) + f_2(y+\bar{\alpha} x) + f_3(y+\beta x) + f_4(y+\bar{\beta} x).$ where $\alpha = (1+i)/\sqrt{2}$ and $\beta = (-1+i)/\sqrt{2}.$		
14.	$z = f_1(y) + x f_2(y) + f_3(x) + y f_4(x) + f_5(y-x).$		

10.6 Determination of the Particular Integral (P.I.) of a Homogeneous Linear Partial Differential Equation with Constant Coefficients

A homogeneous linear partial differential equation is given by

$$f(D, D') z = \phi(x, y)$$

where $f(D, D')$ is a homogeneous function of D and D' of degree n

Then
$$\text{P.I.} = \frac{1}{f(D, D')} \cdot \phi(x, y).$$

To evaluate the above P.I., the symbolic function $f(D, D')$ can be treated as an algebraic function of D and D' . It can be resolved into partial fractions or can be expanded in ascending powers of D or D' .

Before we give general method of finding above P.I., we shall give short methods in some special cases of function $\phi(x, y)$.

10.7 Short Method to Find P.I. when $\phi(x, y)$ is a Function of the form $x^m y^n$ or a Rational Integral Algebraic Function of x and y

We have
$$\text{P.I.} = \frac{1}{f(D, D')} \cdot \phi(x, y).$$

If $\phi(x, y)$ is a function of the form $x^m y^n$ or is a rational integral algebraic function of x and y then we expand $\frac{1}{f(D, D')}$ in an infinite series either in ascending powers of D or in ascending powers of D' and then evaluate. The P.I. obtained by expanding $\frac{1}{f(D, D')}$ in ascending powers of D may be different than that obtained by expanding $\frac{1}{f(D, D')}$ in ascending powers of D' . But this difference is immaterial as it can be incorporated in the arbitrary functions occurring in C.F.

In general for the expansion of $\frac{1}{f(D, D')}$ in infinite series in ascending powers of D or D' we observe the following rule.

If $n < m$, then expand $\frac{1}{f(D, D')}$ in ascending powers of D'/D and

if $m < n$, then expand $\frac{1}{f(D, D')}$ in ascending powers of D/D' .

Solved Examples

Example 1: Solve $(D^2 - 2DD' + D'^2)z = 12xy$.

[Meerut 2011 (Sem. I)]

Solution: Here A.E. is $m^2 - 2m + 1 = 0$,

or $(m-1)^2 = 0 \quad \therefore m = 1, 1.$

$\therefore$ C.F. = $f_1(y+x) + x f_2(y+x)$

Now
$$\text{P.I.} = \frac{1}{D^2 - 2DD' + D'^2} 12xy$$

$$= \frac{1}{(D-D')^2} 12xy = \frac{1}{D^2} \left(1 - \frac{D'}{D}\right)^{-2} 12xy$$

$$= \frac{1}{D^2} \left(1 + \frac{2D'}{D} + \frac{3D'^2}{D^2} + \dots\right) 12xy$$

$$\begin{aligned}
 &= \left(\frac{1}{D^2} + \frac{2D'}{D^3} + \frac{3D'^2}{D^4} + \dots \right) 12xy = \frac{1}{D^2} (12xy) + \frac{2}{D^3} D' (12xy) + \frac{3}{D^4} D'^2 (12xy) + \dots \\
 &= 2x^3y + \frac{2}{D^3} (12x) + 0 = 2x^3y + x^4.
 \end{aligned}$$

∴ The general solution is

$$z = \text{C.F.} + \text{P.I.} = f_1(y+x) + x f_2(y+x) + 2x^3y + x^4.$$

Example 2: Solve $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$.

[Jabalpur 2003]

Solution: Here A.E. is $m^2 - 6m + 9 = 0$

$$\text{or } (m-3)^2 = 0 \quad \therefore m = 3, 3$$

$$\text{C.F.} = f_1(y+3x) + x f_2(y+3x)$$

$$\begin{aligned}
 \text{Now P.I.} &= \frac{1}{D^2 - 6DD' + 9D'^2} \cdot (12x^2 + 36xy) = \frac{1}{(D-3D')^2} (12x^2 + 36xy) \\
 &= \frac{1}{D^2} \left(1 - \frac{3D'}{D} \right)^{-2} (12x^2 + 36xy) \\
 &= \frac{1}{D^2} \left(1 + 2 \cdot \frac{3D'}{D} + 3 \cdot \frac{9D'^2}{D^2} + \dots \right) \cdot (12x^2 + 36xy) \\
 &= \frac{1}{D^2} (12x^2 + 36xy) + \frac{6}{D^3} \cdot D' (12x^2 + 36xy) + \frac{27}{D^4} D'^2 (12x^2 + 36xy) + \dots \\
 &= x^4 + 6x^3y + \frac{6}{D^3} (36x) + \frac{27}{D^4} \cdot 0 + 0 = x^4 + 6x^3y + 9x^4 = 10x^4 + 6x^3y
 \end{aligned}$$

∴ The general solution is

$$z = f_1(y+3x) + x f_2(y+3x) + 10x^4 + 6x^3y.$$

Example 3: Solve $r + (a+b)s + abt = xy$.

Solution: The given equation can be written as

$$[D^2 + (a+b)DD' + abD'^2]z = xy$$

$$\therefore \text{A.E. is } m^2 + (a+b)m + ab = 0 \quad \text{or } (m+a)(m+b) = 0 \therefore m = -a, -b$$

$$\text{C.F.} = f_1(y-ax) + f_2(y-bx)$$

$$\begin{aligned}
 \text{Now } \text{P.I.} &= \frac{1}{[D^2 + (a+b)DD' + abD'^2]} xy \\
 &= \frac{1}{D^2} \left[1 + \left\{ (a+b) \frac{D'}{D} + ab \frac{D'^2}{D^2} \right\} \right]^{-1} xy \\
 &= \frac{1}{D^2} \left[1 - (a+b) \frac{D'}{D} + \dots \right] xy \\
 &= \frac{1}{D^2} xy - (a+b) \frac{1}{D^3} D'(xy) + \dots = \frac{1}{6} x^3 y - (a+b) \cdot \frac{1}{D^3} x + 0 = \frac{1}{6} x^3 y - \frac{1}{24} (a+b) x^4
 \end{aligned}$$

Hence, the general solution is

$$z = f_1(y-ax) + f_2(y-bx) + \frac{1}{6} x^3 y - \frac{1}{24} (a+b) x^4.$$

Example 4: Solve $\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} = x^3 y^3$.

Solution: The given equation can be written as

$$(D^3 - D'^3)z = x^3 y^3$$

$$\therefore \text{A.E. is } m^3 - 1 = 0 \quad \text{or} \quad (m-1)(m-\omega)(m-\omega^2) = 0 \therefore m = 1, \omega, \omega^2.$$

Where ω is cube root of unity.

$$\text{C.F.} = f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x)$$

$$\begin{aligned}
 \text{Now } \text{P.I.} &= \frac{1}{D^3 - D'^3} x^3 y^3 = \frac{1}{D^3} \left(1 - \frac{D'^3}{D^3} \right)^{-1} x^3 y^3 \\
 &= \frac{1}{D^3} \left(1 + \frac{D'^3}{D^3} + \frac{D'^6}{D^6} + \dots \right) x^3 y^3 = \frac{1}{D^3} (x^3 y^3) + \frac{1}{D^6} D'^3 (x^3 y^3) + \dots \\
 &= \frac{x^6}{4.5.6} y^3 + \frac{1}{D^6} (6x^3) + 0 = \frac{x^6 y^3}{120} + \frac{6x^9}{4.5.6.7.8.9} = \frac{x^6 y^3}{120} + \frac{x^9}{10080}
 \end{aligned}$$

Hence, the general solution is

$$V = f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x) + \frac{x^6 y^3}{120} + \frac{x^9}{10080}.$$

Example 5: (a) Solve $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -4\pi(x^2 + y^2)$.

(b) Find a real function V of x and y reducing to zero when $y = 0$ and satisfying

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -4\pi(x^2 + y^2). \quad [\text{Nagpur 2005}]$$

Solution: (a) The given equation can be written as $(D^2 + D'^2)V = -4\pi(x^2 + y^2)$.

$\therefore$ A.E. is $m^2 + 1 = 0 \Rightarrow m = i, -i$

$$\text{C.F.} = f_1(y + ix) + f_2(y - ix).$$

$$\begin{aligned} \text{Now P.I.} &= \frac{1}{D^2 + D'^2} \cdot (-4\pi)(x^2 + y^2) = (-4\pi) \cdot \frac{1}{D^2} \left(1 + \frac{D'^2}{D^2} \right)^{-1} (x^2 + y^2) \\ &= (-4\pi) \frac{1}{D^2} \left(1 - \frac{D'^2}{D^2} + \frac{D'^4}{D^4} \dots \right) (x^2 + y^2) \\ &= (-4\pi) \left[\frac{1}{D^2} (x^2 + y^2) - \frac{1}{D^4} D'^2 (x^2 + y^2) + 0 \right] \\ &= (-4\pi) \left[\frac{x^4}{12} + \frac{x^2 y^2}{2} - \frac{1}{D^4} (2) \right] = -4\pi \left[\frac{x^4}{12} + \frac{x^2 y^2}{2} - \frac{2x^4}{2 \cdot 3 \cdot 4} \right] \\ &= -4\pi \cdot \frac{x^2 y^2}{2} = -2\pi x^2 y^2 \end{aligned}$$

Hence, the general solution is

$$z = f_1(y + ix) + f_2(y - ix) - 2\pi x^2 y^2.$$

(b) Proceeding as in (a)

$$V = f_1(y + ix) + f_2(y - ix) - 2\pi x^2 y^2$$

Since V reduces to zero when $y = 0$

$$\therefore f_1(y + ix) = 0 = f_2(y - ix)$$

$$\text{Hence } V = -2\pi x^2 y^2.$$

Example 6: Solve $(D^2 + 3DD' + 2D'^2)z = x + y$, by expanding the particular integral in ascending powers of D as well as in ascending powers of D' . [Bhopal 2000, 03; Rewa 2002, 03]

Solution: Here A.E. is $m^2 + 3m + 2 = 0$ or $(m+1)(m+2) = 0 \therefore m = -2, -1$

$$\therefore \text{C.F.} = f_1(y-2x) + f_2(y-x)$$

Now, by expanding in ascending powers of D ,

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D^2 + 3DD' + 2D'^2)} \cdot (x + y) \\ &= \frac{1}{2D'^2} \left[1 + \left(\frac{D^2}{2D'^2} + \frac{3D}{2D'} \right) \right]^{-1} (x + y) \\ &= \frac{1}{2D'^2} \left[1 - \left(\frac{D^2}{2D'^2} + \frac{3D}{2D'} \right) + \dots \right] (x + y) \\ &= \frac{1}{2D'^2} (x + y) - \frac{1}{4D'^4} D^2 (x + y) - \frac{3}{4D'^3} D (x + y) + \dots \\ &= \frac{1}{2} \left(\frac{1}{2} x y^2 + \frac{1}{6} y^3 \right) - 0 - \frac{3}{4D'^3} \cdot 1 = \frac{1}{4} x y^2 + \frac{1}{12} y^3 - \frac{3}{4} \cdot \frac{1}{6} y^3 \\ &= \frac{1}{4} x y^2 - \frac{1}{24} y^3 \end{aligned}$$

Again, by expanding in ascending powers of D'

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D^2 + 3DD' + 2D'^2)} (x + y) = \frac{1}{D^2} \left[1 + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2} \right) \right]^{-1} (x + y) \\ &= \frac{1}{D^2} \left[1 - \left(\frac{3D'}{D} + \frac{2D'^2}{D^2} \right) + \dots \right] (x + y) \\ &= \frac{1}{D^2} (x + y) - \frac{3}{D^3} D' (x + y) + \dots = \frac{1}{6} x^3 + \frac{1}{2} x^2 y - \frac{3}{D^3} + 0 \\ &= \frac{1}{6} x^3 + \frac{1}{2} x^2 y - \frac{3}{6} x^3 = \frac{1}{2} x^2 y - \frac{1}{3} x^3. \end{aligned}$$

Hence, the required general solution is given by

$$z = f_1(y-2x) + f_2(y-x) + \frac{1}{4} x y^2 - \frac{1}{24} y^3$$

or
$$z = f_1(y-2x) + f_2(y-x) + \frac{1}{2} x^2 y - \frac{1}{3} x^3.$$

Exercise 10.2

Solve the following partial differential equations.

1. $(D^2 - a^2 D'^2)z = x.$

2. $(D^2 + 3DD' + 2D'^2)z = 12x y.$

3. $(D^2 - 2DD' - 15D'^2)z = 12x y.$

[Kurukshetra 2004; Meerut 2006]

4. $\frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2} = x^2.$

[Ravishankar 2000, 04]

5. $(D^2 - DD' - 6D'^2)z = x y.$

Answers 10.2

1.	$z = f_1(y + ax) + f_2(y - ax) + x^3/6.$
2.	$z = f_1(y - x) + f_2(y - 2x) + 2x^3 y - 3x^4/2.$
3.	$z = f_1(y + 5x) + f_2(y - 3x) + 2x^3 y + x^4.$
4.	$z = f_1(y + ax) + f_2(y - ax) + x^4/12.$
5.	$z = f_1(y + 3x) + f_2(y - 2x) + \frac{1}{6}x^3 y + \frac{1}{24}x^4.$

10.8 Short Methods of Finding P.I. When $\phi(x, y)$ is a Function of $ax + by$.

If $\phi(x, y)$ is a function of $ax + by$, then we can use short methods to find P.I. in two special cases.

Case I: If $f(D, D')$ is a homogeneous function of D and D' of degree n and $\phi(x, y) = \psi^n(ax + by)$, then $\text{P.I.} = \frac{1}{f(D, D')} \psi^n(ax + by) = \frac{1}{f(a, b)} \psi(ax + by)$ provided $f(a, b) \neq 0$ and $\psi^n(ax + by)$ is n -th derivative of $\psi(ax + by)$ w.r.t. $ax + by$.

Proof: We have, $D^r \psi(ax + by) = a^r \psi^r(ax + by)$

and $D^{s'} \psi(ax + by) = b^{s'} \psi^{s'}(ax + by)$

$\therefore f(D, D')$ is a homogeneous function of D and D' of degree n ,

$$\therefore f(D, D') \psi(ax + by) = f(a, b) \psi^n(ax + by)$$

$$\Rightarrow \text{P.I.} = \frac{1}{f(D, D')} \psi^n(ax + by) = \frac{1}{f(a, b)} \psi(ax + by).$$

Provided $f(a, b) \neq 0$.

Working Method : To find P.I. = $\frac{1}{f(D, D')} \psi^n(ax + by)$, where $f(D, D')$ is a rational integral homogeneous function of degree n . substituting $ax + by = t$ in $\psi^n(ax + by)$, which becomes $\psi^n(t)$, integrate it n times i.e., $\int \int \int \dots \int \psi^n(t) dt dt \dots dt = \psi(t)$, then replacing t (n times)

by $ax + by$ in the result, divided it i.e., $\psi(ax + by)$ by $f(a, b)$ obtained from $f(D, D')$ replacing D and D' by a (coeff. of x) and b (coeff. of y) respectively.

$$\text{Thus } \text{P.I.} = \frac{1}{f(D, D')} \cdot \psi^n(ax + by) = \frac{\psi(ax + by)}{f(a, b)}.$$

Provided $f(a, b) \neq 0$

Case II: Exceptional case when $f(a, b) = 0$.

If $\phi(x, y)$ is a function of $ax + by$ i.e., $\phi(x, y) = \psi(ax + by)$ and $f(D, D') = (bD - aD')^n$, so that $f(a, b) = 0$, then

$$\text{P.I.} = \frac{1}{f(D, D')} \phi(x, y) = \frac{1}{(bD - aD')^n} \psi(ax + by) = \frac{x^n}{b^n n!} \psi(ax + by).$$

Proof: Consider the equation

$$(bD - aD') z = x^r \psi(ax + by) \quad \dots(1)$$

$$\text{or } bp - aq = x^r \psi(ax + by)$$

which is of Lagrange's form, so Lagrange's subsidiary (auxiliary) equations one

$$\frac{dx}{b} = \frac{dy}{-a} = \frac{dz}{x^r \psi(ax + by)}$$

Taking 1st and 2nd fractions, we get

$$a dx + b dy = 0 \Rightarrow ax + by = c_1$$

Taking 1st and 3rd fractions, we have

$$dx = b \frac{dz}{x^r \psi(c_1)} \Rightarrow dz = \frac{x^r \psi(c_1)}{b} dx$$

Integrating,
$$z = \frac{\psi(c_1) \cdot x^{r+1}}{b(r+1)} = \frac{x^{r+1}}{b(r+1)} \psi(ax+by) \quad \dots(2)$$

$\therefore$ From (1) and (2), we have

$$z = \frac{1}{(bD - aD')} \cdot x^r \psi(ax+by) = \frac{x^{r+1}}{b(r+1)} \psi(ax+by) \quad \dots(3)$$

$$\begin{aligned} \therefore \text{P.I.} &= \frac{1}{f(D, D')} \phi(x, y) = \frac{1}{(bD - aD')^n} \psi(ax+by) \\ &= \frac{1}{(bD - aD')^{n-1}} \cdot \frac{1}{(bD - aD')} \psi(ax+by) \\ &= \frac{1}{(bD - aD')^{n-1}} \cdot \frac{x}{b} \psi(ax+by) \\ &= \frac{1}{(bD - aD')^{n-2}} \cdot \frac{1}{(bD - aD')} \frac{1}{b} x \psi(ax+by) \\ &= \frac{1}{(bD - aD')^{n-2}} \cdot \frac{1}{b} \cdot \frac{x^2}{2b} \psi(ax+by) = \frac{1}{(bD - aD')^{n-2}} \cdot \frac{x^2}{b^2 2!} \psi(ax+by) \\ &\dots\dots\dots \\ &= \frac{x^n}{b^n n!} \psi(ax+by), \quad \text{Proceeding similarly } (n-2) \text{ times more.} \end{aligned}$$

$$\therefore \text{P.I.} = \frac{1}{(bD - aD')^n} \psi(ax+by) = \frac{x^n}{b^n n!} \psi(ax+by)$$

Working Method : If $f(a, b) = 0$, then

$$\text{P.I.} = \frac{1}{f(D, D')} \phi(x, y) = \frac{1}{(bD - aD')^n} \cdot \psi(ax+by) = \frac{x^n}{b^n n!} \psi(ax+by)$$

where b is coefficient of D in $f(D, D')$ i.e., $(bD - aD')^n$ or coefficient of y in $\psi(ax+by)$ i.e., $\phi(x, y)$.

Solved Examples

Example 1: Find P.I. of $(D^3 - 10D^2D' + D'^3)z = \cos(2x + 3y)$.

Solution:

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^3 - 10D^2D' + D'^3} \cdot \cos(2x + 3y) \\ &= \frac{1}{2^3 - 10 \cdot 2^2 \cdot 3 + 3^3} \cdot \iiint \cos v (dv)^3, \quad \text{Taking } 2x + 3y = v \\ &= -\frac{1(-\sin v)}{85} = \frac{1}{85} \sin(2x + 3y) \end{aligned}$$

Example 2: Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \cos mx \cos ny$. [Kanpur 2007]

Solution: The given equation can be written as

$$(D^2 + D'^2)z = \frac{1}{2} [\cos(mx + ny) + \cos(mx - ny)]$$

Here A.E. is $m^2 + 1 = 0 \quad \therefore m = i, -i$

$$\text{C.F.} = f_1(y + ix) + f_2(y - ix)$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{2} \cdot \frac{1}{D^2 + D'^2} \cos(mx + ny) + \frac{1}{2} \cdot \frac{1}{D^2 + D'^2} \cos(mx - ny) \\ &= \frac{1}{2} \cdot \frac{1}{m^2 + n^2} \cdot \iint \cos v (dv)^2 + \frac{1}{2} \cdot \frac{1}{m^2 + n^2} \cdot \iint \cos t (dt)^2 \\ &\quad \text{Taking } mx + ny = v, \quad mx - ny = t \\ &= \frac{1}{2} \cdot \frac{1}{m^2 + n^2} \cdot \{-\cos(mx + ny)\} + \frac{1}{2} \cdot \frac{1}{m^2 + n^2} \cdot \{-\cos(mx - ny)\} \\ &= \frac{-1}{2(m^2 + n^2)} [\cos(mx + ny) + \cos(mx - ny)] \end{aligned}$$

Hence, the solution is

$$z = f_1(y + ix) + f_2(y - ix) - \frac{1}{2(m^2 + n^2)} [\cos(mx + ny) + \cos(mx - ny)]$$

Example 3: Solve $(D^3 - 4D^2D' + 4DD'^2)z = 4\sin(2x + y)$.

[Indore 2004; MDU Rohtak 2004]

Solution: Here A.E. is $m^3 - 4m^2 + 4m = 0$

or $m(m-2)^2 = 0 \quad \therefore m = 0, 2, 2.$

$$\text{C.F.} = f_1(y) + f_2(y + 2x) + x f_3(y + 2x).$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^3 - 4D^2D' + 4DD'^2} \cdot 4\sin(2x + y) \\ &= \frac{1}{D(D-2D')^2} \cdot 4\sin(2x + y) \\ &= \frac{1}{(D-2D')^2} \cdot \frac{1}{D} 4\sin(2x + y) = \frac{1}{(D-2D')^2} \cdot (-2\cos(2x + y)) \\ &= (-2) \cdot \frac{x^2}{2! \cdot 1^2} \cdot \cos(2x + y), \quad \because f(a, b) = 0 \\ &= -x^2 \cos(2x + y). \end{aligned}$$

$\therefore$ The solution is

$$z = f_1(y) + f_2(y + 2x) + x f_3(y + 2x) - x^2 \cos(2x + y).$$

Example 4: Solve $2r - s - 3t = 5e^x/e^y$.

[Indore 2003; Jiwaji 2003]

Solution: The given equation can be written as

$$(2D^2 - DD' - 3D'^2)z = 5e^{x-y}.$$

A.E. is $2m^2 - m - 3 = 0$

or $(2m-3)(m+1) = 0$

$\therefore$ C.F. = $f_1(2y + 3x) + f_2(y - x)$

$$\begin{aligned} \text{P.I.} &= \frac{1}{(2D^2 - DD' - 3D'^2)} \cdot 5e^{x-y} \\ &= \frac{1}{(D+D')} \cdot \frac{1}{(2D-3D')} \cdot 5e^{x-y} \\ &= \frac{1}{D+D'} \cdot \frac{1}{2+3} 5e^{x-y} = \frac{1}{D+D'} e^{x-y} \end{aligned}$$

$$= \frac{x}{1!} e^{x-y} = x e^{x-y} \quad \because f(a, b) = 0$$

$\therefore$ The solution is $z = f_1(2y + 3x) + f_2(y - x) + x e^{x-y}$.

Example 5: Solve $(D^2 - 6DD' - 9D'^2)z = 6x + 2y$.

Solution: Here A.E. is $m^2 - 6m + 9 = 0$

or $(m-3)^2 = 0$ or $m = 3, 3$.

$$\text{C.F.} = f_1(y + 3x) + x f_2(y + 3x)$$

$$\text{P.I.} = \frac{1}{(D-3D')^2} \cdot (6x + 2y) = 2 \cdot \frac{1}{(D-3D')^2} (3x + y)$$

$$= 2 \cdot \frac{x^2}{2!} \cdot \frac{(3x + y)}{1^2} = x^2(3x + y) \quad \because f(a, b) = 0$$

The solution is $z = f_1(y + 3x) + x f_2(y + 3x) + x^2(3x + y)$.

Example 6: Solve $(D^2 - 5DD' + 4D'^2)z = \sin(4x + y)$.

[Meerut 2006]

Solution: Here A.E. is $m^2 - 5m + 4 = 0$

or $(m-1)(m-4) = 0 \quad \therefore m = 1, 4$.

$$\text{C.F.} = f_1(y + x) + f_2(y + 4x)$$

$$\text{P.I.} = \frac{1}{D^2 - 5DD' + 4D'^2} \sin(4x + y) \quad \because f(a, b) = 0$$

$$= \frac{1}{(D-4D')} \cdot \frac{1}{(D-D')} \sin(4x + y)$$

$$= \frac{1}{(D-4D')} \cdot \frac{1}{4-1} \{-\cos(4x + y)\}$$

$$= \frac{1}{(D-4D')} \cdot \left\{ -\frac{1}{3} \cos(4x + y) \right\}$$

$$= -\frac{x}{3} \cos(4x + y), \quad \because f(a, b) = 0$$

$\therefore$ The solution is $z = f_1(y + x) + f_2(y + 4x) - \frac{1}{3} x \cos(4x + y)$.

Example 7: Solve $4r - 4s + t = 16 \log(x + 2y)$.

[Indore 2001; Ravishankar 2000]

Solution: The equation can be written as

$$(4D^2 - 4DD' + D'^2)z = 16 \log(x + 2y)$$

A.E. is $4m^2 - 4m + 1 = 0$

or $(2m - 1)^2 = 0$

$$\text{C.F.} = f_1(2y + x) + x f_2(2y + x)$$

$$\text{P.I.} = \frac{1}{4D^2 - 4DD' + D'^2} 16 \log(x + 2y)$$

$$= \frac{1}{(2D - D')^2} 16 \log(x + 2y)$$

$$= \frac{x^2}{2^2 \cdot 2!} \cdot 16 \log(x + 2y) = 2x^2 \log(x + 2y) \quad \because f(a, b) = 0$$

$\therefore$ The solution is

$$z = f_1(2y + x) + x f_2(2y + x) + 2x^2 \log(x + 2y).$$

Example 8: Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 12(x + y)$.

Solution: The given equation can be written as

$$(D^2 + D'^2)z = 12(x + y).$$

A.E. is $m^2 + 1 = 0, \therefore m = \pm i$

$$\text{C.F.} = f_1(y + ix) + f_2(y - ix)$$

$$\text{P.I.} = \frac{1}{D^2 + D'^2} 12(x + y) = \frac{12}{1^2 + 1^2} \iint v (dv)^2, \text{ Taking } x + y = v$$

$$= \frac{1}{1^2 + 1^2} \cdot 12 \cdot \frac{(x + y)^3}{6} = (x + y)^3.$$

$\therefore$ The solution is

$$z = f_1(y + ix) + f_2(y - ix) + (x + y)^3.$$

Example 9: Solve $(D^2 + 3DD' + 2D'^2)z = x + y$.

[Meerut 2005, 07]

Solution: Here, A.E. is $m^2 + 3m + 2 = 0$ or $(m+1)(m+2) = 0 \therefore m = -2, -1$

$$\text{C.F.} = f_1(y-2x) + f_2(y-x).$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D^2 + 3DD' + 2D'^2)}(x+y) = \frac{1}{(1^2 + 3.1.1 + 2.1^2)} \iint v (dv)^2 \quad \text{Taking } x+y=v \\ &= \frac{1}{6} \cdot \frac{v^3}{6} = \frac{1}{36}(x+y)^3. \end{aligned}$$

$\therefore$ The solution is

$$z = f_1(y-2x) + f_2(y-x) + \frac{1}{36}(x+y)^3.$$

2nd Method : See example 6 on page 441.

Example 10: Solve $(D^2 - 2DD' + D'^2)z = e^{x+2y} + x^3$.

Solution: Here A.E. is $m^2 - 2m + 1 = 0$ or $(m-1)^2 = 0$ or $m = 1, 1$.

$$\text{C.F.} = f_1(y+x) + x f_2(y+x)$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^2 - 2DD' + D'^2} e^{x+2y} + \frac{1}{(D^2 - 2DD' + D'^2)} x^3 \\ &= \frac{1}{1^2 - 2.1.2 + 2^2} \iint e^v (dv)^2 + \frac{1}{D^2} \left(1 - \frac{D'}{D}\right)^{-2} \cdot x^3 \end{aligned}$$

Taking $x+2y = v$

$$\begin{aligned} &= e^v + \frac{1}{D^2} \left(1 + \frac{2D'}{D} + 3 \cdot \frac{D'^2}{D^2} + \dots\right) \cdot x^3 \\ &= e^{x+2y} + \frac{1}{D^2} x^3 + \frac{2}{D^3} D'x^3 + \dots = e^{x+2y} + \frac{1}{20} x^5. \end{aligned}$$

$\therefore$ The solution is

$$z = f_1(y+x) + x f_2(y+x) + e^{x+2y} + \frac{1}{20} x^5.$$

Example 11: Solve $r + 5s + 6t = 1/(y-2x)$.

[Garhwal 2005]

Solution: The given equation can be written as

$$(D^2 + 5DD' + 6D'^2)z = (y-2x)^{-1}$$

A.E. is $m^2 + 5m + 6 = 0$ or $(m+2)(m+3) = 0 \therefore m = -2, -3$

$$\therefore \text{C.F.} = f_1(y-2x) + f_2(y-3x)$$

$$\text{Now P.I.} = \frac{1}{(D^2 + 5DD' + 6D'^2)} \cdot (y-2x)^{-1}$$

$$= \frac{1}{(D+2D')(D+3D')} \cdot (y-2x)^{-1} = \frac{1}{(D+2D')} \cdot \left(\frac{1}{(D+3D')} \cdot (y-2x)^{-1} \right)$$

$$= \frac{1}{(D+2D')} \cdot \left(\frac{1}{-2+3.1} \int \frac{1}{v} dv \right) \quad \text{Taking } y-2x = v$$

$$= \frac{1}{(D+2D')} \cdot \log(y-2x) = \frac{x}{1!} \cdot \log(y-2x) \quad \because f(a,b) = 0$$

$\therefore$ The solution is

$$z = f_1(y-2x) + f_2(y-3x) + x \log(y-2x).$$

Exercise 10.3

Solve the following partial differential equations

1. $(D^2 + D'^2)z = 30(2x + y).$

[Sagar 2004]

2. $(D^2 - 2DD' + D'^2)z = e^{x+2y}.$

[Kanpur 2005]

3. $(D^2 + 2DD' + D'^2)z = e^{2x+3y}.$

4. $r + s - 2t = (2x + y)^{1/2}.$

5. $(D^2 + D'^2)z = \cos mx \sin ny.$

[Ravishankar 2001]

6. $(D^3 - 4D^2D' + 4DD'^2)z = 2 \sin(3x + 2y).$

[I.A.S. 2006]

7. $(D^3 - 3DD'^2 + 2D'^3)z = (x - 2y)^{1/2}.$

8. $(r + s - 2t) = e^{x+y}.$

9. $(D^2 - 6DD' + 9D'^2)z = \tan(y + 3x).$

[Delhi 2007; Ravishankar 2004]

10. $(2D^2 - 5DD' + 2D'^2)z = 5 \sin(2x + y).$

[M.D.U. Rohtak 2005]

11. $r - t = x - y.$

12. $(D^3 - 4D^2D' + 4DD'^2)z = \cos(2x + y).$

13. $(D^2 - 2DD' + D'^2)z = \tan(x + y).$

14. $(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = e^{x+y}.$

[Meerut 2007; Jabalpur 2004; Bhopal 2000, 03]

15. $(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)z = e^{y+2x} + (y+x)^{1/2}.$

16. $(D^2 - 3DD' + 2D'^2)z = e^{2x-y} + \cos(x+2y).$

17. $(D - 3D')^2(D + 3D')z = e^{3x+y}.$

[Agra 2005; Delhi Maths (Hons.) 2007]

18. $(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y) + e^{3x+y}.$

Answers 10.3

1.	$z = f_1(y+ix) + f_2(y-ix) + (2x+y)^3.$
2.	$z = f_1(y+x) + x f_2(y+x) + e^{x+2y}.$
3.	$z = f_1(y-x) + x f_2(y-x) + (1/25)e^{2x+3y}.$
4.	$z = f_1(y+x) + f_2(y-2x) + (1/15)(2x+y)^{5/2}.$
5.	$z = f_1(y+ix) + f_2(y-ix) - 2(\cos mx \sin ny)/(m^2 + n^2).$
6.	$z = f_1(y) + f_2(y+2x) + x f_3(y+2x) + (2/3) \cos(3x+2y).$
7.	$z = f_1(y+x) + x f_2(y+x) + f_3(y+2x) - (8/2835)(x-2y)^{7/2}.$
8.	$z = f_1(y+x) + f_2(y-2x) + (x/3)e^{x+y}.$
9.	$z = f_1(y+3x) + x f_2(y+3x) + (x^2/2) \tan(y+3x).$
10.	$z = f_1(2y+x) + f_2(y+2x) - (5x/3) \cos(2x+y).$
11.	$z = f_1(y+x) + f_2(y-x) + (x/4)(x-y)^2.$
12.	$z = f_1(y) + f_2(y+2x) + x f_3(y+2x) + (x^2/4) \sin(2x+y).$
13.	$z = f_1(y+x) + x f_2(y+x) + (x^2/2) \tan(y+x).$
14.	$z = f_1(y+2x) + f_2(y+x) + f_3(y-x) - (x/2)e^{x+y}.$
15.	$z = f_1(y+x) + x f_2(y+x) + f_3(y+2x) + x e^{y+2x} - (x^2/3)(x+y)^{3/2}.$
16.	$z = f_1(y+x) + f_2(y+2x) + (1/2)e^{2x-y} - (1/3) \cos(x+2y).$
17.	$z = f_1(y+3x) + x f_2(y+3x) + f_3(y-3x) + (x^2/12)e^{3x+y}.$
18.	$z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) - (1/75) \cos(x+2y) + (1/20)x e^{3x+y}.$

10.9 General Method of Finding the P.I. of Homogeneous Linear Differential Equation with Constant Coefficients

If the given equation is $f(D, D')z = \phi(x, y)$... (1)

where $f(D, D')$ is a homogeneous function of D and D' of degree n such that it can be factorised in n linear factors, i.e.,

$$f(D, D') = (D - m_1 D')(D - m_2 D') \dots (D - m_n D'),$$

then P.I. = $\frac{1}{f(D, D')} \phi(x, y) = \frac{1}{(D - m_1 D')(D - m_2 D') \dots (D - m_n D')} \phi(x, y)$... (2)

Now consider the equation

$$(D - m D')z = \phi(x, y) \quad \dots (3)$$

$$\Rightarrow p - mq = \phi(x, y)$$

which is of the form of Lagrange's equation

$\therefore$ Its subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi(x, y)}$$

Taking 1st and 2nd fractions, $dy + m dx = 0 \Rightarrow y + m x = a$... (4)

Again taking 1st and 3rd fractions

$$dz = \phi(x, y) dx = \phi(x, a - m x) dx \quad \text{substituting from (4)}$$

Integrating, $z = \int \phi(x, a - m x) dx$

$$\text{i.e., } z = \frac{1}{(D - m D')} \phi(x, y) = \int \phi(x, a - m x) dx \quad \dots (5)$$

where $a = y + mx$

After integration of R.H.S. replace $a = y + mx$

$\therefore$ P.I. given by (2) can be obtained by applying (5) by factors in succession, starting from right.

Working Method

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{(D - m_1 D')(D - m_2 D') \dots (D - m_{n-1} D')(D - m_n D')} \phi(x, y) \\
 &= \frac{1}{(D - m_1 D')(D - m_2 D') \dots (D - m_{n-1} D')} \cdot \int \phi(x, a_1 - m_n x) dx \\
 &\quad \text{where } y + m_n x = a_1 \\
 &= \frac{1}{(D - m_1 D')(D - m_2 D') \dots (D - m_{n-1} D')} \cdot \phi_1(x, y) \\
 &\quad \text{where } \phi_1(x, y) \text{ is obtained by replacing } a_1 \text{ by } y + m_n x \\
 &\quad \text{after evaluation of integral } \int \phi(x, a_1 - m_n x) dx \\
 &= \frac{1}{(D - m_1 D')(D - m_2 D') \dots (D - m_{n-2} D')} \cdot \int \phi_1(x, a_2 - m_{n-1} x) dx \\
 &\quad \text{where } a_2 = y + m_{n-1} x \\
 &= \frac{1}{(D - m_1 D')(D - m_2 D') \dots (D - m_{n-2} D')} \cdot \phi_2(x, y)
 \end{aligned}$$

where $\phi_2(x, y)$ is obtained by replacing a_2 by $y + m_{n-1} x$ after the evaluation of integral $\int \phi_1(x, a_2 - m_{n-1} x) dx$

Proceeding in this way n times (In total) we shall obtain P.I.

Solved Examples

Example 1: Solve $(D^2 - 2DD' - 15D'^2)z = 12xy$.

Solution: A.E. is $m^2 - 2m - 15 = 0$

or $(m - 5)(m + 3) = 0. \quad \therefore m = 5, -3$

C.F. = $f_1(y + 5x) + f_2(y - 3x)$

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{D^2 - 2DD' - 15D'^2} \cdot 12xy = \frac{1}{(D + 3D')} \cdot \frac{1}{(D - 5D')} \cdot 12xy \\
 &= \frac{12}{(D + 3D')} \int x(a - 5x) dx \quad \text{where } y + 5x = a \\
 &= \frac{12}{(D + 3D')} \left(\frac{1}{2} a x^2 - \frac{5}{3} x^3 \right)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{12}{(D+3D')} \left\{ \frac{1}{2} (y+5x)x^2 - \frac{5}{3} x^3 \right\} \\
&= \frac{2}{(D+3D')} \cdot (3x^2y + 5x^3) \\
&= 2 \int [3x^2(3x+b) + 5x^3] dx \quad \text{where } y-3x=b \\
&= 2 \int (14x^3 + 3x^2b) dx = 2 \left(\frac{14}{4} x^4 + x^3b \right) \\
&= 7x^4 + 2x^3 (y-3x) = x^4 + 2x^3y.
\end{aligned}$$

$\therefore$ The solution is

$$z = f_1(y+5x) + f_2(y-3x) + x^4 + 2x^3y.$$

Aliter.

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 - 2DD' - 15D'^2} 12xy \\
&= \frac{1}{D^2} \left(1 - \frac{2D'}{D} - \frac{15D'^2}{D^2} \right)^{-1} \cdot 12xy \\
&= \frac{1}{D^2} \left(1 + \frac{2D'}{D} + \dots \right) \cdot 12xy \\
&= \frac{1}{D^2} (12xy) + \frac{2}{D^3} \cdot 12x + 0 = 2x^3y + x^4.
\end{aligned}$$

Example 2: Solve $(D^2 - DD' - 2D'^2)z = (y-1)e^x$.

[Bhopal 2004; Delhi Maths (Hons.) 2004; Rewa 2003; Vikram 2002, 04]

Solution: Here A.E. is $m^2 - m - 2 = 0$

or

$$(m-2)(m+1) = 0. \quad \therefore m = 2, -1.$$

$$\text{C.F.} = f_1(y+2x) + f_2(y-x).$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2 - DD' - 2D'^2} \cdot (y-1)e^x \\
&= \frac{1}{(D-2D')} \cdot \frac{1}{(D+D')} \cdot (y-1)e^x \\
&= \frac{1}{(D-2D')} \cdot \int (x+a-1)e^x dx \quad \text{where } y-x=a. \\
&= \frac{1}{(D-2D')} \left[(a-1) \int e^x dx + \int x e^x dx \right] \\
&= \frac{1}{(D-2D')} \cdot [(a-1)e^x + (x-1)e^x]
\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{(D-2D')} \cdot (a+x-2) e^x = \frac{1}{(D-2D')} (y-2) e^x \\
 &= \int (b-2x-2) e^x \cdot dx \quad \text{where } y+2x=b \\
 &= (b-2) \int e^x dx - 2 \int x e^x dx \\
 &= (b-2) e^x - 2(x-1) e^x = (b-2x) e^x = y e^x.
 \end{aligned}$$

∴ The solution is

$$z = f_1(y+2x) + f_2(y-x) + y e^x$$

Example 3: Solve $(D^2 + 2DD' + D'^2)z = 2 \cos y - x \sin y$.

[Indore 2004; Bilaspur 2002; Rewa 2002]

Solution: A.E. is

$$m^2 + 2m + 1 = 0.$$

or

$$(m+1)^2 = 0 \quad \therefore m = -1, -1,$$

$$\text{C.F.} = f_1(y-x) + x f_2(y-x)$$

$$\begin{aligned}
 \text{P.I.} &= \frac{1}{(D+D')} \cdot \frac{1}{(D+D')} \cdot (2 \cos y - x \sin y) \\
 &= \frac{1}{(D+D')} \int [2 \cos(x+a) - x \sin(x+a)] dx \quad \text{where } y-x=a \\
 &= \frac{1}{(D+D')} [2 \sin(x+a) - \{-x \cos(x+a) + \sin(x+a)\}] \\
 &= \frac{1}{(D+D')} [\sin(x+a) + x \cos(x+a)] \\
 &= \frac{1}{(D+D')} \cdot (\sin y + x \cos y) \\
 &= \int [\sin(x+b) + x \cos(x+b)] dx \quad \text{where } y-x=b \\
 &= -\cos(x+b) + x \sin(x+b) + \cos(x+b) \\
 &= x \sin(x+b) = x \sin y.
 \end{aligned}$$

∴ The solution is $z = f_1(y-x) + x f_2(y-x) + x \sin y$

Example 4: Solve $(r + s - 6t) = y \cos x$.

[Meerut 2000; 02; Indore 2002; Garhwal 2005; Purvanchal 2007; Bilaspur 2002; Delhi Maths (Hons.) 2007; I.A.S. 2008]

Solution: The given equation can be written as

$$(D^2 + DD' - 6D'^2)z = y \cos x$$

A.E. is $m^2 + m - 6 = 0$

or $(m+3)(m-2) = 0 \quad \therefore m = -3, 2.$

C.E. = $f_1(y-3x) + f_2(y+2x)$

P.I. = $\frac{1}{D^2 + DD' - 6D'^2} \cdot y \cos x = \frac{1}{(D-2D')} \cdot \frac{1}{(D+3D')} \cdot y \cos x$

= $\frac{1}{(D-2D')} \cdot \int (3x+a) \cos x \, dx \quad \text{where } y-3x = a$

= $\frac{1}{(D-2D')} \cdot [a \sin x + 3x \sin x + 3 \cos x]$

= $\frac{1}{(D-2D')} \cdot [(y-3x) \sin x + 3x \sin x + 3 \cos x]$

= $\frac{1}{(D-2D')} \cdot (y \sin x + 3 \cos x)$

= $\int [(b-2x) \sin x + 3 \cos x] \, dx, \quad \text{where } y+2x = b$

= $-b \cos x - 2(-\cos x + \sin x) + 3 \sin x$

= $-(y+2x) \cos x + 2x \cos x + \sin x = -y \cos x + \sin x$

$\therefore$ The solution is

$z = f_1(y-3x) + f_2(y+2x) - y \cos x + \sin x$

Exercise 10.4

Solve the following partial differential equations by finding the P.I. by general method

1. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \sin x.$

2. $(D-D')(D+2D')z = (y+1)e^x.$

[I.A.S. 2004]

3. $(D^2 + DD' - 6D'^2)z = y \sin x.$

4. $(2D^2 - 5DD' + 2D'^2)z = 5 \sin(2x+y).$

5. $r - 2s + t = \sin(2x+3y).$

6. $(D^3 - 4D^2D' + 4DD'^2)z = \cos(2x+y).$

7. $(D^3 - 4D^2D' + 4DD'^2)z = \sin(2x+y).$

8. $(D^3 - 3DD'^2 - 2D'^3)z = \cos(x+2y) - e^y(3+2x).$

9. $(D^3 + D^2D' - DD'^2 - D'^3)z = e^y \cos 2x.$

Answers 10.4

1.	$z = f(y-x) - \cos x.$
2.	$z = f_1(y+x) + f_2(y-2x) + y e^x.$
3.	$z = f_1(y-3x) + f_2(y+2x) - (y \sin x + \cos x).$
4.	$z = f_1(2y+x) + f_2(y+2x) - (5x/3) \cos(2x+y).$
5.	$z = f_1(x+y) + x f_2(x+y) - \sin(2x+3y).$
6.	$z = f_1(y) + f_2(y+2x) + x f_3(y+2x) + (x^2/4) \sin(2x+y).$
7.	$z = f_1(y) + f_2(y+2x) + x f_3(y+2x) - (x^2/4) \cos(2x+y).$
8.	$z = f_1(y-x) + x f_2(y-x) + f_3(y+2x) + (1/27) \sin(y+2x) + x e^x.$
9.	$z = f_1(y+x) + f_2(y-x) + x f_3(y-x) - (e^y/25)(\cos 2x + 2 \sin 2x).$

10.10 Non-Homogeneous Linear Differential Equations with Constant Coefficients

A linear partial differential equation which is not homogeneous is called a non-homogeneous linear partial differential equation.

Thus in a non-homogeneous linear partial differential equation with constant coefficients all the derivative appearing in it are not of the same order and the coefficients of the dependent variable and all its derivatives are constant.

Consider non-homogeneous partial differential equation with constant coefficients,

$$f(D, D') z = \phi(x, y) \quad \dots(1)$$

where $f(D, D')$ is not homogeneous function of D and D' .

These equations are classified into two main types **reducible** and **irreducible** differential equations.

[Delhi Maths (Hons.) 2001; 04]

1. Differential equation (1) is known as **reducible differential equation** if $f(D, D')$ is reducible i.e., if $f(D, D')$ can be written as the product of linear factors of the form $aD + bD' + c$, whose a, b, c are constants.

For example $(D^2 - D'^2 + 3D + D' + 2)z = x^2 + xy$ is reducible non-homogeneous linear equation with constant coefficients, as

$$f(D, D') = D^2 - D'^2 + 3D + D' + 2 = (D + D' + 1)(D - D' + 2)$$

2. Differential equation (1) is known as **irreducible differential equation** if $f(D, D')$ is irreducible *i.e.*, if $f(D, D')$ cannot be written as the product of linear factors of the form $aD + bD' + c$, where a, b, c are constants.

$$\text{For example } (D^2 - D'^2 + D + 2)z = x^2 + xy$$

is irreducible non-homogeneous linear equation with constant coefficients, as $f(D, D') = D^2 - D'^2 + D + 2$ cannot be written as the product of linear factors.

10.11 Methods of Finding the Complementary Function (C.F.) of Reducible Non-homogeneous Linear Partial Differential Equation with Constant Coefficients

Let the given partial differential equation be

$$f(D, D')z = \phi(x, y) \quad \dots(1)$$

where $f(D, D')$ is reducible non-homogeneous function of D, D' .

The complementary function (C.F.) of (1) is the solution of the equation

$$f(D, D')z = 0$$

$$\text{or } (a_1D - b_1D' - c_1)(a_2D - b_2D' - c_2) \dots (a_nD - b_nD' - c_n)z = 0 \quad \dots(2)$$

where a_r, b_r, c_r are constants $r = 1, 2, \dots, n$.

Case I : Distinct factors *i.e.*, when all factors in (2) are distinct.

Obviously solution of the equation

$$(a_rD - b_rD' - c_r)z = 0, \quad r = 1, 2, \dots, n \quad \dots(3)$$

is a solution of (2)

$$\text{Equation (3)} \Rightarrow a_r p - b_r q = c_r z$$

Which is Lagrange's equation.

Whose subsidiary equations are

$$\frac{dx}{a_r} = \frac{dy}{-b_r} = \frac{dz}{c_r z}$$

Taking the 1st and 2nd fractions, we get

$$a_r dy + b_r dx = 0 \Rightarrow a_r y + b_r x = \lambda_1 \text{ (arbitrary constant)} \quad \dots(4)$$

Again taking 1st and 3rd fractions, we get

$$\frac{dz}{z} = \frac{c_r}{a_r} dx, \quad \text{if } a_r \neq 0$$

$$\text{Integrating,} \quad \log z = \frac{c_r}{a_r} x + \log \lambda_2 \Rightarrow z = \lambda_2 e^{(c_r/a_r)x} \quad \dots(5)$$

where λ_2 is arbitrary constant

Also taking 2nd and 3rd fractions, we get

$$z = \lambda_2' e^{-(c_r/b_r)y}, \quad \text{if } b_r \neq 0 \quad \dots(6)$$

From (4) and (5) or from (4) and (6), the solution of (3) is

$$z = e^{(c_r/a_r)x} \cdot f_r(a_r y + b_r x), \quad \text{if } a_r \neq 0$$

$$\text{or} \quad z = e^{-(c_r/b_r)y} \cdot \psi_r(a_r y + b_r x), \quad \text{if } b_r \neq 0$$

where f_r and ψ_r are arbitrary functions.

$\therefore$ Solution of (2) i.e., C.F. of (1) is the sum of solutions of the equations of type (3) for each factor of (2).

$$\therefore \text{C.F.} = \sum_{r=1}^n e^{(c_r/a_r)x} \cdot f_r(a_r y + b_r x), \quad \text{if } a_r \neq 0$$

$$\text{or} \quad \text{C.F.} = \sum_{r=1}^n e^{-(c_r/b_r)y} \cdot \psi_r(a_r y + b_r x), \quad \text{if } b_r \neq 0$$

Case II : Repeated factors i.e., when all factors in (2) are not distinct.

Let a factor $(aD - bD' - c)$ be repeated two times in (2).

Then consider the equation

$$(aD - bD' - c)(aD - bD' - c)z = 0 \quad \dots(7)$$

$$\text{or} \quad (aD - bD' - c)v = 0 \quad \dots(8)$$

$$\text{where} \quad (aD - bD' - c)z = v \quad \dots(9)$$

As in case I, solution of (8) is

$$v = e^{(c/a)x} \Psi (ay + bx), \quad \text{if } a \neq 0 \quad \dots(10)$$

$$\text{and} \quad v = e^{-(c/b)y} \Phi (ay + bx), \quad \text{if } b \neq 0 \quad \dots(11)$$

when $a \neq 0$, from (9) and (10), we have

$$(aD - bD' - c)z = e^{(c/a)x} \Psi (ay + bx)$$

$$\text{or} \quad ap - bq = cz + e^{(c/a)x} \Psi (ay + bx) \quad \dots(12)$$

Which is Lagrange's equation.

$\therefore$ The subsidiary equations are

$$\frac{dx}{a} = \frac{dy}{-b} = \frac{dz}{cz + e^{(c/a)x} \Psi (ay + bx)}$$

$$\text{Taking 1st and 2nd fractions } a dy + b dx = 0 \Rightarrow ay + bx = \lambda_3 \text{ (Arbitrary constant)} \quad \dots(13)$$

Taking 1st and 3rd fractions, we get

$$\frac{dz}{dx} - \frac{c}{a}z = \frac{1}{a}e^{(c/a)x} \Psi (\lambda_3), \quad \text{if } a \neq 0 \text{ using (13)}$$

which is a linear differential equation, whose

$$\text{I.F.} = e^{\int (-c/a) dx} = e^{-(c/a)x}$$

$\therefore$ Its solution is

$$\begin{aligned} z \cdot e^{-(c/a)x} &= \lambda_4 + \int e^{-(c/a)x} \cdot \frac{1}{a} e^{(c/a)x} \Psi (\lambda_3) dx \\ &= \lambda_4 + \frac{1}{a} x \Psi (\lambda_3) \end{aligned}$$

$$\Rightarrow \quad z \cdot e^{-(c/a)x} - (x/a) \Psi (ay + bx) = \lambda_4 \quad \dots(14)$$

From (13) and (14), the solution of (12) or (7) is

$$z e^{-(c/a)x} - (x/a) \Psi (ay + bx) = f_1(ay + bx)$$

$$\text{or} \quad z = e^{(c/a)x} \cdot [f_1(ay + bx) + x f_2(ay + bx)] \quad \text{where } \frac{1}{a} \Psi (ay + bx) = f_2(ay + bx)$$

where f_1 and f_2 are arbitrary constants. Similarly if $b \neq 0$, then taking 2nd and 3rd fractions and proceeding as above we get

$$z = e^{-(c/b)y} [\psi_1(ay + bx) + y \psi_2(ay + bx)]$$

where ψ_1 and ψ_2 one arbitrary constants.

Hence in general if the factor $(aD - bD' - c)$ is repeated r times in (2), then the part of C.F. corresponding to this factor is

$$e^{(c/a)x} [f_1(ay + bx) + x f_2(ay + bx) + \dots + x^{r-1} f_r(ay + bx)], \quad \text{if } a \neq 0$$

or
$$e^{-(c/b)y} [\psi_1(ay + bx) + y \psi_2(ay + bx) + \dots + y^{r-1} \psi_r(ay + bx)], \quad \text{if } b \neq 0$$

10.12 Working Method of Finding C.F. of Reducible Non-homogeneous Linear Partial Differential Equation with Constant Coefficients

Let the given partial differential equation be

$$f(D, D') z = \phi(x, y)$$

Then factorise $f(D, D')$ into linear factors of the form $(aD - bD' - c)$. Then write the part of C.F. for each factor according to the following rules. The required C.F. of (1) is obtained by adding all these parts of C.F. for all factors

Factors		Part of C.F.
1.	For non-repeated factor $(a_r D - b_r D' - c_r)$	$e^{(c_r/a_r)x} \cdot f_r(a_r y + b_r x), \quad \text{if } a_r \neq 0$ $e^{-(c_r/b_r)y} \cdot \psi_r(a_r y + b_r x), \quad \text{if } b_r \neq 0$
2.	For repeated factor $(aD - bD' - c)^r$	$e^{(c/a)x} [f_1(ay + bx) + x f_2(ay + bx) + \dots + x^{r-1} f_r(ay + bx)], \quad \text{if } a \neq 0$ $e^{-(c/b)y} [\psi_1(ay + bx) + y \psi_2(ay + bx) + \dots + y^{r-1} \psi_r(ay + bx)], \quad \text{if } b \neq 0$

10.13 Method of Finding C.F. of Irreducible Non-homogeneous Linear Partial Differential Equation with Constant Coefficients

Let the differential equation be given by

$$f(D, D') z = \phi(x, y) \quad \dots(1)$$

where $f(D, D')$ is irreducible

Let $(aD^l + bD'^m)$ be a factor of $f(D, D')$ which cannot be factorised in linear factors.

The complementary function (C.F.) corresponding to $(aD^l + bD'^m)$ i.e., solution of

$$(aD^l + bD'^m)z = 0 \quad \dots(2)$$

is taken as
$$z = \sum A e^{hx+ky} \quad \dots(3)$$

where A, h, k are arbitrary constants. Substituting in, (2), we get $ah^l + bk^m = 0 \quad \dots(4)$

From (4) we find the values of h in terms of k or value of k in terms of h (If possible). Substituting in (3) we obtain the C.F. corresponding to the factor $(aD^l + bD'^m)$ in which we have only two arbitrary constants A and h or A and k .

10.14 Working Method of Finding C.F. of Irreducible Non-homogeneous Linear Partial Differential Equation with Constant Coefficients

Let the given differential equation be

$$f(D, D')z = \phi(x, y) \quad \dots(1)$$

where $f(D, D')$ is irreducible function of D and D' .

To find the C.F. of this equation we proceed as follows.

- (i) Factorise $f(D, D')$ as the product of linear and irreducible factors.
- (ii) Find parts of C.F. for all linear factors by the methods given in 10.12.
- (iii) Find parts of C.F. for all irreducible factors by the method given in 10.13.
- (iv) C.F. of (1) is obtained by adding all the parts of C.F. for linear and irreducible factors of $f(D, D')$.

10.15 Determination of the Particular Integral (P.I.) of Non-homogeneous Linear Partial Differential Equation (Reducible or Irreducible) with Constant Coefficients

The methods of determination of P.I. of non-homogeneous partial differential equation with constant coefficients are very similar to those of ordinary differential equation with constant coefficients. Here we are giving few cases only.

Let the given differential equation be

$$f(D, D')z = \phi(x, y)$$

$$\therefore \text{P.I.} = \frac{1}{f(D, D')} \cdot \phi(x, y)$$

Case I : When $\phi(x, y) = e^{ax+by}$ and $f(a, b) \neq 0$.

$$\text{Then} \quad \text{P.I.} = \frac{1}{f(D, D')} \cdot e^{ax+by} = \frac{1}{f(a, b)} e^{ax+by}$$

provided $f(a, b) \neq 0$.

i.e., in this case we put $D = a$ and $D' = b$ in $f(D, D')$

Case II : When $\phi(x, y) = \sin(ax + by)$ or $\cos(ax + by)$.

$$\begin{aligned} \text{Then} \quad \text{P.I.} &= \frac{1}{f(D, D')} \sin(ax + by) = \frac{1}{f(D^2, DD', D'^2)} \sin(ax + by) \\ &= \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax + by) \end{aligned}$$

$$\begin{aligned} \text{and} \quad \text{P.I.} &= \frac{1}{f(D, D')} \cos(ax + by) = \frac{1}{f(D^2, DD', D'^2)} \cos(ax + by) \\ &= \frac{1}{f(-a^2, -ab, -b^2)} \cos(ax + by) \end{aligned}$$

$$\text{Provided } f(-a^2, -ab, -b^2) \neq 0$$

i.e., in this case we put $D^2 = -a^2, DD' = -ab, D'^2 = -b^2$.

Case III : When $\phi(x, y) = x^m y^n$, where m, n are positive integers.

$$\text{Then} \quad \text{P.I.} = \frac{1}{f(D, D')} \cdot x^m y^n$$

which is evaluated by expanding $1/f(D, D')$ in powers of

1. D, D' if there is constant term in $f(D, D')$
2. D/D' or D'/D if there is no constant term in $f(D, D')$. In general we expand in powers of D'/D if $n < m$ and expand in powers of D/D' if $m < n$

Note : The P.I. obtained by expanding $1/f(D, D')$ in powers of D'/D and by expanding in powers of D/D' looks to be in different forms. But both forms are correct because each form can be transformed to other with the help of the C.F. of the equation.

Case IV : When $\phi(x, y) = Ve^{ax+by}$, where V is a function of x and y .

Then
$$\text{P.I.} = \frac{1}{f(D, D')} V \cdot e^{ax+by} = e^{ax+by} \frac{1}{f(D+a, D'+b)} \cdot V$$

Case V : When $\phi(x, y) = e^{ax+by}$ and $f(a, b) = 0$.

Then
$$\text{P.I.} = \frac{1}{f(D, D')} e^{ax+by} = e^{ax+by} \frac{1}{f(D+a, D'+b)} \cdot 1$$

Now proceed in case (III).

Case VI : When $\phi(x, y) = \sin(ax+by)$ or $\cos(ax+by)$, when $f(-a^2, -ab, -b^2) = 0$.

Then
$$\begin{aligned} \text{P.I.} &= \frac{1}{f(D, D')} \sin(ax+by) \\ &= \text{I.P. of } \frac{1}{f(D, D')} e^{i(ax+by)} = \text{I.P. of } e^{i(ax+by)} \frac{1}{f(D+ia, D'+ib)} \cdot 1 \end{aligned}$$

and
$$\begin{aligned} \text{P.I.} &= \frac{1}{f(D, D')} \cos(ax+by) = \text{R.P. of } \frac{1}{f(D, D')} e^{i(ax+by)} \\ &= \text{R.P. of } e^{i(ax+by)} \frac{1}{f(D+ia, D'+ib)} \cdot 1 \end{aligned}$$

Further proceed as in case III.

Solved Examples

Example 1: If $\alpha D + \beta D' + \gamma$ is a factor of $f(D, D')$ and $\phi(\xi)$ is an arbitrary function of the single variable ξ and $\alpha \neq 0$, then show that

$$u = \exp(-\gamma x/\alpha) \cdot \phi(\beta x - \alpha y)$$

is a solution of partial differential equation $f(D, D')z = 0$.

[Meerut 2010 (Sem. I)]

Solution: Since $\alpha D + \beta D' + \gamma$ is a factor of $f(D, D')$, so the solution of the equation

$$(\alpha D + \beta D' + \gamma)z = 0 \quad \dots(1)$$

is also the solution of the partial differential equation $f(D, D')z = 0 \quad \dots(2)$

$$\text{Equation (1)} \Rightarrow \alpha Dz + \beta D'z + \gamma z = 0 \Rightarrow \alpha p + \beta q = -\gamma z$$

which is the Lagrange's equation,

whose subsidiary equations one

$$\frac{dx}{\alpha} = \frac{dy}{\beta} = \frac{dz}{-\gamma z}$$

Taking 1st and 2nd fractions, we get

$$\beta dx - \alpha dy = 0 \Rightarrow \beta x - \alpha y = c_1 \quad \dots(3)$$

where c_1 is arbitrary constant.

Again taking 1st and 3rd fractions, we get

$$\frac{dz}{z} = -\frac{\gamma}{\alpha} dx$$

$$\text{Integrating,} \quad \log z = -\frac{\gamma x}{\alpha} + \log c_2 \Rightarrow \log(z/c_2) = -\gamma x/\alpha$$

$$\Rightarrow z = c_2 e^{-(\gamma x/\alpha)} \quad \dots(4)$$

where c_2 is an arbitrary constant

From (3) and (4), the solution of (1) or (2) is

$$z = e^{-(\gamma x/\alpha)} \cdot \phi(\beta x - \alpha y)$$

or $u = \exp(-\gamma x/\alpha) \phi(\beta x - \alpha y)$ is solution of (2).

Example 2: Solve $(D^2 - D'^2 + D - D')z = 0$.

Solution: The given equation is

$$(D - D')(D + D' + 1)z = 0.$$

$$\therefore z = f_1(y + x) + e^{-x} f_2(y - x).$$

Example 3: Solve $DD'(D - 2D' - 3)z = 0$.

$$\begin{aligned} \text{Solution: Here } z &= e^{0 \cdot x} f_1(y) + e^{0 \cdot x} f_2(0 - x) + e^{3x} f_3(y + 2x) \\ &= f_1(y) + \phi(x) + e^{3x} f_3(y + 3x) \end{aligned}$$

Example 4: Solve $r + 2s + t + 2p + 2q + z = 0$.

Solution: The given equation can be written as

$$(D^2 + 2DD' + D'^2 + 2D + 2D' + 1)z = 0$$

$$\text{or } (D + D' + 1)^2 z = 0$$

$$\therefore z = e^{-x} f_1(y - x) + x e^{-x} f_2(y - x).$$

Example 5: Solve $(2D^4 - 3D^2D' + D'^2)z = 0$.

Solution: The given equation can be written as

$$(2D^2 - D')(D^2 - D')z = 0 \quad \dots(1)$$

Let
$$z = \sum A e^{hx+ky}$$

be the C.F. corresponding to $(D^2 - D')z = 0$

$$\therefore (D^2 - D')z = \sum A h^2 e^{hx+ky} - \sum A k e^{hx+ky} = \sum A (h^2 - k) e^{hx+ky}$$

$\therefore$ The given equation (1) will be satisfied by this substitution

if
$$h^2 - k = 0.$$

$\therefore$ The C.F. corresponding to $(D^2 - D')z = 0$ is

$$\sum A e^{hx+h^2y}$$

Similarly C.F. corresponding to $(2D^2 - D')z$ is

$$\sum B e^{h'x+k'y} \quad \text{where} \quad 2h'^2 - k' = 0$$

i.e.,
$$\sum B e^{h'x+2h'^2y}.$$

$\therefore$ The most general solution of the given equation is

$$z = \sum A e^{hx+h^2y} + \sum B e^{h'x+2h'^2y}.$$

Example 6: Solve $(D - 2D' - 1)(D - 2D'^2 - 1)z = 0$.

Solution: C.F. corresponding to the first factor is

$$z = e^x f_1(y + 2x)$$

and C.F. corresponding to 2nd factor is

$$\sum A e^{hx+ky}$$

where
$$h - 2k^2 - 1 = 0 \quad \text{or} \quad h = 2k^2 + 1.$$

$$\therefore z = e^x f_1(y + 2x) + \sum A e^{ky + (2k^2 + 1)x}.$$

Example 7: Solve $(D - D' - 1)(D - D' - 2)z = e^{2x-y} + x$.

Solution: Here C.F. = $e^x f_1(y+x) + e^{2x} f_2(y+x)$

and P.I. corresponding to e^{2x-y}

$$\begin{aligned} &= \frac{1}{(D - D' - 1)(D - D' - 2)} \cdot e^{2x-y} \\ &= \frac{1}{(2+1-1)(2+1-2)} \cdot e^{2x-y} = \frac{1}{2} \cdot e^{2x-y} \end{aligned}$$

$$\begin{aligned} \text{P.I. corresponding to } x &= \frac{1}{(D - D' - 1)(D - D' - 2)} \cdot x \\ &= \frac{1}{2} (1 - D + D')^{-1} \left(1 - \frac{1}{2} D + \frac{1}{2} D' \right)^{-1} \cdot x \\ &= \frac{1}{2} (1 + D - D' \dots) \left(1 + \frac{D}{2} - \frac{D'}{2} \dots \right) \cdot x \\ &= \frac{1}{2} \left(1 + D + \frac{D}{2} - \frac{3}{2} D' + \dots \right) \cdot x \\ &= \frac{1}{2} x + \frac{1}{2} \cdot \frac{3}{2} D x - \frac{3}{2} D' x + \dots = \frac{1}{2} x + \frac{3}{4} \cdot \end{aligned}$$

$\therefore$ the solution is

$$z = e^x f_1(y+x) + e^{2x} f_2(y+x) + \frac{x}{2} + \frac{3}{4} + \frac{1}{2} e^{2x-y}$$

Example 8: Solve $(D^3 - 3DD' + D + 1)z = e^{2x+3y}$.

[Kanpur 2006]

Solution: Here $D^3 - 3DD' + D + 1$ cannot be resolved into factors linear in D and D' .

$\therefore$ let C.F. = $\sum A e^{hx+ky}$ where $h^3 - 3hk + h + 1 = 0$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D^3 - 3DD' + D + 1} \cdot e^{2x+3y} \\ &= \frac{e^{2x+3y}}{2^3 - 3 \cdot 2 \cdot 3 + 2 + 1} = -\frac{1}{7} e^{2x+3y}. \end{aligned}$$

$\therefore$ The solution is $z = -\frac{1}{7} e^{2x+3y} + \sum A e^{hx+ky}$ where $h^3 - 3hk + h + 1 = 0$

Example 9: Solve $r - s + p = 1$.

[Sagar 2003; Vikram 2004]

Solution: The given equation can be written as

$$(D^2 - DD' + D)z = 0 \quad \text{or} \quad D(D - D' + 1)z = 0$$

$$\begin{aligned} \therefore \quad \text{C.F.} &= e^{0x} f_1(y) + e^{-x} f_2(y+x) \\ &= f_1(y) + e^{-x} f_2(y+x). \end{aligned}$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{D(D - D' + 1)} \cdot 1 = \frac{1}{D} (1 + D - D')^{-1} \cdot 1 \\ &= \frac{1}{D} \cdot 1 = x. \end{aligned}$$

$$\therefore \quad \text{The solution is} \quad z = f_1(y) + e^{-x} f_2(y+x) + x$$

Example 10: Solve $s + p - q = z + xy$.

Solution: The given equation can be written as

$$(DD' + D - D' - 1)z = xy \quad \text{or} \quad (D - 1)(D' + 1)z = xy$$

$$\begin{aligned} \therefore \quad \text{C.F.} &= e^x f_1(y) + e^{-y} f_2(x) \\ \text{P.I.} &= \frac{1}{(D - 1)(D' + 1)} \cdot xy \\ &= -(1 - D)^{-1} (1 + D')^{-1} xy = -(1 + D + \dots) (1 - D' + \dots) xy \\ &= -(1 + D - D' - DD' \dots) xy = -(xy + y - x - 1) \end{aligned}$$

$$\therefore \quad \text{The solution is } z = e^x f_1(y) + e^{-y} f_2(x) - xy - y + x + 1.$$

Example 11: Solve $(D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}$.

[Delhi Maths (Hons.) 2005, 07; Kurukshetra 2004]

Solution: The given equation can be written as

$$(D - D')(D + D' - 3)z = xy + e^{x+2y}$$

$$\therefore \quad \text{C.F.} = f_1(y+x) + e^{3x} f_2(y-x),$$

$$\text{P.I. corresponding to } xy = \frac{1}{(D - D')(D + D' - 3)} xy$$

$$\begin{aligned}
 &= -\frac{1}{3D} \left(1 - \frac{D'}{D}\right)^{-1} \left(1 - \frac{D}{3} - \frac{D'}{3}\right)^{-1} \cdot x \cdot y \\
 &= -\frac{1}{3D} \left(1 + \frac{D'}{D} \dots\right) \left(1 + \frac{D}{3} + \frac{D'}{3} + \frac{2DD'}{9} \dots\right) x \cdot y \\
 &= -\frac{1}{3D} \left(1 + \frac{D}{3} + \frac{D'}{3} + \frac{D'}{D} + \frac{D'}{3} + \frac{2DD'}{9} \dots\right) x \cdot y \\
 &= -\frac{1}{3} \left(\frac{1}{D} + \frac{1}{3} + \frac{2}{3} \frac{D'}{D} + \frac{D'}{D^2} + \frac{2}{9} D' \dots\right) x \cdot y \\
 &= -\frac{1}{3} \left(\frac{x^2}{2} \cdot y + \frac{x \cdot y}{3} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{2}{9} \cdot x\right)
 \end{aligned}$$

$$\begin{aligned}
 \text{P.I. corresponding to } e^{x+2y} &= \frac{1}{(D+D'-3)(D-D')} \cdot e^{x+2y} \\
 &= \frac{1}{(D+D'-3)} \cdot \frac{1}{1-2} e^{x+2y} \\
 &= -\frac{1}{(D+D'-3)} e^{x+2y} = -e^{x+2y} \frac{1}{(D+1+D'+2-3)} \cdot 1 \\
 &= -e^{x+2y} \frac{1}{D+D'} \cdot 1 = -e^{x+2y} \frac{1}{D} \left(1 + \frac{D'}{D}\right)^{-1} \cdot 1 \\
 &= -e^{x+2y} \cdot \frac{1}{D} \left(1 - \frac{D'}{D} \dots\right) \cdot 1 = -e^{x+2y} \cdot \left(\frac{1}{D} - \frac{D'}{D^2} \dots\right) \cdot 1 \\
 &= -x e^{x+2y}
 \end{aligned}$$

∴ The solution is $z = f_1(y+x) + e^{3x} f_2(y-x)$

$$-\frac{1}{3} \left(\frac{x^2}{2} y + \frac{x \cdot y}{3} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{2}{9} x\right) - x e^{x+2y}$$

Note : Using another method the P.I. corresponding to e^{x+2y} may also be obtained as $-ye^{x+2y}$

Example 12: Solve $(D^2 - DD' + D' - 1)z = \cos(x+2y) + e^y$.

[M.D.U. Rohtak 2004; Delhi Maths (Hons.) 2001]

Solution: The given equation is

$$(D-1)(D-D'+1)z = \cos(x+2y) + e^y$$

$$\therefore \text{C.F.} = e^x f_1(y) + e^{-x} f_2(y+x)$$

P.I. corresponding to $\cos(x+2y)$

$$\begin{aligned} &= \frac{1}{D^2 - DD' + D' - 1} \cos(x+2y) \\ &= \frac{1}{-1^2 - (-1.2) + D' - 1} \cdot \cos(x+2y) \\ &= \frac{1}{D'} \cos(x+2y) = \frac{D'}{D'^2} \cos(x+2y) \\ &= \frac{D'}{-2^2} \cos(x+2y) = -\frac{1}{4} \{-2 \sin(x+2y)\} = \frac{1}{2} \sin(x+2y). \end{aligned}$$

P.I. corresponding to e^y

$$\begin{aligned} &= \frac{1}{(D - D' + 1)(D - 1)} \cdot e^y = \frac{1}{(D - D' + 1)(0 - 1)} e^y \\ &= -e^y \cdot \frac{1}{(D - (D' + 1) + 1)} \cdot 1 = -e^y \cdot \frac{1}{D - D'} \cdot 1 \\ &= -e^y \cdot \frac{1}{D} \left(1 - \frac{D'}{D}\right)^{-1} \cdot 1 = -e^y \cdot \frac{1}{D} \cdot 1 = -x e^y. \end{aligned}$$

$\therefore$ The solution is

$$z = e^x f_1(y) + e^{-x} f_2(y+x) + \frac{1}{2} \sin(x+2y) - x e^y.$$

Example 13: Solve $(D - 3D' - 2)^2 z = 2e^{2x} \tan(y+3x)$. [Delhi Maths (Hons.) 2004]

Solution: Here C.F. = $e^{2x} f_1(y+3x) + x \cdot e^{2x} f_2(y+3x)$,

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D - 3D' - 2)^2} 2e^{2x} \tan(y+3x) \\ &= 2e^{2x} \cdot \frac{1}{(D + 2 - 3D' - 2)^2} \tan(y+3x) \\ &= 2e^{2x} \cdot \frac{1}{(D - 3D')^2} \cdot \tan(y+3x) \\ &= 2e^{2x} \cdot \frac{x^2}{2} \tan(y+3x) \quad \text{By Case II 10.8.} \\ &= x^2 e^{2x} \tan(y+3x) \end{aligned}$$

$\therefore$ The solution is $z = e^{2x} f_1(y+3x) + x e^{2x} f_2(y+3x) + x^2 e^{2x} \tan(y+3x)$.

Exercise 10.5

Solve the following partial differential equations.

1. $(D + D' - 1)(D + 2D' - 2)z = 0.$

2. $(D + 1)(D + D' - 1)z = 0.$

3. $(D^2 - DD' + D' - 1)z = 0.$

4. $t + s + q = 0.$

5. $(3D - 2)(7D' + 2)DD'(2D + 5D' + 7)z = 0.$

6. $(2D - 3)(3D - 5D' - 8)^2 z = 0.$

7. $(D - D'^2)z = 0.$

8. $(D^2 + D + D')z = 0.$

9. $(D + 2D' - 3)(D^2 + D')z = 0.$

10. $(2D - 3D' + 7)^2 (D^2 + 3D')z = 0.$

11. $(DD' + aD + bD' + ab)z = e^{mx + ny}.$

12. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x + 3y}.$

13. $(D^2 - D'^2 + D - D')z = e^{2x + 3y}.$

[Ravishankar 2005]

14. $(D^2 - D'^2 - 3D')z = e^{x + 2y}.$

15. $(D^2 + DD' + D' - 1)z = \sin(x + 2y).$ [Meerut 2001 (B.P.); Ravishankar 2004; Bilaspur 2003]

16. $(D - D' - 1)(D - D' - 2)z = \sin(2x + 3y).$

[Kurukshetra 2005]

17. $(2DD' + D'^2 - 3D')z = 3 \cos(3x - 2y).$

[M.D.U. Rohtak 2005]

18. $(D^2 - DD' - 2D)z = \sin(3x + 4y) - e^{2x + y}.$

19. $(D^2 - DD' - 2D)z = \sin(3x + 4y) + x^2 y.$

[Meerut 2000 (P)]

20. $(D - D'^2)z = \cos(x - 3y).$

[Delhi Maths (Hons.) 2007]

21. $(D + D' - 1)(D + 2D' - 3)z = 2x + 3y.$

[Bhopal 2000, 03, 04]

22. $(D - 1)(D - D' + 1)z = 1 + xy.$

23. $(D^2 - D')z = 2y - x^2.$

[Agra 2005; Delhi Maths (Hons.) 2004]

24. $r - s + 2q - z = x^2 y^2.$

25. $(D - 1)(D - D' + 1)z = e^{-y}.$

26. $(D^2 - D')z = x e^{ax + a^2 y}.$

27. $(3D^2 - 2D'^2 + D - 1)z = 4 e^{x + y} \cos(x + y).$

[Delhi Maths (Hons.) 2008]

28. $(D - 3D' - 2)^2 z = 2 e^{2x} \sin(y + 3x).$

[I.A.S. 2005]

Answers 10.5

1.	$z = e^x f_1(y-x) + e^{2x} f_2(y-2x).$	2.	$z = e^{-x} f_1(y) + e^x f_2(y-x).$
3.	$z = e^x f_1(y) + e^{-x} f_2(y+x).$	4.	$z = f_1(x) + e^{-x} f_2(y-x).$
5.	$z = e^{2x/3} f_1(2y) + e^{-(2y/7)} f_2(7x) + f_3(y) + f_4(x) + e^{-(7x/2)} f_5(2y-5x).$		
6.	$z = e^{3x/2} f_1(2y) + e^{(8x/3)} \{f_2(3y+5x) + x f_3(3y+5x)\}.$		
7.	$z = \sum A e^{k^2 x + k y}. A, k \text{ are arbitrary constants.}$		
8.	$z = \sum A e^{h x - (h^2 + h) y}. A, h \text{ are arbitrary constants.}$		
9.	$z = \sum A e^{h x - h^2 y} + e^{3x} \phi(y-2x). A, h \text{ are arbitrary constant.}$		
10.	$z = e^{-(7x/2)} \{f_1(2y+3x) + x f_2(2y+3x)\} + \sum A e^{h x - h^2 y/3}.$		
11.	$z = e^{-b x} f_1(y) + e^{-a y} f_2(x) + [(m+b)(n+a)]^{-1} e^{m x + n y}.$		
12.	$z = f_1(y-x) + e^{-2x} f_2(y+2x) - (1/10) e^{2x+3y}.$		
13.	$z = f_1(y+x) + e^{-x} f_2(y-x) - (1/6) e^{2x+3y}.$		
14.	$z = \sum A e^{(k^2+3k)^{1/2} x + k y} - (1/9) e^{x+2y}. A, h \text{ are arbitrary constant.}$		
15.	$z = e^{-x} f_1(y) + e^x f_2(y-x) - (1/20)[2 \cos(x+2y) + 4 \sin(x+2y)].$		
16.	$z = e^x f_1(y+x) + e^{2x} f_2(y+x) + (1/10)[\sin(2x+3y) - 3 \cos(2x+3y)].$		
17.	$z = f_1(x) + e^{3x/2} f_2(2y-x) + (3/50)[4 \cos(3x-2y) + 3 \sin(3x-2y)].$		
18.	$z = f_1(y) + e^{2x} f_2(y+x) + (1/15) \sin(3x+4y) + (2/15) \cos(3x+4y) + (1/2) e^{2x+y}.$		
19.	$z = f_1(y) + e^{2x} f_2(y+x) + (1/15)[\sin(3x+4y) + 2 \cos(3x+4y)].$ $-x^3 y/6 - x^2 y/4 + x^3/12 - x y/4 + x^2/4 + 3x/8.$		
20.	$z = \sum A e^{k(kx+y)} + (1/82)[\sin(x-3y) + 9 \cos(x-3y)]$ A and h are arbitrary constant.		
21.	$z = e^x f_1(y-x) + e^{3x} f_2(y-2x) + 2x/3 + y + 23/9.$		
22.	$z = e^x f_1(y) + e^{-x} f_2(y+x) - xy - x.$		
23.	$z = \sum A e^{h x + h^2 y} + x^2 y. A \text{ and } h \text{ are arbitrary constant.}$		
24.	$z = \sum A e^{h x + k y} - x^2 y^2 - 2y^2 + 4xy - 4x^2 y - 4 - 8x^2 - 16y + 16x.$ where $k = (h^2 - 1)/(h - 2).$		
25.	$z = e^x f_1(y) + e^{-x} f_2(y+x) - x e^y.$		
26.	$z = \sum A e^{h x + h^2 y} + e^{a x + a^2 y} (x^2/4a - x/4a^2). A \text{ and } h \text{ are arbitrary constants.}$		
27.	$z = \sum A e^{h x + k y} + (4/3) e^{x+y} \sin(x+y) \text{ where } 3h^2 - 2k^2 + h - 1 = 0.$		
28.	$z = e^{2x}[f_1(y+3x) + x f_2(y+3x)] + x^2 e^{2x} \sin(y+3x).$		

10.16 Equation Reducible to Homogeneous Linear Form

An equation in which the coefficient of derivative of any order is a multiple of the variables of the same degree, may be transformed into the partial differential equations with constant coefficients.

For this we substitute $x = e^X$, $y = e^Y$

so that, $X = \log x$ and $Y = \log y$.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{1}{x} \frac{\partial z}{\partial X} \quad \text{or} \quad x \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \quad \therefore \quad x \frac{\partial}{\partial x} \equiv \frac{\partial}{\partial X} \equiv D \quad (\text{say})$$

$$\text{Now} \quad x \frac{\partial}{\partial x} \left(x^{n-1} \frac{\partial^{n-1} z}{\partial x^{n-1}} \right) = x^n \frac{\partial^n z}{\partial x^n} + (n-1) x^{n-1} \frac{\partial^{n-1} z}{\partial x^{n-1}}$$

$$\therefore \quad x^n \frac{\partial^n z}{\partial x^n} = \left(x \frac{\partial}{\partial x} - n + 1 \right) x^{n-1} \frac{\partial^{n-1} z}{\partial x^{n-1}}$$

Putting $n = 2, 3, \dots$, we have

$$x^2 \frac{\partial^2 z}{\partial x^2} = (D-1) x \frac{\partial z}{\partial x} = D(D-1) z$$

$$x^3 \frac{\partial^3 z}{\partial x^3} = (D-2) x^2 \frac{\partial^2 z}{\partial x^2} = (D-2)(D-1) Dz, \text{ etc.}$$

$$\text{Similarly} \quad y \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} = D' z, \quad y^2 \frac{\partial^2 z}{\partial y^2} = D'(D'-1) z, \text{ etc.}$$

$$\text{Where} \quad y \frac{\partial}{\partial y} \equiv \frac{\partial}{\partial Y} \equiv D' \quad (\text{say})$$

$$\text{and} \quad xy \frac{\partial^2 z}{\partial x \partial y} = DD' z \text{ etc.}$$

Substituting in the given equation, it reduces to the form

$$F(D, D') z = V$$

which is an equation having constant coefficients and can easily be solved by the methods discussed above.

Solved Examples

Example 1: Solve $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0$.

[Purvanchal 2007]

Solution: Substituting $x = e^X$, $y = e^Y$

so that $X = \log x, Y = \log y$

and denoting $\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial Y}$ by D and D' respectively the given equation reduces to

$$[D(D-1) + 2DD' + D'(D'-1)]z = 0$$

or $(D + D')(D + D' - 1)z = 0$

Hence the solution is

$$\begin{aligned} z &= f_1(Y - X) + e^X f_2(Y - X) \\ &= f_1(\log y - \log x) + x f_2(\log y - \log x) \\ &= f_1\left(\log \frac{y}{x}\right) + x f_2\left(\log \frac{y}{x}\right) = \phi_1\left(\frac{y}{x}\right) + x \phi_2\left(\frac{y}{x}\right). \end{aligned}$$

Example 2: Solve $x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} - y \frac{\partial z}{\partial y} + x \frac{\partial z}{\partial x} = 0$.

Solution: Substituting $x = e^X, y = e^Y$

so that $X = \log x, Y = \log y$,

and denoting $\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial Y}$ by D and D' respectively the given equation reduces to

$$[D(D-1) - D'(D'-1) - D' + D]z = 0$$

or $(D^2 - D'^2)z = 0$ or $(D - D')(D + D')z = 0$

$$\begin{aligned} \therefore z &= f_1(Y + X) + f_2(Y - X) \\ &= f_1(\log y + \log x) + f_2(\log y - \log x) \\ &= f_1(\log xy) + f_2\left\{\log \frac{y}{x}\right\} \\ &= \phi_1(xy) + \phi_2(y/x) \end{aligned}$$

Example 3: Solve $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} - nx \frac{\partial z}{\partial x} - ny \frac{\partial z}{\partial y} + nz = x^2 + y^2$.

Solution: Substituting $x = e^X, y = e^Y$ so that $X = \log x, Y = \log y$,

if D and D' denote $\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial Y}$, the given equation reduces to

$$[D(D-1) + 2DD' + D'(D'-1) - nD - nD' + n]z = e^{2X} + e^{2Y}$$

or $(D + D' - 1)(D + D' - n)z = e^{2X} + e^{2Y}$

$$\begin{aligned}\therefore \text{C.F.} &= e^X f_1(Y - X) + e^{nX} f_2(Y - X) \\ &= x f_1(\log y - \log x) + x^n f_2(\log y - \log x) \\ &= x f_1\left(\log \frac{y}{x}\right) + x^n f_2\left(\log \frac{y}{x}\right) \\ &= x \phi_1\left(\frac{y}{x}\right) + x^n \phi_2\left(\frac{y}{x}\right) \\ \text{P.I.} &= \frac{1}{(D + D' - 1)(D + D' - n)} \cdot e^{2x} + \frac{1}{(D + D' - 1)(D + D' - n)} \cdot e^{2Y} \\ &= \frac{e^{2X}}{(2 + 0 - 1)(2 + 0 - n)} + \frac{e^{2Y}}{(0 + 2 - 1)(0 + 2 - n)} \\ &= \frac{e^{2X} + e^{2Y}}{2 - n} = \frac{x^2 + y^2}{2 - n}\end{aligned}$$

$\therefore$ The solution is $z = x \phi_1\left(\frac{y}{x}\right) + x^n \phi_2\left(\frac{y}{x}\right) + \frac{x^2 + y^2}{2 - n}$.

Example 4: Solve $x^2 \frac{\partial^2 z}{\partial x^2} - 4xy \frac{\partial^2 z}{\partial x \partial y} + 4y^2 \frac{\partial^2 z}{\partial y^2} + 6y \frac{\partial z}{\partial y} = x^3 y^4$.

[Jabalpur 2004; Vikram 2004]

Solution: Substituting $x = e^X$, $y = e^Y$, so that $X = \log x$, $Y = \log y$,

and denoting $\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial Y}$ by D and D' respectively, the equation reduces to

$$[D(D - 1) - 4DD' + 4D'(D' - 1) + 6D']z = e^{3X+4Y}$$

or $(D - 2D')(D - 2D' - 1)z = e^{3X+4Y}$

$$\begin{aligned}\text{C.F.} &= f_1(Y + 2X) + e^X f_2(Y + 2X) \\ &= f_1(\log y + 2 \log x) + x f_2(\log y + 2 \log x) \\ &= f_1(\log y x^2) + x f_2(\log y x^2) \\ &= \phi_1(y x^2) + x \phi_2(y x^2).\end{aligned}$$

$$\text{P.I.} = \frac{1}{(D - 2D')(D - 2D' - 1)} \cdot e^{3X+4Y}$$

$$= \frac{e^{3X+4Y}}{(3-8)(3-8-1)} = \frac{1}{30} x^3 y^4$$

∴ The solution is $z = \phi_1(y x^2) + x \phi_2(y x^2) + \frac{1}{30} x^3 y^4$.

Example 5: Solve $x^2 r - 3xys + 2y^2 t + p x + 2q y = x + 2y$.

Solution: The given equation can be written as

$$x^2 \frac{\partial^2 z}{\partial x^2} - 3xy \frac{\partial^2 z}{\partial x \partial y} + 2y^2 \frac{\partial^2 z}{\partial y^2} + x \frac{\partial z}{\partial x} + 2y \frac{\partial z}{\partial y} = x + 2y$$

Substituting $x = e^X$, $y = e^Y$ so that $X = \log x$, $Y = \log y$ and denoting $\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial Y}$ by D and D' the given equation reduces to

$$[D(D-1) - 3DD' + 2D'(D'-1) + D + 2D']z = e^X + 2e^Y$$

$$\text{or } (D-D')(D-2D')z = e^X + 2e^Y$$

$$\begin{aligned} \therefore \text{C.F.} &= f_1(Y+X) + f_2(Y+2X) \\ &= f_1(\log y + \log x) + f_2(\log y + 2 \log x) \\ &= f_1(\log x y) + f_2(\log x^2 y) \\ &= \phi_1(x y) + \phi_2(x^2 y) \end{aligned}$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D-D')(D-2D')} \cdot e^X + \frac{1}{(D-D')(D-2D')} \cdot 2e^Y \\ &= \frac{e^X}{(1-0)(1-0)} + \frac{2e^Y}{(0-1)(0-2)} = e^X + e^Y = x + y \end{aligned}$$

∴ The solution is

$$z = \phi_1(x y) + \phi_2(x^2 y) + x + y$$

10.17 Solution of Linear Partial Differential Equation with Constant Coefficients under given Geometrical Conditions

The solutions of the different equations solved so far involve some arbitrary functions. The values of these functions can be obtained if the surfaces represented by the above solutions have to satisfy some geometrical conditions. Now here we shall find the surfaces (solutions) satisfying the given differential equation under the given geometrical conditions.

Solved Examples

Example 1: Find a surface passing through the two lines $z = x = 0$, $z - 1 = x - y = 0$, satisfying $r - 4s + 4t = 0$. [Meerut 2011 (Sem. I), 12 (Sem. I)]

Solution: The given equation can be written as

$$(D^2 - 4DD' - 4D'^2)z = 0$$

$$\text{A.E. is } m^2 - 4m + 4 = 0. \quad \text{or} \quad (m-2)^2 = 0. \quad \therefore \quad m = 2, 2$$

$$\therefore \quad z = f(y+2x) + x\phi(y+2x) \quad \dots(1)$$

Since the surface passes through the lines

$$z = x = 0 \quad \text{and} \quad z - 1 = x - y = 0$$

$$\therefore \quad \text{From (1),} \quad 0 = f(y+2x) \quad \dots(2)$$

$$\text{and} \quad 1 = f(y+2x) + x\phi(y+2x) \quad \dots(3)$$

From (2) and (3), we have

$$\phi(y+2x) = \frac{1}{x} = \frac{3}{3x} = \frac{3}{2x+x} = \frac{3}{2x+y} \quad \because \quad y-x=0$$

$$\text{Hence from (1), the required solution is } z = x \cdot \frac{3}{2x+y}$$

$$\text{or} \quad z(2x+y) = 3x.$$

Example 2: A surface is drawn satisfying $r + t = 0$ and touching $x^2 + z^2 = 1$ along its section by $y = 0$. Obtain its equation in the form

$$z^2(x^2 + z^2 - 1) = y^2(x^2 + z^2)$$

Solution: The given equation is

$$r + t = 0$$

$$\text{or} \quad (D^2 + D'^2)z = 0 \quad \text{or} \quad (D - iD')(D + iD')z = 0$$

$$\therefore \quad z = f(y+ix) + \phi(y-ix), \quad \dots(1)$$

$$\text{Now} \quad p = \frac{\partial z}{\partial x} = if'(y+ix) - i\phi'(y-ix)$$

$$\text{and} \quad q = \frac{\partial z}{\partial y} = f'(y+ix) + \phi'(y-ix),$$

Also form $x^2 + z^2 = 1$... (2)

$$z = \sqrt{(1-x^2)}$$

$$\therefore p = \frac{\partial z}{\partial x} = \frac{-x}{\sqrt{(1-x^2)}} \quad \text{and} \quad q = \frac{\partial z}{\partial y} = 0.$$

Now under the given conditions for the surface the values of p and q for any point on $y=0$, should be equal.

$$\therefore i f'(y+ix) - i \phi'(y-ix) = \frac{-x}{\sqrt{(1-x^2)}} \quad \dots (3)$$

$$f'(y+ix) + \phi'(y-ix) = 0 \quad \dots (4)$$

and $y=0$... (5)

From (3) and (4), we have

$$2i f'(y+ix) = -\frac{x}{\sqrt{(1-x^2)}}$$

or $f'(y+ix) = \frac{ix}{2\sqrt{\{1+(ix)^2\}}}$

or $f'(y+ix) = \frac{(y+ix)}{2\sqrt{\{1+(y+ix)^2\}}}$ $\because y=0$ from (5)

$$\therefore f(y+ix) = \frac{1}{2} \sqrt{\{1+(y+ix)^2\}} + K_1$$

from (4) $\phi'(y-ix) = -f'(y+ix) = \frac{-ix}{2\sqrt{(1-x^2)}}$

$$= \frac{(y-ix)}{2\sqrt{\{1+(y-ix)^2\}}} \quad \because y=0 \text{ from (5)}$$

$$\therefore \phi(y-ix) = \frac{1}{2} \sqrt{\{1+(y-ix)^2\}} + K_2$$

$\therefore$ from (1), we have

$$z = \frac{1}{2} \left\{ \sqrt{[1+(y+ix)^2]} + \sqrt{[1+(y-ix)^2]} \right\} + K \quad \dots (6)$$

where $K_1 + K_2 = K$.

Equating the two values of z from (2) and (6) when $y=0$, we have

$$\sqrt{(1-x^2)} = \sqrt{(1-x^2)} + K \quad \therefore K = 0$$

$\therefore$ from (6), we have

$$2z = \sqrt{\{1 + (y + ix)^2\}} + \sqrt{\{1 + (y - ix)^2\}}.$$

or
$$2z - \sqrt{\{1 + (y + ix)^2\}} = \sqrt{\{1 + (y - ix)^2\}}.$$

Squaring both sides, we have

$$4z^2 + 1 + (y + ix)^2 - 4z\sqrt{\{1 + (y + ix)^2\}} = 1 + (y - ix)^2$$

$$z^2 + ixy = z\sqrt{\{1 + (y + ix)^2\}}.$$

Again squaring both sides, we have

$$z^4 + 2ixyz^2 + i^2 x^2 y^2 = z^2 \{1 + (y + ix)^2\}$$

or
$$z^4 - x^2 y^2 = z^2 + z^2 y^2 - z^2 x^2$$

or
$$z^2 (z^2 + x^2 - 1) = y^2 (z^2 + x^2).$$

Exercise 10.6

Solve the following partial differential equations.

1. $x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} = x y$ [Jiwaji 2003, 04; Jabalpur 2003; Vikram 2004; Delhi Maths (Hons.) 2004, 06]

2. $x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} = x^2 y.$ [Meerut 2002 (O)]

3. $\frac{\partial^2 z}{\partial x^2} - 4 \frac{\partial^2 z}{\partial x \partial y} + 4 \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} - 2 \frac{\partial z}{\partial y} = e^{x+y}.$

4. $x^2 r - y^2 t + x p - y q = \log x.$ [Kurukshetra 2004]

5. $(x^2 D^2 + 2xy DD' + y^2 D'^2) z = x^m y^n,$ where $m + n \neq 0, 1.$ [Kurukshetra 2004]

6. $(x^2 D^2 + 2xy DD' + y^2 D'^2) z = x^2 y^2.$ [Delhi Maths [H] 2007]

7. $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} - x \frac{\partial z}{\partial x} = x^3 / y^2.$

8. $(D^2 - D') z = 2y - x^2.$

9. Find the surface satisfying the equation $r + t - 2s = 0$ and the conditions that $bz = y^2$ when $x = 0$ and $9z = x^2$ when $y = 0.$

Answers 10.6

1.	$z = x f_1(y/x) + f_2(xy) + xy \log x.$	2.	$z = x f_1(y/x) + f_2(xy) + (1/2)x^2 y.$
3.	$z = f_1(y+2x) + e^{-x} f_2(y+2x) + (1/2)y e^{x+y}.$ or $z = f_1(y+2x) + e^{-x} f_2(y+2x) + (1/2)x e^{x+y}.$		
4.	$z = f_1(xy) + f_2(y/x) + (1/6)(\log x)^3.$		
5.	$z = f_1(y/x) + x f_2(y/x) + [1/\{(m+n)(m+n-1)\}]x^m y^n.$		
6.	$z = f_1(y/x) + x f_2(y/x) + (1/2)x^2 y^2.$	7.	$z = f_1(y) + x^2 f_2(y/x^2) - (1/9)(x^3/y^2).$
8.	$z = \sum A e^{hx+h^2 y} - y^2 - x^4/12.$	9.	$z = (y+x)(x/a + y/b).$

Some Miscellaneous Examples

Example 1: Solve $\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial y^3} + \frac{\partial^3 u}{\partial z^3} - 3 \frac{\partial^3 u}{\partial x \partial y \partial z} = x^3 + y^3 + z^3 - 3xyz.$... (1)

Solution: The given equation can be written as

$$(D_1^3 + D_2^3 + D_3^3 - 3D_1 D_2 D_3)u = x^3 + y^3 + z^3 - 3xyz$$

$$\text{or } (D_1 + D_2 + D_3)(D_1^2 + D_2^2 + D_3^2 - D_1 D_2 - D_2 D_3 - D_3 D_1)u = x^3 + y^3 + z^3 - 3xyz$$

$$\text{or } (D_1 + D_2 + D_3)(D_1 + \omega D_2 + \omega^2 D_3)(D_1 + \omega^2 D_2 + \omega D_3)u = x^3 + y^3 + z^3 - 3xyz$$

where ω is imaginary cube root of unity.

Now let us consider

$$(D_1 + \omega D_2 + \omega^2 D_3)u = 0 \quad \dots (2)$$

Subsidiary equations are $\frac{dx}{1} = \frac{dy}{\omega} = \frac{dz}{\omega^2}.$

$$\therefore y - x\omega = a, \quad z - \omega^2 x = b.$$

$\therefore$ Solution of (2) i.e., C.F. corresponding to $(D_1 + \omega D_2 + \omega^2 D_3)$ is given by

$$f_1(y - \omega x, z - \omega^2 x).$$

Similarly C.F.'s corresponding to other factors are given by

$$f_2(y-x, z-x) \text{ and } f_3(y-\omega^2 x, z-\omega x).$$

$$\therefore \text{C.F.} = f_1(y-\omega x, z-\omega^2 x) + f_2(y-x, z-x) + f_3(y-\omega^2 x, z-\omega x)$$

Now P.I. corresponding to x^3 is

$$\begin{aligned} & \frac{1}{D_1^3 + D_2^3 + D_3^3 - 3D_1 D_2 D_3} x^3 \\ &= \frac{1}{D_1^3} \left(1 + \frac{D_2^3}{D_1^3} + \dots \right)^{-1} \cdot x^3 + \dots = \frac{1}{D_1^3} \cdot x^3 + \dots = \frac{x^6}{4 \cdot 5 \cdot 6} = \frac{x^6}{120} \end{aligned}$$

Similarly particular integrals corresponding to y^3 and z^3 are $\frac{y^6}{120}$ and $\frac{z^6}{120}$

and P.I. corresponding to $-3xyz$

$$\begin{aligned} &= \frac{1}{D_1^3 + D_2^3 + D_3^3 - 3D_1 D_2 D_3} (-3xyz) \\ &= \frac{1}{3D_1 D_2 D_3} \cdot \left(1 - \frac{D_1^2}{3D_2 D_3} \dots \right)^{-1} \cdot (-3xyz) \\ &= \frac{1}{3D_1 D_2 D_3} (-3xyz) + 0 = \frac{x^2 y^2 z^2}{8} \end{aligned}$$

Hence the solution is

$$u = f_1(y-\omega x, z-\omega^2 x) + f_2(y-x, z-x) + f_3(y-\omega^2 x, z-\omega x) + \frac{x^6 + y^6 + z^6}{120} + \frac{x^2 y^2 z^2}{8}$$

Example 2: Solve $\frac{\partial^2 z}{\partial x^2} - 4 \frac{\partial^2 z}{\partial y^2} = \frac{4x}{y^2} - \frac{y}{x^2}$.

[Delhi Maths (Hons.) 2004, 08]

Solution: The given equation can be written as

$$(D^2 - 4D'^2)z = \frac{4x}{y^2} - \frac{y}{x^2} \quad \text{or} \quad (D + 2D')(D - 2D')z = \frac{4x}{y^2} - \frac{y}{x^2}.$$

$$\therefore \text{C.F.} = f_1(y-2x) + f_2(y+2x).$$

$$\text{P.I.} = \frac{1}{(D+2D')} \cdot \frac{1}{(D-2D')} \cdot \left(\frac{4x}{y^2} - \frac{y}{x^2} \right)$$

$$= \frac{1}{(D+2D')} \int \left\{ \frac{4x}{(a-2x)^2} - \frac{a-2x}{x^2} \right\} dx \quad \text{where } y+2x = a.$$

$$\begin{aligned}
&= \frac{1}{(D+2D')} \int \left[-\frac{2}{(a-2x)} + \frac{2a}{(a-2x)^2} - \frac{a}{x^2} + \frac{2}{x} \right] dx \\
&= \frac{1}{(D+2D')} \left[\log(a-2x) + \frac{a}{(a-2x)} + \frac{a}{x} + 2 \log x \right] \\
&= \frac{1}{(D+2D')} \left[\log y + 2 \log x + \frac{y+2x}{y} + \frac{y+2x}{x} \right] \\
&= \frac{1}{(D+2D')} \left[\log y + 2 \log x + \frac{2x}{y} + \frac{y}{x} + 3 \right] \\
&= \int \left[\log(b+2x) + 2 \log x + \frac{2x}{b+2x} + \frac{b+2x}{x} + 3 \right] dx \quad \text{where } y-2x=b \\
&= \int \left[\log(b+2x) + 2 \log x + \frac{2x}{b+2x} + \frac{b}{x} + 5 \right] dx \\
&= x \cdot \log(b+2x) - \int \frac{2x}{(b+2x)} dx + 2x \log x - 2 \int \frac{1}{x} \cdot x dx + \int \frac{2x}{(b+2x)} dx + b \log x + 5x \\
&= x \log(b+2x) + (2x+b) \log x + 3x = x \log y + y \log x + 3x.
\end{aligned}$$

∴ The solution is

$$z = f_1(y-2x) + f_2(y+2x) + x \log y + y \log x + 3x.$$

Example 3: Solve $r - s - 2t = (2x^2 + xy - y^2) \sin xy - \cos xy$. [Delhi Maths (Hons.) 2007]

Solution: The given equation can be written as

$$(D^2 - DD' - 2D'^2)z = (2x^2 + xy - y^2) \sin xy - \cos xy.$$

A.E. is $m^2 - m - 2 = 0.$

or $(m-2)(m+1) = 0. \quad \therefore m = 2, -1.$

C.F. = $f_1(y+2x) + f_2(y-x)$

$$\text{P.I.} = \frac{1}{(D^2 - DD' - 2D'^2)} \cdot \{(2x^2 + xy - y^2) \sin xy - \cos xy\}$$

$$= \frac{1}{(D+D')} \cdot \frac{1}{(D-2D')} \cdot \{(2x^2 + xy - y^2) \sin xy - \cos xy\}$$

$$= \frac{1}{(D+D')} \int [\{2x^2 + x(a-2x) - (a-2x)^2\} \sin x(a-2x) - \cos x(a-2x)] dx$$

where $y+2x = a$

$$\begin{aligned}
&= \frac{1}{(D+D')} [(a-4x)(x-a) \sin(ax-2x^2) - \cos(ax-2x^2)] dx \\
&= \frac{1}{(D+D')} \left[\int (-a+x) \cdot \{(a-4x) \sin(ax-2x^2)\} dx - \int \cos(ax-2x^2) dx \right] \\
&= \frac{1}{(D+D')} \left[(-a+x) \cos(ax-2x^2) + \int \cos(ax-2x^2) dx - \int \cos(ax-2x^2) dx \right] \\
&= \frac{1}{(D+D')} (-x+a) \cos(ax-2x^2) \\
&= \frac{1}{(D+D')} (x+y) \cos xy \\
&= \int (2x+b) \cos x(x+b) dx \quad \text{where } y-x=b \\
&= \int (2x+b) \cos(x^2+xb) dx = \sin(x^2+xb) = \sin xy.
\end{aligned}$$

$\therefore$ The solution is

$$z = f_1(y+2x) + f_2(y+x) + \sin xy.$$

Example 4: Solve $r - t = \tan^3 x \tan y - \tan x \tan^3 y$.

[Delhi Maths 2006]

Solution: The given equation can be written as

$$(D^2 - D'^2)z = \tan^3 x \tan y - \tan x \tan^3 y.$$

A.E. is $m^2 - 1 = 0$

or $(m-1)(m+1) = 0, \quad \therefore m = 1, -1,$

C.F. = $f_1(y+x) + f_2(y-x)$

P.I. = $\frac{1}{(D^2 - D'^2)} \cdot (\tan^3 x \tan y - \tan x \tan^3 y)$

$$= \frac{1}{(D+D')} \cdot \frac{1}{(D-D')} \cdot (\tan^3 x \tan y - \tan x \tan^3 y)$$

$$= \frac{1}{(D+D')} \cdot \int [\tan^3 x \cdot \tan(a-x) - \tan x \tan^3(a-x)] dx \quad \text{where } y+x=a$$

$$= \frac{1}{(D+D')} \cdot \int [(-1 + \sec^2 x) \tan x \tan(a-x) - \tan x \cdot \tan(a-x) \{-1 + \sec^2(a-x)\}] dx$$

$$= \frac{1}{(D+D')} \left[\int \tan(a-x) \cdot \tan x \sec^2 x dx - \int \tan x \tan(a-x) \sec^2(a-x) dx \right]$$

$$\begin{aligned}
&= \frac{1}{(D+D')} \left[\tan(a-x) \cdot \frac{\tan^2 x}{2} + \frac{1}{2} \int \sec^2(a-x) \cdot \tan^2 x \, dx + \tan x \cdot \frac{\tan^2(a-x)}{2} \right. \\
&\quad \left. - \frac{1}{2} \int \sec^2 x \cdot \tan^2(a-x) \, dx \right] \\
&= \frac{1}{2} \cdot \frac{1}{(D+D')} \left[\tan^2 x \tan(a-x) + \tan x \tan^2(a-x) \right. \\
&\quad \left. - \int [\sec^2 x \{\sec^2(a-x) - 1\} - \sec^2(a-x) \cdot (\sec^2 x - 1)] \, dx \right] \\
&= \frac{1}{2} \frac{1}{(D+D')} [\tan^2 x \tan(a-x) + \tan x \tan^2(a-x) + \int \{\sec^2 x - \sec^2(a-x)\} \, dx] \\
&= \frac{1}{2} \frac{1}{(D+D')} [\tan^2 x \tan y + \tan x \tan^2 y + (\tan x + \tan y)] \\
&= \frac{1}{2} \frac{1}{(D+D')} [\tan x \cdot \sec^2 y + \tan y \sec^2 x] \\
&= \frac{1}{2} \int [\tan x \sec^2(b+x) + \tan(b+x) \sec^2 x] \, dx \quad \text{where } y-x=b \\
&= \frac{1}{2} \tan x \tan(b+x) - \int \sec^2 x \cdot \tan(b+x) \, dx + \int \tan(b+x) \sec^2 x \, dx \\
&= \frac{1}{2} \tan x \tan y.
\end{aligned}$$

∴ The solution is

$$z = f_1(y+x) + f_2(y-x) + \frac{1}{2} \tan x \tan y.$$

Example 5: Find a surface satisfying

$$2x^2r - 5xys + 2y^2t + 2(px + qy) = 0$$

and touching the hyperbolic paraboloid $z = x^2 - y^2$ along its section by the plane $y = 1$.

Solution: The given equation can be written as

$$2x^2 \frac{\partial^2 z}{\partial x^2} - 5xy \frac{\partial^2 z}{\partial x \partial y} + 2y^2 \frac{\partial^2 z}{\partial y^2} + 2x \frac{\partial z}{\partial x} + 2y \frac{\partial z}{\partial y} = 0.$$

Substituting $x = e^X$, $y = e^Y$ so that $X = \log x$, $Y = \log y$ and denoting $\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial Y}$ by D and D' respectively, the given equation reduces to

$$[2D(D-1) - 5DD' + 2D'(D'-1) + 2D + 2D']z = 0$$

or $(2D - D')(D - 2D')z = 0$

$$\begin{aligned}
 \therefore \quad z &= f_1(2Y + X) + f_2(Y + 2X) \\
 &= f_1(2 \log y + \log x) + f_2(\log y + 2 \log x) \\
 &= f_1(\log xy^2) + f_2(\log x^2 y) \\
 z &= \phi_1(xy^2) + \phi_2(x^2 y) \quad \dots(1)
 \end{aligned}$$

Now
$$p = \frac{\partial z}{\partial x} = y^2 \phi_1'(xy^2) + 2xy \phi_2'(x^2 y)$$

and
$$q = \frac{\partial z}{\partial y} = 2xy \phi_1'(xy^2) + x^2 \phi_2'(x^2 y)$$

Also from $z = x^2 - y^2$ (2)

$$p = 2x \text{ and } q = -2y.$$

Now under the given conditions for the two surfaces (1) and (2) the values of p and q for any point on the plane $y = 1$, should be equal.

$$\therefore \quad y^2 \phi_1'(xy^2) + 2xy \phi_2'(x^2 y) = 2x \quad \dots(3)$$

$$2xy \phi_1'(xy^2) + x^2 \phi_2'(x^2 y) = -2y \quad \dots(4)$$

and $y = 1$ (5)

From (3) and (4), we have

$$3xy^2 \phi_1'(xy^2) = -4y^2 - 2x^2$$

$$\text{or} \quad \phi_1'(xy^2) = -\frac{4y^2}{3xy^2} - \frac{2x^2}{3y^2x} = -\frac{4}{3x} - \frac{2x}{3}, \quad \therefore y = 1 \text{ from (5)}$$

$$\text{or} \quad \phi_1'(xy^2) = -\frac{4}{3xy^2} - \frac{2}{3}xy^2 \quad \therefore y = 1 \text{ from (5)}$$

$$\therefore \quad \phi_1(xy^2) = -\frac{4}{3} \log(xy^2) - \frac{1}{3}(xy^2)^2 + K_1 \quad \dots(6)$$

and $3x^2 y \phi_2'(x^2 y) = 4x^2 + 2y^2$

$$\text{or} \quad \phi_2'(x^2 y) = \frac{4}{3y} + \frac{2y^2}{3x^2 y} = \frac{4}{3} + \frac{2}{3x^2 y} \quad \therefore y = 1 \text{ from (5)}$$

$$\therefore \quad \phi_2(x^2 y) = \frac{4}{3}x^2 y + \frac{2}{3} \log(x^2 y) + K_2 \quad \dots(7)$$

Substituting these values from (6) and (7), in (1), we have

$$z = -\frac{4}{3} \log xy^2 - \frac{1}{3} x^2 y^4 + \frac{4}{3} x^2 y + \frac{2}{3} \log x^2 y + K \quad \text{where } K = K_1 + K_2$$

or
$$z = \frac{4}{3} x^2 y - \frac{1}{3} x^2 y^4 + \frac{1}{3} \log \frac{(x^2 y)^2}{(xy^2)^4} + K$$

or
$$z = \frac{4}{3} x^2 y - \frac{1}{3} x^2 y^4 - 2 \log y + K. \quad \dots(8)$$

Now equating the value of z from (2) and (8) when $y = 1$, we have

$$x^2 - 1 = \frac{4}{3} x^2 - \frac{1}{3} x^2 + K. \quad \therefore K = -1$$

Hence the required surface is

$$z = \frac{4}{3} x^2 y - \frac{1}{3} x^2 y^4 - 2 \log y - 1$$

or
$$3z = 4x^2 y - x^2 y^4 - 6 \log y - 3.$$

Example 6: Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = x^2 \sin(x + y).$

Solution: The given equation can be written as

$$(D^2 + DD' - 6D'^2)z = x^2 \sin(x + y)$$

or
$$(D - 2D')(D + 3D')z = x^2 \sin(x + y)$$

$\therefore$ C.F. = $f_1(y + 2x) + f_2(y - 3x).$

Now P.I. = $\frac{1}{D^2 + DD' - 6D'^2} x^2 \sin(x + y)$

$$= \text{I.P. of } \frac{1}{D^2 + DD' - 6D'^2} e^{i(x+y)} \cdot x^2$$

$$= \text{I.P. of } e^{i(x+y)} \frac{1}{(D+i)^2 + (D+i)(D'+i) - 6(D'+i)^2} \cdot x^2$$

$$= \text{I.P. of } e^{i(x+y)} \frac{1}{D^2 + 3iD + DD' - 11D'i - 6D'^2 + 4} x^2$$

$$= \text{I.P. of } e^{i(x+y)} \frac{1}{4} \left[1 + \left(\frac{D^2}{4} + \frac{3iD}{4} + \frac{DD'}{4} - \frac{11iD'}{4} - \frac{9D'^2}{4} \right) \right]^{-1} \cdot x^2$$

$$= \text{I.P. of } e^{i(x+y)} \cdot \frac{1}{4} \left[1 - \frac{D^2}{4} - \frac{3iD}{4} - \frac{DD'}{4} + \frac{11iD'}{4} + \frac{9D'^2}{4} + \frac{9i^2 D'^2}{16} \dots \right] \cdot x^2$$

$$\begin{aligned}
 &= \text{I.P. of } e^{i(x+y)} \cdot \frac{1}{4} \left[x^2 - \frac{1}{2} - \frac{3ix}{2} - \frac{9}{8} \right] \\
 &= \text{I.P. of } \frac{1}{4} [\cos(x+y) + i\sin(x+y)] \left(x^2 - \frac{13}{8} - \frac{3ix}{2} \right) \\
 &= \frac{1}{4} \left(x^2 - \frac{13}{8} \right) \sin(x+y) - \frac{3}{8} x \cos(x+y)
 \end{aligned}$$

∴ The solution is

$$z = f_1(y+2x) + f_2(y-3x) + \frac{1}{4} \left(x^2 - \frac{13}{8} \right) \sin(x+y) - \frac{3}{8} x \cos(x+y)$$

Example 7: Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 2 \frac{\partial^2 z}{\partial y^2} + 2 \frac{\partial z}{\partial x} + 2 \frac{\partial z}{\partial y} = e^{2x+3y} + \sin(2x+y) + xy$.

[Delhi 2008]

Solution: The given equation can be written as

$$(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y} + \sin(2x+y) + xy$$

or $(D + D')(D - 2D' + 2)z = e^{2x+3y} + \sin(2x+y) + xy$.

∴ C.F. = $f_1(y-x) + e^{-2x} f_2(y+3x)$

P.I. corresponding to e^{2x+3y}

$$\begin{aligned}
 &= \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'} \cdot e^{2x+3y} \\
 &= \frac{e^{2x+3y}}{2^2 - 2 \cdot 3 - 2 \cdot 3^2 + 2 \cdot 2 + 2 \cdot 3} = -\frac{1}{10} e^{2x+3y}
 \end{aligned}$$

P.I. corresponding to $\sin(2x+y)$

$$\begin{aligned}
 &= \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'} \cdot \sin(2x+y) \\
 &= \frac{1}{-2^2 - (-2 \cdot 1) - 2 \cdot (-1)^2 + 2D + 2D'} \sin(2x+y) \\
 &= \frac{1}{2(D+D')} \sin(2x+y) = \frac{(D-D')}{2(D^2 - D'^2)} \sin(2x+y) \\
 &= \frac{(D-D')}{2(-2^2 + 1^2)} \sin(2x+y) \\
 &= -\frac{1}{6} [2 \cos(2x+y) - \cos(2x+y)] = -\frac{1}{6} \cos(2x+y)
 \end{aligned}$$

$$\begin{aligned}
 \text{and P.I. corresponding to } xy &= \frac{1}{(D+D')(D-2D'+2)} \cdot xy \\
 &= \frac{1}{2(D+D')} \cdot \left(1 + \frac{D}{2} - D'\right)^{-1} \cdot xy \\
 &= \frac{1}{2(D+D')} \cdot \left(1 + \frac{(D-2D')}{2}\right)^{-1} \cdot xy \\
 &= \frac{1}{2(D+D')} \cdot \left\{1 - \frac{D-2D'}{2} + \left(\frac{D-2D'}{2}\right)^2 - \dots\right\} \cdot xy \\
 &= \frac{1}{2(D+D')} \left(1 - \frac{D}{2} + D' - DD' \dots\right) \cdot xy \\
 &= \frac{1}{2(D+D')} \cdot \left(xy - \frac{y}{2} + x - 1\right) \\
 &= \frac{1}{2D} \left(1 + \frac{D'}{D} \dots\right)^{-1} \cdot \left(xy - \frac{y}{2} + x - 1\right) \\
 &= \frac{1}{2D} \left(1 - \frac{D'}{D} \dots\right) \cdot \left(xy - \frac{y}{2} + x - 1\right) \\
 &= \frac{1}{2D} \left(xy - \frac{1}{2}y + x - 1 - \frac{1}{D}x + \frac{1}{D} \cdot \frac{1}{2}\right) \\
 &= \frac{1}{2D} \left(xy - \frac{1}{2}y + \frac{3}{2}x - 1 - \frac{x^2}{2}\right) \\
 &= \frac{1}{2} \left(\frac{x^2}{2}y - \frac{1}{2}xy + \frac{3}{4}x^2 - x - \frac{1}{6}x^3\right) \\
 &= \frac{x}{24} (6xy - 6y + 9x - 12 - 2x^2).
 \end{aligned}$$

∴ The solution is

$$z = f_1(y-x) + e^{-x} f_2(y+2x) - \frac{1}{10} e^{2x+3y} - \frac{1}{6} (2x+y) + \frac{x}{24} (6xy - 6y + 9x - 12 - 2x^2).$$

Example :8 Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} + 3 \frac{\partial z}{\partial y} - 2z = e^{x-y} - x^2 y$.

Solution: The given equation can be written as

$$[D^2 - D'^2 + D + 3D' - 2]z = e^{x-y} - x^2 y$$

or $[(D-D')(D+D') + 2(D+D') - (D-D'+2)]z = e^{x-y} - x^2 y$

$$\text{or} \quad (D - D' + 2)(D + D' - 1)z = e^{x-y} - x^2 y.$$

$$\therefore \quad \text{C.F.} = e^{-2x} f_1(y+x) + e^x f_2(y-x)$$

Now P.I. corresponding to e^{x-y}

$$\begin{aligned} &= \frac{1}{(D - D' + 2)(D + D' - 1)} \cdot e^{x-y} \\ &= \frac{1}{(1+1+2)(1-1-1)} \cdot e^{x-y} = -\frac{1}{4} e^{x-y} \end{aligned}$$

$$\text{and P.I. corresponding to } (-x^2 y) = \frac{1}{(D - D' + 2)(D + D' - 1)} \cdot x^2 y$$

$$\begin{aligned} &= \frac{1}{2} \left(1 + \frac{D - D'}{2} \right)^{-1} \{ 1 - (D + D') \}^{-1} \cdot x^2 y \\ &= \frac{1}{2} \left\{ 1 - \left(\frac{D - D'}{2} \right) + \left(\frac{D - D'}{2} \right)^2 - \left(\frac{D - D'}{2} \right)^3 + \dots \right\} \\ &\quad \times \{ 1 + (D + D') + (D + D')^2 + (D + D')^3 + \dots \} \cdot x^2 y \\ &= \frac{1}{2} \left\{ 1 - \frac{D}{3} + \frac{D'}{2} + \frac{D^2}{4} - \frac{DD'}{2} + \frac{D^2}{4} + \frac{3D^2 D'}{8} + \dots \right\} \\ &\quad \times \{ 1 + D + D' + D^2 + 2DD' + D'^2 + 3D^2 D' + \dots \} \cdot x^2 y \\ &= \frac{1}{2} \left(1 - \frac{D}{2} + \frac{D'}{2} + \frac{D^2}{4} - \frac{DD'}{8} + \frac{3D^2 D'}{8} + D - \frac{D^2}{2} + \frac{DD'}{2} - \frac{D^2 D'}{2} \right. \\ &\quad \left. + D' - \frac{DD'}{2} + \frac{D^2 D'}{4} + D^2 + \frac{D^2 D'}{2} + 2DD' - D^2 D' + 3D^2 D' - \dots \right) x^2 y \\ &= \frac{1}{2} \left(1 + \frac{1}{2} D + \frac{3}{2} D' + \frac{3}{4} D^2 + \frac{3}{2} DD' + \frac{21}{8} D^2 D' + \dots \right) x^2 y \\ &= \frac{1}{2} \left(x^2 y + xy + \frac{3}{2} x^2 + \frac{3}{2} y + 3x + \frac{21}{4} \right). \end{aligned}$$

$\therefore$ The solution is

$$y = e^{-2x} f_1(y+x) + e^x f_2(y-x) + \frac{1}{2} \left(x^2 y + xy + \frac{3}{2} x^2 + \frac{3}{2} y + 3x + \frac{21}{4} \right).$$

Example 9: Solve $\frac{1}{x^2} \frac{\partial^2 z}{\partial x^2} - \frac{1}{x^3} \frac{\partial z}{\partial x} = \frac{1}{y^2} \frac{\partial^2 z}{\partial y^2} - \frac{1}{y^3} \frac{\partial z}{\partial y}$.

Solution: Putting $\frac{x^2}{2} = X, \frac{y^2}{2} = Y$

so that $x dx = dX, y dy = dY$

$$\therefore \frac{\partial z}{\partial X} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial X} = \frac{1}{x} \frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial y} = \frac{1}{y} \frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial X^2} = \frac{\partial}{\partial x} \left(\frac{1}{x} \frac{\partial z}{\partial x} \right) \cdot \frac{\partial x}{\partial X} = -\frac{1}{x^3} \frac{\partial z}{\partial x} + \frac{1}{x^2} \frac{\partial^2 z}{\partial x^2}$$

Similarly $\frac{\partial^2 z}{\partial Y^2} = -\frac{1}{y^3} \frac{\partial z}{\partial y} + \frac{1}{y^2} \frac{\partial^2 z}{\partial y^2}.$

The given equation reduces to

$$\frac{\partial^2 z}{\partial X^2} - \frac{\partial^2 z}{\partial Y^2} = 0$$

or $(D^2 - D'^2)z = 0$ or $(D - D')(D + D')z = 0$

$$\begin{aligned} \therefore z &= f_1(Y + X) + f_2(Y - X) \\ &= f_1\left(\frac{y^2}{2} + \frac{x^2}{2}\right) + f_2\left\{\frac{y^2}{2} - \frac{x^2}{2}\right\} = \phi_1(y^2 + x^2) + \phi_2(y^2 - x^2). \end{aligned}$$

Example 10: Solve $yt - q = xy$.

Solution: The given equation can be written as

$$y \frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial y} = xy \quad \text{or} \quad y^2 \frac{\partial^2 z}{\partial y^2} - y \frac{\partial z}{\partial y} = xy^2. \quad \dots(1)$$

Substituting $x = e^X$, $y = e^Y$, so that $X = \log x$, $Y = \log y$

and denoting $\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial Y}$ by D and D' , the given equation (2) reduces to

$$[D'(D' - 1) - D]z = e^{X+2Y} \quad \text{or} \quad D'(D' - 2)z = e^{X+2Y}$$

$$\begin{aligned} \therefore \text{C.F.} &= f_1(X) + e^{2Y} f_2(X) \\ &= f_1(\log x) + y^2 f_2(\log x) = \phi_1(x) + y^2 \phi_2(x) \end{aligned}$$

and
$$\begin{aligned} \text{P.I.} &= \frac{1}{D'(D' - 2)} \cdot e^{X+2Y} = \frac{1}{2} \cdot \frac{1}{D' - 2} e^{X+2Y} \\ &= \frac{1}{2} e^{X+2Y} \frac{1}{D' + 2 - 2} \cdot 1 = \frac{1}{2} e^{X+2Y} \cdot \frac{1}{D'} \cdot 1 \\ &= \frac{1}{2} Y e^{X+2Y} = \frac{1}{2} xy^2 \log y. \end{aligned}$$

$\therefore$ The solution is

$$z = \phi_1(x) + y^2 \phi_2(x) + \frac{1}{2} xy^2 \log y.$$

Exercise 10.7

Solve the following partial differential equations.

1. $(D^2 - DD' - 2D'^2)z = 2x + 3y + e^{3x+4y}$

2. $(2D^2 - 5DD' + 2D'^2)z = 5(y - x)$

[Kurukshetra 2005]

3. $\frac{\partial^2 z}{\partial x^2} + 3 \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial^2 z}{\partial y^2} = 2x + 3y$

4. $\frac{\partial^3 z}{\partial x^2 \partial y} - 2 \frac{\partial^3 z}{\partial x \partial y^2} + \frac{\partial^3 z}{\partial y^3} = \frac{1}{x^2}$

5. $(D^3 - 7DD'^2 - 6D'^3)z = x^2 + xy^2 + y^3 + \cos(x - y)$

[Kurukshetra 2004]

6. $(D^3 + 2D^2D' - DD'^2 - 2D'^3)z = (y + 2)e^x$

7. $(D^2 - 2aDD' + a^2D'^2)z = f(y + ax)$

8. $(D^2 - D'^2 + D + 3D' - 2)z = e^{x-y}$

9. $(2D^2 - D'^2 + D)z = x^2 - y$

10. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = x y$

11. $(D^2 - DD' + D' - 1)z = e^x$

12. $(D^2 - DD' + D' - 1)z = e^y + x y$

[Delhi Maths (Hons.) 2005]

13. $(D^2 - D')z = e^{x+y} + 5 \cos(x + 2y)$

14. $x^2 \frac{\partial^2 z}{\partial x^2} - 3xy \frac{\partial^2 z}{\partial x \partial y} + 2y^2 \frac{\partial^2 z}{\partial y^2} + 5y \frac{\partial z}{\partial y} - 2z = 0$

15. Show that the surface satisfying $r - 2s + t = 6$ and touching the hyperbolic paraboloid $z = x y$ along its section by the plane $y = x$ is $z = x^2 - x y + y^2$.

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter (a), (b), (c) or (d).

1. The part of E.F. of a $p.d$ equation with constant coefficient corresponding to a root m (repeated r times) is given by :

- (a) $f\{y + m(x + x^2 + \dots + x^{r-1})\}$ (b) $f\{y + (m + m^2 + \dots + m^{r-1})x\}$

- (c) $f_1(y+mx) + xf_2(y+mx) + x^2/3(y+mx) + \dots + x^{r-1}f_r(y+mx)$
 (d) None of these
2. The solution of the *p.d* equation $(D^3 - 7DD'^2 + 6D'^3)z = 0$ is :
- (a) $f_1(y+x) + f_2(y+2x) + f_3(y-3x)$
 (b) $f_1(y-x) + f_2(y-2x) + f_3(y+3x)$
 (c) $f_1(y+x) + f_2(y+2x) + f_3(y+3x)$
 (d) None of these
3. For the *p.d* equation $(D^2 - 2DD' + D'^2)z = \tan(x+y)$, P.I. is given by :
- (a) $\tan(x+y)$ (b) $x \tan(x+y)$
 (c) $(x^2/2) \tan(x+y)$ (d) None of these
4. P.I. of the *p.d* equation $(D^2 + 5DD' + 6D'^2)z = (y-m)^{-1}$ is given by
- (a) $\log(y-2x)$ (b) $x \log(y-2x)$
 (c) $x^2 \log(y-2x)$ (d) None of these
5. The solution of partial differential equation $r^2 + 2s + t + 4p + 4q + 4z = 0$ is :
- (a) $z = e^{-2x}f_1(y+2x) + xe^{-2x}f_2(y+2x)$
 (b) $z = e^{-x}f_1(y+2x) + xe^{-x}f_2(y+2x)$
 (c) $z = e^{-2x}f_1(y-2x) + xe^{-2x}f_2(y-2x)$
 (d) None of these
6. The C.F. of the equation $(D + D'^2)z = 0$ is :
- (a) $\Sigma A e^{k^2x+ky}$ (b) $\Sigma A e^{-k^2x+ky}$
 (c) $\Sigma A e^{-k^2x-ky}$ (d) None of these
7. The P.F. of the equation $(D^2 - D')z = x e^{ax+a^2y}$
- (a) $e^{ax+a^2y}/2a^2$ (b) $e^{ax+a^2y}/(a^4-a)$
 (c) $e^{ax+a^2y}(x^2/4a - x/4a^2)$ (d) None of these
8. The solution of the equation $x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} - y \frac{\partial z}{\partial y} + x \frac{\partial z}{\partial x} = 0$ is
- (a) $z = \phi_1(xy) + \phi_2(y/x)$ (b) $z = \phi_1(x/y) + \phi_2(y/x)$
 (c) $z = \phi_1(xy) + \phi_2(x^2/y)$ (d) None of these

9. The C.F. of the equation $yt - q = xy$ is :

- (a) $\phi_1(x) + y^2\phi_2(x)$ (b) $\phi_1(x) + y\phi_2(x)$
 (c) $\phi_1(x) + y^3\phi_2(x)$ (d) None of these

10. The P.I. of the equation $(D^3 + 2D^2D' - DD'^2 - 2D'^3)z = (y+2)e^x$ is :

- (a) e^y (b) xe^y
 (c) ye^y (d) None of these

Fill in the Blank

Fill in the blank's "....." so that the following statements are complete and correct.

- For the equation $f(D, D')z = \phi(x, y)$, the C.F. is the solution of
- The complete solution of $f(D, D')z = \phi(x, y)$ is given by $z = C.F. + \dots$
- For the equation $f(D, D')z = \phi^n(ax + by)$, the P.I. is given by

$$P.I. = \frac{1}{f(D, D')} \phi^n(ax + by) = \frac{1}{f(a, b)} \phi(ax + by) \text{ provided } \dots$$

- P.I. for the equation $(D - D'^2)z = \cos(x - 2y)$ is
- The solution of the equation $(D^2 + D')z = 2y - x^2$ is $z = \Sigma A e^{hx - h^2 y} + \dots$
- The general solution of $(D^2 + D')z = e^{x-y}$ is $z = \Sigma A e^{hx - h^2 y} + \dots$

Answers

Exercise 10.7

1.	$z = f_1(y+2x) + f_2(y-x) - (1/60)(2x+3y)^3 - (1/35)e^{3x+4y}.$
2.	$z = f_1(2y+x) + f_2(y+2x) + 5(y-x)^3/54.$
3.	$z = f_1(y-x) + f_2(y-2x) + 3x^2y/2 - 7x^3/6.$
4.	$z = f_1(y+x) + x f_2(y+x) + f_3(x) - y \log x.$
5.	$z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) + 5x^6/72 + x^5/60 + 7x^5y/20$ $+ x^4y^2/24 + x^3y^3/6 + (x/4) \cos(x-y)$
6.	$z = f_1(y+x) + f_2(y-x) + f_3(y-2x) + ye^y.$

7.	$z = f_1(y + ax) + x f_2(y + ax) + (x^2/2)f(y + ax).$
8.	$z = e^{-2x} f_1(y + x) + e^x f_2(y - x) - (1/4)e^{x-y}.$
9.	$z = \sum A e^{hx+ky} - x^2 y^2 / 2 + y^3 / 6 - x y^4 / 12 - y^4 / 6 - y^6 / 360.$ where $2h^2 - k^2 + h = 0.$
10.	$z = f_1(y - x) + e^{-2x} f_2(y + 2x) + (x^2 y - xy - 2x) / 4 + 3x^2 / 8 - x^3 / 12.$
11.	$z = e^x f_1(y) + e^{-x} f_2(y + x) - (x e^x) / 2.$
12.	$z = e^x f_1(y) + e^{-x} f_2(y + x) + y e^y - x y - x + 1.$
13.	$z = \sum A e^{hx+h^2 y} + (1/2)x e^{x+y} - [2 \sin(x + 2y) - \cos(x + 2y)].$
14.	$z = x^2 f_1(xy) + x f_2(x^2 y).$

Objective Type Questions

Multiple Choice Questions

1.	(c)	2.	(a)	3.	(c)	4.	(b)
5.	(a)	6.	(b)	7.	(c)	8.	(a)
9.	(a)	10.	(c)				

Fill in the Blank

1.	$f(D, D') z = 0$	2.	P.I.
3.	$f(a, b) \neq 0$	4.	$\sum A e^{k(kx+y)}$
5.	$y^2 + x^4 / 12$	6.	$y e^{x+y}$



UNIT- 5

CHAPTER 11: Reduction of Second Order Partial Differential Equation into Canonical Forms (Non-linear Equations of Second Order)

CHAPTER 12 : Wave Equations
(By Method of Separation of Variables)

CHAPTER 13 : Heat and Diffusion Equations
(By Method of Separation of Variables)

CHAPTER 14 : Laplace Equations
(By Method of Separation of Variables)



Reduction of Second Order Partial Differential Equation into Canonical Forms (Non-linear Equations of Second Order)

11.1 Laplace Transformation (Canonical Forms)

[Avadh 2001, Delhi Maths (Hons.) 2004]

Consider the linear partial differential equation of second order of the form

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad \dots(1)$$

in which R, S, T are continuous functions of x and y possessing continuous partial derivatives of as high order as necessary.

This equation can be reduced to a more simple form by a suitable change of the independent variables. The simpler forms which result in this way are called **canonical forms** of the equation (1).

Let the independent variables x and y be changed to u and v by means of the transformations

$$u = u(x, y) \text{ and } v = v(x, y) \quad \dots(2)$$

Now, we have

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \quad \text{and} \quad q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$$

$$\therefore \frac{\partial}{\partial x} \equiv \frac{\partial u}{\partial x} \frac{\partial}{\partial u} + \frac{\partial v}{\partial x} \frac{\partial}{\partial v} \quad \text{and} \quad \frac{\partial}{\partial y} \equiv \frac{\partial u}{\partial y} \frac{\partial}{\partial u} + \frac{\partial v}{\partial y} \frac{\partial}{\partial v}$$

$$= \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} \left(\frac{\partial u}{\partial y} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} \right) + \frac{\partial z}{\partial u} \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial x} \cdot \frac{\partial v}{\partial y} \cdot \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v} \frac{\partial^2 v}{\partial x \partial y}$$

Putting these values of p , q , r , s and t in (1) and simplifying, we get

$$A \frac{\partial^2 z}{\partial u^2} + 2B \frac{\partial^2 z}{\partial u \partial v} + C \frac{\partial^2 z}{\partial v^2} + F\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) = 0 \quad \dots(3)$$

where,
$$A = R \left(\frac{\partial u}{\partial x} \right)^2 + S \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + T \left(\frac{\partial u}{\partial y} \right)^2 \quad \dots(4)$$

$$B = R \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{1}{2} S \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right) + T \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} \quad \dots(5)$$

$$C = R \left(\frac{\partial v}{\partial x} \right)^2 + S \cdot \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + T \left(\frac{\partial v}{\partial y} \right)^2 \quad \dots(6)$$

and $F\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$ is the function of $u, v, z, \frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

Now, we determine u and v so that the equation (3) takes the simplest possible form.

The equation (3) reduces to the simplest integrable form when the discriminant, $S^2 - 4RT$ of the quadratic equation

$$R\lambda^2 + S\lambda + T = 0 \quad \dots(7)$$

is either positive, negative or zero everywhere.

The three cases are discussed separately as follows.

Case I : If $S^2 - 4RT > 0$ (i.e., if equation (1) is hyperbolic)

In this case the roots λ_1, λ_2 (say) of the equation (7) are real and distinct. If we choose u and v such that

$$\frac{\partial u}{\partial x} = \lambda_1 \frac{\partial u}{\partial y} \quad \dots(8)$$

and
$$\frac{\partial v}{\partial x} = \lambda_2 \frac{\partial v}{\partial y} \quad \dots(9)$$

then
$$A = (R\lambda_1^2 + S\lambda_1 + T) \left(\frac{\partial u}{\partial y} \right)^2 = 0$$

$\therefore R\lambda_1^2 + S\lambda_1 + T = 0$ as λ_1 is a root of equation (7).

Similarly, $C = 0$

Now equation (8), can be written as

$$\frac{\partial u}{\partial x} - \lambda_1 \frac{\partial u}{\partial y} = 0$$

which is Lagrange's form and its auxiliary equations, are

$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{du}{0}$$

Which gives $du = 0 \quad \therefore u = c_1$ (constant)

$$\text{and} \quad \frac{dy}{dx} + \lambda_1 = 0 \quad \dots(10)$$

Let $f_1(x, y) = c_2$ (constant) be the solution of equation (10).

$\therefore$ the solution of (8), can be taken as

$$u = f_1(x, y) \quad \dots(11)$$

which is a suitable choice for u .

Similarly, the solution of (9) i.e., $f_2(x, y) = c_2$, can be taken as

$$v = f_2(x, y) \quad \dots(12)$$

is a suitable choice for v .

Now it can be shown that

$$AC - B^2 = \frac{1}{4}(4RT - S^2) \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right)^2 \quad \dots(13)$$

Since $A = 0$, $C = 0$ when u and v are chosen such that (8) and (9) are satisfied, therefore from (13), we get

$$B^2 = \frac{1}{4}(S^2 - 4RT) \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} \right)^2$$

$$\therefore S^2 - 4RT > 0, \quad \therefore B^2 > 0$$

$$\therefore A = 0 = C, \quad \therefore \text{equation (3) reduces to}$$

$$2B \frac{\partial^2 z}{\partial u \partial v} + F\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) = 0$$

Since $B^2 > 0$, i.e., $B \neq 0$, so we can divide this equation by B .

∴ dividing by $2B$, it reduces to

$$\frac{\partial^2 z}{\partial u \partial v} = \phi \left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v} \right)$$

Hence, if we make the substitutions of u and v defined by (11) and (12), the given equation (1) reduces to the form

$$\frac{\partial^2 z}{\partial u \partial v} = \phi \left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v} \right) \quad \dots(14)$$

which is the **canonical form of given equation** (1) and is simpler to solve, than the given equation (1).

Case II : If $S^2 - 4RT = 0$ (i.e., if equation (1) is parabolic)

In this case the roots of equation (7) i.e., $R\lambda^2 + S\lambda + T = 0$, are real and equal

$$\text{i.e.,} \quad \lambda_2 = \lambda_1$$

Here we choose u as in case I, such that

$$\frac{\partial u}{\partial x} = \lambda_1 \frac{\partial u}{\partial y} \text{ which gives } u = f(x, y) \quad \dots(15)$$

Also we take v to be any function of x and y , which is independent of u .

∴ As in case I, $A = 0$. Also from (13), $B^2 = 0 \therefore S^2 = 4RT$.

$$\text{i.e.,} \quad B = 0$$

Here C cannot be zero, otherwise v would be a function of u .

Putting $A = 0, B = 0$ in (3) and dividing by $C \neq 0$, equation (3) becomes

$$\frac{\partial^2 z}{\partial v^2} = \phi \left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v} \right)$$

Hence, if we make the substitution of u as defined by (15) and v any function of x and y , the given equation (1) in this case reduces to the form

$$\frac{\partial^2 z}{\partial v^2} = \phi \left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v} \right) \quad \dots(16)$$

which is another **canonical form of given equation** (1) and is simpler to solve than the given equation.

Case III : If $S^2 - 4RT < 0$. (i.e., if given equation (1) is elliptic)

In this case the roots of equation (7) i.e., $R\lambda^2 + S\lambda + T = 0$, are complex conjugates.

Proceeding as in case I, here the equation (1) will reduce to the same canonical form [equation (14)] as in case I but here the variables u and v are not real but are infact the complex conjugates.

To find the real canonical form.

$$\text{Let} \quad u = \alpha + i\beta \quad \text{and} \quad v = \alpha - i\beta$$

$$\therefore \quad \alpha = \frac{1}{2}(u + v) \quad \text{and} \quad \beta = \frac{1}{2}i(v - u).$$

Now we transform the independent variables u and v to α and β with the help of these relations.

$$\text{We have,} \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial u} + \frac{\partial z}{\partial \beta} \cdot \frac{\partial \beta}{\partial u} = \frac{1}{2} \left(\frac{\partial z}{\partial \alpha} - i \frac{\partial z}{\partial \beta} \right)$$

$$\text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial v} + \frac{\partial z}{\partial \beta} \cdot \frac{\partial \beta}{\partial v} = \frac{1}{2} \left(\frac{\partial z}{\partial \alpha} + i \frac{\partial z}{\partial \beta} \right)$$

$$\therefore \quad \frac{\partial^2 z}{\partial u \partial v} = \frac{\partial}{\partial u} \left(\frac{\partial z}{\partial v} \right) = \frac{1}{4} \left(\frac{\partial}{\partial \alpha} - i \frac{\partial}{\partial \beta} \right) \left(\frac{\partial z}{\partial \alpha} + i \frac{\partial z}{\partial \beta} \right) = \frac{1}{4} \left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} \right).$$

Substituting in (14), the **canonical form** of equation (1), in this case is

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \phi \left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta} \right) \quad \dots(17)$$

Summary : Here we list the canonical forms of the second order partial differential equation (1), obtained in different cases, in the following table.

Type of the Equation	Canonical Form
Hyperbolic $S^2 - 4RT > 0$	$\frac{\partial^2 z}{\partial u \partial v} = \phi \left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v} \right)$
Parabolic $S^2 - 4RT = 0$	$\frac{\partial^2 z}{\partial v^2} = \phi \left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v} \right)$
Elliptic $S^2 - 4RT < 0$	$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \phi \left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta} \right)$ Here $u = \alpha + i\beta$, $v = \alpha - i\beta$

11.2 Working Method of Reducing a Hyperbolic Equation to Canonical Form

Let the second order partial differential equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad \dots(1)$$

be *hyperbolic* i.e., $S^2 - 4RT > 0$.

To reduce (1) to canonical form, proceed as follows.

Step 1 : Write λ -quadratic equation

$$R\lambda^2 + S\lambda + T = 0 \quad \dots(2)$$

whose roots will be real and distinct $\because S^2 - 4RT > 0$

Solve (2) and let λ_1 , and λ_2 be its two distinct real roots.

Step 2 : Write the corresponding characteristic equations

$$\text{i.e.,} \quad \frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0$$

Solve these equations and let

$$f_1(x, y) = c_1 \quad \text{and} \quad f_2(x, y) = c_2 \quad \text{be the solutions.}$$

where c_1 and c_2 are the arbitrary constants which are known as the characteristic curves or simply the characteristics of the equation (1).

Step 3 : Let u and v be two functions of x and y , such that

$$u = f_1(x, y) \quad \text{and} \quad v = f_2(x, y) \quad \dots(3)$$

Step 4 : Using the relations (3), find p, q, r, s and t as needed for (1) in terms of u and v as in § 11.1

Step 5 : Substituting these values of p, q, r, s, t obtained in step (4) in (1), simplify to get

the *canonical form* of (1), which will be of the form $\frac{\partial^2 z}{\partial u \partial v} = \phi \left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v} \right)$.

Solved Examples

Example 1: Reduce the following equation to canonical form.

$$\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$$

[Agra 2005; Meerut 2009 (B.P.), 11 (Sem. I), 12 (Sem. I);
Delhi Maths (Hons.) 2002, 06; Himachal 2005; Kurukshetra 2004; Ravishankar 2004; I.A.S. 2008]

Solution: The given equation can be written as

$$r - x^2 t = 0 \quad \dots(1)$$

Comparing (1) with the equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \text{ we have}$$

$$R = 1, S = 0, T = -x^2 \quad \therefore S^2 - 4RT = 4x^2 > 0$$

and so (1) is a hyperbolic equation

$\therefore$ λ - quadratic equation, $R\lambda^2 + S\lambda + T = 0$, becomes

$$\lambda^2 - x^2 = 0, \text{ giving } \lambda = \pm x$$

Let $\lambda_1 = x$ and $\lambda_2 = -x$. (Real and distinct roots)

$\therefore$ The characteristic equations $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$, becomes

$$\frac{dy}{dx} + x = 0 \quad \text{and} \quad \frac{dy}{dx} - x = 0$$

Integrating them, we get the following characteristics of (1)

$$y + \frac{1}{2}x^2 = c_1 \quad \text{and} \quad y - \frac{1}{2}x^2 = c_2$$

$\therefore$ To change the independent variable x, y , to u, v , in the given equation (1), we take

$$\therefore \quad u = y + \frac{1}{2}x^2 \quad \text{and} \quad v = y - \frac{1}{2}x^2$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = x \cdot \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left\{ x \cdot \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \right\}$$

$$\begin{aligned}
&= x \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) + 1 \cdot \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \\
&= x \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \cdot \frac{\partial v}{\partial x} \right] + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \\
&= x^2 \left(\frac{\partial^2 z}{\partial u^2} - 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right) + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}
\end{aligned}$$

and

$$\begin{aligned}
t &= \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) \\
&= \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}
\end{aligned}$$

∴ Substituting the values of r and t in (1), we get

$$x^2 \left(\frac{\partial^2 z}{\partial u^2} - 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right) + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} - x^2 \left(\frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right) = 0$$

or

$$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4x^2} \cdot \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right)$$

or

$$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4(u-v)} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right)$$

which is the required *canonical form* of the given equation.

Example 2: Reduce the equation

$$(y-1)r - (y^2-1)s + y(y-1)t + p - q = 2ye^{2x}(1-y)^3$$

to canonical form and hence solve it.

[Rohilkhand 2001; Delhi 2008]

Solution: Comparing the given equation with the equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \text{ we have}$$

$$R = y-1, S = -(y^2-1), T = y(y-1)$$

$$\therefore S^2 - 4RT = (y-1)^4 > 0 \Rightarrow \text{Given equation is hyperbolic.}$$

∴ The quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$(y-1)\lambda^2 - (y^2-1)\lambda + y(y-1) = 0$$

or

$$\lambda^2 - (y+1)\lambda + y = 0$$

or

$$(\lambda-1)(\lambda-y) = 0, \quad \text{giving } \lambda = 1, y$$

i.e., $\lambda_1 = 1$ and $\lambda_2 = y$ (Real and distinct roots)

∴ The characteristic equations $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$ becomes

$$\frac{dy}{dx} + 1 = 0 \quad \text{and} \quad \frac{dy}{dx} + y = 0$$

Integrating these equations, we get the following characteristic curves (or characteristics) of the given equation.

$$x + y = c_1 \quad \text{and} \quad ye^x = c_2$$

∴ To change the independent variables x, y to u, v , in the given equation, we take

$$u = x + y \quad \text{and} \quad v = ye^x.$$

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + ye^x \frac{\partial z}{\partial v} = \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v}$$

$$r = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \left(\frac{\partial}{\partial u} + v \frac{\partial}{\partial v} \right) \left(\frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} \right)$$

$$= \frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} \right) + v \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} \right)$$

$$= \frac{\partial^2 z}{\partial u^2} + 2v \frac{\partial^2 z}{\partial u \partial v} + v^2 \frac{\partial^2 z}{\partial v^2} + v \frac{\partial z}{\partial v}$$

$$s = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v} \right)$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} \right) + e^x \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial v} \right) + e^x \frac{\partial z}{\partial v}$$

$$= \left(\frac{\partial}{\partial u} + v \frac{\partial}{\partial v} \right) \left(\frac{\partial z}{\partial u} \right) + e^x \left(\frac{\partial}{\partial u} + v \frac{\partial}{\partial v} \right) \left(\frac{\partial z}{\partial v} \right) + e^x \frac{\partial z}{\partial v}$$

$$= \frac{\partial^2 z}{\partial u^2} + (e^x + v) \frac{\partial^2 z}{\partial u \partial v} + ve^x \frac{\partial^2 z}{\partial v^2} + e^x \frac{\partial z}{\partial v}$$

$$\text{and} \quad t = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v} \right) = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial u} \right) + e^x \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial v} \right)$$

$$= \frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} \right) \frac{\partial u}{\partial y} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} \right) \frac{\partial v}{\partial y} + e^x \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial v} \right) \frac{\partial u}{\partial y} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial v} \right) \frac{\partial v}{\partial y} \right]$$

$$= \frac{\partial^2 z}{\partial u^2} + 2e^x \frac{\partial^2 z}{\partial u \partial v} + e^{2x} \frac{\partial^2 z}{\partial v^2}$$

Substituting the values of p, q, r, s, t in the given equation, we get

$$\begin{aligned} & (y-1) \left(\frac{\partial^2 z}{\partial u^2} + 2v \frac{\partial^2 z}{\partial v^2} + v^2 \frac{\partial^2 z}{\partial v^2} + v \frac{\partial z}{\partial v} \right) \\ & - (y^2-1) \left[\frac{\partial^2 z}{\partial u^2} + (e^x + v) \frac{\partial^2 z}{\partial u \partial v} + v e^x \frac{\partial^2 z}{\partial v^2} + e^x \frac{\partial z}{\partial v} \right] \\ & + y(y-1) \left[\frac{\partial^2 z}{\partial u^2} + 2e^x \frac{\partial^2 z}{\partial u \partial v} + e^{2x} \frac{\partial^2 z}{\partial v^2} \right] + (v - e^x) \frac{\partial z}{\partial v} = 2ye^{2x}(1-y)^3 \end{aligned}$$

or
$$(1-y)^3 e^x \frac{\partial^2 z}{\partial u \partial v} = 2ye^{2x}(1-y)^3$$

or
$$\frac{\partial^2 z}{\partial u \partial v} = 2ye^x \quad \text{or} \quad \frac{\partial^2 z}{\partial u \partial v} = 2v \quad \dots(1)$$

which is the required *canonical form*.

To Find the Solution : Integrating (1) partially, w.r.t. v , we have

$$\frac{\partial z}{\partial u} = v^2 + f(u). \quad \dots(2)$$

where $f(u)$ is an arbitrary function of u .

Now integrating (3) partially w.r.t. u , we have

$$z = uv^2 + \phi(u) + \psi(v) \quad \text{where} \quad \phi(u) = \int f(u) du.$$

Where $\psi(v)$ is an arbitrary function of v .

or
$$z = (x+y)y^2 e^{2x} + \phi(x+y) + \psi(y e^x)$$

which is the solution of the given equation.

Example 3: Reduce the equation $yr + (x+y)s + xt = 0$ to canonical form and hence find its general solution. [Delhi Maths (Hons.) 2007]

Solution: Given, $yr + (x+y)s + xt = 0 \quad \dots(1)$

Comparing (1) with the equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \text{ we have}$$

$$R = y, S = x + y, T = x,$$

$$\therefore S^2 - 4RT = (x+y)^2 - 4xy = (x-y)^2 > 0 \text{ if } x \neq y$$

which shows that (1) is hyperbolic equation if $x \neq y$.

$\therefore$ λ -quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$y\lambda^2 + (x+y)\lambda + x = 0 \quad \text{or} \quad (y\lambda + x)(\lambda + 1) = 0$$

$$\therefore \quad \lambda = -x/y, -1$$

$\therefore$ The characteristic equations $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$ becomes,

$$\frac{dy}{dx} - \frac{x}{y} = 0 \quad \text{and} \quad \frac{dy}{dx} - 1 = 0$$

$$\text{Solving } y^2/2 - x^2/2 = c_1 \quad \text{and} \quad y - x = c_2$$

where c_1 and c_2 are arbitrary constants are the characteristics of (1)

To change the independent variables x, y to u, v , in (1)

$$\text{we choose} \quad u = y^2/2 - x^2/2 \quad \text{and} \quad v = y - x$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = - \left(x \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right)$$

$$\text{and} \quad q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = y \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$\begin{aligned} r &= \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left(- \left(x \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) \right) \\ &= -1 \frac{\partial z}{\partial u} - x \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} \right) - \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial v} \right) \\ &= - \frac{\partial z}{\partial u} - x \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} \right) \cdot \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} \right) \cdot \frac{\partial v}{\partial x} \right] - \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial v} \right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial v} \right) \frac{\partial v}{\partial x} \right] \\ &= - \frac{\partial z}{\partial u} - x \left[-x \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} \right] - \left[-x \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v^2} \right] \\ &= x^2 \frac{\partial^2 z}{\partial u^2} + 2x \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial u} \\ s &= \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left(y \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) \\ &= y \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} \right) + \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial v} \right) \end{aligned}$$

$$\begin{aligned}
&= y \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} \right) \cdot \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} \right) \frac{\partial v}{\partial x} \right] + \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial v} \right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial v} \right) \cdot \frac{\partial v}{\partial x} \right] \\
&= y \left[-x \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} \right] + \left[-x \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v^2} \right] = - \left[xy \frac{\partial^2 z}{\partial u^2} + (x+y) \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right]
\end{aligned}$$

and

$$\begin{aligned}
t &= \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} \left(y \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) \\
&= 1 \cdot \frac{\partial z}{\partial u} + y \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial u} \right) + \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial v} \right) \\
&= \frac{\partial z}{\partial u} + y \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} \right) \frac{\partial u}{\partial y} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} \right) \frac{\partial v}{\partial y} \right] + \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial v} \right) \cdot \frac{\partial u}{\partial y} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial v} \right) \frac{\partial v}{\partial y} \right] \\
&= \frac{\partial z}{\partial u} + y \left[y \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u} \right] + \left[y \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right] \\
&= y^2 \frac{\partial^2 z}{\partial u^2} + 2y \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial u}
\end{aligned}$$

Substituting the values of r , s and t in (1), we get

$$\begin{aligned}
&y \left[x^2 \frac{\partial^2 z}{\partial u^2} + 2x \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial u} \right] - (x+y) \left[xy \frac{\partial^2 z}{\partial u^2} + (x+y) \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right] \\
&\quad + x \left[y^2 \frac{\partial^2 z}{\partial u^2} + 2y \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial u} \right] = 0
\end{aligned}$$

$$\text{or} \quad [4xy - (x+y)^2] \frac{\partial^2 z}{\partial u \partial v} - y \frac{\partial z}{\partial u} + x \frac{\partial z}{\partial u} = 0$$

$$\text{or} \quad -(y-x)^2 \frac{\partial^2 z}{\partial u \partial v} - (y-x) \frac{\partial z}{\partial u} = 0 \quad \text{or} \quad (y-x) \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial u} = 0 \text{ as } y \neq x.$$

$$\text{or} \quad v \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial u} = 0 \quad \dots(2)$$

which is the required *canonical form*.

To Find the Solution : Multiplying both sides of (2) by u , we get

$$uv \frac{\partial^2 z}{\partial u \partial v} + u \frac{\partial z}{\partial u} = 0 \quad \text{or} \quad (uvDD' + uD)z = 0 \quad \dots(3)$$

$$\text{where} \quad D \equiv \frac{\partial}{\partial u} \quad \text{and} \quad D' \equiv \frac{\partial}{\partial v}.$$

To reduce (2) to linear differential equation with constant coefficients, let $u = e^X$ and $v = e^Y$ so that $X = \log u$ and $Y = \log v$.

$$\therefore \quad u \frac{\partial}{\partial u} \equiv \frac{\partial}{\partial X} = D_1 \text{ (say)} \quad \text{i.e.,} \quad uD \equiv D_1, \quad v \frac{\partial}{\partial v} \equiv \frac{\partial}{\partial Y} \equiv D'_1 \text{ (say)}$$

$$\text{and} \quad uv \frac{\partial^2}{\partial u \partial v} = \frac{\partial^2}{\partial X \partial Y} \equiv D_1 D'_1 \quad \text{i.e.,} \quad uvDD' = D_1 D'_1$$

Substituting in (2), it reduces to

$$(D_1 D'_1 + D_1)z = 0 \quad \text{or} \quad D_1(D'_1 + 1)z = 0$$

whose general solution is

$$z = \phi_1(Y) + e^{-Y} \phi_2(X) = \phi_1(\log v) + v^{-1} \phi_2(\log u)$$

$$\text{or} \quad z = \psi_1(v) + v^{-1} \psi_2(u)$$

$$\text{or} \quad z = \psi_1(y-x) + (y-x)^{-1} \psi_2(y^2 - x^2)$$

where ψ_1 and ψ_2 are arbitrary functions.

Example 4: Reduce the following equation to canonical form and hence solve it.

$$y(x+y)(r-s) - xp - yq - z = 0.$$

[Meerut 2012 (O)]

Solution: Comparing the given equation with the equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \text{ we have}$$

$$R = y(x+y), \quad S = -y(x+y), \quad T = 0$$

$$S^2 - 4RT = y^2(x+y)^2 > 0 \Rightarrow \text{Given equation is hyperbolic}$$

Here the quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$y(x+y)\lambda^2 - y(x+y)\lambda + 0 = 0$$

$$\text{or} \quad \lambda(\lambda - 1) = 0 \quad \therefore \quad \lambda = 0, 1$$

$$\therefore \quad \text{Let} \quad \lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 1$$

$$\therefore \quad \text{The characteristic equations } \frac{dy}{dx} + \lambda_1 = 0 \text{ and } \frac{dy}{dx} + \lambda_2 = 0, \text{ becomes}$$

$$\frac{dy}{dx} = 0 \quad \text{and} \quad \frac{dy}{dx} + 1 = 0 \quad \text{or} \quad \frac{dy}{dx} = 0 \text{ and } dy + dx = 0$$

Integrating, $y = c_1$ and $y + x = c_2$, which are the characteristics of the given equation.

$\therefore$ To change the independent variables x and y to u and v in the given equation, we take $u = y$ and $v = x + y$

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial z}{\partial v}$$

$$\text{and } q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$\therefore \frac{\partial}{\partial x} \equiv \frac{\partial}{\partial v} \quad \text{and} \quad \frac{\partial}{\partial y} \equiv \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial v} \right) = \frac{\partial^2 z}{\partial v^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) = \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$\begin{aligned} \text{and } t &= \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) \\ &= \frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) = \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \end{aligned}$$

Substituting the values of p, q, r, s, t in the given equation, we have

$$-y(x+y) \cdot \frac{\partial^2 z}{\partial u \partial v} - x \frac{\partial z}{\partial v} - y \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) - z = 0$$

$$\text{or } -y(x+y) \cdot \frac{\partial^2 z}{\partial u \partial v} - y \frac{\partial z}{\partial u} - (x+y) \frac{\partial z}{\partial v} - z = 0$$

$$\text{or } uv \frac{\partial^2 z}{\partial u \partial v} + u \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} + z = 0$$

$$\text{or } \frac{\partial^2 z}{\partial u \partial v} + \frac{1}{v} \frac{\partial z}{\partial u} + \frac{1}{u} \frac{\partial z}{\partial v} + \frac{1}{uv} z = 0 \quad \dots(1)$$

which is the *canonical form* of the given equation.

To Find the Solution : Equation (1), can be written as

$$\frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} + \frac{1}{u} z \right) + \frac{1}{v} \left(\frac{\partial z}{\partial u} + \frac{1}{u} z \right) = 0 \quad \dots(2)$$

$$\text{or } \frac{\partial t}{\partial v} + \frac{1}{v} t = 0, \quad \text{Putting } \frac{\partial z}{\partial u} + \frac{1}{u} z = t$$

which is L.D.E. in t with v as independent variable, $\therefore$ I.F. = $e^{\int \left(\frac{1}{v} \right) dv} = e^{\log v} = v$

$$\therefore \quad v \cdot t = \int 0 \cdot dv + f(u) \quad \text{or} \quad v \left(\frac{\partial z}{\partial u} + \frac{1}{u} z \right) = f(u)$$

$$\text{or} \quad \frac{\partial z}{\partial u} + \frac{1}{u} z = \frac{1}{v} f(u)$$

which is L.D.E. in z , with u as independent variable (treating v constant)

$$\therefore \quad \text{I.F.} = e^{\int \left(\frac{1}{u} \right) du} = e^{\log u} = u$$

$$\therefore \quad u \cdot z = \int u \cdot \frac{1}{v} f(u) du + \psi(v) = \frac{1}{v} \phi(u) + \psi(v), \text{ where } \phi(u) = \int u f(u) du$$

$$\text{or} \quad z = \frac{1}{uv} \phi(u) + \frac{1}{u} \psi(v)$$

$$\text{or} \quad z = \frac{1}{y(x+y)} \phi(y) + \frac{1}{y} \psi(x+y)$$

where ϕ and ψ are arbitrary functions.

which is the required solution.

Note : If we write equation (1) as

$$\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial v} + \frac{1}{v} z \right) + \frac{1}{u} \left(\frac{\partial z}{\partial v} + \frac{1}{v} z \right) = 0$$

and proceed as above, then the solution of the equation will be

$$z = \frac{1}{y(x+y)} \phi(x+y) + \frac{1}{(x+y)} \psi(y).$$

Example 5: Reduce the equation

$$(n-1)^2 \frac{\partial^2 z}{\partial x^2} - y^{2n} \frac{\partial^2 z}{\partial y^2} = ny^{2n-1} \frac{\partial z}{\partial y}$$

to canonical form, and find its general solution.

[Delhi Maths (Hons.) 2000, 01, 05; Himachal 2004]

Solution: The given equation can be written as

$$(n-1)^2 r - y^{2n} t - ny^{2n-1} q = 0 \quad \dots(1)$$

Comparing with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have

$$R = (n-1)^2, \quad S = 0, \quad T = -y^{2n}.$$

$$\therefore S^2 - 4RT = 4(n-1)^2 y^{2n} > 0 \Rightarrow (1) \text{ is a hyperbolic equation.}$$

$\therefore$ The quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$(n-1)^2 \lambda^2 - y^{2n} = 0 \quad \text{or} \quad \lambda^2 = \frac{1 y^{2n}}{(n-1)^2}$$

Giving $\lambda = \pm \frac{1}{(n-1)} y^n$ i.e., $\lambda_1 = \frac{1}{(n-1)} y^n$, $\lambda_2 = -\frac{1}{(n-1)} y^n$ (Two real distinct roots)

$\therefore$ The characterise equations $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$ becomes

$$\frac{dy}{dx} + \frac{1}{(n-1)} y^n = 0 \quad \text{and} \quad \frac{dy}{dx} - \frac{1}{(n-1)} y^n = 0$$

$$\text{or} \quad (n-1)y^{-n}dy + dx = 0 \quad \text{and} \quad (n-1)y^{-n}dy - dx = 0$$

integrating, we have

$$x - y^{-n+1} = c_1 \quad \text{and} \quad x + y^{-n+1} = c_2$$

which are the characteristic curves of (1).

$\therefore$ To change the independent variables x, y to u, v in the given equation (1), we take

$$u = x - y^{-n+1} \quad \text{and} \quad v = x + y^{-n+1}$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad \therefore \quad \frac{\partial}{\partial x} \equiv \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = (n-1)y^{-n} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right)$$

$$r = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) = \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$\begin{aligned} \text{and} \quad t &= \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} \left[(n-1)y^{-n} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \right] \\ &= -n(n-1)y^{-n-1} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) + (n-1)y^{-n} \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \\ &= -n(n-1)y^{-n-1} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) + (n-1)y^{-n} \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \frac{\partial u}{\partial y} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \frac{\partial v}{\partial y} \right] \\ &= -n(n-1)y^{-n-1} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) + (n-1)^2 y^{-2n} \left[\frac{\partial^2 z}{\partial u^2} - 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right]. \end{aligned}$$

Substituting these values in (1) and simplifying, we have

$$\frac{\partial^2 z}{\partial u \partial v} = 0 \quad \dots(2)$$

which is the required, *canonical form*.

To Find the Solution : Integrating, (2) w.r.t. v , we have

$$\frac{\partial z}{\partial u} = f(u), \text{ where } f(u) \text{ is an arbitrary function of } u.$$

Again integrating partially w.r.t. u , we have

$$z = \phi(u) + \psi(v) \quad \text{where } \phi(u) = \int f(u) du$$

where $\phi(u)$ and $\psi(v)$ is arbitrary functions.

Hence the required solution is

$$z = \phi(x - y^{-n+1}) + \psi(x + y^{-n+1}).$$

Example 6: Reduce the following equation to canonical form and hence solve

$$x(xy-1)r - (x^2y^2-1)s + y(xy-1)t + (x-1)p + (y-1)q = 0.$$

Solution: Comparing the given equation with the equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \text{ we have}$$

$$R = x(xy-1), S = -(x^2y^2-1), T = y(xy-1)$$

$$\therefore S^2 - 4RT = (xy-1)^4 > 0 \Rightarrow \text{The given equation is hyperbolic.}$$

Here the quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$x(xy-1)\lambda^2 - (x^2y^2-1)\lambda + y(xy-1) = 0$$

$$\text{or } (xy-1)(x\lambda-1)(\lambda-y) = 0 \quad \text{or } \lambda = 1/x, y$$

$$\text{Let } \lambda_1 = 1/x \text{ and } \lambda_2 = y$$

$$\therefore \text{ The characteristic equations } \frac{dy}{dx} + \lambda_1 = 0 \text{ and } \frac{dy}{dx} + \lambda_2 = 0, \text{ becomes}$$

$$\frac{dy}{dx} + \frac{1}{x} = 0 \quad \text{and} \quad \frac{dy}{dx} + y = 0$$

$$\text{or } dy + \frac{dx}{x} = 0 \quad \text{and} \quad \frac{dy}{y} + dx = 0$$

$$\text{Integrating } y + \log x = \log c_1 \text{ and } \log y + x = \log c_2$$

$$\therefore x e^y = c_1 \quad \text{and} \quad y e^x = c_2$$

Which are the characterise curves of the given equation.

$$\therefore \text{ To change the independent variables } x \text{ and } y \text{ to } u \text{ and } v \text{ in the given equation, we take } u = x e^y \text{ and } v = y e^x$$

$$\begin{aligned}
\therefore \quad p &= \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = e^y \frac{\partial z}{\partial u} + ye^x \frac{\partial z}{\partial v} = e^y \frac{\partial z}{\partial u} + v \frac{\partial z}{\partial v} \\
q &= \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = x e^y \frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v} = u \frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v} \\
r &= \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(e^y \frac{\partial z}{\partial u} + ye^x \frac{\partial z}{\partial v} \right) = e^y \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} \right) + ye^x \frac{\partial z}{\partial v} + ye^x \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial v} \right) \\
&= e^y \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} \right) \cdot \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} \right) \frac{\partial v}{\partial x} \right] + ye^x \frac{\partial z}{\partial v} + ye^x \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial v} \right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial v} \right) \frac{\partial v}{\partial x} \right] \\
&= e^y \left[\frac{\partial^2 z}{\partial u^2} e^y + \frac{\partial^2 z}{\partial u \partial v} ye^x \right] + ye^x \frac{\partial z}{\partial v} + ye^x \left[\frac{\partial^2 z}{\partial u \partial v} e^y + \frac{\partial^2 z}{\partial v^2} ye^x \right] \\
&= e^2 y \frac{\partial^2 z}{\partial u^2} + 2 ye^{x+y} \frac{\partial^2 z}{\partial u \partial v} + ye^x \frac{\partial z}{\partial v} + y^2 e^{2x} \frac{\partial^2 z}{\partial v^2} \\
s &= \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left(x e^y \frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v} \right) \\
&= e^y \frac{\partial z}{\partial u} + x e^y \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} \right) + e^x \frac{\partial z}{\partial v} + e^x \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial v} \right) \\
&= e^y \frac{\partial z}{\partial u} + x e^y \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} \right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} \right) \frac{\partial v}{\partial x} \right] \\
&\quad + e^x \frac{\partial z}{\partial v} + e^x \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial v} \right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial v} \right) \frac{\partial v}{\partial x} \right] \\
&= e^y \frac{\partial z}{\partial u} + x e^y \left[\frac{\partial^2 z}{\partial u^2} e^y + \frac{\partial^2 z}{\partial v \partial u} ye^x \right] + e^x \frac{\partial z}{\partial v} + e^x \left[\frac{\partial^2 z}{\partial u \partial v} e^y + \frac{\partial^2 z}{\partial v^2} ye^x \right] \\
&= x e^2 y \frac{\partial^2 z}{\partial u^2} + (1 + xy) e^{x+y} \frac{\partial^2 z}{\partial u \partial v} + e^y \frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v} + ye^{2x} \frac{\partial^2 z}{\partial v^2} \\
t &= \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} \left(x e^y \frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v} \right) \\
&= x e^y \frac{\partial z}{\partial u} + x e^y \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial u} \right) + e^x \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial v} \right) \\
&= x e^y \frac{\partial z}{\partial u} + x e^y \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} \right) \frac{\partial u}{\partial y} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} \right) \frac{\partial v}{\partial y} \right] + e^x \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial v} \right) \frac{\partial u}{\partial y} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial v} \right) \frac{\partial v}{\partial y} \right] \\
&= x e^y \frac{\partial z}{\partial u} + x e^y \left[\frac{\partial^2 z}{\partial u^2} x e^y + \frac{\partial^2 z}{\partial v \partial u} e^x \right] + e^x \left[\frac{\partial^2 z}{\partial v \partial u} x e^y + \frac{\partial^2 z}{\partial v^2} e^x \right]
\end{aligned}$$

$$= x^2 e^{2y} \frac{\partial^2 z}{\partial u^2} + 2x e^{x+y} \frac{\partial^2 z}{\partial u \partial v} + e^{2x} \frac{\partial^2 z}{\partial v^2} + x e^y \frac{\partial z}{\partial u}$$

Substituting the values of p, q, r, s, t in the given equation, we have,

$$\begin{aligned} x(x-y-1) & \left[e^{2y} \frac{\partial^2 z}{\partial u^2} + 2y e^{x+y} \frac{\partial^2 z}{\partial u \partial v} + y e^x \frac{\partial z}{\partial v} + y^2 e^{2x} \frac{\partial^2 z}{\partial v^2} \right] \\ & - (x^2 y^2 - 1) \left[x e^{2y} \frac{\partial^2 z}{\partial u^2} + (x-y+1) e^{x+y} \frac{\partial^2 z}{\partial u \partial v} + e^y \frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v} + y e^{2x} \frac{\partial^2 z}{\partial v^2} \right] \\ & + y(x-y-1) \left[x^2 e^{2y} \frac{\partial^2 z}{\partial u^2} + 2x e^{x+y} \frac{\partial^2 z}{\partial u \partial v} + e^{2x} \frac{\partial^2 z}{\partial v^2} + x e^y \frac{\partial z}{\partial u} \right] \\ & + (x-1) \left[e^y \frac{\partial z}{\partial u} + y e^x \frac{\partial z}{\partial v} \right] + (y-1) \left[x e^y \frac{\partial z}{\partial u} + e^x \frac{\partial z}{\partial v} \right] = 0 \end{aligned}$$

$$\text{or } \frac{\partial^2 z}{\partial u \partial v} = 0 \quad \dots(1)$$

which is the *canonical form* of the given equation.

To Find the Solution : Integrating (1) partially w.r.t., u , we get $\frac{\partial z}{\partial v} = f(v)$

Again integrating partially w.r.t., v , we get

$$z = \phi(v) + \psi(u), \quad \text{where } \phi(v) = \int f(v) dv$$

$$\text{or } z = \phi(y e^x) + \psi(x e^y)$$

where ϕ and ψ are arbitrary functions.

Example 7: Reduce the equation

$$x(y-x)r - (y^2 - x^2)s + y(y-x)t + (y+x)(p-q) = 2x + 2y + 2$$

to canonical form and hence solve.

Solution: Comparing the given equation with the equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \text{ we have}$$

$$R = x(y-x), S = -(y^2 - x^2), T = y(y-x)$$

$$\begin{aligned} \therefore S^2 - 4RT &= (y^2 - x^2)^2 - 4xy(y-x)^2 = (y-x)^2[(y+x)^2 - 4xy] = (y-x)^4 > 0 \\ &\Rightarrow \text{given equation is hyperbolic.} \end{aligned}$$

Here the quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$x(y-x)\lambda^2 - (y^2 - x^2)\lambda + y(y-x) = 0$$

$$\text{or } x\lambda^2 - (y+x)\lambda + y = 0 \text{ or } (x\lambda - y)(\lambda - 1) = 0 \quad \therefore \lambda = y/x, 1$$

$$\therefore \text{ Let } \lambda_1 = y/x \quad \text{and} \quad \lambda_2 = 1$$

$$\therefore \text{ The characteristic equations } \frac{dy}{dx} + \lambda_1 = 0 \text{ and } \frac{dy}{dx} + \lambda_2 = 0, \text{ becomes}$$

$$\frac{dy}{dx} + \frac{y}{x} = 0 \quad \text{and} \quad \frac{dy}{dx} + 1 = 0$$

$$\text{or } \frac{dy}{y} + \frac{dx}{x} = 0 \quad \text{and} \quad dy + dx = 0$$

$$\text{Integrating } \log y + \log x = \log c_1 \text{ i.e., } x y = c_1 \quad \text{and} \quad y + x = c_2$$

Which are the characterises of the given equation

$\therefore$ To change the independent variables x and y to u and v , in the given equation, we take

$$u = x y \quad \text{and} \quad v = x + y$$

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = y \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = x \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left(y \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) = y \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} \right) + \frac{\partial}{\partial x} \frac{\partial z}{\partial v}$$

$$= y \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} \right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} \right) \frac{\partial v}{\partial x} \right] + \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial v} \right) \cdot \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial v} \right) \frac{\partial v}{\partial x} \right]$$

$$= y \left[\frac{\partial^2 z}{\partial u^2} \cdot y + \frac{\partial^2 z}{\partial v \partial u} \cdot 1 \right] + \left[\frac{\partial^2 z}{\partial v \partial u} \cdot y + \frac{\partial^2 z}{\partial v^2} \cdot 1 \right]$$

$$= y^2 \frac{\partial^2 z}{\partial u^2} + 2 y \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left(x \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) = x \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} \right) + \frac{\partial z}{\partial u} + \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial v} \right)$$

$$= x \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} \right) \cdot \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} \right) \frac{\partial v}{\partial x} \right] + \frac{\partial z}{\partial u} + \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial v} \right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial v} \right) \cdot \frac{\partial v}{\partial x} \right]$$

$$= x \left[\frac{\partial^2 z}{\partial u^2} y + \frac{\partial^2 z}{\partial v \partial u} \cdot 1 \right] + \frac{\partial z}{\partial u} + \left[\frac{\partial^2 z}{\partial u \partial v} y + \frac{\partial^2 z}{\partial v^2} \cdot 1 \right]$$

$$= x y \frac{\partial^2 z}{\partial u^2} + (x + y) \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial u}$$

and
$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(x \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) = x \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial u} \right) + \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial v} \right)$$

$$= x \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} \right) \cdot \frac{\partial u}{\partial y} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} \right) \frac{\partial v}{\partial y} \right] + \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial v} \right) \cdot \frac{\partial u}{\partial y} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial v} \right) \frac{\partial v}{\partial y} \right]$$

$$= x \left[\frac{\partial^2 z}{\partial u^2} x + \frac{\partial^2 z}{\partial v \partial u} \cdot 1 \right] + \left[\frac{\partial^2 z}{\partial u \partial v} x + \frac{\partial^2 z}{\partial v^2} \cdot 1 \right] = x^2 \frac{\partial^2 z}{\partial u^2} + 2x \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

Substituting the values of p, q, r, s, t in the given equation, we have

$$\begin{aligned} & x(y-x) \left[y^2 \frac{\partial^2 z}{\partial u^2} + 2y \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right] \\ & - (y^2 - x^2) \left[x y \frac{\partial^2 z}{\partial u^2} + (x+y) \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial u} \right] \\ & + y(y-x) \left[x^2 \frac{\partial^2 z}{\partial u^2} + 2x \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right] \\ & + (y+x) \left[y \frac{\partial z}{\partial u} - x \frac{\partial z}{\partial v} \right] = 2x + 2y + 2 \end{aligned}$$

or
$$-(y-x)^3 \frac{\partial^2 z}{\partial u \partial v} = 2x + 2y + 2$$

or
$$\frac{\partial^2 z}{\partial u \partial v} = \frac{2(x+y)+2}{[(x+y)^2 - 4xy]^{3/2}}$$

or
$$\frac{\partial^2 z}{\partial u \partial v} = \frac{2(v+1)}{(v^2 - 4u)^{3/2}} \quad \dots(1)$$

which is the *canonical form* of the given equation.

To Find the Solution : Integrating (1) partially w.r.t. 'u', (treating v constant), we get

$$\frac{\partial z}{\partial v} = \frac{v+1}{\sqrt{(v^2 - 4u)}} + f(v)$$

or
$$\frac{\partial z}{\partial v} = \frac{v}{\sqrt{(v^2 - 4u)}} + \frac{1}{\sqrt{(v^2 - 4u)}} + f(v)$$

Integrating partially w.r.t. 'v', (treating u constant)

$$z = \sqrt{(v^2 - 4u)} + \log [v + \sqrt{(v^2 - 4u)}] + \phi(v) + \psi(u), \text{ where, } \phi(v) = \int f(v)dv$$

or
$$z = \sqrt{\{(x + y)^2 - 4xy\}} + \log [x + y + \sqrt{\{(x + y)^2 - 4xy\}}] + \phi(x + y) + \psi(xy)$$

or
$$z = (x - y) + \log(2x) + \phi(x + y) + \psi(xy)$$

where ϕ and ψ are arbitrary functions.

which is the solution of the given equation.

Example 8: Reduce the following equation to canonical form and hence solve it

$$xyr - (x^2 - y^2)s - xyt + py - qx = 2(x^2 - y^2). \quad [\text{Delhi Maths (Hons.) 2006}]$$

Solution: Comparing the given equation with the equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \text{ we have}$$

$$R = xy, S = -(x^2 - y^2), T = -xy. \therefore S^2 - 4RT = (x^2 + y^2)^2 > 0$$

$\Rightarrow$ Given equation is hyperbolic.

Here the quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$xy\lambda^2 - (x^2 - y^2)\lambda - xy = 0$$

or
$$(x\lambda + y)(y\lambda - x) = 0 \quad \text{or} \quad \lambda = -y/x, x/y$$

Let
$$\lambda_1 = -y/x \quad \text{and} \quad \lambda_2 = x/y$$

$\therefore$ The characteristic equations $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$, becomes

$$\frac{dy}{dx} - \frac{y}{x} = 0 \quad \text{and} \quad \frac{dy}{dx} + \frac{x}{y} = 0$$

or
$$\frac{dy}{y} - \frac{dx}{x} = 0 \quad \text{and} \quad y dy + x dx = 0$$

Integrating $\log y - \log x = \log c_1$ and $y^2 + x^2 = c_2$

$\therefore \quad y/x = c_1 \quad \text{and} \quad y^2 + x^2 = c_2$

Which are the characteristics of the given equation.

∴

To change the independent variables x and y to u and v , in the given equation we take $u = y/x$ and $v = x^2 + y^2$.

$$\begin{aligned}
 \therefore \quad p &= \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} \left(-\frac{y}{x^2} \right) + \frac{\partial z}{\partial v} \cdot 2x \\
 q &= \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} \left(\frac{1}{x} \right) + \frac{\partial z}{\partial v} \cdot 2y \\
 r &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left(-\frac{y}{x^2} \frac{\partial z}{\partial u} + 2x \frac{\partial z}{\partial v} \right) \\
 &= \left(\frac{2y}{x^3} \right) \frac{\partial z}{\partial u} - \frac{y}{x^2} \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} \right) + 2 \frac{\partial z}{\partial v} + 2x \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial v} \right) \\
 &= \left(\frac{2y}{x^3} \right) \frac{\partial z}{\partial u} - \frac{y}{x^2} \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} \right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} \right) \cdot \frac{\partial v}{\partial x} \right] + 2 \frac{\partial z}{\partial v} + 2x \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial v} \right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial v} \right) \frac{\partial v}{\partial x} \right] \\
 &= \left(\frac{2y}{x^3} \right) \frac{\partial z}{\partial u} - \frac{y}{x^2} \left[\frac{\partial^2 z}{\partial u^2} \cdot \left(-\frac{y}{x^2} \right) + \frac{\partial^2 z}{\partial v \partial u} \cdot 2x \right] + 2 \frac{\partial z}{\partial v} + 2x \left[\frac{\partial^2 z}{\partial u \partial v} \cdot \left(-\frac{y}{x^2} \right) + \frac{\partial^2 z}{\partial v^2} \cdot 2x \right] \\
 &= \left(\frac{y^2}{x^4} \right) \frac{\partial^2 z}{\partial u^2} - \frac{4y}{x} \frac{\partial^2 z}{\partial u \partial v} + 4x^2 \frac{\partial^2 z}{\partial v^2} + \frac{2y}{x^3} \frac{\partial z}{\partial u} + 2 \frac{\partial z}{\partial v} \\
 s &= \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{1}{x} \frac{\partial z}{\partial u} + 2y \frac{\partial z}{\partial v} \right) \\
 &= \left(-\frac{1}{x^2} \right) \frac{\partial z}{\partial u} + \frac{1}{x} \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} \right) + 2y \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial v} \right) \\
 &= \left(-\frac{1}{x^2} \right) \frac{\partial z}{\partial u} + \frac{1}{x} \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} \right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} \right) \cdot \frac{\partial v}{\partial x} \right] + 2y \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial v} \right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial v} \right) \frac{\partial v}{\partial x} \right] \\
 &= -\frac{1}{x^2} \frac{\partial z}{\partial u} + \frac{1}{x} \left[\frac{\partial^2 z}{\partial u^2} \cdot \left(-\frac{y}{x^2} \right) + \frac{\partial^2 z}{\partial v \partial u} \cdot 2x \right] + 2y \left[\frac{\partial^2 z}{\partial u \partial v} \cdot \left(-\frac{y}{x^2} \right) + \frac{\partial^2 z}{\partial v^2} \cdot 2x \right] \\
 &= -\frac{y}{x^3} \frac{\partial^2 z}{\partial u^2} + \left(2 - \frac{2y^2}{x^2} \right) \frac{\partial^2 z}{\partial u \partial v} + 4xy \frac{\partial^2 z}{\partial v^2} - \frac{1}{x^2} \frac{\partial z}{\partial u}
 \end{aligned}$$

and $t = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{1}{x} \frac{\partial z}{\partial u} + 2y \frac{\partial z}{\partial v} \right)$

$$\begin{aligned}
&= \frac{1}{x} \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial u} \right) + 2 \frac{\partial z}{\partial v} + 2y \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial v} \right) \\
&= \frac{1}{x} \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} \right) \cdot \frac{\partial u}{\partial y} + \frac{\partial}{\partial v} \frac{\partial z}{\partial u} \frac{\partial v}{\partial y} \right] + 2 \frac{\partial z}{\partial v} + 2y \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial v} \right) \cdot \frac{\partial u}{\partial y} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial v} \right) \frac{\partial v}{\partial y} \right] \\
&= \frac{1}{x} \left[\frac{\partial^2 z}{\partial u^2} \cdot \frac{1}{x} + \frac{\partial^2 z}{\partial v \partial u} \cdot 2y \right] + 2 \frac{\partial z}{\partial v} + 2y \left[\frac{\partial^2 z}{\partial u \partial v} \cdot \frac{1}{x} + \frac{\partial^2 z}{\partial v^2} \cdot 2y \right] \\
&= \frac{1}{x^2} \frac{\partial^2 z}{\partial u^2} + \frac{4y}{x} \frac{\partial^2 z}{\partial u \partial v} + 4y^2 \frac{\partial^2 z}{\partial v^2} + 2 \frac{\partial z}{\partial v}
\end{aligned}$$

Substituting the values of p, q, r, s, t in the given equation, we have

$$\begin{aligned}
&x \cdot y \left[\frac{y^2}{x^4} \frac{\partial^2 z}{\partial u^2} - \frac{4y}{x} \frac{\partial^2 z}{\partial u \partial v} + 4x^2 \frac{\partial^2 z}{\partial v^2} + \frac{2y}{x^3} \frac{\partial z}{\partial u} + 2 \frac{\partial z}{\partial v} \right] \\
&\quad - (x^2 - y^2) \left[-\frac{y}{x^3} \frac{\partial^2 z}{\partial u^2} + 2 \left(1 - \frac{y^2}{x^2} \right) \frac{\partial^2 z}{\partial u \partial v} + 4x \cdot y \frac{\partial^2 z}{\partial v^2} - \frac{1}{x^2} \frac{\partial z}{\partial u} \right] \\
&\quad - xy \left[\frac{1}{x^2} \frac{\partial^2 z}{\partial u^2} + 4 \frac{y}{x} \frac{\partial^2 z}{\partial u \partial v} + 4y^2 \frac{\partial^2 z}{\partial v^2} + 2 \frac{\partial z}{\partial v} \right] + \left[-\frac{y}{x^2} \frac{\partial z}{\partial u} + 2x \frac{\partial z}{\partial v} \right] y \\
&\quad - \left[\frac{1}{x} \frac{\partial z}{\partial u} + 2y \frac{\partial z}{\partial v} \right] x = 2(x^2 - y^2)
\end{aligned}$$

or
$$\frac{-2(x^2 + y^2)^2}{x^2} \frac{\partial^2 z}{\partial u \partial v} = 2(x^2 - y^2)$$

or
$$\frac{\partial^2 z}{\partial u \partial v} = \frac{(y^2 - x^2)x^2}{(x^2 + y^2)^2} = \frac{(y/x)^2 - 1}{(1 + y^2/x^2)}$$

or
$$\frac{\partial^2 z}{\partial u \partial v} = \frac{u^2 - 1}{(1 + u^2)^2} \quad \dots(1)$$

which is the *canonical form* of the given equation.

To Find the Solution : Integrating (1) partially w.r.t. ' u ' (treating v constant), we get

$$\begin{aligned}
\frac{\partial z}{\partial v} &= \int \frac{u^2 - 1}{(u^2 + 1)^2} du + f(v) = \int \frac{(1 - 1/u^2)}{(u + 1/u)^2} du + f(v) \\
&= \int \frac{dt}{t^2} + f(v) = -\frac{1}{t} + f(v) = -\frac{u}{u^2 + 1} + f(v), \quad \text{Putting } u + \frac{1}{u} = t
\end{aligned}$$

Integrating again partially w.r.t. 'v' (treating u constant), we get

$$\begin{aligned} z &= \left(-\frac{u}{u^2 + 1} \right) v + \int f(v) dv + \psi(u) \\ &= -\frac{uv}{u^2 + 1} + \phi(v) + \psi(u) \quad \text{where } \phi(v) = \int f(v) dv \end{aligned}$$

or $z = -x y + \phi(x^2 + y^2) + \psi(y/x)$, where ϕ and ψ are arbitrary functions.

11.3 Working Method of Reducing a Parabolic Equation to Canonical Form

Let the second order partial differential equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad \dots(1)$$

be parabolic i.e., $S^2 - 4RT = 0$.

To reduce (1) to canonical form, proceed as follows.

Step 1 : Write λ quadratic equation $R\lambda^2 + S\lambda + T = 0$...(2)

whose roots will be real and equal as $S^2 - 4RT = 0$

Solve (2) and let λ_1, λ_1 be two equal roots.

Step 2 : Write the corresponding *characteristic equation*

i.e., $\frac{dy}{dx} + \lambda_1 = 0$. Solve this equation and let $f_1(x, y) = c_1$ be its solution i.e., the characteristic curve (or characteristics) of (1).

where c_1 is an arbitrary constant.

Step 3 : Let u and v be two arbitrary functions of x and y .

such that $u = f_1(x, y)$. Choose $v = f_2(x, y)$...(3)

such that, the chosen arbitrary function $f_2(x, y)$ is independent of $f_1(x, y)$.

For this verify that the Wronskian

$$W(u, v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \neq 0$$

Step 4 : Using the relations (3), find p, q, r, s, t (as needed for (1)) in terms of u and v as in § 11.1.

Step 5 : Substituting these values of p, q, r, s, t (obtained in step 4) in (1), simplify to get the *canonical form* of (1), which will be of the form $\frac{\partial^2 z}{\partial u \partial v} = \phi\left(u, v, z \frac{\partial u}{\partial u}, \frac{\partial v}{\partial v}\right)$.

Solved Examples

Example 1: Reduce the equation

$$\frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

to *canonical form* and hence solve it.

[Meerut 2006 (B.P.) 2010 (Sem. I), 13 (Sem. I);

Delhi Maths (Hons.) 2000, 04, 06, 08; Himachal 2001, 05; Rajasthan 2003; Jabalpur 2004]

Solution: The given equation can be written as

$$r + 2s + t = 0 \quad \dots(1)$$

Comparing this equation with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \text{ we have}$$

$$R = 1, S = 2 \text{ and } T = 1. \quad \therefore S^2 - 4RT = 0 \Rightarrow (1) \text{ is a parabolic equation}$$

$\therefore$ The quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$\lambda^2 + 2\lambda + 1 = 0 \quad \text{or} \quad (\lambda + 1)^2 = 0.$$

Giving $\lambda = -1$, i.e. $\lambda_1 = \lambda_2 = -1$. (Equal roots)

$\therefore$ The characterise equation $\frac{dy}{dx} + \lambda = 0$, becomes $\frac{dy}{dx} - 1 = 0$

Integrating, we have $x - y = c$ (constant), which is the characteristics of (1).

$\therefore$ To change the independent variables x, y to u, v we take $u = x - y$.

Also we have to take v as some function of x and y independent of u ,

$\therefore$ Let $v = x + y$.

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad \therefore \frac{\partial}{\partial x} \equiv \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \quad \therefore \quad \frac{\partial}{\partial y} \equiv -\frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$r = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) = \frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$s = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) \left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) = -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}$$

and
$$t = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) \left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) = \frac{\partial^2 z}{\partial u^2} - 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

Substituting in (1), we have
$$\frac{\partial^2 z}{\partial v^2} = 0 \quad \dots(2)$$

which is required *canonical form*.

To Find the Solution : Integrating (2) partially w.r.t. v , we have

$$\frac{\partial z}{\partial v} = \phi(u)$$

where $\phi(u)$ is some arbitrary function of u .

Again integrating partially w.r.t. v , we have

$$z = v \phi(u) + \psi(u)$$

where $\psi(u)$ is another arbitrary function of u .

Hence, the solution is

$$z = (x + y) \phi(x - y) + \psi(x - y).$$

Example 2: Reduce the equation

$$y^2 \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + x^2 \frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x} \frac{\partial z}{\partial x} + \frac{x^2}{y} \frac{\partial z}{\partial y}$$

to *canonical form*, and hence solve it. [Meerut 2006, 07, 09 (BP); Delhi Maths (Hons.) 2001, 05; GNDU Amritsar 2005; Nagpur 2005]

Solution: The given equation can be written as

$$y^2 r - 2xys + x^2 t - \frac{y^2}{x} p - \frac{x^2}{y} q = 0 \quad \dots(1)$$

Comparing with, $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have

$$R = y^2, S = -2xy, T = x^2 \quad \therefore S^2 - 4RT = 0 \Rightarrow (1) \text{ is a parabolic equation.}$$

The quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$y^2\lambda^2 - 2xy\lambda + x^2 = 0 \quad \text{or} \quad (y\lambda - x)^2 = 0$$

Giving $\lambda = \frac{x}{y}, \frac{x}{y} \quad \text{i.e., } \lambda_1 = \lambda_2 = \frac{x}{y} \text{ (equal roots)}$

$\therefore$ The characterise equation $\frac{dy}{dx} + \lambda = 0$, becomes $\frac{dy}{dx} + \frac{x}{y} = 0$

or $y dy + x dx = 0$. Integrating $x^2 + y^2 = c$ (constant), which is the characteristics of (1).

$\therefore$ To change the independent variables x, y to u, v in the given equation (1), we take

$$u = x^2 + y^2$$

also we have to take v as some function of x and y independent of u ,

$\therefore$ Let $v = x^2 - y^2$

$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} \cdot 2x + \frac{\partial z}{\partial v} \cdot 2x = 2x \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = 2y \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right)$$

$$r = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left\{ 2x \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) \right\}$$

$$= 2 \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) + 2x \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right)$$

$$= 2 \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) + 2x \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) \cdot \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) \cdot \frac{\partial v}{\partial x} \right]$$

$$= 2 \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) + 4x^2 \left[\frac{\partial^2 z}{\partial u^2} + 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right]$$

$$s = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left\{ 2y \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \right\} = 2y \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right)$$

$$= 2y \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \cdot \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \cdot \frac{\partial v}{\partial x} \right] = 4xy \left[\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2} \right]$$

$$\begin{aligned}
 \text{and} \quad t &= \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} \left[2y \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \right] \\
 &= 2 \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) + 2y \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \\
 &= 2 \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) + 2y \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \cdot \frac{\partial u}{\partial y} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \frac{\partial v}{\partial y} \right] \\
 &= 2 \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) + 4y^2 \left[\frac{\partial^2 z}{\partial u^2} - 2 \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right]
 \end{aligned}$$

Substituting in (1) and simplifying, we have

$$\frac{\partial^2 z}{\partial v^2} = 0 \quad \dots(2)$$

which is the required *canonical form*.

To Find the Solution : Integrating (2) partially w.r.t. v , $\frac{\partial z}{\partial v} = \phi(u)$

Again integrating w.r.t. v , we have

$$z = v \phi(u) + \psi(u)$$

where $\phi(u)$, $\psi(u)$ are arbitrary functions of u .

$\therefore$ The solution is

$$z = (x^2 - y^2) \phi(x^2 + y^2) + \psi(x^2 + y^2)$$

which is the required solution.

11.4 Working Method of Reducing Elliptic Equation to Canonical Form

Let the second order partial differential equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad \dots(1)$$

be elliptic i.e., $S^2 - 4RT < 0$.

Step 1 : Write quadratic equation $R\lambda^2 + S\lambda + T = 0 \quad \dots(2)$

Whose roots will be complex conjugates as $S^2 - 4RT < 0$. Solve (2) and let λ_1, λ_2 be complex conjugates.

Step 2 : Write the corresponding characteristic equations

$$\text{i.e., } \frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0.$$

Solving these equations we shall get the solutions of the form

$$f_1(x, y) + i f_2(x, y) = c_1 \quad \text{and} \quad f_1(x, y) - i f_2(x, y) = c_2$$

where c_1 and c_2 are arbitrary constants.

Step 3 : Let u and v be two arbitrary functions of x and y such that

$$u = f_1(x, y) + i f_2(x, y) \quad \text{and} \quad v = f_1(x, y) - i f_2(x, y) \quad \dots(3)$$

Take α and β be two new real independent variables such that $u = \alpha + i\beta$ and $v = \alpha - i\beta$

$$\Rightarrow \quad \alpha = f_1(x, y) \quad \text{and} \quad \beta = f_2(x, y). \quad \dots(4)$$

Step 4 : Using the relations (4), find p, q, r, s, t in terms of α and β (as needed for (1)) as in § 11.1.

Step 5 : Substituting these values of p, q, r, s, t (obtained in step 4) in (1), simplify to get the canonical form of (1), which will be of the form

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \phi \left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta} \right) \quad \text{where } u = \alpha + i\beta, v = \alpha - i\beta.$$

Solved Examples

Example 1: Reduce $x \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x^2$ ($x > 0$) to canonical form.

[Delhi Maths (Hons.) 2007]

Solution: Given differential equation can be written as

$$x r + t - x^2 = 0, \quad (x > 0) \quad \dots(1)$$

Comparing (1) with the equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \text{ we have}$$

$$R = x, S = 0, T = 1. \quad \therefore \quad S^2 - 4RT = -4x < 0 \quad \because \quad x > 0$$

which shows that (1) is an elliptic equation

λ -quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$x\lambda^2 + 1 = 0 \quad \text{or} \quad \lambda = i/\sqrt{x}, -i/\sqrt{x}.$$

$\therefore$ The corresponding characteristic equations

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0, \text{ becomes}$$

$$\frac{dy}{dx} + \frac{i}{\sqrt{x}} = 0 \quad \text{and} \quad \frac{dy}{dx} - \frac{i}{\sqrt{x}} = 0$$

Solving, we get $y + 2i\sqrt{x} = c_1$ and $y - 2i\sqrt{x} = c_2$

where c_1 and c_2 are arbitrary constants. Let u and v be two arbitrary functions of x and y such that $u = y + 2i\sqrt{x} = \alpha + i\beta$ and $v = y - 2i\sqrt{x} = \alpha - i\beta$

$$\Rightarrow \quad \alpha = y \quad \text{and} \quad \beta = 2\sqrt{x} \quad \dots(2)$$

which are new independent variables.

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta} \frac{\partial \beta}{\partial x} = x^{-1/2} \frac{\partial z}{\partial \beta} \quad \dots(3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta} \frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha} \quad \dots(4)$$

$$\begin{aligned} r = \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left(x^{-1/2} \frac{\partial z}{\partial \beta} \right) = -\frac{1}{2} x^{-3/2} \frac{\partial z}{\partial \beta} + x^{-1/2} \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial \beta} \right) \\ &= -\frac{1}{2} x^{-3/2} \frac{\partial z}{\partial \beta} + x^{-1/2} \left[\frac{\partial}{\partial \alpha} \left(\frac{\partial z}{\partial \beta} \right) \cdot \frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta} \left(\frac{\partial z}{\partial \beta} \right) \frac{\partial \beta}{\partial x} \right] \\ &= -\frac{1}{2} x^{-3/2} \frac{\partial z}{\partial \beta} + x^{-1/2} \left[\frac{\partial^2 z}{\partial \alpha \partial \beta} \cdot 0 + \frac{\partial^2 z}{\partial \beta^2} \cdot x^{-1/2} \right] = -\frac{1}{2} x^{-3/2} \frac{\partial z}{\partial \beta} + x^{-1} \frac{\partial^2 z}{\partial \beta^2} \end{aligned}$$

$$\text{and} \quad t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial \alpha} \right) = \frac{\partial}{\partial \alpha} \left(\frac{\partial z}{\partial \alpha} \right) = \frac{\partial^2 z}{\partial \alpha^2} \quad \because \quad \frac{\partial}{\partial y} \equiv \frac{\partial}{\partial \alpha} \text{ from (4)}$$

Substituting the values of r and t in (1), we get

$$x \cdot \left[-\frac{1}{2} x^{-3/2} \frac{\partial z}{\partial \beta} + x^{-1} \frac{\partial^2 z}{\partial \beta^2} \right] + \frac{\partial^2 z}{\partial \alpha^2} - x^2 = 0$$

$$\text{or} \quad \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} - \frac{1}{2} x^{-1/2} \frac{\partial z}{\partial \beta} - x^2 = 0$$

$$\text{or } \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \left(\frac{1}{\beta}\right) \frac{\partial z}{\partial \beta} + \frac{\beta^4}{16}$$

which is the required canonical form.

Example 2: Reduce the following equation to canonical form

$$\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0. \quad [\text{Delhi Maths (Hons.) 2006, 08}]$$

Solution: The given equation can be written as

$$r + x^2 t = 0 \quad \dots(1)$$

Comparing it with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \text{ we have}$$

$$R = 1, S = 0, T = x^2 \quad \therefore S^2 - 4RT = -4x^2 < 0 \Rightarrow (1) \text{ is an elliptic equation}$$

The quadratic equation, $R\lambda^2 + S\lambda + T = 0$, becomes

$$\lambda^2 + x^2 = 0, \quad \text{giving } \lambda = \pm ix$$

$$\text{i.e.,} \quad \lambda_1 = ix, \quad \lambda_2 = -ix \quad (\text{Complex roots})$$

$$\therefore \text{ The characteristic equations, } \frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0 \text{ becomes}$$

$$\frac{dy}{dx} + ix = 0 \quad \text{and} \quad \frac{dy}{dx} - ix = 0$$

Integrating them, we have

$$y + \frac{i}{2}x^2 = \text{constant and } y - \frac{i}{2}x^2 = \text{constant}$$

$\therefore$ To change the independent variables x, y to u, v we take

$$u = y + \frac{i}{2}x^2 = \alpha + i\beta \text{ (say)}$$

$$\text{and} \quad v = y - \frac{i}{2}x^2 = \alpha - i\beta$$

$$\text{so that} \quad \alpha = y \quad \text{and} \quad \beta = \frac{1}{2}x^2.$$

Now we transform the independent variables x and y to α and β . With the help of these relations

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta} \cdot \frac{\partial \beta}{\partial x} = x \frac{\partial z}{\partial \beta}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta} \cdot \frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha} \Rightarrow \frac{\partial}{\partial y} \equiv \frac{\partial}{\partial \alpha}$$

$$r = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left(x \cdot \frac{\partial z}{\partial \beta} \right) = 1 \cdot \frac{\partial z}{\partial \beta} + x \cdot \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial \beta} \right)$$

$$= \frac{\partial z}{\partial \beta} + x \left[\frac{\partial}{\partial \alpha} \left(\frac{\partial z}{\partial \beta} \right) \frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta} \left(\frac{\partial z}{\partial \beta} \right) \cdot \frac{\partial \beta}{\partial x} \right] = \frac{\partial z}{\partial \beta} + x^2 \frac{\partial^2 z}{\partial \beta^2}$$

$$t = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial \alpha} \left(\frac{\partial z}{\partial \alpha} \right) = \frac{\partial^2 z}{\partial \alpha^2}$$

Substituting in (1), we have

$$\left(\frac{\partial z}{\partial \beta} + x^2 \frac{\partial^2 z}{\partial \beta^2} \right) + x^2 \frac{\partial^2 z}{\partial \alpha^2} = 0$$

or
$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{x^2} \frac{\partial z}{\partial \beta}$$

or
$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{2\beta} \frac{\partial z}{\partial \alpha}$$

which is the required *canonical form*.

Example 3: Reduce the equation

$$\frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + 5 \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} - 2 \frac{\partial z}{\partial y} - 3z = 0 \quad \text{to canonical form.}$$

Solution: The given equation can be written as

$$r + 2s + 5t + p - 2q - 3z = 0 \quad \dots(1)$$

Comparing the given equation (1) with the equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \text{ we have}$$

$$R = 1, S = 2, T = 5 \quad \therefore \quad S^2 - 4RT = -16 < 0 \Rightarrow (1) \text{ is an elliptic equation.}$$

Here the quadratic equation $R\lambda^2 + S\lambda + T = 0$, becomes

$$\lambda^2 + 2\lambda + 5 = 0 \quad \therefore \quad \lambda = -1 \pm 2i$$

∴ Let $\lambda_1 = -1 + 2i$, $\lambda_2 = -1 - 2i$ (Complex roots)

∴ The characterise equations $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$, becomes

$$\frac{dy}{dx} - 1 + 2i = 0 \quad \text{and} \quad \frac{dy}{dx} - 1 - 2i = 0$$

Integrating $y - x + 2xi = c_1$ and $y - x - 2xi = c_2$

∴ To change the independent variables x and y in the given equation, we take

$$u = y - x + 2xi = \alpha + i\beta \quad \text{and} \quad v = y - x - 2xi = \alpha - i\beta$$

so that $\alpha = y - x$ and $\beta = 2x$.

Now we transform the independent variables x and y to α and β such that

$$\alpha = y - x \quad \text{and} \quad \beta = 2x.$$

$$\begin{aligned} \therefore p &= \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha} \frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta} \frac{\partial \beta}{\partial x} = -\frac{\partial z}{\partial \alpha} + 2 \frac{\partial z}{\partial \beta} \\ q &= \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta} \cdot \frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha} \Rightarrow \frac{\partial}{\partial y} \equiv \frac{\partial}{\partial \alpha} \\ r &= \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left(-\frac{\partial z}{\partial \alpha} + 2 \frac{\partial z}{\partial \beta} \right) \\ &= \frac{\partial}{\partial \alpha} \left(-\frac{\partial z}{\partial \alpha} + 2 \frac{\partial z}{\partial \beta} \right) \cdot \frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta} \left(-\frac{\partial z}{\partial \alpha} + 2 \frac{\partial z}{\partial \beta} \right) \frac{\partial \beta}{\partial x} \\ &= \left(-\frac{\partial^2 z}{\partial \alpha^2} + 2 \frac{\partial^2 z}{\partial \alpha \partial \beta} \right) (-1) + \left(-\frac{\partial^2 z}{\partial \beta \partial \alpha} + 2 \frac{\partial^2 z}{\partial \beta^2} \right) \cdot 2 \\ &= \frac{\partial^2 z}{\partial \alpha^2} - 4 \frac{\partial^2 z}{\partial \alpha \partial \beta} + 4 \frac{\partial^2 z}{\partial \beta^2} \\ s &= \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial \alpha} \right) \\ &= \frac{\partial}{\partial \alpha} \left(\frac{\partial z}{\partial \alpha} \right) \cdot \frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta} \left(\frac{\partial z}{\partial \alpha} \right) \cdot \frac{\partial \beta}{\partial x} = -\frac{\partial^2 z}{\partial \alpha^2} + 2 \frac{\partial^2 z}{\partial \alpha \partial \beta} \\ t &= \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial \alpha} \left(\frac{\partial z}{\partial \alpha} \right) = \frac{\partial^2 z}{\partial \alpha^2} \end{aligned}$$

Substituting the values of p, q, r, s, t in the given equation, we have

$$\left(\frac{\partial^2 z}{\partial \alpha^2} - 4 \frac{\partial^2 z}{\partial \alpha \partial \beta} + 4 \frac{\partial^2 z}{\partial \beta^2}\right) + 2 \left(-\frac{\partial^2 z}{\partial \alpha^2} + 2 \frac{\partial^2 z}{\partial \alpha \partial \beta}\right) + 5 \frac{\partial^2 z}{\partial \alpha^2} + \left(-\frac{\partial z}{\partial \alpha} + 2 \frac{\partial z}{\partial \beta}\right) - 2 \frac{\partial z}{\partial \alpha} - 3z = 0$$

or $4 \frac{\partial^2 z}{\partial \alpha^2} + 4 \frac{\partial^2 z}{\partial \beta^2} - 3 \frac{\partial z}{\partial \alpha} + 2 \frac{\partial z}{\partial \beta} - 3z = 0$

or $\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{3}{4} \frac{\partial z}{\partial \alpha} - \frac{1}{2} \frac{\partial z}{\partial \beta} + \frac{3}{4} z$

which is the required *canonical form*.

Exercise

Numerical Questions

Reduce the following equations to canonical form

1. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0.$ [Delhi Maths (Hons.) 2002; Sagar 2004]

2. $\frac{\partial^2 z}{\partial x^2} + 4 \frac{\partial^2 z}{\partial x \partial y} - 5 \frac{\partial^2 z}{\partial y^2} + 6 \frac{\partial z}{\partial x} + 3 \frac{\partial z}{\partial y} - 9z = 0.$

3. $6 \frac{\partial^2 z}{\partial x \partial y} + 3 \frac{\partial^2 z}{\partial y^2} + 2 \frac{\partial z}{\partial x} = 0.$

4. $\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} - 8 \frac{\partial^2 z}{\partial y^2} + 9 \frac{\partial z}{\partial x} = 0.$

5. $y^2 \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x^2} = 0.$ [Delhi Maths (Hons.) 2005]

6. $\frac{\partial^2 z}{\partial x^2} - 5 \frac{\partial^2 z}{\partial x \partial y} + 6 \frac{\partial^2 z}{\partial y^2} = 0.$

7. $\frac{\partial^2 z}{\partial x^2} - 6 \frac{\partial^2 z}{\partial x \partial y} + 9 \frac{\partial^2 z}{\partial y^2} + 2 \frac{\partial z}{\partial x} + 3 \frac{\partial z}{\partial y} - z = 0.$

8. $\frac{\partial^2 z}{\partial x^2} - 4 \frac{\partial^2 z}{\partial x \partial y} + 13 \frac{\partial^2 z}{\partial y^2} - 9 \frac{\partial z}{\partial y} = 0.$

9. $\frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + 5 \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} - 2 \frac{\partial z}{\partial y} - 3z = 0.$

10. $t - s + p - q(1 + 1/x) + z/x = 0.$ [Delhi Maths (Hons.) 2004]

Reduce the following equations to canonical form and hence solve.

11. $x^2r - y^2t + px - qy = x^2$.

[Kurukshetra 2003]

12. $x^2(y-1)r - x(y^2-1)s + y(y-1)t + xyp - q = 0$.

13. $x^2r - xys + px + qy - z = 2x^2y$.

14. $x^2r - 2xys + y^2t - xp + 3yq = 8y/x$.

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter (a), (b), (c) or (d).

1. The equation $\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} - 8 \frac{\partial^2 z}{\partial y^2} + 9 \frac{\partial z}{\partial x} = 0$ is

- (a) Elliptic (b) Hyperbolic
(c) Parabolic (d) None of these

2. The characteristic equations of $\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} - 8 \frac{\partial^2 z}{\partial y^2} + 9 \frac{\partial z}{\partial x} = 0$ are :

- (a) $\frac{dy}{dx} - 2 = 0, \frac{dy}{dx} + 4 = 0$ (b) $\frac{dy}{dx} + 2 = 0, \frac{dy}{dx} - 4 = 0$
(c) $\frac{dy}{dx} - 2x = 0, \frac{dy}{dx} + 4x = 0$ (d) None of these

3. The characteristic equations of $y^2r - x^2t = 0$ are :

- (a) $\frac{dy}{dx} + \frac{x}{y} = 0, \frac{dy}{dx} - \frac{x}{y} = 0$ (b) $\frac{dy}{dx} + \frac{y}{x} = 0, \frac{dy}{dx} - \frac{y}{x} = 0$
(c) $\frac{dy}{dx} + y = 0, \frac{dy}{dx} + x = 0$ (d) None of these

4. The characteristics of $y^2r - x^2t = 0$ are :

- (a) $x^2 + y^2 = c_1, x^2 + 2y^2 = c_2$ (b) $x^2 + y^2 = c_1, x^2 - y^2 = c_2$
(c) $2x^2 + y^2 = c_1, x^2 + y^2 = c_2$ (d) None of these

5. The characteristic equations of $4r + 5s + t + 2p + q - 3 = 0$ are :

- (a) $\frac{dy}{dx} - 1 = 0, \frac{dy}{dx} + \frac{1}{4} = 0$ (b) $\frac{dy}{dx} + 1 = 0, \frac{dy}{dx} + \frac{1}{4} = 0$
(c) $\frac{dy}{dx} - 1 = 0, \frac{dy}{dx} - \frac{1}{4} = 0$ (d) None of these

6. The characteristics of equations of $4r + 5s + t + p + q - 2 = 0$ are :

- (a) $y - x = c_1, 4y - x = c_2$ (b) $y + x = c_1, 4y - x = c_2$
 (c) $y + x = c_1, 4y + x = c_2$ (d) None of these

7. The number of characterise equations of

$$\frac{\partial^2 z}{\partial x^2} - 6 \frac{\partial^2 z}{\partial x \partial y} + 9 \frac{\partial^2 z}{\partial y^2} + \frac{2 \partial z}{\partial x} + 3 \frac{\partial z}{\partial y} - z = 0 \text{ is :}$$

- (a) 3 (b) 2
 (c) 1 (d) 0

8. The characteristic of $\frac{\partial^2 z}{\partial x^2} = (1 + y)^2 \frac{\partial^2 z}{\partial y^2}$ are :

- (a) $\log(1 + y) + x = c_1; \log(x + y) - x = c_2$
 (b) $\log(1 - y) + x = c_1, \log(1 + y) - x = c_2$
 (c) $\log(1 - y) - x = c_1, \log(1 + y) + y = c_2$
 (d) None of these

9. If $y - \cos x + x = c_1$ is one of the characteristics of the equation $r - (2 \sin x)s - (\cos^2 x)t - (\cos x)q = 0$, then the other characteristics is

- (a) $y + \cos x + x = c_2$ (b) $y - \cos x - x = c_2$
 (c) $y - \cos x + x = c_2$ (d) None of these

10. The characteristic equations of equation $xy s - x^2 r - px - qy + z = -2xy^2$ are :

- (a) $\frac{dy}{dx} - \frac{y}{x} = 0, \frac{dy}{dx} + \frac{y}{x} = 0$ (b) $\frac{dy}{dx} + \frac{y}{x} = 0, \frac{dy}{dx} = 0$
 (c) $\frac{dy}{dx} - \frac{y}{x} = 0, \frac{dy}{dx} + x = 0$ (d) None of these

Fill in the Blank(s)

Fill in the blanks "....." so that the following statements are complete and correct.

1. The characteristic equations of $x^2 r - y^2 t = 0$ are $\frac{dy}{dx} + \dots\dots\dots = 0$ and

$$\frac{dy}{dx} + \dots\dots\dots = 0$$

2. The characteristics of $x^2 r - y^2 t = 0$ is/are and

3. The characteristics of

$$\frac{\partial^2 z}{\partial x^2} - 6 \frac{\partial^2 z}{\partial x \partial y} + 9 \frac{\partial^2 z}{\partial y^2} + 2 \frac{\partial z}{\partial x} + 3 \frac{\partial z}{\partial y} - 3 = 0 \text{ is/are } \dots\dots\dots$$

4. The characteristic equation of $x^2r - 2xys + y^2t - xp + 3yq = 8y/x$ is/are
5. The characteristic equation and characteristics of the equation $r + 2xs + x^2t = 0$ are and respectively.
6. To reduce the equation $r - 2s + t + p - q = e^x(2y - 3) - e^y$ to canonical form, choose $u = \dots\dots\dots$ and $v = \dots\dots\dots$.

Answers

Numerical Questions

1. $\frac{\partial^2 z}{\partial u \partial v} = 0$	2. $\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4} \frac{\partial z}{\partial u} - \frac{3}{4} \frac{\partial z}{\partial v} - \frac{1}{4} z$
3. $\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{6} \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right)$	4. $\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial z}{\partial u} - \frac{1}{2} \frac{\partial z}{\partial v}$
5. $\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{2\beta} \frac{\partial z}{\partial \beta}$	6. $\frac{\partial^2 z}{\partial u \partial v} = 0$
7. $\frac{\partial^2 z}{\partial v^2} = -\frac{\partial z}{\partial u} - \frac{1}{3} \frac{\partial z}{\partial v} + \frac{1}{9} u$	8. $\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{\partial z}{\partial \alpha}$
9. $\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{3}{4} \frac{\partial z}{\partial \alpha} - \frac{1}{2} \frac{\partial z}{\partial \beta} + \frac{3}{4} z$	10. $\frac{\partial^2 z}{\partial u \partial v} - \frac{\partial z}{\partial v} + \frac{1}{v} \frac{\partial z}{\partial u} - \frac{z}{u} = 0$
11. $\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4}, z = (1/4)x^2 + \phi(xy) + \psi(x/y)$	12. $\frac{\partial^2 z}{\partial u \partial v} = 0, z = \phi(xy) + \phi(xe^y)$
13. $\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{v} \frac{\partial z}{\partial u} + \frac{1}{u} \frac{\partial z}{\partial v} - \frac{2u}{v^2} - \frac{z}{uv}, z = x^2y + xy\phi(y) + y\psi(xy)$	
14. $\frac{\partial^2 z}{\partial v^2} = \frac{2}{v} - \frac{2}{v} \frac{\partial z}{\partial v}, z = \frac{y}{x} + x^2\phi(xy) + \psi(xy)$	

Objective Type Questions

Multiple Choice Questions

1.	(b)	2.	(a)	3.	(a)	4.	(b)
5.	(a)	6.	(a)	7.	(c)	8.	(a)
9.	(b)	10.	(b)				

Fill in the Blank(s)

1.	$y/x, -y/x$	2.	$xy = c_1, y/x = c_2$
3.	$y + 3x = c$	4.	$xdy + ydx = 0$
5.	$\frac{dy}{dx} - x = 0, y - x^2/2 = c_1$	6.	$x + y, y$

○○○

Wave Equations (By Method of Separation of Variables)

12.1 Introduction

In this chapter we shall discuss Wave equation, its derivation and solution in different coordinate systems as it is of great importance in science and engineering.

12.2 Wave Equation

12.2.1 One Dimensional Wave Equation

[Meerut 2002]

Let us consider small transverse vibrations of an elastic string, which is stretched to length l and then fixed at the end points. Suppose that the string is distorted and then at a certain instant, say $t = 0$ it is released and allowed to vibrate. Here we have to find its deflection $u(x, t)$ at any point x and at any time $t > 0$.

For this let us make the following assumptions :

1. The string is homogeneous *i.e.* the mass of the string per unit length is constant. The string is perfectly elastic and does not offer any resistance to bending.
 2. The tension caused by stretching the string before fixing it at the end points is so large that the action of the gravitational force on the string can be neglected.
 3. The motion of the string is a small transverse vibration in a vertical plane *i.e.*, each particle of the string moves strictly vertically, and the deflection and the slope $\frac{\partial u}{\partial x}$ at any point of the string are small in absolute value.
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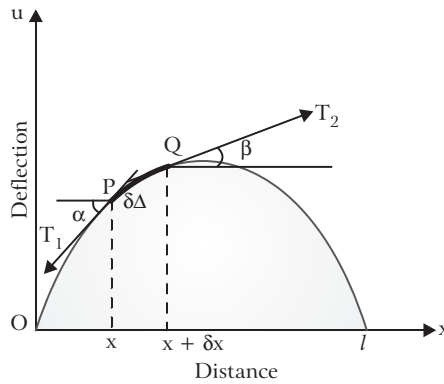


Fig. 12.1

Consider the forces acting on a small portion PQ of the string. Since the string does not offer resistance to bending (according to our assumption), the tensions T_1 and T_2 at P and Q respectively are tangential to the curve of the string. Since there is no motion in horizontal direction.

$$\therefore T_1 \cos \alpha = T_2 \cos \beta = T \quad (\text{constant}). \quad \dots(1)$$

If δs is the length of PQ then its mass is $\rho \delta s$, where ρ is the mass per unit length of the string.

Resultant vertical force acting on PQ is $T_2 \sin \beta - T_1 \sin \alpha$ (negative sign appears because the vertical component of T_1 at P is directed downwards).

By Newton's second law of motion, $P = mf$, we have

$$T_2 \sin \beta - T_1 \sin \alpha = (\rho \delta s) \frac{\partial^2 u}{\partial t^2}$$

(since $\frac{\partial^2 u}{\partial t^2}$ is the acceleration of PQ , vertically upwards)

$$\text{or} \quad \frac{T_2 \sin \beta}{T_2 \cos \beta} - \frac{T_1 \sin \alpha}{T_1 \cos \alpha} = \frac{\rho \delta s}{T} \frac{\partial^2 u}{\partial t^2} \quad [\text{by using (1)}]$$

$$\text{or} \quad \tan \beta - \tan \alpha = \frac{\rho \delta s}{T} \frac{\partial^2 u}{\partial t^2}$$

[since the slope of the curve of the string is small, we may replace δs by δx]

$$\text{or} \quad \left(\frac{\partial u}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial u}{\partial x} \right)_x = \frac{\rho \delta x}{T} \frac{\partial^2 u}{\partial t^2}$$

[since $\tan \alpha$ and $\tan \beta$ are the slopes of curve of the string at x and $x + \delta x$]

$$\text{or} \quad \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\text{or} \quad \frac{u_x(x+\delta x, t) - u_x(x, t)}{\delta x} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

Now as $\delta x \rightarrow 0$, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 u}{\partial t^2}$$

$$\text{which may be written as} \quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad \text{where } c^2 = \frac{T}{\rho}$$

This is known as **one dimensional wave equation**.

- Note : 1.** Above partial derivatives are used because u depends on ' x ' and also on ' t '.
2. For the physical constant T/ρ the notation c^2 has been chosen to indicate that this constant is positive.

12.2.2 Two Dimensional Wave Equation

Let us consider the motion of a stretched membrane. The membrane is stretched and then fixed along its entire boundary in the xy -plane.

Here we have to find the deflection $u(x, y, t)$ of the membrane at any point (x, y) and at any time $t > 0$.

For which we make the following assumptions :

1. The membrane, is homogeneous, *i.e.*, the mass of the membrane, per unit area is constant. The membrane is perfectly flexible and is so thin that it does not offer any resistance to bending.
2. The tension T per unit length caused by stretching the membrane is the same at all points and in all directions and does not change during motion.
3. The deflection $u(x, y, t)$ of the membrane during the motion is small compared with the size of the membrane, and all angles of inclination are small.

Consider the forces acting on a small portion of the membrane as shown in the Fig. 12.2.

Since the deflection of the membrane and the angles of inclination are small, the sides of the portion are approximately equal to δx and δy . If T is the tension (force) per unit length then the force acting on the edges of the portion are $T \delta x$ and $T \delta y$ approximately. These forces are tangential to the membrane (since according to our assumption the membrane is perfectly flexible).

The horizontal components of the forces at one pair of opposite edges are

$$T \delta y \cos \alpha \quad \text{and} \quad T \delta y \cos \beta$$

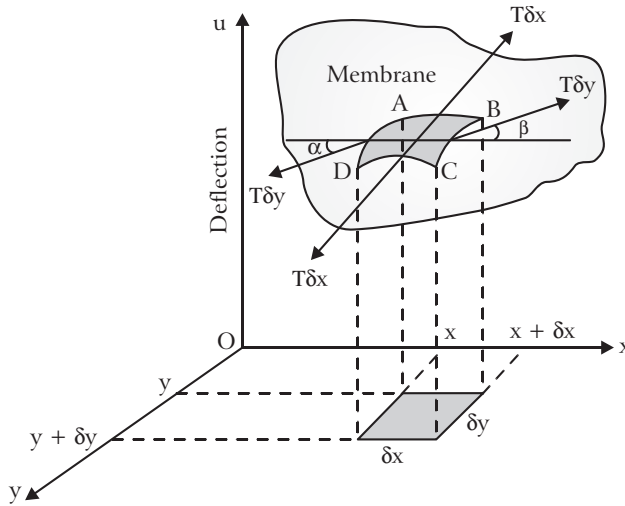


Fig. 12.2

which are approximately equal since α and β are small.

i.e., $\cos \alpha$ and $\cos \beta$ are close to 1.

Hence the horizontal components of the forces at opposite edges are approximately equal. Therefore, the motion of the particles of the membrane in horizontal direction will be negligibly small. Therefore we may assume that each particle of the membrane moves vertically.

Now the resultant of the vertical components of the forces along the edges, parallel to the yu -plane are

$$T \delta y \sin \beta - T \delta y \sin \alpha$$

(minus sign with the second component appears because it is directed downwards)

Since α and β are small, therefore this resultant is equivalent to

$$T \delta y (\tan \beta - \tan \alpha) = T \delta y [u_x(x + \delta x, y_1) - u_x(x, y_2)]$$

where the subscripts x denote partial derivatives and y_1 and y_2 are values between y and $y + \delta y$.

Similarly, the resultant of the vertical components of the forces acting on the other two edges of the portion is

$$T \delta x [u_y(x_1, y + \delta y) - u_y(x_2, y)]$$

where the subscripts denote partial derivatives and x_1 and x_2 are values between x and $x + \delta x$

Hence the resultant of the vertical components of the forces acting on the portion under consideration is

$$T \delta y [u_x(x + \delta x, y_1) - u_x(x, y_2)] + T \delta x [u_y(x_1, y + \delta y) - u_y(x_2, y)]$$

which by Newton's second law of motion is equal to

$$\rho \delta x \delta y \cdot \frac{\partial^2 u}{\partial t^2}$$

where ρ is the mass per unit area of the membrane and $\delta x \delta y$ the area of the portion under consideration and $\frac{\partial^2 u}{\partial t^2}$ the acceleration. Therefore,

$$\rho \delta x \delta y \frac{\partial^2 u}{\partial t^2} = T \delta y [u_x(x + \delta x, y_1) - u_x(x, y_2)] + T \delta x [u_y(x_1, y + \delta y) - u_y(x_2, y)]$$

$$\text{or} \quad \frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \left[\frac{u_x(x + \delta x, y_1) - u_x(x, y_2)}{\delta x} + \frac{u_y(x_1, y + \delta y) - u_y(x_2, y)}{\delta y} \right]$$

Now as $\delta x \rightarrow 0$ and $\delta y \rightarrow 0$

$$\text{We have} \quad \frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\text{or} \quad \frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \dots(1)$$

$$\text{where} \quad c^2 = \frac{T}{\rho}$$

This equation is called **two-dimensional wave equation**.

This equation (1), may also be written as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

where ∇^2 is the Laplacian operator for two-dimension.

12.3 Solution of One Dimensional Wave Equation by Using the Method of Separation of Variables

[Rajasthan 2001; U.P. (Tech) Lucknow 2003; Meerut 2006, 06 (B.P.), 08, 11 (Sem. I); Indore 2006]

One dimensional wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots(1)$$

To find the solution of (1) by using the method of separation of variables, let

$$u(x, t) = F(x) \cdot T(t) \quad \text{or} \quad u = FT \quad \dots(2)$$

Where F is a function of x alone and T is function of t alone.

Since F and T are functions of a single variable only.

$$\begin{aligned} \therefore \quad \frac{\partial u}{\partial t} &= F \frac{dT}{dt}, & \frac{\partial u}{\partial x} &= T \frac{dF}{dx} \\ \frac{\partial^2 u}{\partial t^2} &= F \frac{d^2 T}{dt^2} & \text{and} & \quad \frac{\partial^2 u}{\partial x^2} = T \frac{d^2 F}{dx^2} \end{aligned}$$

Substituting these values in the equation (1), we have

$$T \frac{d^2 T}{dt^2} = c^2 T \frac{d^2 F}{dx^2}$$

$$\text{or} \quad \frac{1}{F} \frac{d^2 F}{dx^2} = \frac{1}{c^2 T} \frac{d^2 T}{dt^2} \quad (\text{By separating the variables})$$

Here the left hand side is a function of the independent variable x alone while the right hand side is function of the independent variable t alone. The two sides cannot be equal to each other, unless each reduce to a constant value.

$$\therefore \quad \frac{1}{F} \frac{d^2 F}{dx^2} = \frac{1}{c^2 T} \frac{d^2 T}{dt^2} = \lambda \quad (\text{say})$$

where λ is a constant.

Thus, we get the following pair of ordinary differential equations

$$\frac{d^2 F}{dx^2} - \lambda F = 0 \quad \text{and} \quad \frac{d^2 T}{dt^2} - \lambda c^2 T = 0$$

Now we have the following cases :

Case I : If $\lambda = 0$, then the two equations reduces to $\frac{d^2 F}{dx^2} = 0$ and $\frac{d^2 T}{dt^2} = 0$,

whose solutions are

$$F = A_1 x + A_2 \quad \text{and} \quad T = A_3 t + A_4$$

$$\therefore \quad u = FT = (A_1 x + A_2)(A_3 t + A_4)$$

Case II : If $\lambda = k^2 > 0$, then the two equations reduces to

$\frac{d^2 F}{dx^2} - k^2 F = 0$ and $\frac{d^2 T}{dt^2} - k^2 c^2 T = 0$, whose solutions are

$$F = B_1 e^{kx} + B_2 e^{-kx} \quad \text{and} \quad T = B_3 e^{ckt} + B_4 e^{-ckt}$$

$$\therefore u = FT = (B_1 e^{kx} + B_2 e^{-kx})(B_3 e^{ckt} + B_4 e^{-ckt}).$$

Case III : If $\lambda = -k^2 < 0$, then the two equations reduces to

$\frac{d^2 F}{dx^2} + k^2 F = 0$ and $\frac{d^2 T}{dt^2} + k^2 c^2 T = 0$, whose solutions are

$$F = C_1 \cos kx + C_2 \sin kx \quad \text{or} \quad F = C e^{\pm ikx}$$

$$\text{and} \quad T = C_3 \cos ckt + C_4 \sin ckt \quad \text{or} \quad T = D e^{\pm ickt}$$

$$\therefore u = FT = (C_1 \cos kx + C_2 \sin kx)(C_3 \cos ckt + C_4 \sin ckt)$$

$$\text{or} \quad u = FT = CD e^{\pm ikx \pm ickt} = A e^{\pm ikx \pm ickt}$$

where $CD = A$ is an arbitrary constant.

Here we are dealing with the wave equation whose solution will be periodic (in time). Since only the solution given in case III *i.e.*, when $\lambda = -k^2 < 0$, is periodic (in time), so a solution of the wave equation (one dimensional) can be taken as

$$u(x, t) = (C_1 \cos kx + C_2 \sin kx)(C_3 \cos ckt + C_4 \sin ckt)$$

The solution can also be taken as

$$u(x, t) = A e^{\pm ikx \pm ickt}.$$

12.4 Solution of One Dimensional Wave Equation Under the Given Conditions

[Meerut 2011 (B.P.)]

Consider the vibration of an elastic string stretched between the fixed points $(0, 0)$ and $(l, 0)$. If $f(x)$ is the initial deflection and $g(x)$ the initial velocity of the string then to find the deflection of the string at any time t .

The vibrations of an elastic are governed by one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots(A)$$

where $u(x, t)$ is the deflection of the string at time $t \geq 0$.

Here we require to find the solution of one dimensional wave equation (A), satisfying the following initial and boundary conditions.

Boundary conditions

$$\text{and } \left. \begin{array}{l} u(0, t) = 0 \\ u(l, t) = 0 \end{array} \right\} \quad \dots(B)$$

and **Initial conditions :**

$$u(x, 0) = f(x) \quad \dots(C)$$

$$\text{and } \left(\frac{\partial u}{\partial t} \right)_{t=0} = g(x) \quad \dots(D)$$

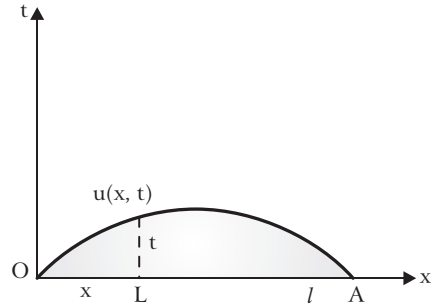


Fig. 12.3

12.4.1 Solution by Using the Method of Separation of Variables

It is obvious from (A) that $u(x, t)$, the solution of (A) is a function of x and t .

$$\therefore \text{ Let } u(x, t) = F(x) \cdot T(t) = F \cdot T \text{ (say)} \quad \dots(1)$$

where F is a function of x only and T a function of t only. Differentiating (1) and substituting in (A), we get

$$F \frac{d^2 T}{dt^2} = c^2 T \frac{d^2 F}{dx^2}$$

$$\text{or } \frac{1}{F} \frac{d^2 F}{dx^2} = \frac{1}{c^2 T} \frac{d^2 T}{dt^2}$$

Here the left hand side is a function of independent variable x alone while the right hand side is a function of the independent variable t alone. The two sides cannot be equal to each other, unless each reduce to a constant value.

$$\therefore \frac{1}{F} \frac{d^2 F}{dx^2} = \frac{1}{c^2 T} \frac{d^2 T}{dt^2} = \lambda \text{ (say)} \quad \text{where } \lambda \text{ is a constant.}$$

Thus, we get the following pair of ordinary differential equations

$$\frac{d^2 F}{dx^2} - \lambda F = 0 \quad \text{and} \quad \frac{d^2 T}{dt^2} - \lambda c^2 T = 0$$

Now we have the following cases :

Case I : If $\lambda = 0$, then the two equations reduce to $\frac{d^2 F}{dx^2} = 0$ and $\frac{d^2 T}{dt^2} = 0$,

whose solutions are

$$\therefore F = A_1 x + A_2 \quad \text{and} \quad T = A_3 t + A_4$$

$$\therefore u(x, t) = FT = (A_1 x + A_2)(A_3 t + A_4)$$

Case II : If $\lambda = k^2 > 0$, then the two equations reduces to

$$\frac{d^2 F}{dx^2} - k^2 F = 0 \quad \text{and} \quad \frac{d^2 T}{dt^2} - k^2 c^2 T = 0, \text{ whose solutions are}$$

$$F = B_1 e^{kx} + B_2 e^{-kx} \quad \text{and} \quad T = B_3 e^{ckt} + B_4 e^{-ckt}$$

$$\therefore u(x, t) = FT = (B_1 e^{kx} + B_2 e^{-kx})(B_3 e^{ckt} + B_4 e^{-ckt})$$

Case III : If $\lambda = -k^2 < 0$, then the two equations reduces to

$$\frac{d^2 F}{dx^2} + k^2 F = 0 \quad \text{and} \quad \frac{d^2 T}{dt^2} + k^2 c^2 T = 0, \text{ whose solutions are}$$

$$F = C_1 \cos kx + C_2 \sin kx \quad \text{and} \quad T = C_3 \cos ckt + C_4 \sin ckt$$

$$\therefore u(x, t) = FT = (C_1 \cos kx + C_2 \sin kx)(C_3 \cos ckt + C_4 \sin ckt)$$

Now from the boundary conditions (B) and (1), we have

$$u(0, t) = F(0)T(t) = 0$$

$$\text{and} \quad u(l, t) = F(l)T(t) = 0$$

which give either $T(t) = 0$, or $F(0) = 0$ and $F(l) = 0$

But $T(t) = 0$ leads to $u(x, t) = 0$, which is not admissible solution.

$$\therefore F(0) = 0 \quad \text{and} \quad F(l) = 0$$

In case I, $F(x) = A_1 x + A_2$

$$\therefore F(0) = 0 \quad \text{and} \quad F(l) = 0 \Rightarrow A_2 = 0 \quad \text{and} \quad A_1 l + A_2 = 0 \Rightarrow A_1 = 0 = A_2$$

$\Rightarrow F(x) = 0 \Rightarrow u(x, t) = FT = 0$, which is not admissible so we reject the solution in case I.

In case II, $F(x) = B_1 e^{kx} + B_2 e^{-kx}$

$$\therefore F(0) = 0 \quad \text{and} \quad F(l) = 0 \Rightarrow B_1 + B_2 = 0 \quad \text{and} \quad B_1 e^{kl} + B_2 e^{-kl} = 0$$

$$\Rightarrow B_1 = 0 = B_2$$

$\Rightarrow F(x) = 0 \Rightarrow u(x, t) = FT = 0$, which is not admissible. So we reject the solution in case II also.

Thus, the solutions given in case I and II do not constitute the required solution of the wave equation (A).

∴ We take the solution in case III, which is periodic (in time)

$$\text{i.e.,} \quad u(x, t) = (C_1 \cos kx + C_2 \sin kx)(C_3 \cos ckt + C_4 \sin ckt)$$

$$\therefore \quad u(0, t) = 0 \Rightarrow C_1(C_3 \cos ckt + C_4 \sin ckt) = 0 \Rightarrow C_1 = 0$$

$$\text{and} \quad u(l, t) = 0 \Rightarrow (0 + C_2 \sin kl)(C_3 \cos ckt + C_4 \sin ckt) = 0$$

$$\Rightarrow \quad \sin kl = 0 \quad \because \quad C_2 = 0 \text{ or } C_3 \cos ckt + C_4 \sin ckt = 0 \text{ will lead to } u(x, t) = 0$$

$$\Rightarrow \quad kl = n\pi \text{ i.e., } k = n\pi/l, \text{ where } n \in I.$$

Hence, the solutions of equation (A) satisfying boundary conditions (B) are given by

$$u(x, y) = u_n(x, t) = \left(C_n \cos \frac{n\pi ct}{l} + D_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \quad \dots(2)$$

Here $C_n = C_2 C_3$ and $D_n = C_2 C_4$ are the new arbitrary constants which depend on n .

Now from (2) and initial conditions (C) and (D), we have

$$u(x, 0) = C_n \sin \frac{n\pi x}{l} = f(x)$$

$$\text{and} \quad \left(\frac{\partial u}{\partial t} \right)_{t=0} = \frac{n\pi c}{l} D_n \sin \frac{n\pi x}{l} = g(x)$$

Evidently these will not be satisfied if we take a single term as our solution. Thus, we consider the more general solution

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t)$$

$$\text{i.e.,} \quad u(x, t) = \sum_{n=1}^{\infty} \left(C_n \cos \frac{n\pi ct}{l} + D_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l} \quad \dots(3)$$

which also satisfy the boundary conditions (B).

Now from (3) and initial conditions (C) and (D), we have

$$u(x, 0) = f(x) \Rightarrow \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{l} = f(x) \quad \dots(4)$$

$$\text{and} \quad \left(\frac{\partial u}{\partial t} \right)_{t=0} = g(x) \Rightarrow \sum_{n=1}^{\infty} \frac{n\pi c}{l} D_n \sin \frac{n\pi x}{l} = g(x) \quad \dots(5)$$

the left hand sides in (4) and (5) can be considered as the fourier sine series of the functions on their right hand sides

$$\therefore C_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx \quad \dots(6)$$

$$\text{and} \quad \frac{n\pi c}{l} D_n = \frac{2}{l} \int_0^l g(x) \sin \frac{n\pi x}{l} dx$$

$$\text{i.e.,} \quad D_n = \frac{2}{n\pi c} \int_0^l g(x) \sin \frac{n\pi x}{l} dx \quad \dots(7)$$

Hence the required solution of wave equation (A) is given by (3) where C_n and D_n are given by (6) and (7) respectively.

Particular Case I : If the initial velocity $g(x)$ of the string is zero *i.e.*, if the string is displaced to the position $f(x)$ and released gently with zero velocity, *i.e.*, if

$$\left(\frac{\partial u}{\partial t} \right)_{t=0} = g(x) = 0$$

$$\text{then from (7)} \quad D_n = 0$$

In this case the solution of wave equation is given by

$$u(x, t) = \sum_{n=1}^{\infty} C_n \cos \frac{n\pi c t}{l} \sin \frac{n\pi x}{l}$$

$$\text{where} \quad C_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

Particular Case II : If the initial displacement $f(x)$ of the string is zero *i.e.*, if the string start vibrating with velocity $g(x)$ from its initial position, *i.e.*, if

$$u(x, 0) = f(x) = 0$$

$$\text{then from (6)} \quad C_n = 0$$

In this case the solution of the wave equation is given by

$$u(x, t) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi c t}{l} \sin \frac{n\pi x}{l}$$

$$\text{where} \quad D_n = \frac{2}{n\pi c} \int_0^l g(x) \sin \frac{n\pi x}{l} dx$$

Note : The functions given by (2) are called the **eigen functions** or **characteristic functions** and the values $\lambda_n = c n \pi / l$ are called the **eigen values** or **characteristic values** of the vibrating string, for $n=1, 2, 3, \dots$. The set of the eigen values $\lambda_1, \lambda_2, \lambda_3, \dots$ is called the **spectrum**.

12.4.2 Solution by D'Alembert's Method

Let us introduce two new variables v and w , such that $v = x + ct$ and $w = x - ct$ so that u becomes a function of v and w

$$\therefore \quad \frac{\partial v}{\partial x} = 1 \quad \text{and} \quad \frac{\partial w}{\partial x} = 1$$

$$\text{Therefore,} \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial x} = \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}$$

$$\Rightarrow \quad \frac{\partial}{\partial x} \equiv \frac{\partial}{\partial v} + \frac{\partial}{\partial w}$$

$$\text{Now} \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \left(\frac{\partial}{\partial v} + \frac{\partial}{\partial w} \right) \left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w} \right)$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial v^2} + 2 \frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2} \quad \dots(1)$$

$$\text{Again} \quad \frac{\partial v}{\partial t} = c \quad \text{and} \quad \frac{\partial w}{\partial t} = -c$$

$$\therefore \quad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial t} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial t} = c \left(\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w} \right) \Rightarrow \frac{\partial}{\partial t} \equiv c \left(\frac{\partial}{\partial v} - \frac{\partial}{\partial w} \right)$$

$$\begin{aligned} \text{Therefore} \quad \frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} \right) = c^2 \left(\frac{\partial}{\partial v} - \frac{\partial}{\partial w} \right) \left(\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w} \right) \\ &= c^2 \left(\frac{\partial^2 u}{\partial v^2} - 2 \frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2} \right) \quad \dots(2) \end{aligned}$$

Substituting from (1) and (2) in (A), we have

$$c^2 \left(\frac{\partial^2 u}{\partial v^2} - 2 \frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2} \right) = c^2 \left(\frac{\partial^2 u}{\partial v^2} + 2 \frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2} \right)$$

$$\text{or} \quad \frac{\partial^2 u}{\partial v \partial w} = 0$$

$$\text{Integrating w.r.t. 'w' we have} \quad \frac{\partial u}{\partial v} = F(v)$$

where $F(v)$ is an arbitrary function of v .

Integrating this w.r.t. 'v' we have

$$u = \phi(v) + \psi(w) \quad \text{where} \quad \int F(v) dv = \phi(v)$$

and $\psi(w)$ is an arbitrary function of w .

$$\therefore \quad u(x, t) = \phi(x + ct) + \psi(x - ct). \quad \dots(3)$$

This is known as **D'Alembert's solution** of the wave equation (A).

$$\text{Now} \quad \frac{\partial u}{\partial t} = c \phi'(x + ct) - c \psi'(x - ct)$$

$\therefore$ using the initial conditions (C) and (D), from (3) we have

$$u(x, 0) = \phi(x) + \psi(x) = f(x), \quad \dots(4)$$

$$\text{and} \quad \left(\frac{\partial u}{\partial t} \right)_{t=0} = c \phi'(x) - c \psi'(x) = 0 \quad \dots(5)$$

Taking the particular case when $g(x) = 0$

From (5), we have

$$\psi'(x) = \phi'(x).$$

$$\therefore \quad \psi(x) = \phi(x) + A.$$

$$\therefore \quad \text{from (4), we have } \phi(x) + [\phi(x) + A] = f(x)$$

$$\text{or} \quad 2\phi(x) + A = f(x)$$

$$\therefore \quad \phi(x) = \frac{1}{2} [f(x) - A]$$

$$\text{and} \quad \psi(x) = \phi(x) + A = \frac{1}{2} [f(x) + A].$$

Hence from (3), we have

$$u(x, t) = \frac{1}{2} [f(x + ct) + f(x - ct)] \quad \dots(6)$$

Now using boundary conditions (B) and (C), we have

$$u(0, t) = \frac{1}{2} [f(ct) + f(-ct)] = 0$$

$$u(l, t) = \frac{1}{2} [f(l + ct) + f(l - ct)] = 0$$

Which shows that the function f must be odd and have period $2l$.

Thus (6) is the solution of wave equation (A) satisfying the boundary conditions (B), (C) and (D).

Solved Examples

Example 1: Find the deflection $u(x, t)$ of the vibrating string (length $l = \pi$, ends fixed and $c^2 = 1$) corresponding to zero initial velocity and initial deflection

$$f(x) = k(\sin x - \sin 2x).$$

Solution: The vibrations of the string are governed by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad (\because c^2 = 1 \text{ and } l = \pi) \quad \dots(A)$$

Here $u(x, t)$ is the deflection of the string, at any time $t > 0$ and the initial and boundary conditions are

$$\text{Boundary conditions : } \begin{cases} u(0, t) = 0 \\ u(\pi, t) = 0 \end{cases} \quad \dots(B)$$

$$\text{Initial conditions : } u(x, 0) = f(x) = k(\sin x - \sin 2x) \quad \dots(C)$$

$$\text{and } \left(\frac{\partial u}{\partial t} \right)_{t=0} = 0 \quad \dots(D)$$

The solution of the wave equation (A) is of the form (See 12.3 on page 541 here $l = \pi$)

$$u(x, t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos pt + C_4 \sin pt) \quad \dots(1)$$

$$\text{Now } u(0, t) = 0 \Rightarrow C_1(C_3 \cos pt + C_4 \sin pt) = 0$$

$$\therefore C_3 \cos pt + C_4 \sin pt \neq 0, \text{ as } u(x, t) = 0 \text{ is the trivial solution if we take } C_3 \cos pt + C_4 \sin pt = 0$$

$$\begin{aligned} \therefore u(x, t) &= C_2 \sin px \cdot (C_3 \cos pt + C_4 \sin pt) \\ &= \sin px \cdot (C_2 C_3 \cos pt + C_2 C_4 \sin pt) \end{aligned} \quad \dots(2)$$

Now

$$u(\pi, t) = 0 \Rightarrow \sin p\pi \cdot (C_2 C_3 \cos pt + C_2 C_4 \sin pt) = 0$$

$$\Rightarrow \sin p\pi = 0, \because C_2 C_3 \cos pt + C_2 C_4 \sin pt \neq 0, \text{ otherwise the solution will be trivial.}$$

$$\Rightarrow p\pi = n\pi \Rightarrow p = n \text{ where } n \text{ is any integer}$$

$$\therefore \text{ from (2), the solutions of wave equation (A) satisfying the boundary conditions (B) are}$$

$$u_n(x, t) = (C_n \cos nt + D_n \sin nt) \sin nx \quad \dots(3)$$

For all integral values of n , where $C_2 C_3 = C_n$, $C_2 C_4 = D_n$

Now in order that (3) may also satisfy the conditions (C) and (D) taking $n=1, 3, \dots$ the general solution of (A) is

$$\therefore u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} (C_n \cos nt + D_n \sin nt) \sin nx \quad \dots(4)$$

which also satisfy the conditions (B)

$$\text{Now} \quad \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} (-n C_n \sin nt + n D_n \cos nt) \sin nx$$

$$\therefore \left(\frac{\partial u}{\partial t} \right)_{t=0} = 0 \Rightarrow n D_n \sin nx = 0 \Rightarrow D_n = 0 \quad \because \sin nx \neq 0$$

$$\therefore \text{from (4), } u(x, t) = \sum_{n=1}^{\infty} C_n \cos nt \sin nx \quad \dots(5)$$

$$\therefore u(x, 0) = f(x) \Rightarrow \sum_{n=1}^{\infty} C_n \sin nx = f(x) = k(\sin x - \sin 2x) \quad \dots(6)$$

Equating coefficients of $\sin nx$ for $n=1, 2, \dots$, we get

$$C_1 = k, C_2 = -k \text{ and } C_3 = 0 = C_4 = \dots$$

$$\therefore \text{from (5), } u(x, t) = C_1 \cos t \sin x + C_2 \cos 2t \sin 2x \\ = k(\cos t \sin x - \cos 2t \sin 2x)$$

which is the required solution.

Note : After equation (6) we can also proceed as follows. In equation (6), considering L.H.S. as the fourier sine expansion of $f(x)$, we have

$$C_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} k \int_0^{\pi} (\sin x - \sin 2x) \sin nx \, dx$$

$$\therefore C_1 = \frac{2}{\pi} k \int_0^{\pi} (\sin x - \sin 2x) \sin x \, dx = \frac{2}{\pi} k \int_0^{\pi} (\sin^2 x - \sin 2x \sin x) \, dx$$

$$= \frac{k}{\pi} \int_0^{\pi} [1 - \cos 2x - (\cos x - \cos 3x)] \, dx$$

$$= \frac{k}{\pi} \left[x - \frac{1}{2} \sin 2x - \sin x + \frac{1}{3} \sin 3x \right]_0^{\pi} = k$$

and
$$C_2 = \frac{2}{\pi} k \int_0^{\pi} (\sin x - \sin 2x) \sin 2x \, dx = \frac{2}{\pi} k \int_0^{\pi} (\sin x \sin 2x - \sin^2 2x) dx$$

$$= \frac{k}{\pi} \int_0^{\pi} [(\cos x - \cos 3x) - (1 - \cos 4x)] dx$$

$$= \frac{k}{\pi} \left[\sin x - \frac{1}{3} \sin 3x - x + \frac{1}{4} \sin 4x \right]_0^{\pi} = -k$$

Obviously $C_3 = C_4 = C_5 = \dots = 0$ (each)

$\therefore$ from (5), $u(x, t) = C_1 \cos t \sin x + C_2 \cos 2t \sin 2x = k (\cos t \sin x - \cos 2t \sin 2x)$.

Example 2: Find the solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

such that $y = P_0 \cos pt$, (P_0 is a constant) when $x = l$ and $y = 0$ when $x = 0$.

[Delhi 2004; Meerut 2001, 07 (B.P.)]

Solution: The solution of the given wave equation is of the form (See 12.3 on page 541)

$$y(x, t) = (C_1 \cos kx + C_2 \sin kx) (C_3 \cos ckt + C_4 \sin ckt) \quad \dots(1)$$

Given $y = 0$, when $x = 0 \Rightarrow C_1 (C_3 \cos ckt + C_4 \sin ckt) = 0$

$\Rightarrow C_1 = 0$ as $C_3 \cos ckt + C_4 \sin ckt \neq 0$, otherwise

$y(x, t) = 0$, which is trivial solution.

$\therefore y(x, t) = C_2 \sin kx \cdot (C_3 \cos ckt + C_4 \sin ckt)$

or $y(x, t) = (C_2 C_3 \cos ckt + C_2 C_4 \sin ckt) \sin kx \quad \dots(2)$

Now $y = P_0 \cos pt$ when $x = l$,

$\therefore$ from (2), $P_0 \cos pt = C_2 C_3 \cos ckt \sin kl + C_2 C_4 \sin ckt \sin kl$

Equating the coefficients of $\sin$ and $\cos$ of various angles on both sides, we get

$$P_0 = C_2 C_3 \sin kl, \quad p = ck \text{ and } 0 = C_2 C_4 \sin kl$$

$$\Rightarrow k = \frac{p}{c}, \quad C_2 C_3 = \frac{P_0}{\sin kl} = \frac{P_0}{\sin \frac{p}{c} l} \text{ and } C_2 C_4 = 0$$

∴ from (2), we get

$$y(x, t) = \frac{P_0}{\sin \frac{p}{c} l} \cos pt \sin \frac{p}{c} x$$

which is the required solution.

Example 3: A string is stretched between two fixed points at a distance l apart. Motion is started by displacing the string in the form $y = a \sin(\pi x/l)$ from which it is released at time $t = 0$. Show that the displacement of any point at a fixed distance x from one end at time t is given by $y(x, t) = a \sin(\pi x/l) \cos(\pi c t/l)$. [Meerut 2012 (Sem. I); Kurukshetra 2004]

Solution: Let an elastic string be tightly stretched between two points O and A . Let O be the origin and OA along x -axis. Giving the string a small displacement perpendicular to its length (parallel to y -axis), let $y(x, t)$ be its displacement at the point $P(x, y)$ at any time t . The vibration of the string will be governed by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

where c is a constant

Since the ends O and A of the string are fixed for all time

∴ the **boundary conditions** are

$$y(0, t) = 0 \text{ and } y(l, t) = 0 \quad \dots(B)$$

According to the question, the **initial conditions** are

$$y(x, 0) = a \sin(\pi x/l) \quad \dots(C)$$

$$\text{and } \left(\frac{\partial y}{\partial t} \right)_{t=0} = 0 \quad \dots(D)$$

Thus we have to solve (A), subject to the conditions (B), (C) and (D).

Since the vibration of the string are periodic, so the solution of (A) will be of the form (See 12.3 page 541)

$$y(x, t) = (C_1 \cos kx + C_2 \sin kx) (C_3 \cos ckt + C_4 \sin ckt) \quad \dots(1)$$

$$\text{Now } y(0, t) = 0 \Rightarrow C_1(C_3 \cos ckt + C_4 \sin ckt) = 0 \Rightarrow C_1 = 0$$

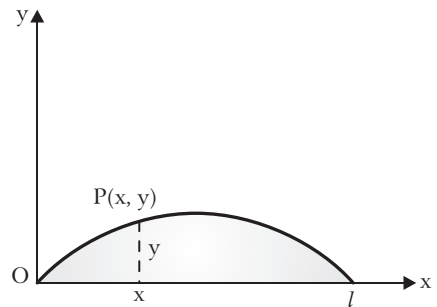


Fig. 12.4

...(A)

$\therefore C_3 \cos ckt + C_4 \sin ckt \neq 0$, otherwise we get $y(x, t) = 0$, which is trivial solution.

$$\therefore \text{from (1), } y(x, t) = \sin kx \cdot (C_2 C_3 \cos ckt + C_2 C_4 \sin ckt) \quad \dots(2)$$

$$\text{Also } y(l, t) = 0 \Rightarrow \sin kl \cdot (C_2 C_3 \cos ckt + C_2 C_4 \sin ckt) = 0$$

$\Rightarrow \sin kl = 0$, $\therefore C_2 C_3 \cos ckt + C_2 C_4 \sin ckt \neq 0$, otherwise we get $y(x, t) = 0$, which is trivial solution.

$$\Rightarrow kl = n\pi \Rightarrow k = n\pi/l, n \in I$$

$\therefore$ from (2), the solutions of (A) satisfying the boundary conditions (B) are given by

$$y_n(x, t) = [C_n \cos(c n\pi t/l) + D_n \sin(c n\pi t/l)] \sin(n\pi x/l) \quad \dots(3)$$

$n \in I$, $C_2 C_3 = C_n$ and $C_2 C_4 = D_n$ are new arbitrary constants

$\therefore$ Taking $n = 1, 2, \dots$, the general solution of (A) can be taken as

$$\begin{aligned} y(x, t) &= \sum_{n=1}^{\infty} y_n(x, t) \\ &= \sum_{n=1}^{\infty} [C_n \cos(c n\pi t/l) + D_n \sin(c n\pi t/l)] \cdot \sin(n\pi x/l) \end{aligned} \quad \dots(4)$$

which also satisfy the conditions (B).

$$\text{Now } \frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} [-C_n \cdot (c n\pi/l) \sin(c n\pi t/l) + D_n \cdot (c n\pi/l) \cdot \cos(c n\pi t/l)] \cdot \sin(n\pi x/l)$$

$$\therefore \text{Condition (D), } \left(\frac{\partial y}{\partial t} \right)_{t=0} = 0 \Rightarrow \sum_{n=1}^{\infty} D_n \cdot (c n\pi/l) \sin(n\pi x/l) = 0$$

$$\Rightarrow D_n = 0$$

$$\therefore \text{from (4), we have, } y(x, t) = \sum_{n=1}^{\infty} C_n \cdot \cos(c n\pi t/l) \sin(n\pi x/l) \quad \dots(5)$$

Now the condition (C), $y(x, 0) = a \sin(\pi x/l)$

$$\Rightarrow a \sin(\pi x/l) = \sum_{n=1}^{\infty} C_n \sin(n\pi x/l)$$

$$\Rightarrow C_1 = a, C_2 = C_3 = \dots = 0 \text{ (each) (Equating coefficients)}$$

Putting in (5), we get

$$y(x, t) = C_1 \cos(c \pi t/l) \sin(\pi x/l) = a \cos(c \pi t/l) \sin(\pi x/l)$$

which is the required solution.

Example 4: A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3(\pi x/l)$. If it is released from rest from this position, find the displacement $y(x, t)$.

Solution: The vibration of the string is governed by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \dots(A)$$

Here the **boundary conditions** are

$$y(0, t) = 0 \text{ and } y(l, t) = 0 \quad \dots(B)$$

and the **initial conditions** are

$$y(x, 0) = y_0 \sin^3(\pi x/l) \quad \dots(C)$$

$$\text{and } \left(\frac{\partial y}{\partial t} \right)_{t=0} = 0 \quad \dots(D)$$

Thus, we have to find the solution of (A) subject to the conditions (B), (C) and (D).

Since the vibrations of the string are periodic, so the solution of (A) will be of the form (See 12.3 page 541)

$$y(x, t) = (C_1 \cos kx + C_2 \sin kx) (C_3 \cos kct + C_4 \sin kct) \quad \dots(1)$$

$$\text{Now } y(0, t) = 0 \Rightarrow C_1(C_3 \cos kct + C_4 \sin kct) = 0$$

$$\Rightarrow C_1 = 0$$

$$\therefore C_3 \cos kct + C_4 \sin kct \neq 0, \text{ otherwise we get trivial solution } y(x, t) = 0$$

$$\therefore \text{ From (1) we get}$$

$$y(x, t) = \sin kx \cdot (C_2 C_3 \cos kct + C_2 C_4 \sin kct) \quad \dots(2)$$

$$\text{Now } y(l, t) = 0 \Rightarrow \sin kl \cdot (C_2 C_3 \cos kct + C_2 C_4 \sin kct) = 0$$

$$\Rightarrow \sin kl = 0, \therefore C_2 C_3 \cos kct + C_2 C_4 \sin kct \neq 0 \text{ otherwise we get trivial solution } y(x, t) = 0$$

$$\Rightarrow kl = n\pi, k = n\pi/l, n \in I.$$

$$\therefore \text{ from (2), the solutions satisfying the boundary conditions (B) are}$$

$$y_n(x, t) = [C_n \cos(n\pi ct/l) + D_n \sin(n\pi ct/l)] \sin(n\pi x/l) \quad \dots(3)$$

where $n \in I$, $C_2 C_3 = C_n$, $C_2 C_4 = D_n$, are new arbitrary constants.

$\therefore$ For $n = 1, 2, \dots$, the general solution of (A) satisfying conditions (B) can be taken as

$$\begin{aligned} y(x, t) &= \sum_{n=1}^{\infty} y_n(x, t) \\ &= \sum_{n=1}^{\infty} \left[C_n \cos\left(\frac{c n \pi t}{l}\right) + D_n \sin\left(\frac{c n \pi t}{l}\right) \right] \sin\left(\frac{n \pi x}{l}\right) \quad \dots(4) \end{aligned}$$

$$\text{Now} \quad \left(\frac{\partial y}{\partial t} \right)_{t=0} = 0 \Rightarrow \sum_{n=1}^{\infty} D_n \cdot (c n \pi / l) \sin(n\pi x/l) = 0$$

$$\Rightarrow D_n = 0$$

$$\therefore \text{ from (4),} \quad y(x, t) = \sum_{n=1}^{\infty} C_n \cos(c n \pi t / l) \cdot \sin(n\pi x / l) \quad \dots(5)$$

$$\text{Now} \quad y(x, 0) = y_0 \sin^3(\pi x / l)$$

$$\Rightarrow y_0 \sin^3(\pi x / l) = \sum_{n=1}^{\infty} C_n \cdot \sin(n\pi x / l)$$

$$\Rightarrow \frac{1}{4} y_0 \left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l} \right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n \pi x}{l}\right)$$

$$\Rightarrow C_1 = \frac{3}{4} y_0, C_3 = -\frac{1}{4} y_0, C_2 = C_4 = \dots = 0 \text{ (each)}$$

$\therefore$ from (5), we get

$$y(x, t) = C_1 \cos \frac{c \pi t}{l} \sin \frac{\pi x}{l} + C_3 \cos \frac{3 \pi ct}{l} \cdot \sin \frac{3 \pi x}{l}$$

$$\text{or} \quad y(x, t) = \frac{y_0}{4} \left[3 \cos \frac{c \pi t}{l} \sin \frac{\pi x}{l} - \cos \frac{3c \pi t}{l} \sin \frac{3 \pi x}{l} \right]$$

which is the required solution.

Example 5: If a string is released from rest in the position $y = 4 \in x(l-x)/l^2$, show that its motion is described by the equation

$$y = \frac{32 \in}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin \frac{(2n+1)\pi x}{l} \cos \frac{(2n+1)\pi ct}{l}.$$

Solution: Let the string be tightly stretched between two points O and A at a distance l apart. Let O be the origin and OA lie along x -axis. The vibration of the string will be governed by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \dots(A)$$

where c is a constant.

Here the **boundary conditions** are

$$y(0, t) = 0 \quad \text{and} \quad y(l, t) = 0 \quad \dots(B)$$

and **initial conditions** are

$$y(x, 0) = 4 \in x(l-x)/l^2 \quad \dots(C)$$

$$\text{and} \quad \left(\frac{\partial y}{\partial t} \right)_{t=0} = 0 \quad \dots(D)$$

Thus, we have to solve (A) subject to the conditions (B), (C) and (D).

Since the vibration of string are periodic, so the solution of equation (A) will be of the form (See 12.3 on page 519).

$$y(x, t) = (C_1 \cos kx + C_2 \sin kx)(C_3 \cos ckt + C_4 \sin ckt) \quad \dots(1)$$

$$\text{Now } y(0, t) = 0 \Rightarrow C_1(C_3 \cos ckt + C_4 \sin ckt) = 0$$

$$\Rightarrow C_1 = 0, \because C_3 \cos ckt + C_4 \sin ckt = 0 \text{ will lead to a trivial solution } y(x, t) = 0$$

$\therefore$ from (1), we get

$$y(x, t) = (C_2 C_3 \cos ckt + C_2 C_4 \sin ckt) \sin kx \quad \dots(2)$$

$$\text{Now } y(l, t) = 0 \Rightarrow (C_2 C_3 \cos ckt + C_2 C_4 \sin ckt) \sin kl = 0$$

$$\Rightarrow \sin kl = 0 \because C_2 C_3 \cos ckt + C_2 C_4 \sin ckt = 0 \text{ will be lead to a trivial solution } y(x, t) = 0$$

$$\Rightarrow kl = n\pi \text{ or } k = n\pi/l, n \text{ is any integer}$$

$\therefore$ from (2), the solutions of (A) satisfying the boundary conditions (B) are given by

$$y_n(x, t) = [C_n \cos(c n \pi t/l) + D_n \sin(c n \pi t/l)] \sin kx \quad \dots(3)$$

where $C_2 C_3 = C_n, C_2 C_4 = D_n$, are the new arbitrary constants and n is any integer

$\therefore$ Taking $n = 1, 2, \dots$, the general solution of (A) satisfying boundary conditions (B) can be taken as

$$y(x, t) = \sum_{n=1}^{\infty} y_n(x, t)$$

or
$$y(x, t) = \sum_{n=1}^{\infty} \left(C_n \cos \frac{c n \pi t}{l} + D_n \sin \frac{c n \pi t}{l} \right) \sin \frac{n \pi x}{l} \quad \dots(4)$$

Now
$$\left(\frac{\partial y}{\partial t} \right)_{t=0} = 0 \Rightarrow \sum_{n=1}^{\infty} D_n \left(\frac{c n \pi}{l} \right) \sin \frac{n \pi x}{l} = 0 \Rightarrow D_n = 0$$

$\therefore$ from (4), $y(x, t) = \sum_{n=1}^{\infty} C_n \cos \frac{c n \pi t}{l} \sin \frac{n \pi x}{l} \quad \dots(5)$

The condition (C), $y(x, 0) = 4 \in x(l-x)/l^2$

$$\Rightarrow \frac{4 \in (lx - x^2)}{l^2} = \sum_{n=1}^{\infty} C_n \sin \frac{n \pi x}{l} \quad \dots(6)$$

The right hand side of (6) can be considered as the fourier sine expansion of L.H.S.

$\therefore C_n = \frac{2}{l} \int_0^l \frac{4 \in}{l^2} (lx - x^2) \sin \frac{n \pi x}{l} dx$

or
$$C_n = \frac{8 \in}{l^3} \left[\left\{ (lx - x^2) \left(-\frac{l}{n\pi} \cos \frac{n \pi x}{l} \right) - (l-2x) \left(-\frac{l^2}{n^2 \pi^2} \sin \frac{n \pi x}{l} \right) \right\} \right]_0^l$$

$$+ \int_0^l (-2) \left(-\frac{l^2}{n^2 \pi^2} \sin \frac{n \pi x}{l} \right) dx$$

$$= \frac{8 \in}{l^3} \left[-\frac{2l^2}{n^2 \pi^2} \cdot \frac{l}{n\pi} \cos \frac{n \pi x}{l} \right]_0^l = \frac{16 \in}{\pi^3 n^3} (1 - \cos n\pi)$$

$$= \frac{32 \in}{\pi^3 n^3} \text{ when } n \text{ is odd and } 0 \text{ when } n \text{ is even.}$$

Putting in (5),

$$y(x, t) = \frac{32 \in}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \cos \frac{c n \pi t}{l} \sin \frac{n \pi x}{l}, \text{ when } n \text{ is odd.}$$

$$= \frac{32 \in}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin \frac{(2n+1) \pi x}{l} \cos \frac{(2n+1) c \pi t}{l}$$

which is the required solution.

Note : The above solution can be also be written as

$$y(x, t) = \frac{32 \epsilon}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin \frac{(2n-1)\pi x}{l} \cos \frac{(2n-1)c \pi t}{l}$$

Example 6: A uniform string of line density ρ is stretched to tension ρc^2 and executes a small transverse vibration in a plane through the undisturbed line of the string. The ends $x = 0, l$ of string are fixed. The string is at rest, with the point $x = b$ drawn aside through a small distance ϵ and released at time $t = 0$. show that at any subsequent time t the transverse displacement y is given by the fourier expansion

$$y = \frac{2 \epsilon l^2}{\pi^2 b(l-b)} \sum_{s=1}^{\infty} \frac{1}{s^2} \sin \left(\frac{s \pi b}{l} \right) \sin \left(\frac{s \pi x}{l} \right) \cos \left(\frac{s \pi c t}{l} \right).$$

Solution: Let OA be the equilibrium position of a string stretched to tension ρc^2 fixed at O and A such that $OA = l$. Let L the point of the string at distance $x = b$ from end O be drawn aside to L' through a small distance $LL' = \epsilon$ and released at time $t = 0$. The string will start vibrating about OA .

Let us take x -axis Ox along OA and y -axis Oy perpendicular to Ox .

The transverse vibrations of the string will be governed by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \dots(A)$$

where $y(x, t)$ is the deflection (displacement) of any point at a distance x from the end O of the string at any time t .

The coordinates of L' are (b, ϵ) and those of A are $(l, 0)$.

$\therefore$ Equation of line OL' is $y - 0 = \frac{\epsilon - 0}{b - 0}(x - 0)$ i.e., $y = \epsilon x/b$

and equation of line $L'A$ is $y - 0 = \frac{\epsilon - 0}{b - l}(x - l)$ i.e., $y = -\frac{\epsilon}{(l-b)}(x - l)$.

At time $t = 0$, the string rests in the position $OL'A$.

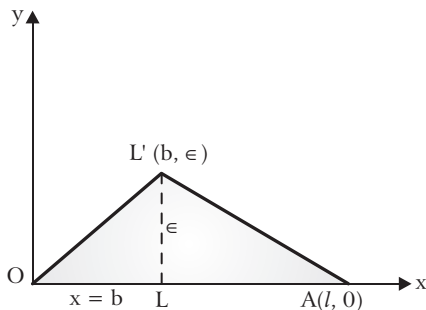


Fig. 12.5

∴ The initial and boundary conditions are as follows :

Boundary conditions : $y(0, t) = 0, y(l, t) = 0$... (B)

Initial conditions : $y(x, 0) = f(x) = \begin{cases} \frac{x}{b} & 0 \leq x \leq b \\ -\frac{x}{(l-b)} & b \leq x \leq l \end{cases}$

and $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$

Proceeding similarly as in 12.3 (Particular case I), the transverse displacement of the string at distance x at any time t is given by

$$y(x, t) = \sum_{s=1}^{\infty} C_s \cos \frac{s \pi c t}{l} \sin \frac{s \pi x}{l} \quad \dots (1)$$

(Taking s for n)

where

$$\begin{aligned} C_s &= \frac{2}{l} \int_0^l f(x) \sin \frac{s \pi x}{l} dx \\ &= \frac{2}{l} \left[\int_0^b \frac{x}{b} \sin \frac{s \pi x}{l} dx - \int_b^l \frac{x}{(l-b)} \sin \frac{s \pi x}{l} dx \right] \\ &= \frac{2}{l} \left[\frac{x}{b} \cdot \left\{ -x \frac{l}{s \pi} \cos \frac{s \pi x}{l} + \frac{l^2}{s^2 \pi^2} \sin \frac{s \pi x}{l} \right\} \right]_0^b \\ &\quad - \frac{x}{(l-b)} \left\{ -(x-l) \cdot \frac{l}{s \pi} \cos \frac{s \pi x}{l} + \frac{l^2}{s^2 \pi^2} \sin \frac{s \pi x}{l} \right\} \right]_b^l \\ &= \frac{2}{l} \left[\frac{x}{b} \cdot \left\{ -\frac{bl}{s \pi} \cos \frac{s \pi b}{l} + \frac{l^2}{s^2 \pi^2} \sin \frac{s \pi b}{l} \right\} \right. \\ &\quad \left. - \frac{x}{(l-b)} \left\{ (b-l) \cdot \frac{l}{s \pi} \cos \frac{s \pi b}{l} - \frac{l^2}{s^2 \pi^2} \sin \frac{s \pi b}{l} \right\} \right] \\ &= \frac{2}{l} \left[\left(-\frac{x}{s \pi} + \frac{x}{s \pi} \right) \cos \frac{s \pi b}{l} + \frac{l^2}{\pi^2 s^2} \left(\frac{1}{b} + \frac{1}{(l-b)} \right) \sin \frac{s \pi b}{l} \right] \\ &= \frac{2}{\pi^2 b (l-b) s^2} \sin \frac{s \pi b}{l} \end{aligned}$$

Putting in (1), the required transverse displacement is given by the Fourier expansion.

$$y(x, t) = \frac{2 \epsilon l^2}{\pi^2 b(l-b)} \sum_{s=1}^{\infty} \frac{1}{s^2} \sin\left(\frac{s \pi b}{l}\right) \cdot \cos\left(\frac{s \pi c t}{l}\right) \sin\left(\frac{s \pi x}{l}\right)$$

Example 7: The points of trisection of a string are pulled aside through a distance b on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at any subsequent time and show that the mid point of the string always remains at rest. [Meerut 2008]

Solution: Let OA be the equilibrium position of a string fixed at O and A s.t. $OA = l = 3a$ (length of the string). Let the x -axis lie along OA and y -axis Oy perpendicular to Ox . Let L and M be the points of trisection of the string (i.e., $OL = LM = MA = a$) be pulled through a distance b on opposite sides to L' and M' respectively and then released from rest. The string will start vibrating about OA and the displacement $y(x, t)$ of any point of the string at distance x at any time t will be governed by one dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \dots(A)$$

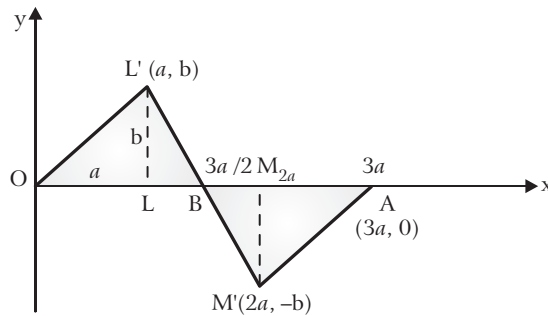


Fig. 12.6

At time $t = 0$, the string rests in the position $OL' M'A$.

The coordinates of L' are (a, b) , those of M' are $(2a, -b)$.

Equation of line OL' is $y - 0 = \frac{b-0}{a-0}(x-0)$ i.e., $y = \frac{b}{a}x$

Equation of line $L'M'$ is $y - b = \frac{-b-b}{2a-a}(x-a)$ i.e., $y = \frac{b(3a-2x)}{a}$

and equation of line $M'A$ is $y + b = \frac{0+b}{3a-2a}(x-2a)$ i.e., $y = \frac{b(x-3a)}{a}$

∴ The initial and boundary conditions are as follows :

Boundary conditions : $y(0, t) = 0, y(3a, t) = 0$...(B)

Initial conditions :

$$y(x, 0) = f(x) = \begin{cases} \frac{bx}{a}, & 0 \leq x \leq a \\ \frac{b(3a-2x)}{a}, & a \leq x \leq 2a \\ \frac{b(x-3a)}{a}, & 2a \leq x \leq 3a \end{cases}$$

and

$$\left(\frac{\partial y}{\partial t} \right)_{t=0} = 0$$

Proceeding similarly as in 12.3 (Particular case I), the displacement of the string at distance x at any time t is given by

$$y(x, t) = \sum_{n=1}^{\infty} C_n \cos \frac{n\pi c t}{3a} \sin \frac{n\pi x}{3a} \quad (\text{Here } l = 3a) \quad \dots(1)$$

where

$$\begin{aligned} C_n &= \frac{2}{3a} \int_0^{3a} f(x) \sin \frac{n\pi x}{3a} dx \\ &= \frac{2}{3a} \left[\int_0^a \frac{b}{a} x \sin \frac{n\pi x}{3a} dx + \int_a^{2a} \frac{b(3a-2x)}{a} \sin \frac{n\pi x}{3a} dx \right. \\ &\quad \left. + \int_{2a}^{3a} \frac{b(x-3a)}{a} \sin \frac{n\pi x}{3a} dx \right] \\ &= \frac{2b}{3a^2} \left[\left\{ -x \cdot \frac{3a}{n\pi} \cos \frac{n\pi x}{3a} + \frac{9a^2}{n^2\pi^2} \sin \frac{n\pi x}{3a} \right\}_0^a \right. \\ &\quad + \left\{ -(3a-2x) \cdot \frac{3a}{n\pi} \cos \frac{n\pi x}{3a} - \frac{18a^2}{n^2\pi^2} \sin \frac{n\pi x}{3a} \right\}_a^{2a} \\ &\quad \left. + \left\{ -(x-3a) \cdot \frac{3a}{n\pi} \cos \frac{n\pi x}{3a} + \frac{9a^2}{n^2\pi^2} \sin \frac{n\pi x}{3a} \right\}_{2a}^{3a} \right] \\ &= \frac{2b}{3a^2} \left[\left\{ -\frac{3a^2}{n\pi} \cos \frac{n\pi}{3} + \frac{9a^2}{n^2\pi^2} \sin \frac{n\pi}{3} \right\} \right. \\ &\quad + \left\{ \frac{3a^2}{n\pi} \cos \frac{2n\pi}{3} - \frac{18a^2}{n^2\pi^2} \sin \frac{2n\pi}{3} + \frac{3a^2}{n\pi} \cos \frac{n\pi}{3} + \frac{18a^2}{n^2\pi^2} \sin \frac{n\pi}{3} \right\} \\ &\quad \left. + \left\{ -0 + 0 - \frac{3a^2}{n\pi} \cos \frac{2n\pi}{3} - \frac{9a^2}{n^2\pi^2} \sin \frac{2n\pi}{3} \right\} \right] \end{aligned}$$

$$\begin{aligned}
 &= \frac{18b}{n^2 \pi^2} \left[\sin \frac{n\pi}{3} - \sin \frac{2n\pi}{3} \right] = \frac{18b}{n^2 \pi^2} \left[\sin \frac{n\pi}{3} - \sin \left(n\pi - \frac{n\pi}{3} \right) \right] \\
 &= \frac{18b}{n^2 \pi^2} [1 + (-1)^n] \sin \frac{n\pi}{3} \quad \because \sin \left(n\pi - \frac{n\pi}{3} \right) = (-1)^{n-1} \sin \frac{n\pi}{3}
 \end{aligned}$$

$\therefore C_n = 0$, when n is odd and $C_n = \frac{18b}{n^2 \pi^2} \cdot 2 \sin \frac{n\pi}{3}$, when n is even.

$\therefore$ Taking $n = 2m$, from (1)

$$y(x, t) = \frac{36b}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{(2m)^2} \sin \frac{2m\pi}{3} \cdot \cos \frac{2m\pi c t}{3a} \cdot \sin \frac{2m\pi x}{3a}.$$

or
$$y(x, t) = \frac{9b}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \sin \frac{2m\pi}{3} \cdot \cos \frac{2m\pi c t}{3a} \cdot \sin \frac{2m\pi x}{3a}$$

2nd Part : At the middle point B of the string $x = OB = 3a/2$

$\therefore$ Putting $x = 3a/2$ in the above expression, the displacement of the mid-point of the string at any time t .

$$y\left(\frac{3a}{2}, t\right) = \frac{9b}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \sin \frac{2m\pi}{3} \cdot \cos \frac{2m\pi c t}{3a} \cdot \sin m\pi = 0$$

$$\because \sin m\pi = 0$$

Hence, the mid-point of the string always remains at rest.

Example 8: A tightly stretched string with fixed ends at $x = 0$ and $x = l$ is initially in equilibrium position. It is set vibrating by giving to each of its point an initial velocity $b \sin^3(\pi x/l)$. Find the displacement of the string at any time t .

[Delhi 2005; Meerut 2010, 12(O)]

Solution: Let a tightly stretched string be in equilibrium position along x -axis with the ends fixed at $x = 0$ and $x = l$.

The vibrations of the string are governed by one dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \dots(A)$$

Here we have to solve (A) subject to the following initial and boundary conditions.

Boundary conditions : $y(0, t) = 0$ and $y(l, t) = 0$...(B)

Initial conditions : $y(x, 0) = 0$...(C)

and
$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = b \sin^3\left(\frac{\pi x}{l}\right) \quad \dots(D)$$

where $y(x, t)$ is the deflection of the point of the string at distance x from the end $x = 0$, at any time t .

Since the vibrations of the string are periodic, so the solution of (A) will be of the form (Proceed as in 12.3 on page 541).

$$y(x, t) = (C_1 \cos kx + C_2 \sin kx)(C_3 \cos ckt + C_4 \sin ckt) \quad \dots(1)$$

Now $y(0, t) = 0 \Rightarrow C_1(C_3 \cos ckt + C_4 \sin ckt) = 0$

$$\Rightarrow C_1 = 0$$

$\therefore C_3 \cos ckt + C_4 \sin ckt \neq 0$, otherwise we get trivial solution $y(x, t) = 0$

$\therefore$ From (1), we get

$$y(x, t) = \sin kx.(C_2 C_3 \cos ckt + C_2 C_4 \sin ckt) \quad \dots(2)$$

Now $y(l, t) = 0 \Rightarrow \sin kl.(C_2 C_3 \cos ckt + C_2 C_4 \sin ckt) = 0$

$$\Rightarrow \sin kl = 0,$$

$\therefore C_2 C_3 \cos ckt + C_2 C_4 \sin ckt \neq 0$, otherwise we get trivial solution $y(x, t) = 0$

$$\Rightarrow kl = n\pi \text{ or } k = n\pi/l, n \in I.$$

$\therefore$ From (2), the solutions of (A) satisfying the boundary conditions are

$$y_n(x, t) = [C_n \cos(n\pi ct/l) + D_n \sin(n\pi ct/l)] \sin(n\pi x/l) \quad \dots(3)$$

where $C_2 C_3 = C_n$, $C_2 C_4 = D_n$, are new arbitrary constants and $n \in I$.

$\therefore$ Taking $n = 1, 2, \dots$ the general solution of (A) satisfying the boundary condition (B) can be taken as

$$\begin{aligned} y(x, t) &= \sum_{n=1}^{\infty} u_n(x, t) \\ &= \sum_{n=1}^{\infty} [C_n \cos(n\pi ct/l) + D_n \sin(n\pi ct/l)] \cdot \sin(n\pi x/l) \quad \dots(4) \end{aligned}$$

Now $y(x, 0) = 0 \Rightarrow \sum_{n=1}^{\infty} C_n \sin(n\pi x/l) = 0 \Rightarrow C_n = 0$

$\therefore$ from (4),
$$y(x, t) = \sum_{n=1}^{\infty} D_n \sin(n\pi ct/l) \cdot \sin(n\pi x/l) \quad \dots(5)$$

Now
$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} D_n \cdot \left(\frac{n\pi c}{l} \right) \cos\left(\frac{n\pi c t}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore \left(\frac{\partial y}{\partial t} \right)_{t=0} = b \sin^3\left(\frac{\pi x}{l}\right)$$

$$\Rightarrow b \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} \left(\frac{n\pi c D_n}{l} \right) \sin\left(\frac{n\pi x}{l}\right)$$

$$\Rightarrow \frac{1}{4} b \left\{ 3 \sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right) \right\} = \sum_{n=1}^{\infty} \left(\frac{\pi c}{l} \right) \cdot n D_n \sin \frac{n\pi x}{l}$$

Equating coefficients of sines of various angles, we get

$$\frac{3}{4} b = \left(\frac{1 \cdot \pi c D_1}{l} \right), \quad -\frac{1}{4} b = \left(\frac{3 \pi c D_3}{l} \right), \quad D_2 = D_4 = \dots = 0 \text{ (each)}$$

$$\Rightarrow D_1 = \frac{3bl}{4\pi c}, \quad D_3 = -\frac{bl}{12\pi c}, \quad D_2 = D_4 = \dots = 0 \text{ (each)}$$

$\therefore$ From (5) we get

$$u(x, t) = \frac{bl}{12\pi c} \left\{ 9 \sin\left(\frac{\pi c t}{l}\right) \sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi c t}{l}\right) \sin\left(\frac{3\pi x}{l}\right) \right\}$$

which is the required solution.

Example 9: A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $kx(l-x)$, find its displacement function. [IAS 2000; Meerut 2010]

Solution: Let a tightly stretched string be in equilibrium position along x -axis with the ends fixed at $x = 0$ and $x = l$.

The vibrations of the string are governed by one dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \dots(A)$$

where $y(x, t)$ is the deflection of the point of the string at distance x from the end $x = 0$ at any time t .

Here we have to solve (A) subject to the following initial and boundary conditions.

Boundary conditions : $y(0, t) = 0$ and $y(l, t) = 0$ $\dots(B)$

Initial conditions : $y(x, 0) = 0$... (C)

and $\left(\frac{\partial y}{\partial t}\right)_{t=0} = k(lx - x^2)$... (D)

Since the vibrations of the string are periodic, so the solution of (A) will be of the form (See 12.3 page 541).

$$y(x, t) = (C_1 \cos kx + C_2 \sin kx)(C_3 \cos ckt + C_4 \sin ckt) \quad \dots (1)$$

Now proceeding as in last example we get

$$C_1 = 0, k = n\pi/l, C_n = 0 \text{ and}$$

$$y(x, t) = \sum_{n=1}^{\infty} D_n \sin(n\pi ct/l) \cdot \sin(n\pi x/l)$$

Now
$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} D_n \left(\frac{n\pi c}{l}\right) \cos\left(\frac{n\pi ct}{l}\right) \cdot \sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore \left(\frac{\partial y}{\partial t}\right)_{t=0} = k(lx - x^2)$$

$$\Rightarrow k(lx - x^2) = \sum_{n=1}^{\infty} D_n \left(\frac{n\pi c}{l}\right) \sin\left(\frac{n\pi x}{l}\right)$$

Obviously the right hand side is the fourier sine series of L.H.S.

$$\therefore \left(\frac{n\pi c}{l}\right) D_n = \frac{2}{l} \int_0^l k(lx - x^2) \cdot \sin\left(\frac{n\pi x}{l}\right) dx$$

$$\begin{aligned} \text{or } D_n &= \frac{2k}{n\pi c} \left[\left\{ (lx - x^2) \left(-\frac{1}{n\pi}\right) \cos\left(\frac{n\pi x}{l}\right) - (l-2x) \left(-\frac{l^2}{n^2\pi^2}\right) \sin\left(\frac{n\pi x}{l}\right) \right\}_0^l \right. \\ &\quad \left. + \int_0^l (-2) \left(-\frac{l^2}{n^2\pi^2}\right) \sin\left(\frac{n\pi x}{l}\right) dx \right] \\ &= \frac{2k}{n\pi c} \left[0 - \frac{2l^3}{n^3\pi^3} \left\{ \cos\frac{n\pi x}{l} \right\}_0^l \right] = \frac{4kl^3}{c\pi^4 n^4} (1 - \cos n\pi) \\ &= 0 \quad \text{if } n \text{ is even} \quad \text{and } \frac{8kl^3}{c\pi^4 n^4} \quad \text{if } n \text{ is odd.} \end{aligned}$$

$$\therefore y(x, t) = \frac{8kl^3}{c\pi^4} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^4} \sin \frac{(2m-1)\pi ct}{l} \sin \frac{(2m-1)\pi x}{l}$$

Note : The above solution can also be written as

$$y(x, t) = \frac{8kl^3}{c\pi^4} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^4} \sin \frac{(2m+1)\pi ct}{l} \sin \frac{(2m+1)\pi x}{l}.$$

Example 10: Obtain the general solution of the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$, given that initial deflection is $f(x)$ and initial velocity is $g(x)$.

Solution: The given wave equation can be written as

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$\text{or} \quad (D^2 - c^2 D'^2)u = 0 \quad \dots(1)$$

$$\text{where } D \equiv \frac{\partial}{\partial t} \text{ and } D' \equiv \frac{\partial}{\partial x}$$

The auxiliary equation of (1) is $m^2 - c^2 = 0 \Rightarrow m = c, -c$

$$\therefore u(x, t) = \phi(x + ct) + \psi(x - ct) \quad \dots(2)$$

where ϕ and ψ are arbitrary functions. This solution (2) is known as the D'Alembert's solution of the wave equation (1).

Now we proceed to find the functions ϕ and ψ using the given initial conditions.

$$\therefore \text{Initial deflection} = u(x, 0) = f(x), \text{ for all } x \text{ (Given)}$$

$$\Rightarrow \phi(x) + \psi(x) = f(x) \quad \dots(3)$$

$$\text{From (2), we have } \frac{\partial u}{\partial t} = c\phi'(x + ct) - c\psi'(x - ct)$$

$$\therefore \text{Initial velocity} = \left(\frac{\partial u}{\partial t} \right)_{t=0} = g(x), \text{ for all } x \text{ (Given)}$$

$$\Rightarrow \phi'(x) - \psi'(x) = (1/c)g(x) \quad \dots(4)$$

$$\text{Integrating (4) we get, } \phi(x) - \psi(x) = \frac{1}{c} \int_a^x g(\theta) d\theta \quad \dots(5)$$

where a is arbitrary.

Solving (3) and (5), we get

$$\phi(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_a^x g(\theta) d\theta \quad \text{and} \quad \psi(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_a^x g(\theta) d\theta$$

$$\therefore \phi(x+ct) = \frac{1}{2} f(x+ct) + \frac{1}{2c} \int_a^{x+ct} g(\theta) d\theta$$

$$\text{and} \quad \psi(x-ct) = \frac{1}{2} f(x-ct) - \frac{1}{2c} \int_a^{x-ct} g(\theta) d\theta$$

Substituting in (2), we get

$$u(x, t) = \frac{1}{2} [f(x+ct) + f(x-ct)] + \frac{1}{2c} \left[\int_a^{x+ct} g(\theta) d\theta - \int_a^{x-ct} g(\theta) d\theta \right]$$

$$\text{or} \quad u(x, t) = \frac{1}{2} [f(x+ct) + f(x-ct)] + \frac{1}{2c} \left[\int_a^{x+ct} g(\theta) d\theta + \int_{x-ct}^a g(\theta) d\theta \right]$$

$$\text{or} \quad u(x, t) = \frac{1}{2} [f(x+ct) + f(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\theta) d\theta \quad \dots(6)$$

which is the required solution.

Particular case : If the string is at rest initially *i.e.*, when $t = 0$, $g(x) = 0$, then the solution (6) reduces to

$$u(x, t) = \frac{1}{2} [f(x+ct) + f(x-ct)]$$

Exercise 12.1

1. A string is stretched between the fixed points (0, 0) and (l, 0) and released from rest from the position $u = A \sin \pi x$. Find the formula for the subsequent displacement $u(x, t)$. [Nagpur 2002]
2. A string is stretched between the fixed points (0, 0) and (l, 0) and released from rest from the position $u(x, 0) = A \sin 2\pi x$. Find the displacement $u(x, t)$.
3. The vibrations of an elastic string is governed by the partial differential equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$. The Length of the string is π and the ends are fixed. The initial velocity is zero and the initial deflection is $f(x) = 2(\sin x + \sin 3x)$. Find the deflection $u(x, t)$ of the vibrating string for $t > 0$.

4. A tightly stretched elastic string of length l with fixed end points $x=0$ and $x=l$ is initially in the position giving $y=c \sin (\pi x/l)$, c being constant. It is released from the position of rest. Find the displacement $y(x, t)$. [Kurukshetra 2004]
5. A tightly stretched elastic string of length π with fixed end points $x=0$ and $x=l$ is initially in the position given by $y=c \sin^3 x$, c being constant. It is released from the position of rest. Find the displacement $y(x, t)$.
6. Find the deflection $u(x, t)$ of the vibrating string (length $l=\pi$, ends fixed and $c=1$) corresponding to zero initial velocity and initial deflection $f(x)$, given by
 - (i) $f(x)=k x(\pi-x)$
 - (ii) $f(x)=k x\left(\pi^2-x^2\right)$.
7. A string is stretched and fastened to two points distance l apart. Motion is started by displacing the string in to the form $y=k(l x-x^2)$ from which it is released at time $t=0$. Find the displacement of any point of the string at a distance x from one end at any time t .
8. A string is stretched between the fixed points $(0, 0)$ and $(l, 0)$ and released at rest from the initial deflection given by

$$f(x)=\begin{cases} (2 k / l) x, & \text { when } 0 < x < l / 2 \\ (2 k / l)(l-x) & \text { when } l / 2 < x < l \end{cases}$$

Find the deflection of the string at any time t .

9. Solve $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}$ under the boundary conditions $y=0$ for $x=0$ and for all values of t , $\frac{\partial y}{\partial t}=0$ for $t=0$ and for all values of x , $y=0$ for $x=\pi$ and for all values of t , $y=h x / d$ for $0 \leq x \leq d$ and $t=0$ and $y=h(\pi-x) /(\pi-d)$ for $d \leq x \leq \pi$ and $t=0$.
10. A string of length l is slightly stretched between two end points. Obtain the solution of this equation if the string is originally plucked at the middle point by giving it an initial displacement h from the mean position.
11. A tightly stretched violin string of length l and fixed at both ends is plucked at $x=l / 3$ and assumes initially the shape of a triangle of height a . Find the displacement $y(x, t)$ at any distance x at any time t after the string is released from rest.
12. A string of length l is initially at rest in its equilibrium position and motion is started by giving each of its points a velocity v given by $v=k x$ if $0 \leq x \leq l / 2$ and $v=k(l-x)$ if $l / 2 \leq x \leq l$. Find the displacement function.
13. Show that a general solution of wave equation

$$\frac{\partial^2 \phi}{\partial t^2}=c^2 \frac{\partial^2 \phi}{\partial x^2} \text { is } \phi(x, t)=f(x+c t)+g(x-c t) .$$
[Nagpur 2005; Meerut 2002, 03; Kurukshetra 2001, 05]

[Hint : See 12.4.2]

14. By separating the variables, show that one dimensional wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$ has solution of the form $A \exp (\pm c n x \pm i n c t)$, where A and n are constants.

Answers 12.1

1.	$u(x, t) = A \cos c \pi t \sin \pi x$	2.	$u(x, t) = A \sin 2x \cos 2\pi c t$
3.	$u(x, t) = 4 \cos x \cos 2t \sin 2x$	4.	$y(x, t) = c \cos(\pi at/l) \sin(\pi x/l)$
5.	$y(x, t) = \frac{c}{4} (3 \cos at \sin x - \cos 3at \sin 3x)$		
6.	(i) $u(x, t) = \frac{8k}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \cos(2n-1)t \sin(2n-1)x$ (ii) $u(x, t) = 12k \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \cos nt \sin nx$		
7.	$y(x, t) = \frac{8kl^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \cos \frac{(2n-1)\pi ct}{l} \sin \frac{(2n-1)\pi x}{l}$		
8.	$u(x, t) = \frac{8k}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \cos \frac{(2n-1)\pi ct}{l} \sin \frac{(2n-1)\pi x}{l}$		
9.	$y(x, t) = \frac{2h}{d(\pi-d)} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin nd. \cos nct \sin nx$		
10.	$u(x, t) = \frac{8h}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \cos \frac{(2n-1)\pi ct}{l} \sin \frac{(2n-1)\pi x}{l}$		
11.	$y(x, t) = \frac{9a}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{3}\right) \sin\left(\frac{n\pi x}{l}\right) \cos\left(\frac{n\pi ct}{l}\right)$		
12.	$y(x, t) = \frac{4kl^2}{c\pi^3} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^3} \sin \frac{(2m-1)\pi ct}{l} \sin \frac{(2m-1)\pi x}{l}$		

12.5 Some Important and Useful Differential Equations and Their Solutions

Here we give few important differential equations and their general solutions which are of great use in further discussion of this chapter.

Equation	General Solution
1. Bessel's Equation	
(i) $\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0$ when n is not an integer	$y(x) = A_n J_n(x) + B_n J_{-n}(x)$ J_n and J_{-n} are Bessel's function of first kind and $J_{-n} \rightarrow \infty$ as $x \rightarrow 0$.
(ii) $\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(m^2 - \frac{n^2}{x^2}\right)y = 0$ when n is not an integer	$y(x) = A_{mn} J_n(mx) + B_{mn} J_{-n}(mx)$ J_n and J_{-n} are Bessel's function of first kind and $J_{-n} \rightarrow \infty$ as $x \rightarrow 0$.
(iii) $\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0$ when n is an integer	$y(x) = A_n J_n(x) + B_n Y_n(x)$ where J_n and Y_n are Bessel's function of first and second kind respectively and $Y_n \rightarrow \infty$ as $x \rightarrow 0$.
(iv) $\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(m^2 - \frac{n^2}{x^2}\right)y = 0$ when n is an integer	$y(x) = A_{mn} J_n(mx) + B_{mn} Y_n(mx)$ where J_n and Y_n are Bessel's functions of first and second kind respectively and $Y_n \rightarrow \infty$ as $x \rightarrow 0$.
(v) $\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + y = 0$ Bessel's equation of zeroth order	$y(x) = A J_0(x) + B Y_0(x)$ where J_0 and Y_0 are Bessel's functions of zeroth order of first and second kind respectively and $Y_0 \rightarrow \infty$ as $x \rightarrow 0$.
(vi) $\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + m^2 y = 0$ Bessel's equation of zeroth order	$y(x) = A_m J_0(mx) + B_m Y_0(mx)$ where J_0 and Y_0 are Bessel's function of zeroth order of first and second kind respectively and $Y_0 \rightarrow \infty$ as $x \rightarrow 0$.
(vii) $\frac{d^2 y}{dx^2} + \frac{2}{r} \frac{dy}{dx} + \left\{m^2 - \frac{n(n+1)}{x^2}\right\}y = 0$	$y(x) = x^{-1/2} [A_{mn} J_{n+1/2}(mx) + B_{mn} J_{-(n+1/2)}(mx)]$ where $J_{\pm(n+1/2)}$ are spherical Bessel's functions, $n+1/2$ is neither zero nor an integer.
2. Legendre's Equation	
(i) $(1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)y = 0$	$y(x) = A_n P_n(x) + B_n Q_n(x)$ where P_n and Q_n are Legendre's functions of first and second kind respectively and $Q_n \rightarrow \infty$ as $x \rightarrow 0$.
(ii) $\frac{d^2 y}{d\theta^2} + \cot \theta \frac{dy}{d\theta} + n(n+1)y = 0$	$y(\theta) = A_n P_n(\cos \theta) + B_n Q_n(\cos \theta)$
(iii) $(1-\mu^2) \frac{d^2 y}{d\mu^2} - 2\mu \frac{dy}{d\mu} + n(n+1)y = 0$ where $\mu = \cos \theta$	$y(\mu) = A_n P_n(\mu) + B_n Q_n(\mu)$ where P_n and Q_n are Legendre's functions of first and second kind respectively and $Q_n \rightarrow \infty$ as $\mu \rightarrow 0$.

3. Homogeneous equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - n^2 y = 0$$

$$x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} - n(n+1)y = 0$$

$$y(x) = A_n r^n + B_n r^{-n}$$

$$y(x) = A_n r^n + B_n r^{-(n+1)}$$

12.6 Solution of Two Dimensional Wave Equation by the Method of Separation of Variables

[Meerut 2003, 10 (Sem. I)]

Consider a thin vibrating membrane stretched to a uniform tension T and bounded by rectangle $0 \leq x \leq a, 0 \leq y \leq b$ in the xy plane. Suppose the entire boundary of the membrane is held fixed for all $t \geq 0$.

The vibrations of the membrane are governed by two dimensional wave equation (see 12.2.2).

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \dots(A)$$

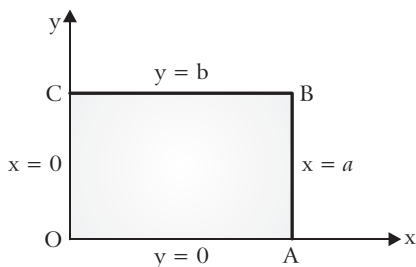


Fig. 12.7

where $c^2 = T/\rho$

$u(x, y, t)$ is the deflection of the membrane at the point (x, y) at any time $t \geq 0$. Let $f(x, y)$ be the initial deflection and $g(x, y)$ the initial velocity of the membrane. Then the initial and boundary conditions are as follows :

Boundary conditions :

$$\text{and} \quad \left. \begin{aligned} u(0, y, t) = 0, \quad u(a, y, t) = 0 \\ u(x, 0, t) = 0, \quad u(x, b, t) = 0 \end{aligned} \right\} \quad \dots(B)$$

and initial conditions

$$u(x, y, 0) = f(x, y) \quad \dots(C)$$

$$\left(\frac{\partial u}{\partial t} \right)_{t=0} = g(x, y) \quad \dots(D)$$

Here we require to find a solution of (A) satisfying the conditions (B), (C) and (D).

It is obvious from (A) that $u(x, y, t)$ is a function of x, y and t . Therefore, suppose that (A) has a solution of the form.

$$u(x, y, t) = F(x) Y(y) T(t) = FYT(\text{say}) \quad \dots(1)$$

where F is a function of x only, Y a function of y only and T a function of T only.

Differentiating (1) and substituting in (A), we get

$$FY \frac{d^2 T}{dt^2} = c^2 \left(YT \frac{d^2 F}{dx^2} + FT \frac{d^2 Y}{dy^2} \right)$$

or
$$\frac{1}{c^2 T} \frac{d^2 T}{dt^2} = \frac{1}{F} \frac{d^2 F}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2}$$

Here the left hand side is a function of independent variable t alone, while the right hand side is a function of independent variables x and y only. The two cannot be equal to each other unless both sides reduce to a constant values.

$\therefore \frac{1}{c^2 T} \frac{d^2 T}{dt^2} = \frac{1}{F} \frac{d^2 F}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} = \lambda$ (say) where λ is a constant

Again the two terms $\frac{1}{F} \frac{d^2 F}{dx^2}$ and $\frac{1}{Y} \frac{d^2 Y}{dy^2}$ on the right hand side are functions of x and y

respectively and their sum cannot be equal to a constant unless each of these is a constant. Thus, we have following three cases.

Case I : If $\lambda = 0$, then we get the following ordinary differential equations

$$\frac{1}{c^2 T} \frac{d^2 T}{dt^2} = 0, \quad \frac{1}{F} \frac{d^2 F}{dx^2} = 0 \quad \text{and} \quad \frac{1}{Y} \frac{d^2 Y}{dy^2} = 0$$

or
$$\frac{d^2 T}{dt^2} = 0, \quad \frac{d^2 F}{dx^2} = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} = 0$$

whose solutions are

$$F = A_1 x + A_2, Y = A_3 y + A_4 \text{ and } T = A_5 T + A_6$$

$\therefore u(x, y, t) = FYT = (A_1 x + A_2)(A_3 y + A_4)(A_5 T + A_6)$

Case II : If $\lambda = k^2 > 0$, then we get the following ordinary differential equations.

$$\frac{1}{c^2 T} \frac{d^2 T}{dt^2} = k^2, \quad \frac{1}{F} \frac{d^2 F}{dx^2} = k_1^2 \text{ and } \frac{1}{Y} \frac{d^2 Y}{dy^2} = k_2^2$$

where $k^2 = k_1^2 + k_2^2$

or
$$\frac{d^2 T}{dt^2} - k^2 c^2 T = 0, \quad \frac{d^2 F}{dx^2} - k_1^2 F = 0 \text{ and } \frac{d^2 Y}{dy^2} - k_2^2 Y = 0$$

whose solutions are

$$F = B_{k_1} e^{k_1 x} + C_{k_1} e^{-k_1 x}, Y = B_{k_2} e^{k_2 y} + C_{k_2} e^{-k_2 y} \text{ and } T = B_k e^{c k t} + C_k e^{-c k t}$$

$$\therefore u(x, y, z) = FYT = (B_{k_1} e^{k_1 x} + C_{k_1} e^{-k_1 x})(B_{k_2} e^{k_2 y} + C_{k_2} e^{-k_2 y})(B_k e^{c k t} + C_k e^{-c k t})$$

Case III : If $\lambda = -k^2 < 0$, then we get the following ordinary differential equations

$$\frac{1}{c^2 T} \frac{d^2 T}{dt^2} = -k^2, \frac{1}{F} \frac{d^2 F}{dx^2} = -k_1^2 \text{ and } \frac{1}{Y} \frac{d^2 Y}{dy^2} = -k_2^2$$

$$\text{where } k^2 = k_1^2 + k_2^2$$

$$\text{or } \frac{d^2 T}{dt^2} + k^2 c^2 T = 0, \frac{d^2 F}{dx^2} + k_1^2 F = 0 \text{ and } \frac{d^2 Y}{dy^2} + k_2^2 Y = 0$$

whose solutions are

$$F = B_{k_1} \cos k_1 x + C_{k_1} \sin k_1 x, Y = B_{k_2} \cos k_2 y + C_{k_2} \sin k_2 y$$

$$\text{and } T = B_k \cos c k t + C_k \sin c k t$$

$$\begin{aligned} \therefore u(x, y, z) &= FYT \\ &= (B_{k_1} \cos k_1 x + C_{k_1} \sin k_1 x)(B_{k_2} \cos k_2 y + C_{k_2} \sin k_2 y) \\ &\quad (B_k \cos c k t + C_k \sin c k t) \end{aligned}$$

Now from the boundary conditions (B) and (1), we have

$$u(0, y, t) = F(0) Y(y) T(t) = 0$$

$$\text{and } u(a, y, t) = F(a) Y(y) T(t) = 0$$

From which it follows that $F(0) = 0$ and $F(a) = 0$

$\therefore Y(y) = 0$ or $T(t) = 0$ leads to a trivial solution $u(x, y, t) = 0$ which is not admissible.

$$\text{Also } u(x, 0, t) = F(x) Y(0) T(t) = 0$$

$$u(x, b, t) = F(x) Y(b) T(t) = 0$$

From which it follows that $Y(0) = 0$ and $Y(b) = 0$

$\therefore F(x) = 0$ or $T(t) = 0$ leads to a trivial solution $u(x, y, t) = 0$ which is not admissible.

In case I : $F(x) = A_1 x + A_2$

$$\therefore F(0) = 0 \text{ and } F(a) = 0 \Rightarrow A_2 = 0 \text{ and } A_1 a + A_2 = 0 \Rightarrow A_1 = 0 = A_2$$

$$\Rightarrow F(x) = 0$$

which is not admissible as it will lead to a trivial solution $u(x, y, t) = 0$

So we reject the solution in case I.

In case II : $F(x) = B_{k_1} e^{k_1 x} + C_{k_1} e^{-k_1 x}$

$$\therefore F(0) = 0 \text{ and } F(a) = 0 \Rightarrow B_{k_1} + C_{k_1} = 0 \text{ and } B_{k_1} e^{k_1 a} + C_{k_1} e^{-k_1 a} = 0$$

$$\Rightarrow B_{k_1} = 0 = C_{k_1} \Rightarrow F(x) = 0$$

which is not admissible as it also lead to a trivial solution $u(x, y, t) = 0$. So we reject the solution in case II also.

Thus, the solutions given in case I and II do not constitute the required solution of the wave equation (A).

$\therefore$ We take the solution given in case III, which is periodic in time, *i.e.*, we take the solution of the form

$$u(x, y, t) = (B_{k_1} \cos k_1 x + C_{k_1} \sin k_1 x)(B_{k_2} \cos k_2 y + C_{k_2} \sin k_2 y) \\ (B_k \cos ckt + C_k \sin ckt) \quad \dots(2)$$

which will satisfy the differential equation (A).

Now we proceed to find the values of the constants so that the solution (2) satisfy the boundary conditions (B) also.

$$\text{Now } u(0, y, t) = 0$$

$$\Rightarrow (B_{k_1} + 0)(B_{k_2} \cos k_2 y + C_{k_2} \sin k_2 y)(B_k \cos ckt + C_k \sin ckt) = 0 \Rightarrow B_{k_1} = 0$$

$$\therefore u(a, y, t) = 0$$

$$\Rightarrow (0 + C_{k_1} \sin k_1 a)(B_{k_2} \cos k_2 y + C_{k_2} \sin k_2 y)(B_k \cos ckt + C_k \sin ckt) = 0$$

$$\Rightarrow \sin k_1 a = 0 \Rightarrow k_1 a = m\pi \quad \text{or} \quad k_1 = m\pi/a, \quad m \in I.$$

Similarly, using the conditions $u(x, 0, t) = 0$ and $u(x, b, t) = 0$

We get $B_{k_2} = 0$ and $k_2 = n\pi/b, \quad n \in I.$

Hence, from (2) the solutions of equation (A) satisfying the boundary conditions (B) are given by

$$u_{mn}(x, y, t) = (C_{mn} \cos c k_{mn} t + D_{mn} \sin c k_{mn} t) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad \dots(3)$$

where $k^2 = k_{mn}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$ and $m, n \in I.$

Here $C_{mn} = C_{k_1} C_{k_2} B_k$ and $D_{mn} = C_{k_1} C_{k_2} C_k$ are the new arbitrary constants.

Now to obtain a solution which also satisfy the initial conditions (C) and (D), we consider the more general solution.

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn}(x, y, t)$$

$$\text{i.e., } u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [C_{mn} \cos(c k_{mn} t) + D_{mn} \sin(c k_{mn} t)] \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} \quad \dots(4)$$

$$\text{where } k_{mn}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$$

which also satisfy the boundary conditions (B).

$$\text{Now } \frac{\partial u}{\partial t} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c k_{mn} [-C_{mn} \sin(c k_{mn} t) + D_{mn} \cos(c k_{mn} t)] \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b}$$

$\therefore$ Using the initial conditions (C) and (D) we have

$$u(x, y, 0) = f(x, y) \Rightarrow \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} = f(x, y) \quad \dots(5)$$

$$\text{and } \left(\frac{\partial u}{\partial t} \right)_{t=0} = g(x, y) \Rightarrow \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c k_{mn} D_{mn} \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} = g(x, y) \quad \dots(6)$$

The left hand side in (5) and (6) can be considered as the *double Fourier sine series* of the functions on their right hand sides.

$$\therefore C_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_{x=0}^a \int_{y=0}^b f(x, y) \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} dx dy$$

$$\text{or } C_{mn} = \frac{4}{ab} \int_{x=0}^a \int_{y=0}^b f(x, y) \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} dx dy \quad \dots(7)$$

$$\text{and } c k_{mn} D_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_{x=0}^a \int_{y=0}^b g(x, y) \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} dx dy$$

$$\text{or } D_{mn} = \frac{4}{a b c k_{mn}} \int_{x=0}^a \int_{y=0}^b g(x, y) \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} dx dy \quad \dots(8)$$

Hence, the required solution of wave equation (A) is given by (4) where C_{mn} and D_{mn} are given by (7) and (8) respectively.

Particular Case I : If the initial velocity $g(x, y)$ of the membrane is zero *i.e.*, if

$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x, y) = 0$$

then from (8) $D_{mn} = 0$

In this case the solution of wave equation (A) is given by

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \cos(c k_{mn} t) \cdot \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b}$$

where
$$C_{mn} = \frac{4}{ab} \int_{x=0}^a \int_{y=0}^b f(x, y) \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} dx dy$$

and
$$k_{mn}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$$

Particular Case II : If the initial deflection $f(x, y)$ of the membrane is zero *i.e.*, if $u(x, y, 0) = f(x, y) = 0$, then from (7) $C_{mn} = 0$

In this case the solution of wave equation (A) is given by

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} D_{mn} \sin(c k_{mn} t) \cdot \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b}$$

where
$$D_{mn} = \frac{4}{a b c k_{mn}} \int_{x=0}^a \int_{y=0}^b g(x, y) \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{b} dx dy$$

and
$$k_{mn}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$$

Note : The functions given by (3) are called the **eigen functions** or **characteristics functions** and the values of k_{mn} given by $k_{mn} = \pi \sqrt{(m^2/a^2 + n^2/b^2)}$ are called the **eigen values** or **characteristic values** of the vibrating membrane.

12.7 Vibration of a Circular Membrane (Solution of Two Dimensional Wave Equation in Polar Coordinates)

Consider a thin vibrating membrane stretched to a uniform tension T and bounded by a circle $x^2 + y^2 = a^2$. Suppose the entire boundary of the membrane is held fixed for all $t \geq 0$.

The vibrations of the membrane are governed by two dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad c^2 = T/\rho$$

Changing into polars by the substitution $x = r \cos \theta$, $y = r \sin \theta$, the two dimensional wave equation in polar form is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right) \quad \dots(A)$$

Obviously u will be a function of r , θ and t . So $u(r, \theta, t)$ is the deflection of the membrane at the point (r, θ) at any time $t \geq 0$.

Let $f(r, \theta)$ be the initial displacement (*i.e.*, deflection) and $g(r, \theta)$ the initial velocity of the membrane. Then the initial and boundary conditions are as follows.

Boundary conditions :

$$u(a, \theta, t) = 0 \quad \dots(B)$$

$$(-\pi \leq \theta \leq \pi, \quad t \geq 0)$$

Initial conditions :

$$u(r, \theta, 0) = f(r, \theta) \quad \dots(C)$$

$$\text{and} \quad \left(\frac{\partial u}{\partial t} \right)_{t=0} = g(r, \theta) \quad \dots(D)$$

$$(0 < r < a, \quad -\pi \leq \theta \leq \pi)$$

Here we require to find the solution of (A) satisfying the conditions (B), (C) and (D).

Suppose (A) has a solution of the form

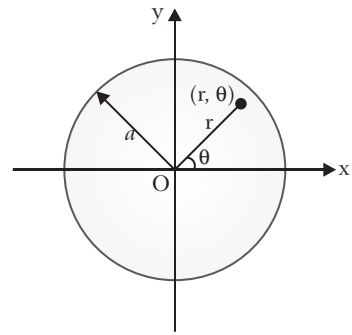


Fig. 12.8

$$u(r, \theta, t) = R(r) F(\theta) T(t) = RFT \text{ (say)} \quad \dots(1)$$

where R is a function of r only, F function of θ only and T function of t only.

Differentiating (1) and substituting in (A), we get

$$RF \frac{d^2 T}{dt^2} = c^2 \left(FT \frac{d^2 R}{dr^2} + \frac{1}{r} FT \frac{dR}{dr} + \frac{1}{r^2} RT \frac{d^2 F}{d\theta^2} \right)$$

or
$$\frac{1}{c^2 T} \frac{d^2 T}{dt^2} = \frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{rR} \frac{dR}{dr} + \frac{1}{r^2} \frac{1}{F} \frac{d^2 F}{d\theta^2}$$

Here left hand side is a function of independent variable t alone, while the right hand side is a function of independent variables R and θ only. The two cannot be equal to each other unless both reduce to a constant value.

$$\therefore \frac{1}{c^2 T} \frac{d^2 T}{dt^2} = \frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{rR} \frac{dR}{dr} + \frac{1}{r^2 F} \frac{d^2 F}{d\theta^2} = -k^2 \text{ (constant)} \quad \dots(2)$$

We separate variable again in equations (2) and write $\frac{1}{F} \frac{d^2 F}{d\theta^2} = -m^2$.

Thus, the ordinary differential equations for R , F and T are

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left(k^2 - \frac{m^2}{r^2} \right) R = 0 \quad \dots(3)$$

$$\frac{d^2 F}{d\theta^2} + m^2 F = 0 \quad \dots(4)$$

and
$$\frac{d^2 T}{dt^2} + c^2 k^2 T = 0 \quad \dots(5)$$

Equation (4) has solution of the form

$$F = Ae^{\pm(im\theta)} \quad m = 0, 1, 2, \dots$$

Introducing the new independent variable $s = kr$ in (3) so that $\frac{1}{r} = \frac{k}{s}$, we have

$$\frac{dR}{dr} = \frac{dR}{ds} \cdot \frac{ds}{dr} = k \frac{dR}{ds} \quad \text{and} \quad \frac{d^2 R}{dr^2} = k^2 \frac{d^2 R}{ds^2}$$

Substituting in (3), it reduces to

$$\frac{d^2 R}{ds^2} + \frac{1}{s} \frac{dR}{ds} + \left(1 - \frac{m^2}{s^2} \right) R = 0$$

which is Bessel's equation, whose general solution is

$$R = C_1 J_m(s) + C_2 Y_m(s)$$

or
$$R = C_1 J_m(kr) + C_2 Y_m(kr)$$

Since the deflection of the membrane is always finite while $Y_m(kr)$ becomes infinite as r approaches zero, we cannot use Y_m and must choose $C_2 = 0$.

$$\therefore R = C_1 J_m(kr)$$

Now using the boundary condition (B),

$$u(a, \theta, t) = R(a) F(\theta) T(t) = 0$$

$\Rightarrow R(a) = 0$ otherwise if $f(\theta) = 0$, or $T(t) = 0$, then $u(r, \theta, t) = 0$, which is a trivial solution and is not admissible.

$$\Rightarrow R(a) = C_1 J_m(ka) = 0$$

$$\therefore J_m(ka) = 0. \quad (\because C_1 \neq 0) \quad \dots(6)$$

Let the positive roots of (6) be k_{mn} , $n = 1, 2, \dots$

The corresponding general solution of (5) is

$$T = a_{mn} \cos(c k_{mn} t) + b_{mn} \sin(c k_{mn} t) \text{ or } T = a'_{mn} e^{\pm i k_{mn} c t}$$

$\therefore$ From (1), the solutions of equation (A), satisfying the boundary conditions (B) are given by

$$u_{mn}(r, \theta, t) = [A_{mn} \cos(c k_{mn} t) + B_{mn} \sin(c k_{mn} t)] e^{\pm i m \theta} J_m(k_{mn} r)$$

$$m, n = 1, 2, 3, \dots$$

where $A C_1 a_{mn} = A_{mn}$ and $A C_1 b_{mn} = B_{mn}$, are new arbitrary constants.

Which can also be written as

$$u_{mn}(r, \theta, t) = C_{mn} J_m(k_{mn} r) \cdot e^{(\pm i m \theta \pm i k_{mn} c t)}$$

where $C_{mn} = a'_{mn} \cdot A C_1$ is arbitrary constant.

The more general solution of (A) is given by

$$u(r, \theta, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn}(r, \theta, t)$$

$$\text{or} \quad u(r, \theta, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [A_{mn} \cos(c k_{mn} t) + B_{mn} \sin(c k_{mn} t)] e^{\pm i m \theta} J_m(k_{mn} r) \quad \dots(7)$$

which can also be written as

$$u(r, \theta, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} J_m(k_{mn} r) \cdot e^{(\pm i m \theta \pm i k_{mn} c t)} \quad \dots(8)$$

These solutions also satisfy the boundary conditions (B).

12.7.1 Symmetrical Case

Considering the solution of the wave equation (A) which are symmetrical radially *i.e.*, when the solution is independent of θ , we get the general solution as

$$u(r, t) = \sum_{n=1}^{\infty} [A_n \cos(c k_n t) + B_n \sin(c k_n t)] J_0(k_n r) \quad \dots(9)$$

$$\text{or} \quad u(r, t) = \sum_{n=1}^{\infty} A_n J_0(k_n r) \cdot e^{\pm i c k_n t}$$

where $k_1, k_2, \dots$ are the positive roots of the equation $J_0(k a) = 0$.

Now from (9) and the initial condition (C) when $t = 0$, we have

$$u(r, 0) = \sum_{n=1}^{\infty} A_n J_0(k_n r) = f(r) \quad f(r, \theta) \text{ becomes } f(r) \text{ when independent of } \theta.$$

Hence A_n must be the coefficients of the Fourier Bessel series which represent $f(r)$ in terms of $J_0(k_n r)$, *i.e.*,

$$A_n = \frac{2}{a^2 J_1^2(k_n a)} \int_0^a r f(r) J_0(k_n r) dr \quad \dots(10)$$

$r = 1, 2, \dots$

Also from (D).

$$\left(\frac{\partial u}{\partial t} \right)_{t=0} = \sum_{n=1}^{\infty} c k_n B_n J_0(k_n r) = g(r)$$

$g(r, \theta)$ becomes $g(r)$ when independent of θ .

$\therefore$ From Fourier Bessel series.

$$c k_n B_n = \frac{2}{a^2 J_1^2(k_n a)} \int_0^a r g(r) J_0(k_n r) dr, \quad n = 1, 2, \dots$$

or
$$B_n = \frac{2}{c k_n a^2 J_1^2(k_n a)} \int_0^a r g(r) J_0(k_n r) dr. \quad \dots(11)$$

Hence (9) is the solution of the wave equation which is radially symmetrical, with the coefficients given by the equations (10) and (11).

Particular Case

If
$$\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(r) = 0$$

then
$$B_n = 0$$

and hence the solution of the wave equation (A) is given by

$$u(r, t) = \sum_{n=1}^{\infty} A_n J_0(k_n r) \cos(c k_n t) \quad \dots(12)$$

where A_n is given by the equation (8).

Solved Examples

Example 1: Find the deflection $u(x, y, t)$ of the square membrane with $a = b = 1$ and $c = 1$, if the initial velocity is zero and the initial deflection is $f(x, y) = A \sin \pi x \sin 2 \pi y$.

[Kurukshetra 2004]

Solution: Here the vibration of the square membrane are governed by two dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \dots(A)$$

$$\therefore c^2 = 1$$

Here the **boundary conditions** are

$$\left. \begin{aligned} u(0, y, t) = 0 \quad u(1, y, t) = 0 \\ u(x, 0, t) = 0 \quad u(x, 1, t) = 0 \end{aligned} \right\} \quad \dots(B)$$

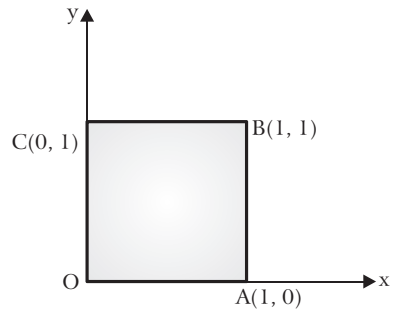


Fig. 12.9

and the **initial conditions** are

$$\begin{aligned} u(x, y, 0) &= f(x, y) \\ &= A \sin \pi x \sin 2 \pi y \end{aligned} \quad \dots(C)$$

and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$... (D)

Let $u(x, y, t) = F(x) Y(y) T(t) = FYT$

be the solution of (A)

Proceeding as in 12.6 particular case I, we have

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \cos(k_{mn}t) \cdot \sin m\pi x \sin n\pi y \quad \dots (1)$$

where $C_{mn} = \frac{4}{1.1} \int_{x=0}^{a=1} \int_{y=0}^{b=1} f(x, y) \sin m\pi x \sin n\pi y \, dx dy$... (2)

and $k_{mn}^2 = \pi^2(m^2/a^2 + n^2/b^2) = \pi^2(m^2 + n^2), \quad \because a = b = 1.$

Now $C_{mn} = 4A \int_{x=0}^1 \int_{y=0}^1 \sin \pi x \sin 2\pi y \cdot \sin(m\pi x) \sin(n\pi y) dy$

or $C_{mn} = 4A \left\{ \int_0^1 \sin \pi x \cdot \sin(m\pi x) dx \cdot \int_0^1 \sin 2\pi y \cdot \sin(n\pi y) dy \right\}$

Obviously $\int_0^1 \sin \pi x \cdot \sin(m\pi x) dx = 0$ if $m \neq 1$

when $m = 1$, $\int_0^1 \sin \pi x \cdot \sin(m\pi x) dx = \int_0^1 \sin^2 \pi x dx$

$$= \frac{1}{2} \int_0^1 (1 - \cos 2\pi x) dx = \frac{1}{2} \left[x - \frac{1}{2\pi} \sin 2\pi x \right]_0^1 = \frac{1}{2}$$

and $\int_0^1 \sin 2\pi y \cdot \sin(n\pi y) dy = 0$ if $n \neq 2$

and when $n = 2$, $\int_0^1 \sin 2\pi y \cdot \sin(n\pi y) dy = \int_0^1 \sin^2 2\pi y dy = \frac{1}{2}$ (As above)

Thus, when $m = 1, n = 2, C_{12} = 4A \cdot \frac{1}{2} \cdot \frac{1}{2} = A$

and $C_{mn} = 0$ (each) when $m \neq 1$ and $n \neq 2$

i.e., $C_{11} = C_{13} = \dots = C_{21} = C_{22} = C_{23} = \dots = 0$ (each)

and $k_{12}^2 = \pi^2(l^2 + 2^2)$ $m = 1, n = 2$, or $k_{12} = \sqrt{5}\pi$

Hence from (1), the required solution is

$$u(x, y, t) = C_{12} \cos(\sqrt{5}\pi t) \sin \pi x \sin(2\pi y)$$

or $u(x, y, t) = A \cos(\sqrt{5}\pi t) \sin \pi x \sin(2\pi y).$

Example 2: Discuss the solution of transverse vibration of a thin membrane bounded by a circle of radius a in xy place described by the function $u(x, y, t)$, satisfying the wave equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

with conditions $u = 0$ or $r = a$ and $u = f(r), \frac{\partial z}{\partial t} = 0$ at $t = 0$.

Solution: Here due to the axial (radial) symmetry, $\frac{\partial^2 u}{\partial \theta^2} = 0$

$\therefore$ the deflection $u(r, t)$ satisfy the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad \dots(A)$$

Here the initial and boundary conditions are

Boundary condition : $u(a, t) = 0 \quad \dots(B)$

and **Initial conditions :** $u(r, 0) = f(r) \quad \dots(C)$

and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(r)$ (say) $\dots(D)$

Thus, we require to find the solution of (A), satisfying the conditions (B), (C) and (D).

Let the solution of (A) be of the form

$$u(r, t) = R(r) T(t) = RT \text{ (say)} \quad \dots(1)$$

where $R(r)$ is a function of r only and $T(t)$ is a function of t only.

Differentiating (1) and substituting in (A), we get

$$T \frac{d^2 R}{dr^2} + \frac{T}{r} \frac{dR}{dr} = \frac{R}{c^2} \frac{d^2 T}{dt^2}$$

or
$$\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{Rr} \frac{dR}{dr} = \frac{1}{c^2 T} \frac{d^2 T}{dt^2}$$

Here left hand side is a function of independent variable r only while right hand side is a function of independent variable t only. The two cannot be equal to each other unless both reduce to a constant value.

$$\therefore \frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{Rr} \frac{dR}{dr} = \frac{1}{c^2 T} \frac{d^2 T}{dt^2} = -k^2 \text{ (constant)}$$

(Negative constant is chosen, so that the solution will be periodic in time)

Thus, the ordinary differential equations for R and T are

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + k^2 R = 0 \quad \dots(2)$$

and
$$\frac{d^2 T}{dt^2} + c^2 k^2 T = 0 \quad \dots(3)$$

To solve (2), we introduce the new independent variable $s = kr$ so that $\frac{1}{r} = \frac{k}{s}$, we have

$$\frac{dR}{dr} = \frac{dR}{ds} \cdot \frac{ds}{dr} = k \frac{dR}{ds} \quad \text{and} \quad \frac{d^2 R}{dr^2} = k^2 \frac{d^2 R}{ds^2}$$

Substituting in (2), it reduces to

$$\frac{d^2 R}{ds^2} + \frac{1}{s} \frac{dR}{ds} + R = 0$$

which is Bessel's equation of zeroth order, whose general solution is

$$R = C_1 J_0(s) + C_2 Y_0(s)$$

or
$$R = C_1 J_0(kr) + C_2 Y_0(kr)$$

Since the deflection of the membrane is always finite while $Y_0(kr)$ becomes infinite as r approaches to zero we cannot use Y_0 and must choose $C_2 = 0$.

$$\therefore R = C_1 J_0(kr)$$

The solution of equation (3) is

$$T = C_3 \cos(ckt) + C_4 \sin(ckt)$$

$\therefore$ from (1), the solution of (A) is of the form

$$u(r, t) = RT = J_0(kr) (C_1 C_3 \cos ckt + C_1 C_4 \sin ckt) \quad \dots(4)$$

Now by the boundary conditions (B) and (4), we have

$$u(a, t) = J_0(ka) (C_1 C_3 \cos ckt + C_1 C_4 \sin ckt) = 0$$

$$\Rightarrow J_0(ka) = 0 \quad \dots(5)$$

$\therefore C_1 C_3 \cos(ckt) + C_1 C_4 \sin(ckt) = 0$ will lead to a trivial solution $u(r, t) = 0$, which is not admissible.

Let $k_n, n = 1, 2, 3, \dots$ be the positive roots of (5).

Hence, from (4) the solution of equation (A) satisfying the boundary conditions (B) are given by

$$u_n(r, t) = [C_n \cos(c k_n t) + D_n \sin(c k_n t)] J_0(k_n r), \quad n = 1, 2, 3, \dots$$

Here $C_n = C_1 C_3, D_n = C_1 C_4$ are the new arbitrary constants.

Now to obtain a solution which also satisfy the initial conditions (C) and (D), we consider the more general solution

$$u(r, t) = \sum_{n=1}^{\infty} u_n(r, t)$$

$$\text{or } u(r, t) = \sum_{n=1}^{\infty} [C_n \cos(c k_n t) + D_n \sin(c k_n t)] J_0(k_n r) \quad \dots(6)$$

Which also satisfy the boundary conditions (B).

$$\text{Now } \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} (c k_n) [-C_n \sin(c k_n t) + D_n \cos(c k_n t)] J_0(k_n r)$$

$$u(r, 0) = f(r) \Rightarrow \sum_{n=1}^{\infty} C_n J_0(k_n r) = f(r) \quad \dots(7)$$

$$\text{and } \left(\frac{\partial u}{\partial t} \right)_{t=0} = g(r) \Rightarrow \sum_{n=1}^{\infty} c k_n D_n J_0(k_n r) = g(r) \quad \dots(8)$$

The left hand sides in (7) and (8) can be considered as the Fourier-Bessel series of the functions on the right hand sides

$$\therefore C_n = \frac{2}{a^2 J_1^2(k_n a)} \int_0^a r f(r) J_0(k_n r) dr \quad \dots(9)$$

and
$$c k_n D_n = \frac{2}{a^2 J_1^2(k_n a)} \int_0^a r g(r) J_0(k_n r) dr$$

or
$$D_n = \frac{2}{c k_n a^2 J_1^2(k_n a)} \int_0^a r g(r) J_0(k_n r) dr \quad \dots(10)$$

Hence the required solution of the wave equation (A) is given by (6) where C_n and D_n are given by (9) and (10) respectively.

Particular Case : If the initial velocity $g(r)$ of the membrane is zero *i.e.*, if

$$\left(\frac{\partial u}{\partial t} \right)_{t=0} = g(r) = 0,$$

then from (10) $D_n = 0$

In this case the solution of wave equation (A) is given by

$$u(r, t) = \sum_{n=1}^{\infty} C_n \cos(c k_n t) \cdot J_0(k_n r)$$

where
$$C_n = \frac{2}{a^2 J_1^2(k_n a)} \int_0^a r f(r) J_0(k_n r) dr$$

Exercise 12.2

1. Solve the differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2}$$

with the condition that $V = 0$ along the lines

$$x = 0, y = 0, x = a, y = a \quad \text{for } t \geq 0.$$

2. Solve the two dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

subject to the following conditions

$$u = 0 \text{ along the lines } x = 0, y = 0, x = a, y = a, \text{ for } t \geq 0$$

and when $t = 0$, $u = f(x, y)$ and $\frac{\partial u}{\partial t} = 0$

for $0 \leq x \leq a$, $0 \leq y \leq a$.

- Find the deflection $u(x, y, t)$ of the square membrane with $a = b = 1$ and $c = 1$, if the initial velocity is zero and the initial deflection is $f(x, y) = A \sin 3 \pi x \cdot \sin 4 \pi y$.
- Discuss the solution of transverse vibrations of a thin membrane bounded by a circle of radius a in xy plane described by the function satisfying

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \frac{\partial z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} = \frac{1}{c^2} \frac{\partial^2 z}{\partial t^2}$$

with the conditions $z = 0$ on $r = a$ and $z = f(r)$, $\left(\frac{\partial z}{\partial r}\right) = 0$ at $t = 0$.

Answers 12.2

1.	$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [C_{mn} \cos(c k_{mn} t) + D_{mn} \sin(c k_{mn} t)] \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{a}$ where $k_{mn}^2 = \pi^2 (m^2 + n^2) / a^2$
2.	$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} \cos(c k_{mn} t) \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{a}$ where $C_{mn} = \frac{4}{a^2} \int_0^a \int_0^a f(x, y) \sin \frac{m \pi x}{a} \sin \frac{n \pi y}{a} dx dy$ and $k_{mn}^2 = \pi^2 (m^2 + n^2) / a^2$
3.	$u(x, y, t) = A \cos(5 \pi t) \sin 3 \pi x \cdot \sin 4 \pi y$
4.	$u(r, t) = \sum_{n=1}^{\infty} C_n J_0(k_n r) \cos(c k_n t)$ where $C_n = \frac{2}{a^2 J_1^2(k_n a)} \int_0^a r f(r) J_0(k_n r) dr$

12.8 Solution of Three Dimensional Wave Equation by the Method of Separation of Variables

Three dimensional wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad \dots(1)$$

$$\text{Let } u(x, y, z, t) = F(x) Y(y) Z(z) T(t) = FYZT \text{ (say)} \quad \dots(2)$$

where F is a function of x alone, Y function of y alone, Z function of z alone and T function of t alone, be the solution of wave equation (1).

Differentiating (2) and substituting in (1), we get

$$\text{or } FYZ \frac{d^2 T}{dt^2} = c^2 \left[YZT \frac{d^2 F}{dx^2} + FZT \frac{d^2 Y}{dy^2} + FYT \frac{d^2 Z}{dz^2} \right]$$

$$\text{or } \frac{1}{c^2 T} \frac{d^2 T}{dt^2} = \frac{1}{F} \frac{d^2 F}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2}$$

Here the left hand side is a function of the independent variable t alone while the right hand side is a function of independent variables x, y and z . The two sides cannot be equal to each other, unless each reduce to a constant value.

$$\therefore \frac{1}{c^2 T} \frac{d^2 T}{dt^2} = \frac{1}{F} \frac{d^2 F}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = -k^2 \text{ (constant)}$$

(Negative constant is chosen so that the solution will be periodic in time)

Also the three terms of right hand side are functions of different independent variable x, y and z respectively. Thus, each of these three must be constant so that their sum is $-k^2$. Taking these constants $-k_1^2, -k_2^2, -k_3^2$, the following ordinary differential equations are obtained.

$$\frac{d^2 T}{dt^2} + c^2 k^2 T = 0, \quad \frac{d^2 F}{dx^2} + k_1^2 F = 0$$

$$\frac{d^2 Y}{dy^2} + k_2^2 Y = 0 \quad \text{and} \quad \frac{d^2 Z}{dz^2} + k_3^2 Z = 0$$

$$\text{where } k^2 = k_1^2 + k_2^2 + k_3^2$$

whose solutions are

$$T = A_k \cos (c k t) + B_k \sin (c k t) \quad \text{or} \quad T = A'_k e^{\pm i c k t}$$

$$F = C_{k_1} \cos k_1 x + D_{k_1} \sin k_1 x \quad \text{or} \quad F = B'_{k_1} e^{\pm i k_1 x}$$

$$Y = E_{k_2} \cos k_2 y + F_{k_2} \sin k_2 y \quad \text{or} \quad Y = C'_{k_2} e^{\pm i k_2 y}$$

$$Z = G_{k_3} \cos k_3 z + H_{k_3} \sin k_3 z \quad \text{or} \quad Z = D'_{k_3} e^{\pm i k_3 z}$$

∴ from (2), the most general solution of the wave equation (1) can be taken as

$$u(x, y, z, t) = \sum_{k_1, k_2, k_3} [\{ A_k \cos (c k t) + B_k \sin (c k t) \} (C_{k_1} \cos k_1 x + D_{k_1} \sin k_1 x)] \\ (E_{k_2} \cos k_2 y + F_{k_2} \sin k_2 y) (G_{k_3} \cos k_3 z + H_{k_3} \sin k_3 z)]$$

The solution can also be taken as

$$u(x, y, z, t) = \sum A'_k B'_{k_1} C'_{k_2} D'_{k_3} e^{\pm i c k t \pm i k_1 x \pm i k_2 y \pm i k_3 z} \\ = \sum h_{k_1 k_2 k_3} e^{\pm i c k t \pm i k_1 x \pm i k_2 y \pm i k_3 z}$$

where $k^2 = k_1^2 + k_2^2 + k_3^2$

and $H_{k_1 k_2 k_3}$ is an arbitrary constant.

12.9 Wave Equation in Cylindrical Coordinates

If (x, y, z) are the cartesian coordinates of a point p and (ρ, ϕ, z) its cylindrical coordinates, then

$$x = \rho \cos \phi, \quad y = \rho \sin \phi \quad \text{and} \quad z = z$$

$$\Rightarrow \quad x^2 + y^2 = \rho^2, \quad \phi = \tan^{-1}(y/x)$$

$$\frac{\partial \rho}{\partial x} = \frac{x}{\rho} = \cos \phi, \quad \frac{\partial \rho}{\partial y} = \frac{y}{\rho} = \sin \phi$$

$$\frac{\partial \phi}{\partial x} = \frac{1}{1 + y^2/x^2} \cdot \left(-\frac{y}{x^2} \right) = \frac{-y}{x^2 + y^2} = \frac{-\sin \phi}{\rho}$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{1 + y^2/x^2} \cdot \left(\frac{1}{x} \right) = \frac{x}{x^2 + y^2} = \frac{\cos \phi}{\rho}$$

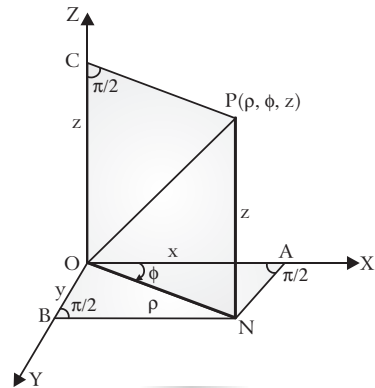


Fig. 12.10

Then
$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial U}{\partial \phi} \cdot \frac{\partial \phi}{\partial x} + \frac{\partial U}{\partial z} \frac{\partial z}{\partial x} = \frac{\partial U}{\partial \rho} \cos \phi + \frac{\partial U}{\partial \phi} \cdot \left(-\frac{\sin \phi}{\rho} \right)$$

$$\therefore \frac{\partial}{\partial x} \equiv \cos \phi \cdot \frac{\partial}{\partial \rho} - \frac{\sin \phi}{\rho} \frac{\partial}{\partial \phi}$$

$$\begin{aligned} \therefore \frac{\partial^2 U}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial U}{\partial x} \right) = \left(\cos \phi \frac{\partial}{\partial \rho} - \frac{\sin \phi}{\rho} \frac{\partial}{\partial \phi} \right) \left(\frac{\partial U}{\partial \rho} \cos \phi - \frac{\sin \phi}{\rho} \frac{\partial U}{\partial \phi} \right) \\ &= \cos \phi \frac{\partial}{\partial \rho} \left(\frac{\partial U}{\partial \rho} \cos \phi - \frac{\sin \phi}{\rho} \frac{\partial U}{\partial \phi} \right) - \frac{\sin \phi}{\rho} \frac{\partial}{\partial \phi} \left(\frac{\partial U}{\partial \rho} \cos \phi - \frac{\sin \phi}{\rho} \frac{\partial U}{\partial \phi} \right) \\ &= \cos \phi \left(\frac{\partial^2 U}{\partial \rho^2} \cos \phi + \frac{\sin \phi}{\rho^2} \frac{\partial U}{\partial \phi} - \frac{\sin \phi}{\rho} \frac{\partial^2 U}{\partial \rho \partial \phi} \right) \\ &\quad - \frac{\sin \phi}{\rho} \left(\frac{\partial^2 U}{\partial \phi \partial \rho} \cos \phi - \frac{\partial U}{\partial \rho} \sin \phi - \frac{\cos \phi}{\rho} \frac{\partial U}{\partial \phi} - \frac{\sin \phi}{\rho} \frac{\partial^2 U}{\partial \phi^2} \right) \\ &= \cos^2 \phi \frac{\partial^2 U}{\partial \rho^2} - 2 \frac{\sin \phi \cos \phi}{\rho} \frac{\partial^2 U}{\partial \rho \partial \phi} + 2 \frac{\sin \phi \cos \phi}{\rho^2} \frac{\partial U}{\partial \phi} \\ &\quad + \frac{\sin^2 \phi}{\rho} \frac{\partial U}{\partial \rho} + \frac{\sin^2 \phi}{\rho^2} \frac{\partial^2 U}{\partial \phi^2} \quad \dots(1) \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial^2 U}{\partial y^2} &= \sin^2 \phi \frac{\partial^2 U}{\partial \rho^2} + \frac{2 \cos \phi \sin \phi}{\rho} \frac{\partial^2 U}{\partial \rho \partial \phi} - \frac{2 \cos \phi \sin \phi}{\rho^2} \frac{\partial U}{\partial \phi} \\ &\quad + \frac{\cos^2 \phi}{\rho} \frac{\partial U}{\partial \rho} + \frac{\cos^2 \phi}{\rho^2} \frac{\partial^2 U}{\partial \phi^2} \quad \dots(2) \end{aligned}$$

and
$$\frac{\partial^2 U}{\partial z^2} = \frac{\partial^2 U}{\partial z^2} \quad \dots(3)$$

Adding (1), (2) and (3)

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial U}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} \quad \dots(4)$$

Hence the wave equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2}$$

is transformed to the equation

$$\frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial U}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} \quad \text{in cylindrical coordinates.}$$

12.10 Solution of Wave Equation in Cylindrical Coordinates by the Method of Separation of Variables

The wave equation in cylindrical coordinates (ρ, ϕ, z) is given by

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad \dots(1)$$

Let the solution of (1) be of the form

$$u(\rho, \phi, z, t) = R(\rho) F(\phi) Z(z) T(t) = RFZT \text{ (say)} \quad \dots(2)$$

where R is a function of ρ alone, F function of ϕ alone, Z function of z alone and T function of t alone.

Differentiating (2) and substituting in (1), we get

$$FZT \frac{d^2 R}{d\rho^2} + \frac{FZT}{\rho} \frac{dR}{d\rho} + \frac{RZT}{\rho^2} \frac{d^2 F}{d\phi^2} + RFT \frac{d^2 Z}{dz^2} = \frac{RFZ}{c^2} \frac{d^2 T}{dt^2}$$

or

$$\frac{1}{R} \frac{d^2 R}{d\rho^2} + \frac{1}{\rho R} \frac{dR}{d\rho} + \frac{1}{\rho^2} \frac{1}{F} \frac{d^2 F}{d\phi^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = \frac{1}{c^2 T} \frac{d^2 T}{dt^2} \quad \dots(3)$$

Here in (3) the right hand side is a function of independent variable t alone while left hand side is a function of independent variables ρ, ϕ and z only. The two sides cannot be equal to each other, unless each reduce to a constant say $-k^2$. Then from (3), we get

$$\frac{1}{c^2 T} \frac{d^2 T}{dt^2} = -k^2 \quad \text{or} \quad \frac{d^2 T}{dt^2} + c^2 k^2 T = 0 \quad \dots(4)$$

and

$$\frac{1}{R} \frac{d^2 R}{d\rho^2} + \frac{1}{\rho R} \frac{dR}{d\rho} + \frac{1}{\rho^2} \cdot \frac{1}{F} \frac{d^2 F}{d\phi^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = -k^2$$

or

$$\frac{1}{R} \frac{d^2 R}{d\rho^2} + \frac{1}{\rho R} \frac{dR}{d\rho} + \frac{1}{\rho^2 F} \frac{d^2 F}{d\phi^2} + k^2 = -\frac{1}{Z} \frac{d^2 Z}{dz^2} \quad \dots(5)$$

Here in (5), the right hand side is a function of independent variable z alone while the left hand side is a function of independent variables ρ and ϕ . Therefore the two sides will be equal to each other only if both reduce to the same constant say λ^2 .

$\therefore$ From (5) we get

$$-\frac{1}{Z} \frac{d^2 Z}{dz^2} = \lambda^2 \quad \text{or} \quad \frac{d^2 Z}{dz^2} + \lambda^2 Z = 0 \quad \dots(6)$$

and
$$\frac{1}{R} \frac{d^2 R}{d\rho^2} + \frac{1}{\rho R} \frac{dR}{d\rho} + \frac{1}{\rho^2 F} \frac{d^2 F}{d\phi^2} + k^2 = \lambda^2$$

or
$$\frac{\rho^2}{R} \frac{d^2 R}{d\rho^2} + \frac{\rho}{R} \frac{dR}{d\rho} + (k^2 - \lambda^2) \rho^2 = -\frac{1}{F} \frac{d^2 F}{d\phi^2} \quad \dots(7)$$

Again in (7) right hand side is function of independent variable ϕ alone and left hand side is function of independent variable ρ , therefore, both sides will be equal to each other if both are equal to the same constant values say m^2 .

$\therefore$ from (7), we have

$$-\frac{1}{F} \frac{d^2 F}{d\phi^2} = m^2 \quad \text{or} \quad \frac{d^2 F}{d\phi^2} + m^2 F = 0 \quad \dots(8)$$

and
$$\frac{\rho^2}{R} \frac{d^2 R}{d\rho^2} + \frac{\rho}{R} \frac{dR}{d\rho} + \gamma^2 \rho^2 = m^2 \quad \text{where } k^2 - \lambda^2 = \gamma^2 (\text{constant})$$

or
$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + (\gamma^2 - m^2/\rho^2) R = 0 \quad \dots(9)$$

Introducing the new variable $s = \gamma\rho$, so that

$$\frac{dR}{d\rho} = \frac{dR}{ds} \cdot \frac{ds}{d\rho} = \gamma \frac{dR}{ds} \quad \text{and} \quad \frac{d^2 R}{d\rho^2} = \gamma^2 \frac{d^2 R}{ds^2}$$

Substituting in (9), it reduces to

$$\frac{d^2 R}{ds^2} + \frac{1}{s} \frac{dR}{ds} + \left(1 - \frac{m^2}{s^2}\right) R = 0 \quad \dots(10)$$

Which is Bessel's equation.

Now solution of (4) is $T = T(t) = A_k \cos(c k t) + B_k \sin(c k t) \quad \text{or} \quad A'_k e^{\pm i c k t}$

Solution of (6) is $Z = Z(z) = C_\lambda \cos \lambda z + D_\lambda \sin \lambda z \quad \text{or} \quad C'_\lambda e^{\pm i \lambda z}$

Solution of (8) is $F = F(\phi) = E_m \cos m \phi + F_m \sin m \phi \quad \text{or} \quad E'_m e^{\pm i m \phi}$

Solution of (10) is
$$R = R(r) = G_{k\lambda m} J_m(s) + H_{k\lambda m} Y_m(s)$$

$$= G_{k\lambda m} J_m(\gamma\rho) + H_{k\lambda m} Y_m(\gamma\rho) \quad \text{when } m \text{ is an integer}$$

and

$$R = R(r) = G_{k\lambda m} J_m(\gamma\rho) + H_{k\lambda m} J_{-m}(\gamma\rho)$$

when m is not an integer

Hence from (2), the more general solution of wave equation (10) in cylindrical coordinates is

$$u(\rho, \phi, z, t) = \sum_{k, \lambda, m} [(A_k \cos c k t + B_k \sin c k t)(C_\lambda \cos \lambda z + D_\lambda \sin \lambda z).$$

$$(E_m \cos m\phi + F_m \sin m\phi).(G_{k\lambda m} J_m(\gamma\rho) + H_{k\lambda m} Y_m(\gamma\rho)] \quad \text{when } m \text{ is an integer}$$

or
$$u(\rho, \phi, z, t) = \sum_{k, \lambda, m} [(A_k \cos c k t + B_k \sin c k t)(C_\lambda \cos \lambda z + D_\lambda \sin \lambda z).$$

$$(E_m \cos m\phi + F_m \sin m\phi)(G_{k\lambda m} J_m(\gamma\rho) + H_{k\lambda m} J_{-m}(\gamma\rho)]$$

where $\gamma^2 = k^2 - \lambda^2$, when m is not an integer.

These solution can also be taken as

$$u(\rho, \phi, z, t) = \sum_{k, \lambda, m} [A_{k\lambda m} J_m(\gamma\rho) + B_{k\lambda m} Y_m(\gamma\rho)] e^{\pm i c k t \pm i \lambda z \pm i m \phi}$$

when m is an integer

or
$$u(\rho, \phi, z, t) = \sum_{k, \lambda, m} [A'_{k\lambda m} J_m(\gamma\rho) + B'_{k\lambda m} J_{-m}(\gamma\rho)]. e^{\pm i c k t \pm i \lambda z \pm i m \phi}$$

when m is not an integer.

Particular Case : As $\rho \rightarrow 0$ (along the z-axis), $Y_m(\gamma\rho) \rightarrow \infty$, $J_{-m}(\gamma\rho) \rightarrow \infty$

$\therefore$ To get a finite values of $u(\rho, \phi, z, t)$ along the z-axis, we choose $B_{k\lambda m} = 0$, $B'_{k\lambda m} = 0$

Hence, the finite solution $u(\rho, \phi, z, t)$ is given by

$$u(\rho, \phi, z, t) = \sum_{k, \lambda, m} A_{k\lambda m} J_m(\lambda\rho).(A_k \cos c k t + B_k \sin c k t).(C_\lambda \cos \lambda z + D_\lambda \sin \lambda z).$$

$$(E_m \cos m\phi + F_m \sin m\phi)$$

or
$$u(\rho, \phi, z, t) = \sum_{k, \lambda, m} A'_{k\lambda m} J_m(\gamma\rho) e^{\pm i c k t \pm i \lambda z \pm i m \phi}$$

where $\gamma^2 = k^2 - \lambda^2$

12.11 Wave Equation in Spherical Coordinates

If (x, y, z) are the cartesian coordinates of a point P and (r, θ, ϕ) its spherical coordinates then

$$x = OA = ON \cos \phi = r \sin \theta \cos \phi$$

$$y = OB = AN = ON \sin \phi = r \sin \theta \sin \phi$$

$$z = OC = NP = r \cos \theta$$

$$\therefore r^2 = x^2 + y^2 + z^2, \tan \phi = \frac{y}{x} \text{ i.e., } \phi = \tan^{-1} \frac{y}{x}$$

$$\text{and } \tan \theta = \frac{\sqrt{(x^2 + y^2)}}{z} \text{ i.e., } \theta = \tan^{-1} \frac{\sqrt{(x^2 + y^2)}}{z}$$

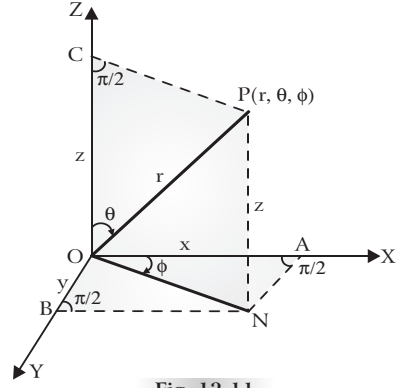


Fig. 12.11

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r} = \sin \theta \cos \phi, \frac{\partial r}{\partial y} = \frac{y}{r} = \sin \theta \sin \phi, \frac{\partial r}{\partial z} = \frac{z}{r} = \cos \theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + (x^2 + y^2)/z^2} \cdot \frac{2x}{2z\sqrt{(x^2 + y^2)}} = \frac{xz}{(x^2 + y^2 + z^2)\sqrt{(x^2 + y^2)}} = \frac{\cos \theta \cos \phi}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + (x^2 + y^2)/z^2} \cdot \frac{2y}{2z\sqrt{(x^2 + y^2)}} = \frac{yz}{(x^2 + y^2 + z^2)\sqrt{(x^2 + y^2)}} = \frac{\cos \theta \sin \phi}{r}$$

$$\frac{\partial \theta}{\partial z} = \frac{1}{1 + (x^2 + y^2)/z^2} \cdot \left(\frac{-\sqrt{(x^2 + y^2)}}{z^2} \right) = -\frac{\sqrt{(x^2 + y^2)}}{x^2 + y^2 + z^2} = -\frac{\sin \theta}{r}$$

$$\frac{\partial \phi}{\partial x} = \frac{1}{1 + y^2/x^2} \left(-\frac{y}{x^2} \right) = \frac{-y}{x^2 + y^2} = \frac{-\sin \phi}{r \sin \theta}$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{1 + y^2/x^2} \left(\frac{1}{x} \right) = \frac{x}{x^2 + y^2} = \frac{\cos \phi}{r \sin \theta}, \quad \frac{\partial \phi}{\partial z} = 0$$

$$\begin{aligned} \text{Then } \frac{\partial U}{\partial x} &= \frac{\partial U}{\partial r} \cdot \frac{dr}{dx} + \frac{\partial U}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} + \frac{\partial U}{\partial \phi} \cdot \frac{\partial \phi}{\partial x} \\ &= \frac{\partial U}{\partial r} \cdot \sin \theta \cos \phi + \frac{\partial U}{\partial \theta} \cdot \frac{\cos \theta \cos \phi}{r} + \frac{\partial U}{\partial \phi} \cdot \left(-\frac{\sin \phi}{r \sin \theta} \right) \end{aligned}$$

$$\therefore \frac{\partial}{\partial x} \equiv \sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{\cos \theta \cos \phi}{r} \frac{\partial}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \phi}$$

$$\therefore \frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial U}{\partial x} \right)$$

$$\begin{aligned}
&= \left(\sin \theta \cos \phi \frac{\partial}{\partial r} + \frac{\cos \theta \cos \phi}{r} \frac{\partial}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \\
&\quad \left(\sin \theta \cos \phi \frac{\partial U}{\partial r} + \frac{\cos \theta \cos \phi}{r} \frac{\partial U}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial U}{\partial \phi} \right) \\
&= \sin \theta \cos \phi \frac{\partial}{\partial r} \left(\sin \theta \cos \phi \frac{\partial U}{\partial r} + \frac{\cos \theta \cos \phi}{r} \frac{\partial U}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial U}{\partial \phi} \right) \\
&\quad + \frac{\cos \theta \cos \phi}{r} \frac{\partial}{\partial \theta} \left(\sin \theta \cos \phi \frac{\partial U}{\partial r} + \frac{\cos \theta \cos \phi}{r} \frac{\partial U}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial U}{\partial \phi} \right) \\
&\quad - \frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \left(\sin \theta \cos \phi \frac{\partial U}{\partial r} + \frac{\cos \theta \cos \phi}{r} \frac{\partial U}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial U}{\partial \phi} \right) \\
&= \sin \theta \cos \phi \left[\sin \theta \cos \phi \frac{\partial^2 U}{\partial r^2} + \frac{\cos \theta \cos \phi}{r} \frac{\partial^2 U}{\partial r \partial \theta} - \frac{\cos \theta \cos \phi}{r^2} \frac{\partial U}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial^2 U}{\partial r \partial \phi} \right. \\
&\quad \left. + \frac{\sin \phi}{r^2 \sin \theta} \frac{\partial U}{\partial \phi} \right] + \frac{\cos \theta \cos \phi}{r} \left[\sin \theta \cos \phi \frac{\partial^2 U}{\partial \theta \partial r} + \cos \theta \cos \phi \frac{\partial^2 U}{\partial r^2} \right. \\
&\quad \left. + \frac{\cos \theta \cos \phi}{r} \frac{\partial^2 U}{\partial \theta^2} - \frac{\sin \theta \cos \phi}{r} \frac{\partial U}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial^2 U}{\partial \theta \partial \phi} + \frac{\sin \phi \cdot \cos \theta}{r \sin^2 \theta} \frac{\partial U}{\partial \phi} \right] \\
&\quad - \frac{\sin \phi}{r \sin \theta} \left[\sin \theta \cos \phi \frac{\partial^2 U}{\partial \phi \partial r} - \sin \theta \sin \phi \frac{\partial U}{\partial r} + \frac{\cos \theta \cos \phi}{r} \frac{\partial^2 U}{\partial \phi \partial \theta} \right. \\
&\quad \left. - \frac{\cos \theta \sin \phi}{r} \frac{\partial U}{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial^2 U}{\partial \phi^2} - \frac{\cos \phi}{r \sin \theta} \frac{\partial U}{\partial \phi} \right] \\
&= \sin^2 \theta \cos^2 \phi \frac{\partial^2 U}{\partial r^2} + 2 \frac{\sin \theta \cos \theta \cos^2 \phi}{r} \frac{\partial^2 U}{\partial r \partial \theta} \\
&\quad - 2 \frac{\sin \theta \cos \theta \cos^2 \phi}{r^2} \frac{\partial U}{\partial \theta} - 2 \frac{\sin \phi \cos \phi}{r} \frac{\partial^2 U}{\partial r \partial \phi} + \frac{\sin \phi \cos \phi}{r^2} \frac{\partial U}{\partial \phi} \\
&\quad + \frac{\cos^2 \theta \cos^2 \phi}{r} \frac{\partial U}{\partial r} + \frac{\cos^2 \theta \cos^2 \phi}{r^2} \frac{\partial^2 U}{\partial \theta^2} - 2 \frac{\cos \theta \sin \phi \cos \phi}{r^2 \sin \theta} \frac{\partial^2 U}{\partial \theta \partial \phi} \\
&\quad + \frac{\cos^2 \theta \sin \phi \cos \phi}{r^2 \sin^2 \theta} \frac{\partial U}{\partial \phi} + \frac{\sin^2 \phi}{r} \frac{\partial U}{\partial r} + \frac{\cos \theta \sin^2 \phi}{r^2 \sin \theta} \frac{\partial U}{\partial \theta} \\
&\quad + \frac{\sin^2 \phi}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \phi^2} + \frac{\sin \phi \cos \phi}{r^2 \sin^2 \theta} \frac{\partial U}{\partial \phi} \dots (1)
\end{aligned}$$

Again $\frac{\partial U}{\partial y} = \frac{\partial U}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial U}{\partial \theta} \cdot \frac{\partial \theta}{\partial y} + \frac{\partial U}{\partial \phi} \cdot \frac{\partial \phi}{\partial y}$

$$= \frac{\partial U}{\partial r} \cdot \sin \theta \sin \phi + \frac{\partial U}{\partial \theta} \cdot \frac{\cos \theta \sin \phi}{r} + \frac{\partial U}{\partial \phi} \cdot \frac{\cos \phi}{r \sin \theta}$$

$$\begin{aligned}
 \therefore \quad \frac{\partial}{\partial y} &\equiv \sin \theta \sin \phi \frac{\partial}{\partial r} + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} + \frac{\cos \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \\
 \therefore \quad \frac{\partial^2 U}{\partial y^2} &= \left(\sin \theta \sin \phi \frac{\partial}{\partial r} + \frac{\cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} + \frac{\cos \phi}{r \sin \theta} \frac{\partial}{\partial \phi} \right) \\
 &\quad \left(\sin \theta \sin \phi \frac{\partial U}{\partial r} + \frac{\cos \theta \sin \phi}{r} \frac{\partial U}{\partial \theta} + \frac{\cos \phi}{r \sin \theta} \frac{\partial U}{\partial \phi} \right) \\
 &= \sin^2 \theta \sin^2 \phi \frac{\partial^2 U}{\partial r^2} + 2 \frac{\sin \theta \cos \theta \sin^2 \phi}{r} \frac{\partial^2 U}{\partial r \partial \theta} \\
 &\quad - 2 \frac{\sin \theta \cos \theta \sin^2 \phi}{r^2} \frac{\partial U}{\partial \theta} + 2 \frac{\sin \phi \cos \phi}{r} \frac{\partial^2 U}{\partial r \partial \phi} - \frac{\sin \phi \cos \phi}{r^2} \frac{\partial U}{\partial \phi} \\
 &\quad + \frac{\cos^2 \theta \sin^2 \phi}{r} \frac{\partial U}{\partial r} + \frac{\cos^2 \theta \sin^2 \phi}{r^2} \frac{\partial^2 U}{\partial \theta^2} + 2 \frac{\cos \theta \sin \phi \cos \phi}{r^2 \sin \theta} \frac{\partial^2 U}{\partial \theta \partial \phi} \\
 &\quad - \frac{\cos^2 \theta \sin \phi \cos \phi}{r^2 \sin^2 \theta} \frac{\partial U}{\partial \phi} + \frac{\cos^2 \phi}{r} \frac{\partial U}{\partial r} + \frac{\cos \theta \cos^2 \phi}{r^2 \sin \theta} \frac{\partial U}{\partial \theta} \\
 &\quad + \frac{\cos^2 \phi}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \phi^2} - \frac{\sin \phi \cos \phi}{r^2 \sin^2 \theta} \frac{\partial U}{\partial \phi} \dots (2)
 \end{aligned}$$

$$\begin{aligned}
 \text{Also } \frac{\partial U}{\partial z} &= \frac{\partial U}{\partial r} \cdot \frac{\partial r}{\partial z} + \frac{\partial U}{\partial \theta} \cdot \frac{\partial \theta}{\partial z} + \frac{\partial U}{\partial \phi} \cdot \frac{\partial \phi}{\partial z} \\
 &= \frac{\partial U}{\partial r} \cdot \cos \theta + \frac{\partial U}{\partial \theta} \left(-\frac{\sin \theta}{r} \right) \quad \therefore \frac{\partial}{\partial z} \equiv \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \quad \frac{\partial^2 U}{\partial z^2} &= \left(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) \left(\cos \theta \frac{\partial U}{\partial r} - \frac{\sin \theta}{r} \frac{\partial U}{\partial \theta} \right) \\
 &= \cos \theta \frac{\partial}{\partial r} \left(\cos \theta \frac{\partial U}{\partial r} - \frac{\sin \theta}{r} \frac{\partial U}{\partial \theta} \right) - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\cos \theta \frac{\partial U}{\partial r} - \frac{\sin \theta}{r} \frac{\partial U}{\partial \theta} \right) \\
 &= \cos^2 \theta \frac{\partial^2 U}{\partial r^2} - 2 \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 U}{\partial r \partial \theta} + 2 \frac{\sin \theta \cos \theta}{r^2} \frac{\partial U}{\partial \theta} + \frac{\sin^2 \theta}{r} \frac{\partial U}{\partial r} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 U}{\partial \theta^2} \\
 &\quad \dots (3)
 \end{aligned}$$

Adding (1), (2) and (3),

$$\begin{aligned}
 \nabla^2 U &= \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \\
 &= \frac{\partial^2 U}{\partial r^2} + \frac{2}{r} \frac{\partial U}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial U}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \phi^2}
 \end{aligned}$$

Hence the wave equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2}$$

is transformed to the equation

$$\frac{\partial^2 U}{\partial r^2} + \frac{2}{r} \frac{\partial U}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial U}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \phi^2} = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2}$$

in spherical coordinates.

12.12 Solution of Wave Equation in Spherical Polar Coordinates by the Method of Separation of Variables

The wave equation in spherical polar coordinates (r, θ, ϕ) is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad \dots(1)$$

Let the solution of (1) be of the form

$$u(r, \theta, \phi, t) = R(r) F_1(\theta) F_2(\phi) T(t) = R F_1 F_2 T(\text{say}) \quad \dots(2)$$

where R is a function of r alone, F_1 function of θ alone, F_2 function of ϕ alone and T function of t alone.

Differentiating (2) and substituting in (1), we get

$$F_1 F_2 T \frac{d^2 R}{dr^2} + \frac{2}{r} F_1 F_2 T \frac{dR}{dr} + \frac{R F_2 T}{r^2} \frac{d^2 F_1}{d\theta^2} + \frac{\cot \theta}{r^2} R F_2 T \frac{dF_1}{d\theta} + \frac{R F_1 T}{r^2 \sin^2 \theta} \frac{d^2 F_2}{d\phi^2} = \frac{R F_1 F_2}{c^2} \frac{d^2 T}{dt^2}$$

$$\text{or} \quad \frac{1}{R} \frac{d^2 R}{dr^2} + \frac{2}{r} \frac{1}{R} \frac{dR}{dr} + \frac{1}{r^2} \frac{1}{F_1} \frac{d^2 F_1}{d\theta^2} + \frac{\cot \theta}{r^2} \frac{1}{F_1} \frac{dF_1}{d\theta} + \frac{1}{r^2 \sin^2 \theta} \frac{1}{F_2} \frac{d^2 F_2}{d\phi^2} = \frac{1}{c^2 T} \frac{d^2 T}{dt^2} = -k^2 \text{ (say)} \quad \dots(3)$$

Here in (3), the right hand side is a function of independent variable t alone while left hand side is a function of other independent variables r, θ, ϕ only. Therefore, the two sides will be equal to each other only if both sides are equal to the same constant values say $-k^2$. Then from (3), we get

$$\frac{1}{c^2 T} \frac{d^2 T}{dt^2} = -k^2 \quad \text{or} \quad \frac{d^2 T}{dt^2} + c^2 k^2 T = 0 \quad \dots(4)$$

$$\text{and} \quad \frac{1}{R} \frac{d^2 R}{dr^2} + \frac{2}{r} \frac{1}{R} \frac{dR}{dr} + \frac{1}{r^2} \cdot \frac{1}{F_1} \frac{d^2 F_1}{d\theta^2} + \frac{\cot \theta}{r^2} \cdot \frac{1}{F_1} \frac{dF_1}{d\theta} + \frac{1}{r^2 \sin^2 \theta} \cdot \frac{1}{F_2} \frac{d^2 F_2}{d\phi^2} = -k^2$$

$$\begin{aligned} \text{or} \quad & \left[\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{2}{rR} \frac{dR}{dr} + \frac{1}{r^2 F_1} \frac{d^2 F_1}{d\theta^2} + \frac{\cot \theta}{r^2} \cdot \frac{1}{F_1} \frac{dF_1}{d\theta} + k^2 \right] r^2 \sin^2 \theta \\ & = -\frac{1}{F_2} \frac{d^2 F_2}{d\phi^2} = m^2 \quad \dots(5) \end{aligned}$$

Here in (5), the right hand side is a function of independent variable ϕ while the left hand side is a function of other independent variables r and θ . Therefore, the two sides will be equal to each other only if both reduce to the same constant values say m^2 .

$\therefore$ from (5), we get

$$-\frac{1}{F_2} \frac{d^2 F_2}{d\phi^2} = m^2 \quad \text{or} \quad \frac{d^2 F_2}{d\phi^2} + m^2 F_2 = 0 \quad \dots(6)$$

$$\begin{aligned} \text{and} \quad & \left[\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{2}{rR} \frac{dR}{dr} + \frac{1}{r^2 F_1} \frac{d^2 F_1}{d\theta^2} + \frac{\cot \theta}{r^2} \cdot \frac{1}{F_1} \frac{dF_1}{d\theta} + k^2 \right] r^2 \sin^2 \theta = m^2 \\ & \frac{r^2}{R} \frac{d^2 R}{dr^2} + \frac{2r}{R} \frac{dR}{dr} + k^2 r^2 = - \left[\frac{1}{F_1} \frac{d^2 F_1}{d\theta^2} + \frac{\cot \theta}{F_1} \frac{dF_1}{d\theta} - \frac{m^2}{\sin^2 \theta} \right] \\ & = n(n+1) \text{ say} \quad \dots(7) \end{aligned}$$

In (7), left hand side is a function of independent variable r alone while right hand side is a function of other independent variable θ . Therefore, the two will be equal to each other only if both reduce to the same constant values say $n(n+1)$. Then from (7), we get

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} + \{k^2 r^2 - n(n+1)\} R = 0$$

$$\text{or} \quad \frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} + \left\{ k^2 - \frac{n(n+1)}{r^2} \right\} R = 0 \quad \dots(8)$$

$$\text{and} \quad \frac{d^2 F_1}{d\theta^2} + \cot \theta \frac{dF_1}{d\theta} + \left[n(n+1) - \frac{m^2}{\sin^2 \theta} \right] F_1 = 0 \quad \dots(9)$$

Now let $\cos \theta = \mu$,

$$\therefore \quad \frac{dF_1}{d\theta} = \frac{dF_1}{d\mu} \cdot \frac{d\mu}{d\theta} = (-\sin \theta) \frac{dF_1}{d\mu}$$

$$\begin{aligned} \text{and } \frac{d^2 F_1}{d\theta^2} &= \frac{d}{d\theta} \left(-\sin \theta \frac{dF_1}{d\mu} \right) = -\cos \theta \frac{dF_1}{d\mu} - \sin \theta \cdot \frac{d^2 F_1}{d\mu^2} \cdot \frac{d\mu}{d\theta} \\ &= -\cos \theta \frac{dF_1}{d\mu} + \sin^2 \theta \frac{d^2 F_1}{d\mu^2} = -\mu \frac{dF_1}{d\mu} + (1-\mu^2) \frac{d^2 F_1}{d\mu^2} \end{aligned}$$

Putting in (9), we get

$$(1-\mu^2) \frac{d^2 F_1}{d\mu^2} - \mu \frac{dF_1}{d\mu} + \cot \theta (-\sin \theta) \frac{dF_1}{d\mu} + \left\{ n(n-1) - \frac{m^2}{1-\mu^2} \right\} F_1 = 0$$

$$\text{or } (1-\mu^2) \frac{d^2 F_1}{d\mu^2} - 2\mu \frac{dF_1}{d\mu} + \left\{ n(n-1) - \frac{m^2}{1-\mu^2} \right\} F_1 = 0 \quad \dots(10)$$

which is associated Legendre's equation.

Now, solution of (4) is

$$T = T(t) = A_k \cos ckt + B_k \sin ckt \quad \text{or } A' e^{\pm ickt} \quad \dots(11)$$

Solution of (6) is

$$F_2 = F_2(\phi) = C_m \cos m\phi + D_m \sin m\phi \quad \text{or } C'_m e^{\pm im\phi} \quad \dots(12)$$

To find solution of (8). Let $R = (kr)^{-1/2} \psi$

$$\begin{aligned} \therefore \frac{dR}{dr} &= (kr)^{-1/2} \frac{d\psi}{dr} - \frac{1}{2} \cdot k^{-1/2} r^{-3/2} \psi, \\ \frac{d^2 R}{dr^2} &= (kr)^{-1/2} \frac{d^2 \psi}{dr^2} - \frac{2}{2} k^{-1/2} r^{-3/2} \frac{d\psi}{dr} + \frac{3}{4} k^{-1/2} r^{-5/2} \psi \end{aligned}$$

Putting in (8), it reduces to

$$\frac{d^2 \psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} + \left\{ k^2 - \frac{(n+1/2)^2}{r^2} \right\} \psi = 0$$

whose solution is

$$\psi = E_{kn} J_{n+1/2}(kr) + F_{kn} J_{-(n+1/2)}(kr)$$

where $n+1/2$ is neither zero nor an integer.

$$\therefore R = R(r) = (kr)^{-1/2} [E_{kn} J_{n+1/2}(kr) + F_{kn} J_{-(n+1/2)}(kr)] \quad \dots(13)$$

$$\text{or } R = E'_{kn} (kr)^{-1/2} J_{\pm(n+1/2)}(kr)$$

and solution of (10) is,

$$F_1 = F_1(\theta) = G_{mn} P_n^m(\mu) + H_{mn} Q_{nm}(\mu) = G_{mn} P_n^m(\cos \theta) + H_{mn} Q_n^m(\cos \theta) \dots (14)$$

Hence, from (2), for all values of k, m, n the solution of the wave equation (1) in spherical polar coordinates are given by

$$u(r, \theta, \phi, t) = R(r) F_1(\theta) F_2(\phi) T(t)$$

where $R(r)$, $F_1(\theta)$, $F_2(\phi)$ and $T(t)$ are given by (11), (12), (13) and (14).

Hence the more general solution is given by

$$\begin{aligned} u(r, \theta, \phi, t) &= \sum_{k,m,n} R(r) F_1(\theta) F_2(\phi) T(t) \\ &= \sum_{k,m,n} (kr)^{-1/2} [(A_k \cos ckt + B_k \sin ckt) \cdot (C_m \cos m\phi + D_m \sin m\phi) \cdot \\ &\quad \{E_{kn} J_{n+1/2}(kr) + F_{kn} J_{-(n+1/2)}(kr)\} \{G_{mn} P_n^m(\cos \theta) + H_{mn} Q_n^m(\cos \theta)\}] \end{aligned}$$

where $n+1/2$ is neither zero nor an integer.

Which can also be written as

$$\begin{aligned} u(r, \theta, \phi, t) &= \sum_{k,m,n} (kr)^{-1/2} K_{k,m,n} [G_{mn} P_n^m(\cos \theta) \\ &\quad + H_{mn} Q_n^m(\cos \theta)] J_{\pm(n+1/2)}(kr) e^{\pm im\phi \pm ickt} \end{aligned}$$

Solved Examples

Example 1: A gas is contained in a cubical box of side a . Show that if c is the velocity of the sound in the gas, the period of free oscillations are

$$\frac{2a}{c\sqrt{n_1^2 + n_2^2 + n_3^2}}$$

where n_1, n_2, n_3 are integers.

Solution: The three dimensional wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \dots (A)$$

Here $u = u(x, y, z, t)$, $0 \leq x, y, z \leq a$

The boundary conditions are

$$\frac{\partial u}{\partial x} = 0 \text{ on the boundary of cubes.}$$

$$i.e., \quad \left(\frac{\partial u}{\partial x}\right)_{x=0} = 0, \left(\frac{\partial u}{\partial x}\right)_{x=a} = 0, \left(\frac{\partial u}{\partial y}\right)_{y=0} = 0, \left(\frac{\partial u}{\partial y}\right)_{y=a} = 0,$$

$$\left(\frac{\partial u}{\partial z}\right)_{z=0} = 0, \left(\frac{\partial u}{\partial z}\right)_{z=a} = 0 \text{ and } \left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$$

(As $x=0, x=a, y=0, y=a, z=0, z=a$ and six faces of the cube)

$$\text{Let } u(x, y, z, t) = F(x) Y(y) Z(z) T(t) = FYZT \quad \dots(1)$$

be the solution of equation (A)

Proceeding as in 12.8 the solution of (A) is of the form

$$u(x, y, z, t) = (A_{k_1} \cos k_1 x + B_{k_1} \sin k_1 x)(C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y) \\ (F_{k_3} \cos k_3 z + F_{k_3} \sin k_3 z)(G_k \cos ckt + H_k \sin ckt)$$

$$\text{where } k^2 = k_1^2 + k_2^2 + k_3^2$$

$$\text{or } u(x, y, z, t) = (A_{k_1} \cos k_1 x + B_{k_1} \sin k_1 x) YZT \quad (\text{In short})$$

$$\text{Now } \frac{\partial u}{\partial x} = -k_1(A_{k_1} \sin k_1 x + B_{k_1} \cos k_1 x) YZT$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_{x=0} = 0 \Rightarrow -k_1 B_{k_1} YZT = 0 \Rightarrow B_{k_1} = 0,$$

$$\therefore Y = 0 \text{ or } T = 0 \text{ or } Z = 0 \text{ will lead to trivial solution.}$$

$$\text{and } \left(\frac{\partial u}{\partial x}\right)_{x=a} = 0 \Rightarrow -k_1(A_{k_1} \sin k_1 a + 0) YZT = 0$$

$$\Rightarrow \sin k_1 a = 0 \quad \therefore A_{k_1} = 0 \text{ or } Y = 0 \text{ or } Z = 0 \text{ or } T = 0 \text{ will lead to trivial solution.}$$

$$\Rightarrow k_1 a = n_1 \pi, \quad n_1 \in I$$

$$\Rightarrow k_1 = n_1 \pi / a, \quad n_1 \in I$$

$$\text{Similarly, condition } \left(\frac{\partial u}{\partial y}\right)_{y=0} = 0 \text{ and } \left(\frac{\partial u}{\partial y}\right)_{y=a} = 0$$

$$\Rightarrow D_{k_2} = 0 \text{ and } k_2 = n_2 \pi / a, \quad n_2 \in I$$

$$\left(\frac{\partial u}{\partial z}\right)_{z=0} = 0 \text{ and } \left(\frac{\partial u}{\partial z}\right)_{z=a} = 0 \Rightarrow F_{k_3} = 0 \text{ and } k_3 = n_3 \pi/a, \quad n_3 \in I$$

$$\text{and } \left(\frac{\partial u}{\partial t}\right)_{t=0} = 0 \Rightarrow H_k = 0$$

$$\therefore k^2 = k_1^2 + k_2^2 + k_3^2 = (n_1^2 + n_2^2 + n_3^2) \pi^2 / a^2 = k_{n_1, n_2, n_3}^2$$

$\therefore$ from (1), the required more general solution is

$$u(x, y, z, t) = \sum_{n_1, n_2, n_3} \left[A_{n_1, n_2, n_3} \cos\left(\frac{n_1 \pi x}{a}\right) \cos\left(\frac{n_2 \pi y}{a}\right) \cos\left(\frac{n_3 \pi z}{a}\right) \right. \\ \left. \cos\left(\sqrt{(n_1^2 + n_2^2 + n_3^2)} \cdot \frac{\pi c t}{a}\right) \right]$$

where n_1, n_2, n_3 are integers.

Obviously the period of free oscillations of the gas are

$$\frac{2\pi}{\sqrt{(n_1^2 + n_2^2 + n_3^2)} \left(\frac{\pi c}{a}\right)} = \frac{2a}{c \sqrt{(n_1^2 + n_2^2 + n_3^2)}}$$

Example 2: Solve three dimensional wave equation in spherical polar coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

if u remains finite for $\theta = 0$.

Solution: From 12.12 on page 598, the solution of the given wave equation is

$$u(r, \theta, \phi, t) = \sum_{k, m, n} r^{-1/2} K_{k, m, n} [G_{mn} P_n^m(\cos \theta) \\ + H_{mn} Q_n^m(\cos \theta)] \cdot J_{\pm(n+1/2)}(kr) \cdot e^{\pm im\phi \pm ick t}$$

Since $Q_n^m(\cos \theta) \rightarrow \infty$ as $\theta \rightarrow 0$, therefore for u to remain finite for $\theta = 0$, we must take $H_{mn} = 0$.

Hence, the required solution is

$$u(r, \theta, \phi, t) = \sum_{k, m, n} r^{-1/2} A_{k, m, n} P_n^m(\cos \theta) \cdot J_{\pm(n+1/2)}(kr) \cdot e^{\pm im\phi \pm ick t}$$

Exercise 12.3

1. Solve three dimensional wave equation in spherical polar coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

If there is an axial symmetry (taking z-axis, as the axis of symmetry), then find the solution which remains finite for $\theta = 0$.

[Hint : In case of axial symmetry about z-axis, u will be independent of ϕ , so $m = 0$ and $P_n^m(\cos \theta) = P_n(\cos \theta)$].

2. Solve three dimensional wave equation in spherical polar coordinates. Also find the solution if there is a spherical symmetry.

[Hint : In case of spherical symmetry u will be independent of θ and ϕ both, $\therefore m = 0 = n$].

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter (a), (b), (c) or (d).

1. Transverse vibration of an elastic string are governed by one dimensional wave equation :

(a) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (b) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial u}{\partial t}$ (c) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

2. Two dimensional wave equation is given by

(a) $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \right)$ (b) $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$
 (c) $\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$ (d) $\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \right)$

3. A string is stretched between the fixed points $(0, 0)$ and $(10, 0)$ and released from rest from the position $u = A \sin 5x$. If $u(x, t)$ is the deflection of sting at any point at anything, then the bounding conditions of the this boundary value problem are :

(a) $u(0, t) = 0 = u(10, t)$ (b) $u(0, 0) = 0, u(10, 0) = 10$
 (c) $u(0, 0) = 0 = u(10, t)$ (d) None of these

4. The initial conditions for the boundary value problem given in alone Q.N. 3 are :

(a) $u(x, 0) = 0, u(x, t) = A \sin 5x$ (b) $u(x, 0) = A \sin 5x, u_t(x, 0) = 0$

- (c) $u(x, 0) = A \sin 5x$, $u_x(x, 0) = 0$ (d) None of these
5. A string of length l is slightly stretched between two end points. The string is originally plucked at the middle point by giving it an initial displacement h from the mean position. If $u(x, t)$ is the deflection string at distance x from one end at anytime t , then boundary conditions of this problem are :
- (a) $u(0, t) = 0 = u(l, 0)$ (b) $u(x, 0) = 0 = u(l, 0)$
 (c) $u(0, t) = 0 = u(l, t)$ (d) None of these
6. The initial conditions for the boundary value problem given in Q.N. 5 are :
- (a) $u(x, 0) = 0 = u(l, 0)$
 (b) $u(x, 0) = \begin{cases} (2h/l)x, & 0 \leq x \leq l/2 \\ 2h(1-x/l), & l/2 \leq x \leq l \end{cases}, u_t(x, 0) = 0$
 (c) $u(x, 0) = h$, $u(l, 0) = 0$ (d) None of these
7. The transverse vibrations of a circular membrane bounded by a circle in xy plane are governed by the equation :
- (a) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$ (b) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$
 (c) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$ (d) None of these
8. Consider the transverse vibrations of a rectangular membrane fixed along its boundary $x=0$, $x=a$, $y=0$, $y=b$. If $A \sin 2\pi x \sin 3\pi y$ be its initial deflection $u(x, y, t)$. Then the boundary conditions of this boundary value problem are :
- (a) $u(x, y, t) = 0$, $u(x, y, t) = A \sin 2\pi x \sin 3\pi y$ for $n=0, a$
 (b) $u(x, y, t) = 0$, $u(x, y, t) = A \sin 2\pi x \sin 3\pi y$ for $y=0, b$
 (c) $u(x, y, t) = 0$ for $n=0, a$; $u(x, y, t) = 0$ for $y=0, b$
 (d) None of these
9. The initial condition for the boundary value problem given Q.N. 8 is :
- (a) $u(x, y, 0) = A \sin 2\pi x \sin 3\pi y$ (b) $u(a, y, 0) = A \sin 2\pi x \sin 3\pi y$
 (c) $u(a, b, 0) = A \sin 2\pi x \sin 3\pi y$ (d) None of these
10. The D'Alembert's solution of wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is given by :
- (a) $u(x, t) = \phi(x) + \psi(t)$ (b) $u(x, t) = \phi(x+ct) + \psi(x-ct)$
 (c) $u(x, t) = \phi(c+t) + \psi(c-t)$ (d) None of these

Fill in the Blank

Fill in the blank's "....." so that the following statements are complete and correct.

1. In the derivation of one dimensional wave equation it is assumed that the tension caused by stretching the string before fixing at the end points is so large that the action of on the string can be neglected.

2. Three dimensional wave equation is given by $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \dots \right)$

3. The wave equation in cylindrical coordinates is given by,

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = \dots\dots\dots$$

4. The wave equation in spherical polar coordinates is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \dots\dots\dots = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

Answers

Exercise 12.3

1.	$u(x, y, z, t) = \sum_{k,m,n} r^{-1/2} K_{k,m,n} [G_{mn} P_n^m(\cos \theta) + H_{mn} Q_n^m(\sin \theta)] \cdot J_{\pm(n+1/2)}(kr) \cdot e^{\pm im\phi \pm ick t}$ $u = \sum_{k,n} r^{-1/2} A_{k,n} \cdot P_n(\cos \theta) \cdot J_{\pm(n+1/2)}(kr) \cdot e^{\pm ick t}$
2.	$u(x, y, z, t) = \sum_{k,m,n} r^{-1/2} K_{k,m,n} [G_{mn} P_n^m(\cos \theta) + H_{mn} Q_n^m(\sin \theta)] \cdot J_{\pm(n+1/2)}(kr) \cdot e^{\pm im\phi \pm ick t}$ $u = \sum_{k,n} r^{-1/2} A_k J_{\pm 1/2}(kr) \cdot e^{\pm ick t}.$

Objective Type Questions

Multiple Choice Questions

1.	(a)	2.	(b)	3.	(a)	4.	(b)
5.	(c)	6.	(b)	7.	(a)	8.	(c)
9.	(a)	10.	(b)				

Fill in the Blank

1.	gravitational force	2.	$\frac{\partial^2 u}{\partial z^2}$
3.	$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$	4.	$\frac{1}{c^2 / \sin^2 \theta} \cdot \frac{\partial^2 u}{\partial \phi^2}$

Heat and Diffusion Equations (By Method of Separation of Variables)

13.1 Introduction

In this chapter we shall discuss heat and diffusion equations, their derivations and solutions in different coordinate systems.

13.2 One Dimensional Heat Equation

[Kanpur 2003]

Consider the flow of heat by conduction in a bar OA lying along x -axis with the end O at the origin. The temperature $U(x, t)$ in the bar at a distance x from O, at time t is a function of x and t . To find $U(x, t)$, we make the following assumptions.

1. The bar is homogeneous *i.e.*, the mass of the bar per unit length *i.e.*, ρ is constant.
2. The sides of the bar are insulated *i.e.*, there is no loss of heat from the sides of the bar by conduction or radiation. Now the amount of heat flowing through any section of the bar in time δt

$$= KA \left(\frac{\partial U}{\partial x} \right) \delta t$$

where A = area of cross section of the bar

$\frac{\partial U}{\partial x}$ = temperature gradient at the section

K = thermal conductivity of the material of the bar.

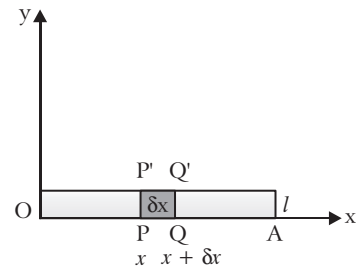


Fig. 13.1

Let the heat flow in the direction OA and let $PP' Q' Q$ be an element of length δx of the bar.

- $\therefore$ the amount of the heat flowing in the element $PP' Q' Q$ through the section PP' in time δt

$$= -KA \left(\frac{\partial U}{\partial x} \right)_x \delta t$$

(The negative sign is taken as the heat flow is decreasing temperature direction)

And the amount of the heat flowing out of the element $PP' Q' Q$ through the section QQ' in time δt

$$= -KA \left(\frac{\partial U}{\partial x} \right)_{x+\delta x} \delta t$$

- $\therefore$ the net quantity of heat in the element $PP' Q' Q$ in time δt

$$= -KA \left(\frac{\partial U}{\partial x} \right)_x \delta t - \left[-KA \left(\frac{\partial U}{\partial x} \right)_{x+\delta x} \delta t \right] = KA \delta t \left[\left(\frac{\partial U}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial U}{\partial x} \right)_x \right] \quad \dots(1)$$

Also the quantity of heat in the element $PP' Q' Q$ in time δt

$$= (A \rho \delta x) \sigma \delta U \quad \dots(2)$$

where σ is the specific heat to the bar.

- $\therefore$ From (1) and (2), we have

$$KA \delta t \left[\left(\frac{\partial U}{\partial x} \right)_{x+\delta x} - \left(\frac{\partial U}{\partial x} \right)_x \right] = (A \rho \delta x) \cdot \sigma \delta U$$

$$\text{or} \quad KA \delta t [U_x(x+\delta x, t) - U_x(x, t)] = (A \rho \delta x) \sigma \delta U$$

$$\text{or} \quad K \frac{U_x(x+\delta x, t) - U_x(x, t)}{\delta x} = \rho \sigma \frac{\delta U}{\delta t}$$

Taking limit as $\delta x \rightarrow 0$ and $\delta t \rightarrow 0$, we get

$$K \frac{\partial}{\partial x} U_x(x, t) = \rho \sigma \frac{\partial U}{\partial t} \quad \text{or} \quad \frac{\partial U}{\partial t} = \frac{K}{\rho \sigma} \frac{\partial^2 U}{\partial x^2}$$

$$\text{or} \quad \frac{\partial U}{\partial t} = c^2 \frac{\partial^2 U}{\partial x^2} \quad \dots(3)$$

where $c^2 = \frac{K}{\rho \sigma}$, is called the diffusivity of the material of the bar. The equation (3) is

known as *one dimensional heat equation*.

13.3 Heat Equation

[Meerut 2000; Agra 2004; Mumbai 2000, 05]

Thermal energy is transferred from warmer to colder regions interior to a solid body by conduction. It is convenient to refer to that transfer as the flow of heat, as if heat were a fluid or gas which diffuses through the body from regions of high concentration into regions of low concentration of that fluid.

Physical experiments show that the ratio of flow is proportion to the gradient of the temperature. Therefore, the velocity $\vec{v}$ of heat flow in a body is given by $\vec{v} = -K \text{ grad } U$ where $U(x, y, z, t)$ is the temperature, t is time, and K is a positive constant, called the **thermal conductivity** of the body.

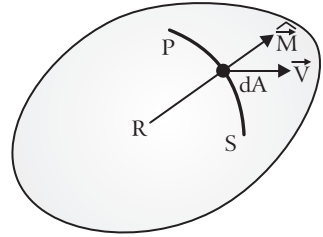


Fig. 13.2

Now let us consider a region R in the body and S be its boundary surface.

Therefore the amount of heat leaving R per unit of time is

$$\iint_S v_M dA$$

where $v_M = \vec{v} \cdot \hat{M}$ is the component of $\vec{v}$ in the direction of the outer unit normal vector $\hat{M}$ of S .

But from "Gauss's divergent theorem we know that, the normal surface integral of a function $\vec{F}$ over the boundary of a closed region is equal to the volume integral of $\text{div } \vec{F}$ taken throughout the region."

$$\begin{aligned} \therefore \iint_S v_M dA &= \iint_S \vec{v} \cdot \hat{M} dA = \iiint_R \text{div } (\vec{v}) dv = -K \iiint_R \text{div } (\text{grad } U) dx dy dz \\ &= -K \iiint_R \nabla^2 U dx dy dz \end{aligned} \quad \dots(1)$$

Also the total amount of heat H in R is

$$H = \iiint_R \sigma \rho U dx dy dz$$

where σ is the specific heat of the material of the body, and ρ is its density.

∴ the time rate of decrease of H is

$$-\frac{\partial H}{\partial t} = - \iiint_R \sigma \rho \frac{\partial U}{\partial t} dx dy dz$$

and this must be equal to the amount of heat leaving, R per unit of time, given by (1)

$$\therefore - \iiint_R \sigma \rho \frac{\partial U}{\partial t} dx dy dz = -K \iiint_R \nabla^2 U dx dy dz$$

$$\text{or} \quad \iiint_R \left(\sigma \rho \frac{\partial U}{\partial t} - K \nabla^2 U \right) dx dy dz = 0$$

Since this holds for any region R in the body, the integrand (if continuous) must be zero everywhere, i.e.,

$$\sigma \rho \frac{\partial U}{\partial t} - K \nabla^2 U = 0 \quad \text{or} \quad \frac{\partial U}{\partial t} = \frac{K}{\sigma \rho} \nabla^2 U$$

$$\text{or} \quad \frac{\partial U}{\partial t} = c^2 \nabla^2 U, \quad \text{where} \quad c^2 = \frac{K}{\sigma \rho}$$

$$\text{i.e.,} \quad \frac{\partial U}{\partial t} = c^2 \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right)$$

This partial differential equation is called **the heat equation**.

If the temperature within a body are independent of z , i.e., if there is no flow of heat in the z direction, the heat equation reduces to the equation for two dimensional flow parallel to the xy plane and is given by

$$\frac{\partial U}{\partial t} = c^2 \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right)$$

which is called **two dimensional heat equation**.

For one dimensional flow parallel to the x -axis, the equation becomes

$$\frac{\partial U}{\partial t} = c^2 \frac{\partial^2 U}{\partial x^2}$$

which is called **one dimensional heat equation**.

13.4 Diffusion Equation

Let us consider the flow of electricity in a long insulated cable. Suppose the flow to be one dimensional, the current i and the voltage E at any point in the cable can be completely specified by one coordinates x and a time variable t .

Consider the fall of potential in a linear element of length δx situated at the point x , we have

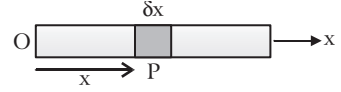


Fig. 13.3

$$-\delta E = iR\delta x + L\delta x \frac{\partial i}{\partial t} \quad \dots(1)$$

where R and L are resistance and inductance per unit length respectively.

If there is a capacitance to earth of C per unit length and conductance G per unit length, then

$$-\delta i = GE\delta x + C\delta x \frac{\partial E}{\partial t} \quad \dots(2)$$

These equation (1) and (2) are equivalent to the differential equations

$$\frac{\partial E}{\partial x} + Ri + L \frac{\partial i}{\partial t} = 0 \quad \dots(3)$$

$$\frac{\partial i}{\partial x} + GE + C \frac{\partial E}{\partial t} = 0 \quad \dots(4)$$

Differentiating (3) w.r.t. x and (4) w.r.t. t we have

$$\frac{\partial^2 E}{\partial x^2} + R \frac{\partial i}{\partial x} + L \frac{\partial^2 i}{\partial x \partial t} = 0 \quad \dots(5)$$

and

$$\frac{\partial^2 i}{\partial x \partial t} + G \frac{\partial E}{\partial t} + C \frac{\partial^2 E}{\partial t^2} = 0 \quad \dots(6)$$

Eliminating $\frac{\partial i}{\partial x}$ and $\frac{\partial^2 i}{\partial x \partial t}$ from (4), (5) and (6), we have

$$\frac{\partial^2 E}{\partial x^2} = CL \frac{\partial^2 E}{\partial t^2} + (CR + GL) \frac{\partial E}{\partial t} + RGE \quad \dots(7)$$

Similarly differentiating (3) w.r.t. t and (4) w.r.t. x , we have

$$\frac{\partial^2 E}{\partial t \partial x} + R \frac{\partial i}{\partial t} + L \frac{\partial^2 i}{\partial t^2} = 0 \quad \dots(8)$$

and
$$\frac{\partial^2 i}{\partial x^2} + G \frac{\partial E}{\partial x} + C \frac{\partial^2 E}{\partial t \partial x} = 0 \quad \dots(9)$$

Eliminating $\frac{\partial E}{\partial x}$ and $\frac{\partial^2 E}{\partial x \partial t}$ from (3), (8) and (9), we have

$$\frac{\partial^2 i}{\partial x^2} = CL \frac{\partial^2 i}{\partial t^2} + (CR + GL) \frac{\partial i}{\partial t} + RGi \quad \dots(10)$$

From (7) and (10), we find that E and i both satisfy the second order partial differential equation

$$\frac{\partial^2 \phi}{\partial x^2} = CL \frac{\partial^2 \phi}{\partial t^2} + (CR + GL) \frac{\partial \phi}{\partial t} + RG \phi \quad \dots(11)$$

Equation (11) is called the **telegraphy equation**.

If the leakage to ground is small, then G and L may be taken to be zero.

From equation (11), we have $\frac{\partial^2 \phi}{\partial x^2} = CR \frac{\partial \phi}{\partial t}$

or
$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{k} \frac{\partial \phi}{\partial t} \quad \dots(12)$$

where $k = \frac{1}{CR}$ is constant.

Equation (12) is known as one **dimensional diffusion equation**.

In three dimensions the diffusion equation is $\nabla^2 \phi = \frac{1}{k} \frac{\partial \phi}{\partial t}$

Note : 1. The equation (12) is also sometimes called the telegraphic equation.

2. The above equation (diffusion equation) is satisfied by the temperature when there is no generation or absorption of heat in the solid, as a result of chemical reactions and as the solid is not under going radioactive decay or is not absorbing radioaction. In the case $k = \frac{k'}{\sigma \rho} = c^2$ where k' is thermal conductivity of the body σ the specific heat of the material of the body and ρ is its density.

And so it reduces to heat equation

$$\frac{\partial \phi}{\partial t} = c^2 \nabla^2 \phi.$$

13.5 Solution of One Dimensional Heat Equation by Separation of Variables

One dimensional heat equation is given by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots(1)$$

where $u(x, t)$ is the temperature at a distance x from one end at any time t .

To find the solution of (1) by using the method of separation of variables, let

$$u(x, t) = F(x) T(t) = FT \text{ (say)} \quad \dots(2)$$

where F is a function of x alone and T the function of t alone,

Differentiating (2) and substituting in (1), we get

$$F \frac{dT}{dt} = c^2 T \frac{d^2 F}{dx^2}$$

or
$$\frac{1}{F} \frac{d^2 F}{dx^2} = \frac{1}{c^2 T} \frac{dT}{dt}$$

Since the left hand side is a function of independent variable x only while right hand side is a function of independent variable t only, so the two sides will be equal to each other if each reduce to a constant value.

$$\therefore \frac{1}{F} \frac{d^2 F}{dx^2} = \frac{1}{c^2 T} \frac{dT}{dt} = \lambda \text{ (say)} \quad \dots(3)$$

where λ is a constant

Now, we have the following three cases :

Case I : If $\lambda = 0$, then from (3), we obtain the following two ordinary differential equations.

$$\frac{d^2 F}{dx^2} = 0 \quad \text{and} \quad \frac{dT}{dt} = 0$$

whose solution are

$$F = A_1 x + A_2 \quad \text{and} \quad T = A_3$$

$$\therefore u = FT = A_1 A_3 x + A_2 A_3 = A x + A' \quad \dots(4)$$

where $A_1 A_3 = A, A_2 A_3 = A'$

Case II : If $\lambda = k^2 > 0$, then from (3), we obtain the following two ordinary differential equations.

$$\frac{d^2 F}{dx^2} - k^2 F = 0 \quad \text{and} \quad \frac{dT}{dt} = c^2 k^2 T$$

whose solutions are

$$F = B_1 e^{kx} + B_2 e^{-kx} \quad \text{and} \quad T = B_3 e^{c^2 k^2 t}$$

$$\begin{aligned} \therefore u = FT &= (B_1 B_3 e^{kx} + B_2 B_3 e^{-kx}) e^{c^2 k^2 t} \\ &= (B e^{kx} + B' e^{-kx}) e^{c^2 k^2 t} \end{aligned} \quad \dots(5)$$

where $B_1 B_3 = B$, $B_2 B_3 = B'$

Case III : If $\lambda = -k^2 < 0$, then from (3), we obtain the following two ordinary differential equations

$$\frac{d^2 F}{dx^2} + k^2 F = 0 \quad \text{and} \quad \frac{dT}{dt} = -c^2 k^2 T$$

whose solution are

$$F = C_1 \cos kx + C_2 \sin kx \quad \text{and} \quad T = C_3 e^{-c^2 k^2 t}$$

$$\begin{aligned} \therefore u = FT &= (C_1 C_3 \cos kx + C_2 C_3 \sin kx) e^{-c^2 k^2 t} \\ &= (C \cos kx + C' \sin kx) e^{-c^2 k^2 t} \end{aligned} \quad \dots(6)$$

where $C_1 C_3 = C$, $C_2 C_3 = C'$

There are three cases depending upon the particular problem we have to choose that solution (or solutions) which is (or are) consistent with the physical nature of the problem. Here we are dealing with heat equation and in such problems temperature $u(x, t)$ decrease with the increase of time t . So the solution of form (6) will be suitable solution.

13.6 Solution of One Dimensional Heat Equation under given Boundary Conditions

Case I : Both Faces (Ends) of the Rod at Zero Temperature

To find the temperature function of the rod (or bar or wire) of length l which is insulated laterally and whose ends are kept at temperature zero and $f(x)$ is the initial temperature in the bar.

[Delhi 2003; Andhra 2003; Agra 2004; Aligarh 2003; Meerut 2010]

Let the rod (bar) be placed along the x -axis with one end at the origin.

The temperature $u(x, t)$ in the rod at any time t at a distance x from the end O (origin) is governed by one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots(A)$$

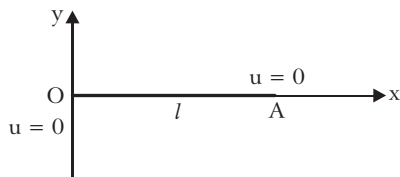


Fig. 13.4

$$c^2 = \frac{K}{\sigma \rho}$$

Boundary Conditions : Since the ends $x = 0$ and $x = l$ are kept at zero temperature, so the boundary condition are

$$u(0, t) = 0 \quad \text{and} \quad u(l, t) = 0, \quad t \geq 0 \quad \dots(B)$$

and **initial condition** is $u(x, 0) = f(x)$ $\dots(C)$

Here we are required to find the solution of heat equation (A) satisfying the conditions (B) and (C).

Let (A) has a solution of the form

$$u(x, t) = F(x) T(t) = F \cdot T \text{ (say)} \quad \dots(1)$$

where F is a function of x alone and T function of t alone.

Differentiating (1) and substituting in (A), we have

$$F \frac{dT}{dt} = c^2 T \frac{d^2 F}{dx^2}$$

or
$$\frac{1}{F} \frac{d^2 F}{dx^2} = \frac{1}{c^2 T} \frac{dT}{dt}$$

Here the left hand side is a function of independent variable x alone while right hand side is a function of independent, variable t alone. Therefore, the two sides will be equal to each other only if each is equal to a constant value.

$$\therefore \frac{1}{F} \frac{d^2 F}{dx^2} = \frac{1}{c^2 T} \frac{dT}{dt} = \lambda \text{ (constant)}$$

Therefore, we obtain the following two ordinary differential equations

$$\frac{d^2 F}{dx^2} = \lambda F \quad \text{and} \quad \frac{dT}{dt} = \lambda c^2 T \quad \dots(2)$$

Now we have the following three cases :

Case (i) : If $\lambda = 0$, then the differential equations (2) reduce to

$$\frac{d^2 F}{dx^2} = 0 \quad \text{and} \quad \frac{dT}{dt} = 0$$

whose solutions are

$$F(x) = A_1 x + A_2 \quad \text{and} \quad T(t) = A_3$$

where A_1, A_2, A_3 are constants.

$$\therefore u(x, t) = F(x) T(t) = (A_1 x + A_2) A_3 = A_1' x + A_2'$$

$$\text{where} \quad A_1' = A_1 A_3 \quad \text{and} \quad A_2' = A_2 A_3$$

$$\text{Now the boundary condition, } u(0, t) = 0 \Rightarrow A_2' = 0$$

$$\text{and } u(l, t) = 0 \Rightarrow A_1' l + A_2' = 0 \Rightarrow A_1' l = 0 \Rightarrow A_1' = 0$$

$\therefore u(x, t) = 0$, which is trivial solution of equation (A) and is inadmissible.

$\therefore$ We reject the solution when $\lambda = 0$.

Case (ii) : If $\lambda = k^2 > 0$, then the differential equations (2) reduce to

$$\frac{d^2 F}{dx^2} - k^2 F = 0 \quad \text{and} \quad \frac{dT}{dt} = k^2 c^2 T$$

Whose solutions are

$$F(x) = B_1 e^{kx} + B_2 e^{-kx} \quad \text{and} \quad T(t) = B_3 e^{k^2 c^2 t}$$

$$\therefore u(x, t) = F(x) \cdot T(t) = (B_1' e^{kx} + B_2' e^{-kx}) e^{k^2 c^2 t}$$

$$\text{where } B_1' = B_1 B_3 \quad \text{and} \quad B_2' = B_2 B_3$$

$\therefore$ The boundary conditions $u(0, t) = 0$ and $u(l, t) = 0$

$$\Rightarrow (B_1' + B_2') e^{k^2 c^2 t} = 0 \quad \text{and} \quad (B_1' e^{kl} + B_2' e^{-kl}) e^{k^2 c^2 t} = 0$$

$$\Rightarrow B_1' + B_2' = 0 \quad \text{and} \quad B_1' e^{kl} + B_2' e^{-kl} = 0$$

$\therefore e^{k^2 c^2 t} = 0$ will lead to trivial solution $u(x, t) = 0$ which is inadmissible.

$$\Rightarrow B_1' = 0 \quad \text{and} \quad B_2' = 0 \Rightarrow u(x, t) = 0$$

which is trivial solution and is inadmissible.

$\therefore$ we reject the solution when $\lambda = k^2 > 0$

Case (iii) : If $\lambda = -k^2 < 0$, then the differential equation (2) reduce to

$$\frac{d^2 F}{dx^2} + k^2 F = 0 \quad \text{and} \quad \frac{dT}{dt} = -k^2 c^2 T$$

whose solution are

$$f(x) = (C_1 \cos kx + C_2 \sin kx) \text{ and } T(t) = C_3 e^{-k^2 c^2 t}$$

$$\therefore u(x, t) = F(x) T(t) = (C_1' \cos kx + C_2' \sin kx) e^{-k^2 c^2 t} \quad \dots(3)$$

where $C_1' = C_1 C_3$ and $C_2' = C_2 C_3$.

Now from (3) and boundary conditions (B), we have

$$u(0, t) = 0 \Rightarrow C_1' e^{-k^2 c^2 t} = 0 \Rightarrow C_1' = 0$$

$\therefore e^{-k^2 c^2 t} = 0$ will lead to inadmissible trivial solution $u(x, t) = 0$

$\therefore$ from (3), we have

$$u(x, t) = (C_2' \sin kx) e^{-k^2 c^2 t} \quad \dots(4)$$

$$\text{Now } u(l, t) = 0 \Rightarrow (C_2' \sin kl) e^{-k^2 c^2 t} = 0$$

$$\Rightarrow \sin kl = 0 \quad \therefore C_2' = 0 \quad \text{or} \quad e^{-k^2 c^2 t} = 0$$

will lead to inadmissible trivial solution $u(x, t) = 0$

$$\Rightarrow kl = n\pi \quad \text{i.e.,} \quad k = \frac{n\pi}{l}, \text{ where } n \text{ is any integer.}$$

For this value of k , from (4) the solutions of equation (A) satisfying conditions (B) are

$$u_n(x, t) = D_n \sin\left(\frac{n\pi x}{l}\right) e^{-n^2 \pi^2 c^2 t / l^2} \quad \dots(5)$$

for all integral values of n

where $D_n = C_2'$ is the new arbitrary constant.

Now in order that (5) may also satisfy the condition (C), taking $n = 1, 2, 3, \dots$, the more general solution of (A) is

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{l}\right) e^{-n^2 \pi^2 c^2 t / l^2} \quad \dots(6)$$

which also satisfy the conditions (B)

$$\therefore \text{Condition (C), } u(x, 0) = f(x) \Rightarrow \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{l}\right) = f(x) \quad \dots(7)$$

The left hand side in (7) can be considered as the Fourier sine series of the function $f(x)$ on its right hand side for $0 < x < l$.

$$\therefore D_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx \quad \dots(8)$$

Hence, the required solution of given equation (A) is given by (6) where D_n is given by (8).

Case II : One End of the Rod at Constant Temperature u_0

To find the temperature function of the rod (or wire) of length l which is insulated laterally, whose end $x = 0$ is kept at zero temperature, end $x = l$ is kept at constant temperature u_0 and v_0 is the initial temperature throughout the rod.

Let the rod (bar) be placed along the x -axis with one end at the origin. The temperature $u(x, t)$ in the rod at any time t at a distance x from the end O (origin) is governed by one dimensional heat (diffusion) equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= c^2 \frac{\partial^2 u}{\partial x^2} \\ c^2 &= \frac{K}{\sigma \rho} \end{aligned}$$

...(A)

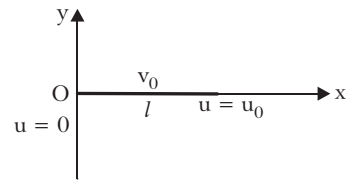


Fig. 13.5

Boundary conditions : Since the end $x = 0$ is kept at zero temperature and $x = l$ is kept at constant temperature u_0 , so the boundary conditions are

$$u(0, t) = 0 \quad \text{and} \quad u(l, t) = u_0, \quad t \geq 0 \quad \dots(B)$$

Initial condition : is

$$u(x, 0) = v_0 \quad (\text{constant}) \quad \dots(C)$$

$$0 < x < l$$

Here we are required to find the solution of heat equation (A) satisfying the conditions (B) and (C).

Let (A) has a solution of the form

$$u(x, t) = F(x). T(t) = FT \text{ (say)} \quad \dots(1)$$

where F is a function of x alone and T function of t alone.

Differentiating (1) and substituting in (A), we have

$$F \frac{dT}{dt} = c^2 T \frac{d^2 F}{dx^2}$$

or
$$\frac{1}{F} \frac{d^2 F}{dx^2} = \frac{1}{c^2 T} \frac{dT}{dt} = 0 \text{ or } k^2 \text{ or } -k^2$$

Since the two sides are functions of different independent variables x and t respectively, so they will be equal to each other only if each is equal to a constant value. This constant may be 0 or $k^2 > 0$ or $-k^2 < 0$.

Thus, we get three sets of two ordinary differential equations.

1. $\frac{d^2 F}{dx^2} = 0$ and $\frac{dT}{dt} = 0$
2. $\frac{d^2 F}{dx^2} - k^2 F = 0$ and $\frac{dT}{dt} = k^2 c^2 T$
3. $\frac{d^2 F}{dx^2} + k^2 F = 0$ and $\frac{dT}{dt} = -k^2 c^2 T$

The general solutions in these three cases are :

(i) $F = A_1 x + A_2$ and $T = A_3$

$$\therefore u = FT = A_1' x + A_2' \text{ where } A_1' = A_1 A_3, A_2' = A_2 A_3$$

(ii) $F = B_1 e^{kx} + B_2 e^{-kx}$ and $T = B_3 e^{k^2 c^2 t}$

$$\therefore u = FT = (B_1' e^{kx} + B_2' e^{-kx}) e^{k^2 c^2 t}, \text{ where } B_1' = B_1 B_3, B_2' = B_2 B_3$$

(iii) $F = C_1 \cos kx + C_2 \sin kx$ and $T = C_3 e^{-k^2 c^2 t}$

$$\therefore u = FT = (C_1' \cos kx + C_2' \sin kx) e^{-k^2 c^2 t}, \text{ where } C_1' = C_1 C_3 \text{ and } C_2' = C_2 C_3.$$

Here, according to the given conditions the solution given by (ii) is inadmissible, as in this case the temperature $u(x, t)$ increases infinitely with time t . The solution given by (iii) is inadequate to give the complete solution, since in this case the temperature $u(x, t)$ will tend to zero as $t \rightarrow \infty$, while constant temperature u_0 is maintained at the end $x = l$ and v_0 (constant) is the initial temperature throughout the rod.

Hence, the solution in this case must be a combination of solutions given in (i) and (iii).

$\therefore$ We can take

$$u(x, t) = u_1(x) + u_2(x, t) \quad \dots(2)$$

Where $u_1(x)$ is the temperature distribution after a long period of time when the rod has reached a steady state of temperature distribution, $u_2(x, t)$ indicate the transient effects which die down with the passage of time. These two must be the solutions of the types (i) and (iii) respectively. Obviously $u_1(x)$ is a solution of type (i) i.e., we can take

$$u_1(x) = A_1'x + A_2'$$

$$\therefore \quad \text{at } x=0 \quad u_1=0 \quad \text{and} \quad \text{at } x=l, u_1=u_0$$

$$\therefore \quad u_1(0) = A_2' = 0 \quad \text{and} \quad u_1(l) = A_1'l + A_2' = u_0 \Rightarrow A_2' = 0 \quad \text{and} \quad A_1' = \frac{u_0}{l}$$

$$\therefore \quad u_1(x) = \frac{u_0}{l}x \quad \dots(3)$$

$$\therefore \quad \text{from (2), we have, } u(x, t) = \frac{u_0}{l}x + u_2(x, t) \quad \dots(4)$$

$$\text{Now } u(0, t) = 0 \quad \therefore (4) \Rightarrow u(0, t) = 0 = u_2(0, t) \Rightarrow u_2(0, t) = 0 \quad \dots(5)$$

$$\text{and } u(l, t) = u_0 \quad \therefore (4) \Rightarrow u(l, t) = u_0 = u_0 + u_2(l, t) \Rightarrow u_2(l, t) = 0 \quad \dots(6)$$

$$\text{and } u(x, 0) = v_0 \quad \therefore (4) \Rightarrow u(x, 0) = v_0 = \frac{u_0}{l}x + u_2(x, 0)$$

$$\Rightarrow \quad u_2(x, 0) = v_0 - \frac{u_0}{l}x \quad \dots(7)$$

Since $u_2(x, t)$ will be a solution of the type given in (C).

$\therefore$ We take

$$u_2(x, t) = (C_1' \cos kx + C_2' \sin kx) e^{-k^2 c^2 t} \quad \dots(8)$$

Now from (5) and (8),

$$u_2(0, t) = 0 = C_1' e^{-k^2 c^2 t} \Rightarrow C_1' = 0 \quad \because e^{-k^2 c^2 t} \neq 0$$

$\therefore$ From (8) we have

$$u_2(x, t) = C_2' \sin kx \cdot e^{-k^2 c^2 t} \quad \dots(9)$$

$$\text{Now from (6) and (9),} \quad u_2(l, t) = 0 = C_2' \sin kl \cdot e^{-k^2 c^2 t}$$

$\Rightarrow \sin kl = 0, \because C_2' \neq 0$ and $e^{-k^2 c^2 t} \neq 0$, otherwise we shall get inadmissible trivial solution.

$$\Rightarrow kl = n\pi, n \text{ is any integer}$$

$$\Rightarrow k = n\pi/l$$

∴ From (8) for integral values of n , the solution $u_2(x, t)$ are given by

$$u_{2n}(x, t) = D_n \sin(n\pi x/l) \cdot e^{-n^2 \pi^2 c^2 t/l^2}$$

where $D_n = C_2'$ is the new arbitrary constant depending on n .

∴ Taking $n = 1, 2, 3, \dots$, we take more general solution as

$$u_2(x, t) = \sum_{n=1}^{\infty} u_{2n}(x, t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{l}\right) \cdot e^{-n^2 \pi^2 c^2 t/l^2} \quad \dots(10)$$

which also satisfy conditions (5) and (6)

Now from (10), and condition

$$\begin{aligned} u_2(x, 0) &= v_0 - \frac{u_0}{l} x \\ \Rightarrow v_0 - \frac{u_0}{l} x &= \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{l}\right) \end{aligned} \quad \dots(11)$$

The right hand side in (11) can be considered as the fourier sine series of the left hand

side function $v_0 - \frac{u_0}{l} x$ for $0 < x < l$.

$$\begin{aligned} \therefore D_n &= \frac{2}{l} \int_0^l \left(v_0 - \frac{u_0}{l} x \right) \sin\left(\frac{n\pi x}{l}\right) dx \\ &= \frac{2}{l} \left[\left(v_0 - \frac{u_0}{l} x \right) \left\{ -\frac{l}{n\pi} \cos\left(\frac{n\pi x}{l}\right) \right\} - \frac{u_0}{l} \left(\frac{l}{n\pi} \right)^2 \sin\left(\frac{n\pi x}{l}\right) \right]_0^l \\ &= \frac{2}{n\pi} [-(v_0 - u_0) \cos n\pi + v_0] = \frac{2}{n\pi} [v_0 - (v_0 - u_0)(-1)^n] \quad \dots(12) \end{aligned}$$

from (2), (3), (10) and (12), we have

$$u(x, t) = \frac{u_0}{l} x + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} [v_0 - (v_0 - u_0)(-1)^n] \cdot \sin\left(\frac{n\pi x}{l}\right) \cdot e^{-(n\pi c/l)^2 t}.$$

Particular case : If the initial temperature of the wire in zero i.e., $v_0 = 0$

$$\text{then } u(x, t) = \frac{u_0}{l} \left[x - \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (-1)^n \cdot \sin\left(\frac{n\pi x}{l}\right) e^{-(n\pi c/l)^2 t} \right]$$

Case III : Case of Indefinite Rod or Bar

To find the temperature function of the rod (or bar) which extends to infinity on both sides and insulated laterally. The initial temperature of the bar is $f(x)$.

Let the rod be placed along x -axis with its centre at the origin.

The temperature $u(x, t)$ in the rod at any time t at a distance x from the centre O (origin) is governed by one dimensional heat (diffusion) equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{K}{\sigma \rho} \quad \dots(A)$$

Here the *initial condition* is that $u(x, 0) = f(x)$...(B)

Let (A) has a solution of the form

$$u(x, t) = F(x) T(t) = F \cdot T(\text{say}) \quad \dots(1)$$

Differentiating (1) and substituting in (A), we get

$$F \frac{dT}{dt} = T c^2 \frac{d^2 F}{dx^2} \quad \text{or} \quad \frac{1}{F} \frac{d^2 F}{dx^2} = \frac{1}{c^2 T} \frac{dT}{dt} = 0 \quad \text{or} \quad p^2 \quad \text{or} \quad -p^2.$$

Thus, we get three sets of two ordinary differential equations.

1. $\frac{d^2 F}{dx^2} = 0$ and $\frac{dT}{dt} = 0$
2. $\frac{d^2 F}{dx^2} - p^2 F = 0$ and $\frac{dT}{dt} = p^2 c^2 T$
3. $\frac{d^2 F}{dx^2} + p^2 F = 0$ and $\frac{dT}{dt} = -p^2 c^2 T$

The general solutions in these cases are

- (i) $F = A_1 x + A_2$ and $T = A_3$
 $\therefore u = FT = A_1' x + A_2'$, where $A_1' = A_1 A_3$ and $A_2' = A_2 A_3$
- (ii) $F = B_1 e^{px} + B_2 e^{-px}$ and $T = B_3 e^{p^2 c^2 t}$
 $\therefore u = FT = (B_1' e^{px} + B_2' e^{-px}) \cdot e^{p^2 c^2 t}$
- (iii) $F = C_1 \cos px + C_2 \sin px$ and $T = e^{-p^2 c^2 t}$
 $\therefore u = FT = (C_1' \cos px + C_2' \sin px) e^{-p^2 c^2 t}$

In this present problem the solutions given in (i) and (ii) will not constitute our solution. Therefore, our solution will be as given in (iii).

Here in this case $f(x)$ is not periodic, so in the present case it is natural to use Fourier integrals instead of fourier series.

Taking $C_1' = D_p$ and $C_2' = E_p$ the new arbitrary constants which are functions of p the general solution of (A) can be taken as

$$u(x, t, p) = (D_p \cos px + E_p \sin px) \cdot e^{-p^2 c^2 t}$$

∴ More general solution of (A) can be taken as

$$u(x, t) = \int_0^{\infty} u(x, t, p) dp$$

$$\text{or} \quad u(x, t) = \int_0^{\infty} (D_p \cos px + E_p \sin px) \cdot e^{-p^2 c^2 t} dp \quad \dots(2)$$

The solution of heat equation is given by (2) provided this integral exists and can be differentiated twice w.r.t., 'x' and once w.r.t., 't'.

$$\therefore \text{ from (2), } u(x, 0) = f(x) \Rightarrow \int_0^{\infty} (D_p \cos px + E_p \sin px) dp = f(x)$$

$$\therefore \quad D_p = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos pv \, dv \quad \text{and} \quad E_p = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin pv \, dv$$

(See Fourier's integrals)

∴ from (2)

$$\begin{aligned} u(x, t) &= \frac{1}{\pi} \int_0^{\infty} \left[\int_{-\infty}^{\infty} f(v) \{ \cos px \cos pv + \sin px \sin pv \} dv \right] \cdot e^{-p^2 c^2 t} dp \\ &= \frac{1}{\pi} \int_0^{\infty} \left[\int_{-\infty}^{\infty} f(v) \cos(px - pv) dv \right] \cdot e^{-p^2 c^2 t} dp \end{aligned}$$

$$\text{or} \quad u(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \left[\int_0^{\infty} e^{-p^2 c^2 t} \cos(x-v)p \, dp \right] \cdot dv \quad \dots(3)$$

Assuming that we may invert the order of integration.

Now
$$\int_0^{\infty} e^{-p^2 c^2 t} \cos(x-v)p dp = \frac{1}{c\sqrt{t}} \int_0^{\infty} e^{-s^2} \cdot \cos \frac{(x-v)s}{c\sqrt{t}} ds$$

Putting $c p \sqrt{t} = s$ so that $dp = \frac{1}{c\sqrt{t}} ds$

$$= \frac{1}{c\sqrt{t}} \cdot \frac{\sqrt{\pi}}{2} \cdot e^{-\{(x-v)/2c\sqrt{t}\}^2} \left(\text{Since } \int_0^{\infty} e^{-s^2} \cos 2bs dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \right)$$

$$= \frac{\sqrt{\pi}}{2c\sqrt{t}} e^{-(x-v)^2/4c^2t}$$

$\therefore$ from (3), we have

$$\begin{aligned} u(x, t) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \frac{\sqrt{\pi} e^{-(x-v)^2/4c^2t}}{2c\sqrt{t}} dv \\ &= \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(v) e^{-(x-v)^2/4c^2t} dv = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x+2c\omega\sqrt{t}) e^{-\omega^2} dv \end{aligned}$$

Putting $\frac{v-x}{2c\sqrt{t}} = \omega$ so that $dv = 2c\sqrt{t} d\omega$

which is the required solution.

Case IV : Insulated Faces of the Rod (or Bar)

To find the temperature function of the rod (or bar) of length l which is perfectly insulated, also at the ends $x = 0$ and $x = l$, assuming that the initial temperature of the bar is $u(x, 0) = f(x)$.

[Meerut 2002 (B.P.)]

Let the rod be placed along x -axis with one-end at the origin.

The temperature $u(x, t)$ in the rod at any time t at a distance x from one end (i.e., origin) is governed by one-dimensional heat (diffusion) equation.

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{K}{\sigma \rho} \quad \dots(A)$$

Since the ends $x = 0$ and $x = l$ are insulated, so the flux of heat i.e., $-K \frac{\partial u}{\partial x}$, K being the thermal conductivity, through, these ends will be zero.

$\therefore$ The **boundary conditions** are

$$u_x(0, t) = 0 \quad \dots(B)$$

$$\text{and} \quad u_x(l, t) = 0 \quad \dots(\text{C})$$

$$t > 0$$

and the **initial condition** is

$$u(x, 0) = f(x), \quad 0 < x < l \quad \dots(\text{D})$$

$$\text{Let} \quad u(x, t) = F(x) T(t) = FT \text{ (say)} \quad \dots(1)$$

be the solution of (A).

Differentiating (1) and substituting in (A), we get

$$F \frac{dT}{dt} = c^2 T \frac{d^2 F}{dx^2} \quad \text{or} \quad \frac{1}{F} \frac{d^2 F}{dx^2} = \frac{1}{c^2 T} \frac{dT}{dt} = 0 \quad \text{or} \quad k^2 \text{ or } -k^2$$

Thus, we get three sets of two ordinary differential equations.

1. $\frac{d^2 F}{dx^2} = 0$ and $\frac{dT}{dt} = 0$
2. $\frac{d^2 F}{dx^2} - k^2 F = 0$ and $\frac{dT}{dt} = k^2 c^2 T$
3. $\frac{d^2 F}{dx^2} + k^2 F = 0$ and $\frac{dT}{dt} = -k^2 c^2 T$

The general solutions in these three cases are :

$$(i) \quad F = A_1 x + A_2 \quad \text{and} \quad T = A_3$$

$$\therefore u = FT = A_1' x + A_2' \quad \text{where} \quad A_1' = A_1 A_3 \quad \text{and} \quad A_2' = A_2 A_3$$

$$(ii) \quad F = B_1 e^{kx} + B_2 e^{-kx} \quad \text{and} \quad T = B_3 e^{k^2 c^2 t}$$

$$\therefore u = FT = (B_1' e^{kx} + B_2' e^{-kx}) e^{k^2 c^2 t}, \quad \text{where} \quad B_1' = B_1 B_3 \quad \text{and} \quad B_2' = B_2 B_3$$

$$(iii) \quad F = C_1 \cos kx + C_2 \sin kx \quad \text{and} \quad T = C_3 e^{-k^2 c^2 t}$$

$$\therefore u = FT = (C_1' e^{kx} + C_2' e^{-kx}) e^{-k^2 c^2 t}$$

Here in the present problem the solution given by (ii) is inadmissible as in this temperature $u(x, t)$ increases infinitely with time. The solution given in (iii) itself is inadequate to give the complete solution since in this case the temperature $u(x, t) \rightarrow 0$ as $t \rightarrow \infty$. Thus, our solution will be combination of solutions given in (i) and (iii)

From the solution in (i), $u(x, t) = A_1'x + A_2'$ $\therefore \frac{\partial u}{\partial x} = A_1'$

Now $u_x(0, t) = 0$ and $u_x(l, t) = 0 \Rightarrow A_1' = 0$

$$\therefore u(x, t) = A_2' \text{ (constant)} \quad \dots(2)$$

is the solution of (A), corresponding to (i).

Again if the solution of the heat equation (A) is given by (iii), then

$$u(x, t) = (C_1' \cos kx + C_2' \sin kx) e^{-k^2 c^2 t} \quad \dots(3)$$

$$\therefore u_x(x, t) = k(-C_1' \sin kx + C_2' \cos kx) e^{-k^2 c^2 t}$$

$$\therefore \text{From (3), } u_x(0, t) = 0 \Rightarrow k C_2' e^{-k^2 c^2 t} = 0 \Rightarrow C_2' = 0 \quad \because e^{-k^2 c^2 t} \neq 0$$

$$\text{and } u_x(l, t) = 0 \Rightarrow -k C_1' \sin kl e^{-k^2 c^2 t} = 0 \quad \because C_2' = 0$$

$$\Rightarrow \sin kl = 0 \quad \because C_1' \neq 0 \text{ and } e^{-k^2 c^2 t} \neq 0 \text{ otherwise } u(x, 0) = 0$$

$$\Rightarrow kl = n\pi, \text{ where } n \text{ is any integer}$$

$$\text{i.e., } k = n\pi/l$$

$\therefore$ From (3), for all integral values of n , the solution of (A) is given by

$$u_n(x, t) = D_n \cos\left(\frac{n\pi x}{l}\right) e^{-n^2 \pi^2 c^2 t/l^2} \quad \dots(4)$$

where $D_n = C_1'$ is new arbitrary constant, depending on n .

From (2) and taking $n = 1, 2, 3, \dots$, in (4), the complete solution of equation (A) satisfying conditions (B) and (C) can be taken as

$$u(x, t) = A_2' + \sum_{n=1}^{\infty} u_n(x, t)$$

$$\text{or } u(x, t) = A_2' + \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{l}\right) e^{-n^2 \pi^2 c^2 t/l^2} \quad \dots(5)$$

Now from (5), the condition (D),

$$\text{i.e., } u(x, 0) = f(x) \Rightarrow f(x) = A_2' + \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{l}\right) \quad \dots(6)$$

Now to find the values of A_2' and D_n we proceed as follows.

Multiplying both sides of (6) by $\cos\left(\frac{n\pi x}{l}\right)$ and integrating between the limits 0 to l , we get

$$D_n \int_0^l \cos^2\left(\frac{n\pi x}{l}\right) dx + A_2' \int_0^l \cos\left(\frac{n\pi x}{l}\right) dx = \int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

or
$$D_n \frac{1}{2} \int_0^l \left(1 + \cos \frac{2n\pi x}{l}\right) dx + A_2' \int_0^l \cos\left(\frac{n\pi x}{l}\right) dx = \int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

or
$$\frac{1}{2} \cdot D_n \cdot l + A_2' \cdot 0 = \int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

or
$$D_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx \quad \dots(7)$$

Again integrating both sides of (6) between the limits 0 to l , we get

$$A_2' \int_0^l dx + \sum_{n=1}^{\infty} D_n \int_0^l \cos \frac{n\pi x}{l} dx = \int_0^l f(x) dx$$

or
$$A_2' \cdot l + 0 = \int_0^l f(x) dx$$

or
$$A_2' = \frac{1}{l} \int_0^l f(x) dx = \frac{1}{2} D_0 \quad \dots(8)$$

Hence from (5) the complete solution of (A) satisfying the conditions (B), (C) and (D) is given by

$$u(x, t) = \frac{1}{2} D_0 + \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{l}\right) e^{-(n\pi c/l)^2 t} \quad \dots(9)$$

where D_n is given by

$$D_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

Solved Examples

Example 1: Solve the equation

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$$

given that $v = 0$, when $t = \infty$ and when $x = 0$ and when $x = l$. [Meerut 2000(P), 02 (P), 07]

Solution: Given differential equation is

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} \quad \dots(A)$$

and the given conditions are

$$v(0, t) = 0 \quad \dots(B)$$

$$v(l, t) = 0 \quad \dots(C)$$

$$\text{and} \quad v(x, \infty) = 0 \quad \dots(D)$$

Now the problem is to find solution of (A) satisfying (B), (C) and (D).

By the method of separation of variables, let

$$v = v(x, t) = F(x) T(t) = FT(\text{say}) \quad \dots(1)$$

where F is a function of x alone and T function of t alone.

Differentiating (1) and substituting in (A), we get

$$T \frac{d^2 F}{dx^2} = F \frac{dT}{dt} \quad \text{or} \quad \frac{1}{F} \frac{d^2 F}{dx^2} = \frac{1}{T} \frac{dT}{dt}$$

Since the two sides are functions of different independent variables, so both sides will be equal to a constant value

$$\therefore \quad \frac{1}{F} \frac{d^2 F}{dx^2} = \frac{1}{T} \frac{dT}{dt} = \lambda \text{ (constant)}$$

Since λ may be equal to 0 or $k^2 > 0$ or $-k^2 < 0$

$\therefore$ We get the following pairs of ordinary differential equations :

$$1. \quad \frac{d^2 F}{dx^2} = 0 \quad \text{and} \quad \frac{dT}{dt} = 0$$

$$2. \quad \frac{d^2 F}{dx^2} - k^2 F = 0 \quad \text{and} \quad \frac{dT}{dt} = k^2 T$$

$$3. \quad \frac{d^2 F}{dx^2} + k^2 F = 0 \quad \text{and} \quad \frac{dT}{dt} = -k^2 T$$

The general solutions in these three cases are :

$$(i) \quad F(x) = A_1 x + A_2 \quad \text{and} \quad T = A_3$$

$$(ii) \quad F(x) = B_1 e^{kx} + B_2 e^{-kx} \quad \text{and} \quad T = B_3 e^{k^2 t}$$

$$(iii) \quad F(x) = C_1 \cos kx + C_2 \sin kx \quad \text{and} \quad T = C_3 e^{-k^2 t}$$

Now the condition (D) implies that as $t \rightarrow \infty$ $v \rightarrow 0$,

which is possible only in case (iii) as $T \rightarrow 0$ as $t \rightarrow \infty$. Thus (i) and (ii) do not constitute the solution of the problem.

$\therefore$ We take the solution (C).

$$i.e., \quad v(x, t) = F(x) T(t) = (C_1 \cos kx + C_2 \sin kx) C_3 e^{-k^2 t}$$

$$i.e., \quad v(x, t) = (C_1' \cos kx + C_2' \sin kx) \cdot e^{-k^2 t} \quad \dots(2)$$

where $C_1' = C_1 C_3$ and $C_2' = C_2 C_3$

$$\therefore \quad \text{from (2) and (B), } v(0, t) = 0 \Rightarrow C_1' e^{-k^2 t} = 0 \Rightarrow C_1' = 0$$

$$\text{from (2) and (C) } v(l, t) = 0 \Rightarrow C_2' \sin kl \cdot e^{-k^2 t} = 0 \quad \because C_1' = 0$$

$$\Rightarrow \sin kl = 0 \quad \because C_2' = 0 \quad \text{or} \quad e^{-k^2 t} = 0 \text{ will give } v(x, t) = 0 \text{ which is inadmissible.}$$

$$\Rightarrow \quad kl = n\pi$$

i.e., $k = n\pi/l$, where n is any integer.

$\therefore$ from (2), solutions of (A) satisfying (B) and (C) are given by

$$v_n(x, t) = D_n \sin\left(\frac{n\pi x}{l}\right) e^{-n^2 \pi^2 t/l^2}$$

where n is any integer and $D_n = C_2'$ is the new arbitrary variable.

Now taking $n = 1, 2, 3, \dots$, the more general solution of (A) satisfying conditions (B), (C) and (D) is given by

$$v(x, t) = \sum_{n=1}^{\infty} v_n(x, t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{l}\right) e^{-n^2 \pi^2 t/l^2}$$

Example 2: Solve one dimensional diffusion equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{k} \frac{\partial \theta}{\partial t}$$

in the region $0 \leq x \leq \pi$, $t \geq 0$, when

- (i) θ remains finite as $t \rightarrow \infty$,
 (ii) $\theta = 0$ if $x = 0$ or π , for all values of t ,
 (iii) At $t = 0$,

$$\theta = x, \quad 0 \leq x \leq \pi/2$$

$$\theta = \pi - x, \quad \pi/2 \leq x \leq \pi.$$

[Meerut 2007 (BP)]

Solution: Here the given equation is

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{k} \frac{\partial \theta}{\partial t}, \quad 0 \leq x \leq \pi \text{ and } t \geq 0 \quad \dots(A)$$

$$\text{Let } \theta = \theta(x, t) = F(x) T(t) = FT(\text{say}) \quad \dots(1)$$

where F is a function of x alone and T function of t alone.

Differentiating (1) and substituting in (A), we get

$$T \frac{d^2 F}{dx^2} = \frac{1}{k} F \frac{dT}{dt} \quad \text{or} \quad \frac{1}{F} \frac{d^2 F}{dx^2} = \frac{1}{kT} \frac{dT}{dt}$$

Since the two sides are functions of different independent variables, so both will be equal to each other if they reduce to a constant value.

$$\therefore \quad \frac{1}{F} \frac{d^2 F}{dx^2} = \frac{1}{kT} \frac{dT}{dt} = \text{constant} = 0, \lambda^2, -\lambda^2.$$

Thus, we get the following pairs of ordinary differential equations.

1. $\frac{d^2 F}{dx^2} = 0$ and $\frac{dT}{dt} = 0$
2. $\frac{d^2 F}{dx^2} - \lambda^2 F = 0$ and $\frac{dT}{dt} = k \lambda^2 T$
3. $\frac{d^2 F}{dx^2} + \lambda^2 F = 0$ and $\frac{dT}{dt} = -k \lambda^2 T$

The general solutions of equations in these cases are :

$$(i) \quad F(x) = A_1 x + A_2 \quad \text{and} \quad T = A_3$$

$$(ii) \quad F(x) = B_1 e^{\lambda x} + B_2 e^{-\lambda x} \quad \text{and} \quad T = A_3 e^{k\lambda^2 t}$$

$$(iii) \quad F(x) = C_1 \cos \lambda x + C_2 \sin \lambda x \quad \text{and} \quad T = C_3 e^{-k\lambda^2 t}$$

Now according to condition (i) θ remains finite as $t \rightarrow \infty$, so the solution given in (ii) cannot be the solution of the problem as here $T \rightarrow \infty$, as $t \rightarrow \infty$. If we take the solution of the problem as given in (i).

$$i.e., \text{ if we take } \theta(x, t) = F(x) T(t) = (A_1 x + A_2) A_3 = A_1' x + A_2'$$

$$\text{where } A_1 A_3 = A_1' \text{ and } A_2 A_3 = A_2'$$

$$\therefore \text{ Conditions given in (ii) } \theta(0, t) = 0 \text{ and } \theta(\pi, t) = 0 \Rightarrow A_2' = 0 \text{ and } A_1' = 0$$

$$\therefore u(x, t) = 0 \text{ which is trivial solution and is inadmissible.}$$

$$\therefore \text{ Taking solution given in (iii)}$$

$$\theta(x, t) = F(x) T(t) = (C_1' \cos \lambda x + C_2' \sin \lambda x) e^{-k\lambda^2 t} \quad \dots(2)$$

$$C_1' = C_1 C_3 \quad \text{and} \quad C_2' = C_2 C_3.$$

Now from the condition given in (ii) and solution (2), we have

$$\theta(0, t) \Rightarrow C_1' e^{-k\lambda^2 t} = 0 \Rightarrow C_1' = 0$$

$$\therefore e^{-k\lambda^2 t} = 0 \text{ will lead to trivial solution } \theta(x, t) = 0.$$

$$\text{and } \theta(\pi, t) = 0 \Rightarrow C_2' \sin \lambda \pi \cdot e^{-k\lambda^2 t} = 0 \Rightarrow \sin \lambda \pi = 0,$$

$$\therefore e^{-k\lambda^2 t} \neq 0 \text{ and } C_2' \neq 0.$$

$$\Rightarrow \lambda \pi = n\pi, \text{ i.e., } \lambda = n, \text{ where } n \text{ is any integer.}$$

$\therefore$ from (2), for all integral values of n , the solutions is equation (A) satisfying condition (i) and (ii) are given by

$$\theta_n(x, t) = D_n \sin nx \cdot e^{-k n^2 t} \quad \therefore \lambda = n$$

where $D_n = C_2'$ is another arbitrary constant depending on n .

Now taking $n = 1, 2, 3, \dots$, the more general solution of (A) is given by

$$\theta(x, t) = \sum_{n=1}^{\infty} \theta_n(x, t) = \sum_{n=1}^{\infty} D_n \sin nx \cdot e^{-k n^2 t} \quad \dots(3)$$

This solution will also satisfy conditions (i) and (ii).

Now from the condition (iii) and solution (3), we have

$$\theta(x, 0) = f(x) \Rightarrow \sum_{n=1}^{\infty} D_n \sin nx = f(x) \quad \dots(4)$$

where
$$f(x) = \begin{cases} x, & 0 \leq x \leq \pi/2 \\ \pi - x, & \pi/2 \leq x \leq \pi \end{cases}$$

The left hand side in (4) can be considered as the Fourier sine series of function $f(x)$ on its right hand side for $0 \leq x \leq \pi$.

$$\begin{aligned} \therefore D_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi} \left[\int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx \right] \\ &= \frac{2}{\pi} \left[\left\{ x \left(-\frac{1}{n} \cos nx \right) + \frac{1}{n^2} \sin nx \right\}_0^{\pi/2} + \left\{ (\pi - x) \left(-\frac{1}{n} \cos nx \right) - \frac{1}{n^2} \sin nx \right\}_{\pi/2}^{\pi} \right] \\ &= \frac{2}{\pi} \left[\left\{ -\frac{\pi}{2n} \cos \left(\frac{n\pi}{2} \right) + \frac{1}{n^2} \sin \left(\frac{n\pi}{2} \right) \right\} + \left\{ \frac{\pi}{2n} \cos \left(\frac{n\pi}{2} \right) + \frac{1}{n^2} \sin \left(\frac{n\pi}{2} \right) \right\} \right] \\ &= \frac{4}{\pi n^2} \sin \left(\frac{n\pi}{2} \right) \end{aligned}$$

$$\therefore \text{ When } n = 2r \text{ (even), then } D_n = \frac{4}{\pi (2r)^2} \sin \left(\frac{2r\pi}{2} \right) = 0$$

and when $n = (2r + 1)$, odd $r = 0, 1, 2, \dots$ Then

$$\begin{aligned} D_n &= \frac{4}{\pi (2r + 1)^2} \sin \left\{ (2r + 1) \frac{\pi}{2} \right\} = \frac{4}{\pi (2r + 1)^2} \sin \left(r\pi + \frac{\pi}{2} \right) \\ &= \frac{4}{\pi (2r + 1)^2} \left\{ \sin r\pi \cos \frac{\pi}{2} + \cos r\pi \sin \frac{\pi}{2} \right\} = \frac{4(-1)^r}{\pi (2r + 1)^2} \end{aligned}$$

$\therefore$ from (3) the required solution is

$$\theta(x, t) = \frac{4}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r + 1)^2} \sin(2r + 1)x \cdot e^{-k(2r + 1)^2 t}$$

which may also be written as

$$\theta(x, t) = \frac{4}{\pi} \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{(2r - 1)^2} \sin(2r - 1)x \cdot e^{-k(2r - 1)^2 t}$$

Example 3: Show that the solution of the equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t}$$

satisfying the conditions

(i) $\theta \rightarrow 0$ as $t \rightarrow \infty$

(ii) $\theta = 0$ when $x = \pm a$ for all values of $t > 0$

(iii) $\theta = x$ when $t = 0$ and $-a < x < a$ is

$$\theta = \frac{2a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin\left(\frac{n\pi x}{a}\right) \exp\left(-\frac{n^2 \pi^2 t}{a^2}\right). \quad [\text{Meerut 2005 (B.P.), 09 (B.P.)}]$$

Solution: Given equation is

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t} \quad \dots(A)$$

$$\text{Let } \theta(x, t) = F(x) T(t) = FT(\text{say}) \quad \dots(1)$$

where F is a function of x alone and T function of t alone.

Proceeding as in Ex. 1 page 604 the solution of (A) satisfying condition (i) can be taken as

$$\theta(x, t) = (C_1' \cos kx + C_2' \sin kx) \cdot e^{-k^2 t} \quad \dots(2)$$

Condition given in (ii) are $\theta(a, t) = 0$ and $\theta(-a, t) = 0$, for $t > 0$

$$\therefore \text{ from (2), } \theta(a, t) = 0 \Rightarrow (C_1' \cos ka + C_2' \sin ka) \cdot e^{-k^2 t} = 0$$

$$\text{and } \theta(-a, t) = 0 \Rightarrow (C_1' \cos ka - C_2' \sin ka) \cdot e^{-k^2 t} = 0$$

$$\text{Adding, } 2C_1' \cos ka \cdot e^{-k^2 t} = 0 \Rightarrow C_1' = 0$$

$$\therefore e^{-k^2 t} = 0 \text{ will lead to inadmissible solution } \theta(x, t) = 0$$

$$\text{Subtracting, } 2C_2' \sin ka \cdot e^{-k^2 t} = 0 \Rightarrow \sin ka = 0$$

$$\therefore C_2' = 0 \text{ or } e^{-k^2 t} = 0 \text{ will lead to inadmissible trivial solution } \theta(x, t) = 0$$

$$\Rightarrow ka = n\pi, n \text{ is any integer}$$

$$\text{i.e., } k = n\pi/a$$

For this value of k , from (2), the solutions of (A) for integral values of n are given by

$$\theta_n(x, t) = D_n \sin\left(\frac{n\pi x}{a}\right) e^{-n^2 \pi^2 t/a^2}$$

where $D_n = C_2'$ is another arbitrary variable depending on n .

Taking $n = 1, 2, 3, \dots$, the more general solution of (A) can be taken as

$$\theta(x, t) = \sum_{n=1}^{\infty} \theta_n(x, t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{a}\right) \cdot e^{-n^2 \pi^2 t/a^2} \quad \dots(3)$$

Condition (iii) is $\theta(x, 0) = x, -a < x < a$

$$\therefore (3) \Rightarrow \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{a}\right) = x$$

Here the left hand side can be considered as the Fourier sine series of function x on its right hand side for $0 < x < a$.

$$\begin{aligned} \therefore D_n &= \frac{2}{a} \int_0^a x \cdot \sin\left(\frac{n\pi x}{a}\right) dx \\ &= \frac{2}{a} \left[x \cdot \left(-\frac{a}{n\pi}\right) \cos\left(\frac{n\pi x}{a}\right) + \frac{a^2}{n^2 \pi^2} \sin\left(\frac{n\pi x}{a}\right) \right]_0^a \\ &= -\frac{2a}{n\pi} \cos n\pi = -\frac{2a}{n\pi} (-1)^n = \frac{2a}{n\pi} (-1)^{n+1} = \frac{2a}{n\pi} (-1)^{n-1} \end{aligned}$$

$\therefore$ from (3), the required solution is

$$\begin{aligned} \theta(x, t) &= \sum_{n=1}^{\infty} \frac{2a}{n\pi} (-1)^{n-1} \sin\left(\frac{n\pi x}{a}\right) \cdot e^{-n^2 \pi^2 t/a^2} \\ &= \frac{2a}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \cdot \sin\left(\frac{n\pi x}{a}\right) \cdot \exp\left(\frac{n^2 \pi^2 t}{a^2}\right) \end{aligned}$$

Example 4: Find that temperature $u(x, t)$ in a bar of silver (length 10 cm, constant cross-section of area 1 cm², density 10.6 gm/cm³, the thermal conductivity 1.04 cal/cm. deg sec, specific heat 0.056 cal/gm deg.) which is perfectly insulated laterally, whose ends are kept at temperature 0°C and whose initial temperature (in °C) is $f(x)$, where

$$(i) \quad f(x) = \begin{cases} x, & \text{if } 0 < x < 5 \\ 10 - x, & \text{if } 5 < x < 10 \end{cases}$$

$$(ii) \quad f(x) = x(10 - x).$$

Solution: The temperature $u(x, t)$ in a bar is governed by the heat equation

$$\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial u}{\partial t} \quad \dots(A)$$

where $c^2 = \frac{k}{\sigma \rho}$

where k = Thermal conductivity = 1.04 cal/cm deg. sec. (Given)

σ = specific heat = 0.056 cal/gm deg. (Given)

and ρ = density = mass per unit length of bar = 10.6 gm/cm² (Given)

$$\therefore c^2 = \frac{1.04}{(0.056)(10.6)} = 1.752$$

Given length of the rod = $l = 10$ cm

The **boundary conditions** of the problem are

$$u(0, t) = 0 \text{ and } u(10, t) = 0 \quad \because l = 10 \quad \dots(B)$$

and **initial condition** is $u(x, 0) = f(x)$...(C)

Let $u(x, t) = F(x) T(t) = FT(\text{say})$...(1)

be the solution of (A) satisfying (B) and (C).

Proceeding as in 13.6 case I page 614, the solution of the problem is given by

$$u(x, t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{10}\right) e^{-n^2 \pi^2 c^2 t / 10^2} \quad \dots(2)$$

$$\because l = 10 \text{ cm}$$

where $D_n = \frac{2}{10} \int_0^{10} f(x) \sin\left(\frac{n\pi x}{10}\right) dx$...(3)

(i) Here $f(x) = \begin{cases} x & \text{if } 0 < x < 5 \\ 10 - x & \text{if } 5 < x < 10 \end{cases}$

$\therefore$ from (3)

$$\therefore D_n = \frac{1}{5} \left[\int_0^5 x \sin\left(\frac{n\pi x}{10}\right) dx + \int_5^{10} (10 - x) \sin\left(\frac{n\pi x}{10}\right) dx \right]$$

$$\begin{aligned}
&= \frac{1}{5} \left[\left\{ x \left(-\frac{10}{n\pi} \cos \left(\frac{n\pi x}{10} \right) \right) + \left(\frac{10}{n\pi} \right)^2 \sin \left(\frac{n\pi x}{10} \right) \right\}_0^5 \right. \\
&\quad \left. + \left\{ (10-x) \left(-\frac{10}{n\pi} \cos \left(\frac{n\pi x}{10} \right) \right) - \left(\frac{10}{n\pi} \right)^2 \sin \frac{n\pi x}{10} \right\}_5^{10} \right] \\
&= \frac{1}{5} \left[\left\{ -\frac{50}{n\pi} \cos \frac{n\pi}{2} + \frac{100}{n^2 \pi^2} \sin \frac{n\pi}{2} \right\} + \left\{ +\frac{50}{n\pi} \cos \frac{n\pi}{2} + \frac{100}{n^2 \pi^2} \sin \frac{n\pi}{2} \right\} \right] \\
&= \frac{40}{n^2 \pi^2} \sin \frac{n\pi}{2}
\end{aligned}$$

$$\therefore \text{ when } n = 2r \text{ (even), then } D_n = \frac{40}{n^2 \pi^2} \sin(r\pi) = 0$$

and when $n = (2r+1)$ odd, then $r = 0, 1, 2, \dots$, then

$$\begin{aligned}
D_n &= \frac{40}{(2r+1)^2 \pi^2} \sin \left((2r+1) \frac{\pi}{2} \right) = \frac{40}{(2r+1)^2 \pi^2} \sin \left(r\pi + \frac{\pi}{2} \right) \\
&= \frac{40}{(2r+1)^2 \pi^2} \left(\sin r\pi \cos \frac{\pi}{2} + \cos r\pi \sin \frac{\pi}{2} \right) = \frac{40(-1)^r}{(2r+1)^2 \pi^2}
\end{aligned}$$

$\therefore$ from (2), the required solution is

$$u(x, t) = \frac{40}{\pi^2} \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)^2} \sin \left(\frac{(2r+1)\pi x}{10} \right) e^{-0.01752 (2r+1)^2 \pi^2 t}$$

$$\therefore c^2 = 1.752$$

which may also be written as

$$u(x, t) = \frac{40}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{r-1}}{(2r-1)^2} \sin \left(\frac{(2r-1)\pi x}{10} \right) e^{-0.01752 (2r-1)^2 \pi^2 t}$$

(ii) Here $f(x) = x(10-x)$

$$\text{From (3), } D_n = \frac{1}{5} \int_0^{10} x(10-x) \cdot \sin \left(\frac{n\pi x}{10} \right) dx$$

$$\begin{aligned}
&= \frac{1}{5} \left[\left\{ x(10-x) \cdot \left(-\frac{10}{n\pi} \cos \frac{n\pi x}{10} \right) - (10-2x) \left(\frac{10^2}{n^2 \pi^2} \sin \frac{n\pi x}{10} \right) \right\}_0^{10} \right. \\
&\quad \left. + \int_0^{10} (-2) \cdot \frac{10^2}{n^2 \pi^2} \sin \left(\frac{n\pi x}{10} \right) dx \right]
\end{aligned}$$

$$= \left[-\frac{200}{5n^2\pi^2} \cdot \left(-\frac{10}{n\pi}\right) \cos\left(\frac{n\pi x}{10}\right) \right]_0^{10} = \frac{400}{n^3\pi^3} (1 - \cos n\pi)$$

$$= 0, \text{ if } n \text{ is even and } \frac{800}{n^3\pi^3} \text{ when } n \text{ is odd.}$$

Putting in (2), and taking $n = 2m - 1$ odd, the required solution is given by

$$u(x, t) = \frac{800}{\pi^3} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \sin\left((2m-1)\frac{\pi x}{10}\right) \cdot e^{-0.01752(2m-1)^2 \pi^2 t}$$

Example 5: Find the solution of $\frac{\partial^2 u}{\partial x^2} = h^2 \frac{\partial u}{\partial t}$ for which $u(0, t) = u(l, t) = 0$,

$u(x, 0) = \sin(\pi x/l)$ by the method of variable separable.

Solution: Given equation is

$$\frac{\partial^2 u}{\partial x^2} = h^2 \frac{\partial u}{\partial t} \quad \dots(A)$$

$$\text{Let } u(x, t) = F(x) \cdot T(t) = FT \text{ (say)} \quad \dots(1)$$

be the solution of (A).

Proceeding as in 13.6 case I page 614, the solution of (A) satisfying the given conditions $u(0, t) = 0 = u(l, t)$ is given by

$$u(x, t) = \sum_{n=1}^n D_n \sin\left(\frac{n\pi x}{l}\right) e^{-n^2 \pi^2 t/l^2 h^2} \quad \dots(2)$$

$$\text{Here } c^2 = 1/h^2$$

$$\text{Now from (2) and the condition } u(x, 0) = \sin\left(\frac{\pi x}{l}\right)$$

$$\sin\left(\frac{\pi x}{l}\right) = \sum_{n=1}^n D_n \sin\left(\frac{n\pi x}{l}\right) \quad \dots(3)$$

Comparing coefficient of sines of same angles from the two sides, we get

$$D_1 = 1 \text{ for } n = 1, \text{ and } D_2 = D_3 = \dots = 0 \text{ (each)}$$

$\therefore$ From (2), the required solution is

$$u(x, t) = \sin\left(\frac{\pi x}{l}\right) e^{-\pi^2 t/l^2 h^2}$$

Aliter : In the above solution, in equation (3), right hand side can be considered as the Fourier sine series of the function $\sin(\pi x/l)$ on its left hand side for $0 < x < l$.

$$\therefore D_n = \frac{2}{l} \int_0^l \sin\left(\frac{\pi x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) dx$$

Obviously $D_n = 0$ if $n = 2, 3, 4, \dots$

$$\text{For } n=1, D_1 = \frac{2}{l} \int_0^l \sin^2\left(\frac{\pi x}{l}\right) dx = \frac{1}{l} \int_0^l [1 - \cos(2\pi x/l)] dx = 1$$

Hence, from (2), the required solution is $u(x, t) = \sin\left(\frac{\pi x}{l}\right) e^{-\pi^2 t/l^2 h^2}$

Example 6: Solve the one dimensional diffusion equation in the range $0 \leq x \leq 2\pi$, $t > 0$, subject to the conditions.

$$\theta(x, 0) = \sin^3 x \quad \text{for } 0 \leq x \leq 2\pi$$

$$\theta(0, t) = \theta(2\pi, t) = 0 \quad \text{for } t \geq 0.$$

[Meerut 2008]

Solution: One dimensional diffusion equation is given by

$$\frac{\partial \theta}{\partial t} = c^2 \frac{\partial^2 \theta}{\partial x^2} \quad \dots(A)$$

$$0 \leq x \leq 2\pi, t \geq 0$$

Here we have to solve (A) subject to the following conditions

$$\theta(0, t) = 0, \theta(2\pi, t) = 0, t \geq 0 \quad \dots(B)$$

$$\text{and} \quad \theta(x, 0) = \sin^3 x \quad \text{for} \quad 0 \leq x \leq 2\pi \quad \dots(C)$$

$$\text{Let} \quad \theta(x, t) = F(x) T(t) = FT \text{ (say)} \quad \dots(1)$$

be the solution of (A), satisfying (B) and (C).

Proceeding as in 13.6 case I, the most general solution of (A) satisfying conditions (B) is given by

$$\theta(x, t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{2}\right) e^{-n^2 c^2 t/4} \quad (\text{Here } l = 2\pi) \quad \dots(2)$$

From (2) and condition (C),

$$\theta(x, 0) = \sin^3 x \Rightarrow \sum_{n=1}^{\infty} D_n \sin\left(\frac{nx}{2}\right) = \sin^3 x$$

or
$$\sum_{n=1}^{\infty} D_n \sin\left(\frac{nx}{2}\right) = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$$

Equating coefficient of sines of like angles, we get

For $n=2$, $D_2 = 3/4$ and for $n=6$, $D_6 = -1/4$ and for other values of n ,

$$D_1 = D_3 = D_4 = D_5 = D_7 = \dots = 0 \text{ (each).}$$

$\therefore$ From (2), the required solution is

$$\theta(x, t) = D_2 \sin\left(\frac{2x}{2}\right) e^{-4c^2 t/4} + D_6 \sin\left(\frac{6x}{2}\right) e^{-36c^2 t/4}$$

or
$$\theta(x, t) = \frac{1}{4} [3 \sin x \cdot e^{-c^2 t} - \sin 3x \cdot e^{-9c^2 t}]$$

Aliter : Proceed upto equation (2).

From condition (C) and solution (2),

$$\theta(x, 0) = \sin^3 x \Rightarrow \sum_{n=1}^{\infty} D_n \sin\left(\frac{nx}{2}\right) = \sin^3 x \quad (\text{Here } l = 2\pi) \dots (3)$$

$\therefore$ Left hand of (3) can be considered as fourier sine series of function $\sin^3 x$ on the right hand side for $0 < x < 2\pi$.

$$\begin{aligned} \therefore D_n &= \frac{2}{2\pi} \int_0^{2\pi} \sin^3 x \cdot \sin\left(\frac{nx}{2}\right) dx \quad \because l = 2\pi \\ &= \frac{1}{\pi} \int_0^{2\pi} \frac{1}{4} (3 \sin x - \sin 3x) \sin(nx/2) dx \\ &= \frac{3}{4\pi} \int_0^{2\pi} \sin x \sin(nx/2) dx - \frac{1}{4\pi} \int_0^{2\pi} \sin 3x \sin(nx/2) dx \end{aligned}$$

Obviously first integral is zero for all of n other than 2 and second integral is zero for all values of n other than 6.

$$\therefore \text{ For } n=2, \quad D_2 = \frac{3}{4\pi} \int_0^{2\pi} \sin^2 x dx - 0 = \frac{3}{4\pi} \int_0^{2\pi} \frac{1}{2} (1 - \cos 2x) dx = \frac{3}{4}$$

and for $n=6$,
$$D_6 = 0 - \frac{1}{4\pi} \int_0^{2\pi} \sin^2 3x \, dx = -\frac{1}{4\pi} \int_0^{2\pi} \frac{1}{2} (1 - \cos 6x) \, dx = -\frac{1}{4}$$

and for other values of n i.e., for $n=1, 3, 4, 5, 7, \dots$ both integrals are zero.

$$\therefore D_1 = D_3 = D_4 = D_5 = D_7 = \dots = 0 \quad (\text{each})$$

$\therefore$ From (2), the required solution is

$$\theta(x, t) = D_2 \sin\left(\frac{2x}{2}\right) e^{-4c^2 t/4} + D_6 \sin\left(\frac{6x}{2}\right) e^{-36c^2 t/4}$$

or
$$\theta(x, t) = \frac{1}{4} (3 \sin x \cdot e^{-c^2 t} - \sin 3x \cdot e^{-9c^2 t})$$

Example 7: An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C , find the temperature at a distance x from A at time t .

[Meerut 2008]

Solution: Let the rod lie along the axis of x and let the end A be at the origin.

The temperature $u(x, t)$ in the rod at any time t and at a distance x from A (origin) is governed by one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots(\text{A})$$

Before the temperature is changed at the end B , in the steady state u is independent of t

$$\therefore \frac{\partial u}{\partial t} = 0$$

$$\therefore \text{from (A), we have } \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u = Ax + B \quad \dots(1)$$

Now at end $A, x=0, u=0$ and at the end $B, x=l, u=100$

$$\therefore 0 = A \cdot 0 + B \text{ and } 100 = Al + B \Rightarrow B = 0 \text{ and } A = 100/l.$$

$\therefore$ from (1), in the steady state the temperature in the rod is given by

$$u = (100/l)x \quad \dots(2)$$

When B is reduced to 0°C , then the boundary conditions of the problem are

$$u(0, t) = 0 \quad \text{and} \quad u(l, t) = 0, \quad t > 0 \quad \dots(\text{B})$$

and the initial condition is

$$u(x, 0) = (100/l)x, \quad 0 < x < l \quad \dots(\text{C})$$

Thus, we have to find the solution of (A) subject to conditions (B) and (C).

$$\text{Let } u(x, t) = F(x) T(t) = FT \text{ (say)} \quad \dots(3)$$

be the solution of (A), where F is a function of x alone and T a function of t alone.

Now proceeding as in 13.6 case I, the solution of the problem is given by

$$u(x, t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{l}\right) e^{-n^2 \pi^2 c^2 t/l^2} \quad \dots(4)$$

$$\text{where } D_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$\text{Here } f(x) = (100 x/l)$$

$$\therefore D_n = \frac{2}{l} \int_0^l \frac{100x}{l} \sin\left(\frac{n\pi x}{l}\right) dx$$

$$\begin{aligned} &= \frac{2}{l} \left[\frac{100x}{l} \left\{ -\frac{l}{n\pi} \cos\left(\frac{n\pi x}{l}\right) \right\} + \left(\frac{l}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{l}\right) \right]_0^l \\ &= \frac{200}{l^2} \cdot \left(-\frac{l^2}{n} \right) \cos n\pi = -\frac{200}{n} (-1)^n = \frac{200}{n} (-1)^{n+1} \end{aligned}$$

$\therefore$ from (4), the required solution is

$$u(x, t) = \frac{200}{n} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{l}\right) e^{-(n\pi c/l)^2 t}$$

Example 8: A bar of length unity, has its end $x = 0$ insulated and its end at $x = 1$ is kept at temperature zero. Find an expression for the temperature $u(x, t)$, if

$$u(x, 0) = \begin{cases} 1 & \text{when } 0 \leq x \leq 1/2 \\ 2(1-x) & \text{when } 1/2 \leq x \leq 1 \end{cases} \quad [\text{Meerut 2011 (B.P.)}]$$

Solution: The temperature $u(x, t)$ in the bar is governed by one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots(A)$$

Here the **boundary conditions** are

$$u_x(0, t) = 0 \quad \text{and} \quad u(1, t) = 0 \quad \dots(B)$$

and **initial condition** is $u(x, 0) = f(x) = \begin{cases} 1 & \text{when } 0 \leq x \leq 1/2 \\ 2(1-x) & \text{when } 1/2 \leq x \leq 1 \end{cases} \quad \dots(C)$

Let $u(x, t) = F(x) \cdot T(t) = FT$ (say) $\dots(1)$

be the solution of (A), where F is a function of x alone and T function of t alone. Differentiating (1) and substituting in (A), we get

$$F \frac{dT}{dt} = c^2 T \frac{d^2 F}{dx^2}$$

or $\frac{1}{F} \frac{d^2 F}{dx^2} = \frac{1}{c^2 T} \frac{dT}{dt} = 0 \text{ or } k^2 \text{ or } -k^2$

Since both sides are functions of different independent variables, so both sides will be equal to a constant value.

Thus, we get three sets of two ordinary differential equations :

1. $\frac{d^2 F}{dx^2} = 0$ and $\frac{dT}{dt} = 0$
2. $\frac{d^2 F}{dx^2} - k^2 F = 0$ and $\frac{dT}{dt} = k^2 c^2 T$
3. $\frac{d^2 F}{dx^2} + k^2 F = 0$ and $\frac{dT}{dt} = -k^2 c^2 T$

The solutions in these three cases are :

(i) $F = A_1 x + A_2$ and $T = A_3$

$\therefore u(x, t) = FT = A_1' x + A_2'$, where $A_1 A_3 = A_1'$ and $A_2 A_3 = A_2'$

(ii) $F = B_1 e^{kx} + B_2 e^{-kx}$ and $T = B_3 e^{k^2 c^2 t}$

$\therefore u(x, t) = FT = (B_1' e^{kx} + B_2' e^{-kx}) \cdot e^{k^2 c^2 t}$, where $B_1 B_3 = B_1'$ and $B_2 B_3 = B_2'$

(iii) $F = C_1 \cos kx + C_2 \sin kx$ and $T = C_3 e^{-k^2 c^2 t}$

$\therefore u(x, t) = FT = (C_1' \cos kx + C_2' \sin kx) e^{-k^2 c^2 t}$ where $C_1 C_3 = C_1'$, $C_2 C_3 = C_2'$

For the solution given in (i), we get

$$u_x(x, t) = A_1'$$

$\therefore$ Boundary conditions $u_x(0, t) = 0$ and $u(l, t) = 0 \Rightarrow A_1' = 0$ and $A_1' + A_2' = 0$

$$\Rightarrow A_1' = 0 = A_2' \Rightarrow u(x, t) = 0$$

which is inadmissible as it is trivial solution.

Solution given in (ii) is also inadmissible as here $u \rightarrow \infty$ as $t \rightarrow \infty$.

$\therefore$ The solution of the problem can be taken as given in (iii)

$$i.e., \quad u(x, t) = (C_1' \cos kx + C_2' \sin kx) e^{-k^2 c^2 t} \quad \dots(2)$$

$$\Rightarrow \quad u_x(x, t) = k(-C_1' \sin kx + C_2' \cos kx) \cdot e^{-k^2 c^2 t}$$

$\therefore$ Boundary condition, $u_x(0, t) = 0$

$$\Rightarrow \quad k C_2' e^{-k^2 c^2 t} = 0 \Rightarrow C_2' = 0$$

$\because \quad e^{-k^2 c^2 t} \neq 0$ otherwise it will lead to a trivial solution.

$\therefore$ from (2), we have

$$u(x, t) = C_1' \cos kx \cdot e^{-k^2 c^2 t} \quad \dots(3)$$

$$\therefore \quad u(1, t) = 0 \quad \Rightarrow \quad C_1' \cos k \cdot e^{-k^2 c^2 t} = 0$$

$$\Rightarrow \quad \cos k = 0, \quad \because \quad C_1' \neq 0, e^{-k^2 c^2 t} \neq 0$$

$$\Rightarrow \quad k = (2n-1)\pi/2, \text{ where } n \text{ is any integer.}$$

$\therefore$ For integral values of n , from (3) the solutions of equation (A) satisfying conditions (B) are given by

$$u_n(x, t) = D_n \cos \left\{ (2n-1) \frac{\pi}{2} x \right\} \cdot e^{-(2n-1)^2 \pi^2 c^2 t/4}$$

$\therefore$ Taking $n = 1, 2, 3, \dots$, more general solution of equation (A) satisfying boundary conditions (B) are given by

$$u_n(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} D_n \cos \frac{(2n-1)\pi x}{2} e^{-(2n-1)^2 \pi^2 c^2 t/4} \quad \dots(4)$$

Now from (3), condition (iii) i.e.,

$$u(x, 0) = f(x) \Rightarrow f(x) = \sum_{n=1}^{\infty} D_n \cos \frac{(2n-1)\pi x}{2} \quad \dots(5)$$

Multiplying both sides of (4) by $2 \cos \frac{(2m-1)\pi x}{2}$ and integrating, between the limits 0 to

1, we get

$$2 \int_0^1 f(x) \cos \frac{(2m-1)\pi x}{2} dx = \sum_{n=1}^{\infty} 2D_n \int_0^1 \cos \frac{(2n-1)\pi x}{2} \cos \frac{(2m-1)\pi x}{2} dx \quad \dots(6)$$

Now in equation (6), L.H.S.

$$\begin{aligned} &= 2 \int_0^{1/2} 1 \cdot \cos \frac{(2m-1)\pi x}{2} dx + 2 \int_{1/2}^1 2(1-x) \cos \frac{(2m-1)\pi x}{2} dx \\ &= \left[\frac{2^2}{(2m-1)\pi} \sin \frac{(2m-1)\pi x}{2} \right]_0^{1/2} + 4 \left[(1-x) \frac{2}{(2m-1)\pi} \sin \frac{(2m-1)\pi x}{2} \right. \\ &\quad \left. - \frac{2^2}{(2m-1)^2 \pi^2} \cos \frac{(2m-1)\pi x}{2} \right]_{1/2}^1 \\ &= \frac{4}{(2m-1)\pi} \sin \frac{(2m-1)\pi}{4} + 4 \left[0 - \frac{4}{(2m-1)^2 \pi^2} \cos (2m-1) \frac{\pi}{2} \right. \\ &\quad \left. - \frac{1}{2} \frac{2}{(2m-1)\pi} \sin \left(\frac{(2m-1)\pi}{4} \right) + \frac{4}{(2m-1)^2 \pi^2} \cos \frac{(2m-1)\pi}{4} \right] \\ &= \frac{16}{(2m-1)^2 \pi^2} \cos \frac{(2m-1)\pi}{4} = \frac{16}{(2n-1)^2 \pi^2} \cos \frac{(2n-1)\pi}{4} \quad \dots(7) \end{aligned}$$

when $m = n$

and for R.H.S. when $m \neq n$,

$$\begin{aligned} 2 \int_0^1 \cos \frac{(2n-1)\pi x}{2} \cos \frac{(2m-1)\pi x}{2} dx &= \int_0^1 [\cos (m+n-1)\pi x + \cos (m-n)\pi x] dx \\ &= \left[\frac{\sin (m+n-1)\pi x}{(m+n-1)\pi} + \frac{\sin (m-n)\pi x}{(m-n)\pi} \right]_0^1 = 0 \end{aligned}$$

$\therefore m+n-1$ is integer for $m, n = 1, 2, \dots$

when $m = n$

$$\begin{aligned} 2 \int_0^1 \cos \frac{(2n-1)\pi x}{2} \cos \frac{(2m-1)\pi x}{2} dx &= 2 \int_0^1 \cos^2 \left(\frac{(2n-1)\pi x}{2} \right) dx \\ &= \int_0^1 [1 - \cos(2n-1)\pi x] dx = \left[x - \frac{\sin(2n-1)\pi x}{(2n-1)\pi} \right]_0^1 = 1 \quad \dots(8) \end{aligned}$$

$\therefore 2n-1$ is integral for $n=1, 2, \dots$

$\therefore$ Using (7) and (8), from (6), we have

$$\frac{16}{(2n-1)^2 \pi^2} \cos \frac{(2n-1)\pi}{4} = D_n$$

Hence, from (4), the required solution is

$$u(x, t) = \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cdot \cos \left(\frac{(2n-1)\pi}{4} \right) \cdot \cos \left(\frac{(2n-1)\pi x}{2} \right) e^{-(2n-1)^2 \pi^2 c^2 t/4}$$

Example 9: Find the temperature $u(x, t)$ in a slab of length π which is insulated laterally and whose ends $x=0$ and $x=\pi$ are also insulated. The initial temperature in the bar is $\sin x$.

Solution: The temperature $u(x, t)$ in the slab are governed by one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots(A)$$

Here the **boundary conditions** are

$$u_x(0, t) = 0 \quad \text{and} \quad u_x(\pi, t) = 0 \quad \dots(B)$$

and the **initial condition** is $u(x, 0) = \sin x \quad \dots(C)$

Let the solution of (A) be given by

$$u(x, t) = F(x) T(t) = FT \text{ (say)} \quad \dots(1)$$

Proceeding as in 13.6 Case IV page 624, the solution of (A) satisfying (B) and (C) as given by

$$u(x, t) = \frac{1}{2} D_0 + \sum_{n=1}^{\infty} D_n \cos nx \cdot e^{-(nc)^2 t} \quad \dots(2)$$

where $D_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx \, dx \quad \because \text{ here } l = \pi$

$$= \frac{1}{\pi} \int_0^{\pi} [\sin(n+1)x - \sin(n-1)x] \, dx = \frac{1}{\pi} \left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right] = 0, \text{ if } n \text{ is odd.}$$

and if $n = 2m$ i.e., even, then $n+1$ and $n-1$ are odd

$$\therefore D_n = D_{2m} = \frac{2}{\pi} \left[\frac{1}{2m+1} - \frac{1}{2m-1} \right] = -\frac{4}{\pi(4m^2-1)}$$

$$\text{and } D_0 = \frac{2}{\pi} \int_0^\pi f(x) dx = \frac{2}{\pi} \int_0^\pi \sin x dx = \frac{2}{\pi} [-\cos x]_0^\pi = \frac{4}{\pi}$$

Here from (2), the required solution is

$$u(x, t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{(4m^2-1)} \cos 2m x \cdot e^{-4m^2 c^2 t}$$

Example 10: The temperature at one end of a bar 50 cm long with insulated sides is kept at 0°C and the other end is kept at 100°C until steady state condition prevails. The two ends are then suddenly insulated, so that the temperature gradient is zero at each end there after. Find the temperature distribution.

Solution: Let the rod lie along the axis of x with one end which is kept at 0°C be at the origin.

The temperature distribution $u(x, t)$ in the bar at any time t at any distance x is governed by one dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots(A)$$

Initially $u = 0$ at $x = 0$ and $u = 100^\circ\text{C}$ at $x = 50$. In the steady state, u is independent of t .

$$\therefore \frac{\partial u}{\partial t} = 0$$

$$\therefore \text{ from (A), we have } \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u = Ax + B$$

$$\therefore \text{ at } x = 0, u = 0 \text{ and at } x = 50, u = 100^\circ\text{C}$$

$$\therefore 0 = 0 + B \text{ and } 100 = 50A + B \Rightarrow B = 0 \text{ and } A = 2$$

$$\therefore u = 2x$$

i.e., initial temperature distribution in the bar is given by

$$u = 2x \text{ i.e., } u(x, 0) = 2x$$

Now let $u(x, t)$ represent the temperature distribution in the rod at time t measured from the instant the two ends of the bar are suddenly insulates after the rod reaches to steady state.

Thus, we are required to solve (A) under the following conditions.

Boundary conditions : $u_x(0, t) = 0$ and $u_x(50, t) = 0$... (B)

and **initial condition** $u(x, 0) = 2x$... (C)

Let $u(x, t) = F(x) T(t) = FT$... (1)

where F is function of x alone and T function of t alone.

Now proceeding similarly, as in 13.6 case IV on page 624 we get

$$u(x, t) = \frac{1}{2} D_0 + \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{50}\right) e^{-(n\pi c/50)^2 t} \quad \dots (2)$$

where $D_n = \frac{2}{50} \int_0^{50} f(x) \cos\left(\frac{n\pi x}{50}\right) dx$ Here $l = 50$ cm

$$= \frac{1}{25} \int_0^{50} 2x \cos\left(\frac{n\pi x}{50}\right) dx = \frac{1}{25} \left[2x \frac{50}{n\pi} \sin\left(\frac{n\pi x}{50}\right) - 2 \left(\frac{50}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{50}\right) \right]_0^{50}$$

$$= \frac{2}{25} \left(\frac{50}{n\pi}\right)^2 (\cos n\pi - 1) = \frac{200}{n^2 \pi^2} [(-1)^n - 1]$$

Obviously $D_n = 0$ when $n = 2m$ (even)

and $D_n = \frac{-400}{n^2 \pi^2}$ when $n = (2m-1)$ odd.

i.e., $D_{2m} = 0$ and $D_{2m-1} = -\frac{400}{(2m-1)^2 \pi^2}$... (3)

and $D_0 = \frac{2}{50} \int_0^{50} f(x) dx = \frac{1}{25} \int_0^{50} 2x dx = \frac{1}{25} [x^2]_0^{50} = 100$... (4)

Now using (3) and (4), the required solution from (2) is given by

$$u(x, t) = \frac{1}{2} (100) + \sum_{n=1}^{\infty} \frac{-400}{(2m-1)^2 \pi^2} \cos \frac{(2m-1) \pi x}{50} \cdot e^{-[(2m-1) \pi c/50]^2 t}$$

$$= 50 - \frac{400}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1) \pi x}{50} e^{-(2n-1)^2 \pi^2 c^2 t/2500}$$

Example 11: The ends A and B of a rod 20 cm long have the temperature 30°C and 80°C until steady state prevails. The temperature of the ends are changed to 40°C and 60°C respectively. Find the temperature distribution in the rod at time t . [Rajasthan 2006]

Solution: Let the rod lie along the axis of x with one end A at the origin. The temperature distribution $u(x, t)$ in the rod at any time t and at any distance x is governed by one dimensional heat equation

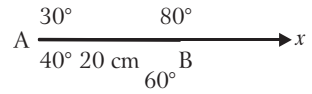


Fig. 13.6

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \dots(A)$$

Initially temperatures at the ends A and B are 30°C and 80°C respectively. In the steady state u is independent of t .

$$\therefore \frac{\partial u}{\partial t} = 0$$

$$\therefore \text{ from (A), we have } \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow u = Ax + B$$

At A , $x=0$, $u=30^\circ$ and at B , $x=20$ and $u=80^\circ\text{C}$

$$\therefore 30 = A \cdot 0 + B \text{ and } 80 = 20A + B \Rightarrow B = 30 \text{ and } A = 5/2$$

$$\therefore u = 30 + (5/2)x.$$

i.e., initial temperature distribution in the rod is given by $u = 30 + (5/2)x$

$$\text{i.e., } u(x, 0) = 30 + (5/2)x.$$

Now let $u(x, t)$ represent the temperature distribution in the rod at time t measured from the instant when the temperature of A and B are changed to 40°C and 60°C respectively.

Thus, we are required to find the solution of (A) under the following conditions.

$$\text{Boundary conditions : } u(0, t) = 40^\circ\text{C} \text{ and } u(20, t) = 60^\circ\text{C} \quad \dots(B)$$

$$\text{and initial condition : } u(x, 0) = 30 + (5/2)x \quad \dots(C)$$

Let the solution of equation (A) be of the form.

$$u(x, t) = F(x) \cdot T(t) = FT(\text{say}) \quad \dots(1)$$

where F is a function of x alone and T function of t alone.

Differentiating (1) and substituting in (1), we get

$$F \frac{dT}{dt} = c^2 T \frac{d^2 F}{dx^2} \text{ or } \frac{1}{F} \frac{d^2 F}{dx^2} = \frac{1}{c^2 T} \frac{dT}{dt} = 0 \text{ or } k^2 \text{ or } -k^2.$$

Since the two sides are functions of different independent variable, so they will be equal if each reduce to a constant value say 0 , $k^2 > 0$ or $-k^2 < 0$.

Thus, we get three sets of two ordinary differential equations :

1. $\frac{d^2 F}{dx^2} = 0$ and $\frac{dT}{dt} = 0$
2. $\frac{d^2 F}{dx^2} - k^2 F = 0$ and $\frac{dT}{dt} = k^2 c^2 T$
3. $\frac{d^2 F}{dx^2} + k^2 F = 0$ and $\frac{dT}{dt} = -k^2 c^2 T$

The general solutions in these cases as follows :

$$(i) \quad F = A_1 x + A_2 \quad \text{and} \quad T = A_3$$

$$\therefore u(x, t) = FT = A_1' x + A_2' \quad \text{where} \quad A_1' = A_1 A_3 \quad \text{and} \quad A_2' = A_2 A_3$$

$$(ii) \quad F = B_1 e^{kx} + B_2 e^{-kx} \quad \text{and} \quad T = B_3 e^{k^2 c^2 t}$$

$$\therefore u(x, t) = F.T = (B_1' e^{kx} + B_2' e^{-kx}) \cdot e^{k^2 c^2 t} \quad \text{where} \quad B_2' = B_1 B_3 \quad \text{and} \quad B_2' = B_2 B_3$$

$$(iii) \quad F = C_1 \cos kx + C_2 \sin kx \quad \text{and} \quad T = C_3 e^{-k^2 c^2 t}$$

$$\therefore u(x, t) = (C_1' \cos kx + C_2' \sin kx) e^{-k^2 c^2 t} \quad \text{where} \quad C_1' = C_1 C_3 \quad \text{and} \quad C_2' = C_2 C_3.$$

The solution given in (ii) approaches to ∞ as t increases indefinitely. So it is inadmissible.

The solution given by (iii) is inadequate as it approaches to 0 as $t \rightarrow \infty$. And the solution given by (i) is independent of time.

Hence, our solution must be a combination of solution of types (i) and (iii). So we can take

$$u(x, t) = u_1(x) + u_2(x, t) \quad \dots(2)$$

Where $u_1(x)$ is the temperature distribution after a long period of time when the rod has reached a steady state of temperature distribution, and $u_2(x, t)$ represented transient effect which die down with the passage of time. These two must be the solutions of type (i) and (iii) respectively.

Thus, $u_1(x)$ is a solution of type (A) i.e., we can take $u_1(x) = A_1' x + A_2'$

Now at A, $x = 0$ and $u_1 = 40^\circ \text{C}$ and at B, $x = 20 \text{ cm}$ and $u_1 = 60^\circ \text{C}$

$$\therefore \quad 40 = A_1' \cdot 0 + A_2' \quad \text{and} \quad 60 = 20A_1' + A_2' \Rightarrow A_2' = 40 \quad \text{and} \quad A_1' = 1$$

$$\therefore \quad u_1(x) = x + 40 \quad \dots(3)$$

$$\therefore \text{ from (2), we have, } u(x, t) = x + 40 + u_2(x, t) \quad \dots(4)$$

Now condition (B), i.e., $u(0, t) = 40 \Rightarrow 40 = 0 + 40 + u_2(0, t)$

$$\Rightarrow u_2(0, t) = 0 \quad \dots(5)$$

and condition (B) i.e.,

$$u(20, t) = 60 \Rightarrow 60 = 20 + 40 + u_2(20, t) \Rightarrow u_2(20, t) = 0 \quad \dots(6)$$

Also initial temperature distribution is $u = 30 + (5/2)x$

$$\text{i.e., } u(x, 0) = 30 + 5x/2 \Rightarrow 30 + 5x/2 = x + 40 + u_2(x, 0)$$

$$\Rightarrow u_2(x, 0) = 3x/2 - 10 \quad \dots(7)$$

Since $u_2(x, t)$ will be a solution of type (C), so we can take

$$u_2(x, t) = (C_1' \cos kx + C_2' \sin kx) e^{-k^2 c^2 t} \quad \dots(8)$$

Now from (5) and (8), $u_2(0, t) = 0 \Rightarrow C_1' = 0, \because e^{-k^2 c^2 t} \neq 0$

$$\therefore \text{ from (6) and (8), } u_2(20, t) = 0 \Rightarrow C_2' \sin(20k) \cdot e^{-k^2 c^2 t} = 0 \quad \because C_1' = 0$$

$$\Rightarrow \sin 20k = 0 \quad \because C_2' \neq 0 \text{ and } e^{-k^2 c^2 t} \neq 0$$

i.e., $20k = n\pi$ or $k = n\pi/20$, k is any integer.

$\therefore$ from (8), for integral values of n , the solutions $u_2(x, t)$ are given by

$$u_{2n}(x, t) = D_n \sin\left(\frac{n\pi x}{20}\right) e^{-(n\pi/20)^2 c^2 t}$$

where $D_n = C_2'$ is the new arbitrary constant depending on n .

Taking $n = 1, 2, 3, \dots$, more general values of $u_2(x, t)$ can be taken as

$$u_2(x, t) = \sum_{n=1}^{\infty} u_{2n}(x, t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{20}\right) e^{-(n\pi/20)^2 c^2 t} \quad \dots(9)$$

Now from (7) $u_2(x, 0) = 3x/2 - 10$

$$\therefore (9) \Rightarrow \frac{3x}{2} - 10 = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{20}\right) \quad \dots(10)$$

The right hand side of (10) can be considered as the Fourier sine series of the function on its L.H.S. for $0 < x < 20$

$$\therefore D_n = \frac{2}{20} \int_0^{20} \left(\frac{3x}{2} - 10 \right) \sin \left(\frac{n\pi x}{20} \right) dx$$

$$\begin{aligned} \text{or } D_n &= \frac{1}{10} \left[\left(\frac{3x}{2} - 10 \right) \left(-\frac{20}{n\pi} \cos \frac{n\pi x}{20} \right) + \frac{3}{2} \left(\frac{20}{n\pi} \right)^2 \sin \frac{n\pi x}{20} \right]_0^{20} \\ &= \frac{1}{10} \left[20 \left(-\frac{20}{n\pi} \right) \cos n\pi - (-10) \left(-\frac{20}{n\pi} \right) \right] = -\frac{20}{n\pi} [2(-1)^n + 1] \end{aligned}$$

Hence, from (2), using (3) and (9) the required solution is given by

$$u(x, t) = x + 40 - \frac{20}{\pi} \sum_{n=1}^{\infty} [2(-1)^n + 1] \sin \frac{n\pi x}{20} e^{-(n\pi c/20)^2 t}$$

Exercise 13.1

1. A rod of length l with insulated sides is initially at a uniform temperature $u_0(x)$. Its ends are suddenly cooled to 0°C and are kept at that temperature. Prove that the temperature function $u(x, t)$ is given by

$$u(x, t) = \frac{2}{l} \sum_{n=1}^{\infty} e^{-c^2 \pi^2 n^2 t / l^2} \cdot \sin \left(\frac{n\pi x}{l} \right) \cdot \int_0^l u_0(x) \sin \left(\frac{n\pi x}{l} \right) dx$$

2. Determine the solution of one dimensional heat equation $\frac{\partial \theta}{\partial t} = a^2 \frac{\partial^2 \theta}{\partial x^2}$, under the following conditions $\theta(0, t) = 0, \theta(l, t) = 0, t \geq 0$ and the initial condition, $\theta(x, 0) = x, 0 < x < l, l$ being the length of the bar.
3. Find by the method of separation of variables the solution $u(x, t)$ of the boundary values problem

$$\frac{\partial u}{\partial t} = 3 \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad 0 < x < 2$$

$$\text{s.t.} \quad u(0, t) = 0, u(2, t) = 0 \text{ and } u(x, 0) = x, \quad 0 < x < 2.$$

4. Solve $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$, under the conditions

$$(i) \quad u \neq \infty \text{ if } t \rightarrow \infty \quad (ii) \quad u(0, t) = u(\pi, t) = 0 \quad (iii) \quad u(x, 0) = \pi x - x^2.$$

5. The heat flow in a bar of length 10 cm of homogeneous material is governed by the partial differential equation $u_t = c^2 u_{xx}$. The ends of the bar are kept at, 0°C , and the initial temperature is $f(x) = x(10 - x)$. Find the temperature in the bar at any instant of time.

6. Find the temperature $u(x, t)$ in a slab whose ends $x=0$ and $x=l$ are kept at temperature zero and whose initial temperature $f(x)$ is given by

$$f(x) = \begin{cases} A, & \text{when } 0 < x < l/2 \\ 0, & \text{when } l/2 < x < l \end{cases} \quad [\text{Meerut 2002 (O)}]$$

7. Find the temperature $u(x, t)$ in a bar of length l , which is perfectly insulated laterally, whose ends are kept at temperature zero and initial temperature is

$$f(x) = \begin{cases} x, & \text{when } 0 < x < l/2 \\ l-x, & \text{when } l/2 < x < l. \end{cases} \quad [\text{Meerut 2000 (P)}]$$

8. Find the temperature $u(x, t)$ in a slab whose ends $x=0$ and $x=l$ are kept at temperature zero and whose initial temperature $f(x) = \sin(\pi x/l)$.

9. Solve $\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = 0$, for $0 < x < \pi$, $t > 0$

$$u(0, t) = u(l, t) = 0 \text{ and } u(x, 0) = \sin^3 x.$$

10. A uniform rod of length 'a' whose surface is thermally insulated is initially at temperature $\theta = \theta_0$. At time $t=0$, one end is suddenly cooled to $\theta=0$ and subsequently maintained at this temperature, the other end remains thermally insulated. Find the temperature $\theta(x, t)$.

11. Solve $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, under the following conditions $u'(0, t) = 0, u'(\pi, t) = 0$ for $t > 0$ and $u(x, 0) = x^2$, for $0 < x < \pi$.

12. The temperature distribution in a bar of length π , which is perfectly insulated at the ends $x=0$ and $x=\pi$ is governed by the partial differential equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$.

Assuming the initial temperature as $u(x, 0) = f(x) = \cos 2x$, find the temperature distribution at any instant of time.

13. Show that the solution of $\frac{\partial u}{\partial t} = k \left(\frac{\partial^2 u}{\partial x^2} \right)$ subject to the conditions (i) u is not infinite

for $t \rightarrow \infty$ (ii) $\frac{\partial u}{\partial x} = 0$ for $x=0$ and $x=l$ (iii) $u = lx - x^2$ for $t=0$ between $x=0$ and $x=l$ is

$$u = \frac{1}{6} l^2 - \frac{l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{n\pi x}{l} e^{-4\pi^2 n^2 kt/l}.$$

14. Obtain temperature distribution $u(x, t)$ in a uniform bar of unit length whose one end is kept at 10°C and the other end is insulated. Further it is given that $u(x, 0) = 1-x$, $0 < x < 1$.

15. An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. Suddenly if the temperature of A

is raised to 20°C and the temperature of other end B is reduced to 80°C . Find the temperature distribution at a distance x from A at time t .

16. Make use of the method of separation of variables to solve $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$,
 $0 \leq x \leq l$, $t > 0$,

$$u(0, t) = 2, u(l, t) = 3, t \geq 0$$

and $u(x, 0) = x(1-x), 0 < x < l$.

[Hint : Solution will be a combination of type (a) and (c)].

17. Find the temperature in a rod (length l and conductivity k), which is at a uniform temperature of 50°C . Suddenly at $t=0$, the end $x=0$ is cooled to 0°C by an application of ice, and the end $x=l$ is heated to 100°C by an application of steam, and these two temperatures are maintained at the ends. Further more, the rod is insulated along its length so that no transfer of heat can occur from the sides.

[Hint : Solution will be a combination of type (a) and (c)].

Answers 13.1

2.	$\theta(x, t) = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin\left(\frac{n\pi x}{l}\right) e^{-n^2 \pi^2 a^2 t/l^2}$
3.	$u(x, t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{2}\right) e^{-3n^2 \pi^2 t/4}$
4.	$u(x, t) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin(2n-1)x \cdot e^{-(2n-1)^2 a^2 t}$
5.	$u(x, t) = \frac{800}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin\left(\frac{(2n-1)\pi x}{10}\right) e^{-(2n-1)^2 \pi^2 c^2 t/100}$
6.	$u(x, t) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin^2(n\pi/4) \cdot \sin(n\pi x/l) \cdot e^{-n^2 \pi^2 c^2 t/l^2}$
7.	$u(x, t) = \frac{4l}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} \sin\left(\frac{(2n-1)\pi x}{l}\right) e^{-(2n-1)^2 \pi^2 c^2 t/l^2}$
8.	$u(x, t) = e^{-\pi^2 c^2 t/l^2} \cdot \sin(n\pi x/l)$
9.	$u(x, t) = \frac{1}{4} [3e^{-kt} \sin x - e^{-9kt} \sin 3x]$

10.	$u(x, t) = \frac{4\theta_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left(\frac{(2n-1)\pi x}{2a}\right) e^{-(2n-1)^2 \pi^2 c^2 t/4a^2}$
11.	$u(x, t) = \frac{1}{3} \pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx \cdot e^{-n^2 c^2 t}$
12.	$u(x, t) = e^{-4t} \cos 2x.$
14.	$u(x, t) = 10 + \sum_{n=1}^{\infty} \left\{ \frac{8(-1)^n}{(2n-1)^2 \pi^2} - \frac{36}{(2n-1)\pi} \right\} e^{-(2n-1)^2 \pi^2 c^2 t/4}$
15.	$u(x, t) = 20 + \frac{40x}{l} - \frac{40}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin\left(\frac{2m\pi x}{l}\right) e^{-c^2 m^2 \pi^2 t/l^2}$
16.	$u(x, t) = x + 2 + 2 \sum_{n=1}^{\infty} \left(\frac{3(-1)^n - 2}{n\pi} - \frac{2\{(-1)^n - 1\}}{n^2 \pi^2} \right) e^{-n^2 \pi^2 c^2 t} \sin n\pi x$
17.	$u(x, t) = \frac{100}{l} x + \frac{100}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin\left(\frac{2m\pi x}{l}\right) e^{-4m^2 c^2 \pi^2 t/l^2}$

13.7 Solution of Two Dimensional Heat Equation in Cartesian Coordinates

[Rajasthan 2001, 03]

Two dimensional heat equation in cartesian coordinates is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial u}{\partial t} \quad \dots(A)$$

$$\text{Let } u(x, y, t) = F(x) Y(y) T(t) = FYT \text{ (say)} \quad \dots(1)$$

be the solution of (A) where F is a function of x alone, Y function of y alone and T function of t alone.

Differentiating (1) and substituting in (A), we get

$$YT \frac{d^2 F}{dx^2} + FT \frac{d^2 Y}{dy^2} = \frac{1}{c^2} FY \frac{dT}{dt}$$

$$\text{or } \frac{1}{F} \frac{d^2 F}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} = \frac{1}{c^2 T} \frac{dT}{dt} \quad \dots(2)$$

Since each term in (2) is a function of different independent variable so the change in any one of these independent variables x , y and t changes one term which is a function of that variable only while other two terms remains constant. Hence, (2) is true only if each term on each side is constant, so let

$$\frac{1}{c^2 T} \frac{dT}{dt} = \lambda \text{ (constant)}, \quad \frac{1}{F} \frac{d^2 F}{dx^2} = \lambda_1 \text{ (constant)} \text{ and } \frac{1}{Y} \frac{d^2 Y}{dy^2} = \lambda_2 \text{ (constant)}$$

$$\text{where } \lambda_1 + \lambda_2 = \lambda$$

Now there are three possibilities :

Case I : $\lambda = 0, \lambda_1 = 0, \lambda_2 = 0$. In this case, we get the following three ordinary differential equations

$$\frac{d^2 F}{dx^2} = 0, \quad \frac{d^2 Y}{dy^2} = 0 \quad \text{and} \quad \frac{dT}{dt} = 0$$

whose general solutions are

$$F = Ax + B, \quad Y = Cy + D \quad \text{and} \quad T = E$$

Case II : $\lambda = k^2 > 0, \lambda_1 = m^2 > 0, \lambda_2 = n^2 > 0$ such that $k^2 = m^2 + n^2$. In this case, we get the following three ordinary differential equations

$$\frac{d^2 F}{dx^2} - m^2 F = 0, \quad \frac{d^2 Y}{dy^2} - n^2 Y = 0 \quad \text{and} \quad \frac{dT}{dt} = c^2 k^2 T$$

whose general solutions are

$$F = A_m e^{mx} + B_m e^{-mx}, \quad Y = C_n e^{ny} + D_n e^{-ny}$$

$$\text{and} \quad T = E_k e^{c^2 k^2 t} = F_{mn} e^{c^2 (m^2 + n^2) t}$$

$$\text{where} \quad k^2 = m^2 + n^2$$

Case III : $\lambda = -k^2 < 0, \lambda_1 = -m^2 < 0$ and $\lambda_2 = -n^2 < 0$ such that $k^2 = m^2 + n^2$

In this case we get the following three ordinary differential equations

$$\frac{d^2 F}{dx^2} + m^2 F = 0, \quad \frac{d^2 Y}{dy^2} + n^2 Y = 0 \quad \text{and} \quad \frac{dT}{dt} = -c^2 k^2 T$$

whose general solutions are

$$F = A_m \cos mx + B_m \sin mx, \quad Y = C_n \cos ny + D_n \sin ny$$

$$\text{and} \quad T = E_k e^{-c^2 k^2 t} = F_{mn} e^{-c^2 (m^2 + n^2) t}$$

or $F = A_m \cos(mx + B_m), Y = C_n \cos(ny + D_n)$

In case I, T remains constant as $t \rightarrow \infty$ and in case II $T \rightarrow \infty$ as $t \rightarrow \infty$. Thus, the solutions given in case I and II do not constitute the solution of equation (A).

In case III, $T \rightarrow 0$ as $t \rightarrow \infty$.

i.e., $u = FYT \rightarrow 0$ as $t \rightarrow \infty$

which is true as the temperature die down with the passage of time t .

Hence, for different values of m and n the solutions of heat equation are given by

$$u(x, y, t) = FYT$$

i.e., $u_{mn}(x, y, t) = (A_m \cos mx + B_m \sin mx) (C_n \cos ny + D_n \sin ny) \cdot F_{mn} e^{-c^2(m^2 + n^2)t}$

For all values of m and n , the more general solution is given by

$$\begin{aligned} u(x, y, t) &= \sum_m \sum_n u_{m,n}(x, y, t) \\ &= \sum_m \sum_n F_{mn} (A_m \cos mx + B_m \sin mx) \\ &\quad (C_n \cos ny + D_n \sin ny) \cdot e^{-c^2(m^2 + n^2)t} \end{aligned}$$

which may also be taken as

$$u(x, y, t) = \sum_m \sum_n H_{mn} \cos(mx + B_m) \cdot \cos(ny + D_n) e^{-c^2(m^2 + n^2)t}$$

where $H_{mn} = A_m \cdot C_n \cdot F_{mn}$

Since $u \rightarrow 0$ as $t \rightarrow \infty$, so the solution given in case III is the most suitable solution for the heat equation.

13.8 Heat Equation in Plane Polar Coordinates

If (x, y) are the Cartesian coordinates of a point P in a plane and (r, θ) its polar coordinates, then $x = r \cos \theta$ and $y = r \sin \theta$

$$\Rightarrow x^2 + y^2 = r^2 \text{ and } \theta = \tan^{-1}(y/x)$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r} = \cos \theta, \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin \theta$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + y^2/x^2} \left(-\frac{y}{x^2} \right) = -\frac{y}{x^2 + y^2} = \frac{-\sin \theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + y^2/x^2} \left(\frac{1}{x} \right) = \frac{x}{x^2 + y^2} = \frac{\cos \theta}{r}$$

Then
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r} \cos \theta + \frac{\partial u}{\partial \theta} \left(-\frac{\sin \theta}{r} \right)$$

$$\therefore \frac{\partial}{\partial x} \equiv \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}$$

$$\begin{aligned} \therefore \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \left(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) \cdot \left(\cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta} \right) \\ &= \cos \theta \frac{\partial}{\partial r} \left(\cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta} \right) - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta} \right) \\ &= \cos \theta \left(\cos \theta \frac{\partial^2 u}{\partial r^2} - \frac{\sin \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin \theta}{r^2} \frac{\partial u}{\partial \theta} \right) \\ &\quad - \frac{\sin \theta}{r} \left(\cos \theta \frac{\partial^2 u}{\partial \theta \partial r} - \sin \theta \frac{\partial u}{\partial r} - \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta} - \frac{\sin \theta}{r} \frac{\partial^2 u}{\partial \theta^2} \right) \\ &= \cos^2 \theta \frac{\partial^2 u}{\partial r^2} - 2 \frac{\sin \theta \cos \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{2 \sin \theta \cos \theta}{r^2} \frac{\partial u}{\partial \theta} \\ &\quad + \frac{\sin^2 \theta}{r} \frac{\partial u}{\partial r} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2} \dots (1) \end{aligned}$$

Similarly,
$$\begin{aligned} \frac{\partial^2 u}{\partial y^2} &= \sin^2 \theta \frac{\partial^2 u}{\partial r^2} + \frac{2 \cos \theta \sin \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} - \frac{2 \cos \theta \sin \theta}{r^2} \frac{\partial u}{\partial \theta} \\ &\quad + \frac{\cos^2 \theta}{r} \frac{\partial u}{\partial r} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2} \dots (2) \end{aligned}$$

Adding (1) and (2)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

Hence the two dimensional heat equation in Cartesian coordinates *i.e.*,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}$$

is transformed to the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}$$

in plane polar coordinates.

13.9 Solution of Heat Equation in Plane Polar Coordinates by Separation of Variables

The heat equation in plane polar coordinates is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = \frac{1}{c^2} \frac{\partial u}{\partial t} \quad \dots(A)$$

Let the solution of (A) which will be a function of r, θ and t be given by

$$u(r, \theta, t) = R(r) F(\theta) T(t) = RFT \text{ (say)} \quad \dots(1)$$

where R is a function of r alone, F function of θ alone and T function of t alone.

Differentiating (1) and substituting in (A), we get

$$FT \frac{d^2 R}{dr^2} + \frac{FT}{r} \frac{dR}{dr} + \frac{RT}{r^2} \frac{d^2 F}{d\theta^2} = \frac{RF}{c^2} \frac{dT}{dt}$$

or
$$\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{rR} \frac{dR}{dr} + \frac{1}{r^2} \frac{1}{F} \frac{d^2 F}{d\theta^2} = \frac{1}{c^2} \frac{dT}{T} \frac{dt}{dt}$$

Here right hand side is a function of independent variable t only while left hand side is a function of independent variables r and θ only. The two sides will be equal to each other only if they reduce to a constant values.

$$\therefore \frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{rR} \frac{dR}{dr} + \frac{1}{r^2} \frac{1}{F} \frac{d^2 F}{d\theta^2} = \frac{1}{c^2} \frac{dT}{T} \frac{dt}{dt} = -k^2 \text{ (constant)}$$

$$\therefore \frac{dT}{dt} = -c^2 k^2 T \quad \dots(2)$$

and
$$\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{rR} \frac{dR}{dr} + \frac{1}{r^2} \frac{1}{F} \frac{d^2 F}{d\theta^2} = -k^2$$

or
$$r^2 \left(\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{rR} \frac{dR}{dr} + k^2 \right) = -\frac{1}{F} \frac{d^2 F}{d\theta^2} = m^2 \text{ (constant)} \quad \dots(3)$$

With the same arguments as above two sides of (3) will be equal to a constant value say m^2 . Thus, from (3), we get

$$\therefore -\frac{1}{F} \frac{d^2 F}{d\theta^2} = m^2 \text{ or } \frac{d^2 F}{d\theta^2} + m^2 F = 0 \quad \dots(4)$$

and
$$r^2 \left(\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{1}{rR} \frac{dR}{dr} + k^2 \right) = m^2$$

or
$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left(k^2 - \frac{m^2}{r^2} \right) R = 0 \quad \dots(5)$$

Now the general solutions of (2), (4) and (5) respectively, are as follows

$$T = T(t) = A_k e^{-c^2 k^2 t}$$

$$F = F(\theta) = B_m \cos m\theta + C_m \sin m\theta \quad \text{or} \quad F(\theta) = B'_m e^{\pm im\theta}$$

and $R = R(r) = D_{km} J_m(kr) + E_{km} J_{-m}(kr) \quad \text{or} \quad R(r) = D'_{km} J_{\pm m}(kr)$

$\therefore$ from (1), the solution of (A) for all values of k and m is given by

$$u_{km}(r, \theta, t) = (D_{km} J_m(kr) + E_{km} J_{-m}(kr)) \cdot (B_m \cos m\theta + C_m \sin m\theta) \cdot A_k e^{-c^2 k^2 t} \quad \dots(6)$$

or $u_{km}(r, \theta, t) = A_k \cdot B'_m \cdot D'_{km} J_{\pm m}(kr) e^{\pm im\theta - c^2 k^2 t} = F_{km} J_{\pm m}(kr) e^{\pm im\theta - c^2 k^2 t} \quad \dots(7)$

where $F_{km} = A_k B'_m D'_{km}$ is the new arbitrary constant.

Hence the more general solution of (A) is given by

$$u(r, \theta, t) = \sum_{k, m} u_{km}(r, \theta, t)$$

$\therefore$ from (6)

$$i.e., \quad u(r, \theta, t) = \sum_{k, m} [D_{km} J_m(kr) + E_{km} J_{-m}(mr)] \cdot (G'_{mk} \cos m\theta + H'_{mk} \sin m\theta) \cdot e^{-c^2 k^2 t}$$

where $G'_{mk} = B_m \cdot A_k$ and $H'_{mk} = C_m \cdot A_k$ are other arbitrary constants.

and from (7), $u(r, \theta, t) = \sum_{k, m} F_{km} J_{\pm m}(kr) e^{\pm im\theta - c^2 k^2 t}$

Solved Examples

Example 1: A thin rectangular plate whose surface is impervious to heat flow has at $t = 0$ an arbitrary distribution of temperature $f(x, y)$. Its four edges $x = 0$, $x = a$, $y = 0$, $y = b$ are kept at zero temperature. Determine the temperature at a point of the plate as t increases.

[Meerut 2001, 03, 06; Delhi B.Sc. (Hons.) 05]

Solution: The temperature in the plate are governed by two dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial u}{\partial t} \quad \dots(A)$$

where $c^2 = k/\sigma\rho$

and $u(x, y, t)$ is the temperature at the point (x, y) at time t .

Here the four faces of the plate are kept at zero temperature and the initial temperature of the plate is $f(x, y)$. Therefore, the function $u(x, y, t)$ is required to satisfy (A) and the following boundary and initial conditions.

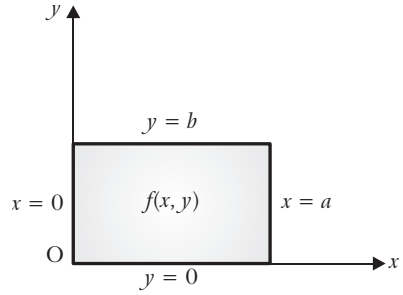


Fig. 13.7

Boundary conditions are,

$$\left. \begin{aligned} u(0, y, t) = 0, \quad u(a, y, t) = 0 \\ u(x, 0, t) = 0, \quad u(x, b, t) = 0 \end{aligned} \right\} \quad \dots(B)$$

and **initial condition** is $u(x, y, 0) = f(x, y)$

Let the solution of (A) be of the form

$$u(x, y, t) = F(x)Y(y)T(t) = FYT \text{ (say)} \quad \dots(1)$$

Where F is a function of x alone, Y function of y alone and T function of t alone.

Differentiating (1) and substituting in (A), we get

$$YT \frac{d^2 F}{dx^2} + FT \frac{d^2 Y}{dy^2} = \frac{1}{c^2} FY \frac{d^2 T}{dt^2}$$

$$\text{or} \quad \frac{1}{F} \frac{d^2 F}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} = \frac{1}{c^2 T} \frac{d^2 T}{dt^2} \quad \dots(2)$$

Here each term is a function of different independent variable, so the change in any one of the variables x, y and t changes one term which is a function of that variable only while other two terms remains unchanged.

Hence (2) is true only if each term on each side is a constant, so let

$$\frac{1}{c^2 T} \frac{dT}{dt} = \lambda \text{ (constant)}, \quad \frac{1}{F} \frac{d^2 F}{dx^2} = \lambda_1 \text{ (constant)} \text{ and } \frac{1}{Y} \frac{d^2 Y}{dy^2} = \lambda_2 \text{ (constant)}$$

where $\lambda = \lambda_1 + \lambda_2$

Now there are three possibilities

$$\lambda = 0 \quad \text{or} \quad k^2 > 0 \quad \text{or} \quad -k^2 < 0$$

Thus, we get the following three sets of three ordinary differential equations :

1. $\frac{d^2 F}{dx^2} = 0, \quad \frac{d^2 Y}{dy^2} = 0 \quad \text{and} \quad \frac{dT}{dt} = 0$
2. $\frac{d^2 F}{dx^2} - k_1^2 F = 0, \quad \frac{d^2 Y}{dy^2} - k_2^2 Y = 0 \quad \text{and} \quad \frac{dT}{dt} = k^2 c^2 T, \quad k^2 = k_1^2 + k_2^2$
3. $\frac{d^2 F}{dx^2} + k_1^2 F = 0, \quad \frac{d^2 Y}{dy^2} + k_2^2 Y = 0 \quad \text{and} \quad \frac{dT}{dt} = -k^2 c^2 T, \quad k^2 = k_1^2 + k_2^2$

The general solutions in these cases are :

- (i) $F = Ax + B, Y = Cy + D \quad \text{and} \quad T = E$
- (ii) $F = A_{k_1} e^{k_1 x} + B_{k_1} e^{-k_1 x}, Y = C_{k_2} e^{k_2 y} + D_{k_2} e^{-k_2 y} \quad \text{and} \quad T = E_k e^{c^2 k^2 t}, \quad k^2 = k_1^2 + k_2^2$
- (iii) $F = A_{k_1} \cos k_1 x + B_{k_1} \sin k_1 x, Y = C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y \quad \text{and} \quad T = E_k e^{-c^2 k^2 t},$
 $k^2 = k_1^2 + k_2^2$

In the solution given in (i), T remains constant as $t \rightarrow \infty$ and in the solution given in case (ii), $T \rightarrow \infty$ as $t \rightarrow \infty$.

Thus, the solutions given in (i) and (ii) do not constitute the solution of equation (A).

In the solution given in (iii), $T \rightarrow 0$ as $t \rightarrow \infty$. Hence taking the solution of (A) in the form given in (iii), we have

$$u(x, y, t) = (A_{k_1} \cos k_1 x + B_{k_1} \sin k_1 x) (C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y) E_k e^{-c^2 k^2 t} \quad \dots(3)$$

$$\therefore \quad \text{from (3), } u(0, y, t) = 0 \Rightarrow A_{k_1} (C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y) E_k e^{-c^2 k^2 t} = 0$$

$$\Rightarrow A_{k_1} = 0, \because C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y \neq 0, E_k \neq 0, e^{-c^2 k^2 t} \neq 0$$

otherwise $u(x, y, t) = 0$ which is inadmissible.

$$\therefore \quad y(a, y, t) = 0 \Rightarrow B_{k_1} \sin k_1 a (C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y) E_k e^{-c^2 k^2 t} = 0$$

$$\Rightarrow \sin k_1 a = 0 \because B_{k_1} \neq 0, C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y \neq 0, E_k \neq 0 \quad \text{and} \quad e^{-c^2 k^2 t} \neq 0$$

$$\Rightarrow k_1 a = m\pi \quad \text{or} \quad k_1 = \frac{m\pi}{a}, \text{ where } m \text{ is any integer.}$$

Similarly, $u(x, 0, t) = 0$ and $u(x, b, t) = 0 \Rightarrow C_{k_2} = 0$ and $k_2 = n\pi/b$ where n is any integer.

Substituting in (3), For different values of m and n the solutions of (A) satisfying conditions (B), are given by

$$U_{mn}(x, y, t) = A_{mn} \cdot e^{-c^2 k_{mn}^2 t} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

Where $A_{mn} = B_{k_1} \cdot D_{k_2} E_k$ is the new variable and $k^2 = k_{mn}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$.

Hence taking $m = 1, 2, 3, \dots$, and $n = 1, 2, 3, \dots$, the more general solution of (A) satisfying conditions (B) is given by

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{-c^2 k_{mn}^2 t} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad \dots(4)$$

Now from (4) and initial condition (C)

$$\text{i.e., } u(x, y, 0) = f(x, y) \Rightarrow f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad \dots(5)$$

The right hand side of (5) can be considered as the double Fourier sine series of $f(x, y)$ on its left hand side

$$\therefore A_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \quad \dots(6)$$

Hence, the solution of heat equation (A) satisfying the boundary conditions (B) and initial condition (C) is given by (4) with the coefficient A_{mn} given by (6) where

$$k_{mn}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$$

Example 2: A rectangular plate bounded by the lines $x = 0, y = 0, x = a, y = b$ has an initial distribution temperature given by

$$V = A \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

The edges are kept at zero temperature and the plane faces are impervious to heat. Find the temperature at any point and time. [Meerut 2000, 02, 08, 12 (O)]

Solution: The temperature $u(x, y, t)$ at any point and time are governed by the two dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \dots(A)$$

Here the boundary and initial conditions are as follows

$$\text{Boundary conditions} \quad \left. \begin{array}{l} u(0, y, t) = 0, \quad u(a, y, t) = 0 \\ u(x, 0, t) = 0, \quad u(x, b, t) = 0 \end{array} \right\} \quad \dots(B)$$

$$\text{Initial condition} \quad u(x, y, 0) = V = A \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \quad \dots(C)$$

$$\text{Let} \quad u(x, y, t) = F(x) Y(y) T(t) \quad \dots(1)$$

Where $F(x)$ is a function of x alone, Y function of y alone and T function of t alone, be the solution of equation (A) satisfying the conditions (B) and (C).

Proceeding similarly as in Ex. 1, we have

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{-c^2 k_{mn}^2 t} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y \quad \dots(2)$$

where $k_{mn}^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$

$$\begin{aligned} \text{and} \quad A_{mn} &= \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \\ &= \frac{4A}{ab} \int_0^a \int_0^b \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \\ &= \frac{4A}{ab} \left[\int_{x=0}^a \sin \frac{\pi x}{a} \sin \frac{m\pi x}{a} dx \right] \left[\int_{y=0}^b \sin \frac{\pi y}{b} \sin \frac{n\pi y}{b} dy \right] \end{aligned}$$

$$\text{Now} \quad \int_{y=0}^b \sin \frac{\pi y}{b} \sin \frac{n\pi y}{b} dy = 0 \quad \text{for } n = 2, 3, \dots$$

But for $n=1$,

$$\int_{y=0}^b \sin \frac{\pi y}{b} \sin \frac{n\pi y}{b} dy = \int_{y=0}^b \left(\sin \frac{\pi y}{b} \right)^2 dy = \frac{1}{2} \int_0^b \left(1 - \cos \frac{2\pi y}{b} \right) dy = \frac{b}{2},$$

$$\therefore A_{m1} = \frac{4A}{ab} \int_0^a \sin \frac{\pi x}{a} \sin \frac{m\pi x}{a} dx = \frac{2A}{a} \int_0^a \sin \frac{\pi x}{a} \sin \frac{m\pi x}{a} dx$$

Which is equal to zero for $m = 2, 3, \dots$

But for $m = 1$.

$$A_{11} = \frac{2A}{a} \int_0^a \sin^2 \frac{\pi x}{a} dx = \frac{A}{a} \int_0^a \left(1 - \cos \frac{2\pi x}{a}\right) dx = A$$

and $k_{11}^2 = \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)$

$$\begin{aligned} \therefore u(x, y, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} e^{-c^2 k_{mn}^2 t} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ &= A_{11} e^{-c^2 k_{11}^2 t} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}, \text{ Since } A_{mn} = 0 \text{ for all integral values of} \end{aligned}$$

$$m > 1, n > 1.$$

$$= A e^{-c^2 \pi^2 (1/a^2 + 1/b^2) t} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}, \text{ where } A_{11} = A.$$

Which give the temperature of the plate at any point and time.

Exercise 13.2

1. A rectangular plate bounded by the lines $x=0, y=0, x=a$ and $y=b$ has an initial distribution temperature given by $V = A \sin \frac{2\pi x}{a} \sin \frac{3\pi y}{b}$.

The edges are kept at zero temperature and the plane faces are impervious to heat. Find the temperature in the plate at any point and time.

2. The edges of a thin square plate of side π cm., are kept at temperature zero and faces are perfectly insulated. The initial temperature $u(x, y, 0) = f(x, y) = xy(\pi - x)(\pi - y)$. Determine the temperature in the plate at time t .

Answers 13.2

1.	$u(x, y, t) = A e^{-c^2 \pi^2 (4/a^2 + 9/b^2) t} \sin \frac{2\pi x}{a} \sin \frac{3\pi y}{b}$
2.	$u(x, y, t) = \frac{64}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(2m-1)^3 (2n-1)^3} e^{-c^2 \{(2m-1)^2 + (2n-1)^2\} t} \sin (2m-1)x \sin (2n-1)y.$

13.10 Solution of Three Dimensional Heat Equation by the Method of Separation of Variables

Three dimensional heat equation is given by

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad \dots(1)$$

$$\text{Let } u(x, y, z, t) = F(x) Y(y) Z(z) T(t) = FYZT \text{ (say)} \quad \dots(2)$$

Where F is a function of x alone, Y function of y alone, Z function of z alone and T function of t alone, be the solution of heat equation (1).

Differentiating (2) and substituting in (1), we get

$$FYZ \frac{dT}{dt} = c^2 \left(YZT \frac{d^2 F}{dx^2} + FZT \frac{d^2 Y}{dy^2} + FYT \frac{d^2 Z}{dz^2} \right)$$

$$\text{or } \frac{1}{F} \frac{d^2 F}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = \frac{1}{c^2 T} \frac{dT}{dt} \quad \dots(3)$$

All the four terms in (3) are functions of different independent variables, therefore the relation (3) will be satisfied if each of the four functions is constant, so let

$$\frac{1}{F} \frac{d^2 F}{dx^2} = \lambda_1, \quad \frac{1}{Y} \frac{d^2 Y}{dy^2} = \lambda_2, \quad \frac{1}{Z} \frac{d^2 Z}{dz^2} = \lambda_3 \quad \text{and} \quad \frac{1}{c^2 T} \frac{dT}{dt} = \lambda \quad \dots(4)$$

such that $\lambda = \lambda_1 + \lambda_2 + \lambda_3$

Now there are the following three cases :

Case I : When $\lambda_1 = 0 = \lambda_2 = \lambda_3 = \lambda$, then the equations in (4), reduce to

$$\frac{d^2 F}{dx^2} = 0, \quad \frac{d^2 Y}{dy^2} = 0, \quad \frac{d^2 Z}{dz^2} = 0 \quad \text{and} \quad \frac{dT}{dt} = 0$$

Whose general solutions are

$$F = Ax + B, \quad Y = Cy + D, \quad Z = Ez + F, \quad T = G$$

$\therefore$ from (2), the solution of (1) is given by

$$u(x, y, z) = (Ax + B)(Cy + D)(Ez + F)G \quad \dots(5)$$

Case II : When $\lambda_1 = k_1^2 > 0$, $\lambda_2 = k_2^2 > 0$, $\lambda_3 = k_3^2 > 0$ and $\lambda = k^2$ where $k_1^2 + k_2^2 + k_3^2 = k^2$,

then equation in (4) reduce to

$$\frac{d^2 F}{dx^2} - k_1^2 F = 0, \quad \frac{d^2 Y}{dy^2} - k_2^2 Y = 0, \quad \frac{d^2 Z}{dz^2} - k_3^2 Z = 0 \quad \text{and} \quad \frac{dT}{dt} = k^2 c^2 T$$

Whose general solutions are

$$\begin{aligned} F &= A_{k_1} e^{x k_1} + B_{k_1} e^{-x k_1}, \quad \text{or} \quad F = A'_{k_1} e^{\pm x k_1} \\ Y &= C_{k_2} e^{y k_2} + D_{k_2} e^{-y k_2} \quad \text{or} \quad Y = B'_{k_2} e^{\pm y k_2} \\ Z &= E_{k_3} e^{z k_3} + F_{k_3} e^{-z k_3} \quad \text{or} \quad Z = C'_{k_3} e^{\pm z k_3} \quad \text{and} \quad T = G_k e^{k^2 c^2 t} \end{aligned}$$

∴ From (2), the most general solution of (1) is given by

$$\begin{aligned} u(x, y, z, t) &= \sum_{k_1, k_2, k_3} (A_{k_1} e^{x k_1} + B_{k_1} e^{-x k_1}) (C_{k_2} e^{y k_2} + D_{k_2} e^{-y k_2}) \\ &\quad (E_{k_3} e^{z k_3} + F_{k_3} e^{-z k_3}) G_k e^{k^2 c^2 t} \dots (6) \end{aligned}$$

where $k^2 = k_1^2 + k_2^2 + k_3^2$.

The solution can also be taken as

$$u(x, y, z, t) = \sum_{k_1, k_2, k_3} H_{k_1, k_2, k_3} e^{\pm x k_1 \pm y k_2 \pm z k_3 + k^2 c^2 t} \dots (7)$$

Where $H_{k_1, k_2, k_3} = A'_{k_1} B'_{k_2} C'_{k_3} G_k$ is the new arbitrary constant.

Case III : When $\lambda_1 = -k_1^2 < 0$, $\lambda_2 = -k_2^2 < 0$, $\lambda_3 = -k_3^2 < 0$ and $\lambda = -k^2 < 0$

Where $k^2 = k_1^2 + k_2^2 + k_3^2$, the equations in (4) reduce to

$$\frac{d^2 F}{dx^2} + k_1^2 F = 0, \quad \frac{d^2 Y}{dy^2} + k_2^2 Y = 0, \quad \frac{d^2 Z}{dz^2} + k_3^2 Z = 0 \quad \text{and} \quad \frac{dT}{dt} = -k^2 c^2 T,$$

where $k^2 = k_1^2 + k_2^2 + k_3^2$.

Whose general solutions are

$$\begin{aligned} F &= A_{k_1} \cos k_1 x + B_{k_1} \sin k_1 x \quad \text{or} \quad F = A'_{k_1} e^{\pm i k_1 x} \\ Y &= C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y \quad \text{or} \quad Y = B'_{k_2} e^{\pm i k_2 y} \\ Z &= E_{k_3} \cos k_3 z + F_{k_3} \sin k_3 z \quad \text{or} \quad Z = C'_{k_3} e^{\pm i k_3 z} \end{aligned}$$

and $T = G_k e^{-k^2 c^2 t}, \quad \text{where} \quad k^2 = k_1^2 + k_2^2 + k_3^2.$

∴ From (2), the most general solution of the heat equation (1) can be taken as

$$u(x, y, z, t) = \sum_{k_1, k_2, k_3} (A_{k_1} \cos k_1 x + B_{k_1} \sin k_1 x) \cdot (C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y) \cdot (E_{k_3} \cos k_3 z + F_{k_3} \sin k_3 z) \cdot G_k e^{-k^2 c^2 t} \dots (8)$$

where $k^2 = k_1^2 + k_2^2 + k_3^2$.

The solution can also be taken as

or $u(x, y, z, t) = \sum_{k_1, k_2, k_3} H_{k_1 k_2 k_3} e^{\pm i k_1 x \pm i k_2 y \pm i k_3 z - k^2 c^2 t}$ where $k^2 = k_1^2 + k_2^2 + k_3^2$

and $H_{k_1 k_2 k_3} = A'_{k_1} \cdot B'_{k_2} \cdot C'_{k_3} \cdot G_k$ is the new arbitrary constant.

13.11 Heat (Diffusion) Equation in Cylindrical Coordinates

Heat equation in Cartesian coordinates (x, y, z) is given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial u}{\partial t} \dots (1)$$

Proceeding as in {12.9 on page 568 in cylindrical coordinates (ρ, ϕ, z) , we have

$$\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2}$$

Hence the heat equation (1), in cylindrical coordinates (ρ, ϕ, z) is transformed to

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}.$$

13.12 Solution of Heat (Diffusion) Equation in Cylindrical Coordinates by the Method of Separation of Variables

[Kurukshetra 2002]

The heat equation in cylindrical coordinates (ρ, ϕ, z) is given by

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial u}{\partial t} \dots (A)$$

Obviously u is a function of ρ, ϕ, z and t .

Let the solution of (A) be of the form

$$u(\rho, \phi, z, t) = R(\rho) F(\phi) Z(z) T(t) = RFZT \text{ (say)} \quad \dots(1)$$

Where R is a function of ρ alone, F function of ϕ alone, Z function of z alone and T function of t alone.

Differentiating (1) and substituting in (A) we get

$$FZT \frac{d^2 R}{d\rho^2} + \frac{FZT}{\rho} \frac{dR}{d\rho} + \frac{RZT}{\rho^2} \frac{d^2 F}{d\phi^2} + RFT \frac{d^2 Z}{dz^2} = \frac{RFZ}{c^2} \frac{dT}{dt}$$

$$\text{or} \quad \frac{1}{R} \frac{d^2 R}{d\rho^2} + \frac{1}{R\rho} \frac{dR}{d\rho} + \frac{1}{\rho^2} \frac{1}{F} \frac{d^2 F}{d\phi^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = \frac{1}{c^2 T} \frac{dT}{dt} = -k^2 \text{ (constant)} \quad \dots(2)$$

In (2) the right hand side is a function of independent variable t alone while left hand side is a function of independent variable ρ, ϕ and z only. The two sides can be equal only if each is equal to the same constant value say $-k^2$. Then from (2), we get

$$\frac{1}{c^2 T} \frac{dT}{dt} = -k^2 \quad \text{or} \quad \frac{dT}{dt} + k^2 c^2 T = 0 \quad \dots(3)$$

$$\text{and} \quad \frac{1}{R} \frac{d^2 R}{d\rho^2} + \frac{1}{R\rho} \frac{dR}{d\rho} + \frac{1}{\rho^2} \frac{1}{F} \frac{d^2 F}{d\phi^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = -k^2$$

$$\text{or} \quad \frac{1}{R} \frac{d^2 R}{d\rho^2} + \frac{1}{R\rho} \frac{dR}{d\rho} + \frac{1}{\rho^2} \frac{1}{F} \frac{d^2 F}{d\phi^2} + k^2 = -\frac{1}{Z} \frac{d^2 Z}{dz^2} = -\lambda^2 \text{ (constant)} \quad \dots(4)$$

With the similar arguments as above, both sides of (4) will be equal to the same constant say $-\lambda^2$. Then from (4), we get

$$-\frac{1}{Z} \frac{d^2 Z}{dz^2} = -\lambda^2 \quad \text{or} \quad \frac{d^2 Z}{dz^2} - \lambda^2 Z = 0 \quad \dots(5)$$

$$\text{and} \quad \frac{1}{R} \frac{d^2 R}{d\rho^2} + \frac{1}{R\rho} \frac{dR}{d\rho} + \frac{1}{\rho^2} \frac{1}{F} \frac{d^2 F}{d\phi^2} + k^2 = -\lambda^2$$

$$\text{or} \quad \rho^2 \left[\frac{1}{R} \frac{d^2 R}{d\rho^2} + \frac{1}{R\rho} \frac{dR}{d\rho} + k^2 + \lambda^2 \right] = -\frac{1}{F} \frac{d^2 F}{d\phi^2} = m^2 \text{ (constant)} \quad \dots(6)$$

With the similar argument as above, both sides of (6) will be equal to the same constant say m^2 , Then from (6), we get

$$-\frac{1}{F} \frac{d^2 F}{d\phi^2} = m^2 \quad \text{or} \quad \frac{d^2 F}{d\phi^2} + m^2 F = 0 \quad \dots(7)$$

$$\text{and} \quad \rho^2 \left[\frac{1}{R} \frac{d^2 R}{d\rho^2} + \frac{1}{R\rho} \frac{dR}{d\rho} + k^2 + \lambda^2 \right] = m^2$$

$$\text{or} \quad \frac{d^2 F}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + (\gamma^2 - m^2/\rho^2) R = 0 \quad \dots(8)$$

$$\text{where} \quad k^2 + \lambda^2 = \gamma^2 \quad (\text{constant})$$

$$\text{Now solution of (3) is} \quad T = T(t) = A_k e^{-k^2 c^2 t}$$

$$\text{Solution of (5) is} \quad Z = Z(z) = B_\lambda e^{\lambda z} + C_\lambda e^{-\lambda z} \quad \text{or} \quad Z = B'_\lambda e^{\pm \lambda z}$$

$$\text{Solution of (7) is } F = F(\phi) = D_m \cos m\phi + E_m \sin m\phi \quad \text{or} \quad F = D'_m e^{\pm im\phi}$$

and solution of (8), which is Bessel's equation is

$$R = R(\rho) = F_{k\lambda m} J_m(\gamma\rho) + G_{k\lambda m} Y_m(\gamma\rho) \quad \text{when } m \text{ is an integer.}$$

$$\text{or} \quad R = R(\rho) = F_{k\lambda m} J_m(\gamma\rho) + G_{k\lambda m} J_{-n}(\gamma\rho) \quad \text{or} \quad R = F'_{k\lambda m} J_{\pm m}(\gamma\rho)$$

when m is not an integer.

Hence, from (1), the general solution of (A) is given by

$$u(\rho, \phi, z, t) = \sum_{k, \lambda, m} A_k (B_\lambda e^{\lambda z} + C_\lambda e^{-\lambda z}) (D_m \cos m\phi + E_m \sin m\phi).$$

$$[F_{k\lambda m} J_m\{\sqrt{(k^2 + \lambda^2)}\rho\} + G_{k\lambda m} Y_m\{\sqrt{(k^2 + \lambda^2)}\rho\}] \cdot e^{-k^2 c^2 t}$$

when m is an integer.

Which can also be taken as

$$u(\rho, \phi, z, t) = \sum_{k, \lambda, m} [H_{k\lambda m} J_m\{\sqrt{(k^2 + \lambda^2)}\rho\} + I_{k\lambda m} Y_m\{\sqrt{(k^2 + \lambda^2)}\rho\}] \cdot e^{(\pm \lambda z \pm im\phi - k^2 c^2 t)}$$

when m is an integer.

$$\text{and } u(\rho, \phi, z, t) = \sum_{k, \lambda, m} A_k (B_\lambda e^{\lambda z} + C_\lambda e^{-\lambda z}) (D_m \cos m\phi + E_m \sin m\phi).$$

$$[F_{k\lambda m} J_m\{\sqrt{(k^2 + \lambda^2)}\rho\} + G_{k\lambda m} J_{-m}\{\sqrt{(k^2 + \lambda^2)}\rho\}] \cdot e^{-k^2 c^2 t}$$

when m is not an integer.

Which can also be written as

$$u(\rho, \phi, z, t) = \sum_{k, \lambda, m} H_{k\lambda m} J_{\pm m}\{\sqrt{(k^2 + \lambda^2)}\rho\} \cdot e^{\pm mz \pm im\phi - k^2 c^2 t}$$

when m is not an integer.

13.13 Heat (Diffusion) Equation in Spherical Polar Coordinates

Heat equation in Cartesian coordinates is given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2} \frac{\partial u}{\partial t} \quad \dots(1)$$

Proceeding as in { 12.11 on page 573 in spherical polar coordinates (r, θ, ϕ) , we have

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2}$$

Hence the heat equation (1), in spherical polar coordinates (r, θ, ϕ) is transformed to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}$$

13.14 Solution of Heat (Diffusion) Equation in Spherical Polar Coordinates by the Method of Separation of Variables

[Kurukshetra 2001]

The heat equation in spherical polar coordinates (r, θ, ϕ) is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = \frac{1}{c^2} \frac{\partial u}{\partial t} \quad \dots(A)$$

Let the solution of (A) be of the form

$$u(r, \theta, \phi, t) = R(r) F_1(\theta) F_2(\phi) T(t) = R F_1 F_2 T \text{ (say)} \quad \dots(1)$$

Where R is a function of r alone, F_1 function of θ alone, F_2 function of ϕ alone and T function of t alone.

Differentiating (1) and substituting in (A), we get

$$\begin{aligned} F_1 F_2 T \frac{d^2 R}{dr^2} + \frac{2}{r} F_1 F_2 T \frac{dR}{dr} + \frac{1}{r^2} R F_2 T \frac{d^2 F_1}{d\theta^2} + \frac{\cot \theta}{r^2} R F_2 T \frac{dF_1}{d\theta} \\ + \frac{1}{r^2 \sin^2 \theta} R F_1 T \frac{d^2 F_2}{d\phi^2} = \frac{R F_1 F_2}{c^2} \frac{dT}{dt} \end{aligned}$$

$$\text{or} \quad \frac{1}{R} \frac{d^2 R}{dr^2} + \frac{2}{r} \cdot \frac{1}{R} \frac{dR}{dr} + \frac{1}{r^2} \cdot \frac{1}{F_1} \frac{d^2 F_1}{d\theta^2} + \frac{\cot \theta}{r^2} \cdot \frac{1}{F_1} \frac{dF_1}{d\theta} + \frac{1}{r^2 \sin^2 \theta} \cdot \frac{1}{F_2} \frac{d^2 F_2}{d\phi^2} = \frac{1}{c^2 T} \frac{dT}{dt} = -k^2 \quad (\text{constant}) \quad \dots(2)$$

Here in (3), right hand side is a function of independent variable t only while left hand side is a function of independent variables r , θ and ϕ only. Therefore, the two sides will be equal to each other only if both sides are equal to the same constant say $-k^2$. Then from (2), we get

$$\frac{1}{c^2 T} \frac{dT}{dt} = -k^2 \quad \text{or} \quad \frac{dT}{dt} = -c^2 k^2 T \quad \dots(3)$$

$$\text{and} \quad \frac{1}{R} \frac{d^2 R}{dr^2} + \frac{2}{r} \cdot \frac{1}{R} \frac{dR}{dr} + \frac{1}{r^2} \cdot \frac{1}{F_1} \frac{d^2 F_1}{d\theta^2} + \frac{\cot \theta}{r^2} \cdot \frac{1}{F_1} \frac{dF_1}{d\theta} + \frac{1}{r^2 \sin^2 \theta} \cdot \frac{1}{F_2} \frac{d^2 F_2}{d\phi^2} = -k^2$$

$$\text{or} \quad \left[\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{2}{rR} \frac{dR}{dr} + \frac{1}{r^2 F_1} \frac{d^2 F_1}{d\theta^2} + \frac{\cot \theta}{r^2} \cdot \frac{1}{F_1} \frac{dF_1}{d\theta} + k^2 \right] r^2 \sin^2 \theta = -\frac{1}{F_2} \frac{d^2 F_2}{d\phi^2} = m^2 \quad (\text{constant}) \quad \dots(4)$$

With the similar arguments as in (2) both sides of (4) will be equal to the same constant say m^2 .

$\therefore$ From (4), we get

$$-\frac{1}{F_2} \frac{d^2 F_2}{d\phi^2} = m^2 \quad \text{or} \quad \frac{d^2 F_2}{d\phi^2} + m^2 F_2 = 0 \quad \dots(5)$$

$$\text{and} \quad \left[\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{2}{rR} \frac{dR}{dr} + \frac{1}{r^2 F_1} \frac{d^2 F_1}{d\theta^2} + \frac{\cot \theta}{r^2} \cdot \frac{1}{F_1} \frac{dF_1}{d\theta} + k^2 \right] r^2 \sin^2 \theta = m^2$$

$$\text{or} \quad \frac{r^2}{R} \frac{d^2 R}{dr^2} + \frac{2r}{R} \frac{dR}{dr} + k^2 r^2 = - \left[\frac{1}{F_1} \frac{d^2 F_1}{d\theta^2} + \frac{\cot \theta}{F_1} \frac{dF_1}{d\theta} - \frac{m^2}{\sin^2 \theta} \right] = n(n+1), \quad (\text{constant}) \quad \dots(6)$$

With the same arguments as in (2), both sides of (6) will be equal to the same constant say $n(n+1)$. Then from (6), we get

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} + \{k^2 r^2 - n(n+1)\} R = 0$$

or
$$\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} + \left\{ k^2 - \frac{n(n+1)}{r^2} \right\} R = 0 \quad \dots(7)$$

and
$$\frac{d^2 F_1}{d\theta^2} + \cot \theta \frac{dF_1}{d\theta} + \left\{ n(n+1) - \frac{m^2}{\sin^2 \theta} \right\} F_1 = 0 \quad \dots(8)$$

Now let $\cos \theta = \mu \quad \therefore \frac{dF_1}{d\theta} = \frac{dF_1}{d\mu} \cdot \frac{d\mu}{d\theta} = (-\sin \theta) \frac{dF_1}{d\mu}$

and
$$\begin{aligned} \frac{d^2 F_1}{d\theta^2} &= \frac{d}{d\theta} \left(-\sin \theta \frac{dF_1}{d\mu} \right) = -\cos \theta \frac{dF_1}{d\mu} - \sin \theta \frac{d^2 F_1}{d\mu^2} \cdot \frac{d\mu}{d\theta} \\ &= -\cos \theta \frac{dF_1}{d\mu} + \sin^2 \theta \frac{d^2 F_1}{d\mu^2} = -\mu \frac{dF_1}{d\mu} + (1-\mu^2) \frac{d^2 F_1}{d\mu^2} \end{aligned}$$

Putting in (8), we get

$$(1-\mu^2) \frac{d^2 F_1}{d\mu^2} - \mu \frac{dF_1}{d\mu} + \cot \theta (-\sin \theta) \frac{dF_1}{d\mu} + \left\{ n(n+1) - \frac{m^2}{1-\mu^2} \right\} F_1 = 0$$

or
$$(1-\mu^2) \frac{d^2 F_1}{d\mu^2} - 2\mu \frac{dF_1}{d\mu} + \left\{ n(n+1) - \frac{m^2}{1-\mu^2} \right\} F_1 = 0 \quad \dots(9)$$

Which is associated Legendre's equation.

Now solution of (3) is

$$T = T(t) = A_k e^{-c^2 k^2 t} \quad \dots(10)$$

Solution of (5) is $F_2 = F_2(\phi) = B_m \cos m\phi + C_m \sin m\phi$ or $F_2(\phi) = C'_m e^{\pm im\phi} \quad \dots(11)$

To find solution of (7), let $R = (kr)^{-1/2} \psi$

$$\therefore \frac{dR}{dr} = (kr)^{-1/2} \frac{d\psi}{dr} - \frac{1}{2} k^{-1/2} r^{-3/2} \psi$$

and
$$\frac{d^2 R}{dr^2} = (kr)^{-1/2} \frac{d^2 \psi}{dr^2} - \frac{2}{2} k^{-1/2} r^{-3/2} \frac{d\psi}{dr} + \frac{3}{4} k^{-1/2} r^{-5/2} \psi$$

Putting in (7), it reduces to

$$\frac{d^2 \psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} + \left\{ k^2 - \frac{(n+1/2)^2}{r^2} \right\} \psi = 0$$

Whose solution is

$$\psi = D_{kn} J_{n+1/2}(kr) + E_{kn} J_{-(n+1/2)}(kr)$$

Where $n+1/2$ is neither zero nor an integer.

$$\therefore R = R(r) = (kr)^{-1/2} [D_{kn} J_{n+1/2}(kr) + E_{kn} J_{-(n+1/2)}(kr)] \quad \dots(12)$$

which can also be taken as $R = D'_{kn}(kr)^{-1/2} J_{\pm(n+1/2)}(kr)$

And solution of (9) is

$$\begin{aligned} F_1 &= F_1(\theta) = F_{mn} P_n^m(\mu) + G_{mn} Q_n^m(\mu) \\ &= F_{mn} P_n^m(\cos \theta) + G_{mn} Q_n^m(\cos \theta) \end{aligned} \quad \dots(13)$$

Hence, from (1), for all values of k, m, n the solution of the heat equation (A) in spherical polar coordinates is given by

$$u_{k,m,n}(r, \theta, \phi, t) = R(r) F_1(\theta) E_2(\phi) T(t)$$

Where $R(r)$, $F_1(\theta)$, $E_2(\phi)$ and $T(t)$ are given by (10), (11), (12) and (13). Hence the more general solution is given by

$$\begin{aligned} u(r, \theta, \phi, t) &= \sum_{k,m,n} R(r) F_1(\theta) E_2(\phi) T(t) \\ &= \sum_{k,m,n} (kr)^{-1/2} \cdot A_k \{B_m \cos m\phi + C_m \sin m\phi\} \cdot \\ &\quad \{D_{kn} J_{n+1/2}(kr) + E_{kn} J_{-(n+1/2)}(kr)\} \cdot \\ &\quad \{F_{mn} P_n^m(\cos \theta) + G_{mn} Q_n^m(\cos \theta)\} \cdot e^{-c^2 k^2 t} \end{aligned}$$

Where $n+1/2$ is neither zero nor integer.

Which can also be written as

$$\begin{aligned} u(r, \theta, \phi, t) &= \sum_{k,m,n} (kr)^{-1/2} H_{kmn} \{F_{mn} P_n^m(\cos \theta) \\ &\quad + G_{mn} Q_n^m(\cos \theta)\} \cdot J_{\pm(n+1/2)}(kr) \cdot e^{\pm im\phi - c^2 k^2 t} \end{aligned}$$

Solved Examples

Example 1: The faces of the solid parallelepiped $0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$ are kept at zero temperature. If, initially the temperature of the solid is given by $\theta(x, y, z, 0) = f(x, y, z)$. Show that at time $t > 0$.

$$\theta(x, y, z, t) = \frac{8}{abc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} F(m, n, q) \cdot e^{-\mu^2 t} \cdot \sin \frac{m\pi x}{a} \cdot \sin \frac{n\pi y}{b} \cdot \sin \frac{q\pi z}{c},$$

$$\text{where } F(m, n, q) = \int_0^a \int_0^b \int_0^c f(x, y, z) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{q\pi z}{c} dx dy dz$$

$$\text{and } \mu^2 = k\pi^2(m^2/a^2 + n^2/b^2 + q^2/c^2).$$

[Meerut 2004 (B.P.), 10]

Solution: The temperature distribution $\theta(x, y, z, t)$ in the solid is governed by the following heat equation (three dimensional)

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} = \frac{1}{k} \frac{\partial \theta}{\partial t} \quad \dots(A)$$

Here the **boundary conditions** are

$$\left. \begin{aligned} \theta(0, y, z, t) &= 0, & \theta(a, y, z, t) &= 0 \\ \theta(x, 0, z, t) &= 0, & \theta(x, b, z, t) &= 0 \\ \theta(x, y, 0, t) &= 0, & \theta(x, y, c, t) &= 0 \end{aligned} \right\} \quad \dots(B)$$

and the **initial condition** is

$$\theta(x, y, z, 0) = f(x, y, z) \quad \dots(C)$$

$$\text{Let } \theta(x, y, z, t) = F(x)Y(y)Z(z)T(t) = FYZT \text{ (say)} \quad \dots(1)$$

Where F is a function of x alone, Y function of y alone, Z function of z alone and T function of t alone, be the solution of heat equation (A).

Differentiating (1) and substituting in (A), we get

$$YZT \frac{d^2 F}{dx^2} + FZT \frac{d^2 Y}{dy^2} + FYT \frac{d^2 Z}{dz^2} = \frac{FYZ}{k} \frac{dT}{dt}$$

$$\text{or } \frac{1}{F} \frac{d^2 F}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = \frac{1}{kT} \frac{dT}{dt} = -\lambda^2 \text{ (constant)} \quad \dots(2)$$

All the four terms in (2) are functions of different independent variable, so equation (2) will be true if each term is equal to constant, so let

$$\frac{1}{kT} \frac{dT}{dt} = -\lambda^2, \quad \frac{1}{F} \frac{d^2 F}{dx^2} = -\lambda_1^2, \quad \frac{1}{Y} \frac{d^2 Y}{dy^2} = -\lambda_2^2 \quad \text{and} \quad \frac{1}{Z} \frac{d^2 Z}{dz^2} = -\lambda_3^2$$

such that $\lambda^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$

Note that negative constants are chosen to suit the conditions of the problem as $T \rightarrow 0$ as $t \rightarrow \infty$.

Thus, we get the following ordinary differential equations

$$\frac{d^2 F}{dx^2} + \lambda_1^2 F = 0, \quad \frac{d^2 Y}{dy^2} + \lambda_2^2 Y = 0, \quad \frac{d^2 Z}{dz^2} + \lambda_3^2 Z = 0 \quad \text{and} \quad \frac{dT}{dt} = -\lambda^2 kT$$

The general solutions of these equations are

$$\begin{aligned} F &= F(x) = A_{\lambda_1} \cos \lambda_1 x + B_{\lambda_1} \sin \lambda_1 x, \\ Y &= Y(y) = C_{\lambda_2} \cos \lambda_2 y + D_{\lambda_2} \sin \lambda_2 y, \\ Z &= Z(z) = E_{\lambda_3} \cos \lambda_3 z + F_{\lambda_3} \sin \lambda_3 z \quad \text{and} \quad T = G_{\lambda} e^{-\lambda^2 k t} \end{aligned}$$

Now using the first two boundary condition in (2), we get

$$\theta(0, y, z, t) = 0 \Rightarrow F(0) Y(y) Z(z) T(t) = 0$$

$$\text{and } \theta(a, y, z, t) = 0 \Rightarrow F(a) Y(y) Z(z) T(t) = 0$$

$$\Rightarrow F(0) = 0 \text{ and } F(a) = 0$$

$$\therefore Y(y) = 0 \text{ or } Z(z) = 0 \quad \text{or} \quad T(t) = 0$$

will lead to inadmissible trivial solution $\theta(x, y, z, t) = 0$

$$\text{Now } F(0) = 0 \Rightarrow A_{\lambda_1} = 0, \therefore F(a) = 0 \Rightarrow B_{\lambda_1} \sin \lambda_1 a = 0$$

$$\Rightarrow \sin \lambda_1 a = 0, \therefore B_{\lambda_1} = 0 \text{ will lead to inadmissible trivial solution.}$$

$$\Rightarrow \lambda_1 a = m\pi, \text{ where } m \text{ is any integer}$$

$$\Rightarrow \lambda_1 = m\pi/a$$

$$\therefore F(x) = B_{\lambda_1} \sin \frac{m\pi x}{a}$$

Similarly, using other four boundary conditions, we get

$$C_{\lambda_2} = 0 \text{ and } \lambda_2 = n\pi/b, E_{\lambda_3} = 0 \text{ and } \lambda_3 = q\pi/c$$

Thus, we get

$$F(x) = B_{\lambda_1} \sin \frac{m\pi x}{a}, Y(y) = D_{\lambda_2} \sin \frac{n\pi y}{b} \text{ and } Z(z) = F_{\lambda_3} \sin \frac{q\pi z}{c}$$

$$\text{and} \quad \lambda^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \pi^2 (m^2/a^2 + n^2/b^2 + q^2/c^2).$$

$\therefore$ Putting in (1), for integral values of m, n and q , the solutions of (A) satisfying the boundary conditions are given by

$$\theta_{m,n,q}(x, y, z, t) = H_{mnq} e^{-\lambda^2 kt} \sin \frac{m\pi x}{a} \cdot \sin \frac{n\pi y}{b} \sin \frac{q\pi z}{c}$$

Where $H_{mnq} = B_{\lambda_1} D_{\lambda_2} \cdot F_{\lambda_3} \cdot G_{\lambda}$ is the new arbitrary constant.

Taking $m = 1, 2, \dots; n = 1, 2, \dots; q = 1, 2, \dots$, the more general solution of (A) is given by

$$\theta(x, y, z, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} H_{mnq} e^{-\lambda^2 kt} \sin \frac{m\pi x}{a} \cdot \sin \frac{n\pi y}{b} \sin \frac{q\pi z}{c} \dots (3)$$

Which also satisfy the boundary conditions.

$\therefore$ The initial condition (C),

$$\text{i.e., } \theta(x, y, z, 0) = f(x, y, z)$$

$$\Rightarrow f(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} H_{mnq} \sin \frac{m\pi x}{a} \cdot \sin \frac{n\pi y}{b} \sin \frac{q\pi z}{c}$$

$$\therefore H_{mnq} = \frac{2}{a} \cdot \frac{2}{b} \cdot \frac{2}{c} \int_{x=0}^a \int_{y=0}^b \int_{z=0}^c f(x, y, z) \sin \frac{m\pi x}{a} \cdot \sin \frac{n\pi y}{b} \sin \frac{q\pi z}{c} dx dy dz$$

$$= \frac{8}{abc} F(m, n, q)$$

Hence, from (3), the required solution is given by

$$\theta(x, y, z, t) = \frac{8}{abc} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} F(m, n, q) e^{-\mu^2 t} \sin \frac{m\pi x}{a} \cdot \sin \frac{n\pi y}{b} \sin \frac{q\pi z}{c}$$

$$\text{where } F(m, n, q) = \int_0^a \int_0^b \int_0^c f(x, y, z) \sin \frac{m\pi x}{a} \cdot \sin \frac{n\pi y}{b} \sin \frac{q\pi z}{c} dx dy dz$$

$$\text{and} \quad \mu^2 = k\lambda^2 = k\pi^2 (m^2/a^2 + n^2/b^2 + q^2/c^2).$$

Example 2: Determine the temperature distribution in the infinite cylinder $0 \leq \rho \leq a$ when the initial temperature is $u(\rho, 0) = f(\rho)$ and the surface $\rho = a$ is maintained at zero temperature.

[Delhi 2005; Meerut 2002 (O), 03 (O), 10 (Sem. I)]

Solution: Heat equation in cylindrical coordinates (ρ, ϕ, z) is given by

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{k} \frac{\partial u}{\partial t} \quad \dots(1)$$

Here the temperature u is independent of z and ϕ so the equation (1) reduces to

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} = \frac{1}{k} \frac{\partial u}{\partial t} \quad \dots(2)$$

$$0 < \rho < a, t > 0$$

Thus, in the present problem the temperature u is governed by equation (2) i.e., $u = u(\rho, t)$.

The **boundary condition** is $u(a, t) = 0, t > 0$... (3)

and **initial condition** is $u(\rho, 0) = f(\rho), (0 \leq \rho < a)$... (4)

The solution of (2) i.e., the temperature u which is a function of ρ and t only, can be taken as

$$u(\rho, t) = R(\rho) T(t) = RT \text{ (say)} \quad \dots(5)$$

where R is a function of ρ alone and T function of t alone.

Differentiating (5) and substituting in (2), we get

$$T \frac{d^2 R}{d\rho^2} + \frac{T}{\rho} \frac{dR}{d\rho} = \frac{R}{k} \frac{dT}{dt}$$

or
$$\frac{1}{R} \frac{d^2 R}{d\rho^2} + \frac{1}{\rho R} \frac{dR}{d\rho} = \frac{1}{kT} \frac{dT}{dt} = -\lambda^2 \text{ (constant)} \quad \dots(6)$$

In (6) right hand side is a function of independent variable t alone while the left hand side is a function of independent variable ρ alone. The two sides will be equal to each other if both sides are equal to a constant say $-\lambda^2$. Then from equation (6), we get

$$\frac{1}{kT} \frac{dT}{dt} = -\lambda^2 \quad \text{or} \quad \frac{dT}{dt} = -\lambda^2 kT \quad \dots(7)$$

and
$$\frac{1}{R} \frac{d^2 R}{d\rho^2} + \frac{1}{\rho R} \frac{dR}{d\rho} = -\lambda^2$$

or
$$\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + \lambda^2 R = 0 \quad \dots(8)$$

The solution of (7) is
$$T = T(t) = A_\lambda e^{-\lambda^2 k t} \quad \dots(9)$$

Equation (8) is a Bessel's equation of zeroth order, so its solution is

$$R = R(\rho) = B_\lambda J_0(\lambda \rho) + C_\lambda Y_0(\lambda \rho)$$

Since $Y_0(\lambda \rho) \rightarrow \infty$ as $\rho \rightarrow 0$, so we take $C_\lambda = 0$

$$\therefore R(\rho) = B_\lambda J_0(\lambda \rho) \quad \dots(10)$$

$$\therefore \text{from (5)} \quad u(\rho, t) = D_\lambda J_0(\lambda \rho) \cdot e^{-\lambda^2 k t} \quad \dots(11)$$

Where $D_\lambda = A_\lambda \cdot B_\lambda$ is the new arbitrary constant.

Now from (11), the boundary condition (3), $u(a, t) = 0 \Rightarrow D_\lambda J_0(\lambda a) e^{-\lambda^2 k t} = 0$

$$\Rightarrow J_0(\lambda a) = 0 \quad \dots(12)$$

$$\therefore D_\lambda \neq 0 \text{ and } e^{-\lambda^2 k t} \neq 0$$

Let $\lambda_1, \lambda_2, \lambda_3, \dots$ be the positive roots of equation (12), then from (11),

$$u_i(\rho, t) = D_{\lambda_i} J_0(\lambda_i \rho) \cdot e^{-\lambda_i^2 k t}$$

are the solutions of (2) for $i = 1, 2, 3, \dots$, satisfying the boundary condition (3). Taking $D_{\lambda_i} = E_i$, the more general solution of (2) can be taken as

$$u(\rho, t) = \sum_{i=1}^{\infty} u_i(\rho, t) = \sum_{i=1}^{\infty} E_i J_0(\lambda_i \rho) \cdot e^{-\lambda_i^2 k t} \quad \dots(13)$$

Which also satisfy the boundary condition (3). Now the initial condition (4),

$$u(\rho, 0) = f(\rho) \Rightarrow \sum_{i=1}^n E_i J_0(\lambda_i \rho) = f(\rho) \quad \dots(14)$$

The left hand side of (14) can be considered as the Fourier Bessel series of $f(\rho)$

$$\therefore E_i = \frac{2}{a^2 J_1^2(\lambda_i a)} \int_0^a \rho f(\rho) J_0(\lambda_i \rho) d\rho \quad \dots(15)$$

Hence, the required temperature distribution is given by (13), where E_i is given by (15).

Note : From (13) and (15) the required temperature distribution $u(\rho, t)$ can be written as

$$u(\rho, t) = \frac{2}{a^2} \sum_{i=1}^{\infty} \frac{J_0(\lambda_i \rho) \cdot e^{-\lambda_i^2 \rho t}}{J_1^2(\lambda_i a)} \int_0^a \rho f(\rho) J_0(\lambda_i \rho) d\rho.$$

Example 3: Find the temperature in a sphere of radius a when its surface is maintained at zero temperature and its initial temperature is $f(r, \theta)$. [Delhi 2003]

Solution: Heat equation in spherical coordinates (r, θ, ϕ) is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = \frac{1}{k} \frac{\partial u}{\partial t} \quad \dots(1)$$

Here due to axial symmetry in a sphere u is independent of ϕ $\therefore \frac{\partial^2 u}{\partial \phi^2} = 0$, so the equation (1) reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} = \frac{1}{k} \frac{\partial u}{\partial t} \quad \dots(2)$$

Thus, in the present problem the temperature u is governed by the equation (2) and so $u = u(r, \theta, t)$

The **boundary condition** is $u(a, \theta, t) = 0, \quad t > 0$... (3)

and **initial condition** is $u(r, \theta, 0) = f(r, \theta), \quad 0 < r < a$... (4)

The solution of (2), i.e., the temperature u which is a function of r, θ and t only, can be taken as

$$u(r, \theta, t) = R(r) F(\theta) T(t) = RFT \text{ (say)} \quad \dots(5)$$

Where R is a function of r alone, F function of θ and T function of t alone.

Differentiating (5) and substituting in (2), we get

$$FT \frac{d^2 R}{dr^2} + \frac{2}{r} FT \frac{dR}{dr} + \frac{RT}{r^2} \frac{d^2 F}{d\theta^2} + \frac{\cot \theta}{r^2} RT \frac{dF}{d\theta} = \frac{RF}{k} \frac{dT}{dt}$$

or
$$\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{2}{rR} \frac{dR}{dr} + \frac{1}{r^2 F} \frac{d^2 F}{d\theta^2} + \frac{\cot \theta}{r^2} \cdot \frac{1}{F} \frac{dF}{d\theta} = \frac{1}{kT} \frac{dT}{dt} = -\lambda^2 \text{ (constant)} \quad \dots(6)$$

In (6) the right hand side is a function of independent variable t alone while the left hand side is a function of independent variables r and θ only. The two sides will be equal if each side reduces to the same constant say $-\lambda^2$. Then from (6), we get

$$\frac{1}{kT} \frac{dT}{dt} = -\lambda^2 \quad \text{or} \quad \frac{dT}{dt} = -k\lambda^2 T \quad \dots(7)$$

and
$$\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{2}{rR} \frac{dR}{dr} + \frac{1}{r^2 F} \frac{d^2 F}{d\theta^2} + \frac{\cot \theta}{r^2} \cdot \frac{1}{F} \frac{dF}{d\theta} = -\lambda^2$$

or
$$r^2 \left(\frac{1}{R} \frac{d^2 R}{dr^2} + \frac{2}{rR} \frac{dR}{dr} + \lambda^2 \right) = - \left(\frac{1}{F} \frac{d^2 F}{d\theta^2} + \frac{\cot \theta}{F} \frac{dF}{d\theta} \right) = n(n+1) \text{ (say)} \quad \dots(8)$$

With the similar arguments as in (6), the two sides of (8) will be equal to the same constant say $n(n+1)$. Then from (8), we get

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} + \{\lambda^2 r^2 - n(n+1)\} R = 0$$

or
$$\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} + \left\{ \lambda^2 - \frac{n(n+1)}{r^2} \right\} R = 0 \quad \dots(9)$$

and
$$\frac{d^2 F}{d\theta^2} + \cot \theta \frac{dF}{d\theta} + n(n+1) F = 0 \quad \dots(10)$$

Now in (9), substituting $R = (\lambda r)^{-1/2} \psi$

so that
$$\frac{dR}{dr} = (\lambda r)^{-1/2} \frac{d\psi}{dr} - \frac{1}{2} \lambda^{-1/2} r^{-3/2} \psi$$

and
$$\frac{d^2 R}{dr^2} = (\lambda r)^{-1/2} \frac{d^2 \psi}{dr^2} - \frac{2}{2} \lambda^{-1/2} r^{-3/2} \frac{d\psi}{dr} + \frac{3}{4} \lambda^{-1/2} r^{-5/2} \psi$$

Putting in (9), it reduces to

$$\frac{d^2 \psi}{dr^2} + \frac{1}{r} \frac{d\psi}{dr} + \left(\lambda^2 - \frac{(n+1/2)^2}{r^2} \right) \psi = 0 \quad \dots(11)$$

And in (10) substituting $\cos \theta = \mu$

$$\therefore \frac{dF}{d\theta} = \frac{dF}{d\mu} \cdot \frac{d\mu}{d\theta} = -\sin \theta \frac{dF}{d\mu}$$

and
$$\begin{aligned} \frac{d^2 F}{d\theta^2} &= \frac{d}{d\theta} \left(-\sin \theta \frac{dF}{d\mu} \right) = -\cos \theta \frac{dF}{d\mu} - \sin \theta \frac{d^2 F}{d\mu^2} \cdot \frac{d\mu}{d\theta} \\ &= -\cos \theta \frac{dF}{d\mu} + \sin^2 \theta \frac{d^2 F}{d\mu^2} = -\mu \frac{dF}{d\mu} + (1-\mu^2) \frac{d^2 F}{d\mu^2} \end{aligned}$$

Putting in (10), we get

$$(1-\mu^2) \frac{d^2 F}{d\mu^2} - \mu \frac{dF}{d\mu} + \cot \theta (-\sin \theta) \frac{dF}{d\mu} + n(n+1) F = 0$$

$$\text{or} \quad (1-\mu^2) \frac{d^2 F}{d\mu^2} - 2\mu \frac{dF}{d\mu} + n(n+1) F = 0 \quad \dots(12)$$

Which is a Legendre's equation.

$$\text{Now the solution of (7) is } T = T(t) = A_{\lambda} e^{-k \lambda^2 t} \quad \dots(13)$$

$$\text{Solution of (11) is} \quad \psi = B_{\lambda n} J_{n+1/2}(\lambda r) + C_{\lambda n} J_{-(n+1/2)}(\lambda r)$$

$$\text{or} \quad R = R(r) = (\lambda r)^{-1/2} [B_{\lambda n} J_{n+1/2}(\lambda r) + C_{\lambda n} J_{-(n+1/2)}(\lambda r)] \quad \dots(14)$$

Where $(n+1/2)$ is neither zero nor an integer.

Solution of (12) is

$$F = F(\theta) = D_n P_n(\mu) + E_n Q_n(\mu) = D_n P_n(\cos \theta) + E_n Q_n(\cos \theta) \quad \dots(15)$$

Now in the sphere $u(r, \theta + 2\pi, t) = u(r, \theta, t)$, so n must be an integer. Also as $J_{-(n+1/2)}(\lambda r) \rightarrow \infty$ as $r \rightarrow 0$, so we take $C_{\lambda n} = 0$ in (14).

Also $Q_n(\cos \theta) \rightarrow \infty$ as $\cos \theta \rightarrow 0$, so we take $E_n = 0$ in (15).

With these values of $C_{\lambda n}$ and E_n , from (14) and (15), we have

$$R(r) = (\lambda r)^{-1/2} B_{\lambda n} J_{n+1/2}(\lambda r) \quad \dots(16)$$

$$\text{and} \quad F(\theta) = D_n P_n(\cos \theta) \quad \dots(17)$$

Now from (5) and boundary condition (3), we have

$$u(a, \theta, t) = 0 \Rightarrow R(a) F(\theta) T(t) = 0 \Rightarrow R(a) = 0,$$

$$\because F(\theta \neq 0, T(t) \neq 0 \text{ otherwise } u = 0.$$

$$\therefore \text{ from (16), } R(a) = (\lambda a)^{-1/2} B_{\lambda n} J_{n+1/2}(\lambda a) = 0$$

$$\Rightarrow J_{n+1/2}(\lambda a) = 0 \because B_{\lambda n} \neq 0, \text{ Otherwise } u = 0$$

Let $\lambda_{n1}, \lambda_{n2}, \dots$, be its positive roots.

Then for different values of λ (i.e., $\lambda = \lambda_{n1}, \lambda_{n2}, \dots$), and integral values of n , the solution of (2), from (5) are given by

$$\begin{aligned} u_{in}(r, \theta, t) &= F_{\lambda_{ni}n} (\lambda_{ni} r)^{-1/2} \cdot e^{-k\lambda_{ni}^2 t} \cdot J_{n+1/2}(\lambda_{ni} r) \cdot P_n(\cos \theta) \\ &= H_{ni} (\lambda_{ni} r)^{-1/2} \cdot e^{-k\lambda_{ni}^2 t} \cdot J_{n+1/2}(\lambda_{ni} r) P_n(\cos \theta) \end{aligned}$$

Taking $F_{\lambda_{ni}n} = H_{ni}$ as the new arbitrary constant.

$\therefore$ for $i=1, 2, \dots$, and $n=1, 2, \dots$, the more general solution, satisfying the boundary condition (3) is given by

$$u(r, \theta, t) = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} H_{ni} (\lambda_{ni} r)^{-1/2} e^{-k\lambda_{ni}^2 t} \cdot J_{n+1/2}(\lambda_{ni} r) P_n(\cos \theta) \quad \dots(18)$$

Now from (18), the initial condition (4), $u(a, \theta, 0) = f(r, \theta)$

$$\Rightarrow f(r, \theta) = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} H_{ni} (\lambda_{ni} r)^{-1/2} J_{n+1/2}(\lambda_{ni} r) P_n(\cos \theta)$$

$\therefore$ from the theory of Bessel's function and Legendre's polynomials,

$$H_{ni} = \frac{(2n+1)\lambda_{ni}^{1/2}}{a^2 [J'_{n+1/2}(\lambda_{ni} a)]^2} \left(\int_0^a r^{3/2} J_{n+1/2}(r \lambda_{ni}) dr \right) \left(\int_{-1}^1 P_n(\mu) f(r, \theta) d\mu \right) \quad \dots(19)$$

where $\mu = \cos \theta$

Hence, the temperature distribution in the sphere is given by (18) where H_{ni} is given by (19).

Exercise 13.3

1. The faces of a cuboid $0 \leq x \leq a, 0 \leq y \leq a, 0 \leq z \leq a$ are kept at zero temperature. If initially the temperature of the solid is given by $\theta(x, y, z, 0) = f(x, y, z)$, then find $\theta(x, y, z, t)$, at time $t > 0$.
2. The faces of the solid parallelepiped $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$ are kept at zero temperature. If initially the temperature of the solid is given by $V = A \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}$, find the temperature distribution in the solid, at $t > 0$.
3. The faces of a cuboid $0 \leq x \leq \pi, 0 \leq y \leq \pi, 0 \leq z \leq \pi$ are kept at zero temperature. If initially the temperature of the solid is given by $f(x, y, z)$, then find temperature distribution in the solid at time $t > 0$.
4. Solve the diffusion equation in case of spherical symmetry.
 [Hint : Write diffusion equation in spherical polar coordinates and take $\frac{\partial u}{\partial \theta} = 0$, $\frac{\partial^2 u}{\partial \theta^2} = 0$ and $\frac{\partial^2 u}{\partial \phi^2} = 0$ as in the case of spherical symmetry u is independent of θ and ϕ].
5. Solve the diffusion equation in case of a axial symmetry.
 [Hint : See Ex. 3 on page 679].
6. A circular plate of radius a has its plane faces insulated. If the initial temperature is $f(\rho)$ and if the rim is kept at temperature zero, find the temperature of the plate at any time.
 [Hint : See Ex. 2 on page 677].

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter (a), (b), (c) or (d).

1. The temperature function $u(x, t)$ in a bar which is insulated laterally are governed by one dimensional heat equation.

(a) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$	(b) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$
(c) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial u}{\partial t^2}$	(d) None of these

2. Let the ends of a bar of length l which is insulated laterally be kept at temperature 0°C and let $f(x)$ (in $^\circ\text{C}$) be the initial temperature of the bar. Then the boundary conditions of this bar, taking $u(x, t)$ as the temperature of the bar at distance x from one end at time t , are given by
 - (a) $u(0, t) = 0, u(l, t) = f(x)$
 - (b) $u(0, t) = 0 = u(l, t)$
 - (c) $u(0, t) = f(x), u(l, t) = 0$
 - (d) None of these
3. The initial condition of the problem given in Q.N. 2 is :
 - (a) $u(x, 0) = 0$
 - (b) $u_x(x, 0) = 0$
 - (c) $u(x, 0) = f(x)$
 - (d) None of these
4. The ends A and B of a rod π cm. long wave temperature 40°C and 90°C until steady state prevails. The temperatures are changed to 30°C and 80°C respectively. If $u(x, t)$ is the temperature distribution of the rod at any time t at distance x , then the boundary conditions of the rod are :
 - (a) $u(0, t) = 40, u(\pi, t) = 90$
 - (b) $u(0, t) = 70, u(\pi, t) = 170$
 - (c) $u(0, t) = 30, u(\pi, t) = 80$
 - (d) None of these
5. The initial condition of the problem in Q.N. 4 is :
 - (a) $u(x, 0) = 80 - 30 = 50$
 - (b) $u(x, 0) = 40 + (50 / \pi) x$
 - (c) $u(x_1) = 50 + (40 / \pi) x$
 - (d) None of these
6. If $u(x, t)$ is the temperature distribution in a uniform bar of length l whose one end is kept at 20°C and the other end is insulated. If $f(x)$ is the initial temperature of the bar, then the boundary conditions of the problem are :
 - (a) $u(0, t) = 20, u_x(l, t) = 0$
 - (b) $u(0, t) = 20, u(l, t) = 0$
 - (c) $u(0, t) = 20, u(l, t) = f(x)$
 - (d) None of these
7. Two dimensional heat equation in cartesian coordinates is given by :
 - (a) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}$
 - (b) $\frac{\partial u}{\partial t} = \frac{1}{c^2} \frac{\partial^2 u}{\partial x^2}$
 - (c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = c^2 \frac{\partial u}{\partial t}$
 - (d) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}$
8. A rectangular plate bounded by the lines $x = 0, x = a, y = 0, y = b$ has an initial temperature $f(x, y)$. If the edges of the plate are kept at zero temperature zero and the plate faces are imprevious to heat. Then the temperature $u(x, y, t)$ in the plate at any point and time is governed by the equation :
 - (a) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{\rho^2} \frac{\partial u}{\partial t}$
 - (b) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}$
 - (c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$
 - (d) None of these

9. Consider the problem given in Q.N. 8. Then its boundary conditions are :

- (a) $u(0, y, t) = 0 = u(a, y, t) = u(x, 0, t) = 0 = u(x, b, t)$
- (b) $u(0, y, t) = 0 = u(a, y, t), u(x, 0, t) = u(x, b, t) = f(x, y)$
- (c) $u(0, y, t) = u(a, y, t) = f(x, y), y(x, 0, t) = 0 = u(x, b, t)$
- (d) None of these

10. Consider the problem given in Q.N. 8. Then the initial condition is :

- (a) $u(x, y, 0) = 0$
- (b) $u(x, y, 0) = f(x, y)$
- (c) $u(x, y, 0) = f_x(x, y)$
- (d) None of these

Fill in the Blank

Fill in the blanks ".....", so that the following statements are complex and correct.

1. In one dimensional heat equation $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$, $c^2 = K/(\sigma\rho)$. Then σ is the of the bar.

2. Two dimensional heat equation in polar coordinates is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \dots = \frac{1}{c^2} \frac{\partial u}{\partial t}.$$

3. The three dimensional heat equation is given by $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \dots$

4. The heat equation in cylindrical coordinates is given by

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \dots = \frac{1}{c^2} \frac{\partial u}{\partial t}$$

5. The heat (diffusion) equation in spherical polar coordinates is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \dots + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}$$

Answers

Exercise 13.3

1.	$\theta(x, y, z, t) = \frac{1}{a^3} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} A_{mnq} e^{-\lambda^2 kt} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} \sin \frac{q\pi z}{a}$ where $A_{mnq} = \frac{8}{a^3} \int_0^a \int_0^a \int_0^a f(x, y, z) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} \sin \frac{q\pi z}{a} dx dy dz$
2.	$u(x, y, z, t) = Ae^{-k\pi^2 (1/a^2 + 1/b^2 + 1/c^2)t} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{\pi z}{c}.$
3.	$u(x, y, z, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{q=1}^{\infty} e^{-k(m^2 + n^2 + q^2)t} \sin mx \sin ny \sin qz$ where $H_{mnq} = \frac{8}{\pi^3} \int_0^{\pi} \int_0^{\pi} \int_0^{\pi} f(x, y, z) \sin mx \sin ny \sin qz$
4.	$u(r, t) = \sum (\lambda r)^{-1/2} [D_{\lambda} J_{1/2}(\lambda r) + E_{\lambda} J_{-1/2}(\lambda r)] \cdot e^{-\lambda^2 kt}$
5.	$u(r, \theta, t) = \sum_{\lambda} \sum_n [B_{\lambda n} J_{n+1/2}(\lambda r) + C_{\lambda n} J_{-(n+1/2)}(\lambda r)] \cdot [F_{m\lambda} P_n(\cos \theta) + G_{n\lambda} Q_n(\cos \theta)] \cdot e^{-k\lambda^2 t}$
6.	$u(\rho, t) = \frac{2}{a^2} \sum_{i=1}^{\infty} \frac{J_0(\lambda_i \rho)}{J_1^2(\lambda_i a)} e^{-\lambda_i^2 kt} \int_0^a \rho f(\rho) J_0(\lambda_i \rho) d\rho.$

Objective Type Questions

Multiple Choice Questions

1.	(a)	2.	(b)	3.	(c)	4.	(c)
5.	(b)	6.	(a)	7.	(d)	8.	(b)
9.	(a)	10.	(b)				

Fill in the Blank

1.	specific heat	2.	$\frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$
3.	$\frac{1}{c^2} \frac{\partial u}{\partial t}$	4.	$\frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2}$
5.	$\frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta}$		

Laplace Equations (By Method of Separation of Variables)

14.1 Introduction

In this chapter we shall discuss laplace equation, its derivations and solutions in different coordinate systems.

14.2 Laplace Equation

In 13.3, we obtained the heat equation as $\frac{\partial U}{\partial t} = c^2 \nabla^2 U$... (1)

When the temperature is in steady state *i.e.*, when U does not vary with time, then the heat equation (1) becomes $\nabla^2 U = 0$

$$i.e., \quad \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$$

This is known as **Laplace equation** in three dimensions.

If U is independent of z , then the **Laplace equation in two dimensions** is given by

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

Note (i) ∇^2 is called the Laplacian operator and is defined as

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- (ii) A function which satisfy Laplace's equation is called the **harmonic function**. So the Laplace's equation is also referred as **harmonic equation**.
- (iii) The function U which occurs in the above Laplace's equation is **potential function** so the Laplace's equation are also referred as the **potential equation**.

14.3 Solution of Two Dimensional Laplace's (Harmonic) Equation by Using the Method of Separation of Variables

Two dimensional Laplace's equation is given

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(A)$$

To find the solution of (A) by using the method of separation of variables, let

$$u(x, y) = F(x) Y(y) = F Y \text{ (say)} \quad \dots(1)$$

Where F is a function of x alone and Y a function of y alone.

Differentiating (1) and substituting in (A), we get

$$Y \frac{d^2 F}{d x^2} + F \frac{d^2 Y}{d y^2} = 0$$

or
$$\frac{1}{F} \frac{d^2 F}{d x^2} = -\frac{1}{Y} \frac{d^2 Y}{d y^2} = \lambda \text{ (say)}$$

Here the left hand side is a function of independent variable x alone while right hand side is a function of independent variable y alone, so the two sides will be equal only if both sides are equal to the same constant say λ . Thus, we get the following two ordinary differential equations.

$$\frac{d^2 F}{d x^2} - \lambda F = 0 \quad \text{and} \quad \frac{d^2 Y}{d y^2} + \lambda Y = 0$$

Now there are three cases as follows :

Case I : If $\lambda = 0$, then the two equations reduce to

$$\frac{d^2 F}{dx^2} = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} = 0$$

Whose solutions are

$$F = A_1 x + A_2 \quad \text{and} \quad Y = A_3 y + A_4$$

$$\therefore u = FY = (A_1 x + A_2)(A_3 y + A_4)$$

Case II : If $\lambda = k^2 > 0$, then the two equations reduce to

$$\frac{d^2 F}{dx^2} - k^2 F = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} + k^2 Y = 0$$

Whose solutions are

$$F = B_1 e^{kx} + B_2 e^{-kx} \quad \text{and} \quad Y = B_3 \cos ky + B_4 \sin ky$$

$$\therefore u = FY = (B_1 e^{kx} + B_2 e^{-kx})(B_3 \cos ky + B_4 \sin ky)$$

Case III : If $\lambda = -k^2 < 0$, then the two equations reduce to

$$\frac{d^2 F}{dx^2} + k^2 F = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} - k^2 Y = 0$$

Whose solutions are

$$F = C_1 \cos kx + C_2 \sin kx \quad \text{and} \quad Y = C_3 e^{ky} + C_4 e^{-ky}$$

$$\therefore u = FY = (C_1 \cos kx + C_2 \sin kx)(C_3 e^{ky} + C_4 e^{-ky})$$

One of these solutions which suits the giving conditions in the problem is taken as the solution of the Laplace equation.

14.4 Solution of Two Dimensional Laplace's Equation under the Given Conditions

Case I

To find the steady state temperature distribution in a thin plate bounded by lines $x = 0$, $x = a$, $y = 0$ and $y = \infty$, assuming that the heat cannot escape from either surface. The sides $x = 0$, $x = a$ being kept at temperature zero, the lower edge $y = 0$ is kept at temperature $f(x)$ and the edge $y = \infty$ at temperature zero.

[Meerut 2010]

The steady state temperature distribution $u(x, y)$ in a thin plate, assuming that the heat cannot escape from either surface is governed by two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(A)$$

Here the **boundary conditions** are

$$u(0, y) = 0 \quad \dots(B_1) \quad x = 0$$

$$u(a, y) = 0 \quad \dots(B_2) \quad x = a$$

$$u(x, 0) = f(x) \quad \dots(B_3) \quad y = 0$$

$$\text{and} \quad u(x, \infty) = 0 \quad \dots(B_4) \quad y = \infty$$

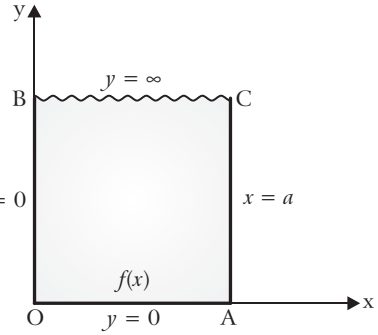


Fig. 14.1

Thus here we are required to find the solution of (A) which satisfy the conditions (B₁), (B₂), (B₃) and (B₄).

Let the solution of (A) be of the form

$$u(x, y) = F(x) Y(y) = FY \text{ (say)} \quad \dots(1)$$

Where F is a function of x alone and Y a function of y alone.

Differentiating (1) and putting in (A), we get

$$Y \frac{d^2 F}{dx^2} + F \frac{d^2 Y}{dy^2} = 0 \quad \text{or} \quad \frac{1}{F} \frac{d^2 F}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2} = \lambda \text{ (say)} \quad \dots(2)$$

Since the two sides of equation (2) are functions of different independent variables, so they will be equal only if each is equal to the same constant say λ .

Thus, we get the following two ordinary differential equations

$$\frac{d^2 F}{dx^2} - \lambda F = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} + \lambda Y = 0 \quad \dots(3)$$

Now three cases arise as follows :

Case (i) : If $\lambda = 0$, then the two equations in (3) reduce to

$$\frac{d^2 F}{dx^2} = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} = 0$$

Whose general solutions are

$$F = A_1 x + A_2 \quad \text{and} \quad Y = A_3 y + A_4$$

$$\therefore \quad u(x, y) = FY = (A_1 x + A_2)(A_3 y + A_4)$$

$$\therefore \quad \text{the boundary conditions (B}_1\text{) and (B}_2\text{) i.e., } u(0, y) = 0 \text{ and } u(a, y) = 0$$

$$\Rightarrow \quad A_2(A_3 y + A_4) = 0 \quad \text{and} \quad (A_1 a + A_2)(A_3 y + A_4) = 0$$

$$\Rightarrow A_2 = 0 \quad \text{and} \quad A_1 a + A_2 = 0$$

$\therefore A_3 y + A_4 = 0$, will lead to trivial solution $u = 0$, which is inadmissible.

$$\Rightarrow A_2 = 0 \quad \text{and} \quad A_1 = 0$$

$\therefore u(x, y) = 0$, which is trivial solution and is inadmissible.

So we reject the case when $\lambda = 0$

Case (ii) : If $\lambda = k^2 > 0$, then the two equations in (3) reduce to

$$\frac{d^2 F}{dx^2} - k^2 F = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} + k^2 Y = 0$$

Whose general solutions are

$$F = B_1 e^{kx} + B_2 e^{-kx} \quad \text{and} \quad Y = B_3 \cos ky + B_4 \sin ky$$

$$\therefore u(x, y) = FY = (B_1 e^{kx} + B_2 e^{-kx})(B_3 \cos ky + B_4 \sin ky)$$

$\therefore$ the boundary conditions (B_1) and (B_2) i.e., $u(0, y) = 0$ and $u(a, y) = 0$

$$\Rightarrow (B_1 + B_2)(B_3 \cos ky + B_4 \sin ky) = 0$$

$$\text{and } (B_1 e^{ka} + B_2 e^{-ka})(B_3 \cos ky + B_4 \sin ky) = 0$$

$$\Rightarrow B_1 + B_2 = 0 \quad \text{and} \quad B_1 e^{ka} + B_2 e^{-ka} = 0$$

$$\therefore B_3 \cos ky + B_4 \sin ky \neq 0$$

$$\Rightarrow B_1 = 0 \quad \text{and} \quad B_2 = 0$$

$\therefore u(x, y) = 0$, which is trivial solution and is inadmissible.

So we reject the case when $\lambda = k^2$ also.

Case (iii) : If $\lambda = -k^2 < 0$, then the two equations in (3) reduce to

$$\frac{d^2 F}{dx^2} + k^2 F = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} - k^2 Y = 0$$

Whose general solutions are

$$F = C_1 \cos kx + C_2 \sin kx \quad \text{and} \quad Y = C_3 e^{ky} + C_4 e^{-ky}$$

$$\therefore u(x, y) = FY = (C_1 \cos kx + C_2 \sin kx)(C_3 e^{ky} + C_4 e^{-ky}) \quad \dots(4)$$

Now the boundary condition (B) i.e., $u(0, y) \Rightarrow C_1(C_3 e^{ky} + C_4 e^{-ky}) = 0$

$$\Rightarrow C_1 = 0, \quad \because C_3 e^{ky} + C_4 e^{-ky} \neq 0$$

$$\therefore \text{ from (4), } u(x, y) = C_2 \sin kx (C_3 e^{ky} + C_4 e^{-ky}) \quad \dots(5)$$

$$\therefore \text{ Boundary condition (B}_2) \text{ i.e., } u(a, y) = 0$$

$$\Rightarrow C_2 \sin ka (C_3 e^{ky} + C_4 e^{-ky}) = 0 \Rightarrow \sin ka = 0$$

$$\because C_2 = 0 \text{ or } C_3 e^{ky} + C_4 e^{-ky} = 0 \text{ will lead to trivial solution } u = 0$$

which is inadmissible.

$$\Rightarrow ka = n\pi \text{ i.e., } k = n\pi/a, \text{ where } n \text{ is any integer.}$$

$$\therefore \text{ From (5), for all integral values of } n, \text{ the solutions of (A) satisfying (B}_1) \text{ and (B}_2) \text{ are given by}$$

$$u_n(x, y) = (E_n e^{n\pi y/a} + F_n e^{-n\pi y/a}) \sin \frac{n\pi x}{a} \quad \dots(6)$$

Where $E_n = C_2 C_3$ and $F_n = C_2 C_4$ are the new arbitrary constants.

Taking $n = 1, 2, 3, \dots$, the more general solution is given by

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} (E_n e^{n\pi y/a} + F_n e^{-n\pi y/a}) \sin \frac{n\pi x}{a} \quad \dots(7)$$

Which also satisfy the boundary conditions (B_1) and (B_2) .

Now the condition $(B_4) \Rightarrow u \rightarrow 0$ when $y \rightarrow \infty$.

$$\therefore \text{ In solution (7), we must take } E_n = 0, \text{ so that this condition may be satisfied, as } e^{n\pi y/a} \rightarrow \infty \text{ as } y \rightarrow \infty.$$

$$\therefore \text{ From (7) the solution of (A) satisfying (B}_1), (B_2) \text{ and (B}_4) \text{ is given by}$$

$$u(x, y) = \sum_{n=1}^{\infty} F_n e^{-n\pi y/a} \sin \frac{n\pi x}{a} \quad \dots(8)$$

Now from (6), the condition (B_3) i.e., $u(x, 0) = f(x)$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} F_n \sin \frac{n\pi x}{a} \quad \dots(9)$$

The right hand side of (9) can be considered as the Fourier sine series of the function $f(x)$ on its left hand side.

$$\therefore F_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx \quad \dots(10)$$

Hence, the required temperature distribution $u(x, y)$ in the plate is given by (8) where F_n is given by (10).

Case II

To find the steady state temperature distribution in a thin rectangular plate bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = h$, assuming that the heat cannot escape from either surface. The edges $x = 0$, $x = a$ and $y = 0$ are kept at temperature zero while the edge $y = h$ is kept at temperature $f(x)$.

The steady state temperature distribution $u(x, y)$ in a thin plate, assuming that the heat cannot escape from either surface is governed by two dimensional Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(A)$$

Here the **boundary conditions** are

$$u(0, y) = 0 \quad \dots(B_1)$$

$$u(a, y) = 0 \quad \dots(B_2)$$

$$u(x, 0) = 0 \quad \dots(B_3)$$

and $u(x, h) = f(x) \quad \dots(B_4)$

Thus, here we are required to find the solution of (A) which satisfy the conditions (B_1) , (B_2) , (B_3) and (B_4) .

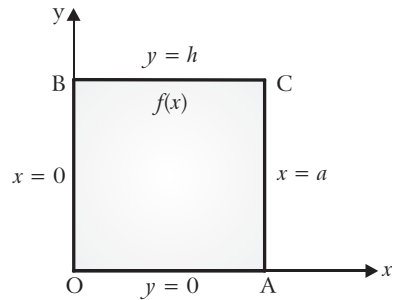


Fig. 14.2

Let the solution of (A) be of the form

$$u(x, y) = F(x) Y(y) = F Y \text{ (say)} \quad \dots(1)$$

Where F is a function of x alone and Y a function of y alone.

Differentiating (1) and putting in (A), we get

$$Y \frac{d^2 F}{dx^2} + F \frac{d^2 Y}{dy^2} = 0 \quad \text{or} \quad \frac{1}{F} \frac{d^2 F}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2} = \lambda \text{ (say)} \quad \dots(2)$$

Since two sides of equation (2) are functions of different independent variables, so they will be equal only if each is equal to the same constant say λ .

Thus, we get the following two ordinary differential equations

$$\frac{d^2 F}{dx^2} - \lambda F = 0 \quad \text{or} \quad \frac{d^2 Y}{dy^2} + \lambda Y = 0 \quad \dots(3)$$

Now three cases arise as follows :

Case (i) : If $\lambda = 0$, then the two equations in (3) reduce to

$$\frac{d^2 F}{dx^2} = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} = 0$$

Whose general solutions are

$$F = A_1 x + A_2 \quad \text{and} \quad Y = A_3 y + A_4$$

$$\therefore u(x, y) = FY = (A_1 x + A_2)(A_3 y + A_4)$$

$\therefore$ the boundary conditions, $u(0, y) = 0$ and $u(a, y) = 0$

$$\Rightarrow A_2(A_3 y + A_4) = 0 \quad \text{and} \quad (A_1 a + A_2)(A_3 y + A_4) = 0$$

$$\Rightarrow A_2 = 0 \quad \text{and} \quad A_1 a + A_2 = 0$$

$\therefore A_3 y + A_4 = 0$, lead to trivial solution $u = 0$ which is inadmissible.

$$\Rightarrow A_1 = 0 \quad \text{and} \quad A_2 = 0$$

$\therefore u(x, y) = 0$, which is trivial solution and is inadmissible.

So we reject the case when $\lambda = 0$

Case (ii) : If $\lambda = k^2 > 0$, then the two equations in (3) reduce to

$$\frac{d^2 F}{dx^2} - k^2 F = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} + k^2 Y = 0$$

Whose general solutions are

$$F = B_1 e^{kx} + B_2 e^{-kx} \quad \text{and} \quad Y = B_3 \cos ky + B_4 \sin ky$$

$$\therefore u(x, y) = FY = (B_1 e^{kx} + B_2 e^{-kx})(B_3 \cos ky + B_4 \sin ky)$$

$\therefore$ The boundary conditions (B_1) and (B_2) i.e., $u(0, y) = 0$ and $u(a, y) = 0$

$$\Rightarrow (B_1 + B_2)(B_3 \cos ky + B_4 \sin ky) = 0$$

$$\text{and } (B_1 e^{ka} + B_2 e^{-ka})(B_3 \cos ky + B_4 \sin ky) = 0$$

$$\Rightarrow B_1 + B_2 = 0 \quad \text{and} \quad B_1 e^{ka} + B_2 e^{-ka} = 0, \quad \therefore B_3 \cos ky + B_4 \sin ky \neq 0$$

$$\Rightarrow B_1 = 0 \quad \text{and} \quad B_2 = 0$$

$\therefore u(x, y) = 0$, which is trivial solution and is inadmissible.

So we also reject the case when $\lambda = k^2 > 0$.

Case (iii) : If $\lambda = -k^2 < 0$, then the two equations reduce to

$$\frac{d^2 F}{dx^2} + k^2 F = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} - k^2 Y = 0$$

Whose general solutions are

$$F = C_1 \cos kx + C_2 \sin kx \quad \text{and} \quad Y = C_3 e^{ky} + C_4 e^{-ky}$$

$$\therefore u(x, y) = FY = (C_1 \cos kx + C_2 \sin kx)(C_3 e^{ky} + C_4 e^{-ky}) \quad \dots(4)$$

Now the boundary condition (B_1) i.e., $u(0, y) = 0 \Rightarrow C_1(C_3 e^{ky} + C_4 e^{-ky}) = 0$

$$\Rightarrow C_1 = 0 \quad \because C_3 e^{ky} + C_4 e^{-ky} \neq 0$$

$$\therefore \text{ from (4), } u(x, y) = C_2 \sin kx (C_3 e^{ky} + C_4 e^{-ky}) \quad \dots(5)$$

$\therefore$ boundary condition (B_2) i.e., $u(a, y) = 0$

$$\Rightarrow C_2 \sin ka (C_3 e^{ky} + C_4 e^{-ky}) = 0$$

$$\Rightarrow \sin ka = 0, \quad \because C_2 = 0 \text{ or } C_3 e^{ky} + C_4 e^{-ky} = 0$$

will lead to trivial solution $u = 0$ which is inadmissible.

$$\Rightarrow ka = n\pi \text{ i.e., } k = n\pi/a, \text{ where } n \text{ is any integer.}$$

$\therefore$ From (5) for all integral values of n , the solutions of (A) satisfying (B_1) and (B_2) are given by

$$u_n(x, y) = (E_n e^{n\pi y/a} + F_n e^{-n\pi y/a}) \sin \frac{n\pi x}{a} \quad \dots(6)$$

Where $E_n = C_2 C_3$ and $F_n = C_2 C_4$ are the new arbitrary constants.

Taking $n = 1, 2, 3, \dots$, the more general solution is given by

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} (E_n e^{n\pi y/a} + F_n e^{-n\pi y/a}) \cdot \sin \frac{n\pi x}{a} \quad \dots(7)$$

Which also satisfy the boundary conditions (B_1) and (B_2).

Now from (7), condition (B_3) i.e., $u(x, 0) = 0$

$$\Rightarrow \sum_{n=1}^{\infty} (E_n + F_n) \sin \frac{n\pi x}{a} = 0$$

$$\Rightarrow E_n + F_n = 0 \quad \because \sin \frac{n\pi x}{a} \neq 0$$

$$\Rightarrow F_n = -E_n$$

Putting in (7), we get

$$u(x, y) = \sum_{n=1}^{\infty} E_n (e^{n\pi y/a} - e^{-n\pi y/a}) \sin \frac{n\pi x}{a}$$

$$\text{or } u(x, y) = \sum_{n=1}^{\infty} H_n \sinh \frac{n\pi y}{a} \cdot \sin \frac{n\pi x}{a} \quad \dots(8)$$

Where $H_n = 2E_n$ is the new arbitrary constant.

Now from (8), condition (B_4) i.e., $u(x, h) = f(x)$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} H_n \sinh \frac{n\pi h}{a} \sin \frac{n\pi x}{a} \quad \dots(9)$$

Here in (9), right hand side can be considered as the Fourier series of the function $f(x)$ on its left hand side.

$$\therefore H_n \sinh \frac{n\pi h}{a} = \frac{2}{h} \int_0^h f(x) \sin \frac{n\pi x}{a} dx$$

$$\text{or } H_n = \frac{2}{h \sinh(n\pi h/a)} \int_0^h f(x) \sin \frac{n\pi x}{a} dx \quad \dots(10)$$

Hence, the required temperature distribution $u(x, y)$ in the plate is given by (9) where H_n is given by (10).

Case III

To find the steady state temperature in a rectangular plate of length a and width b , the sides of which are kept at temperature zero, the lower end is kept at temperature $f(x)$ and the upper edge is kept insulated. [Meerut 2006, 11]

The steady state temperature $u(x, y)$ in a thin plate, assuming that the heat cannot escape from either surface is governed by two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(A)$$

Here the **boundary conditions** are

$$u(0, y) = 0 \quad \dots(B_1)$$

$$u(a, y) = 0 \quad \dots(B_2)$$

$$u(x, 0) = f(x) \quad \dots(B_3)$$

$$u_y(x, h) = 0 \quad \dots(B_4)$$

(Since the edge $y = h$ is insulated)

Thus, here we are required to find the solution of (A) which satisfy the conditions (B_1) , (B_2) , (B_3) and (B_4) .

Let the solution of (A) be of the form

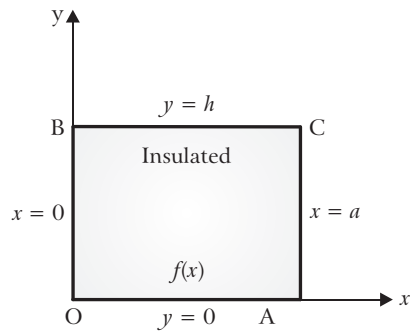


Fig. 14.3

$$u(x, y) = F(x) Y(y) = FY \text{ (say)} \quad \dots(1)$$

Where F is a function of x alone and Y a function of y alone.

Differentiating (1) and substituting in (A), we get

$$Y \frac{d^2 F}{dx^2} + F \frac{d^2 Y}{dy^2} = 0 \quad \text{or} \quad \frac{1}{F} \frac{d^2 F}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2} = \lambda \text{ (say)} \quad \dots(2)$$

Since two sides of equation (2) are functions of different independent variables, so they can be equal only if each is equal to the same constant say λ .

Thus, we get the following two ordinary differential equations :

$$\frac{d^2 F}{dx^2} - \lambda F = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} + \lambda Y = 0 \quad \dots(3)$$

Now three cases arise as follows :

Case (i) : If $\lambda = 0$, then the two equations in (3) reduce to

$$\frac{d^2 F}{dx^2} = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} = 0$$

Whose general solutions are

$$F = A_1 x + A_2 \quad \text{and} \quad Y = A_3 y + A_4$$

$$\therefore u(x, y) = FY = (A_1 x + A_2)(A_3 y + A_4)$$

$\therefore$ Boundary conditions (B_1) and (B_2) i.e., $u(0, y) = 0$ and $u(a, y) = 0$

$$\Rightarrow A_2(A_3 y + A_4) = 0 \quad \text{and} \quad (A_1 a + A_2)(A_3 y + A_4) = 0$$

$$\Rightarrow A_2 = 0 \quad \text{and} \quad A_1 a + A_2 = 0,$$

$\therefore A_3 y + A_4 = 0$ will lead to trivial solution $u(x, y) = 0$, which is inadmissible.

$$\Rightarrow A_2 = 0 \quad \text{and} \quad A_1 = 0$$

$\therefore u(x, y) = 0$, which is trivial solution and is inadmissible.

So we reject the case when $\lambda = 0$.

Case (ii) : If $\lambda = k^2 > 0$, then the two equations in (3), reduce to

$$\frac{d^2 F}{dx^2} - k^2 F = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} + k^2 Y = 0$$

Whose general solutions are

$$F = B_1 e^{kx} + B_2 e^{-kx} \quad \text{and} \quad Y = B_3 \cos k y + B_4 \sin k y$$

$$\therefore u(x, y) = FY = (B_1 e^{kx} + B_2 e^{-kx}) (B_3 \cos ky + B_4 \sin ky)$$

$\therefore$ Boundary conditions (B_1) and (B_2) i.e., $u(0, y) = 0$ and $u(a, y) = 0$

$$\Rightarrow (B_1 + B_2) (B_3 \cos ky + B_4 \sin ky) = 0$$

$$\text{and } (B_1 e^{ka} + B_2 e^{-ka}) (B_3 \cos ky + B_4 \sin ky) = 0$$

$$\Rightarrow B_1 + B_2 = 0 \text{ and } B_1 e^{ka} + B_2 e^{-ka} = 0 \quad \because B_3 \cos ky + B_4 \sin ky \neq 0$$

$$\Rightarrow B_1 = 0 \text{ and } B_2 = 0$$

$\therefore u(x, y) = 0$, which is trivial solution and is inadmissible.

So we reject the case when $\lambda = k^2 > 0$ also.

Case (iii) : If $\lambda = -k^2 < 0$, then the two equations in (3), reduce to

$$\frac{d^2 F}{dx^2} + k^2 F = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} - k^2 Y = 0$$

Whose general solutions are

$$F = C_1 \cos kx + C_2 \sin kx \quad \text{and} \quad Y = C_3 e^{ky} + C_4 e^{-ky}$$

$$\therefore u(x, y) = (C_1 \cos kx + C_2 \sin kx) (C_3 e^{ky} + C_4 e^{-ky}) \quad \dots(4)$$

Now the boundary condition (B_1) i.e., $u(0, y) = 0$

$$\Rightarrow C_1 (C_3 e^{ky} + C_4 e^{-ky}) = 0$$

$$\Rightarrow C_1 = 0 \quad \because C_3 e^{ky} + C_4 e^{-ky} \neq 0$$

$$\therefore \text{from (4), } u(x, y) = C_2 \sin kx (C_3 e^{ky} + C_4 e^{-ky}) \quad \dots(5)$$

$\therefore$ Boundary condition (B_2) i.e., $u(a, y) = 0$

$$\Rightarrow C_2 \sin ka (C_3 e^{ky} + C_4 e^{-ky}) = 0$$

$$\Rightarrow \sin ka = 0 \Rightarrow ka = n\pi \quad \because C_2 = 0 \text{ or } C_3 e^{ky} + C_4 e^{-ky} = 0$$

will lead to trivial solution $u(x, y) = 0$, which is inadmissible.

i.e., $k = n\pi/a$, where n is any integer.

$\therefore$ From (5), for all integral values of n , the solutions of (A) satisfying (B_1) and (B_2) are given by

$$u_n(x, y) = (E_n e^{n\pi y/a} + F_n e^{-n\pi y/a}) \sin \frac{n\pi x}{a} \quad \dots(6)$$

Where $E_n = C_2 C_3$ and $F_n = C_2 C_4$ are the new arbitrary constants.

Taking $n = 1, 2, 3, \dots$, the more general solution is given by

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} (E_n e^{n\pi y/a} + F_n e^{-n\pi y/a}) \sin \frac{n\pi x}{a} \quad \dots(7)$$

Which also satisfy the boundary conditions (B₁) and (B₂).

Now from (7), we have

$$u_y(x, y) = \sum_{n=1}^{\infty} \frac{n\pi}{a} (E_n e^{n\pi y/a} - F_n e^{-n\pi y/a}) \sin \frac{n\pi x}{a}$$

$\therefore$ Condition (B₄) i.e., $u_y(x, h) = 0$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n\pi}{a} (E_n e^{n\pi h/a} - F_n e^{-n\pi h/a}) \sin \frac{n\pi x}{a} = 0$$

$$\Rightarrow E_n e^{n\pi h/a} - F_n e^{-n\pi h/a} = 0 \text{ for all } n = 1, 2, 3, \dots \quad \because \sin \frac{n\pi x}{a} \neq 0$$

$$\Rightarrow F_n = \frac{E_n \cdot e^{n\pi h/a}}{e^{-n\pi h/a}}$$

$$\therefore \text{ from (7), } u(x, y) = \sum_{n=1}^{\infty} E_n [e^{-n\pi(h-y)/a} + e^{n\pi(h-y)/a}] e^{n\pi h/a} \sin \frac{n\pi x}{a}$$

$$\text{or } u(x, y) = \sum_{n=1}^{\infty} H_n \cos h \frac{n\pi(h-y)}{a} \sin \frac{n\pi x}{a} \quad \dots(8)$$

$$\because e^{\theta} + e^{-\theta} = 2 \cosh \theta$$

Where $H_n = 2E_n e^{n\pi h/a}$ is the new arbitrary constant.

From (8), boundary condition (B₃) i.e., $u(x, 0) = f(x)$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} H_n \cos h \frac{n\pi h}{a} \sin \frac{n\pi x}{a} \quad \dots(9)$$

In (9), right hand side can be considered as the Fourier sine series of function $f(x)$ on its left hand side.

$$H_n \cosh \frac{n\pi h}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx$$

or
$$H_n = \frac{2}{a \cosh(n\pi h/a)} \int_0^a f(x) \sin \frac{n\pi x}{a} dx \quad \dots(10)$$

Hence, the required temperature is given by (8) where H_n is given by (10).

Solved Examples

Example 1: A long rectangular plate of width a with insulated surfaces has its temperature u equal to zero on both the long sides and one of the short sides so that $u(0, y) = u(a, y) = u(x, \infty) = 0$, $u(x, 0) = kx$. Show that the steady-state temperature within the plate is given by

$$u(x, y) = \frac{2ak}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n\pi y/a} \sin \frac{n\pi x}{a}.$$

Solution: The steady state temperature $u(x, y)$ within the plate is governed by two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(A)$$

Here the boundary conditions are

$$u(0, y) = 0 \quad \dots(B_1) \quad u(a, y) = 0 \quad \dots(B_2)$$

$$u(x, \infty) = 0 \quad \dots(B_3) \quad \text{and} \quad u(x, 0) = kx = f(x) \text{ (say)} \quad \dots(B_4)$$

Let the solution of (A) be given by

$$u(x, y) = F(x) Y(y) = FY \text{ (say)} \quad \dots(1)$$

Where F is a function of x alone and y function of y alone.

Proceeding similarly as in 14.4 Case I on page 689, we get

$$u(x, y) = \sum_{n=1}^{\infty} F_n e^{-n\pi y/a} \sin \frac{n\pi x}{a} \quad \dots(2)$$

where
$$F_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx$$

$$\begin{aligned}
 \text{or} \quad F_n &= \frac{2}{a} \int_0^a k x \sin \frac{n \pi x}{a} dx \\
 &= \frac{2k}{a} \left[x \left(-\frac{a}{n\pi} \cos \frac{n \pi x}{a} \right) + \left(\frac{a}{n\pi} \right)^2 \sin \frac{n \pi x}{a} \right]_0^a \\
 &= -\frac{2k}{n\pi} a \cos n\pi = \frac{2ak}{n\pi} (-1)^{n+1}
 \end{aligned}$$

Putting in (2), the required steady-state temperature is given by

$$u(x, y) = \frac{2ak}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n\pi y/a} \sin \frac{n \pi x}{a}$$

Example 2: Solve Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ subject to the following boundary conditions : $u = 0$, when y is very large for all x ; $u = 0$ when $x = 0$ and 1 , for all y ; and $u = x(1-x)$ when $y = 0$, for $0 < x < 1$.

Solution: Here we are required to solve the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(A)$$

Subject to the boundary conditions

$$\text{For all } y \quad u(0, y) = 0 \quad \dots(B_1) \quad u(1, y) = 0 \quad \dots(B_2)$$

$$\text{For all } x, \quad u(x, \infty) = 0 \quad \dots(B_3) \quad \text{and} \quad u(x, 0) = x(1-x) = f(x), \text{ say} \quad \dots(B_4)$$

for $0 < x < 1$

Let the required solution be given by

$$u(x, y) = F(x) Y(y) = FY \text{ (say)} \quad \dots(1)$$

Where F is a function of x alone and Y function of y alone.

Proceeding similarly, as in 14.4, Case I on page 689, we get

$$u(x, y) = \sum_{n=1}^{\infty} F_n e^{-n\pi y} \sin n \pi x \quad \dots(2)$$

where $F_n = \frac{2}{1} \int_0^1 f(x) \sin n\pi x \, dx$, Here $a=1$

or
$$F_n = 2 \int_0^1 x(1-x) \sin n\pi x \, dx$$

$$= 2 \left[x(1-x) \left(-\frac{1}{n\pi} \cos n\pi x \right) - (1-2x) \left(-\frac{1}{n^2\pi^2} \sin n\pi x \right) + (-2) \left(\frac{1}{n^3\pi^3} \cos n\pi x \right) \right]_0^1$$

$$= \frac{4}{n^3\pi^3} (1 - \cos n\pi) = \frac{4}{n^3\pi^3} \{1 - (-1)^n\}$$

$$= \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{8}{n^3\pi^3}, & \text{when } n \text{ is odd} \end{cases}$$

Hence from (2), taking $n=2m-1$ (odd)

$$u(x, y) = \frac{8}{\pi^3} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} e^{-(2m-1)\pi y} \sin(2m-1)\pi x$$

or
$$u(x, y) = \frac{8}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} e^{-(2n-1)\pi y} \sin(2n-1)\pi x$$

Note : The above result can also be written as

$$u(x, y) = \frac{8}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} e^{-(2n+1)\pi y} \cdot \sin(2n+1)\pi x.$$

Example 3: A rectangular plate with insulated surfaces is 8 cm wide and so long compared to its width that it may be considered infinite in length. If the temperature along one short edge $y=0$ is given by $u(x, 0) = 100 \sin(\pi x/8)$, $0 < x < 8$, while the two long edges $x=0$ and $x=8$, as well as the other short edge are kept at 0°C , find the steady state temperature $u(x, y)$.

Solution: The steady state temperature $u(x, y)$ in the plate is governed by two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(\text{A})$$

Here the boundary conditions are

$$u(0, y) = 0 \quad \dots(B_1)$$

$$u(8, y) = 0 \quad \dots(B_2)$$

$$u(x, \infty) = 0 \quad \dots(B_3)$$

and $u(x, 0) = 100 \sin(\pi x/8) = f(x)$ say $\dots(B_4)$

$$0 < x < 8$$

Let the solution of (A) be given by

$$u(x, y) = F(x) Y(y) = FY \text{ (say)} \quad \dots(1)$$

Where F is a function of x alone and Y a function of y alone.

Proceeding similarly as in 14.4, Case I on page 689, we get

$$u(x, y) = \sum_{n=1}^{\infty} F_n e^{-n\pi y/8} \sin \frac{n\pi x}{8} \quad \dots(2)$$

where $F_n = \frac{2}{8} \int_0^8 f(x) \sin \frac{n\pi x}{8} dx$, Here $a = 8$ cm

or $F_n = \frac{1}{4} \int_0^8 100 \sin\left(\frac{\pi x}{8}\right) \sin\left(\frac{n\pi x}{8}\right) dx$

$$\begin{aligned} \therefore \text{When } n \neq 1, F_n &= \frac{25}{2} \int_0^8 [\cos\{(n-1)\pi x/8\} - \cos\{(n+1)\pi x/8\}] dx \\ &= \frac{25}{2} \left[\frac{8}{(n-1)\pi} \sin\{(n-1)\pi x/8\} - \frac{8}{(n+1)\pi} \sin\{(n+1)\pi x/8\} \right]_0^8 = 0 \end{aligned}$$

$$\begin{aligned} \text{When } n=1, F_n &= F_1 = 25 \int_0^8 \sin^2(\pi x/8) dx = \frac{25}{2} \{1 - \cos(\pi x/4)\} dx \\ &= \frac{25}{2} \left[x - \frac{4}{\pi} \sin(\pi x/4) \right]_0^8 = \frac{25}{2} (8-0) = 100 \end{aligned}$$

$\therefore$ From (2), we get

$$u(x, y) = F_1 e^{-\pi y/8} \sin(\pi x/8) = 100 \cdot e^{-\pi y/8} \sin(\pi x/8).$$

Example 4: The edge of breadth b of and infinitely long rectangular conducting strip is maintained at constant temperature T_0 , the remaining edges being maintained at zero temperature. Show that the steady temperature distribution in the strip is given by

$$T = \frac{4T_0}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin \{(2n+1)\pi y/b\} e^{-(2n+1)\pi x/b}.$$

Solution: The steady temperature distribution $u(x, y)$ in the strip is governed by two dimensional Laplace equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(A)$$

Here the result to be proved suggests that the edges $y=0$ and $y=b$ are of infinite length.

The **boundary conditions** of the problem are

$$u(0, y) = T_0 \quad \dots(B_1)$$

$$u(\infty, y) = 0 \quad \dots(B_2)$$

$$u(x, 0) = 0 \quad \dots(B_3)$$

and $u(x, b) = 0 \quad \dots(B_4)$

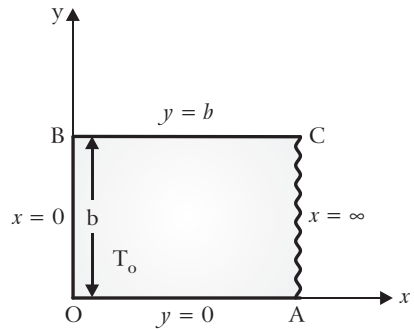


Fig. 14.4

Let $u(x, y) = F(x) Y(y) = FY$ (say) $\dots(1)$

be the solution of (A), where F is a function of x alone and Y function of y alone.

Differentiating (1) and substituting in (A), we get

$$Y \frac{d^2 F}{dx^2} + F \frac{d^2 Y}{dy^2} = 0 \quad \text{or} \quad \frac{1}{F} \frac{d^2 F}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2} = \lambda \text{ (say)} \quad \dots(2)$$

Since the two sides of (2) are functions of different independent variables, so they will be equal only if each is equal to the same constant say λ .

Thus, we get the following two ordinary differential equations

$$\frac{d^2 F}{dx^2} - \lambda F = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} + \lambda Y = 0 \quad \dots(3)$$

Now there are the following three cases :

Case I : If $\lambda = 0$, then the two equations in (3) reduce to

$$\frac{d^2 F}{dx^2} = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} = 0$$

Whose general solutions are

$$F = A_1 x + A_2 \quad \text{and} \quad Y = A_3 y + A_4$$

$$\therefore u(x, y) = FY = (A_1 x + A_2)(A_3 y + A_4)$$

$\therefore$ the boundary conditions (B_3) and (B_4) i.e., $u(x, 0) = 0$ and $u(x, b) = 0$

$$\Rightarrow (A_1 x + A_2) \cdot A_4 = 0 \quad \text{and} \quad (A_1 x + A_2)(A_3 b + A_4) = 0$$

$$\Rightarrow A_4 = 0 \quad \text{and} \quad A_3 b + A_4 = 0 \quad \because A_1 x + A_2 \neq 0$$

$$\Rightarrow A_3 = 0 \quad \text{and} \quad A_4 = 0$$

$$\Rightarrow u(x, y) = 0, \text{ which is trivial solution and is inadmissible.}$$

$\therefore$ We reject the case when $\lambda = 0$.

Case II : If $\lambda = -k^2 < 0$, then the two equations in (3), reduce to

$$\frac{d^2 F}{dx^2} + k^2 F = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} - k^2 Y = 0$$

Whose general solutions are

$$F = B_1 \cos kx + B_2 \sin kx \quad \text{and} \quad Y = B_3 e^{ky} + B_4 e^{-ky}$$

$$\therefore u(x, y) = FY = (B_1 \cos kx + B_2 \sin kx)(B_3 e^{ky} + B_4 e^{-ky})$$

$\therefore$ the boundary conditions (B_3) and (B_4) i.e., $u(x, 0) = 0$ and $u(x, b) = 0$

$$\Rightarrow (B_1 \cos kx + B_2 \sin kx)(B_3 + B_4) = 0$$

$$\text{and} \quad (B_1 \cos kx + B_2 \sin kx)(B_3 e^{kb} + B_4 e^{-kb}) = 0$$

$$\Rightarrow B_3 + B_4 = 0 \quad \text{and} \quad B_3 e^{kb} + B_4 e^{-kb} = 0$$

$$\because B_1 \cos kx + B_2 \sin kx \neq 0$$

$$\Rightarrow B_3 = 0 \quad \text{and} \quad B_4 = 0$$

$$\Rightarrow u(x, y) = 0, \text{ which is trivial solution and is inadmissible.}$$

$\therefore$ We also reject the case when $\lambda = -k^2$.

Case III : If $\lambda = k^2 > 0$, then the two equations in (3), reduce to

$$\frac{d^2 F}{dx^2} = -k^2 F = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} + k^2 Y = 0$$

Whose general solutions are

$$F = C_1 e^{kx} + C_2 e^{-kx} \text{ and } Y = C_3 \cos ky + C_4 \sin ky$$

$$\therefore u = FY = (C_1 e^{kx} + C_2 e^{-kx})(C_3 \cos ky + C_4 \sin ky) \quad \dots(4)$$

$$\therefore \text{ the condition (B}_3) \text{ i.e., } u(x, 0) = 0 \Rightarrow (C_1 e^{kx} + C_2 e^{-kx}) C_3 = 0,$$

$$\Rightarrow C_3 = 0, \because C_1 e^{kx} + C_2 e^{-kx} \neq 0$$

$$\therefore \text{ from (4), } u(x, y) = C_4(C_1 e^{kx} + C_2 e^{-kx}) \sin ky \quad \dots(5)$$

$$\therefore \text{ the condition (B}_4) \text{ i.e., } u(x, b) = 0 \Rightarrow C_4(C_1 e^{kb} + C_2 e^{-kb}) \sin kb = 0$$

$$\Rightarrow \sin kb = 0, \because C_4 \neq 0 \text{ and } C_1 e^{kb} + C_2 e^{-kb} \neq 0$$

$$\Rightarrow kb = n\pi, \text{ } n \text{ is any integer.}$$

$$\text{i.e., } k = n\pi/b$$

$\therefore$ From (5), for all integral values of n , the solutions of (A) satisfying (B₃) and (B₄) are given by

$$u_n(x, y) = (D_n e^{n\pi x/b} + E_n e^{-n\pi x/b}) \sin\left(\frac{n\pi y}{b}\right) \quad \dots(6)$$

Where $D_n = C_4 C_1$ and $E_n = C_4 C_2$ are the new arbitrary constants.

Taking $n = 1, 2, 3, \dots$ the more general solutions of (A) is given by

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} (D_n e^{n\pi x/b} + E_n e^{-n\pi x/b}) \sin\left(\frac{n\pi y}{b}\right) \quad \dots(7)$$

Which also satisfy the conditions (B₃) and (B₄).

Now the condition (B₂) $\Rightarrow u = 0$ when $x \rightarrow \infty$

$$\therefore \text{ in solution (7), we must take } D_n = 0 \because e^{n\pi x/b} \rightarrow \infty \text{ as } x \rightarrow \infty$$

$\therefore$ from (7) the solution of (A) satisfying (B₂), (B₃) and (B₄) is given by

$$u(x, y) = \sum_{n=1}^{\infty} E_n e^{-n\pi x/b} \cdot \sin(n\pi y/b) \quad \dots(8)$$

Now from (8), the condition (B₁) i.e., $u(0, y) = T_0$

$$\Rightarrow T_0 = \sum_{n=1}^{\infty} E_n \sin(n\pi y/b) \quad \dots(9)$$

Obviously, the right hand side in (9) can be considered as the Fourier sine series of function T_0 on the left hand side.

$$\begin{aligned} \therefore E_n &= \frac{2}{b} \int_0^b T_0 \sin(n\pi y/b) dy = \frac{2}{b} T_0 \left[-\frac{b}{n\pi} \cos\left(\frac{n\pi y}{b}\right) \right]_0^b \\ &= \frac{2T_0}{n\pi} (1 - \cos n\pi) = \frac{2T_0}{n\pi} \{1 - (-1)^n\} \\ &= \begin{cases} 0, & \text{when } n \text{ is even} \\ \frac{4T_0}{n\pi}, & \text{when } n \text{ is odd} \end{cases} \end{aligned}$$

$\therefore$ From (8), the required solution is given by

$$u(x, y) = \frac{4T_0}{\pi} \sum_{m=0}^{\infty} \frac{1}{2m+1} e^{-(2m+1)\pi x/b} \sin\left(\frac{(2m+1)\pi y}{b}\right), \quad n = 2m+1 \text{ is odd}$$

$$\text{or } u(x, y) = \frac{4T_0}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin\left\{\frac{(2n+1)\pi y}{b}\right\} e^{-(2n+1)\pi x/b}$$

Note : The above result can also be written as

$$u(x, y) = \frac{4T_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left\{\frac{(2n-1)\pi y}{b}\right\} e^{-(2n-1)\pi x/b}.$$

Example 5: Find the steady state temperature distribution $u(x, y)$ in the uniform square $0 \leq x \leq 1, 0 \leq y \leq 1$ when edge $y = 1$ is kept at temperature u_0 and the remaining three sides are kept at temperature zero.

Solution: The steady state temperature distribution $u(x, y)$ in the square is governed by two dimensional Laplace equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(A)$$

Here the boundary conditions are

$$u(0, y) = 0 \quad \dots(B_1)$$

$$u(1, y) = 0 \quad \dots(B_2)$$

$$u(x, 0) = 0 \quad \dots(B_3)$$

and $u(x, l) = u_0 = f(x)$ say $\dots(B_4)$

Let the solution of (A) be given by

$$u(x, y) = F(x) Y(y) = FY \text{ (say)} \quad \dots(1)$$

Where F is a function of x alone and Y a function of y alone.

Proceeding similarly as in 14.4, case II on page 693 we get

$$u(x, y) = \sum_{n=1}^{\infty} H_n \sinh n\pi y \sin n\pi x \quad \dots(2)$$

where $H_n = \frac{2}{l \sinh n\pi} \int_0^l f(x) \sin n\pi x \, dx$, Here $a = l$, $h = 1$

$$= \frac{2}{\sinh n\pi} \int_0^l u_0 \sin n\pi x \, dx$$

$$= \frac{240}{\sinh n\pi} \left[\frac{\cos n\pi x}{-n\pi} \right]_0^l = \frac{240}{n\pi \sinh n\pi} (1 - \cos n\pi)$$

$H_n = 0$, when n is even

and $H_n = \frac{440}{n\pi \sinh n\pi}$ when n is odd

Hence, from (2), the required solution is

$$u(x, y) = \frac{440}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m-1)} \frac{\sinh (2m-1) \pi y \sin (2m-1) \pi x}{\sinh (2m-1) \pi}$$

Taking $n = 2m-1$ odd.

Example 6: Solve the Laplace's equation. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, which satisfies the conditions,

$u(0, y) = u(l, y) = u(x, 0) = 0$ and $u(x, l) = \sin(n\pi x/l)$.

Solution: Here we are required to find the solution of the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(A)$$

Which satisfies the conditions

$$u(0, y) = 0 \quad \dots(B_1)$$

$$u(l, y) = 0 \quad \dots(B_2)$$

$$u(x, 0) = 0 \quad \dots(B_3)$$

and $u(x, l) = \sin(n\pi x/l) = f(x)$ say $\dots(B_4)$

Let the solution of (A) be given by

$$u(x, y) = F(x) \cdot Y(y) = FY \text{ (say)} \quad \dots(1)$$

Where F is a function of x alone and Y a function of y alone. Proceeding similarly as in 14.4 Case II on page 693, we get

$$u(x, y) = \sum_{m=1}^{\infty} H_m \sinh\left(\frac{m\pi y}{l}\right) \sin\left(\frac{m\pi x}{l}\right) \quad \dots(2)$$

where $H_m = \frac{2}{l \sinh m\pi} \int_0^l f(x) \sin\left(\frac{m\pi x}{l}\right) dx$

Here $a = l = h$, and taking m for n .

or $H_m = \frac{2}{l \sinh m\pi} \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx$

For $m = n$

$$\begin{aligned} \therefore H_m = H_n &= \frac{1}{l \sinh n\pi} \int_0^l \left\{ 1 - \cos\left(\frac{2n\pi x}{l}\right) \right\} dx \\ &= \frac{1}{l \sinh n\pi} \left[x - \frac{l}{2n\pi} \sin\left(\frac{2n\pi x}{l}\right) \right]_0^l = \frac{1}{\sinh n\pi} \end{aligned}$$

and if $m \neq n$, then

$$\begin{aligned} H_m &= \frac{1}{l \sinh m\pi} \int_0^l \left(\cos \frac{(n-m)\pi x}{l} - \cos \frac{(n+m)\pi x}{l} \right) dx \\ &= \frac{1}{l \sinh m\pi} \left[\frac{l}{(n-m)\pi} \sin \frac{(n-m)\pi x}{l} - \frac{l}{(n+m)\pi} \sin \frac{(n+m)\pi x}{l} \right]_0^l = 0 \end{aligned}$$

Hence, from (2), we get

$$u(x, y) = H_n \sinh\left(\frac{n\pi y}{l}\right) \sin\left(\frac{n\pi x}{l}\right) = \frac{\sinh h\left(\frac{n\pi y}{l}\right) \sin\left(\frac{n\pi x}{l}\right)}{\sinh n\pi}$$

Example 7: (A Dirichlet Problem) : Let u be a harmonic function in the interior of a rectangle $0 \leq x \leq a, 0 \leq y \leq b$ in the xy plane, satisfying Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, with $u(0, y) = 0, u(a, y) = 0, u(x, b) = 0$ and $u(x, 0) = f(x)$. Determine u for the above problem.

Solution: Here $u(x, y)$ satisfy the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(A)$$

The given conditions are

$$u(0, y) = 0 \quad \dots(B_1)$$

$$u(a, y) = 0 \quad \dots(B_2)$$

$$u(x, b) = 0 \quad \dots(B_3)$$

and $u(x, 0) = f(x) \quad \dots(B_4)$

Thus, here we are required to find the solution of (A) satisfying the conditions $(B_1), (B_2), (B_3)$ and (B_4) .

Let the solution of (A) be given by

$$u(x, y) = F(x) Y(y) = FY \text{ (say)} \quad \dots(1)$$

Differentiating and substituting in (A), we get

$$Y \frac{d^2 F}{dx^2} + F \frac{d^2 Y}{dy^2} = 0 \quad \text{or} \quad \frac{1}{F} \frac{d^2 F}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2} = \lambda \text{ (say)} \quad \dots(2)$$

Since the two sides of (2) are functions of different independent variable, so the two will be equal to each other if each is equal to the same constant say λ . Thus, we get the following pair of ordinary differential equations.

$$\frac{d^2 F}{dx^2} - \lambda F = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} + \lambda Y = 0 \quad \dots(3)$$

Now there are three cases as follows :

Case I : If $\lambda = 0$, then the two equations in (3), reduce to

$$\frac{d^2 F}{dx^2} = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} = 0$$

Whose general solutions are

$$F = A_1x + A_2 \quad \text{and} \quad Y = A_3y + A_4$$

$$\therefore u(x, y) = (A_1x + A_2)(A_3y + A_4)$$

$$\therefore \text{the conditions } (B_1) \text{ and } (B_2) \text{ i.e., } u(0, y) = 0 \text{ and } u(a, y) = 0$$

$$\Rightarrow A_2(A_3y + A_4) = 0 \quad \text{and} \quad (A_1a + A_2)(A_3y + A_4) = 0$$

$$\Rightarrow A_2 = 0 \quad \text{and} \quad A_1a + A_2 = 0 \quad \because A_3y + A_4 \neq 0$$

$$\Rightarrow A_1 = 0 \quad \text{and} \quad A_2 = 0$$

$$\Rightarrow u(x, y) = 0, \text{ which is trivial solution and is inadmissible.}$$

$$\therefore \text{We reject the case } \lambda = 0.$$

Case II : If $\lambda = k^2 > 0$, then the two equations in (3) reduce to

$$\frac{d^2F}{dx^2} - k^2 F = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} + k^2 Y = 0$$

Whose general solutions are

$$F = B_1 e^{kx} + B_2 e^{-kx} \quad \text{and} \quad Y = B_3 \cos ky + B_4 \sin ky$$

$$\therefore u(x, y) = (B_1 e^{kx} + B_2 e^{-kx})(B_3 \cos ky + B_4 \sin ky)$$

$$\therefore \text{the conditions, } (B_1) \text{ and } (B_2) \text{ i.e., } u(0, y) = 0 \text{ and } u(a, y) = 0$$

$$\Rightarrow (B_1 + B_2)(B_3 \cos ky + B_4 \sin ky) = 0$$

$$\text{and } (B_1 e^{ax} + B_2 e^{-ax})(B_3 \cos ky + B_4 \sin ky) = 0$$

$$\Rightarrow B_1 + B_2 = 0 \quad \text{and} \quad B_1 e^{ax} + B_2 e^{-ax} = 0, \quad \because B_3 \cos ky + B_4 \sin ky \neq 0$$

$$\Rightarrow B_1 = 0 \quad \text{and} \quad B_2 = 0$$

$$\Rightarrow u(x, y) = 0, \text{ which is trivial solution and is inadmissible.}$$

$$\therefore \text{We reject the case } \lambda = k^2 > 0, \text{ also.}$$

Case III : If $\lambda = -k^2 < 0$, then the two equations in (3), reduce to

$$\frac{d^2F}{dx^2} + k^2 F = 0 \quad \text{and} \quad \frac{d^2Y}{dy^2} - k^2 Y = 0$$

Whose general solutions are

$$F = C_1 \cos kx + C_2 \sin kx \text{ and } Y = C_3 e^{ky} + C_4 e^{-ky}$$

$$\therefore u(x, y) = (C_1 \cos kx + C_2 \sin kx)(C_3 e^{ky} + C_4 e^{-ky}) \quad \dots(4)$$

$$\therefore \text{ the conditions (B}_1\text{) i.e., } u(0, y) = 0 \Rightarrow C_1(C_3 e^{ky} + C_4 e^{-ky}) = 0$$

$$\Rightarrow C_1 = 0, \quad \because C_3 e^{ky} + C_4 e^{-ky} \neq 0$$

$$\therefore \text{ from (4) } u(x, y) = C_2 \sin kx (C_3 e^{ky} + C_4 e^{-ky}) \quad \dots(5)$$

$$\therefore \text{ Condition (B}_2\text{) i.e., } u(a, y) = 0 \Rightarrow C_2 \sin ka (C_3 e^{ky} + C_4 e^{-ky}) = 0$$

$$\Rightarrow \sin ka = 0, \quad \because C_2 \neq 0 \text{ and } C_3 e^{ky} + C_4 e^{-ky} \neq 0$$

$$\Rightarrow ka = n\pi, \text{ n is any integer}$$

$$\text{i.e., } k = n\pi/a$$

$\therefore$ From (5), for all integral values of n , the solutions of (A) satisfying (B₂) and (B₃) are given by

$$u_n(x, y) = (D_n e^{n\pi y/a} + E_n e^{-n\pi y/a}) \sin \frac{n\pi x}{a} \quad \dots(6)$$

Where $D_n = C_2 C_3$ and $E_n = C_2 C_4$ are the new arbitrary constants. Taking $n = 1, 2, 3, \dots$, the more general solution of (A) is given by

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x, y) = \sum_{n=1}^{\infty} (D_n e^{n\pi y/a} + E_n e^{-n\pi y/a}) \sin \frac{n\pi x}{a} \quad \dots(7)$$

Which also satisfy the conditions (B₁) and (B₂).

$$\therefore \text{ Condition (B}_3\text{) i.e., } u(x, b) = 0$$

$$\Rightarrow (D_n e^{n\pi b/a} + E_n e^{-n\pi b/a}) \sin \frac{n\pi x}{a} = 0$$

$$\Rightarrow D_n e^{n\pi b/a} + E_n e^{-n\pi b/a} = 0 \quad \because \sin \frac{n\pi x}{a} \neq 0$$

$$\Rightarrow E_n = -\frac{D_n e^{n\pi b/a}}{e^{-n\pi b/a}}$$

Putting in (7), we get

$$u(x, y) = \sum_{n=1}^{\infty} D_n \left(e^{n\pi y/a} - e^{n\pi b/a} \frac{e^{-n\pi y/a}}{e^{-n\pi b/a}} \right) \sin \frac{n\pi x}{a}$$

$$\begin{aligned}
 &= \sum_{n=1}^{\infty} \left[-D_n e^{n\pi b/a} (e^{n\pi(b-y)/a} - e^{-n\pi(b-y)/a}) \cdot \sin \frac{n\pi x}{a} \right] \\
 &= \sum_{n=1}^{\infty} F_n \sinh \frac{n\pi(b-y)}{a} \cdot \sin \frac{n\pi x}{a} \quad \because e^{\theta} - e^{-\theta} = 2 \sinh \theta
 \end{aligned}$$

Where $F_n = -\frac{1}{2} D_n e^{n\pi b/a}$ is the new arbitrary constant.

Now the conditions (B₄) i.e., $u(x,0) = f(x)$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} F_n \sinh \frac{n\pi b}{a} \cdot \sin \frac{n\pi x}{a} \quad \dots(8)$$

In (8), right hand side can be considered as the Fourier sine transform of the function $f(x)$ on its left hand side.

$$\therefore F_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx$$

$$\text{or } F_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \cdot \int_0^a f(x) \sin \frac{n\pi x}{a} dx \quad \dots(9)$$

Hence, the required solution is given by (8) and F_n is given by (9).

Example 8: Find the function $f(x, y)$ which satisfies the Laplace's equation inside the rectangle $0 \leq x \leq \pi$, $0 \leq y \leq a$ subject to the boundary conditions

$$f(0, y) = f(\pi, y) = 0, \quad 0 < y < a$$

$$f(x, a) = 0 \quad \text{for } 0 \leq x \leq \pi \quad \text{and} \quad f(x, 0) = x(\pi - x) \quad \text{for } 0 \leq x \leq \pi.$$

Solution: Here we are required to find the function $f(x, y)$ which satisfies the Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \quad \dots(A)$$

in the rectangle $0 \leq x \leq \pi, 0 \leq y \leq a$

and satisfies the boundary conditions.

$$f(0, y) = 0 \quad \dots(B_1)$$

$$f(\pi, y) = 0 \quad 0 < y < a \quad \dots(B_2)$$

$$f(x, a) = 0 \quad \dots(B_3)$$

and $f(x, 0) = x(\pi - x) = f(x)$ say $\dots(B_4)$

$$0 \leq x \leq \pi$$

Let the solution of (A) be given by

$$f(x, y) = F(x) Y(y) = FY \text{ (say)} \quad \dots(1)$$

Proceeding similarly as in Ex. 7, we get

$$f(x, y) = \sum_{n=1}^{\infty} F_n \sinh n(a-y) \sin nx \quad \dots(2)$$

where $F_n = \frac{2}{\pi \sinh na} \int_0^{\pi} f(x) \sin nx \, dx$ Here $a = \pi$ and $b = a$

$$= \frac{2}{\pi \sinh na} \int_0^{\pi} x(\pi - x) \sin nx \, dx$$

$$= \frac{2}{\pi \sinh na} \left[x(\pi - x) \left(-\frac{1}{n} \cos nx \right) - (\pi - 2x) \left(-\frac{1}{n^2} \sin nx \right) + (-2) \left(\frac{1}{n^3} \cos nx \right) \right]_0^{\pi}$$

$$= \frac{4}{\pi n^3 \sinh na} (1 - \cos n\pi) = \frac{4[1 - (-1)^n]}{\pi n^3 \sinh na}$$

$$= 0 \text{ if } n \text{ is even and } \frac{8}{\pi n^3 \sinh na} \text{ if } n \text{ is odd.}$$

Hence, from (2), the required function $f(x, y)$ is given by

$$f(x, y) = \frac{8}{\pi} \sum_{m=1}^{\infty} \frac{\sinh (2m-1)(a-y)}{(2m-1)^3 \sinh (2m-1)\pi} \sin (2m-1)x$$

Taking $n = 2m - 1$ odd.

Example 9: Find the steady state temperature in a square plate $0 \leq x < \pi, 0 \leq y < \pi$, the sides of which are kept at temperature zero, the lower edge is kept at temperature $f(x) = A \sin x$ and the upper edge is kept insulated.

Solution: The steady state temperature $u(x, y)$ in the plate is governed by two dimensional Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(A)$$

Here the **boundary conditions** are

$$u(0, y) = 0 \quad \dots(B_1)$$

$$u(\pi, y) = 0 \quad \dots(B_2)$$

$$u(x, 0) = f(x) = A \sin \pi x \quad \dots(B_3)$$

$$\text{and } u_y(x, \pi) = 0 \quad \dots(B_4)$$

Let the solution of (A) be of the form

$$u(x, y) = F(x) Y(y) = FY \text{ (say)} \quad \dots(1)$$

Where F is a function of x alone and Y a function of y alone.

Proceeding similarly as in 14.4 Case III, on page 696, we get

$$u(x, y) = \sum_{n=1}^{\infty} H_n \cosh n(\pi - y) \sin nx \quad \dots(2)$$

$$\text{where } H_n = \frac{2}{\pi \cosh n\pi} \int_0^{\pi} f(x) \sin nx \, dx \text{ Here } a = \pi, \, h = \pi$$

$$\text{or } H_n = \frac{2}{\pi \cosh n\pi} \int_0^{\pi} A \sin x \sin nx \, dx$$

$$\begin{aligned} \therefore \text{ When } n \neq 1, \quad H_n &= \frac{2A}{\pi \cosh n\pi} \int_0^{\pi} \sin x \sin nx \, dx \\ &= \frac{A}{\pi \cosh n\pi} \int_0^{\pi} [\cos (n-1)x - \cos (n+1)x] \, dx \\ &= \frac{A}{\pi \cosh n\pi} \left[\frac{\sin (n-1)x}{n-1} - \frac{\sin (n+1)x}{n+1} \right]_0^{\pi} = 0 \end{aligned}$$

$$\begin{aligned} \text{and when } n=1, \quad H_1 &= \frac{2A}{\pi \cosh \pi} \int_0^{\pi} \sin^2 x \, dx = \frac{A}{\pi \cosh \pi} \int_0^{\pi} (1 - \cos 2x) \, dx \\ &= \frac{A}{\pi \cosh \pi} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi} = \frac{A}{\cosh \pi} = A \operatorname{sech} \pi \end{aligned}$$

Hence from (2), the required temperature is given by

$$u(x, y) = H_1 \cosh(\pi - y) \sin x = A \operatorname{sech} \pi \cosh(\pi - y) \sin x$$

Example 10: Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, for $0 < x < \pi$, $0 < y < \pi$

$$u(x, 0) = x^2, \quad u(x, \pi) = 0$$

$$\left[\frac{\partial u}{\partial x} \right]_{(0, y)} = \left[\frac{\partial u}{\partial x} \right]_{(\pi, y)} = 0$$

[Meerut 2010, 12 (O)]

Solution: Given equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$... (A)

Here the boundary conditions are

$$u(x, 0) = x^2, \quad (B_1)$$

$$u(x, \pi) = 0 \quad (B_2)$$

$$u_x(0, y) = 0 \quad (B_3)$$

$$\text{and } u_x(\pi, y) = 0 \quad (B_4)$$

Let the solution of (A) be given by

$$u(x, y) = F(x)Y(y) = FY \text{ (say)} \quad \dots(1)$$

Where F is a function of x alone and Y a function of y alone.

Differentiating (1) and substituting in (A), we get

$$Y \frac{d^2 F}{dx^2} + F \frac{d^2 Y}{dy^2} = 0$$

$$\text{or } \frac{1}{F} \frac{d^2 F}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2} = \lambda \text{ (say)} \quad \dots(2)$$

Since two sides of (2) are functions of different independent variables, so the two can be equal to each other only if each is equal to the same constant say λ .

Thus, from (2), we get the following two ordinary differential equations

$$\frac{d^2 F}{dx^2} - \lambda F = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} + \lambda Y = 0 \quad \dots(3)$$

Now there are the following three cases :

Case I : If $\lambda = 0$, then the equations in (3) reduce to

$$\frac{d^2 F}{dx^2} = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} = 0$$

Whose general solutions are

$$F = A x + B \quad \text{and} \quad Y = C y + D$$

$$\therefore \quad \text{from (1)} \quad u(x, y) = FY = (A x + B)(C y + D) \quad \dots(4)$$

$$\therefore \quad u_x(x, y) = A(C y + D)$$

$$\therefore \quad \text{Condition } (B_3), u_x(0, y) = 0 \Rightarrow 0 = A(C y + D) \Rightarrow A = 0 \quad \because C y + D = 0$$

will lead to trivial solution $u(x, y) = 0$ which is inadmissible.

$$\therefore \quad u(x, y) = B(C y + D)$$

$$\therefore \quad \text{Condition } (B_2), u(x, \pi) = 0 \Rightarrow 0 = B(C \pi + D) \Rightarrow D = -C \pi, \quad \because B \neq 0$$

$$\therefore \quad u(x, y) = B(C y - C \pi) = BC(y - \pi)$$

$\therefore$ from (4) one solution of (A) satisfying conditions (B_2) , (B_3) and (B_4) , can be taken as

$$u(x, y) = A'(y - \pi) \quad \dots(5)$$

Where $A' = BC$ is new arbitrary constant.

Case II : If $\lambda = k^2 > 0$, then the equations in (3) reduce to

$$\frac{d^2 F}{dx^2} - k^2 F = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} + k^2 Y = 0$$

Whose general solutions are

$$F = A_k e^{k x} + B_k e^{-k x} \quad \text{and} \quad Y = C_k \cos k y + D_k \sin k y$$

$$\therefore \quad u(x, y) = FY = (A_k e^{k x} + B_k e^{-k x})(C_k \cos k y + D_k \sin k y) \quad \dots(6)$$

$$\Rightarrow \quad u_x(x, y) = k(A_k e^{k x} - B_k e^{-k x})(C_k \cos k y + D_k \sin k y)$$

$$\therefore \quad \text{Condition } (B_3) \text{ i.e., } u_x(0, y) = 0 \text{ and conditions } (B_4) \text{ i.e., } u_x(\pi, y) = 0$$

$$\Rightarrow \quad k(A_k - B_k)(C_k \cos k y + D_k \sin k y) = 0$$

$$\text{and } k(A_k e^{k \pi} - B_k e^{-k \pi})(C_k \cos k y + D_k \sin k y) = 0$$

$$\Rightarrow \quad A_k - B_k = 0 \quad \text{and} \quad A_k e^{k \pi} - B_k e^{-k \pi} = 0,$$

$$\because \quad k \neq 0 \text{ and } C_k \cos k y + D_k \sin k y \neq 0$$

$$\Rightarrow \quad A_k = 0 \text{ and } B_k = 0$$

∴ from (6) $u(x, y) = 0$, which is trivial solution and is inadmissible.

Hence, we reject the case $\lambda = k^2 > 0$

Case III : If $\lambda = -k^2 < 0$, then the equations in (3), reduce to

$$\frac{d^2 F}{dx^2} + k^2 F = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} - k^2 Y = 0$$

Whose general solutions are

$$F = A_k \cos kx + B_k \sin kx \quad \text{and} \quad Y = C_k \cosh ky + D_k \sinh ky$$

$$\therefore u(x, y) = FY = (A_k \cos kx + B_k \sin kx)(C_k \cosh ky + D_k \sinh ky) \quad \dots(7)$$

$$\text{and} \quad u_x(x, y) = k(-A_k \sin kx + B_k \cos kx)(C_k \cosh ky + D_k \sinh ky) \quad \dots(8)$$

From (7), condition (B_2) , $u(x, \pi) = 0$

$$\Rightarrow 0 = (A_k \cos kx + B_k \sin kx)(C_k \cos k\pi + D_k \sinh k\pi)$$

$$\Rightarrow C_k \cosh k\pi + D_k \sinh k\pi = 0, \quad \because A_k \cos kx + B_k \sin kx \neq 0$$

$$\Rightarrow D_k = -\frac{\cosh k\pi}{\sinh k\pi} C_k$$

from (8) and condition (B_3) $u_x(0, y) = 0 \Rightarrow k B_k (C_k \cosh ky + D_k \sinh ky) = 0$

$$\Rightarrow B_k = 0 \quad \because k \neq 0 \text{ and } C_k \cosh ky + D_k \sinh ky \neq 0$$

Now, putting the values of B_k and D_k in (7), we get

$$u(x, y) = A_k C_k \cos kx \left(\cosh ky - \frac{\cosh k\pi}{\sinh k\pi} \sinh ky \right)$$

$$\text{or} \quad u(x, y) = A_k C_k \cos kx \sinh k(\pi - y) / \sinh k\pi \quad \dots(9)$$

$$\therefore u_x(x, y) = -k A_k C_k \sin kx \sinh k(\pi - y) / \sinh k\pi \quad \dots(10)$$

From (10), condition (B_4) , $u_x(\pi, y) = 0$

$$\Rightarrow -k A_k C_k \sin k\pi \sinh k(\pi - y) / \sinh k\pi = 0$$

$$\Rightarrow \sin k\pi = 0 \quad \because k \neq 0, A_k \neq 0, C_k \neq 0, \sinh k(\pi - y) \neq 0$$

$$\Rightarrow k\pi = n\pi$$

i.e., $k = n$ where n is any integer.

∴ From (10) for different integral values of n , the solutions of (A) satisfying conditions (B_2) , (B_3) and (B_4) are given by

$$u_n(x, y) = A'_n \cos nx \cdot \sinh n(\pi - y) / \sinh n\pi \quad \dots(11)$$

Where $A'_n = A_k C_k$ is the new arbitrary constant.

From (5) and (11), taking $n = 1, 2, 3, \dots$, the more general solution of (A) can be taken as

$$u(x, y) = A'(y - \pi) + \sum_{n=1}^{\infty} u_n(x, y)$$

$$\text{i.e.,} \quad u(x, y) = A'(y - \pi) + \sum_{n=1}^{\infty} A'_n \cos nx \cdot \sinh n(\pi - y) / \sinh n\pi \quad \dots(12)$$

Which also satisfy the conditions (B_2) , (B_3) and (B_4) .

Now from (12), condition (B_1) i.e., $u(x, 0) = x^2$

$$\Rightarrow \quad x^2 = -A' \pi + \sum_{n=1}^{\infty} A'_n \cos nx \quad \dots(13)$$

Now multiplying both sides of (13) by $\cos nx$ and then integrating, both sides from $x = 0$ to $x = \pi$, we get

$$-A' \pi \int_0^{\pi} \cos nx \, dx + A'_n \int_0^{\pi} \cos^2 nx \, dx = \int_0^{\pi} x^2 \cos nx \, dx$$

$$\text{or} \quad 0 + A'_n \int_0^{\pi} \frac{1}{2} (1 - \cos 2nx) \, dx = \left[x^2 \left(\frac{1}{n} \sin nx \right) - 2x \left(-\frac{1}{n^2} \cos nx \right) + 2 \left(-\frac{1}{n^3} \sin nx \right) \right]_0^{\pi}$$

$$\text{or} \quad A'_n \cdot \frac{\pi}{2} = \frac{2\pi}{n^2} \cos n\pi$$

$$\text{or} \quad A'_n = \frac{4}{n^2} \cos n\pi = \frac{4}{n^2} (-1)^n$$

Again integrating both sides of (13) from $x = 0$ to $x = \pi$, we get

$$\int_0^{\pi} x^2 \, dx = -A' \pi \int_0^{\pi} dx + \sum_{n=1}^{\infty} A'_n \int_0^{\pi} \cos nx \, dx$$

$$\text{or} \quad \frac{1}{3} \pi^3 = -A' \pi^2 + 0 \quad \text{or} \quad A' = -\pi/3$$

Putting in (12), the required solution of (A) is given by

$$u(x, y) = \frac{1}{3} \pi (\pi - y) + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx \sinh n(\pi - y)}{n^2 \sinh n\pi}$$

Example 11: Square plate has its faces and its edges $x = 0$ and $x = \pi$ ($0 < y < \pi$) insulated. Its edges $y = 0$ and $y = \pi$ are kept at temperatures zero and $f(x)$ respectively. Prove that the steady state temperature in the plate is given by

$$u(x, y) = \frac{1}{2\pi} a_0 y + \sum_{n=1}^{\infty} a_n \sinh ny \cdot \cos nx \cdot \operatorname{cosech} n\pi$$

where
$$a_n = \frac{2}{a} \int_0^{\pi} f(x) \cos nx \, dx, \quad n = 0, 1, 2, \dots$$

Solution: The steady state temperature distribution in the square plate is governed by the Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(A)$$

Here the boundary conditions are

$$u_x(0, y) = 0 \quad (B_1)$$

$$u_x(\pi, y) = 0 \quad (B_2)$$

$$u(x, 0) = 0 \quad (B_3)$$

and
$$u(x, \pi) = f(x) \quad (B_4)$$

Let the solution of (A) be given by

$$u(x, y) = F(x) Y(y) = FY \text{ (say)} \quad \dots(1)$$

Where F is a function of x alone and Y a function of y alone.

Differentiating (1) and substituting in (A), we get

$$Y \frac{d^2 F}{dx^2} + F \frac{d^2 Y}{dy^2} = 0 \quad \text{or} \quad \frac{1}{F} \frac{d^2 F}{dx^2} = -\frac{1}{Y} \frac{d^2 Y}{dy^2} = \lambda \text{ (say)} \quad \dots(2)$$

Since the two sides of (2) are functions of different independent variables; so the two sides can be equal to each other only if each is equal to the same constant say λ . Thus from (2) we get the following two ordinary differential equations

$$\frac{d^2 F}{dx^2} - \lambda F = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} + \lambda Y = 0 \quad \dots(3)$$

Now there are the following three cases :

Case I : If $\lambda = 0$, then the equations in (3) reduce to

$$\frac{d^2 F}{d x^2} = 0 \quad \text{and} \quad \frac{d^2 Y}{d y^2} = 0$$

Whose general solutions are

$$F = A x + B \quad \text{and} \quad Y = C y + D$$

$$\therefore \quad \text{from (1), } u(x, y) = FY = (Ax + B)(Cy + D) \quad \dots(4)$$

$$\therefore \quad u_x(x, y) = A(Cy + D)$$

$$\therefore \quad \text{Condition (B}_1\text{) i.e., } u_x(0, y) = 0$$

$$\text{and condition (B}_2\text{) i.e., } u_x(\pi, y) = 0$$

$$\Rightarrow A(Cy + D) = 0 \quad \Rightarrow A = 0 \quad \because Cy + D \neq 0$$

$$\text{and condition (B}_3\text{) i.e., } u(x, 0) = 0 \Rightarrow (Ax + B)(D) = 0 \Rightarrow D = 0$$

$$\therefore Ax + B \neq 0$$

$\therefore$ from (4), one solution of (A) satisfying (B₁), (B₂) and (B₃) can be taken as

$$u(x, y) = BCy$$

$$\text{or } u(x, y) = A'y \quad \dots(5)$$

Where $A' = BC$ is the new arbitrary constant.

Case II : If $\lambda = k^2 > 0$, then the equations in (3), reduce to

$$\frac{d^2 F}{d x^2} - k^2 F = 0 \quad \text{and} \quad \frac{d^2 Y}{d y^2} + k^2 Y = 0$$

Whose general solutions are

$$F = A_k e^{kx} + B_k e^{-kx} \quad \text{and} \quad Y = C_k \cos ky + D_k \sin ky$$

$$\therefore \quad u(x, y) = FY = (A_k e^{kx} + B_k e^{-kx})(C_k \cos ky + D_k \sin ky) \quad \dots(6)$$

$$\therefore \quad u_x(x, y) = k(A_k e^{kx} - B_k e^{-kx})(C_k \cos ky + D_k \sin ky)$$

$$\therefore \quad \text{Condition (B}_1\text{), } u_x(0, y) = 0 \text{ and condition (B}_2\text{) } u_x(\pi, y) = 0$$

$$\Rightarrow k(A_k - B_k)(C_k \cos k y + D_k \sin k y) = 0$$

$$\text{and } k(A_k e^{k\pi} - B_k e^{-k\pi})(C_k \cos ky + D_k \sin ky) = 0$$

$$\Rightarrow A_k - B_k = 0 \quad \text{and} \quad A_k e^{k\pi} - B_k e^{-k\pi} = 0$$

$$\therefore C_k \cos ky + D_k \sin ky \neq 0, k \neq 0$$

$$\Rightarrow A_k = 0 \quad \text{and} \quad B_k = 0$$

$\therefore$ From (6) $u(x, y) = 0$, which is trivial solution and is inadmissible.

Hence, we reject the case $\lambda = k^2 > 0$.

Case III : If $\lambda = -k^2 < 0$. then the equations in (3) reduce to

$$\frac{d^2 F}{dx^2} + k^2 F = 0 \quad \text{and} \quad \frac{d^2 Y}{dy^2} - k^2 Y = 0$$

Whose general solutions are

$$F = A_k \cos kx + B_k \sin kx \quad \text{and} \quad Y = C_k \cosh ky + D_k \sinh ky$$

$$\therefore u(x, y) = FY = (A_k \cos kx + B_k \sin kx)(C_k \cosh ky + D_k \sinh ky) \quad \dots(7)$$

$$\therefore u_x(x, y) = k(-A_k \sin kx + B_k \cos kx)(C_k \cosh ky + D_k \sinh ky) \quad \dots(8)$$

From (7) the condition (B_3) , $u(x, 0) = 0$

$$\Rightarrow (A_k \cos kx + B_k \sin kx) \cdot C_k = 0$$

$$\Rightarrow C_k = 0 \quad \therefore A_k \cos kx + B_k \sin kx \neq 0$$

Also from (8) the condition (B_1) , $u_x(0, y) = 0$

$$\Rightarrow k(B_k)(C_k \cosh ky + D_k \sinh ky) = 0$$

$$\Rightarrow B_k = 0, \quad \therefore C_k \cosh ky + D_k \sinh ky \neq 0 \text{ and } k \neq 0$$

Putting $C_k = 0, B_k = 0$ in (7), we get

$$u(x, y) = A_k D_k \cos kx \sinh ky \quad \dots(9)$$

$$\therefore u_x(x, y) = -k A_k D_k \sin kx \sinh ky \quad \dots(10)$$

From (10), condition (B_2) $u_x(\pi, y) = 0 \Rightarrow -k A_k D_k \sin k\pi \sinh ky = 0$

$$\Rightarrow \sin k\pi = 0, \quad \therefore k \neq 0, A_k \neq 0, D_k \neq 0, \sinh ky \neq 0$$

$$\Rightarrow k\pi = n\pi \quad \text{i.e., } k = n, \text{ where } n \text{ is any integer.}$$

$\therefore$ From (9) for different integral values of n , the solutions of (A) satisfying conditions (B_1) , (B_2) and (B_3) are given by

$$u_n(x, y) = A'_n \cos nx \cdot \sinh ny \quad \dots(11)$$

where $A'_n = A_k D_k$ is the new arbitrary constant.

$\therefore$ From (5) and (11), taking $n = 1, 2, 3, \dots$, the more general solution of (A) is given by

$$u_n(x, y) = A'y + \sum_{n=1}^{\infty} u_n(x, y)$$

$$u(x, y) = A'y + \sum_{n=1}^{\infty} A'_n \cos nx \sinh ny \quad \dots(12)$$

which also satisfy the conditions (B_1) , (B_2) and (B_3) .

Now from (12), the condition (B_4) i.e., $u(x, \pi) = f(x)$

$$\Rightarrow f(x) = A'\pi + \sum_{n=1}^{\infty} A'_n \cos nx \sinh n\pi \quad \dots(13)$$

Multiplying both sides of (12) by $\cos nx$ and then integrating, from $x = 0$ to $x = \pi$, we get

$$\begin{aligned} \int_0^{\pi} f(x) \cos nx \, dx &= A'\pi \int_0^{\pi} \cos nx \, dx + A'_n \int_0^{\pi} \cos^2 nx \sinh n\pi \, dx \\ &= A'\pi \left[\frac{1}{n} \sin nx \right]_0^{\pi} + \frac{1}{2} A'_n \int_0^{\pi} (1 - \cos 2nx) \sinh n\pi \, dx \\ &= 0 + \frac{1}{2} A'_n \cdot \pi \sinh n\pi \\ A'_n &= \frac{a_n}{\sinh n\pi}, \quad \text{where } a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \end{aligned} \quad \dots(14)$$

Again integrating both sides of (12) from $x = 0$ to $x = \pi$, we get

$$\int_0^{\pi} f(x) \, dx = A'\pi \int_0^{\pi} dx + \sum_{n=1}^{\infty} A'_n \sinh n\pi \int_0^{\pi} \cos nx \, dx = A'\pi^2 + 0$$

or
$$A' = \frac{1}{\pi^2} \int_0^\pi f(x) dx = \frac{1}{2\pi} \cdot \frac{2}{\pi} \int_0^\pi f(x) dx = \frac{1}{2\pi} a_0$$

or
$$A' = \frac{1}{\pi^2} \int_0^\pi f(x) dx = \frac{1}{2\pi} \cdot \frac{2}{\pi} \int_0^\pi f(x) dx = \frac{1}{2\pi} a_0$$

Hence, from (12) the required solution is

$$u(x, y) = \frac{1}{2\pi} a_0 y + \sum_{n=1}^{\infty} a_n \cos nx \sinh ny / \sinh n\pi$$

where
$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx, \quad n = 0, 1, 2, \dots$$

Exercise 14.1

- Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, given that (i) $u = 0$ when $x = 0$ and $x = c$,
(ii) $u \rightarrow 0$ as $y \rightarrow \infty$ and (iii) $u = U_0$ when $y = l$.
- A long rectangular plate of width π with insulated surfaces has its temperature u equal to zero on both the long sides and one of the short sides, so that $u(0, y) = u(\pi, y) = u(x, \infty) = 0, u(x, 0) = Ax$. Find the steady state temperature within the plate.
- Solve the Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, subject to the conditions $u = 0$ when y is very large for all x ; $u = 0$ when $x = 0$ and π , for all y ; $u = x(\pi - x)$ when $y = 0$ for $0 < x < \pi$.
- An infinite long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π and this end is maintained at temperature u_0 at all points and other edges are at zero temperature. Determine the temperature at any point of the plate in the steady state. [Meerut 2007]
- An infinite long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The short edge $y = 0$ is of length π and the temperature along this edge is given by $u(x, 0) = A \sin 2x, 0 < x < \pi$, while the two long edges $x = 0$ and $x = \pi$ as well as the other short edge are kept at 0°C . Find the steady state temperature $u(x, y)$ in the plate.
- A rectangular plate with insulated surface 8 cm wide and so long compared to its width that it can be considered infinite in the length without introducing an

appreciable error. If the temperature along the short edge $y=0$ is given by $u(x, 0) = 100 \sin(\pi x/8)$ in $0 < x < 8$ while the two long edges $x=0$ and $x=8$ as well as the other short edges are kept at 0°C . Find the steady state temperature function $u(x, y)$.

7. The short edge $x=0$ of unit breadth of an infinite long rectangular plate is maintained at temperature $k y$, the remaining edges being maintained at zero temperature. Find the steady state temperature distribution in the plate.
8. A rectangular plate with insulated surface is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along the short edge $y=0$ is given by

$$\begin{aligned} u(x, 0) &= 20x, & 0 < x \leq 5 \\ &= 20(10 - x), & 5 < x < 10 \end{aligned}$$

while the two long edges $x=0$ and $x=10$ as well as the other short edge is kept at 0°C . Find the steady state temperature at any point (x, y) of the plate.

9. Find the steady state temperature distribution $u(x, y)$ in the uniform square $0 \leq x \leq \pi, 0 \leq y \leq \pi$ whose edge $y=\pi$ is kept at temperature $Ax(\pi-x)$ and the remaining three sides are kept at temperature zero.
10. Find the steady state temperature inside the square $0 \leq x \leq 1, 0 \leq y \leq 1$ subject to the following boundary conditions.

$$u(0, y) = u(1, y) = 0, \quad 0 < y < 1$$

$$u(x, 1) = 0 \text{ for } 0 \leq x \leq 1 \text{ and } u(x, 0) = x(1-x), \text{ for } 0 \leq x \leq 1.$$

11. Find the steady state temperature in a rectangular plate of length a and width b , the sides of which are kept at temperature zero, the lower edge is kept at temperature T_0 and the upper edge is kept insulated.
12. Obtain the steady state temperature distribution in a rectangular metal plate of length a and width b , the sides of which are kept at temperature 0°C , the lower edge is kept at 100°C and the upper edge kept insulated.
13. Find the steady state temperature in a rectangular plate of length a and width b , the sides of which are kept at temperature zero, the lower edge is insulated and the upper edge is kept at temperature $f(x)$.

Answers 14.1

1.	$u(x, y) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} e^{(2n-1)(1-y)\pi/c} \cdot \sin\left(\frac{(2n-1)\pi x}{c}\right)$
2.	$u(x, y) = 2A \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{-ny} \cdot \sin nx.$
3.	$u(x, y) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} e^{-(2n-1)y} \cdot \sin(2n-1)x.$
4.	$u(x, y) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} e^{-(2n-1)y} \cdot \sin(2n-1)x.$
5.	$u(x, y) = A \cdot e^{-2y} \sin 2x.$
6.	$u(x, y) = 100 e^{-\pi y/8} \sin(\pi x/8).$
7.	$u(x, y) = \frac{2A}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} e^{-n\pi x} \sin n\pi y.$
8.	$u(x, y) = \frac{800}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} e^{-(2n-1)\pi y/10} \sin \frac{(2n-1)\pi x}{10}$
9.	$u(x, y) = \frac{8A}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \operatorname{cosech}\{(2n-1)\pi\} \cdot \sinh\{(2n-1)y\} \cdot \sin\{(2n-1)x\}.$
10.	$u(x, y) = \frac{8}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \operatorname{cosech}(2n-1)\pi \cdot \sinh\{(2n-1)\pi(1-y)\} \sin\{(2n-1)\pi x\}.$
11.	$u(x, y) = \frac{4T_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \operatorname{sech}\left\{\frac{(2n-1)\pi b}{a}\right\} \cosh\left\{\frac{(2n-1)\pi(b-y)}{a}\right\} \sin\left\{\frac{(2n-1)\pi x}{a}\right\}$
12.	$u(x, y) = \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \operatorname{sech}\left\{\frac{(2n-1)\pi b}{a}\right\} \cosh\left\{\frac{(2n-1)\pi(b-y)}{a}\right\} \sin\left\{\frac{(2n-1)\pi x}{a}\right\}$
13.	$u(x, y) = \sum_{n=1}^{\infty} H_n \cosh\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$ where $H_n = \frac{2}{a \cosh(n\pi b/a)} \int_0^a f(x) \sin \frac{n\pi x}{a} dx$

14.5 Laplace Equation in Plane Polar Coordinates

[Meerut 2008 (B.P.)]

Laplace equation in two dimensions is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \dots(1)$$

Proceeding as in {13.8 on page 632 in plane polar coordinates (r, θ) , we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.$$

Hence, the Laplace equation (1) in plane polar coordinates (r, θ) is transformed to

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

14.6 Solution of Laplace Equation in Plane Polar Coordinates by Separation of Variables

The Laplace equation in plane polar coordinates is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \dots(A)$$

Let the solution of (A) which is a function of r and θ be given by

$$u(r, \theta) = R(r) F(\theta) = RF \text{ (say)} \quad \dots(1)$$

Where R is a function of r alone and F a function of θ alone.

Differentiating (1) and substituting in (A), we get

$$F \frac{d^2 R}{dr^2} + \frac{F}{r} \frac{dR}{dr} + \frac{R}{r^2} \frac{d^2 F}{d\theta^2} = 0$$

$$\text{or} \quad \frac{1}{R} \left(r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} \right) = -\frac{1}{F} \frac{d^2 F}{d\theta^2} = \lambda \text{ (say)} \quad \dots(2)$$

In (2) the two sides are functions of different independent variables, so the two will be equal to each other if each is equal to the same constant, say λ .

Thus, from (2), we get the following two ordinary differential equations

$$\frac{d^2 F}{d\theta^2} + \lambda F = 0 \quad \dots(3)$$

$$\text{and} \quad r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} - \lambda R = 0 \quad \dots(4)$$

The equation (4) is a homogeneous differential equation with variable coefficients.

Now let $r = e^z$ so that $\frac{dr}{dz} = e^z = r$

$$\therefore \quad \frac{dR}{dr} = \frac{dR}{dz} \cdot \frac{dz}{dr} = \frac{dR}{dz} \frac{1}{r} \quad \text{or} \quad r \frac{dR}{dr} = \frac{dR}{dz} \quad \therefore \quad r \frac{d}{dr} \equiv \frac{d}{dz} \equiv D_1 \text{ (say)}$$

$$\text{i.e.,} \quad r \frac{dR}{dr} = \frac{dR}{dz} = D_1 R$$

$$\text{and} \quad r \frac{d}{dr} \left(r \frac{dR}{dr} \right) = r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} \quad \text{or} \quad r^2 \frac{d^2 R}{dr^2} = r \frac{d}{dr} \left(r \frac{dR}{dr} \right) - r \frac{dR}{dr}$$

$$\text{or} \quad r^2 \frac{d^2 R}{dr^2} = \left(r \frac{d}{dr} - 1 \right) \left(r \frac{dR}{dr} \right) = (D_1 - 1) D_1 R$$

Putting in (4), we get

$$(D_1 - 1) D_1 R + D_1 R - \lambda R = 0 \quad \text{or} \quad (D_1^2 - \lambda) R = 0 \quad \dots(5)$$

Now there are the following three cases :

Case I : If $\lambda = 0$, then the equations (3) and (5), reduce to

$$\frac{d^2 F}{d\theta^2} = 0 \quad \text{and} \quad D_1^2 R = 0 \quad \text{i.e.,} \quad \frac{d^2 R}{dz^2} = 0$$

Whose general solutions are

$$F = A_1 \theta + A_2 \quad \text{and} \quad R = A_3 z + A_4 = A_3 \log r + A_4.$$

$$\therefore \quad u(r, \theta) = FR = (A_1 \theta + A_2)(A_3 \log r + A_4) \quad \dots(6)$$

Case II : If $\lambda = k^2 > 0$, then the equations (3) and (5), reduce to

$$\frac{d^2 F}{d\theta^2} + k^2 F = 0 \quad \text{and} \quad (D_1^2 - k^2) R = 0$$

Whose general solutions are

$$F = B_1 \cos k\theta + B_2 \sin k\theta$$

$$\text{and} \quad R = B_3 e^{kz} + B_4 e^{-kz} = B_3 r^k + B_4 r^{-k}$$

$$\therefore \quad u(r, \theta) = FR = (B_1 \cos k\theta + B_2 \sin k\theta)(B_3 r^k + B_4 r^{-k}) \quad \dots(7)$$

Case III : If $\lambda = -k^2 < 0$, then the equations (3) and (5), reduce to

$$\frac{d^2 F}{d\theta^2} - k^2 F = 0 \quad \text{and} \quad (D_1^2 + k^2) R = 0$$

Whose general solutions are

$$F = C_1 e^{k\theta} + C_2 e^{-k\theta}$$

$$\text{and} \quad R = C_3 \cos kz + C_4 \sin kz = C_3 \cos(k \log r) + C_4 \sin(k \log r)$$

$$\therefore \quad u(r, \theta) = FR = (C_1 e^{k\theta} + C_2 e^{-k\theta})[C_3 \cos(k \log r) + C_4 \sin(k \log r)] \quad \dots(8)$$

Thus, the solutions of the given equation (A) are given by (6), (7) and (8) for different values of λ . Out of these solutions the most suitable solution of (A) is chosen which depends upon the physical nature of the problem under consideration and the given boundary conditions.

Solved Examples

Example 1: A thin semi-circular plate of radius a which is insulated on both the sides has its boundary diameter kept at 0°C and its temperature along the semi-circular boundary is $f(\theta)$. Find the temperature distribution in the plate in the steady state. [Meerut 2011 (B.P.)]

Solution: In steady state, the temperature distribution in the plate is governed by the Laplace equation (in plane polar coordinates)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \dots(A)$$

Here it is given that the temperature $u(r, \theta)$ is zero along the bounding diameter AOB i.e., $u(r, \theta)$ is zero along $OA(\theta = 0)$ and $OB(\theta = \pi)$.

Also temperature $u(r, \theta)$ along the semi-circular boundary $ACB(r = a)$ is $f(\theta)$.

Thus, the boundary conditions of the problem are

$$u(r, 0) = 0 \quad 0 \leq r \leq a, \quad \dots(B_1)$$

$$u(r, \pi) = 0 \quad 0 \leq r \leq a, \quad \dots(B_2)$$

$$\text{and} \quad u(a, \theta) = f(\theta) \quad 0 < \theta < \pi \quad \dots(B_3)$$

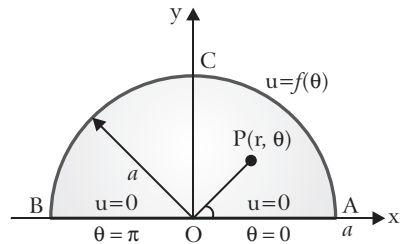


Fig. 14.5

Let the solution of equation (A) be given by

$$u(r, \theta) = R(r) F(\theta) = RF \text{ (say)} \quad \dots(1)$$

where $R(r)$ is a function of r alone and $F(\theta)$ a function of θ alone.

Differentiating (1) and substituting in (A), we get

$$F \frac{d^2 R}{dr^2} + \frac{F}{r} \frac{dR}{dr} + \frac{R}{r^2} \frac{d^2 F}{d\theta^2} = 0$$

$$\text{or} \quad \frac{1}{R} \left(r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} \right) = -\frac{1}{F} \frac{d^2 F}{d\theta^2} = \lambda \text{ (say)} \quad \dots(2)$$

In (2), two sides are functions of different artificial variables, so they will be equal to each other if each is equal to the same constant say λ .

Thus, from (2) we get the following two ordinary differential equations

$$\frac{d^2 F}{d\theta^2} + \lambda F = 0 \quad \dots(3)$$

$$\text{and} \quad r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} - \lambda R = 0 \quad \dots(4)$$

(4) is a homogeneous differential equation.

Let $r = e^z$, so that $r \frac{dR}{dr} = D_1 R$, $r^2 \frac{d^2 R}{dr^2} = (D_1 - 1) D_1 R$, where $D_1 \equiv r \frac{d}{dr} \equiv \frac{d}{dz}$

Putting in (4), it reduces to $(D_1 - 1) D_1 R + D_1 R - \lambda R = 0$

$$\text{or} \quad (D_1^2 - \lambda) R = 0 \quad \dots(5)$$

Now there are three cases as follows :

Case I : If $\lambda = 0$, then the equations in (3) and (5) reduce to

$$\frac{d^2 F}{d\theta^2} = 0 \quad \text{and} \quad D_1^2 R = \frac{d^2 R}{dz^2} = 0$$

Whose general solutions are

$$F = A_1 \theta + A_2 \quad \text{and} \quad R = A_3 z + A_4 = A_3 \log r + A_4$$

$$\therefore u(r, \theta) = FR = (A_1 \theta + A_2)(A_3 \log r + A_4)$$

$\therefore$ The boundary conditions (B_1) and (B_2) i.e., $u(r, 0) = 0$ and $u(r, \pi) = 0$

$$\Rightarrow A_2(A_3 \log r + A_4) = 0 \quad \text{and} \quad (A_1 \pi + A_2)(A_3 \log r + A_4) = 0$$

$$\Rightarrow A_2 = 0 \quad \text{and} \quad A_1 \pi + A_2 = 0 \quad \because A_3 \log r + A_4 \neq 0$$

$$\Rightarrow A_1 = 0 \quad \text{and} \quad A_2 = 0$$

$\therefore u(r, \theta) = 0$, which is trivial solution and is inadmissible.

So we reject the case when $\lambda = 0$

Case II : If $\lambda = -k^2 < 0$, then the equations in (3) and (5) reduce to

$$\frac{d^2 F}{d\theta^2} - k^2 F = 0 \quad \text{and} \quad (D_1^2 + k^2) R = 0$$

Whose general solutions are $F = B_1 e^{k\theta} + B_2 e^{-k\theta}$

$$\text{and} \quad R = B_3 \cos k z + B_4 \sin k z = B_3 \cos(k \log r) + B_4 \sin(k \log r)$$

$$\therefore u = FR = (B_1 e^{k\theta} + B_2 e^{-k\theta}) [B_3 \cos(k \log r) + B_4 \sin(k \log r)] \quad \dots(6)$$

Since on physical grounds of the problem, we must have $u(r, \theta) = u(r, \theta + 2\pi)$, which is possible if $u(r, \theta)$ involve trigonometric functions of θ .

Since here $u(r, \theta)$ in (6) does not contain trigonometrical functions of θ , so we reject the case when $\lambda = -k^2 < 0$.

Case III : If $\lambda = k^2 > 0$, then from (3) and (5) we get the following two ordinary differential equations.

$$\frac{d^2 F}{d\theta^2} + k^2 F = 0 \quad \text{and} \quad \frac{d^2 R}{dz^2} - k^2 R = 0$$

Whose general solutions are

$$F = A \cos k\theta + B \sin k\theta \quad \text{and} \quad R = C e^{kz} + D e^{-kz} = C r^k + D r^{-k}$$

$$\therefore u(r, \theta) = FR = (A \cos k\theta + B \sin k\theta) (C r^k + D r^{-k}) \quad \dots(7)$$

Now from (7), the boundary condition (B_1) i.e., $u(r, 0) = 0$

$$\Rightarrow A(C r^k + D r^{-k}) = 0 \Rightarrow A = 0, \quad \because C r^k + D r^{-k} \neq 0$$

$$\therefore \text{from (7), we get} \quad u(r, \theta) = B \sin k\theta (C r^k + D r^{-k}) \quad \dots(8)$$

Since when $r \rightarrow 0$, $u(r, \theta) \rightarrow 0$, but in (8) when $r \rightarrow 0$, $r^{-k} \rightarrow \infty$ as $k > 0$, so in (8), we must take $D = 0$. Taking $D = 0$ in (8), we get

$$u(r, \theta) = (B \sin k\theta) (C r^k) \quad \dots(9)$$

∴ from (9) and the boundary condition (B_2) i.e., $u(r, \pi) = 0$

$$\Rightarrow (B \sin k\pi)(C r^k) = 0$$

$$\Rightarrow \sin k\pi = 0, \quad \therefore B \neq 0 \text{ and } C r^k \neq 0$$

$$\Rightarrow k\pi = n\pi \quad \text{i.e., } k = n, \text{ where } n \text{ is any integer.}$$

∴ From (9), for integral values of n , the solutions of (A) satisfying the conditions (B_1) and (B_2) are given by

$$u_n(r, \theta) = (E_n r^n) \sin n\theta.$$

Where $E_n = BC$ is the new arbitrary constant.

Taking $n = 1, 2, 3, \dots$, more general solution is given by

$$u(r, \theta) = \sum_{n=1}^{\infty} u_n(r, \theta) = \sum_{n=1}^{\infty} E_n r^n \sin n\theta \quad \dots(10)$$

Which also satisfy conditions (B_1) and (B_2).

Now from (10), the boundary condition (B_3) i.e., $u(a, \theta) = f(\theta)$

$$\Rightarrow f(\theta) = \sum_{n=1}^{\infty} E_n a^n \sin n\theta \quad \dots(11)$$

In (11), right hand side can be considered as the Fourier sine series of function $f(\theta)$ on its left hand side

$$\therefore a^n E_n = \frac{2}{\pi} \int_0^{\pi} f(\theta) \sin n\theta d\theta$$

$$\text{or } E_n = \frac{2}{\pi a^n} \int_0^{\pi} f(\theta) \sin n\theta d\theta \quad \dots(12)$$

Hence, the required temperature distribution, in steady state, in the plate is given by (10), where E_n is given by (12).

Example 2: The boundary diameter of a semi-circular plate of radius 10 cm is kept at 0°C and its temperature along the semi-circular boundary is given by

$$u(10, \theta) = f(\theta) = \begin{cases} 50\theta, & \text{for } 0 \leq \theta \leq \pi/2 \\ 50(\pi - \theta), & \text{for } \pi/2 \leq \theta \leq \pi \end{cases}$$

Find the steady state temperature $u(r, \theta)$ in the plate.

Solution: The steady state temperature in the semi-circular plate is governed by the Laplace equation (in plane polar coordinates)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \dots(A)$$

(Refer to fig. 14.5 in Ex. 1)

As in Ex. 1, the boundary conditions of the problem are

$$u(r, 0) = 0 \quad 0 \leq r \leq 10 \quad \dots(B_1)$$

$$u(r, \pi) = 0 \quad 0 \leq r \leq 10 \quad \dots(B_2)$$

and $u(10, \theta) = f(\theta) \quad 0 < \theta < \pi \quad \dots(B_3)$

Let the solution of equation (A) be given by

$$u(r, \theta) = R(r) F(\theta) = RF \text{ (say)} \quad \dots(1)$$

Where R is a function of r alone and F a function of θ alone.

Now proceeding similarly as in Ex. 1, the required solution is given by

$$u(r, \theta) = \sum_{n=1}^{\infty} E_n r^n \sin n\theta \quad \dots(2)$$

where $E_n = \frac{2}{\pi(10^n)} \int_0^{\pi} f(\theta) \sin n\theta d\theta$

or
$$E_n = \frac{2}{\pi \cdot 10^n} \left[\int_0^{\pi/2} 50\theta \sin n\theta + \int_{\pi/2}^{\pi} 50(\pi - \theta) \sin n\theta d\theta \right]$$

$$= \frac{2}{\pi \cdot 10^n} \left[\left\{ 50\theta \left(-\frac{1}{n} \cos n\theta \right) - 50 \left(-\frac{1}{n^2} \sin n\theta \right) \right\}_0^{\pi/2} \right. \\ \left. + \left\{ 50(\pi - \theta) \left(-\frac{1}{n} \cos n\theta \right) - (-50) \left(-\frac{1}{n^2} \sin n\theta \right) \right\}_{\pi/2}^{\pi} \right]$$

$$= \frac{100}{\pi \cdot 10^n} \left[-\frac{\pi}{2n} \cos \frac{n\pi}{2} + \frac{1}{n^2} \sin \frac{n\pi}{2} + \frac{\pi}{2n} \cos \frac{n\pi}{2} + \frac{1}{n^2} \sin \frac{n\pi}{2} \right]$$

$$= \frac{200}{\pi n^2 \cdot 10^n} \sin \frac{n\pi}{2}$$

$$= \begin{cases} 0, & \text{if } n = 2m \text{ (even), } m = 1, 2, \dots \\ \frac{200(-1)^{m-1}}{\pi(2m-1)^2 \cdot 10^{(2m-1)}} & \text{if } n = 2m-1 \text{ (odd), } m = 1, 2, \dots \end{cases}$$

$$\begin{aligned}\therefore \quad \text{When } n = 2m - 1, \sin \frac{n\pi}{2} &= \sin (2m - 1) \frac{\pi}{2} = \sin (m\pi - \pi/2) \\ &= \sin m\pi \cos \frac{\pi}{2} - \cos m\pi \sin \frac{\pi}{2} = -(1)^m = (-1)^{m+1} = (-1)^{m-1}\end{aligned}$$

Hence, putting in (2) the required solution is

$$u(r, \theta) = \frac{200}{\pi} \sum_{m=1}^{\infty} \left(\frac{r}{10}\right)^{2m-1} (-1)^{m-1} \cdot \frac{\sin (2m-1)\theta}{(2m-1)^2}$$

Example 3: Solve the differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad (\text{Laplace equation in plane polar coordinates}) \text{ in the region}$$

$0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$ and satisfying the boundary conditions.

(i) u remains finite as $r \rightarrow 0$ and

(ii) $u = \sum_n C_n \cos n\theta$, when $r = a$.

Solution: The given differential equation is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \dots(A)$$

Let the solution of (A) be given by

$$u(r, \theta) = R(r) F(\theta) = RF \text{ (say)} \quad \dots(1)$$

Where R is a function of r alone and F a function of θ alone.

Differentiating (1) and substituting in (A), we get

$$F \frac{d^2 R}{dr^2} + \frac{F}{r} \frac{dR}{dr} + \frac{R}{r^2} \frac{d^2 F}{d\theta^2} = 0$$

or

$$\frac{1}{R} \left(r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} \right) = -\frac{1}{F} \frac{d^2 F}{d\theta^2} \quad \dots(2)$$

The two sides of (1) are functions of different independent variables, so they will be equal to each other if each is equal to the same constant. Since there is trigonometric function $\cos n\theta$ in the given conditions, so the solution must involve trigonometric function of θ . Hence, we take each side of (2) equal to the constant $n^2 > 0$ where n is an integer.

$\therefore$ From (2), we get the following two ordinary differential equations

$$\frac{d^2 F}{d\theta^2} + n^2 F = 0 \quad \dots(3)$$

and
$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} - n^2 R = 0 \quad \dots(4)$$

Equation (4) is homogeneous equation, so we substitute $r = e^z$ so that

$$\frac{dR}{dr} = D_1 z \text{ and } r^2 \frac{d^2 R}{dr^2} = (D_1 - 1)D_1 z, \text{ where } D_1 \equiv r \frac{dR}{dr} \equiv \frac{d}{dz}$$

$\therefore$ (4) reduces to

$$\{(D_1 - 1)D_1 + D_1 - n^2\} R = 0 \quad \text{ro} \quad (D_1^2 - n^2) R = 0 \quad \dots(5)$$

The general solutions of (3) and (5) are

$$F = A_n \cos n\theta + B_n \sin n\theta \quad \text{and} \quad R = C_n e^{nz} + D_n e^{-nz} = C_n r^n + D_n r^{-n} \quad \dots(6)$$

$\therefore$ from (1) the solutions of (A) for integral values of n are given by

$$u_n(r, \theta) = FR = (A_n \cos n\theta + B_n \sin n\theta)(C_n r^n + D_n r^{-n}) \quad \dots(7)$$

Taking $n = 1, 2, 3, \dots$, the more general solution of (A) is given by

$$u(r, \theta) = \sum_{n=1}^{\infty} u_n(r, \theta) = \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)(C_n r^n + D_n r^{-n}) \quad \dots(8)$$

According to given condition (i) when $r \rightarrow 0$, u is finite.

But in (7), when $r \rightarrow 0$, $r^{-n} \rightarrow \infty$. $\therefore$ We must take $D_n = 0$

$$\therefore u(r, \theta) = \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta)(C_n r^n)$$

or
$$u(r, \theta) = \sum_{n=1}^{\infty} (E_n \cos n\theta + F_n \sin n\theta) r^n \quad \dots(9)$$

where $E_n = A_n C_n$ and $F_n = B_n C_n$ are the new arbitrary constants.

According to given condition (ii), $u = \sum C_n \cos n\theta$ when $r = a$

i.e.,
$$u(a, \theta) = \sum C_n \cos n\theta$$

$\therefore$ from (9), we get $\sum C_n \cos n\theta = \sum (E_n \cos n\theta + F_n \sin n\theta) a^n$

$\therefore F_n = 0$ and $E_n \cdot a^n = C_n$ i.e., $E_n = C_n / a^n$

Putting in (9) the required solution is

$$u(r, \theta) = \sum_{n=1}^n C_n (r/a)^n \cos n\theta$$

Example 4: Find the steady state temperature in a circular plate of radius a whose circular edge $r = a$ is kept at temperature $f(\theta)$. The plate is insulated so that there is no loss of heat from either surface.

Solution: The steady state temperature in a circular plate with insulated surface is governed by Laplace equation (in plane polar coordinates)

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \dots(A)$$

Here we are required to solve (A) subject to the following boundary condition

$$u(a, \theta) = f(\theta), \quad 0 \leq \theta \leq 2\pi \quad \dots(B)$$

Obviously $u(r, \theta)$ the solution of (A) must be periodic in θ and finite when $r \rightarrow 0$.

Let the solution of (A) be given by

$$u(r, \theta) = R(r) F(\theta) = RF \text{ (say)} \quad \dots(1)$$

Differentiating (1) and substituting in (A), we get

$$F r^2 \frac{d^2 R}{dr^2} + F r \frac{dR}{dr} + R \frac{d^2 F}{d\theta^2} = 0$$

$$\text{or} \quad \frac{1}{R} \left(r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} \right) = -\frac{1}{F} \frac{d^2 F}{d\theta^2} = \lambda \text{ (say)} \quad \dots(2)$$

Two sides of (2) are functions of different artificial variables, so the two sides can be equal to each other if each is equal to the same constant λ (say).

Thus, (2), reduce to the following two ordinary differential equations.

$$\frac{d^2 F}{d\theta^2} + \lambda F = 0 \quad \dots(3)$$

$$\text{and} \quad r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} - \lambda R = 0 \quad \dots(4)$$

Equation (4) is a homogeneous equation with variable coefficients. To solve (4), let $r = e^z$, so that $r \frac{dR}{dr} = D_1 z$ and $r^2 \frac{d^2 R}{dr^2} = (D_1 - 1) D_1 z$, where $D_1 \equiv r \frac{d}{dr} \equiv \frac{d}{dz}$

Putting in (4), it reduces to

$$[(D_1 - 1)D_1 + D_1 - \lambda] R = 0 \quad \text{or} \quad (D_1^2 - \lambda)R = 0 \quad \dots(5)$$

Now there are following cases :

Case I : If $\lambda = 0$, then equation (3) and (5) reduce to

$$\frac{d^2 F}{d\theta^2} = 0 \quad \text{and} \quad D_1^2 R = \frac{d^2 R}{dz^2} = 0$$

Whose general solutions are

$$F = A_1 \theta + A_2 \quad \text{and} \quad R = A_3 z + A_4 = A_3 \log r + A_4$$

$$\therefore \quad u(r, \theta) = FR = (A_1 \theta + A_2)(A_3 \log r + A_4)$$

Since u must be periodic and finite when $r \rightarrow 0$

$$\therefore \quad \text{we take } A_1 = 0 \text{ and } A_3 = 0, \quad \because \log r \rightarrow \infty \text{ as } r \rightarrow 0$$

$$\therefore \quad u(r, \theta) = A_2 A_4 = A \quad (\text{constant}) \quad \dots(6)$$

Case II : If $\lambda = k^2 > 0$, then equations (3) and (5) reduce to

$$\frac{d^2 F}{d\theta^2} + k^2 F = 0 \quad \text{and} \quad (D_1^2 - k^2) R = 0$$

Whose general solutions are

$$F = A_k \cos k\theta + B_k \sin k\theta \quad \text{and} \quad R = C_k e^{kz} + D_k e^{-kz} = C_k r^k + D_k r^{-k}$$

$$\therefore \quad u(r, \theta) = FR = (A_k \cos k\theta + B_k \sin k\theta)(C_k r^k + D_k r^{-k}) \quad \dots(7)$$

$$\because \quad u(r, \theta) \text{ is periodic with period } 2\pi, \text{ so we must take } k = n$$

where $n = 1, 2, 3, \dots$, (an integer)

(Note that we cannot take $\lambda = -k^2 < 0$, since in this case $u(r, \theta)$ will not involve trigonometric function, of θ while $u(r, \theta)$ must involve trigonometric function of θ as it is periodic function of θ of period 2π .)

$\therefore$ For $n = 1, 2, 3, \dots$, (an integer), from (7) the solutions of (A) are given by

$$u_n(r, \theta) = (A_n \cos n\theta + B_n \sin n\theta)(C_n r^n + D_n r^{-n}) \quad \dots(8)$$

From (6) and (8) more general solution of (A) can be taken as

$$\begin{aligned} u(r, \theta) &= A + \sum_{n=1}^{\infty} u_n(r, \theta) \\ &= A + \sum_{n=1}^{\infty} (A_n \cos n\theta + B_n \sin n\theta) (C_n r^n + D_n r^{-n}) \end{aligned} \quad \dots(9)$$

Since $u(r, \theta)$ is finite when $r \rightarrow 0$, but when $r \rightarrow 0$, $r^{-n} \rightarrow \infty$, so in (9), we must take $D_n = 0$. Putting in (9), we get

$$\therefore u(r, \theta) = A + \sum_{n=1}^{\infty} (E_n \cos n\theta + F_n \sin n\theta) r^n \quad \dots(10)$$

Where $E_n = A_n C_n$ and $F_n = B_n D_n$ are new arbitrary constants.

Using conditions (B), i.e., $u(a, \theta) = f(\theta)$, from (10), we get

$$f(\theta) = A + \sum_{n=1}^{\infty} (E_n \cos n\theta + F_n \sin n\theta) a^n \quad \dots(11)$$

Multiplying both sides of (11) by $\cos n\theta$ and integrating between the limits $\theta = 0$ to 2π , we get

$$\begin{aligned} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta &= A \int_0^{2\pi} \cos n\theta \, d\theta + a^n E_n \int_0^{2\pi} \cos^2 n\theta \, d\theta + a^n F_n \int_0^{2\pi} \sin n\theta \cos n\theta \, d\theta \\ &= \left[A \frac{\sin n\theta}{n} \right]_0^{2\pi} + \frac{1}{2} a^n E_n \int_0^{2\pi} (1 + \cos 2n\theta) \, d\theta + \frac{1}{2} a^n F_n \int_0^{2\pi} \sin 2n\theta \, d\theta \\ &= 0 + \frac{1}{2} a^n E_n \left[\theta + \frac{1}{2n} \sin 2n\theta \right]_0^{2\pi} + \frac{1}{2} a^n F_n \left[-\frac{1}{2n} \cos 2n\theta \right]_0^{2\pi} \\ &= \frac{1}{2} a^n E_n 2\pi = \pi a^n E_n \end{aligned}$$

$$\therefore E_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta \quad \dots(12)$$

Similarly, multiplying both sides of (11) by $\sin n\theta$ and integrating between the limits $\theta = 0$ to 2π , we get

$$F_n = \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \sin n\theta d\theta \quad \dots(13)$$

Integrating both sides of (11) between the limits, $\theta = 0$ to 2π , we get

$$\begin{aligned} \int_0^{2\pi} f(\theta) d\theta &= \int_0^{2\pi} A d\theta + \sum_{n=1}^{\infty} [E_n \int_0^{2\pi} \cos n\theta d\theta + F_n \int_0^{2\pi} \sin n\theta d\theta] \\ &= A 2\pi + 0 \end{aligned}$$

$$\therefore A = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta = \frac{1}{2} E_0 \quad \dots(14)$$

$$\text{From (10), } u(r, \theta) = \frac{1}{2} E_0 + \sum_{n=1}^{\infty} (E_n \cos n\theta + F_n \sin n\theta) r^n \quad \dots(15)$$

Hence, the required solution is given by (15) where E_0, E_n are given by (12)

and F_n by (13).

Example 5: Find the steady state temperature in a circular plate of radius a which has half of its circumference at 0°C and the other half at temperature $u_0^\circ\text{C}$.

Solution: The steady state temperature in a circular plate is governed by Laplace equation (in plane polar coordinates)

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \dots(\text{A})$$

Here we are required to solve (A) subject to the boundary condition

$$u(a, \theta) = f(\theta) = \begin{cases} u_0, & 0 < \theta < \pi \\ 0, & \pi < \theta < 2\pi \end{cases}$$

Obviously $u(r, \theta)$, the solution of (A), must be periodic in θ and finite when $r \rightarrow 0$.

Let the solution of (A) be given by

$$u(r, \theta) = R(r) F(\theta) = RF \text{ (say)} \quad \dots(1)$$

Now proceeding similarly as in Ex. 4, we get

$$u(r, \theta) = \frac{1}{2} E_0 + \sum_{n=1}^{\infty} (E_n \cos n\theta + F_n \sin n\theta) r^n \quad \dots(2)$$

where

$$\begin{aligned} E_n &= \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta \\ &= \frac{1}{\pi a^n} \left[\int_0^{\pi} f(\theta) \cos n\theta \, d\theta + \int_{\pi}^{2\pi} f(\theta) \cos n\theta \, d\theta \right] \\ &= \frac{1}{\pi a^n} \left[\int_0^{\pi} u_0 \cos n\theta \, d\theta + 0 \right] = \frac{u_0}{\pi a^n} \left(\frac{1}{n} \sin n\theta \right)_0^{\pi} = 0 \end{aligned}$$

$$\begin{aligned} E_0 &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \, d\theta = \frac{1}{\pi} \left[\int_0^{\pi} f(\theta) \, d\theta + \int_{\pi}^{2\pi} f(\theta) \, d\theta \right] \\ &= \frac{1}{\pi} \left[\int_0^{\pi} u_0 \, d\theta + 0 \right] = \frac{1}{\pi} (\pi u_0) = u_0 \end{aligned}$$

and

$$\begin{aligned} F_n &= \frac{1}{\pi a^n} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta \\ &= \frac{1}{\pi a^n} \left[\int_0^{\pi} f(\theta) \sin n\theta \, d\theta + \int_{\pi}^{2\pi} f(\theta) \sin n\theta \, d\theta \right] \\ &= \frac{1}{\pi a^n} \left[\int_0^{\pi} u_0 \sin n\theta \, d\theta + 0 \right] \\ &= \frac{u_0}{\pi a^n} \left[-\frac{1}{n} \cos n\theta \right]_0^{\pi} = \frac{u_0}{\pi n a^n} (1 - \cos n\theta) = \frac{u_0}{\pi n a^n} [1 - (-1)^n] \\ &= \begin{cases} 0, & \text{if } n = 2m \text{ (even), } m = 1, 2, \dots \\ \frac{2u_0}{\pi (2m-1) a^{(2m-1)}} & \text{if } n = 2m-1 \text{ (odd), } m = 1, 2, \dots \end{cases} \end{aligned}$$

Hence, from (2), the required solution is

$$u(r, \theta) = \frac{1}{2} u_0 + \frac{2u_0}{\pi} \sum_{m=1}^{\infty} \left(\frac{r}{a} \right)^{2m-1} \cdot \frac{\sin (2m-1)\theta}{2m-1}$$

Example 6: A circular sector is determined by $0 \leq r \leq a, 0 \leq \theta \leq \alpha$. The temperature is kept at 0°C along the straight edges and at $f(\theta)$ along the curved edge. Find the steady state temperature at any point of the sector with its surfaces insulated.

Solution: Let OAB be a thin circular sector of radius a and $\angle AOB = \alpha$. The surfaces of the sector are insulated. The steady state temperature $u(r, \theta)$ in the sector is governed by the Laplace equation (in place polar coordinates)

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \dots(A)$$

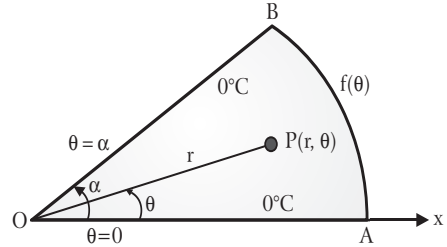


Fig. 14.6

Here the boundary conditions of the problem are

$$u(r, 0) = 0, u(r, \alpha) = 0, \quad 0 < r < a \quad \dots(B_1)$$

and
$$u(a, \theta) = f(\theta), \quad 0 < \theta < \alpha \quad \dots(B_2)$$

Let the solution of (A) be given by

$$u(r, \theta) = R(r) F(\theta) = RF \text{ (say)} \quad \dots(1)$$

Where R is a function of r alone and F function of θ alone.

Differentiating (1) and substituting in (A), we get

$$F r^2 \frac{d^2 R}{dr^2} + F r \frac{dR}{dr} + R \frac{d^2 F}{d\theta^2} = 0$$

or
$$\frac{1}{R} \left(r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} \right) = -\frac{1}{F} \frac{d^2 F}{d\theta^2} = \lambda \text{ (say)} \quad \dots(2)$$

Since each side of (2) is a function of different independent variables, so the two will be equal to each other if each is equal to the same constant say λ .

Thus, from (2), we get the following two ordinary differential equations

$$\frac{d^2 F}{d\theta^2} + \lambda F = 0 \quad \dots(3)$$

and
$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} - \lambda R = 0 \quad \dots(4)$$

Equation (4) is a homogeneous equation with variable coefficients. To solve (4), let $r = e^z$, so that

$$r \frac{dR}{dr} = D_1 z \quad \text{and} \quad r^2 \frac{d^2 R}{dr^2} = (D_1 - 1) D_1 z \quad \text{where} \quad D_1 \equiv r \frac{d}{dr} \equiv \frac{d}{dz}$$

Putting in (4), it reduces to

$$\{(D_1 - 1) D_1 + D_1 - \lambda\} R = 0 \quad \text{or} \quad (D_1^2 - \lambda) R = 0 \quad \dots(5)$$

Now there are following cases :

Case I : If $\lambda = 0$, then (3) and (5), reduce to

$$\frac{d^2 F}{d\theta^2} = 0 \quad \text{and} \quad D_1^2 R = \frac{d^2 R}{dz^2} = 0$$

Whose general solutions are

$$F = A_1 \theta + A_2, \quad R = A_3 z + A_4 = A_3 \log r + A_4$$

$$\therefore u(r, \theta) = FR = (A_1 \theta + A_2)(A_3 \log r + A_4)$$

$$\therefore \text{Condition (B}_1) \text{ i.e., } u(r, 0) = 0 \quad \text{and} \quad u(r, \alpha) = 0$$

$$\Rightarrow A_2(A_3 \log r + A_4) = 0 \quad \text{and} \quad (A_1 \alpha + A_2)(A_3 \log r + A_4) = 0$$

$$\Rightarrow A_2 = 0 \quad \text{and} \quad A_1 \alpha + A_2 = 0, \quad \because A_3 \log r + A_4 \neq 0$$

otherwise we get $u(r, \theta) = 0$, which is inadmissible.

$$\Rightarrow A_1 = 0 \quad \text{and} \quad A_2 = 0$$

$$\Rightarrow u(r, \theta) = 0, \text{ which is inadmissible.}$$

So we reject the case $\lambda = 0$.

Case II : If $\lambda = k^2 > 0$, then (3) and (5), reduce to

$$\frac{d^2 F}{d\theta^2} + k^2 F = 0 \quad \text{and} \quad (D_1^2 - k^2) R = 0$$

Whose general solutions are

$$F = A_k \cos k\theta + B_k \sin k\theta \quad \text{and} \quad R = C_k e^{kz} + D_k e^{-kz} = C_k r^k + D_k r^{-k}$$

$$\therefore u(r, \theta) = F.R = (A_k \cos k\theta + B_k \sin k\theta)(C_k r^k + D_k r^{-k}) \quad \dots(6)$$

Note that we cannot take $\lambda = -k^2 < 0$. Since in this case $u(r, \theta)$ will not involve trigonometric function of θ while $u(r, \theta)$ must involve trigonometric function of θ as it is periodic function of period 2π .

Since $u(r, \theta)$ is finite when $r \rightarrow 0$, but when $r \rightarrow 0$, $r^{-k} \rightarrow \infty \quad \because k > 0$,

So in (6) we must take $D_k = 0$

$$\therefore \quad \text{from (6)} \quad u(r, \theta) = (A_k \cos k\theta + B_k \sin k\theta)(C_k r^k) \quad \dots(7)$$

Now from (7), the boundary condition (B_1) i.e., $u(r, 0) = 0$

$$\Rightarrow \quad A_k(C_k \cdot r^k) = 0 \quad \Rightarrow \quad A_k = 0,$$

$\because r^k = 0$ or $C_k = 0$ will lead to trivial solution $u(r, \theta) = 0$ which is inadmissible.

$\therefore$ from (7), we have

$$u(r, \theta) = (B_k \sin k\theta)(C_k \cdot r^k) \quad \dots(8)$$

Now from (8), the boundary condition (B_1) i.e., $u(r, \alpha) = 0$

$$\Rightarrow \quad B_k C_k \sin k\alpha \cdot r^k = 0$$

$$\Rightarrow \quad \sin k\alpha = 0. \quad \because B_k C_k \neq 0 \text{ and } r^k \neq 0$$

$$\Rightarrow \quad k\alpha = n\pi \text{ i.e., } k = n\pi/\alpha, n \text{ is an integer.}$$

$\therefore$ From (8), for integral values of n , the solutions of (A), satisfying conditions (B_1) are given by

$$u_n(r, \theta) = F_n \sin(n\pi\theta/\alpha) \cdot r^{n\pi/\alpha}$$

Taking $F_n = B_k C_k$ is the new arbitrary constant.

Now taking $n = 1, 2, 3, \dots$, the more general solution of (A) is given by

$$u(r, \theta) = \sum_{n=1}^{\infty} u_n(r, \theta) = \sum_{n=1}^{\infty} F_n \sin(n\pi\theta/\alpha) \cdot r^{n\pi/\alpha} \quad \dots(9)$$

Which also satisfy the boundary conditions (B_1) .

Now from (9), the boundary condition (B_2) i.e., $u(a, \theta) = f(\theta)$

$$\Rightarrow \quad f(\theta) = \sum_{n=1}^{\infty} F_n \sin(n\pi\theta/\alpha) \cdot a^{n\pi/\alpha} \quad \dots(10)$$

The right hand side of (10) can be considered as the Fourier series of the function $f(\theta)$ on its left hand side for $0 < \theta < \alpha$

$$\therefore a^{n\pi/\alpha} \cdot F_n = \frac{2}{\alpha} \int_0^\alpha f(\theta) \sin\left(\frac{n\pi\theta}{\alpha}\right) d\theta$$

$$\text{or } F_n = \frac{2}{\alpha \cdot a^{n\pi/\alpha}} \int_0^\alpha f(\theta) \sin\left(\frac{n\pi\theta}{\alpha}\right) d\theta \quad \dots(11)$$

Hence, the required temperature in the sector is given by (9) where F_n is given by (11).

Note : If $\alpha = \pi$, then the thin circular sector reduces to a semi-circular plate and the result (9) and (11), reduce to the results (10) and (12) of Ex. 1 on page 703.

Example 7: Find the steady state temperature at any point of a circular sector; determined by $0 \leq r \leq a, 0 \leq \theta \leq \pi/2$, if the temperature is maintained at 0°C along the straight edge and $50(\pi\theta - 2\theta^2)$ along the curved edge. Also find the steady state temperature at $(a/2, \pi/4)$.

Solution: The steady state temperature $u(r, \theta)$ in the sector will be governed by the Laplace equation (in plane polar coordinates)

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \dots(A)$$

Here the boundary conditions of the problem are

$$u(r, 0) = 0, \quad u(r, \pi/2) = 0, \quad 0 < r < a \quad \dots(B_1)$$

$$\text{and } u(a, \theta) = f(\theta) = 50(\pi\theta - 2\theta^2) \quad \dots(B_2)$$

Let the solution of (A) be given by

$$u(r, \theta) = R(r) F(\theta) = RF \text{ (say)} \quad \dots(1)$$

Where R is a function of r alone and F a function of θ alone.

Now proceeding similarly as in Ex. 6 page 741, we get

$$u(r, \theta) = \sum_{n=1}^{\infty} F_n \sin(2n\theta) \cdot r^{2n} \quad \text{Here } \alpha = \pi/2 \quad \dots(2)$$

where

$$\begin{aligned} F_n &= \frac{2}{(\pi/2) a^{2n}} \int_0^{\pi/2} f(\theta) \sin 2n\theta d\theta, \\ &= \frac{4}{\pi a^{2n}} \int_0^{\pi/2} 50(\pi\theta - 2\theta^2) \sin 2n\theta d\theta \end{aligned}$$

$$\begin{aligned}
&= \frac{200}{\pi a^{2n}} \left[(\pi\theta - 2\theta^2) \left(-\frac{\cos 2n\theta}{2n} \right) - (\pi - 4\theta) \left(-\frac{\sin 2n\theta}{4n^2} \right) + (-4) \left(\frac{\cos 2n\theta}{8n^3} \right) \right] \Bigg|_0^{\pi/2} \\
&= \frac{200}{\pi a^{2n}} \left[\frac{4}{8n^3} (1 - \cos n\pi) \right] = \frac{100}{\pi n^3 a^{2n}} [1 - (-1)^n] \\
&= \begin{cases} 0, & \text{when } n=2m \text{ (even), } m=1, 2, \dots \\ \frac{200}{\pi (2m-1)^3 a^{2n}}, & \text{when } n=2m-1 \text{ (odd), } m=1, 2, \dots \end{cases}
\end{aligned}$$

Hence, from (2), the steady state temperature at any point (r, θ) in the sector is given by

$$u(r, \theta) = \frac{200}{\pi} \sum_{m=1}^{\infty} \left(\frac{r}{a} \right)^{2(2m-1)} \frac{\sin 2(2m-1)\theta}{(2m-1)^3} \quad \dots(3)$$

2nd Part : Putting $r = a/2$ and $\theta = \pi/4$ in (3), the temperature at $(a/2, \pi/4)$ is given by

$$\begin{aligned}
u(a/2, \pi/4) &= \frac{200}{\pi} \sum_{m=1}^{\infty} \left(\frac{1}{2} \right)^{2(2m-1)} \frac{\sin \{(2m-1)\pi/2\}}{(2m-1)^3} \\
&= \frac{200}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2^{4m-2} (2m-1)^3} \\
&\because \sin \{(2m-1)\pi/2\} = \sin (m\pi - \pi/2) = (-1)^{m+1}.
\end{aligned}$$

Example 8: u is a function of r and θ satisfying the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$$

within the region of the plane bounded by $r = a, r = b, \theta = 0, \theta = \pi/2$. Its value along the boundary $r = a$ is $\theta(\pi/2 - \theta)$ and along the other boundary is zero. Prove that

$$u(r, \theta) = \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(r/b)^{4m-2} - (b/r)^{4m-2}}{(a/b)^{4m-2} - (b/a)^{4m-2}} \frac{\sin(4m-2)\theta}{(2m-1)^3}.$$

Solution: The function $u(r, \theta)$ satisfy the equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \dots(A)$$

The boundary conditions of the problem are

$$u(r, 0) = 0, u(r, \pi/2) = 0, \quad a < r < b \quad \dots(B_1)$$

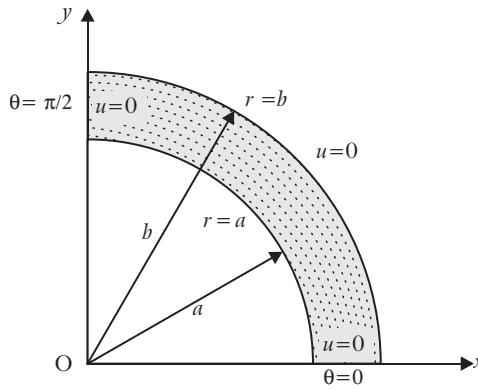


Fig. 14.7

$$u(b, \theta) = 0, 0 < \theta < \pi/2 \quad \dots(B_2)$$

and $u(a, \theta) = \theta(\pi/2 - \theta), 0 < \theta < \pi/2 \quad \dots(B_3)$

Here we are required to find the solution of (A), satisfying the conditions (B_1) , (B_2) and (B_3) .

Let the solution of (A) be given by

$$u(r, \theta) = R(r) F(\theta) = RF \text{ (say)} \quad \dots(1)$$

Where R is a function of r alone and F a function of θ alone.

Differentiating (1) and substituting in (A), we get

$$F \frac{d^2 R}{dr^2} + \frac{F}{r} \frac{dR}{dr} + \frac{R}{r^2} \frac{d^2 F}{d\theta^2} = 0$$

or $\frac{1}{R} \left(r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} \right) = -\frac{1}{F} \frac{d^2 F}{d\theta^2} = \lambda \text{ (say)} \quad \dots(2)$

Now the two sides of (2) are functions of different independent variables, so the two functions will be equal to each other if each is equal to the same constant λ (say).

Thus, from (2), we get the following two ordinary differential equations.

$$\frac{d^2 F}{d\theta^2} + \lambda F = 0 \quad \dots(3)$$

and $r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} - \lambda R = 0 \quad \dots(4)$

The equation (4) is a homogeneous differential equation with variable coefficients. To solve it let $r = e^z$, so that

$$r \frac{dR}{dr} = D_1 z \text{ and } r^2 \frac{d^2 R}{dr^2} = (D_1 - 1) D_1 z, \text{ where } D_1 \equiv r \frac{d}{dr} \equiv \frac{d}{dz}$$

Putting in (4), it reduces to

$$[(D_1 - 1) D_1 + D_1 - \lambda] R = 0 \quad \text{or} \quad (D_1^2 - \lambda) R = 0 \quad \dots(5)$$

Now there are following cases :

Case I : If $\lambda = 0$, then (3) and (5) reduce to

$$\frac{d^2 F}{d\theta^2} = 0 \quad \text{and} \quad D_1^2 R = \frac{d^2 R}{dz^2} = 0$$

Whose general solutions are

$$F = A\theta + B \quad \text{and} \quad R = Cz + D = C \log r + D$$

$$\therefore u(r, \theta) = FR = (A\theta + B)(C \log r + D) \quad \dots(6)$$

From (6), conditions (B_1) i.e., $u(r, 0) = 0$ and $u(r, \pi/2) = 0$

$$\Rightarrow B(C \log r + D) = 0 \quad \text{and} \quad (A \cdot \pi/2 + B)(C \log r + D) = 0$$

$$\Rightarrow B = 0 \quad \text{and} \quad A \pi/2 + B = 0,$$

$\therefore C \log r + D = 0$ will lead to, trivial solution $u(r, \theta) = 0$, which is inadmissible.

$$\Rightarrow A = 0 \quad \text{and} \quad B = 0$$

$$\Rightarrow u(r, \theta) = 0, \text{ which is inadmissible.}$$

Hence, we reject the case $\lambda = 0$.

Case II : If $\lambda = k^2 > 0$, then (3) and (5) reduce to

$$\frac{d^2 F}{d\theta^2} + k^2 F = 0 \quad \text{and} \quad (D_1^2 - k^2) R = 0$$

Whose general solutions are

$$F = A_k \cos k\theta + B_k \sin k\theta \quad \text{and} \quad R = C_k e^{kz} + D_k e^{-kz} = C_k r^k + D_k r^{-k}.$$

$$\therefore u(r, \theta) = F.R = (A_k \cos k\theta + B_k \sin k\theta)(C_k r^k + D_k r^{-k}) \quad \dots(7)$$

Note that $\lambda \neq -k^2 < 0$. Since in the problem $u(r, \theta)$ is a periodic function of period 2π , i.e., $u(r, \theta) = u(r, \theta + 2\pi)$, so $u(r, \theta)$ must involve trigonometric function of θ while if we take $\lambda = -k^2 < 0$, then $u(r, \theta)$ will not involve trigonometric function of θ .

Now condition in (B_1) , i.e., $u(r, 0) = 0$ and (7)

$$\Rightarrow A_k(C_k r^k + D_k r^{-k}) = 0 \Rightarrow A_k = 0 \quad \because C_k r^k + D_k r^{-k} \neq 0$$

Putting $A_k = 0$, in (7), we get

$$u(r, \theta) = (C_k r^k + D_k r^{-k}) B_k \sin k\theta \quad \dots(8)$$

Now condition in (B_1) i.e., $u(r, \pi/2) = 0$ and (8)

$$\Rightarrow (C_k r^k + D_k r^{-k}) B_k \sin(k\pi/2) = 0$$

$$\Rightarrow \sin(k\pi/2) = 0, \quad \because C_k r^k + D_k r^{-k} \neq 0, B_k \neq 0$$

$$\Rightarrow k\pi/2 = n\pi, \text{ where } n \text{ is integer.}$$

$$\Rightarrow k = 2n.$$

$\therefore$ from (8)

$$u_n(r, \theta) = (E_n r^{2n} + F_n r^{-2n}) \sin 2n\theta$$

Where $E_n = C_k B_k$ and $F_n = D_k B_k$ are the new arbitrary constants are solutions of (A) satisfying conditions (B_1) for integral values of n .

Taking $n = 1, 2, 3, \dots$, the more general solution of (A) can be written as

$$u(r, \theta) = \sum_{n=1}^{\infty} u_n(r, \theta) = \sum_{n=1}^{\infty} (E_n r^{2n} + F_n r^{-2n}) \sin 2n\theta \quad \dots(9)$$

Which also satisfy the conditions (B_1) .

Now condition (B_2) i.e., $u(b, \theta) = 0$ and (9)

$$\Rightarrow (E_n b^{2n} + F_n b^{-2n}) \sin 2n\theta = 0$$

$$\Rightarrow E_n b^{2n} + F_n b^{-2n} = 0, \quad \because \sin 2n\theta \neq 0$$

$$\Rightarrow F_n = -E_n b^{4n}$$

Putting in (9), we get

$$u(r, \theta) = \sum_{n=1}^{\infty} E_n (r^{2n} - b^{4n} r^{-2n}) \sin 2n\theta \quad \dots(10)$$

Now condition (B₃) i.e., $u(a, \theta) = \theta(\pi/2 - \theta)$ and (10)

$$\Rightarrow \quad \theta(\pi/2 - \theta) = \sum_{n=1}^{\infty} E_n (a^{2n} - b^{4n} a^{-2n}) \sin 2n\theta \quad \dots(11)$$

The right hand side of (11) can be considered as Fourier series of $\theta(\pi/2 - \theta)$ the function on its left hand side in half range $0 \leq \theta \leq \pi/2$

$$\begin{aligned} \therefore E_n (a^{2n} - b^{4n} a^{-2n}) &= \frac{2}{(\pi/2)} \int_0^{\pi/2} \theta \left(\frac{\pi}{2} - \theta \right) \sin 2n\theta \, d\theta \\ &= \frac{4}{\pi} \left[\theta \left(\frac{\pi}{2} - \theta \right) \left(-\frac{\cos 2n\theta}{2n} \right) - \left(\frac{\pi}{2} - 2\theta \right) \left(-\frac{\sin 2n\theta}{4n^2} \right) + (-2) \left(\frac{\cos 2n\theta}{8n^3} \right) \right]_0^{\pi/2} \\ &= \frac{1}{\pi n^3} [1 - \cos n\pi] = \frac{1}{\pi n^3} [1 - (-1)^n] \\ &= \begin{cases} 0, & \text{when } n = 2m \text{ (even), } m = 1, 2, \dots \\ \frac{2}{\pi (2m-1)^3} & \text{when } n = 2m-1 \text{ (odd), } m = 1, 2, \dots \end{cases} \end{aligned}$$

$$\therefore E_n = \begin{cases} 0, & \text{when } n=2m, \quad m = 1, 2, \dots \\ \frac{2}{\pi (2m-1)^3 \{a^{2(2m-1)} - b^{4(2m-1)} a^{-2(2m-1)}\}} & \text{when } n=2m-1, \quad m = 1, 2, \dots \end{cases}$$

Hence, from (10), we have

$$\begin{aligned} u(r, \theta) &= \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\{r^{2(2m-1)} - b^{4(2m-1)} r^{-2(2m-1)}\} \sin 2(2m-1)\theta}{(2m-1)^3 \{a^{2(2m-1)} - b^{4(2m-1)} a^{-2(2m-1)}\}} \\ &= \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(r/b)^{4m-2} - (b/r)^{4m-2}}{(a/b)^{4m-2} - (b/a)^{4m-2}} \cdot \frac{\sin (4m-2)\theta}{(2m-1)^3} \end{aligned}$$

Exercise 14.2

- Obtain steady state temperature distribution in a semi-circular plate of radius a , insulated on both faces, with its curved boundary kept at a constant temperature u_0 and its bounding diameter kept at temperature zero. [Meerut 2011 (B.P.)]
- A thin semi-circular plate of radius a , insulated on both faces, has its boundary diameter kept at 0°C and its temperature along the semi-circular boundary is 100°C . If $u(r, \theta)$ is the steady state temperature, in the plate, then prove that

$$u(r, \theta) = \frac{400}{\pi} \sum_{m=1}^{\infty} \left(\frac{r}{a}\right)^{2m-1} \frac{\sin(2m-1)\theta}{2m-1}$$

Also find $u(a/4, \pi/2)$.

- Find the steady state temperature in a circular plate of radius a which has one half of its circumference at 0°C and the other half at 100°C .
- Find the steady state temperature at the points in the sector given by $0 \leq \theta \leq \pi/4$, $0 \leq r \leq a$ of a circular plate if the temperature is maintained at 0°C along the side edges and at a constant temperature $u_0^\circ\text{C}$ along the curved edge.

Answers 14.2

1.	$u(r, \theta) = \frac{440}{\pi} \sum_{m=1}^{\infty} \left(\frac{r}{a}\right)^{2m-1} \frac{\sin(2m-1)\theta}{2m-1}$
2.	$u(a/4, \pi/2) = \frac{400}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(2m-1)} \left(\frac{1}{4}\right)^{2m-1}$
3.	$u(r, \theta) = 50 + \frac{200}{\pi} \sum_{m=1}^{\infty} \left(\frac{r}{a}\right)^{2m-1} \frac{\sin(2m-1)\theta}{(2m-1)}$
4.	$u(r, \theta) = \frac{440}{\pi} \sum_{m=1}^{\infty} \left(\frac{r}{a}\right)^{4(2m-1)} \frac{\sin 4(2m-1)\theta}{(2m-1)}$

14.7 Solution of Laplace's Equation in Rectangular Cartesian Coordinates (x, y, z) by the Method of Separation of Variables

Three dimensional Laplace's equation in cartesian coordinates is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \dots(A)$$

Let the solution of (A) be of the form

$$u(x, y, z) = F(x) Y(y) Z(z) = FYZ \text{ (say)} \quad \dots(1)$$

Where F is a function of x alone, Y a function of y alone and Z a function of z alone.

Differentiating (1) and substituting in (A), we get

$$YZ \frac{d^2 F}{dx^2} + FZ \frac{d^2 Y}{dy^2} + FY \frac{d^2 Z}{dz^2} = 0$$

or
$$\frac{1}{F} \frac{d^2 F}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} = -\frac{1}{Z} \frac{d^2 Z}{dz^2} \quad \dots(2)$$

All the three terms in (2) are functions of different independent variables, therefore (2) will be satisfied if each of the three terms is constant. So let

$$\frac{1}{F} \frac{d^2 F}{dx^2} = \lambda_1, \quad \frac{1}{Y} \frac{d^2 Y}{dy^2} = \lambda_2 \quad \text{and} \quad -\frac{1}{Z} \frac{d^2 Z}{dz^2} = \lambda$$

$$\text{where } \lambda_1 + \lambda_2 = \lambda$$

Thus, we get the following three ordinary differential equations

$$\frac{d^2 F}{dx^2} - \lambda_1 F = 0, \quad \frac{d^2 Y}{dy^2} - \lambda_2 Y = 0 \quad \text{and} \quad \frac{d^2 Z}{dz^2} + \lambda Z = 0 \quad \dots(3)$$

$$\text{where } \lambda_1 + \lambda_2 = \lambda$$

Now there are the following three cases :

Case I : When $\lambda_1 = 0 = \lambda_2 = \lambda$, then the equations given in (3) reduce to

$$\frac{d^2 F}{dx^2} = 0, \quad \frac{d^2 Y}{dy^2} = 0 \quad \text{and} \quad \frac{d^2 Z}{dz^2} = 0$$

Whose general solutions are

$$F = Ax + B, Y = Cy + D \text{ and } Z = Ez + F$$

∴ from (1) solution of (A) is given by

$$u(x, y) = FYZ = (Ax + B)(Cy + D)(Ez + F) \quad \dots(4)$$

Case II : When $\lambda_1 = k_1^2 > 0$, $\lambda_2 = k_2^2 > 0$ and $\lambda = k^2 > 0$ s.t., $k_1^2 + k_2^2 = k^2$, then the equations in (3) reduce to

$$\frac{d^2 F}{dx^2} - k_1^2 F = 0, \quad \frac{d^2 Y}{dy^2} - k_2^2 Y = 0 \text{ and } \frac{d^2 Z}{dz^2} + k^2 Z = 0$$

Whose general solutions are

$$F = A_{k_1} e^{xk_1} + B_{k_1} e^{-xk_1} \quad \text{or} \quad F = A'_{k_1} e^{\pm xk_1}$$

$$Y = C_{k_2} e^{yk_2} + D_{k_2} e^{-yk_2} \quad \text{or} \quad Y = B'_{k_2} e^{\pm yk_2}$$

$$\text{and} \quad Z = E_k \cos kz + F_k \sin kz \quad \text{or} \quad Z = C'_k e^{\pm ikz}$$

∴ from (1), the most general solution of (A) is given by

$$u(x, y, z) = \sum_{k_1, k_2} (A_{k_1} e^{xk_1} + B_{k_1} e^{-xk_1}) (C_{k_2} e^{yk_2} + D_{k_2} e^{-yk_2}) (E_k \cos kz + F_k \sin kz) \quad \dots(5)$$

$$\text{where} \quad k^2 = k_1^2 + k_2^2$$

The solution can also be taken as

$$u(x, y, z) = \sum_{k_1, k_2} H_{k_1 k_2} e^{\pm xk_1 \pm yk_2 \pm ikz} \quad \dots(6)$$

$$\text{where} \quad k^2 = k_1^2 + k_2^2$$

and $H_{k_1 k_2} = A'_{k_1} B'_{k_2} C'_k$ is the new arbitrary constant.

Case III : When $\lambda_1 = -k_1^2 < 0$, $\lambda_2 = -k_2^2 < 0$ and $\lambda = -k^2 < 0$ s.t. $k^2 = k_1^2 + k_2^2$, then the equations in (3) reduce to

$$\frac{d^2 F}{dx^2} + k_1^2 F = 0, \quad \frac{d^2 Y}{dy^2} + k_2^2 Y = 0 \quad \text{and} \quad \frac{d^2 Z}{dz^2} - k^2 Z = 0$$

Whose general solutions are

$$F = A_{k_1} \cos k_1 x + B_{k_1} \sin k_1 x \quad \text{or} \quad F = A'_{k_1} e^{\pm ik_1 x}$$

$$Y = C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y \quad \text{or} \quad Y = B'_{k_2} e^{\pm ik_2 y}$$

and $Z = E_k e^{kz} + F_k e^{-kz}$ or $Z = C'_k e^{\pm kz}$

∴ from (1), the most general solution of (A) is given by

$$u(x, y, z) = \sum_{k_1, k_2} (A_{k_1} \cos k_1 x + B_{k_1} \sin k_1 x)(C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y) \cdot (E_k e^{kz} + F_k e^{-kz}) \quad \dots(7)$$

where $k^2 = k_1^2 + k_2^2$

The solution can also be taken as

$$u(x, y, z) = \sum_{k_1, k_2} H_{k_1 k_2} e^{\pm ik_1 x \pm ik_2 y \pm kz}, \text{ where } k^2 = k_1^2 + k_2^2$$

and $H_{k_1 k_2} = A'_{k_1} \cdot B'_{k_2} \cdot C'_k$ is the new arbitrary constant.

14.8 Laplace Equation in Cylindrical Coordinates

Laplace equation in cartesian coordinates (x, y, z) is given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \dots(1)$$

Proceeding as in {12.9 on page 568 in cylindrical coordinates (ρ, ϕ, z) , we have

$$\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2}$$

Hence, the Laplace equation (1) in cylindrical coordinates (ρ, ϕ, z) is transformed to

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

14.9 Solution of Laplace's Equation in Cylindrical Coordinates by the Method of Separation of Variables

[Meerut 2001, 02 (B.P.)]

The Laplace's equation in cylindrical coordinates (ρ, ϕ, z) is given by

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \dots(A)$$

Let the solution of (A) be of the form

$$u(\rho, \phi, z) = R(\rho) F(\phi) Z(z) = RFZ \text{ (say)} \quad \dots(1)$$

Where R is a function of ρ alone, F a function of ϕ alone and Z a function of z alone.

Differentiating (1) and substituting in (A), we get

$$FZ \frac{d^2 R}{d\rho^2} + \frac{FZ}{\rho} \frac{dR}{d\rho} + \frac{RZ}{\rho^2} \frac{d^2 F}{d\phi^2} + RF \frac{d^2 Z}{dz^2} = 0$$

or $\frac{1}{R} \frac{d^2 R}{d\rho^2} + \frac{1}{\rho R} \frac{dR}{d\rho} + \frac{1}{\rho^2 F} \frac{d^2 F}{d\phi^2} = -\frac{1}{Z} \frac{d^2 Z}{dz^2} = -k^2 \text{ (say)} \quad \dots(2)$

Here in (2), right hand side is a function of independent variable z alone while left hand side is a function of ρ and ϕ . So the two sides will be equal to each other if each is equal to the same constant say $-k^2$. Here constant $-k^2$ is chosen, because the solutions obtained from the resulting equations are useful for practical problems.

Thus, from (2) we get the following equations

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \quad \dots(3)$$

and $\frac{1}{R} \frac{d^2 R}{d\rho^2} + \frac{1}{\rho R} \frac{dR}{d\rho} + \frac{1}{\rho^2 F} \frac{d^2 F}{d\phi^2} = -k^2$

or $\frac{\rho^2}{F} \frac{d^2 R}{d\rho^2} + \frac{\rho}{R} \frac{dR}{d\rho} + k^2 \rho^2 = -\frac{1}{F} \frac{d^2 F}{d\phi^2} = m^2 \text{ (say)} \quad \dots(4)$

Here in (4) right hand side is a function of independent variable ϕ only while left hand side is a function of independent variable ρ only, so the two will be equal to each other if each is equal to the same constant. On physical grounds of the problem the solution must involve the trigonometrical functions of ϕ , so we choose this constant equal to m^2 (say). Also the physical condition of the problem *i.e.*, $u(\rho, \phi) = u(\rho, \phi + 2\pi)$ will be satisfied only if we suppose m to be an integer.

Thus, from (4), we get the following equations

$$\frac{d^2 F}{d\phi^2} + m^2 F = 0 \quad \dots(5)$$

and $\frac{\rho^2}{R} \frac{d^2 R}{d\rho^2} + \frac{\rho}{R} \frac{dR}{d\rho} + k^2 \rho^2 = m^2$

or $\rho^2 \frac{d^2 R}{d\rho^2} + \rho \frac{dR}{d\rho} + (k^2 \rho^2 - m^2) R = 0 \quad \dots(6)$

Now let $k\rho = x$, so that $\frac{d\rho}{dx} = \frac{1}{k}$

and
$$\frac{dR}{d\rho} = \frac{dR}{dx} \cdot \frac{dx}{d\rho} = k \frac{dR}{dx} \Rightarrow \frac{d}{d\rho} \equiv k \frac{d}{dx}$$

and
$$\frac{d}{d\rho} \left(\frac{dR}{d\rho} \right) = k \frac{d}{dx} \left(k \frac{dR}{dx} \right)$$

$\therefore \frac{d^2 R}{d\rho^2} = k^2 \frac{d^2 R}{dx^2}$

Putting in (6), it reduces to

$$\frac{x^2}{k^2} k^2 \frac{d^2 R}{dx^2} + \frac{x}{k} \cdot k \frac{dR}{dx} + (x^2 - m^2) R = 0$$

or
$$\frac{d^2 R}{dx^2} + \frac{1}{x} \frac{dR}{dx} + \left(1 - \frac{m^2}{x^2} \right) R = 0 \quad \dots(7)$$

Now general solution of (3) is

or
$$\left. \begin{aligned} Z &= A_k e^{kz} + B_k e^{-kz} \\ Z &= A'_k e^{\pm kz} \end{aligned} \right\} \quad \dots(8)$$

General solution of (5) is

or
$$\left. \begin{aligned} F &= C_m \cos m\phi + D_m \sin m\phi \\ F &= C'_m e^{\pm im\phi} \end{aligned} \right\} \quad \dots(9)$$

Equation (7) is Bessel's equation, so its general solution is

$$R = E_{km} J_m(x) + F_{km} J_{-m}(x) = E_{km} J_m(k\rho) + F_{km} J_{-m}(k\rho),$$

when m is not an integer

and
$$R = E_{km} J_m(k\rho) + F_{km} Y_m(k\rho), \text{ when } m \text{ is an integer.}$$

Hence, from (1), the most general solution of (A) is given by

$$u(\rho, \theta, z) = \sum_{k,m} (A_k e^{kz} + B_k e^{-kz}) (C_m \cos m\phi + D_m \sin m\phi)$$

$$[E_{km} J_m(k\rho) + F_{km} J_{-m}(k\rho)]$$

when m is not an integer.

and
$$u(\rho, \phi, z) = \sum_{k,m} (A_k e^{kz} + B_k e^{-kz})(C_m \cos m\phi + D_m \sin m\phi).$$

$$[E_{km} J_m(k\rho) + F_{km} Y_m(k\rho)] \text{ when } m \text{ is an integer.}$$

Note : 1. The solution can also be written as

$$u(\rho, \phi, z) = \sum_{k,m} e^{\pm kz \pm im\phi} [G_{km} J_m(k\rho) + H_{km} J_{-m}(k\rho)]$$

when m is not an integer.

And
$$u(\rho, \phi, z) = \sum_{k,m} e^{\pm kz \pm im\phi} [G_{km} J_m(k\rho) + H_{km} Y_m(k\rho)]$$

when m is an integer.

Where $G_{km} = A'_k C'_m E_{km}$ and $H_{km} = A'_k C'_m F_{km}$ are the new arbitrary constants.

2. The solution can also be written as

$$u(\rho, \phi, z) = \sum_{k,m} e^{\pm kz \pm im\phi} R(\rho)$$

when $R(\rho)$ is the solution of Bessel's equation (7).

14.10 Laplace Equation in Spherical Coordinates

Laplace equation in cartesian coordinates (x, y, z) is given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \dots(1)$$

Proceeding as in {12.11 on page 573 in spherical polar coordinates (r, θ, ϕ) , we have

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2}$$

Hence, the Laplace equation (1), in spherical coordinates (r, θ, ϕ) is transformed to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0$$

14.11 Solution of Laplace's Equation in Spherical Coordinates by the Method of Separation of Variables

The Laplace's equation in spherical coordinates is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0 \quad \dots(A)$$

Obviously solution of (A) is a function of r , θ and ϕ .

$\therefore$ Let the solution $u(r, \theta, \phi)$ of (A) be given by

$$u(r, \theta, \phi) = R(r) F_1(\theta) F_2(\phi) = R F_1 F_2 \text{ (say)} \quad \dots(1)$$

Where R is a function of r alone, F_1 a function of θ alone and F_2 a function of ϕ alone.

Differentiating (1) and substituting in (A), we get

$$\begin{aligned} F_1 F_2 \frac{d^2 R}{dr^2} + \frac{2 F_1 F_2}{r} \frac{dR}{dr} + \frac{R F_2}{r^2} \frac{d^2 F_1}{d\theta^2} + \frac{R F_2 \cot \theta}{r^2} \frac{dF_1}{d\theta} + \frac{R F_1}{r^2 \sin^2 \theta} \frac{d^2 F_2}{d\phi^2} &= 0 \\ \text{or} \quad \left[\frac{1}{R} \left(\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} \right) + \frac{1}{r^2 F_1} \left(\frac{d^2 F_1}{d\theta^2} + \cot \theta \frac{dF_1}{d\theta} \right) \right] r^2 \sin^2 \theta & \\ &= - \frac{1}{F_2} \frac{d^2 F_2}{d\phi^2} = m^2 \text{ (say)} \quad \dots(2) \end{aligned}$$

In (2), right hand side is a function of independent variable ϕ alone while left hand side is independent of ϕ . So the two sides will be equal only if both are equal to the same constant. We choose this constant equal to m^2 , as in many physical problems we require the solution which involve trigonometric functions of θ .

Thus (2), gives

$$\frac{d^2 F_2}{d\phi^2} + m^2 F_2 = 0 \quad \dots(3)$$

$$\text{and} \quad \left[\frac{1}{R} \left(\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} \right) + \frac{1}{r^2 F_1} \left(\frac{d^2 F_1}{d\theta^2} + \cot \theta \frac{dF_1}{d\theta} \right) \right] r^2 \sin^2 \theta = m^2$$

$$\text{or} \quad \frac{r^2}{R} \left(\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} \right) = \frac{m^2}{\sin^2 \theta} - \frac{1}{F_1} \left(\frac{d^2 F_1}{d\theta^2} + \cot \theta \frac{dF_1}{d\theta} \right) = n(n+1) \text{ say} \quad \dots(4)$$

In (4) two sides are functions of different artificial variable, so they will be equal to each other if each is equal to the same constant say $n(n+1)$. (Note)

Thus, from (4) we get the following two differential equations

$$\frac{r^2}{R} \left(\frac{d^2 R}{dr^2} + \frac{2}{r} \frac{dR}{dr} \right) = n(n+1)$$

$$\text{or} \quad r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} - n(n+1) R = 0 \quad \dots(5)$$

$$\text{and} \quad \frac{m^2}{\sin^2 \theta} - \frac{1}{F_1} \left(\frac{d^2 F_1}{d\theta^2} + \cot \theta \frac{dF_1}{d\theta} \right) = n(n+1)$$

$$\text{or} \quad \frac{d^2 F_1}{d\theta^2} + \cot \theta \frac{dF_1}{d\theta} + \left(n(n+1) - \frac{m^2}{\sin^2 \theta} \right) F_1 = 0 \quad \dots(6)$$

Equation (5) is a homogeneous equation with variable coefficient. So let $r = e^z$ so that

$$r \frac{dR}{dr} = D_1 z \text{ and } r^2 \frac{d^2 R}{dr^2} = (D_1 - 1) D_1 z$$

$$\text{where} \quad r \frac{d}{dr} \equiv \frac{d}{dz} = D_1$$

Putting in (5), we get

$$[(D_1 - 1) D_1 + 2 D_1 - n(n+1)] R = 0$$

$$\text{or} \quad (D_1^2 + D_1 - n^2 - n) R = 0$$

$$\text{or} \quad [(D_1 - n)(D_1 + n) + (D_1 - n)] R = 0$$

$$\text{or} \quad (D_1 - n)(D_1 + n + 1) R = 0 \quad \dots(7)$$

$$\text{Now to solve (6), let } \mu = \cos \theta, \text{ so that } \frac{dF_1}{d\theta} = \frac{dF_1}{d\mu} \frac{d\mu}{d\theta} = -\sin \theta \frac{dF_1}{d\mu}$$

$$\begin{aligned} \text{and} \quad \frac{d^2 F_1}{d\theta^2} &= \frac{d}{d\theta} \left(-\sin \theta \frac{dF_1}{d\mu} \right) = -\cos \theta \frac{dF_1}{d\mu} - \sin \theta \frac{d^2 F_1}{d\mu^2} \cdot \frac{d\mu}{d\theta} \\ &= -\cos \theta \frac{dF_1}{d\mu} + \sin^2 \theta \frac{d^2 F_1}{d\mu^2} = -\cos \theta \frac{dF_1}{d\mu} + (1 - \cos^2 \theta) \frac{d^2 F_1}{d\mu^2} \end{aligned}$$

Putting in (6), we get

$$-\cos \theta \frac{dF_1}{d\mu} + (1 - \cos^2 \theta) \frac{d^2 F_1}{d\mu^2} + \cot \theta (-\sin \theta) \frac{dF_1}{d\mu} + \left[n(n+1) - \frac{m^2}{1 - \cos^2 \theta} \right] F_1 = 0$$

$$\text{or} \quad (1 - \cos^2 \theta) \frac{d^2 F_1}{d\mu^2} - 2 \cos \theta \frac{dF_1}{d\mu} + \left[n(n+1) - \frac{m^2}{1 - \cos^2 \theta} \right] F_1 = 0$$

$$\text{or} \quad (1 - \mu^2) \frac{d^2 F_1}{d\mu^2} - 2\mu \frac{dF_1}{d\mu} + \left[n(n+1) - \frac{m^2}{1 - \mu^2} \right] F_1 = 0 \quad \dots(8)$$

The general solution of (3) is

$$\text{or} \quad \left. \begin{aligned} F &= A_m \cos m\phi + B_m \sin m\phi \\ F &= A'_m e^{\pm im\phi} \end{aligned} \right\} \quad \dots(9)$$

General solution of (7) is

$$R = C_n e^{nz} + D_n e^{-(n+1)z} = C_n r^n + D_n / r^{n+1} \quad \dots(10)$$

Equation (8) is Associated Legendre equation, whose solution is

$$F_1 = E_{mn} P_n^m(\mu) + F_{mn} Q_n^m(\mu)$$

$$\text{or} \quad F_1 = E_{mn} P_n^m(\cos \theta) + F_{mn} Q_n^m(\cos \theta) \quad \dots(11)$$

From (1), using (9), (10) and (11), the solution of (A) for various values of m and n are given by

$$u_{mn}(r, \theta, \phi) = (A_m \cos m\phi + B_m \sin m\phi)(C_n r^n + D_n / r^{n+1}) \cdot [E_{mn} P_n^m(\cos \theta) + F_{mn} Q_n^m(\cos \theta)]$$

A more general solution of (A) is given by

$$\begin{aligned} u(r, \theta, \phi) &= \sum_m \sum_n u_{mn}(r, \theta, \phi) \\ &= \sum_m \sum_n [(A_m \cos m\phi + B_m \sin m\phi)(C_n r^n + D_n / r^{n+1}) \cdot \{E_{mn} P_n^m(\cos \theta) + F_{mn} Q_n^m(\cos \theta)\}] \quad \dots(12) \end{aligned}$$

Which can also be taken as

$$\begin{aligned} u(r, \theta, \phi) &= \sum_m \sum_n A'_m e^{\pm im\phi} (C_n r^n + D_n / r^{n+1}) \cdot [E_{mn} P_n^m(\cos \theta) + F_{mn} Q_n^m(\cos \theta)] \\ &= \sum_m \sum_n [(A_{mn} r^n + B_{mn} / r^{n+1}) \cdot [E_{mn} P_n^m(\cos \theta) + F_{mn} Q_n^m(\cos \theta)] e^{\pm im\phi}] \quad \dots(13) \end{aligned}$$

Note : Along the polar axis (initial line) $\theta = 0$, $\cos \theta = 1 \Rightarrow Q_n^m(\cos \theta) \rightarrow \infty$

$\therefore$ If the solution $u(r, \theta, \phi)$ remains finite along the polar axis $\theta = 0$, then we must take $F_{mn} = 0$ in (12). In this case the solution is given by

$$u(r, \theta, \phi) = \sum_m \sum_n (A_m \cos m\phi + B_m \sin m\phi)(C_n r^n + D_n/r^{n+1}). P_n^m(\cos \theta)$$

which can also be taken as

$$u(r, \theta, \phi) = \sum_m \sum_n (C_n r^n + D_n/r^{n+1}). e^{\pm im\phi} P_n^m(\cos \theta)$$

Solved Examples

Example 1: Find the potential $\phi(x, y, z)$ in the parallelepiped $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$, satisfying the conditions :

(i) $\phi = 0$ on $x = 0, x = a, y = 0, y = b$ and $z = 0$

(ii) $\phi = f(x, y)$ on $z = c, 0 \leq x \leq a, 0 \leq y \leq b$.

[Meerut 2010 (Sem. I)]

Solution : The potential $\phi(x, y, z)$ in the parallelepiped satisfies the Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \dots(A)$$

Here we are required to find the solution of (A), satisfying the following conditions.

$$\phi(0, y, z) = 0, \quad \phi(a, y, z) = 0 \quad \dots(B_1)$$

$$\phi(x, 0, z) = 0, \quad \phi(x, b, z) = 0 \quad \dots(B_2)$$

$$\text{and} \quad \phi(x, y, 0) = 0, \quad \phi(x, y, c) = f(x, y) \quad \dots(B_3)$$

Let the solution of (A) be given by

$$\phi(x, y, z) = F(x) Y(y) Z(z) \quad \dots(1)$$

Where F is a function of x alone, Y a function of y alone and Z a function of z alone.

Differentiating (1) and substituting in (A), we get

$$YZ \frac{d^2 F}{dx^2} + FZ \frac{d^2 Y}{dy^2} + FY \frac{d^2 Z}{dz^2} = 0$$

$$\text{or} \quad \frac{1}{F} \frac{d^2 F}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} = -\frac{1}{Z} \frac{d^2 Z}{dz^2} = \lambda \text{ (say)} \quad \dots(2)$$

Here in (2) each term is function of different independent variable. Thus (2), will be true only if each term is equal to a constant. So let

$$\text{if } -\frac{1}{Z} \frac{d^2 Z}{dz^2} = \lambda, \frac{1}{F} \frac{d^2 F}{dx^2} = \lambda_1 \text{ and } \frac{1}{Y} \frac{d^2 Y}{dy^2} = \lambda_2 \text{ s.t. } \lambda = \lambda_1 + \lambda_2.$$

Thus, we get the following three ordinary differential equations

$$\frac{d^2 F}{dx^2} - \lambda_1 F = 0, \frac{d^2 Y}{dy^2} - \lambda_2 Y = 0 \text{ and } \frac{d^2 Z}{dz^2} + \lambda Z = 0 \quad \dots(3)$$

$$\text{where } \lambda = \lambda_1 + \lambda_2$$

Now the following three cases arise.

Case I : If $\lambda = 0 = \lambda_1 = \lambda_2$, then the equations in (3) reduce to

$$\frac{d^2 F}{dx^2} = 0, \frac{d^2 Y}{dy^2} = 0 \text{ and } \frac{d^2 Z}{dz^2} = 0$$

Whose general solutions are

$$F = Ax + B, Y = Cx + D \text{ and } Z = Ez + F$$

$$\therefore \phi(x, y, z) = (Ax + B)(Cx + D)(Ez + F)$$

Now the conditions (B_1) i.e., $\phi(0, y, z) = 0$ and $\phi(a, y, z) = 0$

$$\Rightarrow 0 = B(Cx + D)(Ez + F) \text{ and } (Aa + B)(Cx + D)(Ez + F)$$

$$\Rightarrow B = 0 \text{ and } Aa + B, \therefore Cx + D = 0 \text{ and } Ez + F = 0,$$

will lead to $\phi(x, y, z) = 0$, which is trivial solution and is inadmissible.

$$\Rightarrow A = 0 \text{ and } B = 0$$

$$\therefore \phi(x, y, z) = 0. \text{ Which is trivial solution and is inadmissible.}$$

So we reject the case when $\lambda = 0 = \lambda_1 = \lambda_2$.

Case II : If $\lambda = k^2 > 0$, $\lambda_1 = k_1^2 > 0$ and $\lambda_2 = k_2^2 > 0$ s.t., $\lambda = \lambda_1 + \lambda_2$ i.e., $k^2 = k_1^2 + k_2^2$,

then the equations in (3) reduce to

$$\frac{d^2 F}{dx^2} - k_1^2 F = 0, \frac{d^2 Y}{dy^2} - k_2^2 Y = 0 \text{ and } \frac{d^2 Z}{dz^2} + k^2 Z = 0$$

Whose general solutions are

$$F = Ae^{k_1 x} + Be^{-k_1 x}, Y = Ce^{k_2 y} + De^{-k_2 y} \text{ and } Z = E \cos kz + F \sin kz$$

$$\therefore \phi(x, y, z) = (Ae^{k_1 x} + Be^{-k_1 x})(Ce^{k_2 y} + De^{-k_2 y})(E \cos kz + F \sin kz)$$

$\therefore$ The conditions (B_1) i.e., $\phi(0, y, z) = 0$ and $\phi(a, y, z) = 0$

$$\Rightarrow 0 = (A + B)(Ce^{k_2 y} + De^{-k_2 y})(E \cos kz + F \sin kz)$$

$$\text{and } 0 = (Ae^{k_1 a} + Be^{-k_1 a})(Ce^{k_2 y} + De^{-k_2 y})(E \cos kz + F \sin kz)$$

$$\Rightarrow A + B = 0 \text{ and } Ae^{k_1 a} + Be^{-k_1 a} = 0,$$

$$\therefore Ce^{k_2 y} + De^{-k_2 y} \neq 0 \text{ and } E \cos kz + F \sin kz \neq 0$$

$$\Rightarrow A = 0 \text{ and } B = 0$$

$\therefore \phi(x, y, z) = 0$, which is trivial solution and is inadmissible.

So we also reject the case when $\lambda = k^2$, $\lambda_1 = k_1^2$ and $\lambda_2 = k_2^2$.

Case III : If $\lambda = -k^2 < 0$, $\lambda_1 = -k_1^2 < 0$, $\lambda_2 = -k_2^2 < 0$, s.t. $k^2 = k_1^2 + k_2^2$, then the equations in (3) reduce to

$$\frac{d^2 F}{dx^2} + k_1^2 F = 0, \frac{d^2 Y}{dy^2} + k_2^2 Y = 0 \text{ and } \frac{d^2 Z}{dz^2} - k^2 Z = 0$$

Whose general solutions are

$$F = A_{k_1} \cos k_1 x + B_{k_1} \sin k_1 x, Y = C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y$$

$$\text{and } Z = E_k e^{kz} + F_k e^{-kz}$$

$$\therefore \phi(x, y, z) = (A_{k_1} \cos k_1 x + B_{k_1} \sin k_1 x)(C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y)(E_k e^{kz} + F_k e^{-kz}) \quad \dots(4)$$

$\therefore$ Conditions (B_1) i.e., $\phi(0, y, z) = 0$

$$\Rightarrow A_{k_1} (C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y)(E_k e^{kz} + F_k e^{-kz}) = 0$$

$$\Rightarrow A_{k_1} = 0,$$

$$\therefore C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y \neq 0 \text{ and } E_k e^{kz} + F_k e^{-kz} \neq 0$$

And $\phi(a, y, z) = 0$

$$\Rightarrow (0 + B_{k_1} \sin k_1 a)(C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y)(E_k e^{kz} + F_k e^{-kz}) = 0$$

$$\Rightarrow \sin k_1 a = 0,$$

$$\therefore B_{k_1} \neq 0, C_{k_2} \cos k_2 y + D_{k_2} \sin k_2 y \neq 0, E_k e^{kz} + F_k e^{-kz} \neq 0$$

$$\Rightarrow k_1 a = m\pi, \text{ where } m \text{ is an integer.}$$

$$\text{i.e., } k_1 = m\pi/a$$

Similarly, the conditions (B₂) i.e., $\phi(x, 0, y) = 0$ and $\phi(x, b, y) = 0$

$$\Rightarrow C_{k_2} = 0 \text{ and } k_2 = n\pi/b, \text{ where } n \text{ is an integer.}$$

$$\therefore k^2 = k_1^2 + k_2^2 \Rightarrow k^2 = k_{mn}^2 = (m^2/a^2 + n^2/b^2)\pi^2.$$

Thus, for integral values of m and n , the solutions of (A) from (1) satisfying the conditions (B₁) and (B₂) are given by

$$\begin{aligned} \phi_{mn}(x, y, z) &= \left(B_{k_2} \sin \frac{m\pi x}{a} \right) \left(D_{k_2} \sin \frac{n\pi y}{b} \right) (E_{k_{mn}} e^{k_{mn} z} + F_{k_{mn}} e^{-k_{mn} z}) \\ &= \sin \frac{m\pi x}{a} \cdot \sin \frac{n\pi y}{b} (G_{mn} e^{k_{mn} z} + H_{mn} e^{-k_{mn} z}) \end{aligned}$$

Where $G_{mn} = B_{k_1} D_{k_2} E_{k_{mn}}$ and $H_{mn} = B_{k_1} D_{k_2} F_{k_{mn}}$ are the new arbitrary constants.

Taking $m = 1, 2, 3, \dots$ and $n = 1, 2, 3, \dots$, the more general solution of (A) is given by

$$\begin{aligned} \phi(x, y, z) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn}(x, y, z) \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \cdot \sin \frac{n\pi y}{b} (G_{mn} e^{k_{mn} z} + H_{mn} e^{-k_{mn} z}) \quad \dots(5) \end{aligned}$$

Which also satisfy the conditions (B₁) and (B₂).

$\therefore$ the conditions (B₃) i.e., $\phi(x, y, 0) = 0$

$$\Rightarrow \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (G_{mn} + H_{mn}) = 0$$

$$\Rightarrow G_{mn} + H_{mn} = 0 \Rightarrow H_{mn} = -G_{mn}$$

$\therefore$ from (5), we get

$$\phi(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} G_{mn} (e^{k_{mn}z} - e^{-k_{mn}z})$$

$$\text{or } \phi(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} I_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sinh(k_{mn}z) \quad \dots(6)$$

Where $I_{mn} = 2 G_{mn}$ is the new arbitrary constant. Finally from (6), the condition (B_3) i.e., $\phi(x, y, c) = f(x, y)$.

$$\Rightarrow f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} I_{mn} \sinh(c k_{mn}) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad \dots(7)$$

Here in (7) right hand side can be considered as the double Fourier sine series of the function $f(x, y)$ on its left hand side.

$$\therefore I_{mn} \sinh(c k_{mn}) = \frac{2}{a} \cdot \frac{2}{b} \int_{x=0}^a \int_{y=0}^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy$$

$$\text{or } I_{mn} = \frac{4}{ab \sinh(c k_{mn})} \int_{x=0}^a \int_{y=0}^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \quad \dots(8)$$

$$\text{where } k_{mn}^2 = (m^2/a^2 + n^2/b^2) \pi^2$$

Hence, the required solution of (A) is given by (6) where I_{mn} is given by (8).

Example 2: Find the potential ϕ in the parallelepiped, $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$, satisfying the conditions :

$$(i) \quad \phi = 0 \text{ on } x = 0, x = a, y = 0, y = b \text{ and } z = 0$$

$$(ii) \quad \phi = A \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \text{ on } z = c, 0 \leq x \leq a, 0 \leq y \leq b.$$

[Meerut 2009 (B.P.)]

Solution: The potential $\phi(x, y, z)$ in the parallelepiped satisfies the Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \dots(A)$$

Here we are required to find the solution of (A) satisfying the following conditions

$$\phi(0, y, z) = 0, \quad \phi(a, y, z) = 0 \quad \dots(B_1)$$

$$\phi(x, 0, z) = 0, \quad \phi(x, b, z) = 0 \quad \dots(B_2)$$

$$\text{and} \quad \phi(x, y, 0) = 0, \quad \phi(x, y, c) = f(x, y) = A \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \quad \dots(B_3)$$

Let the solution of (A) be given by

$$\phi(x, y, z) = F(x) Y(y) Z(z) \quad \dots(1)$$

Where F is a function of x alone, Y a function of y alone and Z a function of z alone. Now proceeding as in previous Ex. 1, the required solution is given by

$$\phi(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} I_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sinh(k_{mn}z) \quad \dots(2)$$

$$\text{where} \quad k_{mn}^2 = (m^2/a^2 + n^2/b^2)\pi^2$$

$$\begin{aligned} \text{and} \quad I_{mn} &= \frac{4}{ab \sinh(c k_{mn})} \int_{x=0}^a \int_{y=0}^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \\ &= \frac{4}{ab \sinh(c k_{mn})} \int_{x=0}^a \int_{y=0}^b A \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \\ &= \frac{4A}{ab \sinh(c k_{mn})} \left[\int_{x=0}^a \sin \frac{\pi x}{a} \sin \frac{m\pi x}{a} dx \cdot \int_{y=0}^b \sin \frac{\pi y}{b} \sin \frac{n\pi y}{b} dy \right] \end{aligned}$$

$$\text{Now} \quad I_1 = \int_{x=0}^a \sin \frac{\pi x}{a} \sin \frac{m\pi x}{a} dx = 0 \quad \text{if } m \neq 1$$

$$\text{and when} \quad m=1, I_1 = \int_{x=0}^a \sin^2 \frac{\pi x}{a} dx = \frac{1}{2} \int_0^{\pi} \left(1 - \cos \frac{2\pi x}{a}\right) dx$$

$$= \frac{1}{2} \left[x - \frac{a}{2\pi} \sin \frac{2\pi x}{a} \right]_0^a = \frac{1}{2} a$$

Similarly,
$$I_2 = \int_{y=0}^a \sin \frac{\pi y}{b} \sin \frac{n \pi y}{b} dy = 0, \text{ where } n \neq 1$$

and for $n=1$,
$$I_2 = \frac{1}{2} b$$

$\therefore$ for $m=1, n=1, k_{11}^2 = k_{11}^2 = (1/a^2 + 1/b^2)\pi^2$

and
$$I_{11} = \frac{4A}{ab \sinh(c k_{11})} \cdot \frac{a}{2} \cdot \frac{b}{2} = \frac{A}{\sinh(c k_{11})}$$

and $I_{mn} = 0$ if $m \neq 1$, and $n \neq 1$.

$\therefore$ From (2), the required solution is

$$\begin{aligned} \phi(x, y, z) &= I_{11} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sinh(k_{11} z) \\ &= \frac{A}{\sinh(c k_{11})} \sin \frac{\pi x}{a} \cdot \sin \frac{\pi y}{b} \sinh(k_{11} z) \end{aligned}$$

where $k_{11}^2 = (1/a^2 + 1/b^2)\pi^2$

Example 3: Find the potential ϕ in the parallelopiped $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$, satisfying the conditions

(i) $\phi = 0$ on $x = 0, y = 0, z = 0; x = a, y = b$

(ii) $\phi = A$ on $z = c, 0 \leq x \leq a, 0 \leq y \leq b$.

Solution: The potential $\phi(x, y, z)$ in the parallelopiped satisfies the Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \dots(A)$$

Here we are required to find the solution of (A) satisfying the following conditions

$$\phi(0, y, z) = 0, \quad \phi(a, y, z) = 0 \quad \dots(B_1)$$

$$\phi(x, 0, z) = 0, \quad \phi(x, b, z) = 0 \quad \dots(B_2)$$

$$\text{and} \quad \phi(x, y, 0) = 0, \quad \phi(x, y, c) = f(x, y) = A \quad \dots(B_3)$$

Let the solution of (A) be given by

$$\phi(x, y, z) = F(x) Y(y) Z(z) \quad \dots(1)$$

Where F is a function of x alone, Y a function of y alone and Z a function of z alone. Now proceeding as in Ex. 1 on page 760, the required solution is given by

$$\phi(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} I_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sinh(k_{mn}z) \quad \dots(2)$$

where $k_{mn}^2 = (m^2/a^2 + n^2/b^2)\pi^2$

and

$$\begin{aligned} I_{mn} &= \frac{4}{ab \sinh(c k_{mn})} \int_{x=0}^a \int_{y=0}^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \\ &= \frac{4}{ab \sinh(c k_{mn})} \int_{x=0}^a \int_{y=0}^b A \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \\ &= \frac{4A}{ab \sinh(c k_{mn})} \left[-\frac{a}{m\pi} \cos \frac{m\pi x}{a} \right]_{x=0}^a \left[-\frac{b}{n\pi} \cos \frac{n\pi y}{b} \right]_{y=0}^b \\ &= \frac{4A}{mn\pi^2 \sinh(c k_{mn})} [1 - (-1)^m] \cdot [1 - (-1)^n] \end{aligned}$$

We have

$$1 - (-1)^m = \begin{cases} 0, & \text{when } m = 2r, \quad \text{even, } r = 1, 2, \dots \\ 2, & \text{when } m = 2r - 1, \quad \text{odd, } r = 1, 2, 3, \dots \end{cases}$$

And $1 - (-1)^n = \begin{cases} 0, & \text{when } n = 2s, \quad \text{even, } s = 1, 2, \dots \\ 2, & \text{when } n = 2s - 1, \quad \text{odd, } s = 1, 2, 3, \dots \end{cases}$

$\therefore I_{mn} = 0$ if one or both of m and n are even.

$\therefore$ Taking $m = 2r - 1$ and $n = 2s - 1, r, s = 1, 2, \dots$, from (2) the required solution of (A) satisfying the given conditions is given by

$$\phi(x, y, z) = \frac{16A}{\pi^2} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{\sinh(c k_{rs})}{(2r-1)(2s-1)} \sin \frac{(2r-1)\pi x}{a} \sin \frac{(2s-1)\pi y}{b}$$

where $k_{rs}^2 = \left(\frac{(2r-1)^2}{a^2} + \frac{(2s-1)^2}{b^2} \right) \pi^2$

Example 4: Show that the solutions of Laplace's equation $\nabla^2 u = 0$ in cylindrical coordinates satisfying the conditions :

(i) $u \rightarrow 0$ as $z \rightarrow \infty$ and

(ii) u is finite as $\rho \rightarrow 0$ are of the form

$$u = \sum_m \sum_n G_{mn} J_n(m\rho) e^{-mz \pm i n \phi}.$$

Solution: The Laplace's equation in cylindrical coordinates is

$$\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \dots(A)$$

Here we are required to find the solution of (A), satisfying the following conditions

$$u \rightarrow 0 \text{ as } z \rightarrow \infty \quad \dots(B_1)$$

$$\text{and } u \text{ is finite as } \rho \rightarrow 0. \quad \dots(B_2)$$

Let the solution of (A) be of the form

$$u(\rho, \phi, z) = R(\rho) F(\phi) Z(z) = RFZ \text{ (say)} \quad \dots(1)$$

Where R is a function of ρ alone, F a function of ϕ alone and Z a function of z alone.

Now proceeding similarly as in 14.9, taking m^2 for k^2 and n^2 for m^2 , the solution of (A) is given by

$$u(\rho, \phi, z) = \sum_m \sum_n (A_m e^{mz} + B_m e^{-mz}) e^{\pm i n \phi} \cdot [E_{mn} J_n(m\rho) + F_{mn} Y_n(m\rho)] \quad \dots(2)$$

Since according to condition (B_1) $u \rightarrow 0$ as $z \rightarrow \infty$, so in (2) we must take $A_m = 0$.

$$\therefore \text{ as } z \rightarrow \infty, e^{mz} \rightarrow \infty.$$

Also according to the condition (B_2) u is finite as $\rho \rightarrow 0$, so in (2), we must take $F_{mn} = 0$.

$$\therefore \text{ as } \rho \rightarrow 0, Y_n(m\rho) \rightarrow \infty.$$

Putting $A_m = 0$ and $F_{mn} = 0$ in (2), the required solution is

$$u(\rho, \phi, z) = \sum_m \sum_n G_{mn} e^{-mz} \cdot e^{\pm i n \phi} \cdot J_n(m\rho) = \sum_m \sum_n G_{mn} J_n(m\rho) \cdot e^{-mz \pm i n \phi}.$$

Where $G_{mn} = B_m \cdot E_{mn}$ is a new arbitrary constants.

Example 5: Obtain the axially symmetrical solution of the three dimensional Laplace's equation.

Solution: Laplace's equation in cylindrical coordinates (ρ, ϕ, z) is given by

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \dots(1)$$

If u is symmetrical about the axis of z , then $\frac{\partial^2 u}{\partial \phi^2} = 0$

$$\therefore \quad (1) \text{ reduces to } \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \dots(2)$$

Let the solution of (2) be given by

$$u(\rho, z) = R(\rho) Z(z) = RZ \text{ (say)} \quad \dots(3)$$

Differentiating (3) and substituting in (2), we get

$$Z \frac{d^2 R}{d\rho^2} + \frac{Z}{\rho} \frac{dR}{d\rho} + R \frac{d^2 Z}{dz^2} = 0$$

$$\text{or} \quad \frac{1}{R} \left(\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} \right) = -\frac{1}{Z} \frac{d^2 Z}{dz^2} = -m^2 \text{ (say)} \quad \dots(4)$$

Since right hand side in (4) is a function of independent variable z alone while left hand side is a function of different independent variable ρ . So the two sides will be equal to each other if each is equal to the same constant say $-m^2$.

Thus, from (4) we get the following two ordinary differential equations

$$\frac{d^2 Z}{dz^2} - m^2 Z = 0 \quad \dots(5)$$

$$\text{and} \quad \frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} + m^2 R = 0 \quad \dots(6)$$

General solution of (5) is

$$Z = A_m e^{mz} + B_m e^{-mz} \quad \text{or} \quad Z = A_m e^{\pm mz} \quad \dots(7)$$

Equation (6) is Bessel's equation of zeroth order whose general solution is

$$R = C_m J_0(m\rho) + D_m Y_0(m\rho) \quad \dots(8)$$

Where J_0 and Y_0 are Bessel's functions of zeroth order of first and second kind respectively.

∴ from (3), the solutions of (2), for different values of m are given by

$$u_m(\rho, z) = (A_m e^{mz} + B_m e^{-mz}) [C_m J_0(m\rho) + D_m Y_0(m\rho)]$$

Hence, more general solution of (2) is given by

$$u(\rho, z) = \sum_m u_m(\rho, z) = \sum_m (A_m e^{mz} + B_m e^{-mz}) [C_m J_0(m\rho) + D_m Y_0(m\rho)] \quad \dots(9)$$

This, solution can also be written as

$$\begin{aligned} u(\rho, z) &= \sum_m A_m e^{\pm mz} \cdot [C_m J_0(m\rho) + D_m Y_0(m\rho)] \\ &= \sum_m e^{\pm mz} [E_m J_0(m\rho) + F_m Y_0(m\rho)] \end{aligned}$$

Where $E_m = A_m C_m$ and $F_m = A_m D_m$ are the new arbitrary constants.

Note : In (9) when $\rho \rightarrow 0$, $Y_0(m\rho) \rightarrow \infty$. Thus for the solution $u(\rho, z)$ given by (9), to remain finite along the line $\rho = 0$ at the origin, we must take $D_m = 0$.

Hence, the solution of the Laplace's equation which is symmetrical about z -axis and remains finite on the line $\rho = 0$ is given by [putting $D_m = 0$ in (9)]

$$u(\rho, z) = \sum_m (G_m e^{mz} + H_m e^{-mz}) \cdot J_0(m\rho)$$

where $G_m = A_m \cdot C_m$ and $H_m = B_m C_m$ are the new arbitrary constants.

This solution can also be written as

$$u(\rho, z) = \sum_m G_m e^{\pm mz} \cdot J_0(m\rho).$$

Exercise 14.3

1. Find the potential ϕ in the cuboid $0 \leq x \leq a, 0 \leq y \leq a, 0 \leq z \leq a$, satisfying the conditions.
 - (i) $\phi = 0$, on $x = 0 = y = z, x = a = y$
 - (ii) $\phi = f(x, y)$ on $z = c, 0 \leq x \leq a, 0 \leq y \leq a$.
2. Find the potential ϕ in the parallelopiped $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$, satisfying the conditions.
 - (i) $\phi = 0$ on $x = 0 = y = z, x = a$ and $y = b$
 - (ii) $\phi = A \sin \frac{\pi x}{a} \sin \frac{2\pi y}{b}$ on $z = c, 0 \leq x \leq a, 0 \leq y \leq b$.
3. Find the potential ϕ in the cuboid $0 \leq x \leq a, 0 \leq y \leq a, 0 \leq z \leq a$, satisfying the conditions.
 - (i) $\phi = 0$, on $x = 0 = y = z, x = a = y$
 - (ii) $\phi = A$, on $z = c, 0 \leq x \leq a, 0 \leq y \leq a$.
4. Obtain the general solution of Laplace's equation in cylindrical coordinates which remain finite on the axis of z and is symmetrical about it.
[Hint : See note of Ex. 5 on page 769].
5. Obtain the solution of Laplace's equation in cylindrical coordinates which is symmetrical about the z -axis and tends to zero as $\rho \rightarrow 0$ and as $z \rightarrow \infty$.
[Hint : Take $G_m = 0$ in Q.N. 4.].

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter (a), (b), (c) or (d).

1. When the temperature is in steady state, the heat equation becomes :

(a) Wave equation	(b) Laplace equation
(c) Bessel's equation	(d) None of these
2. Laplace's equation is also called :

(a) Harmonic equation	(b) Wave equation
(c) Legendre equation	(d) None of these
3. Consider the steady state temperature $u(x, y)$ in a rectangular plate of length a and width b , the sides of which are kept at temperature zero, the lower end is kept at temperature $f(x)$ and the upper edge is insulated. Then $u(x, y)$ satisfy the equation :

- (a) $\frac{\partial^2 u}{\partial x^2} = 0$ (b) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}$
 (c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ (d) None of these

4. The boundary conditions of the boundary value problem given in Q.N. 3. are :

- (a) $u(0, y) = 0 = u(a, y) = u_y(x, b), u(x, 0) = f(x)$
 (b) $u(0, y) = 0 = u(a, y), u(x, 0) = 0 = u(x, b)$
 (c) $u(0, y) = 0 = u(a, y), u(x, 0) = f(x) = u(x, b)$
 (d) None of these

5. Let $u(x, y)$ be steady state temperature function in a square plate $0 \leq x < \pi$, $0 \leq y < \pi$, the sides of which are kept at temperature zero, the lower edge is kept at temperature $f(x)$ and upper edge at temperature v_0 . Then $u(x, y)$ satisfy the equation :

- (a) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$ (b) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
 (c) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}$ (d) None of these

6. The boundary conditions of the problem in Q.N. 5 are :

- (a) $u(0, y) = 1 = u(\pi, y), u(x, 0) = v_0 = u(x, \pi)$
 (b) $u(0, y) = v_0 = u(\pi, y), u(x, 0) = f(x) = u(x, \pi)$
 (c) $u(0, y) = 0 = u(\pi, y), u(x, 0) = f(x), u(x, \pi) = v_0$
 (d) None of these

7. The Laplace equation in polar coordinates is given by

- (a) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial^2 u}{\partial \theta^2} = 0$ (b) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$
 (c) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r^2} \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial^2 u}{\partial \theta^2} = 0$ (d) None of these

8. The boundary diameter of a semi-circular plate of radius a is kept at 0°C and its temperature along the semi-circular boundary is given by $f(\theta)$. Then the steady state temperature function $u(r, \theta)$ satisfy :

- (a) $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$ (b) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$
 (c) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$ (d) $\frac{\partial^2 u}{\partial r^2} - \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0$

9. The boundary conditions of the problem in Q.N. 8. are given by :
- $u(r, 0) = 0, -a \leq r \leq a, u(a, \theta) = f(\theta), 0 < \theta < 2\pi$
 - $u(r, 0) = 0 = u(r, \pi), 0 \leq r \leq a, u(a, \theta) = f(\theta), 0 < \theta < \pi$
 - $u(r, 0) = 0 = u(r, \pi), 0 \leq r \leq a, u(a, 0) = f(\theta)$
 - None of these
10. $u(r, \theta)$ is the steady state temperature distribution in a circular plate of radius 10 cm, which has half of its circumference at 0°C and the other half at temperature $v_0^\circ\text{C}$. Then the boundary conditions of the problem is/are :
- $u(10, \theta) = 0, 0 < \theta < 2\pi$
 - $u(10, \theta) = v_0, 0 < \theta < 2\pi$
 - $u(10, \theta) = v_0, 0 < \theta < \pi, u(10, \theta) = 0, \pi < \theta < 2\pi$
 - None of these

Fill in the Blank

Fill in the blanks "....." so that the following statements are complete and correct.

- The Laplace's equation is given by $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \dots\dots\dots$.
- The Laplace's equation in cylindrical coordinates is given by $\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial u}{\partial \rho} + \dots\dots\dots + \frac{\partial^2 u}{\partial z^2} = 0$.
- The Laplace's equation in spherical coordinates is given by $\frac{\partial^2 u}{\partial r^2} + \dots\dots\dots + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} = 0$.
- The potential ϕ in the parallelopiped $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$, satisfy the equation
- The surfaces of a sector determined by $0 \leq r \leq a, 0 \leq \theta \leq \pi/4$ are insulated. The temperature is kept at 0°C along the straight edges and at $f(\theta)$ along the current edge. Then the boundary conditions of the problem are $u(r, 0) = 0 = u(r, \pi/4), 0 < r < a$ and

Answers

Exercise 14.3

1.	$\phi(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} I_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} \sinh(k_{mn}z)$ where $k_{mn}^2 = (m^2 + n^2)\pi^2/a^2$ and $I_{mn} = \frac{4}{a^2 \sinh(c k_{mn})} \int_{x=0}^a \int_{y=0}^a f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} dx dy$
2.	$\phi(x, y, z) = A \operatorname{cosech}(c k_{12}) \sin \frac{\pi x}{a} \sin \frac{2\pi y}{a} \sinh(k_{12}z)$ where $k_{12}^2 = (1/a^2 + 4/b^2)\pi^2$.
3.	$\phi(x, y, z) = \frac{16A}{\pi^2} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{\sinh(c k_{rs})}{(2r-1)(2s-1)} \sin \frac{(2r-1)\pi x}{a} \sin \frac{(2s-1)\pi y}{a}$ where $k_{rs}^2 = [(2r-1)^2 + (2s-1)^2]\pi^2/a^2$.
4.	$u(\rho, z) = \sum_m (G_m e^{mz} + H_m e^{-mz}) J_0(m\rho)$ or $u(\rho, z) = \sum_m G_m e^{\pm mz} J_0(m\rho).$
5.	$u(\rho, z) = \sum_m H_m e^{-mz} J_0(m\rho).$

Objective Type Questions

Multiple Choice Questions

1.	(b)	2.	(a)	3.	(c)	4.	(a)
5.	(b)	6.	(c)	7.	(b)	8.	(a)
9.	(b)	10.	(c)				

Fill in the Blank

1.	0	2.	$\frac{1}{\rho^2} \frac{\partial^2 u}{\partial \phi^2}$
3.	$\frac{2}{r} \frac{\partial u}{\partial r}$	4.	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$
5.	$u(a, \theta) = f(\theta), 0 < \theta < \pi/4$		

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