Krishna's TEXT BOOK on

Real Analysis

(For B.A. and B.Sc. II[nd] year students of All Colleges affiliated to Siddharth University, Kapilvastu Siddharth Nagar in Uttar Pradesh)

As per Latest Syllabus of Siddharth University, Kapilvastu Siddharth Nagar (U.P.)

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Siddharth Edition

KRISHNA Prakashan Media (P) Ltd., Meerut
KRISHNA HOUSE, 11, Shivaji Road, Meerut-250 001 (U.P.), India

Jai Shri Radhey Shyam

Dedicated

to

Lord

Krishna

Authors & Publishers

Preface

This book on **REAL ANALYSIS** has been specially written according to the **Latest Syllabus** to meet the requirements of the **B.A. and B.Sc. Part-II Students** of all colleges affiliated to **D.D.U. Gorakhpur University, Gorakhpur & Shri Siddhartha University, Siddharth Nagar in U.P.**.

The subject matter has been discussed in such a simple way that the students will find no difficulty to understand it. The proofs of various theorems and examples have been given with minute details. Each chapter of this book contains complete theory and a fairly large number of solved examples. Sufficient problems have also been selected from various university examination papers. At the end of each chapter an exercise containing objective questions has been given.

We have tried our best to keep the book free from misprints. The authors shall be grateful to the readers who point out errors and omissions which, inspite of all care, might have been there.

The authors, in general, hope that the present book will be warmly received by the students and teachers. We shall **indeed** be very thankful to our colleagues for their recommending this book to their students.

The authors wish to express their thanks to **Mr. S.K. Rastogi** *(M.D.)*, **Mr. Sugam Rastogi** *(Executive Director)*, **Mrs. Kanupriya Rastogi** *(Director)* and **entire team** of **KRISHNA Prakashan Media (P) Ltd., Meerut** for bringing out this book in the present nice form.

The authors will feel amply rewarded if the book serves the purpose for which it is meant. Suggestions for the improvement of the book are always welcome.

— Authors

Syllabus

Real Analysis

D.D.U. Gorakhpur University, Gorakhpur &
Shri Siddharth University, Siddharth Nagar

B.A./B.Sc. IInd Year

Section-A

Dedekind's definition of real numbers Addition, subtraction and multiplication of real numbers. Section of sets of real numbers. Lower and upper bounds. Supremum and infimum of the subsets of R. Completeness of R. (2 questions)

Definition of a sequence. Theorems on limits of sequences. Bounded and monotonic sequences. Convergence of a sequence. Cauchy's convergence criteria. Completeness of R. Bolzano Weierstrass criteria. Limit superior and limit inferior.

Convergence of a series. Series of non-negative terms. The number "e" as an irrational number. Comparison test. Cauchy's nth root test. Ratio test. Raabe's test. Logarithmic, De Morgan and Bertrand's tests. Alternating series. Leibnitz test. Absolute and conditional convergences. (2 questions)

Section-B

Continuous functions and their properties. Classification of discontinuities. Sequential continuity and limits. Uniform convergence. Differentiability, Chain rule of differentiability.

Rolle's theorem. Lagrange's and Cauchy's mean value theorems. Darboux theorem. Indeterminate forms. (2 questions)

Riemann integral. Integrability of continuous and monotonic functions. The fundamental theorem of Integral Calculus. Mean Value theorems of Integral calculus. Riemann- Stiltjes integration.

Improper integrals and their convergence Comparison tests. (2 questions)

Brief Contents

Krishna's

REAL ANALYSIS

Chapters

Chapter 1

Real Number System

(Axiomatic Study of Real Numbers)

1 Introduction

The real number system is the foundation on which the whole branch of mathematics known as 'Real Analysis' rests. The beauty of this mathematical system lies in the fact that for many important abstract mathematical systems the structure of real number system serves as a model. Its knowledge is absolutely necessary for any worthwhile student of mathematics. We shall here make a formal description of the real number system with axioms, definitions of some terms and statements of some theorems. There are different ways of introducing the real number system. We shall consider the real numbers as *undefined objects* satisfying certain axioms. These axioms will characterize the real number system.

We assume that the reader is familiar with the real numbers and the binary composition of addition and multiplication possessing a number of properties and the manner of dealing with inequalities involving real numbers. We shall select those properties as axioms concerning the real number system from which all the other properties of the real numbers can be proved. The axioms are divided into three categories :

(1) *Field axioms*, (2) *Order axioms* and

(3) *Completeness axiom*.

2 Field Axioms

Let **R** be the set of real numbers having at least two distinct elements equipped with two fundamental algebraic operations called addition and multiplication and denoted by '+' and '.' respectively. These operations satisfy the following axioms:

A_1. The set **R** is closed with respect to addition *i.e.*, $a + b$ is a unique real number for any two real numbers a and b. **(Gorakhpur 2013)**

A_2. Addition is associative, *i.e.*,

$$(a + b) + c = a + (b + c) \ \forall \ a, b, c \in \mathbf{R}.$$

A_3. Addition is commutative *i.e.*, $a + b = b + a \ \forall \ a, b \in \mathbf{R}$.

A_4. There exists an element 0 in **R** such that $0 + a = a \ \forall \ a \in \mathbf{R}$.

A_5. To each element a in **R** there exists an element $-a$ in **R** such that $-a + a = 0$.

M_1. The set **R** is closed with respect to multiplication *i.e.*, $a \,.\, b$ is a unique real number for any two real numbers a and b.

M_2. Multiplication is associative *i.e.*,

$$a \,.\, (b \,.\, c) = (a \,.\, b) \,.\, c \ \forall \ a, b, c \in \mathbf{R}.$$

M_3. Multiplication is commutative *i.e.*,

$$a \,.\, b = b \,.\, a \ \forall \ a, b, c \in \mathbf{R}.$$

M_4. There exists an element namely $1 \neq 0$ in **R** such that

$$1 \,.\, a = a \ \forall \ a \in \mathbf{R}.$$

M_5. To each element $a \neq 0$ in **R** there exists an element $1/a$ in **R** such that

$$\frac{1}{a} \cdot a = 1.$$

The real number $1 / a$ is also denoted by a^{-1}.

Distributive law: Multiplication is distributive with respect to addition *i.e.*,

$$a \,.\, (b + c) = a \,.\, b + a \,.\, c \ \forall \ a, b, c \in \mathbf{R}.$$

Because of the above properties the algebraic structure **(R, +, .) is called a field. As a matter of fact any mathematical system satisfying the above axioms is called a field. Thus we may speak of the field Q of rational numbers or the field C of complex numbers.**

The real number 0 is the identity element for addition and the number $-a$ is the additive inverse of the real number a and is usually called the negative of a. The real number 1 is the identity element for multiplication and the real number $1 / a$ or a^{-1} is the multiplicative inverse of the real number a and is usually called the *reciprocal* of a.

3 Subtraction and Division in R

Definition 1: *The difference between two real numbers a and b is defined by $a + (-b)$ and is denoted by $a - b$.*

The operation of finding the difference is called **subtraction**.

Definition 2: *The* **quotient** *of a real number a by a real number b ($b \neq 0$) is defined by $a \,.\, b^{-1}$, and is denoted by*

$\frac{a}{b}$ or a / b or $a \div b$.

The operation of finding quotient is called **division**.

Some Remarks: 1. In general $a - b \neq b - a$ and $a / b \neq b / a$.

2. Division by 0 is not allowed.

3. Though a / b has meaning, b / a may not be defined.

4 Some Properties of Real Numbers

We shall state some important consequences of the field properties of real numbers.

1. There exists a *unique* identity element for addition in **R**. **(Gorakhpur 2013)**

2. There exists a *unique* additive inverse for each element in **R**.

3. $a + b = a + c \Rightarrow b = c$.

4. $a + b = a \Rightarrow b = 0$.

5. $a + b = 0 \Rightarrow b = -a$.

6. $-(-a) = a$.

7. There exists a unique identity element for multiplication in **R**.

8. There exists a unique multiplicative inverse for each non-zero element in **R**.

9. $a \neq 0, a \,.\, b = a \,.\, c \Rightarrow b = c$.

10. $a \neq 0, a \,.\, b = a \Rightarrow b = 1$.

11. $a \neq 0, a \,.\, b = 1 \Rightarrow b = 1 / a$.

12. $a \neq 0 \Rightarrow \dfrac{1}{(1 / a)} = a$.

13. $a \,.\, 0 = 0 \;\forall\; a \in \mathbf{R}$.

14. $a \neq 0, b \neq 0 \Rightarrow a \,.\, b \neq 0$.

15. $a \,.\, b = 0 \Leftrightarrow a = 0$ or $b = 0$.

16. $a \,.\, (-b) = -(a \,.\, b)$ and $(-a) \,.\, b = -(a \,.\, b)$.

17. $(-a) \,.\, (-b) = a \,.\, b$.

18. $(-1) \,.\, a = -a$.

19. $-(a + b) = -a - b$.

20. $\dfrac{1}{a \,.\, b} = \left(\dfrac{1}{a}\right) \cdot \left(\dfrac{1}{b}\right), a \neq 0, b \neq 0$.

21. If a and b are any two real numbers, then the equation $x + a = b$ has a unique solution $x = [b + (-a)] = b - a$ in **R**.

22. If a, b are any real numbers and $a \neq 0$, then the equation $ax = b$ has a unique solution $x = (1 / a) \,.\, b = b / a$ in **R**.

5 Integral Powers of a Real Number

Let $a \in \mathbf{R}$.

If n is a positive integer, we define $a^n = a \,.\, a \,.\, a \ldots a$ to n factors. In particular,

$$a^1 = a, a^2 = a \,.\, a, a^3 = a \,.\, a \,.\, a = a^2 \,.\, a \text{ and so on.}$$

We define $a^0 = 1$.

If n is a positive integer, then $-n$ is a negative integer. For $a \neq 0$, we define $a^{-n} = (a^n)^{-1}$ where $(a^n)^{-1}$ is the multiplicative inverse of a^n in **R**.

We are free to write $(a^{-1})^n$ or $(a^n)^{-1}$ in place of a^{-n}.

6 The Order Axioms

The order relation 'greater than' ($>$) between pairs of real numbers satisfies the following axioms :

O$_1$. For any two real numbers a, b one and only one of the following is true :

$$a > b, a = b, b > a.$$

It is known as the *law of trichotomy*.

O$_2$. For $a, b, c \in \mathbf{R}$, $a > b, b > c \Rightarrow a > c$. It is known as the *law of transitivity*.

O$_3$. For all real numbers a, b and $c, a > b \Rightarrow a + c > b + c$. It is known as *monotone property for addition*.

O$_4$. For all real numbers a, b and c, $a > b$ and $c > 0 \Rightarrow ac > bc$. It is known as *monotone property for multiplication*.

Because of these properties the field of real numbers is an *ordered field*.

The system **Q** of all rational numbers is an ordered field while the system **C** of all complex numbers is a field which is not ordered.

7 Some More Definitions

We define some other relations in terms of the relation 'greater than' on the real numbers.

1. The *order relation* '**less than**' ($<$) between the real numbers a and b is defined as
$$a < b \quad \text{if } b > a.$$
2. A real number a is said to be **greater than or equal** to b ($a \geq b$) if either $a > b$ or $a = b$.
3. A real number a is said to be **less than or equal** to b ($a \leq b$) if either $a < b$ or $a = b$.
4. A real number a is said to be **positive** if $a > 0$.
5. A real number a is said to be negative if $a < 0$.

   The sets of all positive real numbers and all negative real numbers are denoted by $\mathbf{R}^+$ and $\mathbf{R}^-$ respectively.

   Hence $\mathbf{R} = \mathbf{R}^+ \cup \{0\} \cup \mathbf{R}^-$

Some properties of order relation

1. For each real number a, one and only one of the following holds :
$$a > 0, a = 0, \; -a > 0.$$

2. For each real number a, one and only one of the following holds :

$$a < 0, a = 0, -a < 0.$$

3. (i) $a \in \mathbf{R}^+ \Leftrightarrow a > 0$ and $a \in \mathbf{R}^- \Leftrightarrow a < 0$.

   (ii) $a \in \mathbf{R}^+, b \in \mathbf{R}^- \Rightarrow a > b$ *i.e.* every positive number is greater than every negative number.

4. $a, b \in \mathbf{R}^+ \Rightarrow a + b \in \mathbf{R}^+$ and $ab \in \mathbf{R}^+$ *i.e.*,

$$a > 0, b > 0 \Rightarrow a + b > 0 \text{ and } ab > 0.$$

5. $a, b \in \mathbf{R}^- \Rightarrow a + b \in \mathbf{R}^-$ and $ab \in \mathbf{R}^+$.
6. $a < b$ and $b < c \Rightarrow a < c$.
7. $a < b \Leftrightarrow a + c < b + c$,

   $a < b$ and $c < 0 \Rightarrow ac > bc$.
8. $a < 0 \Leftrightarrow -a > 0, a > 0 \Leftrightarrow -a < 0$.
9. $a > b \Leftrightarrow a - b > 0, a < b \Leftrightarrow a - b < 0$.
10. $a > b \Leftrightarrow -a < -b$.
11. $a > 0 \Leftrightarrow \dfrac{1}{a} > 0$.
12. (i) $a > b > 0 \Rightarrow \dfrac{1}{b} > \dfrac{1}{a} > 0$.

    (ii) $0 < a < b \Rightarrow \dfrac{1}{a} > \dfrac{1}{b}$.
13. $a \neq 0 \Rightarrow a^2 > 0$. In particular $1 > 0$.
14. $a > b > 0 \Rightarrow a^2 > b^2, a < b < 0 \Rightarrow a^2 > b^2$.

The relations '$\geq$' and '$\leq$' are known as the *weak inequalities* while the relations '>' and '<' are known as the *strict inequalities*.

8 The Extended Real Number System – Finite and Infinite Sets

It is often convenient to extend the system of the real numbers by the addition of two elements ∞ and $-\infty$. The enlarged set is called the set of extended real numbers. We preserve the original order in **R** and define the use of ∞ and $-\infty$ in combination with real numbers by the symbols $+, -, \times, \div, <, >$ as follows :

If a is any real number, then

$$-\infty < a < \infty, a + \infty = \infty + a = -a + \infty = \infty;$$

$$a - \infty = -\infty + a = -\infty - a = -\infty;$$

$$\frac{a}{\infty} = 0; \frac{\infty}{a} = \infty \times a = a \times \infty = \begin{cases} \infty & \text{if } a > 0 \\ -\infty & \text{if } a < 0. \end{cases}$$

Further $\infty \times \infty = (-\infty) \times (-\infty) = \infty + \infty = \infty$.

$\infty \times (-\infty) + (-\infty) \times \infty = -\infty - \infty = -\infty$.

The following combinations remain meaningless.

$$\infty - \infty, -\infty + \infty, 0 \times \infty, \infty \times 0, \frac{\infty}{\infty}.$$

Finite and Infinite Sets: A set is said to be finite if it consists of a specific number of distinct elements, i.e., if in counting the distinct members of the set, the counting process can come to an end. Otherwise, a set is infinite.

Illustrations: **1.** Let **A** be the set of the days of the week. Then **A** is finite.

2. Let $\mathbf{B} = \{1, 3, 5, 7, 9, \ldots\ldots\}$ be the set of odd natural numbers. Then **B** is infinite.

3. The set **Q** of rational numbers is an infinite set.

4. Let $\mathbf{D} = \{x : x \text{ is a river on earth}\}$. Although it is difficult to count the number of rivers on the earth, **D** is still a finite set. If we ever count the number of rivers on the earth, the counting process will definitely come to an end.

5. The set of all points in a plane is an infinite set.

9 Some Important Subsets of *R*

1. The set of natural numbers:

Inductive set: Definition: *A subset S of* **R** *is called an inductive set if*

(i) $1 \in S$ *and* *(ii)* $p \in S \Rightarrow p + 1 \in S.$

For example **R** itself is an inductive set.

The set **N** of natural numbers is the smallest inductive subset of **R**.

By inductive hypothesis, $1 \in N \Rightarrow 1 + 1 \in \mathbf{N}$ and is denoted by 2. Now $2 \in \mathbf{N} \Rightarrow 2 + 1 \in \mathbf{N}$ and is denoted by 3. Continuing in this way, we have

$$\mathbf{N} = \{1, 2, 3, 4, 5, \ldots\}$$

The set of natural numbers is also called the set of positive integers and is also denoted by $\mathbf{Z}^+$.

Principle of Mathematical Induction

A proposition $P(m)$ is true for all $m \in \mathbf{N}$ provided :

(i) $P(1)$ is true *i.e.*, the proposition is true when $m = 1$, and

(ii) $\forall\ k \in \mathbf{N}$, $P(k)$ is true implies $P(k + 1)$ is true *i.e.*, if the proposition is true for any $k \in \mathbf{N}$ then it is also true for $k + 1 \in \mathbf{N}$.

The natural numbers form a limited system because the algebraic operations on **N** do not satisfy the field axioms $\mathbf{A_4}, \mathbf{A_5}$ and $\mathbf{M_5}$.

2. Integers: The subset of **R** containing all the natural numbers, their additive inverses *i.e.*, negatives and the additive identity *i.e.*, 0, is called the set of integers and is denoted by **Z** or **I**.

Thus $\mathbf{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Hence $\mathbf{N} \subset \mathbf{Z} \subset \mathbf{R}.$

The algebraic operations on **Z** do not satisfy the field axiom $\mathbf{M_5}$.

3. Rational numbers: The set of rational numbers contains all the real numbers of the form p / q, where p, q are integers and $q \neq 0$. It is denoted by **Q**. Thus

$$\mathbf{Q} = \{p / q : p, q \in \mathbf{Z}, q \neq 0\}.$$ **(Gorakhpur 2015)**

It should be noted that every integer is a rational number but the converse is not true. For example, 5 is an integer and we can write $5 = \frac{5}{1}$ and so 5 is also a rational number.

On the other hand $\frac{2}{3}$ is a rational number but not an integer.

Hence $\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R}$.

The system $(Q, +, .)$ is an ordered field.

4. Irrational numbers: Any real number which is not rational, is called an irrational number. The set $\mathbf{R} - \mathbf{Q}$ is the set of irrational numbers. **(Gorakhpur 2010)**

Illustrative Examples

Example 1: *Show that* $\sqrt{2}$ *is irrational.*

Solution: Suppose there is a rational number p / q, where $q \neq 0$, whose square is 2. Suppose also that p and q do not have 2 as a common factor because if such a factor exists then it can be cancelled.

Now $(p / q)^2 = 2 \Rightarrow p^2 = 2q^2$.

Since q is an integer, so is q^2 and $2q^2$. Therefore p^2 is an integer divisible by 2.

Now 2 is a prime integer and 2 is a divisor of p^2 *i.e.*, pp.

$\therefore$ 2 must be a divisor of p.

[Note that if a prime number a is a divisor of bc, where b and c are integers, then a must be a divisor of b or a must be a divisor of c].

Since 2 is a factor of p, let $p = 2n \Rightarrow p^2 = 4n^2$.

Now $p^2 = 4n^2 \Rightarrow 2q^2 = 4n^2 \Rightarrow q^2 = 2n^2 \Rightarrow 2$ is a factor of q^2

$\Rightarrow$ 2 is a factor of q.

Thus we conclude that p and q have 2 as a common factor which is contrary to our hypothesis. Therefore there exists no rational number whose square is 2. Hence we cannot put $\sqrt{2}$ in the form p/q. This shows that $\sqrt{2}$ is not rational. Hence $\sqrt{2}$ is irrational.

Example 2: *Prove that no positive integer other than a square number has a square root within the set of rational number.*

Solution: Let m be a positive integer which is not the square of any integer. We are to prove that there exists no rational number x such that $x^2 = m$.

Suppose there is a rational number p / q whose square is m. Suppose also that p and q are positive integers relatively prime to each other because if p and q have any common factor then it can be cancelled.

Now $\qquad (p/q)^2 = m \Rightarrow p^2 = mq^2$.

Let a be any prime factor of q.

Then a is a divisor of p^2.

$\Rightarrow \qquad a$ is a divisor of $pp \Rightarrow a$ is a divisor of p.

Thus a is a prime number which is a common factor of p and q and so p and q are not relatively prime which contradicts our assumption. Therefore no prime number can be a factor of q and so q must be equal to 1. Then

$$p^2 = m$$

i.e., m is the square of the integer p. This contradicts the hypothesis that m is not the square of any integer. Hence there exists no rational number p/q whose square is m.

Example 3: *Show that* $\sqrt{8}$ *is not a rational number.*

Solution: If possible, let $\sqrt{8}$ be the rational number p/q, where $q \neq 0$ and p, q are positive integers prime to each other.

Now $\qquad 2 < \sqrt{8} < 3$.

$\therefore \qquad 2 < p/q < 3 \Rightarrow 2q < p < 3q \Rightarrow 0 < p - 2q < q$.

Thus $p - 2q$ is a positive integer less than q, so that $(\sqrt{8})\,(p-2q)$ *i.e.*, $(p/q)(p-2q)$ is not an integer.

But $\qquad (\sqrt{8})\,(p-2q) = \dfrac{p}{q}(p-2q) = \dfrac{p^2}{q} - 2p = \dfrac{p^2}{q^2}\cdot q - 2p = 8q - 2p,$

which is an integer. Thus we arrive at a contradiction.

Hence $\sqrt{8}$ is not a rational number.

10 Intervals

Interval: Definition: *A subset* S *of* $\mathbf{R}$ *is called an interval if* $a, b \in S$, $x \in \mathbf{R}, a < x < b \Rightarrow x \in S$.

If a and b are any two real numbers such that $a \leq b$, then the set $\{x \in \mathbf{R} : a < x < b\}$ is called an **open interval** and is denoted by $]\,a, b\,[$ or by (a, b). Here both the end points a and b do not belong to the interval. If to the open interval $]\,a, b\,[$, we add the real numbers a and b, we get the **closed interval** $[a, b]$.

Thus $[a, b] = \{\, x \in \mathbf{R} : a \leq x \leq b\}$. It should be noted that

$$[a, b] = \{a\} \cup\,]\,a, b\,[\, \cup \{b\}.$$

If $a = b$, then $]\,a, a\,[\, = \varnothing$ and $[a, a] = \{a\}$.

In the case of the closed interval $[a, b]$ both the end points a and b belong to the interval. Out of the two end point a and b, a is known as the **left end point** whreas b is known as the **right end point**.

The sets $\{\, x \in \mathbf{R} : a < x \leq b\}$ and $\{\, x \in \mathbf{R} : a \leq x < b\}$ are called the **semi-open** or **semi-closed** intervals and are denoted respectively by the symbols $]\,a, b\,]$ and $[a, b\,[$ or by the symbols $(a, b]$ and $[a, b)$. The interval $[a, b[$ is called **a right half open interval** and the interval $]a, b]$ is called *a* **left half open interval**.

The sets $\quad]a, \infty[= \{x \in \mathbf{R} : x > a\}$

and $\quad]-\infty, a[= \{x \in \mathbf{R} : x < a\}$

are called **open rays**.

The sets $\quad [a, \infty[= \{x \in \mathbf{R} : x \geq a\}$

and $\quad]-\infty, a] = \{x \in \mathbf{R} : x \leq a\}$

are called **closed rays**.

Since
$$\begin{aligned}\mathbf{R} &= \{x : x \leq 0\} \cup \{x : x \geq 0\}\\ &=]-\infty, 0] \cup [0, \infty[,\end{aligned}$$

therefore we can write **R** in the form of an open interval as

$$\mathbf{R} =]-\infty, \infty[.$$

Length of an interval: For each interval whose end points are any real numbers a and b such that $a \leq b$, the length of the interval is $b - a$. Obviously the length of each of the intervals $]a, b[$, $[a, b]$, $[a, b[$ and $]a, b]$ is $b - a$. These intervals are called **finite intervals** because the length of each of them is finite. The intervals $]a, \infty[$, $[a, \infty[$, $]-\infty, a[$, $]-\infty, a]$ and $]-\infty, \infty[$ are called **infinite intervals** because the length of each of them is infinite.

11 Finite and Infinite Subsets of R

A subset S of **R** *is said to be finite if either it is empty or there exists a one-to-one mapping from the set* $\{1, 2, \ldots, n\}$ *onto the set S for some natural number n.*

A subset S of **R** *which is not finite is called* **infinite**.

Illustrations: **1.** The sets **N**, **Z**, **Q**, **R** are infinite.

2. The set $\{e, \pi, \sqrt{2}\}$ is a finite set because from the set $\{1, 2, 3\}$ onto the set $\{e, \pi, \sqrt{2}\}$ many one-to-one mappings exist. One such mapping is $1 \to e$, $2 \to \pi, 3 \to \sqrt{2}$.

3. The set $\left\{1, \frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^{100}}\right\}$ is a finite set.

4. The set of all primes less than 10^{100} is a finite set.

5. If $a, b \in \mathbf{R}$ and $a < b$, then all the intervals $]a, b[$, $[a, b]$, $[a, b[$, $]a, b]$, $]-\infty, a[$, $]-\infty, a]$, $]a, \infty[$, $[a, \infty[$ are infinite subsets of **R**. It should be noted that the interval $]a, b[$ is called a finite interval in the sense that its length $b - a$ is finite though as a subset of **R** it is an infinite subset of **R**.

12 Absolute Value (Modulus of a Real Number)

Definition: *If* $x \in \mathbf{R}$, *then its absolute value, denoted by* $|x|$, *is defined as follows :*

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

It should be observed that $|x|$ is defined for every $x \in \mathbf{R}$. Also $a = b \Rightarrow |a| = |b|$ but $|a| = |b|$ does not necessarily imply that $a = b$. For example, $|-5| = |5| = 5$ but $-5 \neq 5$.

Also it is easy to see that $|x| = 0 \Leftrightarrow x = 0$.

From the definition of $|x|$, it is obvious that $|x| = -x$ if $x \leq 0$. For, if $x = 0$, we have $|x| = 0$ and $-0 = 0$.

Theorem 1: *For every* $x \in \mathbf{R}$,

(*i*) $|x| \geq 0$. (*ii*) $|x| = max. \{-x, x\}$.

(*iii*) $|x| \geq x$. (*iv*) $x \geq -|x|$.

(*v*) $|x| = |-x|$. (*vi*) $|x|^2 = x^2 = |-x|^2$.

Proof: (i) If $x \in \mathbf{R}$, then by the law of trichotomy, one and exactly one of the following is true :

$$x > 0, x = 0, x < 0.$$

If $x \geq 0$, then by the definition of $|x|$, we have

$$|x| = x \geq 0. \qquad \text{...(1)}$$

Again if $x < 0$, then by the definition of $|x|$, we have

$$|x| = -x > 0. \qquad \text{...(2)}$$

Note that if $x < 0$ *i.e.*, x is $-$ ive, then $-x > 0$ *i.e.*, $-x$ is $+$ I've.

From (1) and (2), we conclude that

$$|x| \geq 0 \quad \forall \; x \in \mathbf{R}.$$

(ii) If $x \geq 0$, then by the definition of $|x|$, we have

$$|x| = x, \text{ and } x \geq -x.$$

Again if $x < 0$, then by the definition of $|x|$, we have

$$|x| = -x, \text{ and } -x > x.$$

Thus in either case, $|x|$ is the greater of the two numbers x and $-x$.

Hence $\qquad |x| = \max. \{x, -x\} \; \forall \; x \in \mathbf{R}$.

(iii) If $x \geq 0$, then by the definition of $|x|$, we have

$$|x| = x \geq x.$$

Again if $x < 0$, then by the definition of $|x|$, we have

$$|x| = -x > x.$$ [Note that if x is $-$ I've, then $-x$ is $+$ I've and so $-x > x$].

Thus in either case, we have $|x| \geq x$.

Hence $\qquad |x| \geq x \quad \forall \; x \in \mathbf{R}$.

(iv) If $x \geq 0$, then by the definition of $|x|$, we have

$$|x| = x. \quad \text{But } |x| \geq -|x| \text{ and so } x \geq -|x|.$$

Again if $x < 0$, then $|x| = -x$ and so

$$-|x| = -(-x) \; \text{ i.e. } \; x = -|x|.$$

Thus in either case, we have $x \geq -|x|$.

(v) We have $|x| = \max. \{x, -x\}$, as proved in (ii).

$\therefore \qquad |-x| = \max.\{-x, -(-x)\} = \max.\{-x, x\} = |x|.$

The students should prove this result by direct application of the definition of $|x|$.

(vi) If $x \geq 0$, then $|x| = x$ and so $|x|^2 = x^2$.

Again if $x < 0$, then $|x| = -x$ and so $|x|^2 = (-x)^2 = x^2$.

Thus in either case, we have $|x|^2 = x^2$.

$\therefore \qquad |x|^2 = x^2 \ \forall \ x \in \mathbf{R}.$

Applying this result, we have

$$|-x|^2 = (-x)^2 = x^2.$$

Hence $\qquad |x|^2 = x^2 = |-x|^2 \ \forall \ x \in \mathbf{R}.$

Theorem 2: *For all $x, y \in \mathbf{R}$ prove that*

(*i*) $|xy| = |x| . |y|$. (*ii*) $|x + y| \leq |x| + |y|$. **(The triangle inequality)**

(*iii*) $|x - y| \geq ||x| - |y||$.

Proof: (i) We have $|xy|^2 = (xy)^2 = x^2 y^2 = |x|^2 . |y|^2 = (|x| . |y|)^2$.

$\therefore \qquad |xy| = \pm (|x| . |y|).$

But $|xy|$ and $|x| . |y|$ are both non-negative.

So rejecting the – I've sign which is inadmissible, we have

$$|xy| = |x| . |y| .$$

(ii) We have $|x + y|^2 = (x + y)^2 = x^2 + y^2 + 2xy$

$$\leq x^2 + y^2 + 2|xy| \qquad [\because |xy| \geq xy]$$

$$= |x|^2 + |y|^2 + 2|x| . |y| \qquad [\because |xy| = |x| . |y|]$$

$$= (|x| + |y|)^2.$$

Thus $\qquad |x + y|^2 \leq (|x| + |y|)^2. \qquad ...(1)$

Since $|x + y|$ and $|x| + |y|$ are both non-negative, therefore from (1), we have

$$|x + y| \leq |x| + |y| .$$

(iii) We have $|x| = |(x - y) + y|$

$\leq |x - y| + |y|$, by the triangle inequality.

$\therefore \qquad |x| - |y| \leq |x - y| . \qquad ...(1)$

Again $|y| = |(y - x) + x|$

$\leq |y - x| + |x|$, by the triangle inequality.

$\therefore \qquad |y| - |x| \leq |y - x|$

or $\qquad -(|x| - |y|) \leq |x - y|$, since $|y - x| = |x - y|$. $\qquad ...(2)$

From (1) and (2), we see that

$$|x - y| \geq \max.\{(|x| - |y|), -(|x| - |y|)\}$$

i.e., $\qquad |x - y| \geq ||x| - |y||.$

Theorem 3: (*i*) *If x, ε be real numbers and $\varepsilon > 0$, then*

$$|x| < \varepsilon \Leftrightarrow -\varepsilon < x < \varepsilon .$$

(*ii*) *If x, y,* ε *be real numbers and* $\varepsilon > 0$ *, then*

$$|x - y| < \varepsilon \Leftrightarrow y - \varepsilon < x < y + \varepsilon.$$

Proof: (i) We have $|x| < \varepsilon \Leftrightarrow \max.\{x, -x\} < \varepsilon$

$$\Leftrightarrow -x < \varepsilon \text{ and } x < \varepsilon$$
$$\Leftrightarrow x > -\varepsilon \text{ and } x < \varepsilon$$
$$\Leftrightarrow -\varepsilon < x < \varepsilon.$$

(ii) We have $|x - y| < \varepsilon \Leftrightarrow \max.\{(x - y), -(x - y)\} < \varepsilon$

$$\Leftrightarrow -(x - y) < \varepsilon \text{ and } x - y < \varepsilon$$
$$\Leftrightarrow x - y > -\varepsilon \text{ and } x - y < \varepsilon$$
$$\Leftrightarrow x > y - \varepsilon \text{ and } x < y + \varepsilon$$
$$\Leftrightarrow y - \varepsilon < x < y + \varepsilon.$$

Illustrative Examples

Example 4: *For any real numbers x and y, show that*

$$|x - y| \leq |x| + |y|.$$

Solution: We have $|x - y| = |x + (-y)|$

$$\leq |x| + |-y|, \qquad \text{by the triangle inequality}$$
$$= |x| + |y|. \qquad [\because |-y| = |y|]$$

$\therefore$ $$|x - y| \leq |x| + |y| \ \forall\ x, y \in \mathbf{R}.$$

Example 5: *Prove that* $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$, $y \neq 0$.

Solution: We have $\left|\dfrac{x}{y}\right|^2 = \left(\dfrac{x}{y}\right)^2 = \dfrac{x^2}{y^2}$

$$= \frac{|x|^2}{|y|^2}$$
$$= \left(\frac{|x|}{|y|}\right)^2.$$

$\therefore$ $$\left|\frac{x}{y}\right| = \pm\frac{|x|}{|y|}.$$

But $\left|\dfrac{x}{y}\right|$ and $\dfrac{|x|}{|y|}$ are both non-negative.

Therefore rejecting the – I've sign which is inadmissible, we have

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}.$$

Example 6: *If* $k \in \mathbf{R}^+$ *and* $|x - y| < k\varepsilon \;\forall\; \varepsilon > 0$, *then prove that* $x = y$.

Solution: If possible, let $x \neq y$. Then $|x - y| > 0$ and so taking $\varepsilon = \dfrac{1}{2k}|x - y| > 0$ and applying the given result that

$$|x - y| < k\varepsilon \;\forall\; \varepsilon > 0,$$

we have $$|x - y| < k \cdot \frac{1}{2k}|x - y|$$

i.e., $$|x - y| < \frac{1}{2}|x - y|$$

which is not possible.

Hence our initial assumption is wrong and we must have $x = y$.

Comprehensive Exercise 1

1. For any real numbers x and y, show that
$$|x + y| \geq ||x| - |y||.$$
2. If x, y, z are any real numbers, then prove that
$$|x + y + z| \leq |x| + |y| + |z|.$$
3. If $x_1, x_2, \ldots, x_n$ be any real numbers, then prove that
   (i) $|x_1 + x_2 + \ldots + x_n| \leq |x_1| + |x_2| + \ldots + |x_n|$.
   (ii) $|x_1 x_2 \ldots x_n| = |x_1| \cdot |x_2| \ldots . |x_n|$.
4. If x be any real number not equal to zero, then prove that
$$\left|\frac{1}{x}\right| = \frac{1}{|x|}.$$
5. For any $a, b \in \mathbf{R}$, prove that
   (i) $|a - b| = 0 \Leftrightarrow a = b$. (ii) $|a - b| = |b - a|$.
6. Prove that $a < x < b \;\Leftrightarrow \left| x - \frac{1}{2}(a + b)\right| < \frac{1}{2}(b - a)$; $a, x, b \in \mathbf{R}$.
7. Show that there is no rational number whose square is 3.
8. If $0 < \theta < 1, |x| < 1$, show that $\left|\dfrac{x(1 - \theta)}{1 + \theta x}\right| < 1$.

13 Bounded Sets - Bounded and Unbounded Subsets of Real Numbers

1. **Aggregate:** **Definition:** *A non-empty subset S of* **R** *is called an aggregate.*

For example, the set of all positive integers $\mathbf{Z}^+$ is an aggregate. If $S = \{1, 2, 3\}$, then S is an aggregate. The null set $\varnothing$ is not an aggregate.

Boundedness of subsets of R

2. Upper bound of a subset of R:

Definition: *Let S be a subset of real numbers. If there exists a real number u, such that*

$$x \leq u \;\;\forall\;\; x \in S,$$

then u is called an **upper bound** *of S.*

If there exists an upper bound for a set S, then the set S is said to be **bounded above.**

It is not necessary that a set S should be bounded above. If there exists no real number u such that $x \leq u \;\forall\; x \in S$, then the set S is said to be **not bounded above** or **unbounded above.** Thus a set S is unbounded above if however large a real number u we may take, there exists at least one $x \in S$ such that $x > u$.

For example, the set of positive integers $\mathbf{Z}^+ = \{1, 2, 3, 4, \ldots\}$ is not bounded above while the set of negative integers

$$\mathbf{Z}^- = \{\ldots, -3, -2, -1\}$$

is bounded above and -1 is an upper bound. The set of negative real numbers is also bounded above, 0 being an upper bound.

If u is an upper bound of a set S, then every real number greater than u is also an upper bound of S. Thus if a set S is bounded above, then the set of all such numbers that are upper bounds of S is infinite.

3. Least upper bound or supremum or suprema: **(Gorakhpur 2012, 15)**

Definition: *If s is an upper bound of a subset S of* **R** *and any real number less than s is not an upper bound of S, then s is called the* **least upper bound (l.u.b)** or **supremum (sup)** *of S.*

Thus if a set S is bounded above and if the set of all upper bounds of S has a smallest member, say s, then s is called the least upper bound or the supremum of the set S.

If a real number s is the supremum of a subset S of real numbers, then for every $\varepsilon > 0$, there exists a real number $x \in S$ such that

$$s - \varepsilon < x \leq s.$$

Example 7: *Prove that the greatest member of a set, if it exists, is the supremum (g.l.b.) of the set.*

Solution: Let g be the greatest member of the set S. Then $x \leq g, \;\forall\; x \in S$ and so g is an upper bound of S.

Also no number less than g can be an upper bound of S. For if y be any number less than g, there exists at least one member g of S which is greater than y.

Thus g is the least of all the upper bounds of S *i.e.*, g is the supremum of S.

4. Lower bound of a subset of R:

As we have defined an upper bound of a set, in the same way we can also define a lower bound of a set.

Definition: *Let S be a subset of real numbers. If there exists a real number v, such that*

$$x \geq v \ \forall \ x \in S,$$

then v is called a **lower bound** *of S.*

If there exists a lower bound for a set S, then the set S is said to be **bounded below.**

It is not necessary that a set should be bounded below. If there exists no real number v such that $x \geq v \ \ \forall \ \ x \in S$, then the set S is said to be **not bounded below** or **unbounded below**. Thus a set S is unbounded below if however small a real number v we may take, there exists at least one $x \in S$ such that $x < v$.

For example, the set of negative integers

$$\mathbf{Z}^- = \{\ldots, -3, -2, -1\}$$

is not bounded below while the set of positive integers $\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$ is bounded below and 1 is a lower bound. The set of positive real numbers $\mathbf{R}^+$ is also bounded below, 0 being a lower bound.

If v is a lower bound of a set S, then every real number smaller than v is also a lower bound of S. Thus if a set S is bounded below, then the set of all such real numbers that are lower bounds of S is infinite.

5. Greatest lower bound or infimum or infima:

Definition: *If t is a lower bound of a subset S of* **R** *and any real number greater than t is not a lower bound of S, then t is called the* **greatest lower bound (g.l.b.)** *or* **infimum (inf.)** *of S.*

(Kanpur 2011; Gorakhpur 15)

Thus if a set S is bounded below and if the set of all lower bounds of S has a greatest member, say t, then t is called the greatest lower bound or the infimum of the set S.

If a real number t is the infimum of a subset of S of real numbers, then for every $\varepsilon > 0$, there exists a real number $x \in S$ such that

$$t \leq x < t + \varepsilon.$$

Note: Supremum is defined only for the subsets of **R** which are bounded above and infimum for the subsets which are bounded below.

Exercise: *Prove that the smallest member of a set, if it exists, is the infimum (g.l.b.) of the set.*

6. Bounded subsets of real numbers:

Definition: *A subset S of real numbers is said to be* **bounded**, *if it is bounded above as well as bounded below.*

Thus a set S is bounded if and only if there exist two real numbers u, v such that

$$v \leq x \leq u \ \ \forall \ \ x \in S$$

or $\quad S \subset [v, u]$ *i.e.*,

S is a subset of the closed interval $[v, u]$.

It can be easily seen that a set S is bounded if and only if there exists a positive real number k such that

$$|x| < k \ \text{ for all } \ x \in S.$$

Unbounded set of real numbers

Definition: *A subset S of* **R** *is said to be unbounded if it is not bounded above or not bounded below.*

A finite subset of **R** is always bounded. The subsets **Q**, **Z** and $\mathbf{Z}^+$ of **R** are not bounded.

The open interval $]\,3, 5\,[$ is a bounded subset of **R**, 5 being an upper bound and 3 a lower bound.

The null set $\varnothing$ *is bounded but it neither possesses supremum nor infimum.*

The null set $\varnothing$ is bounded above because if u is any real number, then u is an upper bound for $\varnothing$. Obviously the condition $x \le u$ for all $x \in \varnothing$ is vacuously satisfied because $\varnothing$ has no elements.

Thus every real number is an upper bound for $\varnothing$. Since the set of all real numbers has no smallest member, therefore sup $\varnothing$ does not exist.

Also we observe that $\varnothing$ is bounded below. In fact every real number is a lower bound for $\varnothing$. Since the set of all real numbers has no greatest member, therefore inf $\varnothing$ does not exist.

7. Greatest and least members of a subset of R:

If the supremum of a subset S of **R** *is a member of S, then we say that S* **attains** *its supremum and this supremum is called the greatest member of S.*

If the infimum of a subset of S of **R** *is a member of S, then we say that S* **attains** *its infimum and this infimum is called the least member of S.*

The greatest member of a set S, if it exists, is the supremum of S. But the supremum of S need not be the greatest member of S unless S attains this supremum.

For example if S is the open interval

$$]\,2, 3\,[\text{ i.e., } S = \{x : 2 < x < 3\},$$

then $\sup S = 3$ and $\inf S = 2$. Note that 3 is an upper bound for S and no real number less than 3 is an upper bound for S. Similarly 2 is a lower bound for S and no real number greater than 2 is a lower bound for S. Since $2 \notin S$ and $3 \notin S$, therefore S has no least member and no greatest member.

On the other hand if A is the closed interval $[2, 3]$ *i.e.*, $A = \{x : 2 \le x \le 3\}$, then $\sup A = 3$ and $\inf A = 2$. Here both the infimum and the supremum *i.e.*, 2 and 3 are members of A and so 2 is the least member of A and 3 is the greatest member of A.

8. Bounded and unbounded intervals:

Bounded intervals: If a and b are any real numbers such that $b > a$, then the intervals $]\,a, b\,[$, $[a, b]$, $[a, b\,[$ and $]\,a, b]$ are called **bounded intervals** because each of them is a bounded subset of **R**.

If $S_1 = \,]\,a, b\,[$, then sup $S_1 = b$ and inf $S_1 = a$. Here both a and b do not belong to S_1 and so S_1 has no least member and no greatest member.

If $S_2 = [a, b]$, then $\sup S_2 = b$ and $\inf S_2 = a$. Here both a and b are members of S_2 and so a is the least member of S_2 and b is its greatest member.

If $S_3 = [\,a, b\,[$, then $\sup S_3 = b$ and $\inf S_3 = a$. Here $a \in S_3$ and is the least member of S_3 but $b \notin S_3$ and so S_3 has no greatest member.

If $S_4 =]\, a, b\,]$, then $\sup S_4 = b$ and $\inf S_4 = a$. Here $b \in S_4$ and is the greatest member of S_4 but $a \notin S_4$ and so S_4 has no least member.

Unbounded intervals: If a is any real number, then the intervals $]\, a, \infty\, [, [a, \infty\, [\, ,]\, -\infty, a\, [$ and $]\, -\infty, a]$ are called **unbounded intervals** because each of them is not a bounded subset of **R**. The set $\mathbf{R} =]\, -\infty, \infty\, [$ is also an unbounded interval. It is neither bounded above nor bounded below.

If $S_1 =]\, a, \infty\, [$, then S_1 is not bounded above and so the question of $\sup S_1$ does not arise. However S_1 is bounded below and $\inf S_1 = a$. Since $a \notin S_1$, therefore S_1 has no least member.

Similarly we can discuss the cases of the other three unbounded intervals mentioned above.

9. Some important observations about supremum and infimum of a subset of R if they exist:

(i) A non-empty finite subset of **R** is always bounded and has its greatest member as its supremum and its smallest member as its infimum.

(ii) Supremum and infimum of a bounded subset of **R** are unique.

(iii) In the case of a singleton set $S = \{a\}, a \in \mathbf{R}$, supremum and infimum coincide. In this case a is both $\sup S$ and $\inf S$.

(iv) Supremum and infimum of a bounded set need not necessarily belong to the set.

(v) Even in the case of an infinite bounded subset of **R** supremum and infimum may both belong to the set. For example, if S is the closed interval $[4, 9]$, then S is an infinite subset of **R** and $\sup S = 9$ and $\inf S = 4$ are both members of S.

(vi) If s and t are the supremum and the infimum of a non-empty subset S of **R**, then $t \le s$.

14 Some Properties of Supremum and Infimum

Theorem 1. Uniqueness of the supremum.

The supremum of a set $S \subset \mathbf{R}$, if it exists, is unique. **(Gorakhpur 2012)**

Proof: Let S be a non-empty subset of **R** which is bounded above. If possible let s_1 and s_2 be two suprema of S. Then to show that $s_1 = s_2$.

Since both s_1 and s_2 are suprema of S, therefore they are upper bounds of S.

Now s_1 is a supremum and s_2 is an upper bound of S

$$\Rightarrow \qquad s_1 \le s_2. \qquad \text{...(1)}$$

Similarly s_2 is a supremum and s_1 is an upper bound of S

$$\Rightarrow \qquad s_2 \le s_1. \qquad \text{...(2)}$$

From (1) and (2), we have $s_1 \ngtr s_2$ and $s_2 \ngtr s_1$. Therefore by the law of trichotomy, we have $s_1 = s_2$.

Theorem 2: Uniqueness of the infimum.

The infimum of a set $S \subset \mathbf{R}$, if it exists, is unique. **(Gorakhpur 2014)**

Proof: Let S be a non-empty subset of **R** which is bounded below. If possible let t_1 and t_2 be two infima of S. Then to show that $t_1 = t_2$.

Since both t_1 and t_2 are infima of S, therefore they are lower bounds of S.

Now t_1 is an infimum and t_2 is a lower bound of S

$$\Rightarrow t_1 \geq t_2. \qquad ...(1)$$

Similarly t_2 is an infimum and t_1 is a lower bound of S

$$\Rightarrow t_2 \geq t_1. \qquad ...(2)$$

From (1) and (2), we have $t_1 = t_2$.

Theorem 3: A characteristic property of supremum . **(Gorakhpur 2013)**

If S be a non-empty subset of **R** *, then a real number s is the supremum for S if and only if*

(i) $x \leq s$ for all $x \in S$,

and (ii) for each positive real number ε, there exists a real number $x \in S$ such that $x > s - \varepsilon$.

Proof. The '**only if part**' *i.e.*, the given conditions are **necessary** for s to be the supremum of S.

Let s be the supremum for S. Then since s is an upper bound for S, we have

$$x \leq s \;\; \forall \, x \in S.$$

Again take any real number $\varepsilon > 0$. Then $s - \varepsilon < s$ and so $s - \varepsilon$ cannot be an upper bound for S since it is less than the supremum. Hence there must exist some $x \in S$ such that

$$x > s - \varepsilon.$$

Hence the conditions are necessary.

The '**if part**' *i.e.*, the given conditions are **sufficient** for s to be the supremum of S.

Let the conditions (i) and (ii) hold. Then to show that

$$s = \sup S$$

By condition (i) s is an upper bound for S. Now s will be the supremum of S if we show that no real number less than s can be an upper bound for S.

Let s' be any real number less than s, then $s - s' > 0$. Now if we take $\varepsilon = s - s' > 0$, then by the given condition (ii) there exists $x \in S$ such that $x > s - \varepsilon$ *i.e.*, $x > s - (s - s')$ *i.e.*, $x > s'$, showing that s' is not an upper bound for S. Thus we see that s is an upper bound for S and no real number less than s is an upper bound for S. Hence s is the supremum of S.

This completes the proof of the theorem.

Theorem 4: A characterization of infimum.

If S be a non-empty subset of **R**, *then a real number t is the infimum for S if and only if*

(i) $x \geq t$ for all $x \in S$,

and (ii) for each real number $\varepsilon > 0$, there exists a real number $x \in S$ such that $x < t + \varepsilon$.

Proof: The '**only if part**' *i.e.*, the given conditions are **necessary** for t to be the infimum of S.

Let t be the infimum for S. Then since t is a lower bound for S, we have

$$x \geq t \;\; \forall \, x \in S.$$

Again take any real number $\varepsilon > 0$. Then $t + \varepsilon > t$ and so $t + \varepsilon$ cannot be a lower bound for S since it is greater than the greatest lower bound (*i.e.*, inf) t of S. Thus we see that t is a lower bound for S and no real number greater than t is a lower bound for S. Hence there must exist some $x \in S$ such that $x < t + \varepsilon$.

Hence the conditions are necessary.

The '**if part**' *i.e.,* the given conditions are **sufficient** for t to be the infimum of S.

Let the conditions (i) and (ii) hold. Then to show that $t = \inf S$.

By condition (i) t is a lower bound for S.

Now t will be the infimum (*i.e.,* g.l.b.) of S if we show that no real number greater than t can be a lower bound for S.

Let t' be any real number greater than t, then $t' - t > 0$. Now if we take $\varepsilon = t' - t > 0$, then by the given condition (ii) there exists $x \in S$ such that $x < t + \varepsilon$ *i.e.,* $x < t + (t' - t)$ *i.e.,* $x < t'$, showing that t' is not a lower bound for S. Thus we see that t is a lower bound for S and no real number greater than t is a lower bound for S. Hence t is the g.l.b. *i.e.,* the infimum of S.

This completes the proof of the theorem.

Illustrative Examples

Example 8: *Prove that the set* $\mathbf{R}^+$ *of positive real numbers is not bounded above.*

Solution: The set $\mathbf{R}^+$ of positive real numbers is not bounded above. For, suppose that $\mathbf{R}^+$ is bounded above and u is an upper bound.

Since $1 \in \mathbf{R}^+$ and u is an upper bound for $\mathbf{R}^+$, therefore,

$$1 \leq u, \text{ which means } u > 0.$$

$\therefore$ $u + 1 > 0$ and consequently $u + 1 \in \mathbf{R}^+$.

Thus there exists $u + 1 \in \mathbf{R}^+$ such that $u + 1 >$ an upper bound u of $\mathbf{R}^+$. But this is a contradiction because if u is an upper bound for $\mathbf{R}^+$, then we must have $x \leq u$ for all x in $\mathbf{R}^+$.

Hence u is not an upper bound of $\mathbf{R}^+$ and so $\mathbf{R}^+$ is not bounded above.

Example 9: *Let S be a non-empty bounded subset of* $\mathbf{R}$ *such that sup S = inf S. What can be said about the set S ?*

Solution: Let $\sup S = \inf S = u$.

Then u is an upper bound as well as a lower bound for S.

$\therefore$ $x \leq u$ for all $x \in S$ ...(1)

and $x \geq u$ for all $x \in S$. ...(2)

From (1) and (2), we have $x = u$ for all $x \in S$ *i.e.,* u is the only element in S *i.e.,* S is the singleton $\{u\}$.

$\therefore$ if $\sup S = \inf S$, then S is a singleton set.

Example 10: *If u is an upper bound of a set* $S \subset \mathbf{R}$ *and* $u \in S$, *then* $u = \sup S$.

Solution: If possible, let s be $\sup S$.

Since s is sup S, therefore s is an upper bound for S.

$\therefore \quad u \in S \Rightarrow u \leq s.$ ...(1)

Also u is an upper bound of S and s is the l.u.b. of S

$\Rightarrow s \leq u.$ ...(2)

From (1) and (2), we have $s = u$.

Hence $\quad$ sup $S = u$.

Example 11: *Show that every non-empty finite subset of* **R** *is bounded.*

Solution: Let S be a non-empty finite subset of **R**. Then there are only a finite number of elements in S and so by the properties of the order relation in **R** out of these elements one element $a \in S$ shall be the smallest element of S and one element $b \in S$ shall be the greatest element of S.

Thus we have $a \leq x \leq b \ \forall \ x \in S$. Hence the set S is bounded.

Example 12: *Find the supremum and infimum of the singleton*

$\{2\} \subset \mathbf{R}$.

Solution: For $\{2\} \subset \mathbf{R}$, we find that the set of upper bounds of $\{2\}$ is given by $\{x \in \mathbf{R} : x \geq 2\}$ and 2 being the least of these upper bounds, we have, sup $\{2\} = 2$.

Again the set of lower bounds of $\{2\}$ is given by

$\{x \in \mathbf{R} : x \leq \ 2\}$

and 2 being the greatest of these lower bounds, we have

$\inf \{2\} = 2.$

Hence $\quad \sup \{2\} = 2 = \leq \inf\{2\}$.

Example 13: *Find the g.l.b. and l.u.b. of the set*

$S = \{x \in \mathbf{Z} : x^2 \leq 25\}$.

Solution: Writing in tabular form, we have

$S = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$.

Here S is a finite subset of **R**. The smallest member of S is -5 and so it is a lower bound for S. All real numbers less than -5 are also lower bounds for S. Thus the set of all lower bounds of the set S is the set $\{x : x \in \mathbf{R}$ and $x \leq -5\}$. Since -5 is the greatest of all these lower bounds of S, therefore g.l.b. of $S = -5$.

Similarly the greatest member of S is 5 and so it is an upper bound for S. Since an upper bound 5 of the set S is a member of the set S, therefore it is the supremum of S. Hence sup $S = 5$.

Example 14: *Give examples to show that*

(*i*) *every infinite set need not be bounded.*

(*ii*) *every subset of an unbounded set is not necessarily unbounded.*

Solution: (i) Consider the infinite set $S = \{x \in \mathbf{R} : 2 < x < 3\}$. Then S is a bounded subset of **R**, 2 being a lower bound and 3 an upper bound for S.

Again consider the set of integers

$\mathbf{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

The set $\mathbf{Z}$ is an infinite subset of $\mathbf{R}$ and is not bounded. It is neither bounded above nor bounded below.

(ii) The set of integers $\mathbf{Z}$ is unbounded but its subset $S = \{1, 2, 3\}$ is bounded. We see that 1 is a lower bound and 3 an upper bound for S.

Example 15: *Give an example each of a bounded set which contains its*

(*i*) *g.l.b. but does not contain its l.u.b.*

(*ii*) *l.u.b. but does not contain its g.l.b.*

Solution: (i) Consider the set $S = \{x : x \in \mathbf{R} \text{ and } 2 \le x < 5\}$. We have g.l.b. of $S = 2$ which is a member of S and l.u.b. of $S = 5$ which is not a member of S.

(ii) Consider the set $A = \{x : x \in \mathbf{R} \text{ and } 3 < x \le 7\}$. We have g.l.b. of $A = 3$ and $3 \notin A$. Also l.u.b. of $A = 7$ and $7 \in A$.

Example 16: *A, B are sets such that* $a \in A, b \in B \Rightarrow a < b$. *Show that*

$$l.u.b.\ A \le g.l.b.\ B.$$

Solution: Let l.u.b. $A = s$ and g.l.b. $B = t$.

To show that $\quad s \le t$.

Suppose if possible $\quad s > t$.

Since l.u.b. $A = s$ and $t < s$, therefore there exists $x \in A$ such that $x > t$.

Now g.l.b. $B = t$ and $x > t \Rightarrow$ there exists $y \in B$ such that $y < x$.

Thus there exists $x \in A$ and $y \in B$ such that $x > y$ which is against the hypothesis that $a \in A, b \in B \Rightarrow a < b$.

Hence our initial assumption $s > t$ is wrong and we must have $s \le t$.

Example 17: *Find the supremum and infimum, if they exist, of the following sets:*

(*i*) $\left\{\dfrac{1}{n} : n \in \mathbf{N}\right\}$

(*ii*) $\left\{x \in \mathbf{Q} : x = \dfrac{n}{n+1}, n \in \mathbf{N}\right\}$

(Gorakhpur 2011)

(*iii*) $\left\{-2, -\dfrac{3}{2}, -\dfrac{4}{3}, -\dfrac{5}{4}, \ldots\right\}$.

(*iv*) *the set of positive integers* $\mathbf{Z}^+$

(*v*) $\{x : x = (-1)^n\, n, n \in \mathbf{N}\}$.

(*vi*) $\left\{1 + \dfrac{(-1)^n}{n} : n \in \mathbf{N}\right\}$.

(Gorakhpur 2010)

Solution: (i) Let $S = \left\{\dfrac{1}{n} : n \in \mathbf{N}\right\}$

i.e. $$S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}.$$

The set S is bounded above, 1 being an upper bound for S. Note that we have $x \le 1$ for all $x \in S$.

Since $1 \in S$, therefore any number less than 1 is not an upper bound for S.

$\therefore \quad \sup S = 1$. Also we observe that $1 \in S$.

Again 0 is a lower bound for S because $x \ge 0$ for all $x \in S$.

Any real number greater than 0 cannot be a lower bound for S as shown below :
Let $v > 0$. However small $v > 0$ may be, there exists $n \in \mathbf{N}$ such that $1/n < v$.
Thus there exists $1/n \in S$ such that $1/n < v$.
Therefore v is not a lower bound for S.
Thus 0 is a lower bound for S and no real number greater than 0 can be a lower bound for S.

$\therefore$ g.l.b. of S *i.e.*, $\inf S = 0$. Here we observe that $0 \notin S$.

(ii) Let $S = \left\{\dfrac{n}{n+1} : n \in \mathbf{N}\right\} = \left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots\right\}$.

The set S is bounded below, $\frac{1}{2}$ being a lower bound for S. Since $\frac{1}{2} \in S$, therefore any real number greater than $\frac{1}{2}$ cannot be a lower bound for S.

$\therefore$ g.l.b. of S *i.e.*, $\inf S = \frac{1}{2}$. We observe that $\frac{1}{2} \in S$.

Since $\dfrac{n}{n+1} < 1 \;\; \forall\, n \in \mathbf{N}$, therefore 1 is an upper bound for S. Any real number less than 1 cannot be an upper bound for S as shown below.

Let u be any real number < 1. We shall show that there exists some $x \in S$ such that $x > u$ *i.e.*, there exists some $n \in \mathbf{N}$ such that $n/(n+1) > u$.

We have $\dfrac{n}{n+1} = \dfrac{(n+1)-1}{n+1} = 1 - \dfrac{1}{n+1}$.

Now $u < 1 \Rightarrow 1 - u > 0$. However small $1 - u > 0$ may be, there exists $n \in \mathbf{N}$ such that

$$\frac{1}{n} < 1 - u \;\Rightarrow\; \frac{1}{1+n} < 1 - u \qquad \left[\because \frac{1}{1+n} < \frac{1}{n}\right]$$

$$\Rightarrow 1 - \frac{1}{1+n} > u \Rightarrow \frac{n}{n+1} > u.$$

Thus there exists $n \in \mathbf{N}$ such that $\dfrac{n}{n+1} > u$ and hence u cannot be an upper bound for S.

Thus 1 is an upper bound for S and no real number less than 1 can be an upper bound for S.

$\therefore$ $\sup S = 1$. We observe that $1 \notin S$.

(iii) Let
$$S = \left\{-2, -\frac{3}{2}, -\frac{4}{3}, -\frac{5}{4}, \ldots\right\}$$
$$= \left\{-2, -1\frac{1}{2}, -1\frac{1}{3}, -1\frac{1}{4}, \ldots\right\}.$$

We have $\inf S = -2 \in S$

and $\sup S = -1 \notin S$.

(iv) We have $\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$.

The set $\mathbf{Z}^+$ is bounded below, 1 being a lower bound for $\mathbf{Z}^+$. Since $1 \in \mathbf{Z}^+$, therefore no real number greater than 1 can be a lower bound for $\mathbf{Z}^+$.

$$\therefore \qquad \inf Z^+ = 1.$$

But the set $\mathbf{Z}^+$ is not bounded above.

There is no real number u such that

$$x \le u \text{ for all } x \in \mathbf{Z}^+.$$

Since the set $\mathbf{Z}^+$ is not bounded above, therefore sup $\mathbf{Z}^+$ has no meaning.

(v) Let $S = \{x : x = (-1)^n\, n, n \in \mathbf{N}\}$

$$= \{-1, 2, -3, 4, -5, 6, \ldots\}$$
$$= \{\ldots, -5, -3, -1, 2, 4, 6, \ldots\}.$$

Here the set S is not bounded above and so sup S cannot be discussed. Also S is not bounded below and so inf S has no meaning.

(vi) Let $S = \left\{1 + \dfrac{(-1)^n}{n} : n \in \mathbf{N}\right\}$

$$= \left\{0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \frac{7}{6}, \frac{6}{7}, \frac{9}{8}, \frac{8}{9}, \ldots\right\}$$
$$= \left\{\frac{0}{1}, \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \ldots, \frac{2n-2}{2n-1}, \ldots\right\} \cup \left\{\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \frac{9}{8}, \ldots, \frac{2n+1}{2n}, \ldots\right\}.$$

Here we observe that the proper fractions $\dfrac{0}{1}, \dfrac{2}{3}, \dfrac{4}{5}, \dfrac{6}{7}, \ldots$ are increasing and tending to 1.

The improper fractions beginning with $\dfrac{3}{2}$ are decreasing and tending to 1.

We have $\inf S = 0$ and $\sup S = \dfrac{3}{2}$.

Comprehensive Exercise 2

1. Let $S = \{x \in \mathbf{R} : x = n + 3, n \in \mathbf{N}\}$. Show that S is bounded below but not above. Find the g.l.b. of S.
2. Which of the following sets are bounded below, which are bounded above, and which are bounded neither below nor above ?

   (i) $\{-1, -2, -3, -4, -5, \ldots\}$.

   (ii) $\{1, 2, 3, 4, 5, \ldots\}$.

   (iii) $\{2, 2^2, 2^3, \ldots, 2^n, \ldots\}$.

   (iv) $\left\{1, \dfrac{1}{4}, \left(\dfrac{1}{4}\right)^2, \left(\dfrac{1}{4}\right)^3, \ldots, \left(\dfrac{1}{4}\right)^n, \ldots\right\}$

   (v) $\left\{x : x = (-1)^n \dfrac{1}{n}, n \in \mathbf{N}\right\}$

   (vi) $\{x : x = (-2)^n, n \in \mathbf{N}\}$.

3. Find the supremum and infimum, if they exist, of the following sets.

(i) $\{x : x = 1 + (1/n), n \in \mathbf{N}\}$. (ii) $\{x : x = 1 - (1/n), n \in \mathbf{N}\}$.

(iii) $\{x \in \mathbf{R} : -5 < x < 3\}$. (iv) $\{x \in \mathbf{R} : x = 2^n, n \in \mathbf{N}\}$.

(v) $\{\pi + 1, \pi + \frac{1}{2}, \pi + \frac{1}{3}, \ldots\}$. (vi) $\{3, 4, 12, 20\}$.

(vii) $\left\{-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\right\}$. (viii) $\left\{m + \frac{1}{n} : m, n \in \mathbf{N}\right\}$.

(ix) $\left\{\frac{1}{5n} : n \in \mathbf{Z}, n \neq 0\right\}$. (x) $\left\{\frac{3n+2}{2n+1} : n \in \mathbf{N}\right\}$.

4. Given an example of a non-empty bounded subset S of $\mathbf{R}$, whose supremum and infimum both belong to $\mathbf{R} - S$.

5. Prove that the set $\mathbf{R}^-$ of negative real numbers is not bounded below.

6. Prove that the set of all real numbers $\mathbf{R}$ is not bounded.

7. Find the l.u.b. for the following sets :

(i) $(1, 2] \cup [3, 8)$. (ii) The empty set.

(iii) $\left\{\pi + \frac{1}{2}, \pi + \frac{1}{4}, \pi + \frac{1}{8}, \ldots\right\}$.

8. Find the l.u.b. and the g.l.b. of the set

$$S = \left\{\frac{2n+1}{3n+2} : n \in \mathbf{N}\right\}.$$

9. Find g.l.b. and l.u.b. of the following sets S :

(i) $S = \{x \in \mathbf{Q} : x = \frac{(-1)^n}{n}, n \in \mathbf{N}\}$.

(ii) $S = \left\{x \in \mathbf{Q} : x = (-1)^n \left(\frac{1}{4} - \frac{4}{n}\right), n \in \mathbf{N}\right\}$.

(iii) $S = \left\{\left(1 - \frac{1}{n}\right) \sin \frac{n\pi}{2}, n \in \mathbf{N}\right\}$.

(iv) $S = \left\{x \in \mathbf{Q} : x = (-1)^n \left(\frac{1}{n} - \frac{4}{n}\right), n \in \mathbf{N}\right\}$.

10. Give an example of a set which is :

(i) bounded above but not below **(Gorakhpur 2015)**

(iii) bounded below but not above

(iii) neither bounded above nor bounded below **(Gorakhpur 2013)**

(iv) both bounded above and below.

11. If $A \neq \varnothing$ is bounded below and $-A$ denotes the set of all $-x$ for which $x \in A$, then prove that $-A \neq \varnothing$, that $-A$ is bounded above, and that $-\sup(-A) = \inf A$.

12. If $A \neq \varnothing$, $B \neq \varnothing$ and $x \leq y$, $\forall x \in A$ and $\forall y \in B$, then prove that :

(i) $\sup A \leq y, \forall y \in B$ (ii) $\sup A \leq \inf B$.

13. If $B \supseteq A\ (\neq \varnothing)$ and B is bounded, then $\sup B \geq \sup A \geq \inf A \geq \inf B$.

Answers 2

1. $\text{Inf } S = 4$.

2. (i) Bounded above but not bounded below.

(ii) Bounded below but not bounded above.

(iii) Bounded below but not bounded above.

(iv) Bounded below as well as bounded above. The l.u.b. of the set $= 1$ and the g.l.b. $= 0$. Here the l.u.b. is a member of the set while the g.l.b. does not belong to the set.

(v) Bounded below as well as bounded above. Here the l.u.b. of the set $= \dfrac{1}{2}$ and the g.l.b. $= -1$ and both the l.u.b. and the g.l.b. are members of the set.

(vi) Neither bounded below nor bounded above.

3. (i) Sup = 2, inf = 1. (ii) Sup = 1, inf = 0.

(iii) Sup = 3, inf = – 5. (iv) Inf = 2, sup does not exist.

(v) Sup = $\pi + 1$, inf = π. (vi) Sup = 20, inf = 3.

(vii) Sup = 0, inf = – 1. (viii) Inf = 1, sup does not exist.

(ix) Sup = 1/5, inf = – 1/5. (x) Sup = 5/3. inf = 3/2

4. $S = \{x : x \in \boldsymbol{R} \text{ and } 2 < x < 3\}$ *i.e.*, $S =$ the open interval $]2, 3[$

7. (i) 8. (ii) No. (iii) $\pi + \dfrac{1}{2}$

8. l.u.b. $= \dfrac{2}{3} \cdot$ and g.l.b. $= \dfrac{3}{5}$

9. (i) g.l.b. = – 1 and l.u.b. $= \dfrac{1}{2}$

(ii) g.l.b. = – 7/4 and l.u.b. = 15/4

(iii) g.l.b. = –1 and l.u.b. = 3

(iv) g.l.b. = – 3/2 and l.u.b. =3

10. (i) The set of negative integers

(ii) The set of positive integers

(iii) The set of rational numbers

(iv) The set $S = \left\{\dfrac{1}{n} : n \in \boldsymbol{N}\right\}$

15 Completeness

Completeness of a system S of numbers with respect to boundedness is defined as follows :

Definition: *A system S of numbers is said to be* **complete** *if every non-empty subset of S, which is bounded above has a member of S for its supremum.* **(Gorakhpur 2012)**

For example, the set **Z** of all integers is complete with respect to boundedness.

Let S be any non-empty subset of **Z** which is bounded above. Then S must possess a greatest integer. For, if S does not possess a greatest integer, then however great an integer u we may take, there exists an integer $x \in S$ such that $x > u$ and so S cannot be bounded above. But this is a contradiction. Hence S must contain a greatest integer, say s.

Then $s \in S$ and $x \leq s \,\forall\, x \in S$. So s is an upper bound for S and no integer less than s can be an upper bound for S.

$\therefore \qquad \sup S = s \in S.$

Hence for any non-empty subset S of integers which is bounded above, the sup S is an element of S.

Thus every non-empty subset of **Z** which is bounded above has a member of **Z** for its supremum. Therefore the set **Z** of all integers is complete with respect to boundedness.

16 Order-Completeness Axiom for Real Numbers (Completeness Property of R)

Every non-empty subset of real numbers which is bounded above has a supremum.

This property of real numbers is known as *order-completeness* or simply *completeness.*

Roughly speaking, this axiom means that **R** (regarded as a set of points on a line) has no holes in it. If S is any non-empty subset of **R** which is bounded above, then the set of all upper bounds of S must have a smallest member *i.e.* S must possess the supremum which is a member of **R**.

The set of rational numbers **Q** does not satisfy the order completeness axiom as we shall show later. Thus it is the completeness property which enables us to distinguish between the set **Q** of rational numbers and the set **R** of real numbers. Both the sets **Q** and **R** form ordered fields. The field **R** possesses the completeness property while the field **Q** does not possess it.

The set **R** thus satisfies (1) Field axioms (2) Order axioms and (3) Completeness axiom and *hence* **R** *is a complete ordered field.*

Complete ordered field

Definition: *An ordered field F is said to be a* **complete ordered field** *if every non-empty subset S of F which is bounded above has an element of F for its supremum.*

The field **R** of real numbers is a complete ordered field while the field **Q** of rational numbers is an ordered field but is not complete.

In fact, it is a characterisation of the set **R** *of real numbers that it is a* **complete ordered field.** An ordered field F is complete if and only if it is the field of real numbers. Thus we may define the real numbers as the elements which form a complete ordered field. This is a characteristic property of real numbers.

The following theorem asserts the existence of the infimum of any non-empty subset of **R** which is bounded below.

Theorem 1: *Any non-empty subset of real numbers which is bounded below has an infimum.*

Proof: Let $S \subset \mathbf{R}$ and $S \neq \varnothing$.

Let S be bounded below. Then to show that S has an infimum.

Let us denote by T the set of negatives of the members of S *i.e.*,

$$T = \{-x : x \in S\}$$

or $$T = \{y : y \in \mathbf{R} \text{ and } y = -x \text{ for some } x \in S\}.$$

First we shall show that T is bounded above.

Since S is bounded below, therefore let v be a lower bound for S so that

$$x \geq v \quad \forall \; x \in S$$

$\Rightarrow$ $$-x \leq -v \;\forall\; x \in S$$

$\Rightarrow$ $$y \leq -v \;\forall\; y \in T. \qquad [\because\; y \in T \;\Rightarrow y = -x \text{ for some } x \in S]$$

Thus $-v$ is an upper bound for T and so T is bounded above.

$\therefore$ by the completeness axiom, T has the supremum, say t.

We shall show that $-t$ is the infimum of S *i.e.*, $-t$ is the greatest lower bound of S.

Since t is an upper bound of T, therefore $-t$ is a lower bound of S.

Let w be any lower bound of S. Then $-w$ is an upper bound of T.

Now $-w$ is an upper bound of T and t is the supremum of T.

$\therefore$ $$t \leq -w$$

$\Rightarrow$ $$-t \geq w.$$

Thus $-t$ is a lower bound of S and if w is any lower bound of S, then $w \leq -t$.

Hence $-t$ is the greatest lower bound of S *i.e.*, $-t = \inf S$.

Hence the theorem.

Theorem 2: *The set* $\mathbf{N}$ *of natural numbers is not bounded above.*

Proof. If possible, let $\mathbf{N}$ be bounded above.

Since $1 \in \mathbf{N}$, therefore $\mathbf{N} \neq \varnothing$. Thus $\mathbf{N}$ is a non-empty subset of $\mathbf{R}$ which is bounded above. Therefore by order-completeness property $\mathbf{N}$ must have supremum, say, s.

Then $$x \leq s \quad \forall \; x \in \mathbf{N}$$

$\Rightarrow$ $$(x+1) \leq s \quad \forall \; x \in \mathbf{N} \qquad [\because\; x \in \mathbf{N} \;\Rightarrow\; x+1 \in \mathbf{N}]$$

$\Rightarrow$ $$x \leq s-1 \;\forall\; x \in \mathbf{N}$$

$\Rightarrow$ $s-1$ is an upper bound of $\mathbf{N}$.

But $s-1 < s$ and we have assumed that s is the supremum of $\mathbf{N}$. Thus we get an upper bound of $\mathbf{N}$ which is less than the supremum of $\mathbf{N}$. But this contradicts the fact that s is the supremum of $\mathbf{N}$.

Hence $\mathbf{N}$ is not bounded above.

Theorem 3: *The set* $\mathbf{Q}$ *of rational numbers is not order-complete i.e., the ordered field of rational numbers is not a complete ordered field.* **(Kanpur 2009; Gorakhpur 13)**

Proof: We shall show that there exists a non-empty subset of $\mathbf{Q}$ which is bounded above but which does not have any rational number for its supremum.

Let us consider the set S of all those positive rational numbers whose squares are less than 2 *i.e.*, let

$$S = \{x : x \in \mathbf{Q}^+ \text{ and } 0 < x^2 < 2\}.$$

Since $1 \in S$, therefore S is non-empty. Also S is bounded above, 2 being an upper bound for S. Note that $2^2 = 4 > 2$ and so $x \in \mathrm{Q}^+$ and $x^2 < 2 \Rightarrow x < 2$.

Thus S is a non-empty subset of **Q** which is bounded above. We shall show that S does not have any rational number a as its least upper bound. The following cases arise :

Case I: $a \leq 0$. Since every member of S is positive, therefore, a cannot be an upper bound of S and so it cannot be the supremum for S.

Case II: $a > 0$ and $0 < a^2 < 2$.

Let $$b = \frac{4+3a}{3+2a}\cdot \qquad \text{...(1)}$$

Then $$b^2 - 2 = \left(\frac{4+3a}{3+2a}\right)^2 - 2 = \frac{a^2-2}{(3+2a)^2} \qquad \text{...(2)}$$

and $$b - a = \frac{4+3a}{3+2a} - a = \frac{4-2a^2}{3+2a} = \frac{2\,(2-a^2)}{3+2a}\cdot \qquad \text{...(3)}$$

Since a is a positive rational number, it follows from (1) that b is also a positive rational number. Also, since $a^2 < 2$, from (2) we find that $b^2 < 2$ and from (3) we find that $b > a$. Thus b is a positive rational number such that $0 < b^2 < 2$ and so by the definition of the set S, we have $b \in S$. Also, since $b > a$, therefore a cannot be an upper bound for S.

Case III: $a > 0$ and $a^2 = 2$. This is not possible because we know that there is no rational number whose square is 2.

Case IV: $a > 0$ and $a^2 > 2$.

Let us take $b = \dfrac{4+3a}{3+2a}$, as we have taken in case II. But now a^2 being > 2, from (1), (2) and (3) we see that b is a positive rational number such that $b^2 > 2$ and $b < a$ which means that $2 < b^2 < a^2$.

If y be any arbitrary member of S, then we have

$$0 < y^2 < 2 < b^2 < a^2$$

i.e., $$0 < y < b < a.$$

This shows that both a and b are upper bounds for S and that a cannot be the supremum for S because b is an upper bound for S which is less than a.

Since, by the law of trichotomy, the possibilities discussed above are mutually exclusive and exhaustive, therefore, it follows that if a is any rational number, then a cannot be the least upper bound of S.

Hence the system of rational numbers does not satisfy order completeness property.

17 Archimedean Property of Real Numbers

The order-completeness property of real numbers has important consequences. The most important of them is the following *Archimedean property of real numbers.*

Theorem 1: *Let a be any real number and b any positive real number. Then there exists a positive integer n such that*

$$nb > a.$$

Proof: If $a \le 0$, the theorem is obvious because then for every positive integer n we have $nb > a$. [Note that $n > 0, b > 0 \Rightarrow nb > 0$ and so if $a \le 0$, then we have $nb > a$]. So now let a be > 0.

In this case assume that there exists no positive integer n such that $nb > a$.

Then we have $nb \le a \ \forall\ n \in \mathbf{N}$.

It means that a is an upper bound of the non-empty subset S of $\mathbf{R}$ given by

$$S = \{b, 2b, 3b, 4b, \ldots\} = \{nb : n \in \mathbf{N}\}.$$

$\therefore$ by the completeness property of $\mathbf{R}$, S must have the supremum, say s.

Then $nb \le s$ for all $n \in \mathbf{N}$ and so

$$(n+1)\, b \le s \text{ for all } n \in \mathbf{N} \qquad [\because\ n \in \mathbf{N} \Rightarrow n+1 \in \mathbf{N}]$$

i.e., $nb + b \le s$ fo all $n \in \mathbf{N}$ *i.e.,* $nb \le s - b$ for all $n \in \mathbf{N}$.

$\therefore$ $s - b$ is an upper bound of S.

Since $b > 0$, therefore $s - b < s$.

It means that we have an upper bound for S which is less than the supremum s of S. But this contradicts the definition of the supremum.

Hence our initial assumption is wrong and so there must exist some positive integer n such that

$$nb > a.$$

Archimedean ordered field

Definition: *An ordered field F is said to be an Archimedean ordered field if* $\forall\ x, y \in F, y > 0$, *there exists some* $n \in \mathbf{N}$ *such that* $ny > x$.

For example, the field $\mathbf{R}$ of real numbers is an Archimedean ordered field.

Corollary 1: *For every real number a there exists a positive integer n such that*

$$n > a.$$

Proof: Since 1 is a positive real number, therefore by the Archimedean property of real numbers, for every real number a, there exists a positive integer n such that

$$n.1 > a \ \text{ i.e., } n > a.$$

Corollary 2: *For any positive real number x, there exists a positive integer n such that*

$$1/n < x.$$

Proof: By Archimedean property of real numbers, for every positive real number x, there exists a positive integer n such that

$$nx > 1. \qquad \text{[Note that we have taken } b = x \text{ and } a = 1 \text{ in the result of theorem 1]}$$

Since $n > 0$, therefore dividing both sides of the inequality $nx > 1$ by n, we have

$$x > 1/n \quad i.e., \quad 1/n < x.$$

Theorem 2: *For any real number x, there exist two integers m and n such that*

$$m < x < n.$$

Proof: Since 1 is a positive real number, therefore by the Archimedean property of real numbers, for every real number x, there exists a positive integer n such that

$$n.1 > x \quad i.e., \quad n > x. \qquad \text{...(1)}$$

Now $x \in \mathbf{R} \Rightarrow -x \in \mathbf{R}$. Therefore by the result (1), there exists a positive integer m_1 such that

$$m_1 > -x \quad i.e., \quad -m_1 < x. \qquad \text{...(2)}$$

Since $-m_1$ is also an integer, therefore setting $-m_1 = m$, we see from (1) and (2) that there exist two integers m and n such that

$$m < x < n.$$

Hence the theorem is proved.

Theorem 3: *For any real number x, there exists a unique integer n such that*

$$n \le x < n + 1.$$

Proof: Consider the set $S = \{y : y \in \mathbf{Z} \text{ and } y \le x\}$.

Then S is a non-empty subset of integers and is bounded above by x. Therefore S has its supremum, say n, in $\mathbf{Z}$. This n is the greatest integer belonging to the set S.

Thus there exists a greatest integer n such that $n \le x$.

Then $n + 1$ being $> n$, we have $n + 1 > x$. Hence there exists an integer n such that

$$n \le x < n + 1.$$

The uniqueness of the integer n satisfying the inequality

$$n \le x < n + 1$$

follows from the fact that the supremum of a set which is bounded above is unique.

Note: If x is any real number, then we have just shown that there exists one and only one integer n such that $n \le x < n + 1$. The integer n is called the integral part of x and is denoted by $[x]$. Also $x - [x]$ is called the fractional part of x and is always non-negative.

Corollary: *For any $x \in \mathbf{R}$, there exists a unique integer n such that*

$$x - 1 < n \le x.$$

Theorem 4: *For any $x \in \mathbf{R}$ there exists a unique integer n such that*

$$x - 1 \le n < x.$$

Proof: For any $x \in \mathbf{R}$, we know that there exist two integers n_1 and n_2 such that $n_2 < x < n_1$.

Now let n be the largest integer among $n_2, n_2 + 1, n_2 + 2, \ldots, n_1$ such that $n < x$. Then

$$n + 1 \ge x$$

or $$n \ge x - 1$$

or $$x - 1 \le n.$$

Thus we have $n < x$ and $x - 1 \le n$.

Combining these two inequalities, we have

$$x - 1 \le n < x.$$

18 Real Line R (Representation of Real Numbers as Points on a Straight Line)

The geometric representation of real numbers is suggestive and profitably useful to understand more clearly certain properties of real numbers.

Consider any straight line. Take a point O on it. The line is divided into two parts. The portion of the line to the right of O is called the positive part and the portion of the line to the left of O is called the negative part. Let A be any point on the positive part. Let the points O and A represent the numbers 0 and 1 respectively.

Taking OA as unit, we can associate with each real number, exactly one point on the line. Represent the positive real numbers by points to the right of O and the negative real numbers by points to the left of O. Also, each point on the line corresponds to one and only one real number. The line representing the real numbers is called the **real line R**. Thus we talk of a real number as a point of the real line **R**.

Dedikind-Cantor axiom: *To every real number there corresponds a unique point on a directed line and conversely, to every point on a directed line there corresponds a unique real number.*

We can say that there is a one-to-one correspondence between the real numbers and the points of a directed line. That is why the directed line is called the **real line** or **real axis** and **a real number is called a point of the real line.**

Note: If $a, b \in \mathbf{R}, a < b$, then the point a lies to the left of the point b.

If $a, b \in \mathbf{R}, a > b$, then the point a lies to the right of the point b.

The negative numbers lie to the left of O and the positive numbers lie to the right of O.

If $a, b, c \in \mathbf{R}$ and $a < c < b$, then the point c lies between the points a and b.

If $a, b \in \mathbf{R}$, then $|a - b|$ is called the distance between the points a and b.

19 The Denseness Property of The Real Number System

Theorem 1: *Between any two distinct real numbers there always lies a rational number and therefore infinitely many rational numbers.* **(Kanpur 2009)**

Proof: Let a and b be any two distinct real numbers and let $a < b$ so that $b - a > 0$.

Since $b - a > 0$, therefore by the Archimedean property of real numbers, there exists a positive integer n such that

$$n(b-a) > 1 \quad i.e. \quad nb > na + 1. \quad ...(1)$$

Also there exists a unique integer m such that

$$m - 1 \leq na < m$$

so that $$na + 1 \geq m > na. \quad ...(2)$$

From (1) and (2), we have

$$nb > na + 1 \geq m > na.$$

This gives $na < m < nb$

or $$a < \frac{m}{n} < b.$$

Since m and n are integers and $n \neq 0$, therefore m / n is a rational number. If we denote m / n by r, then we have

$$a < r < b$$

and thus there exists a rational number r lying between a and b.

Repeating the above argument for a and r and r and b, we get rational numbers say r_1 and r_2 such that

$$a < r_1 < r \text{ and } r < r_2 < b.$$

Combining these, we get $a < r_1 < r < r_2 < b$.

Continuing to proceed in this way we get infinitely many rational numbers between any two distinct real numbers a and b.

Theorem 2: *Between any two distinct real numbers there always lies an irrational number and therefore infinitely many irrational numbers.*

Proof: Let a and b be any two distinct real numbers and let $a < b$ so that $b - a > 0$.

Let α be any positive irrational number. Since $b - a > 0$, therefore by the Archimedean property of real numbers, there exists a positive integer n such that

$$n(b - a) > \alpha$$

i.e., $$b - a > \frac{\alpha}{n}$$

i.e., $$b > a + \frac{\alpha}{n}\cdot$$

But since α is positive, we have

$$a + \frac{\alpha}{n} > a + \frac{\alpha}{2n} > a.$$

Hence $$b > a + \frac{\alpha}{n} > a + \frac{\alpha}{2n} > a. \qquad \text{...(1)}$$

Now $\left(a + \frac{\alpha}{n}\right) - \left(a + \frac{\alpha}{2n}\right) = \frac{\alpha}{2n}$ which is an irrational number, α being an irrational number. It means that both the numbers $a + \frac{\alpha}{n}$ and $a + \frac{\alpha}{2n}$ cannot be rational otherwise their difference will be a rational number. So at least one of these two numbers is irrational. Let us denote it by x. Then from (1), we have

$$a < x < b$$

and thus there exists an irrational number x lying between a and b.

Repeating the above argument for a and x and x and b, we get irrational numbers say x_1 and x_2 such that

$$a < x_1 < x \text{ and } x < x_2 < b.$$

Combining these, we get

$$a < x_1 < x < x_2 < b.$$

Continuing to proceed in this way we get infinitely many irrational numbers between any two distinct real numbers a and b.

Theorem 3: *Between any two distinct real numbers, there lie an infinite number of real numbers.*

The proof follows from any of the two theorems 1 and 2 above.

Comprehensive Exercise 3

1. Assuming as order-completeness axiom for real numbers that every non-empty subset of real numbers which is bounded below has an infimum, prove that every non-empty subset of real numbers which is bounded above has a supremum.
2. If a and b are any real numbers and $a > 1$, then prove that there exists a positive integer n such that $a^n > b$.
3. Show that the set S of all those positive rational numbers whose square is less than 3 is a non-empty subset of the set **Q** of all rational numbers, which is bounded above. Show also that S has no rational number for its supremum.

   $\left[\text{Hint. Proceed as in theorem 3 of article 6. Take } b = \dfrac{3+2a}{2+a} \cdot \right]$
4. Prove that the system **Q** of rational numbers has the Archimedean property *i.e.*, is an Archimedean ordered field.

20 Neighbourhood of a Point

Definition: *A subset N of* **R** *is said to be a neighbourhood of a point $p \in$* **R** *if there exists a real number $\varepsilon > 0$ such that*

$$]\, p - \varepsilon, p + \varepsilon\, [\, \subset N.$$

Since $\quad p \in\,]\, p - \varepsilon, p + \varepsilon\, [$ and $]\, p - \varepsilon, p + \varepsilon\, [\, \subset N$,

therefore if N is a neighbourhood of p, we must have $p \in$ **N**.

Thus $N \subset$ **R** is a neighbourhood of a point $p \in$ **R** if there exists an open interval contained in N whose centre is the point p.

For example, if N be the closed interval $[3, 5]$, then N is a neighbourhood of the point $4 \in [3, 5]$ because $\varepsilon = \dfrac{1}{4}$ is a positive real number such that

$$\left]\, 4 - \frac{1}{4}, 4 + \frac{1}{4} \right[\subset [3, 5].$$

We shall use the abbreviated form 'nbd' or 'nhd' for the word neighbourhood.

Theorem 1: A characterisation of neighbourhood *i.e.*, an equivalent definition of neighbourhood.

A subset N of **R** *is a neighbourhood of a point $p \in$* **R** *if and only if there exists an open interval $]\, a, b\, [$ containing p and contained in N i.e., if and only if there exists an open interval $]\, a, b\, [$ such that* $\quad p \in\,]\, a, b\, [\, \subset N.$

Proof: 'Only if' part: First suppose that N is a nbd of p. Then by our definition of nbd of a point, there exists a positive real number ε such that

$$]\, p-\varepsilon, p+\varepsilon\, [\, \subset N.$$

Now $]\, p-\varepsilon, p+\varepsilon\, [$ is an open interval containing p and contained in N. Thus if N is a nbd of p, then there exists an open interval $]\, p-\varepsilon, p+\varepsilon\, [$ such that

$$p \in\,]\, p-\varepsilon, p+\varepsilon\, [\, \subset N.$$

'If ' part. Conversely suppose there exists an open interval $]\, a, b\, [$ such that

$$p \in\,]\, a, b\, [\, \subset N.$$

Then to prove that N is a nbd of p.

Let us take a positive real number ε equal to (or even less than) the minimum of the two positive real numbers $p-a$ and $b-p$.

Then $\varepsilon > 0$ is such that

$$\varepsilon \le p-a \text{ and } \varepsilon \le b-p \quad \textit{i.e.,} \quad a \le p-\varepsilon \text{ and } b \ge p+\varepsilon$$

i.e., $\quad a \le p-\varepsilon < p < p+\varepsilon \le b$

i.e., $\quad]\, p-\varepsilon, p+\varepsilon\, [\, \subset\,]\, a, b\, [\, \subset N.$

Thus there exists $\varepsilon > 0$ such that

$$p \in\,]\, p-\varepsilon, p+\varepsilon\, [\, \subset N.$$

Hence N is a nbd of p.

ε-neighbourhood of p. From the above theorem we conclude that if p is any real number, then any open interval I containing p is a neighbourhood of p and consequently all supersets of I are also neighbourhoods of p.

However, for any positive real number ε, $]\, p-\varepsilon, p+\varepsilon\, [$ is an open interval containing p and consequently it is a neighbourhood of p. It is a *symmetric nbd* of p. We often use this form of nbd of p and call it an ε-neighbourhood of p and denote it by $N(p, \varepsilon)$ or by $N_\varepsilon(p)$. We refer to ε as the **radius** of $N(p, \varepsilon)$ and the point p itself is the **mid-point** or the **centre** of $N(p, \varepsilon)$. It is evident that

$$x \in N(p, \varepsilon) \text{ iff } |x-p| < \varepsilon.$$

Thus $\quad N(p, \varepsilon)$ *i.e.,* ε-nbd of $p = \{x : x \in \mathbf{R} \text{ and } |x-p| < \varepsilon\}$

$$= \{x : x \in \mathbf{R} \text{ and } p-\varepsilon < x < p+\varepsilon\} = \,]\, p-\varepsilon, p+\varepsilon\, [.$$

Geometrically speaking an ε-nbd of p is the set of all the points on the real line which are within ε distance of p on either side of it.

Deleted neighbourhood of p. If from a nbd of a point p, the point p itself is deleted or excluded, then we get a **deleted neighbourhood** of p. Thus if N is a nbd of a point p, then the set $N - \{p\}$ is a deleted nbd of p. **(Kanpur 2010)**

A *symmetric deleted nbd* of a point p will be of the form

$$]\, p-\varepsilon, p+\varepsilon\, [\, - \{p\},$$

where ε is any positive real number.

We shall denote it by $N^{(d)}(p, \varepsilon)$.

Thus an ε-nbd of p is a set of the form

$$\{x \in \mathbf{R} : |x-p| < \varepsilon\}$$

and a deleted symmetric nbd of p is a set of the form

$$\{x \in \mathbf{R} : 0 < |x - p| < \varepsilon\}.$$

Precisely speaking a symmetric deleted nbd of p is the union of two bounded intervals

$$]\, p - \varepsilon, p\, [\text{ and }]\, p, p + \varepsilon\, [.$$

Theorem 2: *A non-empty subset A of* $\mathbf{R}$ *is a nbd of* $p \in \mathbf{R}$ *if and only if there exists a positive integer n such that*

$$\left] p - \frac{1}{n}, p + \frac{1}{n} \right[\subset A.$$

Proof: Let a non-empty subset A of $\mathbf{R}$ be a nbd of a point $p \in \mathbf{R}$. Then there exists $\varepsilon > 0$ such that

$$p \in \,]\, p - \varepsilon, p + \varepsilon\, [\subset A.$$

Now given any positive real number ε, we can always choose a positive integer n so large that $\quad 1/n < \varepsilon.$

But $\quad 1/n < \varepsilon \Rightarrow p + (1/n) < p + \varepsilon.$

Also $\quad 1/n < \varepsilon \;\Rightarrow\; -1/n > -\varepsilon$

$\Rightarrow \quad p - (1/n) > p - \varepsilon.$

$\therefore \quad p - \varepsilon < p - (1/n) < p + (1/n) < p + \varepsilon$

$$\Rightarrow \quad \left] p - \frac{1}{n}, p + \frac{1}{n} \right[\subset \,]\, p - \varepsilon, p + \varepsilon\, [.$$

Hence if A is a nbd of p, there exists a positive integer n such that

$$\left] p - \frac{1}{n}, p + \frac{1}{n} \right[\subset A.$$

Conversely suppose there exists a positive integer n such that

$$\left] p - \frac{1}{n}, p + \frac{1}{n} \right[\subset A.$$

Then A is a nbd of p because $\left] p - \frac{1}{n}, p + \frac{1}{n} \right[$ is an open interval containing p and contained in A.

Hence the theorem.

Illustrative Examples

Example 18: *Any open interval is a nbd of each of its points.*

Solution: Let $]\, a, b\, [$ be any open interval and x be any arbitrary point of $]\, a, b\, [$. Then to show that $]\, a, b\, [$ is a nbd of x.

If we take ε as the minimum of the two positive numbers $x - a$ and $b - x$, then $\varepsilon > 0$ is such that

$$x \in \,]\, x - \varepsilon, x + \varepsilon\, [\subset \,]\, a, b\, [.$$

Hence $]\, a, b\, [$ is a nbd of x.

Since x is an arbitrary point of $]\, a, b\, [$, therefore we conclude that any open interval is a nbd of each of its points.

Example 19: *A closed interval $[a, b]$ is a nbd of each of its points except the two end points a and b.*

Solution: Let $x \in]a, b[$.

Take $\varepsilon = \min \{x - a, b - x\}$.

Then $\varepsilon \le x - a \Rightarrow a \le x - \varepsilon$

and $\varepsilon \le b - x \Rightarrow x + \varepsilon \le b$.

$\therefore$ $a \le x - \varepsilon < x + \varepsilon \le b$.

Thus $\varepsilon > 0$ is such that

$$x \in]x - \varepsilon, x + \varepsilon[\subset]a, b[\subset [a, b].$$

$\therefore$ $[a, b]$ is a nbd of $x \in]a, b[$.

$\therefore$ $[a, b]$ is a nbd of each point of $]a, b[$.

Again $[a, b]$ is not a nbd of $a \in [a, b]$. For if we take any positive real number ε, then $]a - \varepsilon, a + \varepsilon[\not\subset [a, b]$. Note that if $x \in]a - \varepsilon, a + \varepsilon[$ and $a - \varepsilon < x < a$, then $x \notin [a, b]$ and therefore $]a - \varepsilon, a + \varepsilon[\not\subset [a, b]$. Thus there exists no $\varepsilon > 0$ such that

$$]a - \varepsilon, a + \varepsilon[\subset [a, b].$$

Hence $[a, b]$ is not a nbd of a.

Similarly $[a, b]$ is not a nbd of $b \in [a, b]$. For if we take any positive real number ε, then $]b - \varepsilon, b + \varepsilon[\not\subset [a, b]$. Note that if $x \in]b - \varepsilon, b + \varepsilon[$ and $b < x < b + \varepsilon$, then $x \notin [a, b]$ and so $]b - \varepsilon, b + \varepsilon[\not\subset [a, b] \; \forall \; \varepsilon > 0$. Hence $[a, b]$ is not a nbd of b.

Example 20: *The set of rational numbers* **Q** *is not a nbd of any of its points.*

Solution: Let $p \in \mathbf{Q}$. For any positive real number ε, $p - \varepsilon$ and $p + \varepsilon$ are two distinct real numbers and we know that between any two distinct real numbers there lie infinite irrational numbers which are not members of **Q**.

$\therefore$ $]p - \varepsilon, p + \varepsilon[\not\subset \mathbf{Q} \quad \forall \; \varepsilon > 0.$

$\therefore$ **Q** is not a nbd of p.

Since p is an arbitrary point of **Q**, therefore **Q** is not a nbd of any of its points.

Example 21: *Is it true that the null set $\varnothing$ is a nbd of each of its points ?*

Solution: Yes. The null set $\varnothing$ is a nbd of each of its points because there is no point at all in $\varnothing$ and so there is no point in $\varnothing$ of which it is not a nbd.

Example 22: *Show that a non-empty finite set cannot be a nbd of any of its points.*

Solution: Let S be any non-empty finite set and let $x \in S$. Then for any positive real number ε, the open interval $]x - \varepsilon, x + \varepsilon[$ is an infinite set *i.e.*, it contains an infinite number of distinct elements and so it cannot be a subset of a finite set S. Thus there exists no $\varepsilon > 0$ such that

$$]x - \varepsilon, x + \varepsilon[\subset S.$$

$\therefore$ S is not a nbd of x and hence S is not a nbd of any of its points.

Example 23: *Show that the set $\mathbf{Z}^+$ of all positive integers is not a nbd of any of its points.*

Or

Show that the set **N** *of all natural numbers is not a neighbourhood of any of its points.*

Solution: Let $x \in \mathbf{Z}^+$. Then

$$]\,x-\varepsilon, x+\varepsilon\,[\not\subset \mathbf{Z}^+ \;\forall\, \varepsilon > 0. \qquad \text{[Give argument as in Ex. 20]}$$

Hence $\mathbf{Z}^+$ is not a nbd of any $x \in \mathbf{Z}^+$.

Example 24: *Show that the set* **R** *of all real numbers is a nbd of each of its points.*

Solution: Let $x \in \mathbf{R}$. Then for every positive real number ε, we have

$$x \in\,]\,x-\varepsilon, x+\varepsilon\,[\subset \mathbf{R}.$$

Hence **R** is a nbd of every $x \in \mathbf{R}$.

Example 25: *Show that any set S cannot be a nbd of any point of the set* $\mathbf{R} - S$.

Solution: Let $x \in \mathbf{R} - S$. Then $x \notin S$.

Since $\forall\, \varepsilon > 0, x \in\,]\,x-\varepsilon, x+\varepsilon\,[$, therefore

$$]\,x-\varepsilon, x+\varepsilon\,[\not\subset S \quad \forall\, \varepsilon > 0.$$

Hence S is not a nbd of any $x \in \mathbf{R} - S$.

Example 26: *Which of the following subsets of* **R** *are nbds of* 3 *? Give reasons.*

(*i*) $]\,2, 4\,[$; (*ii*) $[2, 4\,[$; (*iii*) $]\,2, 4]$; (*iv*) $[2, 4]$

(*v*) $]\,3, 7\,[$; (*vi*) $]\,3, 5\,]$; (*vii*) $[3, 6\,[$ (*viii*) $[2, 4] - \left\{3\frac{1}{4}\right\}$.

Solution: (i) Since $]\,2, 4\,[$ is an open interval and $3 \in\,]\,2, 4\,[$, therefore $]\,2, 4\,[$ is a nbd of 3.

(ii) Since $3 \in$ the open interval $]\,2, 4[$ which is a subset of $[2, 4[$, therefore $[2, 4[$ is a nbd of 3.

(iii) Since $3 \in\,]\,2, 4[\, \subset\,]\,2, 4]$, therefore $]\,2, 4]$ is a nbd of 3.

(iv) Since there exists an open interval $]\,2, 4[$ such that

$$3 \in\,]\,2, 4\,[\, \subset [2, 4],$$

therefore $[2, 4]$ is a nbd of 3.

(v) Since $3 \notin\,]\,3, 7\,[$, therefore $]\,3, 7\,[$ cannot be a nbd of 3.

(vi) Since $3 \notin\,]\,3, 5]$, therefore $]\,3, 5]$ is not a nbd of 3.

(vii) $[3, 6\,[$ is not a nbd of $3 \in [3, 6\,[$ because

$$]\,3-\varepsilon, 3+\varepsilon\,[\not\in [3, 6\,[\;\forall\, \varepsilon > 0.$$

(viii) $[2, 4] - \left\{3\frac{1}{4}\right\}$ is a nbd of 3 since there exists an open interval $\left]3 - \frac{1}{5}, 3 + \frac{1}{5}\right[$ such that

$$3 \in \left]\,3 - \frac{1}{5}, 3 + \frac{1}{5}\right[\subset \left([\,2, 4\,] - \{3\tfrac{1}{4}\}\right).$$

Example 27: *Let* $I_n = \left]-\frac{1}{n}, 1 + \frac{1}{n}\right[$ *be an open interval for each* $n \in \mathbf{N}$, *the set of natural numbers. Find* $\bigcap_{n=1}^{\infty} I_n$ and *show that it is a nbd of each of its points with the exception of two points.*

Solution: First we shall show that

$$\bigcap_{n=1}^{\infty} I_n = [0,1].$$

We have $x \in [0,1] \Rightarrow 0 \le x \le 1$

$\Rightarrow \qquad x > -\dfrac{1}{n}$ and $x < 1 + \dfrac{1}{n}$ for all $n \in \mathbf{N}$

$\Rightarrow \qquad x \in \left]-\dfrac{1}{n}, 1+\dfrac{1}{n}\right[$ for all $n \in \mathbf{N}$

$\Rightarrow \qquad x \in \bigcap_{n=1}^{\infty} I_n.$

$\therefore \qquad [0,1] \subset \bigcap_{n=1}^{\infty} I_n.$ ...(1)

Again let $x \in \bigcap_{n=1}^{\infty} I_n$ *i.e.*, $x \in I_n \;\forall\; n \in \mathbf{N}$. Then to show that $x \in [0,1]$. If possible let $x \notin [0,1]$.

Then either $x > 1$ or $x < 0$.

If $x > 1$, let $x = 1 + b$ where $b > 0$. Then there exists $m \in \mathbf{N}$ such that $1/m < b$ so that $1 + (1/m) < 1 + b$ or $1 + b > 1 + (1/m)$. But then $x = 1 + b \notin \left]-\dfrac{1}{m}, 1+\dfrac{1}{m}\right[$ which contradicts our assumption that $x \in I_n \;\forall\; n \in \mathbf{N}$.

Similarly if $x < 0$, let $x = -b$ where $b > 0$. Then there exists $m \in \mathbf{N}$ such that $1/m < b$ or $-1/m > -b$. But then $x = -b \notin \left]-\dfrac{1}{m}, 1+\dfrac{1}{m}\right[$ which contradicts our assumption that $x \in I_n \;\forall\; n \in \mathbf{N}$.

$\therefore \qquad$ if $x \in \bigcap_{n=1}^{\infty} I_n$, then definitely $x \in [0,1]$,

so that $\qquad \bigcap_{n=1}^{\infty} I_n \subset [0,1].$ ...(2)

From (1) and (2), we conclude that

$$\bigcap_{n=1}^{\infty} I_n = [0,1].$$

Now $[0, 1]$ is a closed interval. It is a nbd of each of its points except its end points 0 and 1.

21 Properties of Neighbourhoods

Theorem 1: *On the real line* $\mathbf{R}$, *for each point* $p \in \mathbf{R}$, *there exists at least one nbd of p.*

Proof: If $p \in \mathbf{R}$, then $\mathbf{R}$ is always a nbd of p because for each $\varepsilon > 0$, we have

$$p \in \,]\,p - \varepsilon, p + \varepsilon\,[\, \subset \mathrm{R}.$$

Theorem 2: *If N is a nbd of any point* $p \in \mathbf{R}$, *then* $p \in N$.

Proof: If N is a nbd of $p \in \mathbf{R}$, then by the definition of a nbd of a point, there exists $\varepsilon > 0$ such that

$$p \in]\, p-\varepsilon, p+\varepsilon\, [\, \subset N.$$

Hence if N is a nbd of p, then $p \in N$.

Theorem 3: *Any superset of a nbd of a point is also a nbd of that point.*

Proof: Let N be a nbd of a point $p \in \mathbf{R}$.

Then there exists $\varepsilon > 0$ such that

$$p \in]\, p-\varepsilon, p+\varepsilon\, [\, \subset N. \qquad ...(1)$$

If $M \supset N$, then from (1) it follows that

$$p \in]\, p-\varepsilon, p+\varepsilon\, [\, \subset N \subset M,$$

so that M is also a nbd of p.

Theorem 4: *The intersection of two nbds of a point is also a nbd of that point.*

Proof: Let M, N be two nbds of a point p. Then there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$]\, p-\varepsilon_1, p+\varepsilon_1\, [\, \subset M,$$

and $$]\, p-\varepsilon_2, p+\varepsilon_2\, [\, \subset N.$$

Take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Then

$$]\, p-\varepsilon, p+\varepsilon\, [\, \subset\,]\, p-\varepsilon_1, p+\varepsilon_1\, [\, \subset M,$$

and $$]\, p-\varepsilon, p+\varepsilon\, [\, \subset\,]\, p-\varepsilon_2, p+\varepsilon_2\, [\, \subset N.$$

It follows that $]\, p-\varepsilon, p+\varepsilon\, [\, \subset M \cap N$ and consequently $M \cap N$ is a nbd of p.

Theorem 5: *On the real line* $\mathbf{R}$ *for each point $p \in \mathbf{R}$ and each nbd N of p, there exists a nbd M of p such that $M \subset N$ and M is a nbd of each of its points.*

Proof: Since N is a nbd of p, therefore there exists $\varepsilon > 0$ such that

$$p \in]\, p-\varepsilon, p+\varepsilon\, [\, \subset N.$$

Let $M =]\, p-\varepsilon, p+\varepsilon\, [$. Then as $]\, p-\varepsilon, p+\varepsilon\, [$ is an open interval containing p, so it is a nbd of p and also a nbd of each of its points.

Hence there exists a nbd M of p such that $M \subset N$ and M is a nbd of each of its points.

22 Some More Theorems on Neighbourhoods

Theorem 1: *If a and b are any two distinct real numbers, then there exist neighbourhoods of a and b which are disjoint.* (*This is known as* **Haousdorff property**.)

Proof: Take $\varepsilon = \frac{1}{3}|b-a|$. Then the nbd $N(a, \varepsilon)$ of a and the nbd $N(b, \varepsilon)$ of b are such that no real number exists which is a member of both these nbds.

Theorem 2: *Let a be any point of the nbd $N(p, \varepsilon)$. Then there exists a nbd of a which is entirely contained in $N(p, \varepsilon)$.*

Proof: $a \in N(p, \varepsilon) \Rightarrow |p-a| < \varepsilon$ so that

$$\varepsilon - |p-a| > 0.$$

Choose a positive real number δ such that

$$\delta < \varepsilon - |p-a|. \qquad ...(1)$$

We shall show that the nbd $N(a, \delta)$ of a is entirely contained in $N(p, \varepsilon)$ *i.e.*, $N(a, \delta) \subset N(p, \varepsilon)$.

Let $\quad x \in N(a, \delta)$. Then

$$|x-a| < \delta < \varepsilon - |p-a|, \text{by (1)}. \quad ...(2)$$

Now $\quad |x-p| = |(x-a)+(a-p)| \le |x-a| + |a-p|$

$< \varepsilon - |p-a| + |a-p|$, by (2)

$= \varepsilon.$

Thus $|x-p| < \varepsilon$ which implies that $x \in N(p, \varepsilon)$.

$\therefore \quad x \in N(a, \delta) \Rightarrow x \in N(p, \varepsilon)$.

Hence $\quad N(a, \delta) \subset N(p, \varepsilon)$.

Theorem 3: *Let a denote any point of the intersection M of the nbds $N(a_1, \varepsilon_1)$ and $N(a_2, \varepsilon_2)$ of a_1 and a_2. Then there exists a nbd of a which is entirely contained in M.*

Proof: Since $M = N(a_1, \varepsilon_1) \cap N(a_2, \varepsilon_2)$, therefore

$a \in M \Rightarrow a \in N(a_1, \varepsilon_1)$ and $a \in N(a_2, \varepsilon_2)$

$\Rightarrow |a-a_1| < \varepsilon_1$ and $|a-a_2| < \varepsilon_2$

$\Rightarrow \varepsilon_1 - |a-a_1| > 0$

and $\quad \varepsilon_2 - |a-a_2| > 0$.

Let ε denote the minimum of the two positive real numbers

$\varepsilon_1 - |a-a_1|$ and $\varepsilon_2 - |a-a_2|$.

We shall show that the ε-nbd of a *i.e.* $N(a, \varepsilon)$ is entirely contained in M.

Let $\quad x \in N(a, \varepsilon)$. Then

$$|x-a| < \varepsilon \le \varepsilon_1 - |a-a_1|. \quad ...(1)$$

Now $\quad |x-a_1| = |(x-a)+(a-a_1)|$

$\le |x-a| + |a-a_1|$

$< \varepsilon_1 - |a-a_1| + |a-a_1|$, by (1)

$= \varepsilon_1$.

Thus $|x-a_1| < \varepsilon_1$ which implies that $x \in N(a_1, \varepsilon_1)$.

$\therefore \quad x \in N(a, \varepsilon) \Rightarrow x \in N(a_1, \varepsilon_1)$.

$\therefore \quad N(a, \varepsilon) \subset N(a_1, \varepsilon_1)$. ...(2)

Similarly we can show that

$N(a, \varepsilon) \subset N(a_2, \varepsilon_2)$. ...(3)

From (2) and (3), we have

$N(a, \varepsilon) \subset N(a_1, \varepsilon_1) \cap N(a_2, \varepsilon_2)$

$\Rightarrow \quad N(a, \varepsilon) \subset M$.

Hence the result.

Comprehensive Exercise 4

1. Show that the closed interval $[1, 3]$ is a nbd of 2 but not of any of its end points 1 and 3.

2. Show that the set **Z** of all integers is not a nbd of any of its points.
3. Give an example of each of the following :
   (i) a set which is a nbd of each of its points.
   (ii) a set which is not a nbd of any of its points.
   (iii) a set which is a nbd of each of its points with the exception of two points.
   (iv) a set which is a nbd of each of its points with the exception of one point.
4. Show that a set $N \subset \mathbf{R}$ is a nbd of a point $p \in \mathbf{R}$ if and only if there exists a positive rational number r such that
$$]\, p - r, p + r\, [\subset N.$$
5. Show that the intersection of the family of all neighbourhoods of an arbitrary point $x \in \mathbf{R}$ is the singleton $\{x\}$.
   [**Hint.** If y be a point different from x, and $\varepsilon = | x - y |$, then obviously $]\, x - \varepsilon, x + \varepsilon\, [$ is a nbd of x that does not contain y.
6. Show that the set $S =]\, 2, 3\, [\cup]\, 5, 6\, [$ is a nbd of each of its points.
7. If $I = \bigcup_{n=1}^{\infty} I_n$, where each I_n is an open interval, then show that I is a nbd of each of its points.
8. Is the set $A = \{1, 2, 3, 4\}$ a nbd of 2 ?
9. Is **Q** the set of rational numbers a nbd of the point 4 ?
10. Show that the right half open interval $[\, 2, 3\, [$ is a nbd of each of its points except that of 2.
11. Show that the set of all irrational numbers is not a nbd of any of its points.

23 Adherent Points and Limit Points (or Cluster Points) of a Set

Adherent point: Definition: *A point $p \in \mathbf{R}$ is said to be an* **adherent point** *of a set $A \subset \mathbf{R}$ if every neighbourhood of p contains a point of A. The set of all adherent points of A is called the adherence of A and is denoted by* **Adh** *A.*

From the above definition it is obvious that every point of A is an adherent point of A *i.e.*, $A \subset \text{Adh } A$.

Limit point or Limiting point or Cluster point

Definition: *A point $p \in \mathbf{R}$ is said to be a* **limit point** (*or* **an accumulation point** *or* **a cluster point** *or* **a condensation point**) *of a set $A \subset \mathbf{R}$ if every neighbourhood of p contains a point of A* **distinct from** *p.* **(Kanpur 2012)**

Thus p will be a limit point of A if and only if for each $\varepsilon > 0$, the open interval $]\, p - \varepsilon, p + \varepsilon\, [$ contains a point of A other than p.

Symbolically, a point $p \in \mathbf{R}$ will be a limit point of a subset A of $\mathbf{R}$ iff
$$(N \cap A) - \{p\} \neq \varnothing \quad \text{for every nbd } N \text{ of } p$$

or $\quad (N - \{p\}) \cap A \neq \varnothing$ for every nbd N of p

or $\quad N \cap (A - \{p\}) \neq \varnothing$ for every nbd N of p

or $\quad (\,]\, p - \varepsilon, p + \varepsilon\, [\, \cap A) - \{p\} \neq \varnothing$ for every $\varepsilon > 0$.

From the definitions of a limit point and an adherent point of a set $A \subset \mathbf{R}$, it is obvious that every limit point of A is also an adherent point of A but the converse is not always true. For example 1 is an adherent point of the set $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ but it is not a limit point of this set.

A limit point of a set A may or may not belong to A.

Closed set: Definition: *A set $A \subset \mathbf{R}$ is said to be a* **closed set** *if it contains all its limit points.*

In order to show that a point p is not a **limit point** of a set A, it is enough to show that there exists a neighbourhood N of p, such that either $N \cap A = \{p\}$ or $N \cap A = \varnothing$. In other words a point p will not be a limit point of a set A if there exists some open interval I containing p which contains no points of A other than p or if there exists a real number $\varepsilon > 0$ such that the open interval $]\, p - \varepsilon, p + \varepsilon\, [$ contains no points of A other than p.

Isolated point: Definition: *A point $a \in A$ is said to be an* **isolated point** *of A if it is not a limit point of A i.e., if there exists a nbd of a which contains no points of A other than a itself. A set A is called a* **discrete set** *if all its points are isolated points.*

For example all the points of the set $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$ are its isolated points and so it is a discrete set.

It should be noted that each point of a set A is either an isolated point of A or a limit point of A.

A dense-in-itself set and a perfect set: Definition: *A subset A of $\mathbf{R}$ is said to be* **dense-in-itself** *if it possesses no isolated points i.e., every point of A is a limit point of A. Further a set A is called a* **perfect set** *if it is dense-in-itself and if it contains all its limit points.*

(Kanpur 2012)

Some important characterisations of limit points or equivalent definitions of a limit point.

Theorem 1: *A point $p \in \mathbf{R}$ is a limit point of a set $A \subset \mathbf{R}$ if and only if every neighbourhood of p contains* **infinitely many** *points of A.*

Proof: First suppose that every nbd of p contains infinitely many points of A. Then obviously every nbd of p contains a point of A which is different from p, and consequently, p is a limit point of A.

Conversely suppose that p is a limit point of A. Then to prove that every nbd of p contains infinitely many points of A.

Suppose there exists a nbd N of p which contains only finitely many points of A. Then there exists $\varepsilon > 0$ such that the open interval $]\, p - \varepsilon, p + \varepsilon\, [$ contains only finitely many points of A. If p is the only point of A which is contained in $]\, p - \varepsilon, p + \varepsilon\, [$, then p is not a limit point of A. If $]\, p - \varepsilon, p + \varepsilon\, [$ contains points of A other than p also, then since their number has been assumed to be finite, let they be $p_1, p_2, \ldots, p_n$. Out of these n points of A let p_r be the point which is nearest to p and let $|\, p_r - p\,| = \varepsilon_1$, so that,

$$\varepsilon_1 = \min\{|p_1 - p|, |p_2 - p|, \ldots, |p_n - p|\}.$$

Then the real number $\varepsilon_1 > 0$ is such that the open interval $]\, p - \varepsilon_1, p + \varepsilon_1\, [$ contains no points of A other than p. Thus there exists a nbd of p which contains no points of A other than p and so p is not a limit point of A.

Thus if there exists a nbd N of p which contains only finitely many points of A, then p is not a limit point of A. But this contradicts the hypothesis that p is a limit point of A. Hence if p is a limit point of A, then every nbd of p must contain infinitely many points of A.

Note: The above characterisation of a limit point will be very useful to solve some problems.

Theorem 2: *A point $p \in \mathbf{R}$ is a limit point of a set $A \subset \mathbf{R}$ iff for each neighbourhood N of p,*

$$(A \cap N) - \{p\} \neq \varnothing.$$

Proof: First suppose that p is a limit point of A. Let N be any nbd of p. Then there exists $\varepsilon > 0$ such that

$$p \in\,]\, p - \varepsilon, p + \varepsilon\, [\subset N. \qquad \ldots(1)$$

Since p is a limit point of A, therefore the open interval $]\, p - \varepsilon, p + \varepsilon\, [$ must contain a point of A other than p.

$\therefore$ $\quad (A \cap\,]\, p - \varepsilon, p + \varepsilon\, [\,) - \{p\} \neq \varnothing$

$\Rightarrow$ $\quad (A \cap N) - \{p\} \neq \varnothing$, from (1).

$\therefore$ if p is a limit point of A, then for each nbd N of p,

$$(A \cap N) - \{p\} \neq \varnothing.$$

Conversely suppose that for every nbd N of p, we have

$$(A \cap N) - \{p\} \neq \varnothing.$$

Then to prove that p is a limit point of A.

Take any positive real number ε. Then $]\, p - \varepsilon, p + \varepsilon\, [$ is a nbd of p and so by hypothesis

$$(\,]\, p - \varepsilon, p + \varepsilon\, [\cap A) - \{p\} \neq \varnothing$$

i.e. $\quad]\, p - \varepsilon, p + \varepsilon\, [$ contains a point of A other than p.

Thus for every $\varepsilon > 0$, the open interval $]\, p - \varepsilon, p + \varepsilon\, [$ contains a point of A other than p. Hence p must be a limit point of A.

Some more theorems on limit points

Theorem 3: *If a non-empty subset S of **R** which is bounded above has no maximum member, then its supremum is a limit point of the set S.*

Proof: Let S be a non-empty subset of **R** which is bounded above. Then by the order-completeness property of real numbers, S has a supremum in **R**.

Let $\quad \sup S = u$.

Since S has no maximum member, therefore $u \notin S$.

Take any positive real number ε. Since $\sup S = u$, therefore $u - \varepsilon$ cannot be an upper bound for S and so there exists $x \in S$ such that $x > u - \varepsilon$.

Since $u + \varepsilon > u$ and u is $\sup S$, therefore $u + \varepsilon$ is also an upper bound for S and so $x \in S \Rightarrow x < u + \varepsilon$.

Thus for every $\varepsilon > 0$, there exists $x \in S$ such that

$$u - \varepsilon < x < u + \varepsilon.$$

Since $u \notin S$, therefore $x \neq u$.

$\therefore$ every ε-nbd $]\, u - \varepsilon, u + \varepsilon\,[$ of u contains a point x of S which is different from u. Hence u is a limit point of S.

Theorem 4: *If a non-empty subset S of* **R** *which is bounded below has no minimum member, then its infimum is a limit point of the set S.*

Proof: Let S be a non-empty subset of **R** which is bounded below. Then by the order-completeness property of real numbers, S has an infimum in **R**.

Let $\inf S = v$.

Since S has no minimum member, therefore $v \notin S$.

Take $\varepsilon > 0$. Since $\inf S = v$, therefore $v + \varepsilon$ cannot be a lower bound for S and so there exists $x \in S$ such that $x < v + \varepsilon$.

Since $v - \varepsilon < v$ and v is $\inf S$, therefore $v - \varepsilon$ is also a lower bound for S and so

$$x \in S \Rightarrow x > v - \varepsilon.$$

Thus for every $\varepsilon > 0$, there exists $x \in S$ such that

$$v - \varepsilon < x < v + \varepsilon.$$

Since $v \notin S$, therefore $x \neq v$.

$\therefore$ every ε-nbd $]\, v - \varepsilon, v + \varepsilon\,[$ of v contains a point x of S which is different from v. Hence v is a limit point of S.

24 Derived Sets

Definition: *The set of all limit points of a set $A \subset$* **R** *is called the derived set of A and is denoted by $D(A)$ or by A'.*

Thus $D(A) = \{x : x \text{ is a limit point of } A\}$.

Again the derived set of $D(A)$ is called the second derived set of A and is denoted by $D^2(A)$ or by A''. In general, the nth derived set of A is denoted by $D^{(n)}(A)$ or by $A^{(n)}$.

It can be easily seen that Adh $A = A \cup D(A)$.

Also it can be easily seen that if a set A is finite, then A has no limit point and consequently $D(A) = \varnothing$.

Definition. *A set is said to be of* **first species** *if it has only a finite number of derived sets. It is said to be of* **second species** *if the number of its derived sets is infinite.*

Note that if a set is of first species, then its last derived set must be empty.

A set whose nth derived set is a finite set so that its $(n + 1)$th derived set is empty is called a set of ***n*th order**.

Illustrative Examples

Example 28: *Find the derived set of the set* **Q** *of all rational numbers.*

Solution: We shall show that every real number is a limit point of the set **Q**.

Let p be any real number and let $\varepsilon > 0$ be given. Then $p - \varepsilon$ and $p + \varepsilon$ are two distinct real numbers and we know that between two distinct real numbers there lie infinitely many rational numbers. Therefore for every $\varepsilon > 0$, the open interval $]\, p - \varepsilon, p + \varepsilon\, [$ contains at least one point of **Q** other than p. Hence p is a limit point of **Q**.

Since p is an arbitrary real number, therefore every real number is a limit point of **Q**. Hence the set of the limit points of **Q** is the set of all real numbers **R**.

$\therefore \qquad D(\mathbf{Q}) = \mathbf{R}.$

Example 29: *Find the limiting points of the set of irrational numbers.*

Solution: We shall show that every real number is a limit point of the set $\mathbf{R} - \mathbf{Q}$ of irrational numbers.

Let $p \in \mathbf{R}$ and $\varepsilon > 0$ be given. Then between two distinct real numbers $p - \varepsilon$ and $p + \varepsilon$ there lie infinitely many irrational numbers. Therefore for every $\varepsilon > 0$, the open interval $]\, p - \varepsilon, p + \varepsilon\, [$ contains at least one point of the set of irrational numbers which is distinct from p. Hence every real number p is a limit point of the set of irrational numbers *i.e.*, $D(\mathbf{R} - \mathbf{Q}) = \mathbf{R}$.

Example 30: *Show that every point of the set* **R** *of real numbers is a limit point of* **R**.

Solution: Let $p \in \mathbf{R}$. Then for every $\varepsilon > 0$, the open interval $]\, p - \varepsilon, p + \varepsilon\, [$ contains infinitely many real numbers. So for every $\varepsilon > 0$, the open interval $]\, p - \varepsilon, p + \varepsilon\, [$ contains at least one point of the set **R** which is distinct from p. Hence every real number p is a limit point of **R** *i.e.*, $D(\mathbf{R}) = \mathbf{R}$.

Note: We have $D(\mathbf{Q}) = \mathbf{R}$, $D^2(\mathbf{Q}) = D(\mathbf{R}) = \mathbf{R}$, $D^3(\mathbf{Q}) = \mathbf{R}$, $D^4(\mathbf{Q}) = \mathbf{R}$ and so on. Thus for every positive integer n, $D^{(n)}(\mathbf{Q}) = \mathbf{R}$. Therefore the number of derived sets of the set **Q** is infinite and so the set **Q** is of the second species.

Since each point of the set **Q** is its limit point, therefore the set **Q** has no isolated point. Hence the set **Q** is dense-in-itself. But the set **Q** is not perfect because it does not contain all its limit points. We have $D(\mathbf{Q}) = \mathbf{R} \not\subset \mathbf{Q}$.

Example 31: *Show that the set* **N** *of natural numbers has no limit points.*

Solution: Let p be any real number. The open interval $\left] p - \frac{1}{4}, p + \frac{1}{4} \right[$ whose length is $\frac{1}{2}$ contains at the most one natural number. Thus $]\, p - \frac{1}{4}, p + \frac{1}{4}\, [$ is a nbd of p which does not contain infinitely many points of the set **N**. Hence p is not a limit point of **N**. Thus no real number p is a limit point of **N** *i.e.*, $D(\mathbf{N}) = \varnothing$.

Note: Since no point of the set **N** is a limit point of **N**, therefore all the points of **N** are isolated points. Hence **N** is a discrete set.

Example 32: *Show that the set* **Z** *of all integers has no limit points.*

Solution: Proceed as in Ex. 30 and show that no real number p is a limit point of the set **Z**. Hence $D(\mathbf{Z}) = \varnothing$.

Alternative Solution: We shall show that every real number p is not a limit point of the set **Z**.

(i) If $p \in \mathbf{Z}$, then $\left] p - \frac{1}{4}, p + \frac{1}{4} \right[\cap \mathbf{Z} = \{p\}$.

(ii) If $p \notin \mathbf{Z}$ *i.e.*, if p is a real number which is not an integer then for some integer j, we have $j < p < j + 1$. Thus $]\, j, j + 1\, [$ is a nbd of p which does not contain any point of **Z**.

From (i) and (ii), we see that whatever real number p we may take, there exists a nbd of p which does not contain any point of **Z** other than p. Hence p is not a limit point of **Z**. Thus every real number p is not a limit point of **Z** and so **Z** has no limit points *i.e.*, $D(\mathbf{Z}) = \varnothing$.

Note: Since $D(\mathbf{Z}) = \varnothing \subset \mathbf{Z}$, therefore the set **Z** is a closed set.

Example 33: *Show that a finite set has no limit points.* **(Kanpur 2009)**

Solution: Let S be any finite set. We shall show that every real number p is not a limit point of the set S.

Since the set S is finite, therefore if we take any real number $\varepsilon > 0$, the open interval $]\, p - \varepsilon, p + \varepsilon\, [$ contains only finitely many points of the set S. Thus $]\, p - \varepsilon, p + \varepsilon\, [$ is a nbd of p which does not contain infinitely many points of S and so p is not a limit point of S.

Thus every real number p is not a limit point of S and so a finite set S has no limit points *i.e.*, the derived set of a finite set is empty.

Note 1: If S is a finite set, then $D(S) = \varnothing\ S$. Hence every finite set is a closed set.

Note 2: The null set $\varnothing$ has no limit points *i.e.*, the derived set of the null set is the null set.

Thus $D(\varnothing) = \varnothing \subset \varnothing$ and so the null set $\varnothing$ is a closed set.

Example 34: *Find the limit points of the open interval* $]\, 0, 1\, [$.

Solution: Let $S = \,]\, 0, 1\, [$.

First we shall show that every point of the closed interval $[0, 1]$ is a limit point of S.

Let p be any point of $[0, 1]$ and ε be any positive real number. Then the open interval $]\, p - \varepsilon, p + \varepsilon\, [$ contains infinitely many points of $S = \,]\, 0, 1\, [$ and so it contains at least one point of $]\, 0, 1\, [$ other than p. Hence p is a limit point of $]\, 0, 1\, [$.

Now we shall show that if $p \notin [0, 1]$ *i..e.*, if $p < 0$ or if $p > 1$, then p is not a limit point of $]\, 0, 1\, [$.

Suppose $p \notin [0, 1]$. If we take a positive real number ε such that ε is less than the distance of the point p from each of the end points 0 and 1 of the closed interval $[0, 1]$ *i.e.*, ε is less than each of the positive real numbers $|p - 0|$ and $|p - 1|$, then the open interval

$]p-\varepsilon, p+\varepsilon[$ does not contain any point of the set $S =]0,1[$ and consequently p is not a limit point of S.

Thus if $p \notin [0,1]$, then p is not a limit point of $]0,1[$ and if $p \in [0,1]$, then p is a limit point of] 0, 1 [. Hence p in a limit point of] 0, 1 [iff $p \in [0,1]$.

$\therefore \qquad D(\,]0,1[\,) = [0,1].$

Note: Here we see that the points 0 and 1 do not belong to $]0,1[$, but they are limit points of] 0, 1 [. The set] 0, 1 [is dense-in-itself but is not perfect.

Example 35: *Find the limit points of the closed interval* $[0,1]$.

Solution: Let $S = [0,1]$.

Proceeding exactly as in Ex. 33, we can show that a real number p is a limit point of $[0,1]$ iff $p \in [0,1]$.

$\therefore \qquad D([0,1]) = [0,1].$

Note 1: Since each point of [0, 1] is its limit point, therefore the set [0, 1] is dense-in-itself. Also no point outside [0, 1] is a limit point of [0, 1] *i.e.*, [0, 1] contains all its limit points and therefore [0, 1] is also a closed set. Since the set [0, 1] is dense-in-itself and is also closed, therefore it is a perfect set.

Also if $S = [0,1]$, then $D(S) = S$, $D^2(S) = S$, and so on. Thus $D^{(n)}(S) = S$ for every positive integer n. Therefore the closed interval $S = [0,1]$ is a set of second species.

Note 2: Proceeding exactly as in Ex. 33, we can show that

$$D(\,[0,1[\,) = [0,1] \quad \text{and} \quad D(\,]0,1]\,) = [0,1].$$

Example 36: *Find the set of the limit points of the set*

$$S = \{1/n : n \in \mathbf{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}.$$

Solution: We shall show that the set $S = \{1/n : n \in \mathbf{N}\}$ has only one limit point, namely 0.

First, we show that 0 is a limit point of S.

Let $\varepsilon > 0$ be given. Then by Archimedean property of real numbers there exists a positive integer m such that $1/m < \varepsilon$, so that

$$-\varepsilon < 0 < 1/m < \varepsilon.$$

$\therefore$ for every $\varepsilon > 0$, the open interval $]-\varepsilon, \varepsilon[$ contains a point of S other than 0, namely $1/m$. Therefore 0 is a limit point of S.

Now we shall show that no real number p other than 0 can be a limit point of S. The following cases arise :

(i) $p < 0$. The open interval $]p-1, 0[$ is a nbd of p which contains no point of S and consequently p is not a limit point of S.

(ii) $p > 1$. The open interval $]1, p+1[$ is a nbd of p which contains no point of S and so p is not a limit point of S.

(iii) $0 < p < 1$ but $p \notin S$. In this case $1/p > 1$ and $1/p$ is not an integer. Therefore there exists a positive integer m such that

$$m < \frac{1}{p} < m+1 \;\; i.e., \;\; \frac{1}{m+1} < p < \frac{1}{m}.$$

Thus $\left]\dfrac{1}{m+1}, \dfrac{1}{m}\right[$ is a nbd of p which contains no point of S and so p is not a limit point of S.

(iv) $p = 1$. The open interval $]\frac{1}{2}, 2[$ is a nbd of 1 which contains no point of S other than 1 and so 1 is not a limit point of S.

(v) $p \neq 1$ and $p \in S$. If $p = 1/m$, where $m \in \mathbf{N}$ and $m \neq 1$, then $\left]\dfrac{1}{m+1}, \dfrac{1}{m-1}\right[$ is a nbd of p which contains no point of S other than p and so p is not a limit point of S.

Hence no real number other than 0 is a limit point of S. Therefore 0 is the only limit point of S.

We have $D(S) = \{0\}$. Also $D^2(S) = D\{0\} = \varnothing$.

$\therefore$ the set S is of the first species and of first order.

Here we observe that the limit point 0 of S is not a point of the set S and so the set S is not closed.

Example 37: *Find the limit points of the set*

$$S = \left\{\frac{n}{n+1} : n \in \mathbf{N}\right\} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots\right\}.$$

Solution: We shall show that the given set S has only one limit point, namely 1.

First we show that 1 is a limit point of S.

We observe that $\quad \dfrac{n}{n+1} = \dfrac{(n+1)-1}{n+1} = 1 - \dfrac{1}{n+1}$.

Let $\varepsilon > 0$ be given. By Archimedean property of real numbers there exists a positive integer m such that

$$m > 1/\varepsilon \quad \text{or} \quad m+1 > \frac{1}{\varepsilon}$$

$$\text{or} \quad \frac{1}{m+1} < \varepsilon \quad \text{or} \quad -\frac{1}{m+1} > -\varepsilon$$

$$\text{or} \quad 1 - \frac{1}{m+1} > 1 - \varepsilon \quad \text{or} \quad \frac{m}{m+1} > 1 - \varepsilon$$

$$\text{or} \quad 1 - \varepsilon < \frac{m}{m+1} < 1 + \varepsilon.$$

$\therefore$ for every $\varepsilon > 0$, the open interval $]1-\varepsilon, 1+\varepsilon[$ contains a point of S other than 1, namely $m/(m+1)$. Therefore 1 is a limit point of S.

Now we shall show that no real number p other than 1 can be a limit point of S. The following cases arise :

(i) $p > 1$. The open interval $]1, p+1[$ is a nbd of p which contains no point of S and so p is not a limit point of S.

(ii) $p < 1$ and $p \notin S$. In this case let x_0 be the point of S which is nearest to p. If we choose a positive real number ε such that $\varepsilon < |p - x_0|$, then the open interval $]p-\varepsilon, p+\varepsilon[$ contains no point of S and so p is not a limit point of S.

(iii) $p < 1$ and $p \in S$.

Let $\quad p = \dfrac{n}{n+1}$, where $n \in \mathbf{N}$.

Since $\quad \dfrac{1}{2} < \dfrac{2}{3} < \dfrac{3}{4} < \ldots$, *i.e.*, $\dfrac{n}{n+1} < \dfrac{n+1}{(n+1)+1}$ for all $n \in \mathbf{N}$,

therefore the open interval

$$\left] \frac{n-1}{(n-1)+1}, \frac{n+1}{(n+1)+1} \right[$$

is a nbd of $n/(n+1)$ which contains no point of S other than $n/(n+1)$ and so p is not a limit point of S.

Hence no real number other than 1 is a limit point of S. Therefore 1 is the only limit point of S.

We have $\quad D(S) = \{1\}$.

Note: Since the only limit point of S is 1 which is not a point of the set S, therefore each point of S is an isolated point and so the set S is a discrete set.

Example 38: *Find the derived set of the set*

$$S = \left\{ \frac{1}{m} + \frac{1}{n} : m \in \mathbf{N}, n \in \mathbf{N} \right\}.$$

Solution: To find the derived set of S *i.e.*, to find the set of all limit points of S, we proceed as follows :

Keep m fixed and vary n. Then as n gets bigger, $1/n$ gets nearer to 0 so that $1/m + 1/n$ gets nearer to $1/m$ and consequently $1/m$ is a limit point of S. Since m is any member of $\mathbf{N}$, therefore all the points of the set $\{1/m : m \in \mathbf{N}\}$ are limit points of the given set S.

Again vary both m and n so that $1/m + 1/n$ gets nearer and nearer to 0 and consequently 0 is also a limit point of S. Thus

$$D(S) = \left\{ \frac{1}{m} : m \in \mathbf{N} \right\} \cup \{0\}.$$

Note that $\left\{ \dfrac{1}{m} : m \in \mathbf{N} \right\}$ and $\left\{ \dfrac{1}{n} : n \in \mathbf{N} \right\}$ are the same sets so that it is immaterial if we keep n fixed and vary m.

Here $\quad D^2(S) = \{0\}$, so that $D^3(S) = \varnothing$.

The set S is therefore of the first species and of second order.

25 Existence of Limit Points of a Set Bolzano-Weierstrass Theorem

We have seen that a finite set has no limit points. Also we have observed that an infinite set may or may not have a limit point. For example the infinite set $\mathbf{N}$ of natural numbers has no limit point whereas the infinite set $S = \{1/n : n \in \mathbf{N}\}$ has 0 as its limit point. We observe that the set S is bounded. Now we shall give a theorem which gives us a set of sufficient conditions for a set to have a limit point. This theorem is known as Bolzano-Weierstrass theorem and it is one of the most important theorems of Real Analysis.

Bolzano-Weierstrass Theorem: *Every infinite bounded set of real numbers has a limit point.* **(Kanpur 2009; Gorakhpur 13)**

Proof. Let S be an infinite bounded set of real numbers and let h and k be its infimum and supremum respectively. Note that both h and k exist by order-completeness property of real numbers.

Let us define a set H as follows :

A real number x belongs to H iff it exceeds at most a finite number of members of S *i.e.*,

$H = \{x : x \in \mathbf{R}$ and the number of elements of S which are less than x is finite$\}$.

Since $h = \inf S$, therefore $x \geq h, \forall\ x \in S$.

$\therefore$ h exceeds no member of S and so by our definition of $H, h \in H$. Thus $H \neq \varnothing$.

Also no number greater than k is in H. For let $y > k = \sup S$. Then $x < y, \forall\ x \in S$. Therefore y exceeds all the members of S which are infinite because S is an infinite set. Therefore if $y > k$, then by our definition of H, $y \notin H$.

$\therefore$ the set H is bounded above with k as an upper bound.

Thus H is a non-empty subset of $\mathbf{R}$ bounded above by k. Hence by order-completeness property, H must have a supremum in $\mathbf{R}$.

Let $\sup H = p$. We shall show that p is a limit point of S.

Let $\varepsilon > 0$ be given.

Since $\sup H = p$, therefore $p - \varepsilon$ cannot be an upper bound for H. Therefore there exists $x \in H$ such that $x > p - \varepsilon$ *i.e.*, $p - \varepsilon < x$.

Since $x \in H$, therefore, it exceeds at most a finite number of members of S and consequently $p - \varepsilon$ also exceeds at most a finite number of members of S.

Again since $p = \sup H$, $p + \varepsilon$ cannot belong to H and so $p + \varepsilon$ exceeds an infinite number of members of S.

Now $p - \varepsilon$ exceeds only a finite number of members of S and $p + \varepsilon$ exceeds an infinite number of members of S. It means that the nbd $]\, p - \varepsilon, p + \varepsilon\, [$ of p contains infinitely many points of S.

Since ε is arbitrary, therefore every ε-nbd of p contains infinitely many points of S. Hence p is a limit point of S. This proves the theorem.

Remark: The conditions in the above theorem are only sufficient conditions for a set S to have a limit point. These conditions are not necessary for a set S to have a limit point. Even an infinite unbounded set may have a limit point. For example the set $\mathbf{Q}$ of rational numbers is unbounded and still it has limit points. We know that every real number is a limit point of the set $\mathbf{Q}$.

26 Some Theorems on Derived Sets

Theorem 1: *If $\varnothing$ be the empty set, then its derived set $D(\varnothing) = \varnothing$.*

Proof: If p be any point of $\mathbf{R}$, then $\mathbf{R}$ is a nbd of p and $\mathbf{R} \cap \varnothing = \varnothing$ *i.e.*, $\mathbf{R}$ contains no point of $\varnothing$ because the empty set $\varnothing$ has no points at all. Therefore p is not a limit point of $\varnothing$. Thus no point of $\mathbf{R}$ is a limit point of $\varnothing$ and hence

$$D(\varnothing) = \varnothing.$$

Theorem 2: *If A and B are any subsets of* $\mathbf{R}$, *then*

$$A \subset B \Rightarrow D(A) \subset D(B).$$

Proof: It is given that $A \subset B$.

Let $x \in D(A)$ *i.e.*, let x be any limit point of A. Then

$x \in D(A) \Rightarrow$ every nbd N of x contains a point of A other than x

$\Rightarrow$ every nbd N of x contains a point of B other than x [$\because B \supset A$]

$\Rightarrow$ x is a limit point of B

$\Rightarrow$ $x \in D(B)$.

$\therefore A \subset B \Rightarrow D(A) \subset D(B)$.

Theorem 3: *If A and B are any subsets of* $\mathbf{R}$, *then*

$$D(A \cap B) \subset D(A) \cap D(B).$$

Proof. We know that $A \subset B \Rightarrow D(A) \subset D(B)$. [See theorem 2 above.]

Now $A \cap B \subset A$ and $A \cap B \subset B$.

$\therefore$ $D(A \cap B) \subset D(A)$ and $D(A \cap B) \subset D(B)$.

Hence $D(A \cap B) \subset D(A) \cap D(B)$.

Theorem 4. *If A and B are any subsets of* $\mathbf{R}$, *then*

$$D(A \cup B) = D(A) \cup D(B).$$

Proof: We know that $A \subset B \Rightarrow D(A) \subset D(B)$.

Now $A \subset A \cup B$ and $B \subset A \cup B$.

$\therefore$ $D(A) \subset D(A \cup B)$ and $D(B) \subset D(A \cup B)$.

$\therefore$ $D(A) \cup D(B) \subset D(A \cup B)$. ...(1)

Now we shall show that $D(A \cup B) \subset D(A) \cup D(B)$.

Let $x \in D(A \cup B)$. Then to show that $x \in D(A) \cup D(B)$.

For this we shall show that $x \notin D(A) \cup D(B) \Rightarrow x \notin D(A \cup B)$.

Suppose $x \notin D(A) \cup D(B)$. Then $x \notin D(A)$ and $x \notin D(B)$ *i.e.*, x is neither a limit point of A nor a limit point of B.

$\therefore$ there exists a nbd N of x which contains no point of A other than x and there exists a nbd M of x which contains no point of B other than x.

Since the intersection of two nbds of x is also a nbd of x, therefore $N \cap M$ is a nbd of x which contains no point of A other than x and no point of B other than x *i.e.*, which contains no point of $A \cup B$ other than x. Consequently x is not a limit point of $A \cup B$ *i.e.*, $x \notin D(A \cup B)$.

Thus $x \notin D(A) \cup D(B) \Rightarrow x \notin D(A \cup B)$.

$\therefore$ $x \in D(A \cup B) \Rightarrow x \in D(A) \cup D(B)$.

$\therefore \qquad D(A \cup B) \subset D(A) \cup D(B).$ ...(2)

From (1) and (2), we conclude that

$$D(A \cup B) = D(A) \cup D(B).$$

Theorem 5: *If A be any subset of* **R**, *then*

$$x \in D(A) \Rightarrow x \in D(A - \{x\}).$$

Proof: We have $x \in D(A) \Rightarrow x$ is a limit of point A

$\Rightarrow$ every nbd N of x contains a point of A other than x

$\Rightarrow$ every nbd N of x contains a point of $A - \{x\}$ other than x

$\Rightarrow$ x is a limit point of $A - \{x\}$

$\Rightarrow$ $x \in D(A - \{x\})$.

$\therefore$ $D(A) \subset D(A - \{x\})$.

Theorem 6: *The derived set of any bounded set is again a bounded set.*

Proof. Let S be a bounded set.

Then there exist $h, k \in \mathbf{R}$ such that $S \subset [h, k]$.

We shall show that no member of the derived set $D(S)$ can be less than h or greater than k.

Let $p < h$. Then we shall show that p cannot be a limit point of S.

Let $\varepsilon = h - p$, so that $\varepsilon > 0$. Also then $h = p + \varepsilon$.

Since h is a lower bound for S, therefore $x \in S \Rightarrow x \geq h$

i.e., $\qquad x \in S \Rightarrow x \nless p + \varepsilon.$ $\qquad [\because\ h = p + \varepsilon]$

$\therefore \varepsilon > 0$ is such that the open interval $]\, p - \varepsilon, p + \varepsilon\, [$ contains no point of S.

$\therefore$ if $p < h$, then p is not a limit point of S.

Similarly we can show that if $p > k$, then p is not a limit point of S.

$\therefore$ all the limit points of S lie in the closed interval $[h, k]$.

Hence $D(S) \subset [h, k]$ *i.e.,* $D(S)$ is bounded.

Theorem 7: *Every bounded infinite set has the greatest and the smallest limit points i.e., the derived set of any infinite bounded set attains its bounds.*

Proof: Let S be an infinite bounded set. Since S is bounded, therefore there exist $h, k \in \mathbf{R}$ such that $S \subset [h, k]$.

Now $S \subset [h, k] \Rightarrow D(S) \subset D([h, k])$.

But the derived set of the closed interval $[h, k]$ is $[h, k]$ *i.e.,* $D([h, k]) = [h, k]$.

$\therefore D(S) \subset [h, k]$, so that $D(S)$ is bounded.

Since S is an infinite bounded set, therefore by Bolzano-Weierstrass theorem, S has at least one limit point and so $D(S) \neq \emptyset$.

Thus $D(S)$ is a non-empty bounded subset of **R**.

$\therefore$ by order-completeness property of **R**, $D(S)$ has infimum as well as supremum in **R**.

Let $\qquad \inf D(S) = p$ and $\sup D(S) = q$.

We shall show that both p and q belong to $D(S)$ *i.e.,* both p and q are limit points of S.

Let $\varepsilon > 0$ be given.

Now $\quad p = \inf$ of $D(S) \Rightarrow$ there exists some $x \in D(S)$

such that $\quad p \le x < p + \varepsilon$

$\Rightarrow \quad p - \varepsilon < x < p + \varepsilon$

$\Rightarrow \quad x \in \,]\, p - \varepsilon, p + \varepsilon\, [$

$\Rightarrow \quad]\, p - \varepsilon, p + \varepsilon\, [$ is a nbd of some $x \in D(S)$

$\Rightarrow \quad]\, p - \varepsilon, p + \varepsilon\, [$ is a nbd of x which is a limit point of S

$\Rightarrow \quad]\, p - \varepsilon, p + \varepsilon\, [$ contains infinitely many points of S.

Thus for every $\varepsilon > 0$, the open interval $]\, p - \varepsilon, p + \varepsilon\, [$ contains infinitely many points of S and consequently p is a limit point of S.

Similarly we can show that q is also a limit point of S.

Thus both p and q *i.e.*, the inf and sup of $D(S)$ belong to $D(S)$. So p is the smallest and q the greatest member of $D(S)$.

$\therefore$ the set S has smallest and greatest limit points.

Definition: *The smallest and the greatest limit points of an infinite bounded set S, as they do exist by the above theorem, are called the* **lower limit (limit inferior)** and the **upper limit (limit superior)** *of S respectively.*

We usually denote these limits by $\quad p = \underline{\lim}\, S$ and $q = \overline{\lim}\, S$.

Not that $\underline{\lim}\, S \le \overline{\lim}\, S$.

27 Interior of a Set

Interior point of a set: Definition: *Let S be a subset of* **R**. *A point $p \in S$ is said to be an interior point of S if there exists a nbd $]\, p - \varepsilon, p + \varepsilon\, [$ of p which is entirely contained in S i.e., if there exists $\varepsilon > 0$ such that*

$$]\, p - \varepsilon, p + \varepsilon\, [\subset S. \qquad \textbf{(Kanpur 2012)}$$

Interior of a set: Definition

If S is a subset of **R**, *then the set of all interior points of S is called the interior of S and is denoted by S° or int (S). Obviously $S^{\circ} \subset S$.*

Illustrations:

1. *Every point of an open interval is an interior point of the interval.*

   Let S denote the open interval $]\, a, b\, [$ and let $p \in S$ so that $a < p < b$. If we take $\varepsilon = \min\{p - a, b - p\}$ then $\varepsilon > 0$ is such that $]\, p - \varepsilon, p + \varepsilon\, [\subset S$. Therefore p is an interior point of S.

2. *Every point of a closed interval except the end points of the interval is an interior point of the interval.*

   Let S denote the closed interval $[a, b]$.

   We have $a \in S$. There exists no $\varepsilon > 0$ such that

   $$]\, a - \varepsilon, a + \varepsilon\, [\subset S.$$

   Hence a is not an interior point of S.

   Similarly b is also not an interior point of S.

But every point $p \in \,]a, b[$ is an interior point of $[a, b]$ as can be easily shown. Hence if $S = [a, b]$, then $S^o = \,]a, b[$.

3. Let S denote the set of rational numbers $\mathbf{Q}$. Then $S^o = \varnothing$.

   If $p \in \mathbf{Q}$, then every nbd of p contains irrationals also. So there exists no $\varepsilon > 0$ such that $]p - \varepsilon, p + \varepsilon[\subset \mathbf{Q}$. Therefore p is not an interior point of $\mathbf{Q}$. Hence $\mathbf{Q}$ has no interior points so that $\mathbf{Q}^o = \varnothing$.

   In a similar manner we can show that if $S = \mathbf{Z}^+$, then $S^o = \varnothing$. Also $\mathbf{Z}^o = \varnothing$.

4. If S denotes the set of real numbers $\mathbf{R}$, then $S^o = \mathbf{R}$ because every real number is an interior point of $\mathbf{R}$.

28 Open Sets

Definition: *Let S be a subset of $\mathbf{R}$. Then S is said to be open if every point of S is an interior point of S.*

The following two characterizations of an open set immediately follow from its definition.

(*i*) *A set S is open iff $S^o = S$*

(*ii*) *A set S is open iff it contains a neighbourhood of each of its points.*

Illustrations:

1. *Every open interval is an open set, but the converse is not true.*

Let S denote the open interval $]a, b[$. Let $p \in S$ so that $a < p < b$.

Take $\varepsilon = \min\{p - a, b - p\}$. Then $\varepsilon > 0$ is such that

$$]p - \varepsilon, p + \varepsilon[\subset S.$$

Thus to each $p \in S$, there exists $\varepsilon > 0$ such that

$$]p - \varepsilon, p + \varepsilon[\subset S$$

i.e., S contains a nbd of each of its points.

Hence every open interval is an open set.

The converse need not be true *i.e., an open set need not be an open interval.* For example, if $S = \,]2, 3[\cup]5, 7[$, then S is an open set but S is not an open interval.

2. *A closed or semi-closed interval is not an open set.*

Let S denote the closed interval $[a, b]$. We have $a \in S$, but there exists no $\varepsilon > 0$ such that $]a - \varepsilon, a + \varepsilon[\subset S$. Thus S does not contain a nbd of the point $a \in S$. Hence S is not an open set.

Similarly, if $S = [a, b[$, then S is not open because $a \in S$ and there exists no $\varepsilon > 0$ such that $]a - \varepsilon, a + \varepsilon[\subset S$.

Also if $S = \,]a, b]$, then S is not open because $b \in S$ and there exists no $\varepsilon > 0$ such that $]b - \varepsilon, b + \varepsilon[\subset S$.

3. *Every non-empty finite set is not an open set.*

Let S denote any non-empty finite subset of **R**. If $p \in S$, then for every $\varepsilon > 0$, the open interval $]\, p - \varepsilon, p + \varepsilon\, [$ contains infinitely many points of **R** and so $]\, p - \varepsilon, p + \varepsilon\, [$ cannot be a subset of a finite set S. Thus there exists no nbd of p which is contained in S. Hence S is not an open set.

4. *The set of rational numbers* **Q** *is not an open set.*

We have $\mathbf{Q}^0 = \varnothing$, so that $\mathbf{Q}^0 \neq \mathbf{Q}$. Hence **Q** is not an open set.

5. *The set of all real numbers* **R** *and the empty set* $\varnothing$ *are open.*

The set **R** of all real numbers is an open set. If $p \in \mathbf{R}$, then for every $\varepsilon > 0$, we have $]\, p - \varepsilon, p + \varepsilon\, [\; \subset \mathbf{R}$. Thus **R** contains a neighbourhood of each of its points. Hence **R** is an open set.

Now we show that the null set $\varnothing$ is an open set.

Since $\varnothing$ has no points the condition that $\varnothing$ contains a neighbourhood of each of its points is vacuously satisfied. Hence $\varnothing$ is open.

Some important properties of open sets

Theorem 1: *The union of an arbitrary family of open sets is open.*

Proof: Let $\{G_\lambda : \lambda \in \Lambda\}$ be an arbitrary family of open sets. Here Λ is an index set and is such that for every $\lambda \in \Lambda$, G_λ is an open set.

Let $G = \cup \{G_\lambda : \lambda \in \Lambda\}$. Then in order to show that G is an open set, we shall show that every point of G is an interior point of G.

Let $p \in G$. Since G is the union of the family $\{G_\lambda\}$, therefore $p \in G_\lambda$ for some $\lambda \in \Lambda$.

Since G_λ is an open set and $p \in G_\lambda$, therefore there must exist some $\varepsilon > 0$ such that $]\, p - \varepsilon, p + \varepsilon\, [\subset G_\lambda \subset G$. Note that G is the union of the family $\{G_\lambda\}$.

Thus $]\, p - \varepsilon, p + \varepsilon\, [\subset G$ and so p is an interior point of G.

Since every point of G is an interior point of G, therefore G is an open set.

Hence the union of an arbitrary family of open sets is an open set.

Note: In particular the union of two open sets is an open set.

Theorem 2: *The intersection of a finite collection of open sets is an open set.*

(Kanpur 2009)

Proof: Let $G = \bigcap_{i=1}^{n} G_i$, where each G_i is an open set.

If $G = \varnothing$, then G is an open set and the proof is complete. So let $G \neq \varnothing$.

Let $p \in G$. Then $p \in G_i$ for each $i = 1, 2, \ldots, n$.

Since each G_i is an open set, therefore for every $i = 1, 2, \ldots, n$, there exists $\varepsilon_i > 0$ such that

$$]\, p - \varepsilon_i\, , p + \varepsilon_i\, [\subset G_i.$$

Let $\qquad \varepsilon = \min\, [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n]$.

Then $\qquad]\, p - \varepsilon, p + \varepsilon\, [\subset\,]\, p - \varepsilon_i, p + \varepsilon_i\, [\subset G_i$ for each i

$\Rightarrow \qquad]\, p - \varepsilon, p + \varepsilon\, [\subset \bigcap_{i=1}^{n} G_i$

$\Rightarrow \quad]\, p-\varepsilon, p+\varepsilon\, [\, \subset G$

$\Rightarrow \quad p$ is an interior point of G.

Thus every point p of G is an interior point of G and so G is an open set.

Hence a finite intersection of open sets is an open set.

Remark: The intersection of an infinite collection of open sets is not necessarily an open set. The following example supports this statement.

Let $\quad G_n =]-1/n, 1/n\,[\,, n \in \mathbf{N}$.

Then each G_n is an open set because every open interval is an open set.

Now $\bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty}]-1/n, 1/n\,[= \{0\}$ which is not open because there exists no $\varepsilon > 0$ such that $]-\varepsilon, \varepsilon\,[\, \subset \{0\}$.

Hence the intersection of an infinite collection of open sets is not necessarily an open set.

29 Closed Sets

Definition: *Let F be a subset of* $\mathbf{R}$. *Then F is said to be closed if its complement F' is open.*

Note that $F' = \mathbf{R} - F = \{x : x \in \mathbf{R}$ and $x \notin F\}$. **(Kanpur 2010)**

Illustrations:

1. *Every closed interval is a closed set.*

Let $\quad F = [a, b]$. Then $F' =]-\infty, a\,[\, \cup\,]\, b, \infty\,[$.

Here $]-\infty, a\,[$ and $]\, b, \infty\,[$ are both open sets and we know that the union of two open sets is also an open set.

$\therefore \quad F' =]-\infty, a\,[\, \cup\,]\, b, \infty\,[$ is an open set. Hence by definition F is closed.

2. *The right half open interval $[a, b\,[$ is not closed.*

Let $\quad F = [a, b\,[$. Then $F' =]-\infty, a\,[\, \cup [b, \infty\,[$.

Obviously F' is not an open set because $b \in F'$ and b is not an interior point of F'. Note that there exists no $\varepsilon > 0$ such that $]\, b-\varepsilon, b+\varepsilon\,[\, \subset F'$.

Since F' is not open, therefore F is not closed.

Remark: Similarly we can show that every open interval $]\, a, b\,[$ is not a closed set and the left half open interval $]\, a, b\,]$ is also not a closed set.

3. *Every singleton set in* $\mathbf{R}$ *is closed.*

Let $F = \{a\}$. Then $F' =]-\infty, a\,[\, \cup\,]\, a, \infty\,[$ which is open being the union of two open sets. Hence by definition F is closed.

4. *The set* $\mathbf{Q}$ *of all rationals is not closed.*

We have $\mathbf{Q}' = \mathbf{R} - \mathbf{Q} =$ the set of all irrational numbers.

The set $\mathbf{Q}'$ is not open because if $p \in \mathbf{Q}'$, then for every $\varepsilon > 0$ the open interval $]\, p-\varepsilon, p+\varepsilon\,[$ contains rational numbers also and so is not a subset of $\mathbf{Q}'$. Thus if $p \in \mathbf{Q}'$, then p is not an interior point of $\mathbf{Q}'$ and so $\mathbf{Q}'$ is not open. Hence $\mathbf{Q}$ is not closed.

5. *The set of all real numbers* $\mathbf{R}$ *and the null set $\varnothing$ are closed.*

We have $\mathbf{R}' = \varnothing$. Since $\varnothing$ is open, therefore $\mathbf{R}'$ is open. Hence $\mathbf{R}$ is closed.

Again $\varnothing' = \mathbf{R}$. Since $\mathbf{R}$ is open, therefore $\varnothing'$ is open. Hence $\varnothing$ is closed.

Some important properties of closed sets

Theorem 1: *The union of a finite collection of closed sets is closed.*

Proof: Let $F_1, F_2, \ldots, F_n$ be n closed sets. Then to prove that $\bigcup_{i=1}^{n} F_i$ is closed.

Since each F_i $(i = 1, 2, \ldots, n)$ is a closed set, therefore each F_i' is an open set.

By theorem 2 of article 28, we know that the intersection of a finite collection of open sets is open.

$\therefore \qquad \bigcap_{i=1}^{n} F_i'$ is open.

But by De Morgan's law $\left(\bigcup_{i=1}^{n} F_i\right)' = \bigcap_{i=1}^{n} F_i'$.

$\therefore \qquad \left(\bigcup_{i=1}^{n} F_i\right)'$ is an open set. Hence by definition of a closed set, $\bigcup_{i=1}^{n} F_i$ is a closed set.

Theorem 2: *The intersection of an arbitrary collection of closed sets is closed.*

Proof: Let $\{F_\lambda : \lambda \in \Lambda\}$ be an arbitrary family of closed sets. Here Λ is an index set and is such that for every $\lambda \in \Lambda$, F_λ is a closed set.

Now to prove that $\cap \{F_\lambda\}$ is a closed set. Since each F_λ is a closed set, therefore each F_λ' is an open set.

By theorem 1 of article 28, we know that the union of an arbitrary family of open sets is open.

$\therefore \qquad \cup \{F_\lambda' : \lambda \in \Lambda\}$ is open.

But by De Morgan's law, $[\cap \{F_\lambda : \lambda \in \Lambda\}]' = \cup \{F_\lambda' : \lambda \in \Lambda\}$.

$\therefore \qquad [\cap \{F_\lambda : \lambda \in \Lambda\}]'$ is an open set.

Hence $\cap \{F_\lambda : \lambda \in \Lambda\}$ is a closed set.

Example 39: *Every finite subset of* $\mathbf{R}$ *is a closed set.*

Solution: Let $F = \{a_1, a_2, \ldots, a_n\}$ be a finite subset of $\mathbf{R}$. Then to prove that F is closed.

We know that every singleton in $\mathbf{R}$ is a closed set. Also the union of a finite collection of closed sets is closed. The set F can be expressed as the union of n singletons *i.e.*, we can write

$$F = \{a_1\} \cup \{a_2\} \cup \ldots \cup \{a_n\}.$$

Hence F is a closed set.

Example 40: *Give examples to show that the union of an infinite collection of closed sets is not necessarily closed.*

Solution: Let $F_n = [1/n, 1], n \in \mathbf{N}$.

Then each F_n is a closed set in $\mathbf{R}$ because each closed interval is a closed set.

Now $\cup \{F_n : n \in \mathbf{N}\}$

$$= \{1\} \cup [\frac{1}{2}, 1] \cup [\frac{1}{3}, 1] \cup [\frac{1}{4}, 1] \cup \ldots =]\,0, 1].$$

Since] 0, 1] is not a closed set in **R**, therefore it follows that the union of an infinite collection of closed sets is not necessarily closed.

As an other illustration consider the set **Q** of all rational numbers.

We can write $\mathbf{Q} = \cup \{\, r\}. r \in \mathbf{Q}$

Since each singleton set in **R** is closed, therefore **Q** is expressed as an infinite union of closed sets. But we know that **Q** is not closed since its complement **Q′** is not open. Hence it follows that an infinite union of closed sets is not necessarily closed.

30 Closure of a Set

Definition: *Let A be any subset of* **R**. *Then the closure of A is defined as the smallest closed set containing A and is denoted by* $\overline{A}$ *or cl* (A).

The following two results are quite obvious :

(*i*) $\overline{A}$ *is the intersection of the family of all closed sets containing A.*

(*ii*) *A is closed iff* $\overline{A} = A$.

Some theorems connecting closed sets and limit points

Theorem 1: A characterization of closed sets. *Let A be a subset of* **R**. *Then A is closed iff* $D(A) \subset A$ *i.e., A is closed iff A contains all its limit points.*

Proof: Let A be closed so that A' is open. If $D(A) = \varnothing$, then obviously $D(A) \subset A$. So let $D(A) \neq \varnothing$. Let $p \in D(A)$ *i.e.*, let p be a limit point of A. We wish to show that $p \in A$.

Suppose, if possible, $p \notin A$. Then $p \in A'$ and, since A' is open, there exists $\varepsilon > 0$ such that $]\, p - \varepsilon, p + \varepsilon\, [\subset A'$. Consequently $]\, p - \varepsilon, p + \varepsilon\, [\cap A = \varnothing$ *i.e.*, $]\, p - \varepsilon, p + \varepsilon\, [$ contains no point of A, contradicting the fact that p is a limit point of A. Hence we must have $p \in A$.

Thus if A is closed, then $D(A) \subset A$.

Conversely, we assume that $D(A) \subset A$ *i.e.*, A contains all its limit points. Then we wish to show that A is closed. By definition A will be closed if A' is open.

Let $p \in A'$. Then $p \notin A$, and so $p \notin D(A)$ because by hypothesis $D(A) \subset A$.

Now $p \notin D(A) \Rightarrow p$ is not a limit point of A

$\Rightarrow$ there exists $\varepsilon > 0$ such that $]\, p - \varepsilon, p + \varepsilon\, [$ contains no point of A other than p

$\Rightarrow$ there exists $\varepsilon > 0$ such that $]\, p - \varepsilon, p + \varepsilon\, [$ contains no point of A because $p \notin A$

$\Rightarrow$ there exists $\varepsilon > 0$ such that $]\, p - \varepsilon, p + \varepsilon\, [\subset A'$.

Thus A' contains a neighbourhood of each of its points and so A' is open. Hence A is closed.

Illustrations:

1. Every finite set is closed. Note that a finite set has no limit points and so the condition that a finite set contains all its limit points is vacuously satisfied.

2. The set of all rational numbers $\mathbf{Q}$ is not closed because $D(\mathbf{Q}) = \mathbf{R}$ is not contained in $\mathbf{Q}$.

Theorem 2: *Let A be any subset of $\mathbf{R}$. Then $D(A)$ is a closed set.*

Proof: In order to prove that $D(A)$ is closed we shall show that $D(A)$ contains all its limit points. [Refer theorem 1 above.]

Let p be any limit point of $D(A)$. Then every ε-nbd of p contains infinitely many points of $D(A)$ and since each point of $D(A)$ is a limit point of A, every ε-nbd of p must contain infinitely many points of A. Thus, p is also a limit point of A and so $p \in D(A)$. Therefore $D(A)$ contains all its limit points and so $D(A)$ is closed.

Theorem 3: *Let A be any subset of $\mathbf{R}$. Then $\overline{A} = A \cup D(A)$ i.e., $\overline{A}$ is the set of all adherent points of A.*

Proof: We shall first prove that $A \cup D(A)$ is a closed set.

Let p be any limit point of $A \cup D(A)$. Then either p is a limit point of A or a limit point of $D(A)$. If p is a limit point of A, then $p \in D(A)$. If p is a limit point of $D(A)$, then since $D(A)$ is closed, $p \in D(A)$. So in either case $p \in D(A)$ and surely then $p \in A \cup D(A)$. Thus $A \cup D(A)$ contains all its limit points and hence $A \cup D(A)$ is closed.

Now $A \cup D(A)$ is a closed set containing A, and since $\overline{A}$ is the smallest closed set containing A, we have

$$\overline{A} \subset A \cup D(A). \qquad ...(1)$$

Also $$A \subset \overline{A} \Rightarrow D(A) \subset D(\overline{A}). \qquad ...(2)$$

[Refer theorem 2 of article 26]

Since $\overline{A}$ is closed, we have $D(\overline{A}) \subset \overline{A}$. ...(3)

[Refer theorem 1 of article 30]

$\therefore$ from (2) and (3), we have $D(A) \subset \overline{A}$.

Moreover, $A \subset \overline{A}$, and $D(A) \subset \overline{A} \Rightarrow A \cup D(A) \subset \overline{A}$. ...(4)

$\therefore$ from (1) and (4), we have $\overline{A} = A \cup D(A)$.

Illustration 1: If $A =]1, 2[$, then $D(A) = [1, 2]$.

$$\therefore \quad \overline{A} = A \cup D(A) =]1, 2[\cup [1, 2] = [1, 2].$$

Illustration 2: If $\mathbf{Z}$ is the set of all integers, then $\overline{\mathbf{Z}} = \mathbf{Z} \cup D(\mathbf{Z}) = \mathbf{Z} \cup \varnothing = \mathbf{Z}$.

Illustration 3: If $\mathbf{Q}$ is the set of all rational numbers, then

$$\overline{\mathbf{Q}} = \mathbf{Q} \cup D(\mathbf{Q}) = \mathbf{Q} \cup \mathbf{R} = \mathbf{R}.$$

Illustration 4: Consider the finite set $A = \{1, 3, 7\}$. We have

$$\overline{A} = A \cup D(A) = A \cup \varnothing = A.$$

Illustrative Examples

Example 41: *Show that the union of two closed sets is also a closed set.*

Solution: Let F_1, F_2 be two closed sets. Then to prove that $F_1 \cup F_2$ is also a closed set *i.e.*, $(F_1 \cup F_2)'$ is an open set.

Since both F_1 and F_2 are closed sets, therefore both F_1' and F_2' are open sets. But the intersection of two open sets is an open set.

$\therefore$ $F_1' \cap F_2'$ is an open set

$\Rightarrow$ $(F_1 \cup F_2)'$ is an open set

[$\because$ by De Morgan's law $(F_1 \cup F_2)' = F_1' \cap F_2'$]

$\Rightarrow$ $F_1 \cup F_2$ is a closed set.

Example 42: *Show that the union of two open sets is an open set.*

Solution: Let G_1, G_2 be two open sets. Then to prove that $G = G_1 \cup G_2$ is also an open set.

In order to show that G is an open set, we shall show that every point of G is an interior point of G.

Let $p \in G = G_1 \cup G_2$. Then $p \in G_n$ for some $n \in \{1, 2\}$.

Since each G_n is an open set and $p \in G_n$, therefore there must exist some $\varepsilon > 0$ such that $]\, p - \varepsilon, p + \varepsilon\, [\subset G_n \subset G$. Note that $G_1 \subset G$ and $G_2 \subset G$.

Thus $]\, p - \varepsilon, p + \varepsilon\, [\subset G$ and so p is an interior point of G.

Since every point of G is an interior point of G, therefore G is an open set. Hence the union of two open sets is an open set.

31 Countability of Sets

Denumerable Sets. Definition

A set A is said to be **denumerable (or countably infinite)** *if there exists a one-to-one correspondence between the set A and the set* **N** *of natural numbers i.e., if there exists a one-one mapping from the set* **N** *of natural numbers onto the set A.*

Example: Consider the set $A = \{1/2, 2/3, 3/4, \ldots\}$. Then A is denumerable because the mapping $f : \mathbf{N} \Rightarrow A$ defined by

$$f(n) = \frac{n}{n+1} \quad \forall\, n \in \mathbf{N}$$

is bijective *i.e.*, one-one and onto.

32 Countable and Uncountable Sets

Definition: *A set which is either finite or denumerable is called a* **countable set.**

Thus a set A is countable $\Leftrightarrow$ A is finite or there exists a one-one mapping f of $\mathbf{N}$ onto A.

A set which is neither finite nor denumerable is said to be **uncountable** or **non-denumerable.**

Thus there are many sizes of infinite sets. The smallest size is called countable.

Example 43: *The set* $\mathbf{Z}$ *of all integers is countable i.e., there exists a one-one onto map between the set of integers and the set of natural numbers.*

Solution: Let $\mathbf{N}$ denote the set of natural numbers and $\mathbf{Z}$ denote the set of integers. Consider the mapping $f : \mathbf{N} \to \mathbf{Z}$ defined by

$$f(r) = (r-1)/2 \qquad \text{when } r \text{ is odd and}$$
$$f(r) = -r/2 \qquad \text{when } r \text{ is even.}$$

f is one-one. We have $f(1) = 0, f(2) = -1, f(3) = 1, f(4) = -2, f(5) = 2,$ and so on.

Thus we easily observe that under the mapping f distinct elements of $\mathbf{N}$ have distinct f-images in $\mathbf{Z}$. Therefore f is one-one.

f is onto. Let $y \in \mathbf{Z}$. If $y \geq 0$, then

$$2y + 1 \in \mathbf{N} \quad \text{and} \quad f(2y+1) = \{(2y+1) - 1\}/2 = y.$$

Again if $y < 0$, then $-2y \in \mathbf{N}$ and $f(-2y) = \{-(-2y)\}/2 = y$.

Thus $y \in \mathbf{Z} \Rightarrow$ there exists some element $x \in \mathbf{N}$ such that $f(x) = y$. Therefore f is onto.

Thus the mapping f gives us a one-to-one correspondence between $\mathbf{N}$ and $\mathbf{Z}$. Hence $\mathbf{Z}$ is denumerable.

Theorem 1: *Every subset of a countable set is countable.*

Proof: Let A be any countable set and let $B \subset A$. If B is finite, there is nothing to prove.

Therefore we suppose that A is an infinite countable set and B is an infinite subset of A.

Since A is countably infinite *i.e.*, denumerable, therefore we can write A as an infinite sequence

$$< a_1, a_2, a_3, \ldots, a_n, \ldots > .$$

Let n_1 be the smallest positive integer such that $a_{n_1} \in B$. Again let n_2 be the next smallest positive integer such that $n_2 > n_1$ and $a_{n_2} \in B$ and so on.

Then $\qquad B = \{a_{n_1}, a_{n_2}, \ldots\}.$

Obviously the mapping $f : \mathbf{N} \to B$ such that

$$f(k) = a_{n_k}$$

is one-one and onto.

Hence B is countable.

Theorem 2: *Every superset of an uncountable set is uncountable.*

Proof: Let A be an uncountable set and $B \supset A$. Then to prove that B is also uncountable.

Suppose B is countable. Since $A \subset B$, therefore A must be countable which is against our hypothesis.

Hence B must be uncountable.

Theorem 3: *Union of a finite number of countable sets is countable.*

Proof: Let A_1 and A_2 be two countable sets. The members of A_1 and A_2 can be arranged in a definite order. Thus we can write

$$A_1 = \{x_1, x_2, x_3, \ldots\}$$

and $$A_2 = \{y_1, y_2, y_3, \ldots\}.$$

Since subsets of countable sets are countable, therefore we can assume that A_1 and A_2 have no common members. For if A_1 and A_2 have common members, we can take subsets of A_1 and A_2 so that they have no common members.

$\therefore$ The set $A_1 \cup A_2$ is the same as if the union of A_1 and A_2 is taken without having any common member.

Let us write $\quad A_1 \cup A_2 = \{z_1, z_2, z_3, \ldots\}$,

where $\quad z_{2n-1} = x_n, z_{2n} = y_n, n = 1, 2, 3, \ldots$.

Then we find that every member of $A_1 \cup A_2$ occupies a definite place in this arrangement.

$\therefore \quad A_1 \cup A_2$ is countable.

Repeating the above argument with a finite number of countable sets, we conclude that the union of a finite number of countable sets is countable.

Theorem 4: *The union of a countable family of countable sets is countable.* **(Kanpur 2010)**

Proof: Let $\{A_n\}$ be a countable family of sets such that each A_n is countable. We may enumerate, the elements of each $A_n, n = 1, 2, 3, \ldots$ in an array as follows :

$$\begin{array}{cccccc}
A_1 & a_{11} \rightarrow & a_{12} & a_{13} \rightarrow & a_{14} & a_{15} \ldots\ldots\ldots \\
A_2 & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \ldots\ldots\ldots \\
A_3 & a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \ldots\ldots\ldots \\
A_4 & a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \ldots\ldots\ldots \\
A_5 & a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \ldots\ldots\ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \quad \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \quad \ldots
\end{array}$$

If we run out of elements in any set, *i.e.*, if any of the sets is finite, we just put down x's in the spot where an element should go. In the above array a_{ij} stands for the j th element of the i th set. Let us define the height of the element a_{ij} to be $i + j$. With this definition, the height of a_{11} is 2. Similarly a_{12}, a_{21} are the only elements of height 3. There are exactly n elements namely $a_{n1}, a_{(n-1)2}, \ldots, a_{1n}$ of height $n + 1$. Therefore we arrange the elements of $\cup A_n$ according to their height as follows :

$$a_{11};$$
$$a_{21}, a_{12};$$
$$a_{31}, a_{22}, a_{13};$$

$$
\begin{array}{l}
a_{41}, a_{32}, a_{23}, a_{14}\ ; \\
\dots \quad \dots \quad \dots \quad \dots \\
\dots \quad \dots \quad \dots \quad \dots \\
a_{n1}, a_{(n-1)\,2}, \dots, a_{1n}\ ; \\
\dots \qquad \dots \quad \dots \quad \dots \\
\dots \qquad \dots \quad \dots \quad \dots
\end{array}
$$

From the above scheme it is evident that a_{pq} is the qth element of $(p+q-1)$th row. Thus all the elements of $\cup\, A_n$ have been arranged in an infinite sequence as

$$\{a_{11}, a_{21}, a_{12}, a_{31}, a_{22}, a_{13}, a_{41}, a_{32}, a_{23}, a_{14}, \dots\}.$$

In fact, the map $f : \cup\, A_n \to \mathbf{N}$

defined by $f\,(a_{pq}) = \dfrac{(p+q-2)\,(p+q-1)}{2} + q$

gives an enumeration of $\cup\, A_n$.

Hence the union of a countable family of countable sets is countable.

Theorem 5: *The set* $\mathbf{N} \times \mathbf{N}$ *is countable.* **(Kanpur 2011)**

Proof: We have $\mathbf{N} \times \mathbf{N} = \{(m, n) : m, n \in \mathbf{N}\}$. To prove that $\mathbf{N} \times \mathbf{N}$ is countable. We may arrange the set $\mathbf{N} \times \mathbf{N}$ as shown below :

$$
\begin{array}{llll}
(1,1) \to (1,2) & (1,3) \to (1,4) \dots\dots\dots\dots \\
(2,1) \quad (2,2) & (2,3) \quad (2,4) \dots\dots\dots \\
\downarrow & \\
(3,1) \quad (3,2) & (3,3) \quad (3,4) \dots\dots\dots \\
(4,1) \quad (4,2) & (4,3) \quad (4,4) \dots\dots\dots \\
\downarrow & \\
\dots \quad \dots \quad \dots & \dots \quad \dots \quad \dots \\
\dots \quad \dots \quad \dots & \dots \quad \dots \quad \dots
\end{array}
$$

We observe that in the above arrangement every member of $\mathbf{N} \times \mathbf{N}$ is placed in a definite position such that no member escapes. For example the member (i, j) of $\mathbf{N} \times \mathbf{N}$ occurs in the ith row and in the jth column.

Hence the set $\mathbf{N} \times \mathbf{N}$ is countable.

33 Dedikind's Theorem or Dedikind's Axiom or Dedikind's Property of Real Numbers

Let L and U be two subsets of $\mathbf{R}$ *such that*

(i) $L \ne \varnothing, U \ne \varnothing$ *(i.e., each set has at least one element)*

(ii) $L \cup U = \mathbf{R}$ *(i.e., each real number is either in L or in U) and*

(iii) $x \in L, y \in U \Rightarrow x < y$ *(i.e., each member of L is smaller than each member of U);*

then the subset L has the greatest member or the subset U has the smallest member i.e., there exists $\alpha \in \mathbf{R}$ *such that*

$$x < \alpha \Rightarrow x \in L, y > \alpha \Rightarrow y \in U.$$

In the following theorem we shall establish the equivalence between the Order-completeness property and Dedikind's theorem.

Theorem: *To prove that Order-completeness axiom* $\Leftrightarrow$ *Dedikind's theorem.*

Proof: First we shall prove that

Completeness axiom $\Rightarrow$ Dedikind's axiom.

Let L and U be two subsets of $\mathbf{R}$ such that

(i) $L \neq \varnothing, U \neq \varnothing$ (ii) $L \cup U = \mathbf{R}$

and (iii) $x \in L, y \in U \Rightarrow x < y$.

Then we wish to prove that there exists $\alpha \in \mathbf{R}$ such that

$$x < \alpha \Rightarrow x \in L, y > \alpha \Rightarrow y \in U.$$

From (i), $L \neq \varnothing$.

Also $U \neq \varnothing$, so let $y \in U$.

Then from (iii), we have

$$x < y, \ \forall \ x \in L.$$

$\therefore$ y is an upper bound for L.

Thus L is a non-empty subset of $\mathbf{R}$ which is bounded above. So by order-completeness property of real numbers, L must have its supremum in $\mathbf{R}$.

Let $\sup L = \alpha$.

We shall show that $\alpha \in \mathbf{R}$ is such that

$$x < \alpha \Rightarrow x \in L, y > \alpha \Rightarrow y \in U.$$

Let $y > \alpha$.

Then $y \notin L$ because $\alpha = \sup L$.

But from (ii), every real number is either in L or in U.

So $y \notin L \Rightarrow y \in U$.

Thus $y > \alpha \Rightarrow y \in U$.

Again let $x < \alpha$. Since $\alpha = \sup L$, therefore x cannot be an upper bound for L. So there exists $z \in L$ such that $z > x$ *i.e.*, $x < z$. Now from (iii), $z \in L$ and $x < z \Rightarrow x \notin U \Rightarrow x \in L$.

Thus $x < \alpha \Rightarrow x \in L$.

$\therefore$ there exists $\alpha \in \mathbf{R}$ such that

$$x < \alpha \Rightarrow x \in L \text{ and } y > \alpha \Rightarrow y \in U.$$

Hence completeness axiom $\Rightarrow$ Dedikind's axiom.

Now we shall prove that

Dedikind's axiom $\Rightarrow$ Completeness axiom.

Let S be a non-empty subset of $\mathbf{R}$ which is bounded above.

Then we have to prove that S has its supremum in $\mathbf{R}$.

Since S is bounded above, therefore let u be an upper bound for S *i.e.*,

$$x \le u, \forall \ x \in S.$$

Let $U = \{ s : s \in \mathbf{R}$ and s is an upper bound for $S \}$.

Since u is an upper bound for S, therefore $u \in U$ and so

$U \ne \emptyset$. Let $L = \mathbf{R} - U$.

Then $S \subset L$ and $L \ne \emptyset$. Also $L \cup U = (\mathbf{R} - U) \cup U = \mathbf{R}$.

Now let $x \in L, y \in U$.

Then $x \ne y$ because by definition of L, L and U are disjoint.

Also we cannot have $x > y$. Because if $x > y$, then

$y \in U \Rightarrow y$ is an upper bound for S

$\Rightarrow$ x is also an upper bound for S $[\because x > y]$

$\Rightarrow$ $x \in U$, by definition of U

$\Rightarrow$ $x \in L \cap U$ $[\because \ x \in L]$

$\Rightarrow$ $L \cap U \ne \emptyset$, which is a contradiction.

$\therefore$ $x \in L, y \in U \Rightarrow \ x < y$.

$\therefore$ by Dedikind's axiom, there exists $\alpha \in \mathbf{R}$ such that

$$x < \alpha \Rightarrow x \in L, y > \alpha \Rightarrow y \in U.$$

Now we shall show that $\alpha = \sup S$.

If $y > \alpha$, then $y \in U$ and so $y \notin L$ because L and U are disjoint.

Thus if $y > \alpha$, then y cannot be in L and hence can also not be in S because $S \subset L$.

Thus no real number $y > \alpha$ is in S.

α is an upper bound for S *i.e.*,

$$x \le \alpha, \forall \ x \in S.$$

Now let $\beta < \alpha$. Then

$\beta < \alpha \Rightarrow \beta \in L \ \Rightarrow \beta \notin U$

$\Rightarrow$ β is not an upper bound for S.

Thus α is an upper bound for S and no real number $\beta < \alpha$ can be an upper bound for S.

$\therefore$ $\alpha = \sup S$.

Hence Dedikind's axiom $\Rightarrow$ Completeness axiom.

Comprehensive Exercise 5

1. Give an example of a set which has

(i) no limit point (iii) exactly one limit point

(iii) exactly two limit points (iv) an infinite number of limit points

(v) every point of the set as its limit point.

2. Give an example of a set having the numbers 1 and – 1 as the only limit points.

3. Find the derived set of the set of points in the open interval $] \, a, b \, [$.

4. If a and b are real numbers such that $b > a$, then show that the derived set of each of the intervals $]\,a, b\,[$, $]\,a, b\,]$, $[\,a, b\,[$ and $[a, b]$ is the interval $[a, b]$.
5. Show that the derived set of the open interval $]\,2, 3\,[$ is the closed interval $[2, 3]$.
6. Show that the derived set of the closed interval $[3, 5]$ is the closed interval $[3, 5]$ itself. Hence show that the closed interval $[3, 5]$ is a set of second species.
7. Let $S = \mathbf{R} - \{1/n : n \in \mathbf{N}\}$. Show that each point of $\mathbf{R}$ is a limit point of S.
8. Let $S = \left\{3 + \frac{1}{n} : n \in \mathbf{N}\right\}$. Show that the only limit point of S is 3.
9. If A' denotes the derived set of A, then find a set A such that
   (i) $A \cap A' = \varnothing$. (ii) $A = A'$. (iii) $A' \subset A$. (iv) $A \subset A'$.
10. Find the derived set of the set $S = \{1, 3, 7, 11\}$.
11. Give an example of two subsets A and B of $\mathbf{R}$ such that
$$D(A \cap B) \neq D(A) \cap D(B).$$
12. Show that a point $p \in \mathbf{R}$ is a limit point of a set $S \subset \mathbf{R}$ iff for each positive integer n, the open interval $]\,p - 1/n, p + 1/n\,[$ contains a point of S other than p.
13. Show that a point $p \in \mathbf{R}$ is a limit point of a set $S \subset \mathbf{R}$ iff for each positive rational number r, $]\,p - r, p + r\,[\, \cap S - \{p\} \neq \varnothing$.
14. Find the derived set of each of the following sets :
   (i) The open ray $]\,a, \infty\,[$. (ii) The open ray $]\,-\infty, a\,[$.
   (iii) $\{1 + 3^{-n} : n \in \mathbf{N}\}$. (iv) $\left\{1 - \frac{3}{n} : n \in \mathbf{N}\right\}$.
15. Find the limit points of the set $\left\{1, -1, 1\frac{1}{2}, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{3}, \ldots\right\}$.
16. Find the limit points of the set $S = \left\{\frac{3n + 2}{2n + 1} : n \in \mathbf{N}\right\}$.
17. If $S = \left\{1 + \frac{(-1)^n}{n} : n \text{ is a natural number}\right\}$, find the limit points of the set S.
18. Show that the set $S = \left\{1 + \frac{(-1)^n}{2^n} : n \text{ is a positive integer}\right\}$ is bounded.
   Show that 1 is a limit point of S. Are there any other limit points of S ?
19. (a) Prove that every infinite bounded set of real numbers has at least one limit point. **(Kanpur 2011)**
   (b) Verify the converse of the above theorem in case of the set
$$S = \left\{1, \frac{1}{2}, -3, \frac{1}{4}, 5, \frac{1}{6}, -7, \frac{1}{8}, 9, \ldots\right\}.$$
20. Choose the correct answer :
   If S is closed, then the complement (relative to any open set containing S) of S is
   (A) open (B) closed (C) neither open nor closed.
21. Give examples for the sets (a) which are neither open nor closed
   (b) which are both open and closed.

22. Prove that a set is closed iff its complement is open.
It is given that a set S is not closed, does it imply that it is an open set ? Justify this by giving examples. **(Kanpur 2010)**

Answers 5

1. (i) The set of positive integers $\mathbf{Z}^+$ (ii) The set $\{1/n : n \in \mathbf{Z}^+\}$

(iii) The set $\left\{\frac{1}{2}, -\frac{1}{2}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{4}, -\frac{3}{4}, \dots, \frac{n}{n+1}, \frac{-n}{n+1}, \dots\right\}$

Note that 1 and – 1 are the only limit points of this set

(iv) The open interval $]1, 2[$ (v) The closed interval $[2, 5]$

2. The set given in part (iii) of Ex. 27

3. The closed interval $[a, b]$

9. (i) The set $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}\right\}$

(i) The closed interval $[3, 4]$

(iii) The set $A = \left\{0, \frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \left(\frac{1}{2}\right)^4, \dots\right\}$

(iv) The open interval $]0, 1[$

10. The empty set $\varnothing$ **11.** Take $A = [0, 1[\,, B = \,]1, 2]$

14. (i) The closed ray $[a, \infty[$ (ii) The closed ray $]-\infty, a]$

(iii) The singleton $\{1\}$ (iv) The singleton $\{1\}$

15. 1 and –1 **16.** $\frac{3}{2}$

17. 1 **20.** (B)

21. (a) (i) The set of rational numbers Q

(ii) The right half open interval $[2, 3[$

(iii) The left half open interval $]2, 3\,]$

(b) Set $\varnothing$ and R are both and closed

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. For all $x, y \in \mathbf{R}$, the triangle inequality is

(a) $|x + y| \le |x| + |y|$ (b) $|xy| = |x||y|$

(c) $|x - y| \le |x| + |y|$ (d) $|x + y| = |x| + |y|$.

2. "To every real number there corresponds a unique point on a directed line and conversely, to every point on a directed line there corresponds a unique real number". This is known as:
(a) Cauchy's axiom (b) Weierstrass axiom
(c) Dedikind-Cantor axiom (d) Cantor's axiom

3. If a and b are any two real numbers such that $a \le b$, then the set $\{ x \in \mathbf{R} : a < x < b\}$ is known as :
(a) right-half open interval (b) open interval
(c) left-half open interval (d) closed interval.

4. The set $\{ x : x \in \mathbf{R}, a \le x \le b\}$ is known as :
(a) open interval (b) left-half open interval
(c) closed interval (d) none of these.

5. Which of the following sets is bounded below but not bounded above ?
(a) **N** (b) **Z**
(c) **Q** (d) **R**

6. Let a be any real number and b any positive real number. Then there exists a positive integer n such that
(a) $nb \ge a$ (b) $nb > a$
(c) $nb \le a$ (d) $nb < a$.

7. The least upper bound of the set $\left\{ \frac{1}{n} : n \in \mathbf{N} \right\}$ is
(a) 1 (b) -1
(c) 0 (d) none of these.

8. The greatest lower bound of the set of positive even integers is
(a) 0 (b) 2
(c) 1 (d) none of these.

9. The least upper bound of the set $\{ x : x = 1 - (1/n), n \in \mathbf{N}\}$ is
(a) 0 (b) -1
(c) 1 (d) none of these.

10. The set $S = \{ x : 2 \le x < 4\}$ is such that it contains
(a) g.l.b. but not l.u.b. (b) g.l.b. and l.u.b.
(c) l.u.b. but not g.l.b. (d) none of these.

11. Which of the following set is *nbd* of each of its points ?
(a) Set **N** of natural numbers (b) Set **Q** of rational numbers
(c) Set **Z** of integers (d) open interval $]a, b[$.

12. Neighbourhood of $\frac{1}{2}$ is the set
(a) $]0, \frac{1}{2}[$ (b) $[-\frac{1}{2}, \frac{1}{2}]$
(c) **R** (d) none of these.

13. Derived set of the set **Q** of all rational numbers is
(a) **Q** (b) **R**
(c) **Z** (d) none of these.

14. The set of all rational numbers is
(a) closed (b) open
(c) neither open nor closed (d) none of these.

15. The set of all integers is
(a) countable (b) uncountable
(c) finite (d) none of these.

16. The set of all real numbers is
(a) closed (b) open
(c) both open and closed (d) none of these.

17. The union of infinite number of open sets is
(a) an open set (b) a closed set
(c) need not be an open set (d) none of these.

18. If A and B are any subsets of $\mathbf{R}$, then $A \subset B \Rightarrow$
(a) $D(A) \subset D(B)$ (b) $D(A) \supset D(B)$
(c) $D(A) = D(B)$ (d) none of these.

19. Every singleton set in $\mathbf{R}$ is
(a) open (b) closed
(c) neither open nor closed (d) none of these.

20. If S denotes the set of real numbers $\mathbf{R}$, then $S^\circ =$
(a) $\mathbf{Q}$ (b) $\mathbf{Z}$
(c) $\mathbf{R}$ (d) none of these.

Fill in the Blank(s)

Fill in the blanks "............" so that the following statements are complete and correct.

1. If S be any subset of the set $\mathbf{R}$ of real numbers and if there exists a real number u, such that $x \le u \; \forall \; x \in S$, then u is called an .... of S.

2. Every non-empty set of real numbers which is bounded above has its .... in $\mathbf{R}$.

3. If a and b are any real numbers and $a > 1$, then there exists a positive integer n such that a^n .... b.

4. Any .... interval is a *nbd* of each of its points.

5. The set of all limit points of a set $A \subset \mathbf{R}$ is called the .... of A.

6. If S is a finite set, then $D(S) =$ ....

7. Let F be a subset of $\mathbf{R}$. Then F is said to be closed if its complement F' is ....

8. $\overline{A}$ is the intersection of the family of all closed sets containing ....

9. Every .... of a countable set is countable.

10. For any two real numbers a, b one and only one of the following is true

$$a > b, a = b, a < b.$$

It is known as the law of ....

True or False

Write 'T' for true and 'F' for false statement.

1. The field of real numbers is an ordered field.

2. There is no rational number whose square is 3.
3. For all $x, y \in \mathbf{R}, |x + y| \geq ||x| - |y||$.
4. The set $\mathbf{R}^+$ of positive real numbers is bounded above.
5. Between any two distinct real numbers, there lie an infinite number of real numbers.
6. A closed interval $[a, b]$ is a *nbd* of each of its points.
7. The intersection of two *nbds* of a point need not be a *nbd* of that point.
8. Every point of the set $\mathbf{R}$ of real numbers is a limit point of $\mathbf{R}$.
9. Every open interval is not an open set.
10. Union of a finite number of countable sets is not countable.
11. The intersection of an infinite collection of open sets is an open set.
12. If A and B are any subsets of $\mathbf{R}$, then $D(A \cup B) = D(A) \cup D(B)$.

Answers

Multiple Choice Questions

1. (a)	2. (c)	3. (b)	4. (c)	5. (a)
6. (b)	7. (a)	8. (b)	9. (c)	10. (a)
11. (d)	12. (c)	13. (b)	14. (c)	15. (a)
16. (c)	17. (a)	18. (a)	19. (b)	20. (c)

Fill in the Blank(s)

1. upper bound	2. supremum	3. >	4. open	
5. derived set	6. ϕ	7. open	8. A	9. subset
10. trichotomy				

True or False

1. *T*	2. *T*	3. *T*	4. *F*	5. *T*
6. *F*	7. *F*	8. *T*	9. *F*	10. *F*
11. *F*	12. *T*			

□

Chapter

2

Sequences

1 Introduction

In the present chapter we shall study a special class of functions, namely sequences. The study of sequences plays an important role in Analysis.

2 Sequence

Definition: *Let S be any non-empty set. A function whose domain is the set* **N** *of natural numbers and whose range is a subset of S, is called a sequence in the set S.*

In other words a sequence in a set S is a rule which assigns to each natural number a unique element of S.

Real Sequence: A sequence whose range is a subset of **R** is called a *real sequence* or a sequence of real numbers.

In this chapter we shall study only real sequences. Therefore the term sequence will be used to denote a real sequence.

If s is a sequence, then the image $s(n)$ of $n \in \mathbf{N}$ is usually denoted by s_n. It is customary to denote the sequence s by the symbol $< s_n >$ or by $\{s_n\}$. The image s_n of n is called the nth term of the sequence.

A sequence can be described in several different ways.

1. Listing in order, the first few elements of a sequence, till the rule for writing down different elements becomes clear. For example, $< 1, 8, 27, 64, \ldots >$ is the sequence whose nth term is n^3.
2. Defining a sequence by a formula for its nth term. For example, the sequence $< 1, 8, 27, 64, \ldots >$ can also be written as $< 1, 8, \ldots, n^3, \ldots >$ or as $< n^3 : n \in \mathbf{N} >$ or simply as $< n^3 >$.
3. Defining a sequence by a Recursion formula *i.e.* by a rule which expresses the nth term in terms of the $(n-1)$th term. For example, let

$$a_1 = 1, a_{n+1} = 3a_n, \text{ for all } n \geq 1.$$

These relations define a sequence whose nth term is 3^{n-1}.

Illustrations:

1. $< \frac{1}{n} >$ is the sequence $< 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots >$.
2. $< \frac{n}{n+1} >$ is the sequence $< \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots >$.
3. $< (-1)^n / n >$ is the sequence $< -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \ldots >$.
4. Let $s_1 = 1, s_2 = 1$ and $s_{n+2} = s_{n+1} + s_n$ for all $n \geq 1$.

   From the above formula, $s_3 = s_2 + s_1 = 2$,

$$s_4 = s_3 + s_2 = 3, s_5 = s_4 + s_3 = 5 \text{ and so on.}$$

$\therefore$ $$< s_n > = < 1, 1, 2, 3, 5, \ldots >.$$

Range of a sequence: *The set of all distinct terms of a sequence is called its range.*

$\therefore$ The range of a sequence $< s_n >$ = the set $\{s_1, s_2, \ldots\}$.

Illustration 1: The range of the sequence $< (-1)^n > = \{-1, 1\}$, a finite set.

Illustration 2: The range of the sequence

$$\frac{1}{n} = \left\{ \frac{1}{n} : n \in \mathbf{N} \right\},$$

is an infinite set.

Constant sequence: *A sequence $< s_n >$ defined by $s_n = a$ for all $n \in \mathbf{N}$ is called a constant sequence.*

The sequence $< s_n > = < a, a, a, \ldots >$ is a constant sequence.

Its range = the singleton $\{a\}$ is a finite set.

Equality of two sequences: *Two sequences $< s_n >$ and $< t_n >$ are said to be equal if $s_n = t_n \; \forall \; n \in \mathbf{N}$.*

3 Operations on Sequences

Since sequences of real numbers are real valued functions, we define the sum, difference, product etc. of two sequences as follows :

Let $< s_n >, < t_n >$ be two sequences. Then the sequences having nth terms $s_n + t_n$, $s_n - t_n$ and $s_n t_n$ are respectively called the *sum, difference* and *product* of the sequences $< s_n >$ and $< t_n >$. These sequences are denoted by $< s_n + t_n >$, $< s_n - t_n >$ and $< s_n t_n >$ respectively.

If $t_n \neq 0$ for all $n \in \mathbf{N}$, then the sequence whose nth term is $1 / t_n$ is known as the *reciprocal of the sequence* $< t_n >$ and is denoted by $< 1 / t_n >$. Also the sequence whose nth term is s_n / t_n is called the *quotient* of the sequence $< s_n >$ by the sequence $< t_n >$ and it is denoted by $< s_n / t_n >$.

If $c \in \mathbf{R}$, then the sequence having nth term cs_n is called the *scalar multiple* of $< s_n >$ by c. This sequence is denoted by $< cs_n >$.

4 Subsequences and Order Preservation

Subsequence: *Let $< s_n >$ be any sequence. If $< n_1, n_2, \ldots, n_k, \ldots >$ be a strictly increasing sequence of positive integers i.e., $i > j \Rightarrow n_i > n_j$, then the sequence*

$$< s_{n_1}, s_{n_2}, \ldots\ldots, s_{n_k}, \ldots >$$

is called a subsequence of $< s_n >$.

From the condition $i > j \Rightarrow n_i > n_j$, we conclude that the order of the various terms in the subsequence is the same as it is in the sequence.

Illustrations:

1. Let $< s_n > = < 1, 0, 1, 0, 1, 0, \ldots >$ *i.e.*, $s_1 = 1, s_2 = 0, s_3 = 1, s_4 = 0, \ldots\ldots$.

Take $n_1 = 1, n_2 = 3, n_3 = 5, \ldots..$ Then $< n_r >$ is a sequence of positive integers such that $n_1 < n_2 < n_3 < \ldots..$

Hence $< 1, 1, 1, \ldots >$ is a subsequence of $< s_n >$.

Similarly if we take $n_1 = 2, n_2 = 4, n_3 = 6, \ldots$, then the sequence $< 0, 0, 0, \ldots >$ is also a subsequence of $< s_n >$.

2. The sequence of primes $< 2, 3, 5, 7, 11, \ldots >$ is a subsequence of the sequence of natural numbers $< 1, 2, 3, 4, \ldots.. >$.

3. The sequence $< 7^2, 3^2, 15^2, 11^2, 19^2, \ldots >$ is not a subsequence of the sequence $< 1^2, 2^2, 3^2, 4^2, \ldots >$.

Here $n_1 = 7, n_2 = 3, n_3 = 15, n_4 = 11, n_5 = 19$. But $< 7, 3, 15, 11, 19, \ldots >$ is not a strictly increasing sequence of positive integers.

Order Preservation: Consider the sequence

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots \qquad \ldots (1)$$

There can be found many sequences contained in this sequence, for example the sequences

$$\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\ldots$$

and $$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\ldots \qquad \ldots (2)$$

Now to see if the order of the original sequence is preserved we need to examine the subscripts of the terms of the sequences under consideration.

Let the sequence given in (1) be denoted by $\{a_n\}_{n=1}^{\infty}$. Then the sequence given by (2) can be denoted by $\{a_{2^n}\}_{n=1}^{\infty}$.

Here we say that the order of the terms of sequence (1) is preserved in the sequence (2) as the sequence of subscripts $\{2^n\}_{n=1}^{\infty}$ is strictly increasing.

A sequence is not only a countably infinite set, instead it is defined as a countably infinite set expressed in a specific order.

A sequence is said to be contained in another if the order of the terms is preserved. The sequence $\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{6}, \ldots\ldots$ is not considered to be contained as sequence, in the sequence (1) even though it is contained as a subset.

5 Bounded Sequences

(Kanpur 2008)

Definition 1: *A sequence $< s_n >$ is said to be bounded above if the range set of $< s_n >$ is bounded above i.e., if there exists a real number k_1 such that*

$$s_n \leq k_1 \text{ for } all\ n \in \mathbf{N}.$$

The number k_1 is called an **upper bound** of the sequence $< s_n >$.

Definition 2: *A sequence $< s_n >$ is said to be bounded below if the range set of $< s_n >$ is bounded below i.e., if there exists a real number k_2 such that*

$$s_n \geq k_2 \text{ for } all\ n \in \mathbf{N}.$$

The number k_2 is called a **lower bound** of the sequence $< s_n >$.

Definition 3: *A sequence $< s_n >$ is said to be bounded if the range set of $< s_n >$ is both bounded above and bounded below i.e., if there exist two real numbers k_1 and k_2 such that*

$$k_2 \leq s_n \leq k_1 \text{ for } all\ n \in \mathbf{N}.$$

Equivalently, a sequence $< s_n >$ is bounded if and only if there exists a real number $K > 0$ such that

$$|s_n| \leq K \text{ for all } n \in \mathbf{N}.$$

It is not necessary that a sequence be bounded above or bounded below.

A sequence $< s_n >$ is said to be **unbounded** if it is either unbounded below or unbounded above.

Definition 4: *The least number say, M, if it exists, of the set of the upper bounds of $< s_n >$ is called the* **least upper bound** (**l.u.b.**) *or the* **supremum** (**sup**) *of the sequence $< s_n >$.*

The greatest number say, m, if it exists, of the set of the lower bounds of $< s_n >$ is called the **greatest lower bound** (**g.l.b.**) *or the* **infimum** (**inf.**) *of the sequence $< s_n >$.*

Note 1: If the range of a sequence is a finite set , then the sequence is bounded because a finite set is always bounded.

Note 2: Every subsequence of a bounded sequence is bounded.

Illustrations:

1. The sequence $< \frac{1}{n} >$ is bounded since $\left| \frac{1}{n} \right| \le 1$ for all $n \in \mathbf{N}$.
2. The sequence $< \frac{n}{n+1} >$ is bounded since $\frac{1}{2} \le \frac{n}{n+1} < 1$ for all $n \in \mathbf{N}$.
3. The sequence $< (-1)^n >$ is bounded since $|(-1)^n| \le 1$ for all $n \in \mathbf{N}$. In fact $|(-1)^n| = 1$ for all $n \in \mathbf{N}$.
4. The sequence $< (-1)^n / n >$ is bounded since $|(-1)^n / n| \le 1$ for all $n \in \mathbf{N}$.
5. The sequence $< s_n >$ defined by $s_n = 1 + (-1)^n$ for all $n \in \mathbf{N}$, is bounded since the range set of the sequence is $\{0, 2\}$, which is a finite set.
6. The sequence $< n^2 >$ is bounded below by 1 but not bounded above.
7. The sequence $< -n^2 >$ is bounded above by -1 but not bounded below.
8. The sequence $< s_n > = < (-1)^n n >$ is neither bounded below nor bounded above.

For any positive real number K, there exists a positive integer $2m$ such that $2m > K$. It gives that $s_{2m} > K$. Hence $< s_n >$ is not bounded above.

Similarly it can be shown that $< s_n >$ is not bounded below.

Theorem: *A sequence $< s_n >$ is bounded iff there exist $m \in \mathbf{N}$, $l \in \mathbf{R}$ and $a > 0$ such that*

$|s_n - l| < a$ for *all* $n \ge m$.

Proof: Let $< s_n >$ be a bounded sequence. Then there exist two real numbers k_1, k_2 such that $k_1 < s_n < k_2$ for all $n \in \mathbf{N}$

$$\Rightarrow \quad k_1 - \frac{k_1 + k_2}{2} < s_n - \frac{k_1 + k_2}{2} < k_2 - \frac{k_1 + k_2}{2} \;\; \forall \; n \in \mathbf{N}$$

$$\Rightarrow \quad \frac{k_1 - k_2}{2} < s_n - \frac{k_1 + k_2}{2} < \frac{k_2 - k_1}{2} \;\; \forall \; n \in \mathbf{N}$$

$$\Rightarrow \quad -a < s_n - l < a \;\; \forall \; n \in \mathbf{N} \quad \text{where} \quad l = \frac{k_1 + k_2}{2} \quad \text{and} \quad a = \frac{k_2 - k_1}{2}$$

$$\Rightarrow \quad |s_n - l| < a \;\; \forall \; n \in \mathbf{N}$$

$$\Rightarrow \quad |s_n - l| < a \;\; \forall \; n \ge m, \text{where } m = 1 \in \mathbf{N}, l \in \mathbf{R} \text{ and } a > 0.$$

Conversely, let there exist $l \in \mathbf{R}, a > 0$ and $m \in \mathbf{N}$ such that $|s_n - l| < a$ for all $n \ge m$.

This gives $l - a < s_n < l + a$ for all $n \ge m$.

Choose $k_1 = \min\{s_1, s_2, \ldots, s_{m-1}, l - a\}$.

Then $k_1 \le s_n$, for all $n \in \mathbf{N}$.

Again choose $k_2 = \max\{s_1, s_2, \ldots, s_{m-1}, l + a\}$.

Then $s_n \le k_2$ for all $n \in \mathbf{N}$.

$\therefore \quad k_1 \le s_n \le k_2$ for all $n \in \mathbf{N}$.

Hence $< s_n >$ is a bounded sequence.

Note: In view of the above theorem we conclude that a sequence is bounded even if $k_1 \le s_n \le k_2$ for $n \ge m$. However in such a case it is not necessary that k_1 is a lower bound and k_2 is an upper bound of the sequence $< s_n >$.

6 Convergent Sequences

Definition: *A sequence $< s_n >$ is said to converge to a number l, if for any given $\varepsilon > 0$ there exists a positive integer m such that*

$$|s_n - l| < \varepsilon \text{ for all } n \geq m.$$

The number l is called the **limit** *of the sequence $< s_n >$ and we write $s_n \to l$ as $n \to \infty$ or* $\lim\limits_{n \to \infty} s_n = l$ *or simply* $\lim s_n = l$.

(Gorakhpur 2011)

The positive integer m depends on the value of ε.

The phrase '$|s_n - l| < \varepsilon$ for all $n \geq m$' expresses the fact that the absolute value of the difference between s_n and l can be made less than ε from some stage onwards *i.e.*, $l - \varepsilon < s_n < l + \varepsilon$ from some stage onwards.

The truth of the statement that the sequence $< s_n >$ converges to l depends upon showing that all except a finite number of terms of the sequence must lie in the open interval $]\, l - \varepsilon, l + \varepsilon\, [$, whatever $\varepsilon > 0$ we take. If we can find even one ε for which infinitely many terms of the sequence lie outside $]\, l - \varepsilon, l + \varepsilon\, [$, the sequence will not converge to l. The number of terms lying outside $]\, l - \varepsilon, l + \varepsilon\, [$ depends upon ε. The smaller the ε, the larger the number of terms which lie outside $]\, l - \varepsilon, l + \varepsilon\, [$.

From the above discussion, we conclude that '*a sequence converges to l iff it lies ultimately in each open interval around l*'.

Theorem 1: *If $< s_n >$ is a sequence of non-negative numbers such that $\lim s_n = l$, then $l \geq 0$.*

Proof: Suppose, if possible, $l < 0$. Then $-l > 0$.

Since $\lim s_n = l$, for $\varepsilon = -(l/2) > 0$, there exists $m \in \mathbf{N}$ such that $|s_n - l| < -l/2$ for all $n \geq m$.

In particular, $|s_m - l| < -l/2$

i.e., $$l + \frac{l}{2} < s_m < l - \frac{l}{2}$$

i.e., $s_m < l/2$ or $s_m < 0$, because by assumption $l < 0$.

But by hypothesis we have $s_m \geq 0$ because it is given that $s_n \geq 0$ for all n.

Hence our assumption is wrong. So we must have $l \geq 0$.

Remark: In the above proof ε can be taken any positive real number such that $0 < \varepsilon \leq -l$, say $\varepsilon = -l$ or $\varepsilon = -l/2$ or $\varepsilon = -l/3$ etc.

Note 1: A negative number cannot be the limit of a sequence of non-negative numbers.

2. If $\lim s_n = l$ and $l < 0$ then there exists a positive integer m such that $s_n < 0$ for all $n \geq m$.

3. A sequence $< s_n >$ is called a null sequence if $\lim s_n = 0$.

Theorem 2: *A sequence cannot converge to more than one limit i.e., the limit of a sequence is unique.* **(Gorakhpur 2011)**

Proof: If possible, suppose a sequence $< s_n >$ converges to two distinct numbers l and l'. Since $l \neq l'$, therefore $|l - l'| > 0$.

Let $\varepsilon = \frac{1}{2}|l - l'|$, then $\varepsilon > 0$.

Now $< s_n >$ converges to $l \Rightarrow$ there exists $m_1 \in \mathbf{N}$ such that

$$|s_n - l| < \varepsilon \text{ for all } n \geq m_1. \quad \text{...(1)}$$

Similarly $< s_n >$ converges to $l' \rightarrow$ there exists $m_2 \in \mathbf{N}$ such that

$$|s_n - l'| < \varepsilon \text{ for all } n \geq m_2\ . \quad \text{...(2)}$$

Let $m = \max\{m_1, m_2\}$.

Then (1) and (2) hold for all $n \geq m$.

We have for all $n \geq m$

$$|l - l'| = |(s_n - l) - (s_n - l')| \leq |s_n - l| + |s_n - l'|$$
$$< \varepsilon + \varepsilon = 2\varepsilon = |l - l'|.$$

Thus $|l - l'| < |l - l'|$, which is absurd. Hence our initial assumption that $l \neq l'$ is wrong and we must have $l = l'$ *i.e.*, the limit of a sequence is unique.

Note: After taking $\varepsilon = \frac{1}{2}|l - l'|$, we can also give the following argument :

Since $< s_n >$ converges to both l and l', therefore it lies ultimately in both the intervals $]l - \varepsilon, l + \varepsilon[$ and $]l' - \varepsilon, l' + \varepsilon[$. This is impossible since these two intervals have no real number in common. Hence our assumption is wrong and thus the sequence cannot converge to more than one limit.

Theorem 3: *If $< s_n >$ converges to l, then any subsequence of $< s_n >$ also converges to l.*

Proof: Let $< s_{n_k} >$ be any subsequence of $< s_n >$. Then by definition of subsequence, $n_1, n_2, \ldots, n_k, \ldots.$ are positive integers such that

$$n_1 < n_2 < \ldots < n_k < \ldots.$$

Now $\quad n_1 \geq 1 \Rightarrow n_k \geq k$ (by induction).

Since $< s_n >$ converges to l, so given $\varepsilon > 0$, there exists a positive integer m such that

$$|s_k - l| < \varepsilon \text{ for all } k \geq m.$$

For $k \geq m$ we have $n_k \geq k \geq m$.

$\therefore \quad |s_{n_k} - l| < \varepsilon$ for all $n_k \geq m$.

$\therefore \quad < s_{n_k} >$ converges to l.

Corollary: *All subsequences of a convergent sequence converge to the same limit.*

Proof: By theorem 3, any subsequence of a sequence converges to the same limit as the limit of the sequence and by theorem 2, the limit of a sequence is unique. This shows that all subsequences of a convergent sequence have the same limit.

Note: To show that a given sequence is not convergent it is enough to show that two of its subsequences converge to different limits.

Illustration: The sequence $< (-1)^n >$ is not convergent.

The two subsequences $< 1, 1, 1, \ldots. >$ and $< -1, -1, -1, \ldots >$ of the given sequence converge respectively to 1 and -1 which are different.

Theorem 4: *If the subsequences* $< s_{2n-1} >$ *and* $< s_{2n} >$ *of the sequence* $< s_n >$ *converge to the same limit l, then the sequence* $< s_n >$ *converges to l.*

Proof: Let $\varepsilon > 0$ be given. Then, since $\lim s_{2n-1} = l$, there exists $m_1 \in \mathbf{N}$ such that $| s_{2n-1} - l | < \varepsilon \ \ \forall \ n \geq m_1$.

Similarly $\lim s_{2n} = l \Rightarrow$ for $\varepsilon > 0$, there exists $m_2 \in \mathbf{N}$ such that

$$| s_{2n} - l | < \varepsilon \ \ \forall \ n \geq m_2 \ .$$

Let $\quad m = \max. \{ m_1, m_2 \}$.

Then $\quad | s_{2n-1} - l | < \varepsilon$ and $| s_{2n} - l | < \varepsilon \ \forall \ n \geq m$.

$\therefore \quad | s_n - l | < \varepsilon \ \ \forall \ n \geq 2m - 1$.

Hence $< s_n >$ converges to l.

Theorem 5: *Every convergent sequence is bounded.*

(Kanpur 2008; Gorakhpur 10, 13, 14)

Proof: Let $< s_n >$ be a sequence which converges to l. Take $\varepsilon = 1$. Then there exists a positive integer m such that

$$| s_n - l | < 1, \text{ for all } n \geq m,$$

i.e., $\quad l - 1 < s_n < l + 1$, for all $n \geq m$.

Let $\quad k = \min. \{ s_1, s_2, \ldots\ldots, s_{m-1}, l - 1 \}$,

and $\quad K = \max. \{ s_1, s_2, \ldots\ldots, s_{m-1}, l + 1 \}$.

$\therefore \quad k \leq s_n \leq K$ for all $n \in \mathbf{N}$.

Hence the sequence $< s_n >$ is bounded.

Note: The converse of the above theorem need not be true. That is, a bounded sequence need not be convergent. For example, the sequence $< (-1)^n >$ is bounded but is not convergent. **(Gorakhpur 2013, 15)**

Illustrative Examples

Example 1: *Show that the sequence* $< 1/n >$ *has the limit* 0.

Solution: For any given $\varepsilon > 0$, we have $\left| \frac{1}{n} - 0 \right| < \varepsilon$ when $\frac{1}{n} < \varepsilon$ *i.e.*, when $n > \frac{1}{\varepsilon} \cdot$

Let us choose a positive integer $m > 1/\varepsilon$. Then for all $n \geq m$, we have

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{m} < \varepsilon.$$

$\therefore \quad < \frac{1}{n} >$ converges to 0 *i.e.*, $< \frac{1}{n} >$ has the limit 0.

Example 2: *If* $s_n = k \ (\in \mathbf{R})$ *is a constant sequence, then* $\lim s_n = k$.

Solution: We have $| s_n - k | = | k - k | = 0$ for all $n \in \mathbf{N}$.

Given any $\varepsilon > 0$, $| s_n - k | = 0 < \varepsilon$ for all n

i.e., $\quad | s_n - k | < \varepsilon$ for all $n \geq m = 1$.

Hence $\quad \lim s_n = k$.

Example 3: *The sequence* $< s_n >$ *where* $s_n = 1/2^n$ *converges to '0'.*

Solution: We have $|s_n - 0| = \frac{1}{2^n}$.

$\therefore$ For any $\varepsilon > 0, |s_n - 0| < \varepsilon$ if $\frac{1}{2^n} < \varepsilon$

i.e., if $2^n > \frac{1}{\varepsilon}$ *i.e.*, if $n > \frac{\log(1/\varepsilon)}{\log 2}$.

Let us choose a positive integer $m > \left(\log \frac{1}{\varepsilon}\right) \Big/ \log 2.$

Then for all $n \geq m, |s_n - 0| < \varepsilon.$

$\therefore$ $< s_n >$ converges to 0.

Example 4: *Show that the sequence* $< s_n >$ *defined by* $s_n = r^n$ *converges to zero if* $|r| < 1$.

Solution: If $|r| < 1$, then we can write $|r| = \frac{1}{1+h}$, where $h > 0$. Since $h > 0$, therefore

$$(1+h)^n = 1 + nh + \frac{n(n-1)}{2} h^2 + \ldots + h^n \geq 1 + nh \text{ for all } n.$$

Now $|s_n - 0| = |r^n| = |r|^n = \frac{1}{(1+h)^n} \leq \frac{1}{1+nh}, \ \forall\ n.$

Let $\varepsilon > 0$ be given. Then $|s_n - 0| < \varepsilon$ if $\frac{1}{1+nh} < \varepsilon$ *i.e.*, if $n > \left(\frac{1}{\varepsilon} - 1\right) \Big/ h.$

If we take a positive integer $m > \left(\frac{1}{\varepsilon} - 1\right) \Big/ h$, then for all $n \geq m$, $|s_n - 0| < \varepsilon$.

Hence $< s_n >$ converges to zero.

Example 5: *Let* $< s_n >$ *be a sequence such that* $s_n \neq 0$ *for any n, and* $\frac{s_{n+1}}{s_n} \to l$. *Prove that if* $|l| < 1$, *then* $s_n \to 0$.

Solution: Since $|l| < 1$, hence there exists $\varepsilon_0 > 0$ such that

$$|l| + \varepsilon_0 = h < 1.$$

Now $\frac{s_{n+1}}{s_n} \to l \Rightarrow$ there exists a positive integer m such that

$$\left| \frac{s_{n+1}}{s_n} - l \right| < \varepsilon_0 \text{ for all } n \geq m.$$

We have $$\left|\frac{s_{n+1}}{s_n}\right| = \left|\left(\frac{s_{n+1}}{s_n} - l\right) + l\right| \leq \left|\frac{s_{n+1}}{s_n} - l\right| + |l|,$$

$$< \varepsilon_0 + |l|, \text{ for all } n \geq m$$

i.e., $$\left|\frac{s_{n+1}}{s_n}\right| < h, \text{ for all } n \geq m. \quad \ldots(1)$$

Replacing n by $m, m+1, \ldots, n-1$ successively in (1) and multiplying the corresponding sides of the resulting $n - m$ inequalities, we get

$$\left|\frac{s_{m+1}}{s_m}\right| \cdot \left|\frac{s_{m+2}}{s_{m+1}}\right| \cdots\cdots \left|\frac{s_n}{s_{n-1}}\right| < h^{n-m},$$

or $$\left|\frac{s_{m+1}}{s_m} \cdot \frac{s_{m+2}}{s_{m+1}} \cdots\cdots\cdots \frac{s_n}{s_{n-1}}\right| < h^{n-m},$$

or $$|s_n| < h^n \left(\frac{|s_m|}{h^m}\right), \text{ for all } n > m. \quad \ldots(2)$$

Again, since $0 < h < 1$, therefore, $h^n \to 0$ and hence, given $\varepsilon > 0$, there exists a positive integer m_1 such that

$$|h^n| < h^m \varepsilon / |s_m|, \text{ for all } n \geq m_1. \quad \ldots(3)$$

Let us choose a positive integer p such that $p > \max.\{m, m_1\}$.

From (2) and (3), we get

$$|s_n| < \varepsilon, \text{ for all } n \geq p.$$

Hence $s_n \to 0$.

Example 6: *Find an* $m \in \mathbf{N}$ *such that* $\left|\frac{2n}{n+3} - 2\right| < \frac{1}{5}$ *for all* $n \geq m$.

Solution: We have $\left|\frac{2n}{n+3} - 2\right| < \frac{1}{5} \Rightarrow \left|\frac{2n - 2n - 6}{n+3}\right| < \frac{1}{5}$

$\Rightarrow$ $$\frac{6}{n+3} < \frac{1}{5} \Rightarrow \frac{n+3}{6} > 5 \Rightarrow n > 27.$$

If we take a positive integer $m > 27$, we have

$$\left|\frac{2n}{n+3} - 2\right| < \frac{1}{5}, \text{ for all } n \geq m.$$

Hence for $\varepsilon = 1/5$, the required least value of $m = 28$.

In fact, we can find $m \in \mathbf{N}$ for each $\varepsilon > 0$ such that

$$\left|\frac{2n}{n+3} - 2\right| < \varepsilon \text{ for all } n \geq m.$$

$\therefore$ $$\lim \frac{2n}{n+3} = 2.$$

Example 7: *Show that the sequence* $< s_n >$ *where* $s_n = \frac{2n^2 + 1}{2n^2 - 1}$, $\forall\ n \in \mathbf{N}$ *converges to* 1.

Solution: Let $\varepsilon > 0$ be given.

We have $$|s_n - 1| = \left|\frac{2n^2 + 1}{2n^2 - 1} - 1\right| = \left|\frac{2}{2n^2 - 1}\right| = \frac{2}{2n^2 - 1} < \varepsilon \text{ if } n > \sqrt{\left(\frac{2 + \varepsilon}{2\varepsilon}\right)}.$$

If we choose a positive integer $m > \sqrt{\left(\frac{2 + \varepsilon}{2\varepsilon}\right)}$, then for all $n \geq m$, $|s_n - 1| < \varepsilon$.

$\therefore$ $$\lim s_n = 1.$$

Example 8: *Show that the sequence* $< s_n >$ *where* $s_n = \dfrac{3n}{n + 5n^{1/2}}$ *has the limit* 3.

Solution: Let $\varepsilon > 0$ be given.

We have $$\left|\frac{3n}{n+5n^{1/2}} - 3\right| = \left|\frac{3n - 3n - 15n^{1/2}}{n + 5n^{1/2}}\right| = \frac{15n^{1/2}}{n+5n^{1/2}}$$

which is less than $\dfrac{15n^{1/2}}{n} = \dfrac{15}{n^{1/2}}$.

Thus $$\frac{15n^{1/2}}{n+5n^{1/2}} \text{ will be} < \varepsilon \text{ if } \frac{15}{n^{1/2}} < \varepsilon \text{ i.e. if } n > \frac{225}{\varepsilon^2}.$$

If we choose a positive integer $m > \dfrac{225}{\varepsilon^2}$, then $|s_n - 3| < \varepsilon$ for all $n \ge m$.

Hence $$\lim_{n \to \infty} s_n = 3.$$

Example 9: *Prove that* $\lim (1/n^p) = 0, p > 0$.

Solution: Let $\varepsilon > 0$ be given.

We have $$\left|\frac{1}{n^p} - 0\right| < \varepsilon \Rightarrow \frac{1}{n^p} < \varepsilon \Rightarrow n^p > \frac{1}{\varepsilon}$$
$$\Rightarrow n > \left(\frac{1}{\varepsilon}\right)^{1/p}.$$

By Archimedean property, for $(1/\varepsilon)^{1/p} \in \mathbf{R}$ there exists a positive integer $m > (1/\varepsilon)^{1/p}$. If we choose $m > (1/\varepsilon)^{1/p}$, then we have
$$\left|\frac{1}{n^p} - 0\right| < \varepsilon \text{ for all } n \ge m.$$

Hence $$\lim \frac{1}{n^p} = 0, \text{ when } p > 0.$$

Example 10: *Show that the sequence* $< s_n >$, *where* $s_n = (-1)^{n-1}/n$, *converges to* 0.

Solution: We have $s_n = \dfrac{1}{n}$ if n is odd, $s_n = -\dfrac{1}{n}$ if n is even. The sequences $< s_{2n-1} >$ and $< s_{2n} >$ are subsequences of $< s_n >$.

$$< s_{2n-1} > = < 1, \frac{1}{3}, \frac{1}{5}, \ldots > , < s_{2n} > = < -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, \ldots >.$$

But $< 1, \dfrac{1}{3}, \dfrac{1}{5}, \ldots >$ and $< \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{6}, \ldots >$ are subsequences of the sequence $< 1/n >$ which converges to 0.

$\therefore$ $< s_{2n-1} >$ and $< s_{2n} >$ both converge to the same limit 0.

Hence $< s_n >$ converges to 0.

[By theorem 4, article 6]

Alternative Solution: Take any given $\varepsilon > 0$.

We have $|s_n - 0| = |s_n| = \left|\dfrac{(-1)^{n-1}}{n}\right| = \dfrac{1}{n} < \varepsilon$ if $n > \dfrac{1}{\varepsilon}\cdot$

Now by Archimedean property of real numbers, for a given real number $1/\varepsilon$ there exists a positive integer m such that $m > 1/\varepsilon$ or $1/m < \varepsilon$.

Then for all $n \geq m$, we have

$$|s_n - 0| = \frac{1}{n} \leq \frac{1}{m} < \varepsilon.$$

Hence $< s_n >$ converges to zero.

Example 11: *Show that* $\lim \sqrt[n]{n} = 1$.

Solution: Let $\sqrt[n]{n} = 1 + h_n$, where $h_n \geq 0$.

$$\therefore \quad n = (1 + h_n)^n = 1 + nh_n + \frac{n(n-1)}{1.2} h_n^2 + \ldots + h_n^n$$

$$> \frac{n(n-1)}{2} h_n^2 \text{ for all } n, \text{ since } h_n \geq 0.$$

$$\therefore \quad h_n^2 < \frac{2}{n-1} \text{ for } n \geq 2 \quad i.e. \ |h_n| < \sqrt{\left(\frac{2}{n-1}\right)} \text{ for } n \geq 2.$$

Let $\varepsilon > 0$ be given. Then $|h_n| < \sqrt{\left(\dfrac{2}{n-1}\right)} < \varepsilon$, provided $\dfrac{2}{n-1} < \varepsilon^2$ *i.e.* $n > \dfrac{2}{\varepsilon^2} + 1$.

If we choose $m \in \mathbf{N}$ such that $m > \dfrac{2}{\varepsilon^2} + 1$, then we have

$$|h_n| < \varepsilon \ \forall \ n \geq m$$

i.e. $$|\sqrt[n]{n} - 1| < \varepsilon \ \forall \ n \geq m.$$

$$\therefore \quad \lim \sqrt[n]{n} = 1.$$

Example 12: *Show that the sequence* $< s_n >$ *defined by*

$s_n = \{\sqrt{(n+1)} - \sqrt{n}\}, \forall \ n \in \mathbf{N}$ *is convergent.*

Solution: We have $s_n = \sqrt{(n+1)} - \sqrt{n}$

$$= \{\sqrt{(n+1)} - \sqrt{n}\} \frac{\{\sqrt{(n+1)} + \sqrt{n}\}}{\{\sqrt{(n+1)} + \sqrt{n}\}}$$

$$= \frac{1}{\sqrt{(n+1)} + \sqrt{n}} < \frac{1}{\sqrt{n} + \sqrt{n}} = \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{n}} \quad i.e., \ s_n < \frac{1}{\sqrt{n}}\cdot$$

Let $\varepsilon > 0$ be given. Then $|s_n - 0| < \dfrac{1}{\sqrt{n}} < \varepsilon$, provided $\sqrt{n} > \dfrac{1}{\varepsilon}$ *i.e.,* $n > \dfrac{1}{\varepsilon^2}\cdot$

If m is a positive integer greater than $1/\varepsilon^2$, then

$$|s_n - 0| < \varepsilon \text{ for all } n \geq m.$$

Hence $$\lim s_n = 0.$$

Example 13: *Show that the sequence* $< s_n >$ *where* $s_n = \sin n\pi\theta$ *and* θ *is a rational number such that* $0 < \theta < 1$, *is not convergent.*

Solution: Let $\theta = \frac{p}{q}$, where p and q are integers. Since $0 < \theta < 1$, we must have $q \geq 2$. For $n = q, 2q, 3q, \ldots$, the terms of $< s_n >$ are $\sin \pi p, \sin 2\pi p, \sin 3\pi p, \ldots$, *i.e.*, 0, 0, 0, ... Thus $< s_n >$ contains a subsequence < 0, 0, 0, ...> which converges to 0. Now for $n = q + 1$, $2q + 1, 3q + 1$, ... the terms of $< s_n >$ are

$$\sin\left(\pi p + \frac{\pi p}{q}\right), \sin\left(2\pi p + \frac{\pi p}{q}\right), \sin\left(3\pi p + \frac{\pi p}{q}\right), \ldots.$$

i.e., $$(-1)^p \sin(\pi p / q), (-1)^{2p} \sin(\pi p / q), (-1)^{3p} \sin(\pi p / q), \ldots.$$

All these terms have absolute value $\sin(\pi p / q)$ and do not tend to zero since $0 < \pi p / q < \pi$. $[\because\ 0 < p / q < 1]$.

Thus the sequence $< s_n >$ contains a subsequence whose limit is 0 and a subsequence which (may or may not converge but certainly) does not have the limit zero. Hence we conclude that the given sequence is not convergent. [See corollary to theorem 3]

Remark: The sequence $< \sin n\pi\theta >$ obviously converges to 0 for $\theta = 0$ or $\theta = 1$.

7 Divergent Sequences

Definition 1: *A sequence $< s_n >$ is said to diverge to $+\infty$ if for any given $k > 0$ (however large), there exists $m \in \mathbf{N}$ such that $s_n > k$* for all $n \geq m$.

If $< s_n >$ diverges to infinity, we write $s_n \to \infty$ as $n \to \infty$ or $\lim s_n = +\infty$.

Definition 2: *A sequence $< s_n >$ is said to diverge to $-\infty$ if for any given $k < 0$ (however small), there exists $m \in \mathbf{N}$ such that $s_n < k$ for all $n \geq m$.*

If $< s_n >$ diverges to minus infinity, we write

$$s_n \to -\infty \text{ as } n \to \infty \quad \text{or} \quad \lim s_n = -\infty.$$

A sequence is said to be a divergent sequence if it diverges to either $+\infty$ or $-\infty$.

Illustrations:

1. $< 2, 4, 6, \ldots, 2n, \ldots >$ diverges to $+\infty$.
2. $< 3, 3^2, 3^3, \ldots, 3^n, \ldots >$ diverges to $+\infty$.
3. $< x, x^2, x^3, \ldots, x^n, \ldots >, x > 1$, diverges to $+\infty$.
4. $< -2, -4, -6, \ldots, -2n, \ldots >$ diverges to $-\infty$.
5. $< -3, -3^2, -3^3, \ldots, -3^n, \ldots >$ diverges to $-\infty$.
6. $< -x, -x^2, -x^3, \ldots, -x^n, \ldots >, x > 1$, diverges to $-\infty$.

8 Oscillatory Sequences

Definition: *A sequence $< s_n >$ is said to be an oscillatory sequence if it is neither convergent nor divergent.*

An oscillatory sequence is said to oscillate finitely or infinitely according as it is bounded or unbounded.

Illustrations:

1. The sequence $<(-1)^n>$ oscillates finitely.
2. The sequence $<(-1)^n n>$ oscillates infinitely.

Theorem 1: *If a sequence $< s_n >$ diverges to infinity then any subsequence of $< s_n >$ also diverges to infinity.*

Proof: Let $< s_{n_k} >$ be any subsequence of the sequence $< s_n >$. Then by the definition of a subsequence $< n_1, n_2, \ldots, n_k, \ldots >$ is a strictly increasing sequence of positive integers. This implies $n_1 \geq 1 \Rightarrow n_k \geq k$ (by induction).

Take any given positive real number k_1.

Now $< s_n >$ diverges to $\infty \Rightarrow$ for $k_1 > 0$ there exists $m \in \mathbf{N}$ such that $s_n > k_1$ for all $n \geq m$ *i.e.*, $s_k > k_1 \ \forall \ k \geq m$.

For $k \geq m$, we have $n_k \geq k \geq m$ *i.e.*, $n_k \geq m$.

$\therefore \quad s_{n_k} > k_1$ for all $n_k \geq m$.

$\therefore \quad < s_{n_k} >$ diverges to infinity.

Note: If $s_{2n-1} \to \infty$ as $n \to \infty$ and $s_{2n} \to \infty$ as $n \to \infty$, then $s_n \to \infty$ as $n \to \infty$.

Theorem 2: *If $s_n > 0$ for all $n \in \mathbf{N}$, then*

$$s_n \to \infty \text{ as } n \to \infty \Leftrightarrow \frac{1}{s_n} \to 0 \text{ as } n \to \infty.$$

Proof: Let $s_n \to \infty$ as $n \to \infty$.

Let $\varepsilon > 0$ be given. Since $s_n \to \infty$ as $n \to \infty$, hence for $1/\varepsilon > 0$ there exists $m \in \mathbf{N}$ such that

$$s_n > 1/\varepsilon \text{ for all } n \geq m$$

$$\Rightarrow \quad \frac{1}{s_n} < \varepsilon \ \forall \ n \geq m \Rightarrow \left|\frac{1}{s_n}\right| < \varepsilon \ \forall \ n \geq m \qquad [\because s_n > 0]$$

$$\Rightarrow \quad \left|\frac{1}{s_n} - 0\right| < \varepsilon \ \forall \ n \geq m \ \Rightarrow \ \frac{1}{s_n} \to 0 \text{ as } n \to \infty.$$

Conversely, let $\frac{1}{s_n} \to 0$ as $n \to \infty$.

Take any given $k > 0$.

Now $\frac{1}{s_n} \to 0$ as $n \to \infty \Rightarrow$ for $1/k > 0$ there exists $m \in \mathbf{N}$ such that

$$\left|\frac{1}{s_n} - 0\right| < \frac{1}{k} \text{ for all } n \geq m$$

$$\Rightarrow \quad \frac{1}{s_n} < \frac{1}{k} \ \forall \ n \geq m \Rightarrow s_n > k \ \ \forall \ \ n \geq m$$

$$\Rightarrow \quad s_n \to \infty \text{ as } n \to \infty.$$

Theorem 3: *If the sequences $< s_n >, < t_n >$ diverge to infinity then $< s_n + t_n >$ and $< s_n t_n >$ diverge to infinity.*

Proof: Take any given $k > 0$. The sequence $< s_n >$ diverges to infinity $\Rightarrow$ for $k > 0$, there exists $m_1 \in \mathbf{N}$ such that $s_n > k \; \forall \, n \geq m_1$. Again, the sequence $< t_n >$ diverges to infinity $\Rightarrow$ for $1 > 0$, there exists $m_2 \in \mathbf{N}$ such that $t_n > 1 \; \forall \, n \geq m_2$.

Take $m = \max. \{m_1, m_2\}$.

$\therefore$ $s_n + t_n > k + 1 > k \;\; \forall \, n \geq m$ and $s_n \, t_n > k \,.1 = k \;\; \forall \, n \geq m$.

$\therefore$ Both $< s_n + t_n >$ and $< s_n \, t_n >$ diverge to infinity.

Theorem 4: *If $< s_n >$ diverges to infinity and $< t_n >$ is bounded then $< s_n + t_n >$ diverges to infinity.*

Proof: The sequence $< t_n >$ is bounded $\Rightarrow$ there exists $k_1 > 0$ such that

$$| t_n | < k_1 \; \forall \, n \in \mathbf{N}.$$

The sequence $< s_n >$ diverges to infinity $\Rightarrow$ for $k > 0$ there exists $m \in \mathbf{N}$ such that $s_n > k + k_1 \; \forall \, n \geq m$.

$\therefore$ For all $n \geq m$, we have

$$\begin{aligned} s_n + t_n &\geq s_n - | t_n | && [\because \; x \geq - | x |] \\ &> k + k_1 - k_1 = k. \end{aligned}$$

Thus for $k > 0$, there exists $m \in \mathbf{N}$ such that $s_n + t_n > k$ for all $n \geq m$.

Hence $< s_n + t_n >$ diverges to infinity.

Corollary: *If $< s_n >$ diverges to infinity and $< t_n >$ converges then $< s_n + t_n >$ diverges to infinity.*

Illustrative Examples

Example 14: *Prove that the sequence $< n^p >$ where $p > 0$ diverges to infinity.*

Solution: Let $s_n = n^p$. Then $s_n > 0$ for all n as $n \in \mathbf{N}$ and $p > 0$.

$\therefore$ The sequence $< \dfrac{1}{s_n} > = < \dfrac{1}{n^p} >$ exists.

Since we know that $\dfrac{1}{n^p} \to 0$ as $n \to \infty$,

$\therefore$ $n^p \to \infty$ as $n \to \infty$.

Hence $< n^p >$ diverges to ∞.

Example 15: *Show that the sequence $< \log \dfrac{1}{n} >$ diverges to $- \infty$.*

Solution: Let $s_n = \log \dfrac{1}{n} \cdot$ Take any given $k < 0$.

Then $s_n < k$ if $\log \dfrac{1}{n} < k$ *i.e.*, if $- \log n < k$

i.e., if $\log n > - k$ *i.e.*, if $n > e^{-k}$.

If we take $m \in \mathbf{N}$ such that $m > e^{-k}$, then $s_n < k$ for all $n \geq m$.

Hence $s_n \to - \infty$ as $n \to \infty$.

Example 16: *If $< t_n >$ diverges to ∞ and $s_n > t_n \;\; \forall \; n$, then $< s_n >$ diverges to ∞.*

Solution: Take any given $k > 0$.

Since $< t_n >$ diverges to ∞, therefore, for $k > 0$ there exists $m \in \mathbf{N}$ such that

$$t_n > k \text{ for all } n \geq m$$

$$\Rightarrow \quad s_n > k \text{ for all } n \geq m. \qquad [\because s_n > t_n \ \forall \ \ n \in \mathbf{N}]$$

Hence $< s_n >$ diverges to ∞.

9 Algebra of Convergent Sequences

Theorem 1: *If* $\lim s_n = l$ *and* $\lim t_n = l'$ *then* $\lim (s_n + t_n) = l + l'$. *In other words the limit of the sum of two convergent sequences is the sum of their limits.*

Proof: Take any given $\varepsilon > 0$.

Since $\lim s_n = l$, therefore for a given positive real number $\varepsilon / 2$ there exists $m_1 \in \mathbf{N}$ such that

$$|s_n - l| < \varepsilon / 2 \text{ for all } n \geq m_1.$$

Similarly, since $\lim t_n = l'$, there exists $m_2 \in \mathbf{N}$ such that $|t_n - l'| < \varepsilon / 2$ for all $n \geq m_2$.

Let $m = \max. \{m_1, m_2\}$. Then

$$|s_n - l| < \varepsilon / 2 \text{ and } |t_n - l'| < \varepsilon / 2 \text{ for all } n \geq m.$$

$\therefore$ for all $n \geq m$, we have

$$|(s_n + t_n) - (l + l')| = |(s_n - l) + (t_n - l')|$$

$$\leq |s_n - l| + |t_n - l'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus for any given $\varepsilon > 0$, there exists a positive integer m such that

$$|(s_n + t_n) - (l + l')| < \varepsilon \text{ for all } n \geq m.$$

$\therefore$ The sequence $< s_n + t_n >$ is convergent and

$$\lim (s_n + t_n) = l + l' = \lim s_n + \lim t_n.$$

Note: The converse of the above theorem need not be true.

Let $s_n = (-1)^n$ and $t_n = (-1)^{n+1}$.

Then $s_n + t_n = (-1)^n + (-1)^{n+1} = (-1)^n [1 + (-1)] = 0$.

Hence the sequence $< s_n + t_n >$ converges to 0, while $< s_n >$ and $< t_n >$ oscillate finitely.

Theorem 2: *If* $\lim s_n = l$ and $c \in \mathbf{R}$, *then* $\lim (cs_n) = cl$.

Proof: If $c = 0$, the theorem is obvious because then $\lim (cs_n) = 0 = 0 \,.\, l$.

Let $c \neq 0$. Take any given $\varepsilon > 0$.

Since $\lim s_n = l$, hence for a given positive real number $\varepsilon / |c|$, there exists $m \in \mathbf{N}$ such that

$$|s_n - l| < \frac{\varepsilon}{|c|} \text{ for all } n \geq m.$$

Now for all $n \geq m$, we have

$$|cs_n - cl| = |c\,(s_n - l)| = |c|\,|s_n - l| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$

$\therefore$ $\lim\,(cs_n) = cl.$

Theorem 3: *If* $\lim s_n = l$ *and* $\lim t_n = l'$, *then* $\lim\,(s_n - t_n) = l - l'$.

Proof: By theorem 2,

$$\lim\,(-t_n) = \lim\,[(-1)\,t_n] = (-1)\,l' = -l'.$$

We have $\lim\,(s_n - t_n) = \lim\,[s_n + (-t_n)] = \lim s_n + \lim\,(-t_n)$, [by theorem 1]

$$= l - l'.$$

Corollary: *If* $< s_n >$ *and* $< t_n >$ *are convergent sequences such that* $s_n \le t_n$ *for all* $n \in \mathbf{N}$ *and lim* $s_n = l$, *lim* $t_n = l'$, *then* $l \le l'$.

Proof: By theorem 3, we have $\lim\,(t_n - s_n) = l' - l$. By hypothesis $t_n - s_n \ge 0$ for all $n \in \mathbf{N}$. Hence $l' - l \ge 0$, by theorem 1 of article 6. Thus $l' \ge l$ *i.e.*, $l \le l'$.

Theorem 4: *If* $\lim s_n = 0$ *and the sequence* $< t_n >$ *is bounded then* $\lim\,(s_n\, t_n) = 0$.

Proof: The sequence $< t_n >$ is bounded $\Rightarrow$ there exists $k \in \mathbf{R}^+$ such that

$$|t_n| < k \;\; \forall \;\; n \in \mathbf{N}.$$

Take any given $\varepsilon > 0$.

Since $\lim s_n = 0$, therefore for a given positive real number ε / k there exists $m \in \mathbf{N}$ such that $|s_n - 0| = |s_n| < \dfrac{\varepsilon}{k}\; \forall\, n \ge m$.

Now for all $n \ge m$, we have

$$|s_n\, t_n - 0| = |s_n\, t_n| = |s_n|\,|t_n| < \frac{\varepsilon}{k} \cdot k = \varepsilon.$$

Thus for any given $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that $|s_n\, t_n - 0| < \varepsilon$ for all $n \ge m$.

Hence $\lim\,(s_n\, t_n) = 0.$

Theorem 5: *If* $\lim s_n = l$ *and* $\lim t_n = l'$, *then* $\lim\,(s_n t_n) = ll'$.

Proof: Let $\varepsilon > 0$ be given. We have

$$|s_n\, t_n - ll'| = |s_n t_n - l t_n + l t_n - ll'| = |t_n\,(s_n - l) + l\,(t_n - l')|$$
$$\le |t_n\,(s_n - l)| + |l\,(t_n - l')|$$
$$= |t_n|\,|s_n - l| + |l|\,|t_n - l'|. \qquad \text{...(1)}$$

Since the sequence $< t_n >$ is convergent, therefore it is bounded *i.e.* there exists a positive real number k such that

$$|t_n| \le k \text{ for all } n \in \mathbf{N}.$$

Since the sequences $< s_n >$ and $< t_n >$ are convergent, therefore there exist positive integers m_1 and m_2, such that

$$|s_n - l| < \frac{\varepsilon}{2\,k} \text{ for all } n \ge m_1 \qquad \text{...(2)}$$

and

$$|t_n - l'| < \frac{\varepsilon}{2\,(|l| + 1)} \text{ for all } n \ge m_2\,. \qquad \text{...(3)}$$

Let $m = \max.\{m_1, m_2\}$. From (1), (2) and (3), we have for all $n \ge m$

$$|s_n t_n - ll'| < k \cdot \frac{\varepsilon}{2k} + |l| \frac{\varepsilon}{2(|l|+1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\lim (s_n t_n) = l\, l'$.

Note 1: In the inequality (3) we have taken $(|l|+1)$. Had we not done so, this inequality would have failed in case $l = 0$. Hence to include this case also we used this device.

Note 2: The converse of the above theorem need not be true.

Let $s_n = (-1)^n$ and $t_n = (-1)^{n+1}$.

Then $s_n t_n = (-1)^n (-1)^{n+1} = (-1)^{2n+1} = -1$.

Thus $< s_n t_n >$ is a constant sequence and converges to -1 while $< s_n >$ and $< t_n >$ oscillate finitely.

Theorem 6: *If* $\lim s_n = l$ and $l \neq 0$, *then there exists a positive number k and a positive integer m, such that* $|s_n| > k$ for all $n \geq m$.

Proof: Let us choose $\varepsilon = \frac{1}{2}|l|$. Then $\varepsilon > 0$, since $l \neq 0$. Since $\lim s_n = l$, therefore there must exist a positive integer m, such that $|s_n - l| < \varepsilon$ for all $n \geq m$.

We can write $l = l - s_n + s_n$.

$$\therefore \quad |l| = |(l - s_n) + s_n|$$
$$\leq |l - s_n| + |s_n| < \varepsilon + |s_n| \text{ for all } n \geq m.$$
$$\therefore \quad |s_n| > |l| - \varepsilon \text{ for all } n \geq m$$
or
$$|s_n| > |l| - \frac{1}{2}|l| = \frac{1}{2}|l| \text{ for all } n \geq m.$$

Thus we have found a positive number $k = \frac{1}{2}|l|$ and a positive integer m, such that $|s_n| > k$ for all $n \geq m$.

Theorem 7: *If* $\lim t_n = l', l' \neq 0$ and $t_n \neq 0 \ \forall\ n$, *then* $\lim (1/t_n) = 1/l'$.

Proof: We have
$$\left| \frac{1}{t_n} - \frac{1}{l'} \right| = \frac{|l' - t_n|}{|t_n| . |l'|}. \quad \text{...(1)}$$

Since $l' \neq 0$, therefore by theorem 6, there exists a positive number k and a positive integer m_1, such that

$$|t_n| > k \text{ or } \frac{1}{|t_n|} < \frac{1}{k} \text{ for all } n \geq m_1. \quad \text{...(2)}$$

Take any given $\varepsilon > 0$.

Since $\lim t_n = l'$, therefore for a given positive real number $k|l'|\varepsilon$, there exists $m_2 \in \mathbf{N}$ such that

$$|t_n - l'| < k|l'|\varepsilon \text{ for all } n \geq m_2. \quad \text{...(3)}$$

Let $m = \max. \{m_1, m_2\}$. From (1), (2) and (3), we have

$$\left| \frac{1}{t_n} - \frac{1}{l'} \right| < \frac{1}{|l'|} \cdot \frac{1}{k} \cdot k|l'|\varepsilon \text{ for all } n \geq m$$

i.e., $\left|\dfrac{1}{t_n} - \dfrac{1}{l'}\right| < \varepsilon$ for all $n \geq m$.

Hence $\lim (1/t_n) = 1/l'$.

Theorem 8: *If $\lim s_n = l$ and $\lim t_n = l'$ $(\neq 0)$, $t_n \neq 0$ for all n, then*

$$\lim (s_n / t_n) = l / l'.$$

Proof: Since $t_n \neq 0 \ \forall \ n$ and $l' \neq 0$, therefore, by theorem 7, $\lim (1/t_n) = 1/l'$.

Now $\lim\left(\dfrac{s_n}{t_n}\right) = \lim\left(s_n \cdot \dfrac{1}{t_n}\right) = (\lim s_n) \lim\left(\dfrac{1}{t_n}\right)$ [by theorem 5, article 9]

$$= l \cdot \frac{1}{l'} = \frac{l}{l'} \cdot$$

Hence $\lim \dfrac{s_n}{t_n} = \dfrac{l}{l'} = \dfrac{\lim s_n}{\lim t_n}$.

Theorem 9: Squeeze Theorem (Sandwich Theorem): *If $< s_n >, < t_n >$ and $< u_n >$ are three sequences such that*

(i) for some positive integer k, $s_n \leq u_n \leq t_n$ for $n \geq k$,

(ii) $\lim s_n = \lim t_n = l$, *then* $\lim u_n = l$.

Proof: Let $\varepsilon > 0$ be given.

Since $\lim s_n = l$, therefore, there exists $m_1 \in \mathbf{N}$ such that

$$|s_n - l| < \varepsilon \text{ for all } n \geq m_1$$

i.e., $l - \varepsilon < s_n < l + \varepsilon$ for all $n \geq m_1$.

Similarly $\lim t_n = l$, therefore, there exists $m_2 \in \mathbf{N}$ such that

$$|t_n - l| < \varepsilon \text{ for all } n \geq m_2$$

i.e., $l - \varepsilon < t_n < l + \varepsilon$ for all $n \geq m_2$.

Let $m = \max.\{m_1, m_2, k\}$. Then, for $n \geq m$, we have

$$l - \varepsilon < s_n \leq u_n \leq t_n < l + \varepsilon$$

or $l - \varepsilon < u_n < l + \varepsilon$.

Thus $|u_n - l| < \varepsilon$ for all $n \geq m$.

Hence $\lim u_n = l$.

Corollary: *If $< s_n >$ and $< t_n >$ are two sequences such that $|s_n| \leq |t_n| \ \forall \ n \geq k$ where $k \in \mathbf{N}$ and $\lim t_n = 0$, then $\lim s_n = 0$.*

Proof: Lim $t_n = 0 \Rightarrow \lim |t_n| = 0$ and $\lim (-|t_n|) = 0$.

We have $|s_n| \leq |t_n| \ \forall \ n \geq k$

$\Rightarrow$ $-|t_n| \leq s_n \leq |t_n| \ \forall \ n \geq k$.

Hence by Sandwich theorem, $\lim s_n = 0$.

Note: If $|s_n| \leq \alpha |t_n| \ \forall \ n \geq k$ where $k \in \mathbf{N}$ and α is a positive real number, then

$$\lim t_n = 0 \Rightarrow \lim s_n = 0.$$

For example, let $s_n = \dfrac{\cos n\pi}{n} \cdot$

Then $\quad |s_n| = \left|\dfrac{\cos n\pi}{n}\right| \le \dfrac{1}{n}\cdot$ $\quad [\because \ -1 \le \cos n\pi \le 1]$

$\therefore \quad |s_n| \le \left|\dfrac{1}{n}\right|$ and $\lim \dfrac{1}{n} = 0.$

Hence $\quad \lim s_n = \lim \dfrac{\cos n\pi}{n} = 0.$

Theorem 10: (Cauchy's first theorem on limits)

If $\lim\limits_{n \to \infty} s_n = l,$ *then* $\lim\limits_{n \to \infty} \dfrac{s_1 + s_2 + \ldots + s_n}{n} = l.$ **(Gorakhpur 2015)**

Proof: Define a sequence $< t_n >$ such that

$$s_n = l + t_n \ \forall \ n \in \mathbf{N}.$$

$\therefore \quad \lim t_n = 0$

and
$$\frac{s_1 + s_2 + \ldots + s_n}{n} = l + \frac{t_1 + t_2 + \ldots + t_n}{n}\cdot \quad \ldots(1)$$

In order to prove the theorem we wish to show that

$$\lim_{n \to \infty} \frac{t_1 + t_2 + \ldots + t_n}{n} = 0.$$

Let $\varepsilon > 0$ be given. Since $\lim t_n = 0$, therefore, there exists a positive integer m, such that

$$|t_n - 0| = |t_n| < \varepsilon / 2 \ \ \forall \ n \ge m. \quad \ldots(2)$$

Also, since every convergent sequence is bounded, hence there exists a real number $k > 0$ such that

$$|t_n| \le k \ \forall \ n \in \mathbf{N}. \quad \ldots(3)$$

Now for all $n \ge m$, we have

$$\left|\frac{t_1 + t_2 + \ldots + t_n}{n}\right| = \left|\frac{t_1 + t_2 + \ldots + t_m}{n} + \frac{t_{m+1} + t_{m+2} + \ldots + t_n}{n}\right|$$

$$\le \frac{|t_1| + |t_2| + \ldots + |t_m|}{n} + \frac{|t_{m+1}| + |t_{m+2}| + \ldots + |t_n|}{n}$$

$$\le \frac{mk}{n} + \frac{n-m}{n}\cdot\frac{\varepsilon}{2} \quad \text{[From (2) and (3)]}$$

$$< \frac{mk}{n} + \frac{\varepsilon}{2}\cdot \quad \ldots(4)$$

$$\left[\because \ 0 \le \frac{n-m}{n} < 1\right]$$

If m is fixed, then $\dfrac{mk}{n} < \dfrac{1}{2}\varepsilon$ if $n > \dfrac{2mk}{\varepsilon}\cdot$

Let us choose a positive integer $p > \dfrac{2mk}{\varepsilon}\cdot$ Then

$$\frac{mk}{n} < \frac{1}{2}\varepsilon \text{ for } n \ge p. \quad \ldots(5)$$

Let $M = \max. \{m, p\}$. From (4) and (5), we have

$$\left|\frac{t_1 + t_2 + \ldots + t_n}{n}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \;\forall\; n \geq M.$$

Thus $\lim \dfrac{t_1 + t_2 + \ldots + t_n}{n} = 0$ and consequently (1) gives

$$\lim \frac{s_1 + s_2 + \ldots + s_n}{n} = l.$$

Note: The converse of the above theorem need not be true.

Consider the sequence $< s_n >$ where $s_n = (-1)^n$.

For this sequence,

$$\frac{s_1 + s_2 + \ldots + s_n}{n} = 0, \text{ if } n \text{ is even, and } \quad = -\frac{1}{n}, \text{ if } n \text{ is odd.}$$

$\therefore \quad \lim \dfrac{s_1 + s_2 + \ldots + s_n}{n} = 0$, but $< s_n >$ is not convergent.

Theorem 11: (Cauchy's second theorem on limits). *If $< s_n >$ is a sequence such that $s_n > 0$ for all n and $\lim s_n = l$, then* $\lim (s_1\, s_2 \ldots s_n)^{1/n} = l$.

Proof: Let us define a sequence $< t_n >$ such that

$$t_n = \log s_n \text{ for all } n.$$

Since $\lim s_n = l$, therefore $\lim t_n = \log l$.

By Cauchy's first theorem on limits, we have

$$\lim \frac{t_1 + t_2 + \ldots + t_n}{n} = \log l$$

i.e., $$\lim \frac{\log s_1 + \log s_2 + \ldots + \log s_n}{n} = \log l$$

i.e., $$\lim \log (s_1\, s_2 \ldots s_n)^{1/n} = \log l$$

and hence, $\lim (s_1\, s_2 \ldots s_n)^{1/n} = l$.

Note: While proving this theorem, we have used the following fact :

$$\lim s_n = l \Leftrightarrow \lim \log s_n = \log l,$$

provided $s_n > 0$ for all n and $l > 0$.

Theorem 12: *If $< s_n >$ is a sequence such that $s_n > 0$ for all $n \in \mathbf{N}$ and $\lim \dfrac{s_{n+1}}{s_n} = l$, then* $\lim \sqrt[n]{s_n} = l$.

Proof: Let us define a sequence $< t_n >$ such that

$$t_1 = s_1,\; t_2 = \frac{s_2}{s_1},\; t_3 = \frac{s_3}{s_2}, \ldots,\; t_n = \frac{s_n}{s_{n-1}}, \ldots.$$

$$\therefore \quad t_1\, t_2 \ldots t_n = s_n\,.$$

Also $\lim \dfrac{s_{n+1}}{s_n} = l \Rightarrow \lim \dfrac{s_n}{s_{n-1}} = l \Rightarrow \lim t_n = l$.

Since $s_n > 0$ for all n, hence $t_n > 0$ for all n.

Thus we have a sequence $< t_n >$ such that $t_n > 0$ for all n and $\lim t_n = l$.

Hence by theorem 11 of article 9 , we have $\lim (t_1 \, t_2 \dots t_n)^{1/n} = l$

i.e., $\quad \lim (s_n)^{1/n} = l.$

Theorem 13: (**Cesaro's theorem**)

If $\lim s_n = l$ *and* $\lim t_n = l'$, *then*

$$\lim \frac{s_1 t_n + s_2 t_{n-1} + \dots + s_n \, t_1}{n} = ll'.$$

Proof: Let $s_n = l + x_n$ and $|x_n| = X_n$. Then $\lim x_n = 0$ and hence $\lim X_n = 0$. Therefore by theorem 10 of article 9, we have

$$\lim \frac{1}{n} (X_1 + X_2 + \dots + X_n) = 0.$$

Now

$$\frac{1}{n} (s_1 \, t_n + s_2 \, t_{n-1} + \dots + s_n \, t_1)$$

$$= \frac{l}{n} (t_1 + t_2 + \dots + t_n) + \frac{1}{n} (x_1 \, t_n + x_2 \, t_{n-1} + \dots + x_n \, t_1), \qquad \dots(1)$$

on substituting for $s_1, s_2, \dots, s_n$.

Now the sequence $< t_n >$ is convergent and every convergent sequence is bounded. Therefore there exists a positive real number k, such that

$$|t_n| < k \; \forall \; n.$$

$\therefore$

$$0 \le \left| \frac{1}{n} (x_1 \, t_n + x_2 \, t_{n-1} + \dots + x_n \, t_1) \right|$$

$$\le \frac{1}{n} [|x_1| . |t_n| + |x_2| . |t_{n-1}| + \dots + |x_n| . | \, t_1|]$$

$$< \frac{k}{n} (|x_1| + |x_2| + \dots + |x_n|)$$

$$= \frac{k}{n} (X_1 + X_2 + \dots + X_n)$$

$\to 0$, since k is fixed for all n.

$\therefore$

$$\lim \frac{1}{n} (x_1 t_n + x_2 t_{n-1} + \dots + x_n \, t_1) = 0, \text{ by Sandwich theorem.}$$

Now since $\lim t_n = l'$, we have by theorem 10 of article 9,

$$\lim \frac{1}{n} \left(t_1 + t_2 + \dots + t_n \right) = l'.$$

Hence finally, we get from (1)

$$\lim \frac{1}{n} \left(s_1 t_n + s_2 \, t_{n-1} + \dots + s_n \, t_1 \right) = ll'.$$

Note: Theorems of article 9 provide an easier method for evaluating the limits of sequences than the method for evaluating these limits directly by definition. Later on we shall illustrate the use of the theorems of this section to evaluate the limits of sequences.

10 Monotonic Sequences

Definition 1: *A sequence $< s_n >$ is said to be monotonically increasing (or non-decreasing), if $s_n \le s_{n+1}$ for all n i.e., $s_n \le s_m$ for $n < m$.*

Definition 2: *A sequence $< s_n >$ is said to be strictly increasing if $s_n < s_{n+1}$ for all $n \in \mathbf{N}$.*

Definition 3: *A sequence $< s_n >$ is said to be monotonically decreasing (or non-increasing) if $s_n \ge s_{n+1}$ for all n i.e., $s_n \ge s_m$ for $n < m$.*

Definition 4: *A sequence $< s_n >$ is said to be strictly decreasing if $s_n > s_{n+1}$ for all $n \in \mathbf{N}$.*

Definition 5: *A sequence $< s_n >$ is said to be monotonic if it is either monotonically increasing or monotonically decreasing.*

Illustrations:

1. The sequence $< 1, 2, 3, \ldots, n, \ldots >$ is strictly increasing.
2. $< 2, 2, 4, 4, 6, 6, \ldots >$ is monotonically increasing.
3. $< -\frac{1}{n} >$ is strictly increasing.
4. $< 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots >$ is strictly decreasing.
5. $< 1, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \ldots >$ is monotonically decreasing.
6. $< -2, -4, -6, -8, \ldots >$ is strictly decreasing.
7. $< 0, 1, 0, 1, 0, 1, \ldots >$ is not monotonic.
8. $< -2, 2, -4, 4, -6, 6, \ldots >$ is not monotonic.
9. $< 1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \ldots >$ is not monotonic.

Note: If $< s_n >$ is a sequence of positive terms, then $< s_n >$ is increasing $\Leftrightarrow < 1/s_n >$ is decreasing.

Theorem 1: (Monotone convergence theorem). *Every bounded monotonically increasing sequence converges.* **(Gorakhpur 2011, 13)**

Proof: Let $< s_n >$ be a bounded monotonically increasing sequence.

Let $S = \{ s_n : n \in \mathbf{N} \}$ be the range of the sequence $< s_n >$. Then S is a non-empty set which is bounded above. Hence by the completeness axiom for $\mathbf{R}$, there exists a number $l = \sup S$. We shall show that $< s_n >$ converges to l.

Let $\varepsilon > 0$ be given. Then $l - \varepsilon < l$, so that $l - \varepsilon$ is not an upper bound of S. Hence there exists a positive integer m such that $s_m > l - \varepsilon$. Since $< s_n >$ is monotonically increasing, therefore,

$$s_n \ge s_m > l - \varepsilon \text{ for all } n \ge m. \qquad \ldots(1)$$

Also, since l is the supremum of S, therefore,

$$s_n \le l < l + \varepsilon \text{ for all } n. \qquad \ldots(2)$$

From (1) and (2), we get $l - \varepsilon < s_n < l + \varepsilon$ for all $n \ge m$

i.e., $\quad |s_n - l| < \varepsilon$ for all $n \ge m$.

$\therefore \qquad \lim s_n = l.$

Corollary 1: *Every bounded monotonically decreasing sequence converges.*

Proof: Let $< s_n >$ be a bounded monotonically decreasing sequence. Define a sequence $< t_n >$ such that $t_n = - s_n$, for all $n \in \mathbf{N}$. Then $< t_n >$ is a bounded monotonically increasing sequence and hence by the above theorem, it converges. If $\lim t_n = l$, then $\lim s_n = \lim (- t_n) = - \lim t_n = - l$.

Note: We can prove this result independently by taking infimum of the set S.

Corollary 2: *Every bounded monotonic sequence converges.* **(Kanpur 2012)**

Proof: In order to prove this theorem we are to prove the following two results.

(i) Every bounded monotonically increasing sequence is convergent. (Give complete proof of theorem 1)

(ii) Every bounded monotonically decreasing sequence is convergent. (Deduce it from the result of theorem 1 as deduced in corollary 1)

Theorem 2: *A non-decreasing (i.e., monotonically increasing) sequence which is not bounded above diverges to infinity.*

Proof: Let $< s_n >$ be a non-decreasing sequence which is not bounded above.

Take any real number $k > 0$.

Now $< s_n >$ is not bounded above $\Rightarrow$ there exists $m \in \mathbf{N}$ such that $s_m > k$.

Also $< s_n >$ is non-decreasing $\Rightarrow s_n \geq s_m$ for $n > m$.

$\therefore \qquad s_n \geq s_m > k$ for $n > m$ or $s_n > k$ for $n > m$.

Hence $< s_n >$ diverges to infinity.

Theorem 3: *A non-increasing (i.e., monotonically decreasing) sequence which is not bounded below diverges to minus infinity.*

Proof: Let $< s_n >$ be a non-increasing sequence which is not bounded below.

Take any real number $k < 0$.

Now $< s_n >$ is not bounded below $\Rightarrow$ there exists $m \in \mathbf{N}$ such that $s_m < k$.

Also $< s_n >$ is non-increasing $\Rightarrow s_n \leq s_m$ for $n > m$.

$\therefore \qquad s_n \leq s_m < k$ for $n > m$ or $s_n < k$ for $n > m$.

Hence $< s_n >$ diverges to $- \infty$.

Theorem 4: *Every sequence has a monotonic subsequence.*

Proof: Consider the sequence $a_0 = < s_n >$. Let $a_1, a_2, a_3, \ldots$ denote the subsequences

$$< s_2, s_3, s_4, \ldots >, < s_3, s_4, s_5, \ldots >, < s_4, s_5, s_6, \ldots >, \ldots \text{ respectively.}$$

There arise two different cases.

(i) **Each of the sequences $a_0, a_1, a_2, \ldots$ has a greatest term.** Let $s_{n_1}, s_{n_2}, s_{n_3}, \ldots$ denote the greatest terms of $a_0, a_1, a_2, \ldots$ respectively.

Then $n_1 \leq n_2 \leq n_3 \leq \ldots$ and $s_{n_1} \geq s_{n_2} \geq s_{n_3} \geq \ldots$.

Consequently $< s_{n_1}, s_{n_2}, s_{n_3}, \ldots >$ is a monotonically decreasing subsequence of $< s_n >$.

(ii) **At least one of the sequences $a_0, a_1, a_2, \ldots$ has no greatest term.** Suppose a_m has no greatest term. Then each term of a_m is ultimately followed by some term of a_m that exceeds it. For, if there is a term of a_m which exceeds all the terms following it, then it can be exceeded by finitely many terms at the most and hence, a_m must have a greatest term. Now s_{m+1} is the first term of a_m. Let s_{n_2} be the first term of a_m exceeding s_{m+1}, s_{n_3} the first term of a_m that follows s_{n_2} and exceeds it, s_{n_4} the first term of a_m that follows s_{n_3} and exceeds it, and so on.

Thus $< s_{m+1}, s_{n_2}, s_{n_3}, s_{n_4}, \ldots >$ is a monotonically increasing subsequence of $< s_n >$.

Note: In the above proof, we have used the concept of a greatest term of a sequence.

A term s_k of a sequence $< s_n >$ is said to be a greatest term of $< s_n >$ if $s_n \le s_k$ for all n.

It is not necessary for a sequence to have a greatest term; *e.g.*, the sequence $< 1, 3, 5, \ldots >$ has no greatest term. Also, it is not necessary that a greatest term of a sequence be unique; *e.g.*, for the sequence $< s_n >$ defined by $s_n = (-1)^{n-1}$, each of the terms $s_1, s_3, s_5, \ldots$ is a greatest term.

If $< s_{n_k} >$ be a subsequence of $< s_n >$, s_{n_p} be a greatest term of $< s_{n_k} >$ and s_m be a greatest term of $< s_n >$, then $s_{n_p} \le s_m$, because s_{n_p} is also a term of $< s_n >$, and s_m is a greatest term of $< s_n >$.

11 Limit Points of a Sequence

Definition: *A real number p is said to be a limit point (or a cluster point) of a sequence $< s_n >$ if every neighbourhood of p contains infinite number of terms of the sequence.*

(Gorakhpur 2012)

Since every open interval $]\, p - \varepsilon, p + \varepsilon\, [$, $\varepsilon > 0$, is a neighbourhood of p and also every neighbourhood of p contains an open interval $]\, p - \varepsilon, p + \varepsilon\, [$ for some $\varepsilon > 0$, therefore we can say that a real number p is a limit point of a sequence $< s_n >$ iff given any $\varepsilon > 0$, $s_n \in \,]\, p - \varepsilon, p + \varepsilon\, [$ for infinitely many values of n *i.e.*, $|s_n - p| < \varepsilon$ for infinitely many values of n.

It can be easily seen that a real number p is a limit point of a sequence $< s_n >$ iff given any neighbourhood N of p and $m \in \mathbf{N}$ we can find $k \in \mathbf{N}$ such that $k > m$ and $s_k \in N$.

Remarks: 1. Limit point of a sequence is different from the limit of a sequence. The limit of a sequence is a limit point of the sequence, while a limit point of a sequence need not be the limit of the sequence.

2. Limit point of a sequence need not be a term of the sequence.

3. If $s_n = l$ for infinitely many values of n then l is a limit point of $< s_n >$.

4. A real number p is not a limit point of $< s_n >$ if there exists even one neighbourhood of p containing finite number of terms of the sequence.

Illustrations:

1. The sequence $< (-1)^n >$ has 1 and -1 as limit points. Here $s_n = -1$, if n is odd and $s_n = 1$, if n is even. Any neighbourhood of -1 will contain all the odd terms of the sequence hence -1 is a limit point.

Similarly any neighbourhood of 1 will contain all the even terms of the sequence, so 1 is a limit point.

2. The sequence $< \dfrac{1}{n} >$ has only one limit point, namely 0.

Given $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that $\frac{1}{m} < \varepsilon$.

For $\quad n \geq m, 0 < \frac{1}{n} \leq \frac{1}{m} < \varepsilon \quad$ *i.e.,* $\; -\varepsilon < 0 < \frac{1}{n} < \varepsilon$ for all $n \geq m$.

Hence $\frac{1}{n} \in \,]-\varepsilon, \varepsilon[$ for all $n \geq m$. Thus every ε-nhd of 0 contains infinitely many points of the sequence. Hence 0 is a limit point of this sequence.

3. The sequence $< 1, 2, 3, \ldots, n, \ldots >$ has no limit point.

Let $p \in \mathbf{R}$. Whatever ε we take, the neighbourhood $]\, p - \varepsilon, p + \varepsilon\, [$ of p contains at the most a finite number of terms of this sequence. Hence p is not a limit point of this sequence.

Theorem 1: *If l is a limit point of the range of a sequence $< s_n >$, then l is a limit point of the sequence $< s_n >$.*

Proof: Let S be the range set of the sequence $< s_n >$. Since l is a limit point of S, therefore, every nhd. of l contains infinite number of distinct elements of the set S. But each element of the set S is a term of the sequence $< s_n >$. Hence every nhd of l contains infinite number of terms of the sequence $< s_n >$. Thus l is a limit point of the sequence $< s_n >$.

Note 1: The converse of the above theorem need not be true.

Consider the sequence $< s_n >$ where $s_n = 1 + (-1)^n$.

We have $s_n = 0$, if n is odd and $s_n = 2$, if n is even.

$\therefore \quad$ 0 , 2 are limit points of the sequence $< s_n >$. But the range of this sequence is the set $\{0, 2\}$, which is a finite set.

Now a finite set has no limit points, hence the range of $< s_n >$ has no limit points.

Note 2: If a sequence has all its terms distinct, then the limit points of the sequence and the limit points of the range set are same.

Theorem 2: *If $s_n \to l$, then l is the only limit point of $< s_n >$.*

Proof: First we shall show that l is a limit point of $< s_n >$. Let $\varepsilon > 0$ be given. Since $s_n \to l$, therefore, there exists a positive integer m such that

$$|s_n - l| < \varepsilon \text{ for all } n \geq m$$

i.e.,

$$|s_n - l| < \varepsilon \text{ for infinitely many values of } n.$$

This shows that l is a limit point of $< s_n >$.

Now we shall show that if l' be any limit point of $< s_n >$, then we must have $l' = l$.

Let $\varepsilon > 0$ be arbitrary. Since l is the limit of $< s_n >$, therefore, there exists a positive integer p such that

$$|s_n - l| < \varepsilon / 2 \text{ for all } n \geq p. \qquad \ldots(1)$$

Since l' is a limit point of $< s_n >$, therefore, there must exist a positive integer $q > p$ such that

$$|s_q - l'| < \varepsilon / 2. \qquad \ldots(2)$$

Putting $n = q$ in (1), $|s_q - l| < \varepsilon / 2$. $\qquad \ldots(3)$

Now

$$\begin{aligned} |l - l'| &= |(s_q - l') - (s_q - l)| \leq |s_q - l'| + |s_q - l| \\ &< \varepsilon / 2 + \varepsilon / 2, \text{ from (2) and (3)} \end{aligned}$$

i.e., $|l - l'| < \varepsilon$.

Since ε is arbitrary, hence we must have $|l - l'| = 0$ *i.e.,* $l = l'$.

Note: The converse of the above theorem need not be true; *i.e.,* a sequence having only one limit point may not converge.

Consider the sequence $< s_n >$, where s_n is given by

$$s_n = \begin{cases} \dfrac{1}{n}, & \text{if } n \text{ is even;} \\ n, & \text{if } n \text{ is odd.} \end{cases}$$

There is only one limit point of $< s_n >$, namely 0, and yet the sequence does not converge.

Theorem 3: (Bolzano-Weierstrass Theorem for sequences): *Every bounded sequence has at least one limit point.*

Proof: Let $< s_n >$ be a bounded sequence and S be its range set. Then $S = \{s_n : n \in \mathbf{N}\}$. Since the sequence $< s_n >$ is bounded, therefore, S is a bounded set.

Case I: Let S be a finite set. Then for infinitely many indices n, $s_n = p$, where p is some real number. Obviously p is a limit point of $< s_n >$.

Case II: Let S be an infinite set. Since S is bounded, by Bolzano-Weierstrass theorem for sets of real numbers, S has a limit point, say p. Hence every nbd of p contains infinitely many distinct points of S or every nbd of p contains infinitely many terms of the sequence $< s_n >$ and consequently p is a limit point of the sequence $< s_n >$.

Corollary 1: *If F is a closed and bounded set of real numbers, then every sequence in F has a limit point in F.*

Proof: Let $< s_n >$ be a sequence in F. Then $< s_n >$ is bounded and hence, it has a limit point, say p. Also $p \in F$, since p cannot belong to $\mathbf{R} - F$. If $p \in \mathbf{R} - F$, then $\mathbf{R} - F$ is an open set containing p. Thus it is a neighbourhood of p that contains no term of the sequence $< s_n >$ and hence we get a contradiction to the fact that p is a limit point of the sequence $< s_n >$.

Corollary 2: *If I is a closed interval, then every sequence in I has a limit point in I.*

Proof: Since every closed interval is a closed and bounded set, hence the result follows from corollary 1.

We have proved earlier that every convergent sequence is bounded and also it has only one limit point. Now we shall prove the converse.

Theorem 4: *If a sequence $< s_n >$ is bounded and has only one limit point, say l, then $s_n \to l$.*

Proof: Since $< s_n >$ is bounded, so it has at least one limit point. But l is the only limit point of $< s_n >$. Hence, for any $\varepsilon > 0$, $]\, l - \varepsilon, l + \varepsilon\, [$ contains s_n for all except a finite number of values of n.

Let $s_{m_1}, s_{m_2}, s_{m_3}, \ldots, s_{m_p}$ be the finite number of terms of the sequence $< s_n >$ that lie outside $]\, l - \varepsilon, l + \varepsilon[$.

If $m - 1 = \max.\{m_1, m_2, \ldots, m_p\}$, then $s_n \in \,]\, l - \varepsilon, l + \varepsilon\, [\ \forall\, n \geq m$.

Hence for any $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$|s_n - l| < \varepsilon \quad \forall\, n \geq m.$$

$\therefore$ the sequence $< s_n >$ converges to l.

Theorem 5: *A real number p is a limit point of a sequence* $< s_n >$ *iff there exists a subsequence of* $< s_n >$ *converging to p.*

Proof: First, let p be a limit point of a sequence $< s_n >$.

We shall use the result that a real number p is a limit point of a sequence $< s_n >$ if given any $\varepsilon > 0$ and any positive integer m, there exists a positive integer $k > m$ such that $s_k \in]\, p - \varepsilon, p + \varepsilon\, [$.

By choosing $\varepsilon = 1$ and $m = 1$, there must exist a positive integer $n_1 > 1$, such that

$$|s_{n_1} - p| < 1. \quad \ldots(1)$$

Choosing $\varepsilon = \frac{1}{2}$, $m = n_1$, there must exist a positive integer $n_2 > n_1$, such that

$$|s_{n_2} - p| < \frac{1}{2}. \quad \ldots(2)$$

Continuing in this way, we can inductively define a subsequence $< s_{n_1}, s_{n_2}, \ldots, s_{n_k}, \ldots >$ such that $|s_{n_k} - p| < \frac{1}{k}$.

In fact, if we assume that $s_{n_1}, s_{n_2}, \ldots, s_{n_k}$ have been obtained, by choosing

$$\varepsilon = \frac{1}{k+1}, m = n_k,$$

we can get a positive integer $n_{k+1} > n_k$ such that

$$|s_{n_{k+1}} - p| < \frac{1}{k+1}.$$

But s_{n_1} has already been obtained. Thus the construction of $< s_{n_k} >$ is complete by induction.

We now claim that the sequence $< s_{n_k} > \to p$. In fact, for any $\varepsilon > 0$, we can choose a positive integer j, such that $1/j < \varepsilon$. For this choice of j, we get $| s_{n_k} - p| < 1/j < \varepsilon \, \forall k \geq j$. This shows that the sequence $< s_{n_k} > \to p$.

Conversely, let $< s_{n_k} >$ be a subsequence of $< s_n >$ converging to p. We have to show that p is a limit point of $< s_n >$.

Since $s_{n_k} \to p$, therefore, given any $\varepsilon > 0$ there exists positive integer j such that

$$|s_{n_k} - p| < \varepsilon \text{ for all } k \geq j.$$

Thus every neighbourhood of p contains infinitely many terms of $< s_{n_k} >$ *i.e.,* infinitely many terms of $< s_n >$ and consequently p is a limit point of $< s_n >$.

Theorem 6: *The set of limit points of a bounded sequence is bounded.*

Proof: Let $< s_n >$ be a bounded sequence. Then there exist $k_1, k_2 \in \mathbf{R}$ such that

$$k_1 \leq s_n \leq k_2 \text{ for all } n \in \mathbf{N}$$

i.e., $s_n \notin]-\infty, k_1[$ and $s_n \notin]\, k_2, \infty\, [$ for any n. Hence if $l \in \mathbf{R}$ and $l \notin]-\infty, k_1[\cup]\, k_2, \infty\, [$, then l is not a limit point of the sequence. Thus if $l \in \mathbf{R}$ is a limit point of the sequence, then $l \in [k_1, k_2]$. Consequently the set of limit points of $< s_n >$ is bounded.

Theorem 7: *Every bounded sequence has the greatest and the least limit points.*

Proof: Let $< s_n >$ be a bounded sequence. Then the set L of limit points of $< s_n >$ is bounded.

Now $L \neq \varnothing$ and L is bounded, hence by completeness axiom the set L has infimum and supremum.

If $\inf L = u$ and $\sup L = v$, then we have to show that $u, v \in L$.

For $\varepsilon > 0$, $]\, v - \varepsilon, v + \varepsilon\,[$ is a neighbourhood of v.

Since $v = \sup L$, therefore, there exists some $x \in L$ such that

$$v - \varepsilon < x \leq v < v + \varepsilon \quad i.e., \quad x \in \,]\, v - \varepsilon, v + \varepsilon\,[$$

or $\quad]\, v - \varepsilon, v + \varepsilon\,[$ is a neighbourhood of x.

Since x is a limit point of $< s_n >$, hence $]\, v - \varepsilon, v + \varepsilon\,[$ contains infinite number of terms of the sequence. It holds for every $\varepsilon > 0$. Thus every neighbourhood of v contains infinite number of terms of the sequence $< s_n >$.

$\therefore \quad v$ is a limit point of the sequence $< s_n >$.

$\therefore \quad v \in L$.

Similarly, we can show that $u \in L$.

12 Cauchy Sequences

Cauchy Convergence Criterion for Sequences:

In this section we shall establish an important criterion, known as *Cauchy Convergence Criterion,* which will help us to decide whether a sequence is convergent or divergent without knowing its limit or limit point. It involves only the elements of the sequence to which we wish to apply it.

Definition: *A sequence $< s_n >$ is said to be a Cauchy sequence if given $\varepsilon > 0$ there exists $m \in \mathbf{N}$ such that*

$$|s_n - s_m| < \varepsilon \text{ for all } n \geq m$$

or $$|s_{n+p} - s_n| < \varepsilon \text{ for all } n \geq m \text{ and every } p \geq 0$$

or $$|s_{m+p} - s_m| < \varepsilon \text{ for all } p \geq 0$$

or $$|s_p - s_q| < \varepsilon \text{ for all } p, q \geq m.$$ **(Gorakhpur 2012)**

Remark: $|s_p - s_q| < \varepsilon$ for all $p, q \geq m$ means that s_p and s_q are arbitrarily close together for large values of p and q.

Illustrations:

1. $< 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots >$ is a Cauchy sequence.

Let the given sequence be $< s_n >$, where $s_n = 1/n$.

Take any given $\varepsilon > 0$.

If $n \geq m$, then

$$|s_n - s_m| = \left| \frac{1}{n} - \frac{1}{m} \right| = \left| \frac{m-n}{nm} \right| = \frac{n-m}{nm} = \frac{n-m}{n} \cdot \frac{1}{m} < \frac{1}{m}.$$

$$\left[\because \; 0 \leq \frac{n-m}{n} < 1 \right]$$

If we take $m \in \mathbf{N}$ such that $m > \frac{1}{\varepsilon}$ *i.e,* $\frac{1}{m} < \varepsilon$, then

$$|s_n - s_m| = \left|\frac{1}{n} - \frac{1}{m}\right| < \varepsilon, \forall \; n \geq m.$$

Hence the given sequence is a Cauchy sequence.

2. The sequence $< n^2 >$ is not a Cauchy sequence.

If $n > m$, then $n^2 - m^2 = (n-m)(n+m) > 2m > 1$, for any value of m. Taking $\varepsilon = 1$, we cannot find a positive integer m such that $|n^2 - m^2| < \varepsilon$ for all $n \geq m$.

Theorem 1: *If $< s_n >$ is a Cauchy sequence, then $< s_n >$ is bounded.*

(Kanpur 2008; Gorakhpur 14)

Proof: Let $< s_n >$ be a Cauchy sequence. For $\varepsilon = 1$, there exists $m \in \mathbf{N}$ such that $|s_n - s_m| < 1$ for all $n \geq m$

i.e., $s_m - 1 < s_n < s_m + 1$ for all $n \geq m$.

Let $k_1 = \text{min.} \{s_1, s_2, \ldots, s_{m-1}, s_m - 1\}$

and $k_2 = \text{max.} \{s_1, s_2, \ldots, s_{m-1}, s_m + 1\}$.

$\therefore$ $k_1 \leq s_n \leq k_2$ for all $n \in \mathbf{N}$.

Hence $< s_n >$ is bounded.

Note: The converse of the above theorem need not be true. The sequence $< (-1)^n >$ is bounded but is not a Cauchy sequence. **(Gorakhpur 2014)**

Theorem 2: (Cauchy Convergence Criterion): *A sequence converges if and only if it is a Cauchy sequence.*

Proof: First, let $< s_n >$ be a convergent sequence which converges to, say, l.

Since $s_n \to l$, therefore, for given $\varepsilon > 0$ there must exist $m \in \mathbf{N}$ such that

$$|s_n - l| < \varepsilon / 2 \; \forall \; n \geq m.$$

In particular, $|s_m - l| < \varepsilon / 2$.

Now
$$|s_n - s_m| = |(s_n - l) - (s_m - l)| \leq |s_n - l| + |s_m - l|$$
$$< \varepsilon / 2 + \varepsilon / 2 \text{ for all } n \geq m.$$

Thus $|s_n - s_m| < \varepsilon \; \forall \; n \geq m$, showing that $< s_n >$ is a Cauchy sequence.

Conversely, let $< s_n >$ be a Cauchy sequence. Then $< s_n >$ is bounded. By Bolzano-Weierstrass theorem, $< s_n >$ has a limit point, say l. We shall show that $s_n \to l$.

Let $\varepsilon > 0$ be given. Since $< s_n >$ is a Cauchy sequence, there exists $m \in \mathbf{N}$ such that

$$|s_n - s_m| < \varepsilon / 3 \; \forall \; n \geq m.$$

Since l is a limit point of $< s_n >$, therefore every nbd of l contains infinite terms of the sequence $< s_n >$. In particular the open interval $]\, l - \frac{1}{3}\varepsilon, l + \frac{1}{3}\varepsilon \,[$ contains infinite terms of $< s_n >$. Hence there exists a positive integer $k > m$ such that

$$l - \frac{1}{3}\varepsilon < s_k < l + \frac{1}{3}\varepsilon \quad \textit{i.e.,} \quad |s_k - l| < \varepsilon / 3.$$

Now
$$|s_n - l| = |(s_n - s_m) + (s_m - s_k) + (s_k - l)|$$
$$\leq |s_n - s_m| + |s_m - s_k| + |s_k - l|$$
$$< \varepsilon / 3 + \varepsilon / 3 + \varepsilon / 3 \text{ for all } n \geq m.$$

Thus $|s_n - l| < \varepsilon$ for all $n \geq m$.

$\therefore$ $< s_n >$ converges to l.

13 Limit Superior and Limit Inferior of a Sequence

Let $< s_n >$ be a sequence which is bounded above. Then, for each fixed $n \in \mathbf{N}$, the set $\{s_n, s_{n+1}, \ldots\}$ is bounded above and hence it must have a supremum. Let

$$\bar{s}_n = \sup\{s_n, s_{n+1}, \ldots\}.$$

Since $\{s_{n+1}, s_{n+2}, \ldots\}$ is a subset of $\{s_n, s_{n+1}, \ldots\}$, therefore, it is obvious that $\bar{s}_n \geq \bar{s}_{n+1}$. Thus the sequence $< \bar{s}_n >$ is a monotonically decreasing sequence and consequently, it either converges or else it diverges to $-\infty$.

Similarly, if the sequence $< s_n >$ is bounded below, then the set $\{s_n, s_{n+1}, \ldots\}$ has an infimum. Let $\underline{s_n} = \inf\{s_n, s_{n+1}, \ldots\}$, then the sequence $< \underline{s_n} >$ is monotonically increasing and hence it either converges or diverges to ∞.

Keeping these notations in mind we now define limit superior and limit inferior.

Definition 1: *Let $< s_n >$ be a sequence of real numbers which is bounded above and let*

$$\bar{s}_n = \sup\{s_n, s_{n+1}, \ldots\}.$$

If $< \bar{s}_n >$ converges we define the **limit superior** *of $< s_n >$ by*

$$\lim_{n \to \infty} \sup s_n = \lim_{n \to \infty} \bar{s}_n$$

If $< \bar{s}_n >$ diverges to $-\infty$, we write $\lim\limits_{n \to \infty} \sup s_n = -\infty$

If a sequence $< s_n >$ is not bounded above, we write

$$\lim_{n \to \infty} \sup s_n = \infty$$

Definition 2: *Let $< s_n >$ be a sequence of real numbers which is bounded below and let*

$$\underline{s_n} = \inf\{s_n, s_{n+1}, \ldots\}.$$

If $< \underline{s_n} >$ converges we define the **limit inferior** *of $< s_n >$ by*

$$\lim_{n \to \infty} \inf s_n = \lim_{n \to \infty} \underline{s_n}.$$

If $< \underline{s_n} >$ diverges to ∞, we write $\lim\limits_{n \to \infty} \inf s_n = \infty$.

If a sequence $< s_n >$ is not bounded below, we write

$$\lim_{n \to \infty} \inf s_n = -\infty.$$

Note 1: The notations $\overline{\lim}\, s_n$ and $\underline{\lim}\, s_n$ are also used for lim sup s_n and lim inf s_n respectively. In future, we shall use these notations.

Note 2: The limit superior and the limit inferior are also called the **upper limit** and the **lower limit** of $< s_n >$ respectively.

Note 3: We have $\overline{\lim}\, s_n = \inf\{\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n, \ldots\}$

and $\underline{\lim}\, s_n = \sup\{\underline{s_1}, \underline{s_2}, \ldots, \underline{s_n}, \ldots\}$.

Illustrations:

1. Let $< s_n >$ be the sequence defined by $s_n = (-1)^n \; \forall \, n \in \mathbf{N}$.

It is bounded above by 1 and bounded below by -1. For this sequence, $\overline{s_n} = 1$ and $\underline{s_n} = -1$ for all $n \in \mathbf{N}$.

Hence $\overline{\lim}\, s_n = 1$ and $\underline{\lim}\, s_n = -1$.

2. Let $< s_n >$ be the sequence defined by $s_n = -n \;\forall\; n \in \mathbf{N}$. It is bounded above by -1 but it is not bounded below.

$$\bar{s}_n = \sup \{-n, -n-1, \ldots\} = -n$$

Since $\bar{s}_n \to -\infty$ as $n \to \infty$, hence $\overline{\lim}\, s_n = -\infty$. Also, since $< s_n >$ is not bounded below, by definition $\underline{\lim}\, s_n = -\infty$. Thus in this sequence both the limit superior and the limit inferior are $-\infty$.

3. Let $< s_n >$ be the sequence defined by $s_n = n \;\forall\; n \in \mathbf{N}$. It is bounded below but not bounded above.

$$\underline{s_n} = \inf \{n, n+1, \ldots\} = n.$$

Since $\underline{s_n} \to \infty$ as $n \to \infty$, hence $\underline{\lim}\, s_n = \infty$.

Also, since $< s_n >$ is not bounded above, by definition $\overline{\lim}\, s_n = \infty$. Thus in this sequence both the limit superior and the limit inferior are ∞.

4. Let $< s_n >$ be the sequence defined by $s_n = (-1)^n \left(1 + \dfrac{1}{n}\right)$.

Then $\quad < s_n > = < -2, \dfrac{3}{2}, -\dfrac{4}{3}, \dfrac{5}{4}, -\dfrac{6}{5}, \dfrac{7}{6}, \ldots >.$

In this case $\quad \bar{s}_1 = \dfrac{3}{2}, \bar{s}_2 = \dfrac{3}{2}, \bar{s}_3 = \dfrac{5}{4}, \bar{s}_4 = \dfrac{5}{4}, \bar{s}_5 = \dfrac{7}{6}$ etc.

and $\quad \underline{s_1} = -2, \underline{s_2} = -\dfrac{4}{3}, \underline{s_3} = -\dfrac{4}{3}, \underline{s_4} = -\dfrac{6}{5}$ etc.

Hence $\quad \overline{\lim}\, s_n = \inf \left\{\dfrac{3}{2}, \dfrac{5}{4}, \dfrac{7}{6}, \ldots\right\} = 1$

and $\quad \underline{\lim}\, s_n = \sup \left\{-2, -\dfrac{4}{3}, -\dfrac{6}{5}, \ldots \right\} = -1.$

Theorem 1: *If $< s_n >$ is a convergent sequence of real numbers and if $\lim s_n = l$, then $\overline{\lim}\, s_n = \underline{\lim}\, s_n = l$. Conversely, if*

$$\overline{\lim}\, s_n = \underline{\lim}\, s_n = l \in \mathbf{R},$$

then $< s_n >$ is convergent and $\lim\limits_{n \to \infty} s_n = l$.

Proof: First suppose that the sequence $< s_n >$ converges with $\lim s_n = l$. Let $\varepsilon > 0$ be given. Since $s_n \to l$, therefore, we can find a positive integer m, such that

$$|s_n - l| < \varepsilon \text{ for all } n \geq m$$

i.e., $\quad l - \varepsilon < s_n < l + \varepsilon$ for all $n \geq m$.

This inequality shows that for all $n \geq m$, $l + \varepsilon$ is an upper bound of $\{s_n, s_{n+1}, \ldots\}$ and $l - \varepsilon$ is not an upper bound of $\{s_n, s_{n+1}, \ldots\}$.

Since $\bar{s}_n = \{s_n, s_{n+1}, \ldots\}$, it follows that

$$l - \varepsilon < \bar{s}_n \leq l + \varepsilon, \; n \geq m.$$

Taking limits as $n \to \infty$, we get

$$l - \varepsilon \leq \lim_{n \to \infty} \bar{s}_n \leq l + \varepsilon.$$

Since ε is arbitrary, it follows that $\overline{\lim}\, s_n = l.$

Similarly, we can show that $\underline{\lim}\, s_n = l.$

Thus $\qquad \overline{\lim}\, s_n = \underline{\lim}\, s_n = l.$

Conversely, let $\overline{\lim}\, s_n = \underline{\lim}\, s_n = l.$

Since $l = \lim_{n \to \infty} \bar{s}_n$, given any $\varepsilon > 0$, there exists $m_1 \in \mathbf{N}$ such that

$$|\bar{s}_n - l| < \varepsilon \quad \text{for } n \geq m_1 \qquad i.e., \qquad l - \varepsilon < \bar{s}_n < l + \varepsilon \quad \text{for } n \geq m_1.$$

The definition of $\bar{s}_n$ then gives that

$$s_n < l + \varepsilon \quad \text{for } n \geq m_1. \qquad \ldots(1)$$

Similarly, since $l = \lim_{n \to \infty} \underline{s_n}$, there exists $m_2 \in \mathbf{N}$ such that

$$|\underline{s_n} - l| < \varepsilon \text{ for } n \geq m_2$$

which implies as above that $s_n > l - \varepsilon$ for $n \geq m_2$. $\qquad \ldots(2)$

Let $m = \max. \{m_1, m_2\}$. Then from (1) and (2) we find that

$$|s_n - l| < \varepsilon \text{ for } n \geq m.$$

This proves that the sequence $< s_n >$ converges and that

$$\lim_{n \to \infty} s_n = l.$$

Similar results hold good for divergent sequences. Below we state them without proof.

Theorem 2: *A sequence $< s_n >$ diverges to $+\infty$ iff* $\overline{\lim}\, s_n = \underline{\lim}\, s_n = \infty.$

Theorem 3: *A sequence $< s_n >$ diverges to $-\infty$ iff* $\overline{\lim}\, s_n = \underline{\lim}\, s_n = -\infty.$

Theorem 4: *If $< s_n >$ and $< t_n >$ are bounded sequences of real numbers such that $s_n \leq t_n$ for all $n \in \mathbf{N}$, then $\overline{\lim}\, s_n \leq \overline{\lim}\, t_n$ and $\underline{\lim}\, s_n \leq \underline{\lim}\, t_n$.*

Proof: Since $s_n \leq t_n$, therefore it is easy to see that

$$\bar{s}_n \leq \bar{t}_n \quad \text{and } \underline{s_n} \leq \underline{t_n},$$

where $\bar{s}_n, \bar{t}_n, \underline{s_n}, \underline{t_n}$ have their usual meanings as defined earlier.

Then we have from the corollary to theorem 3 of article 9,

$$\lim \bar{s}_n \leq \lim \bar{t}_n \quad \text{and} \quad \lim \underline{s_n} \leq \lim \underline{t_n}$$

or

$$\overline{\lim}\, s_n \leq \overline{\lim}\, t_n \quad \text{and} \quad \underline{\lim}\, s_n \leq \underline{\lim}\, t_n.$$

Theorem 5: *If $< s_n >$ and $< t_n >$ are bounded sequences of real numbers, then*

(*i*) $\overline{\lim}\,(s_n + t_n) \leq \overline{\lim}\, s_n + \overline{\lim}\, t_n$; $\qquad$ (*ii*) $\underline{\lim}\,(s_n + t_n) \geq \underline{\lim}\, s_n + \underline{\lim}\, t_n$.

Proof: Let $\bar{s}_n = \{s_n, s_{n+1}, \ldots\}$, and $\bar{t}_n = \sup\{t_n, t_{n+1}, \ldots\}$.

Then $\qquad s_k \leq \bar{s}_n, (k \geq n), \; t_k \leq \bar{t}_n, (k \geq n).$

$\therefore \qquad s_k + t_k \leq \bar{s}_n + \bar{t}_n$ for $k \geq n$.

Thus $\bar{s}_n + \bar{t}_n$ is an upper bound for $\{s_n + t_n, s_{n+1} + t_{n+1}, \ldots\}$.

Hence $\overline{(s_n + t_n)} = \sup\{s_n + t_n, s_{n+1} + t_{n+1}, \ldots\} \leq \bar{s}_n + \bar{t}_n.$

$$\therefore \qquad \lim \overline{(s_n + t_n)} \leq \lim(\bar{s}_n + \bar{t}_n) = \lim \bar{s}_n + \lim \bar{t}_n$$

i.e., $\overline{\lim}\,(s_n + t_n) \le \overline{\lim}\, s_n + \overline{\lim}\, t_n.$

Thus the result (i) has been proved. Similarly (ii) can be proved.

Note: It can be shown that there exist sequences for which the inequalities in the above theorem are strict inequalities.

Let $s_n = (-1)^n,\ n \in \mathbf{N}$ and $t_n = (-1)^{n+1},\ n \in \mathbf{N}$.

Then $s_n + t_n = 0,\ n \in \mathbf{N}$. Here $\overline{\lim}\, s_n = 1 = \overline{\lim}\, t_n$ and $\overline{\lim}\,(s_n + t_n) = 0$.

Hence, in this case $\overline{\lim}\,(s_n + t_n) < \overline{\lim}\, s_n + \overline{\lim}\, t_n \cdot$

14 Nested Interval Theorem or Cantor's Intersection Theorem

Definition: *A sequence of sets* $< A_n >$ *is called a nested sequence of sets if* $A_1 \supset A_2 \supset \ldots.\ A_n \supset A_{n+1} \supset \ldots.$

Theorem: *For each* $n \in \mathbf{N}$, *let* $I_n = [\,a_n, b_n]$ *be a (non-empty) closed and bounded interval on* $\mathbf{R}$, *such that* $< I_n >$ *is a nested sequence with* $\lim_{n \to \infty}$ *(length of* I_n*) i.e.,* $\lim_{n \to \infty} (b_n - a_n) = 0$. *Then* $\bigcap_{n=1}^{\infty} I_n$ *contains precisely one point.*

(Gorakhpur 2014)

Proof: Since $< I_n >$ is nested, we have

$$I_n \supset I_{n+1} \text{ for all } n \in \mathbf{N}$$

i.e., $[a_n, b_n] \supset [a_{n+1}, b_{n+1}]$ for all $n \in \mathbf{N}$. ...(1)

It follows from (1) that $a_n \le a_{n+1} \le b_{n+1} \le b_n\ \forall\ n \in \mathbf{N}$.

This shows that the sequence $< a_n >$ is a monotonically increasing sequence bounded above by b_1, and $< b_n >$ is a monotonically decreasing sequence bounded below by a_1. Hence, $< a_n >$ and $< b_n >$, both converge.

Now we have $\lim_{n \to \infty}$ (length of I_n) $= \lim_{n \to \infty} (b_n - a_n) = 0$

$$\Rightarrow \quad \lim_{n \to \infty} b_n = \lim_{n \to \infty} a_n = l, \text{ (say)}.$$

Since a monotonically increasing bounded sequence converges to its supremum, it follows that l is the supremum of the range set of the sequence $< a_n >$. Also, a monotonically decreasing bounded sequence converges to its infimum, hence l is the infimum of the range set of the sequence $< b_n >$.

Thus $a_n \le l \le b_n\ \forall\ n$. Hence, $l \in I_n$ for all n. It follows that $l \in \bigcap_{n=1}^{\infty} I_n$.

Clearly no $l' \ne l$ can belong to $\bigcap_{n=1}^{\infty} I_n$.

For let $l' \in \bigcap_{n=1}^{\infty} I_n$, then $0 \le |l' - l| \le |b_n - a_n|\ \forall\ n$.

Since $|b_n - a_n| \to 0$, hence we get $|l' - l| = 0$ *i.e.*, $l' = l$.

Hence $\bigcap_{n=1}^{\infty}$ consists of exactly one point.

Illustrative Examples

Example 17: *Show that the sequence* $< s_n >$ *defined by*

$$s_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n},$$

converges.

Solution: We have

$$s_{n+1} - s_n = \left(\frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n+2}\right) - \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right)$$

$$= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} > 0 \text{ for all } n.$$

Hence, the sequence $< s_n >$ is monotonically increasing.

Now $$|s_n| = s_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n} < \frac{1}{n} + \ldots + \frac{1}{n} \text{ (upto } n \text{ terms)}$$

$$= n \cdot \frac{1}{n} = 1$$

i.e., $|s_n| < 1$ for all n.

Hence, the sequence $< s_n >$ is bounded.

Since $< s_n >$ is a bounded, monotonically increasing sequence, hence it converges.

Example 18: *Show that the sequence* $< s_n >$ *defined by the relation*

$$s_1 = 2, s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{(n-1)!} \quad (n \geq 2), \textit{ converges.}$$

Solution: We have $s_{n+1} - s_n = \dfrac{1}{n!} > 0$ for all n.

Hence, the sequence $< s_n >$ is monotonically increasing.

Now, we shall show that $< s_n >$ is bounded.

For $n \geq 2$, $n! = 1.2.3 \ldots n$ contains $(n-1)$ factors each of which is greater than or equal to 2. Hence $n! \geq 2^{n-1}$ for all $n \geq 2$.

$\therefore$ $$\frac{1}{n!} \leq \frac{1}{2^{n-1}}, \text{ for all } n \geq 2.$$

Thus $$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{(n-1)!}$$

$$\leq 1 + \frac{1}{1!} + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-2}}$$

$$= 1 + \frac{1 - \left(\frac{1}{2}\right)^{n-1}}{1 - \frac{1}{2}} < 3, \text{ for all } n \geq 2.$$

Also $\quad s_1 = 2 < 3.$

$\therefore \quad 2 \leq s_n < 3$ for all $n \in \mathbf{N}$

i.e., $\quad < s_n >$ is bounded.

Since $< s_n >$ is a bounded, monotonically increasing sequence, consequently, it converges.

Example 19: *Show that* $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ *exists and lies between* 2 *and* 3.

Solution: Here $s_n = \left(1 + \frac{1}{n}\right)^n$. Obviously $s_1 = 2.$

By the binomial theorem, we get

$$s_n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \ldots + \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots$$

$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \ldots \left(1 - \frac{n-1}{n}\right). \quad \ldots(1)$$

Similarly,

$$s_{n+1} = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \frac{1}{3!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) + \ldots$$

$$+ \frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) \ldots \left(1 - \frac{n}{n+1}\right)$$

$$> 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots$$

$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \ldots \left(1 - \frac{n-1}{n}\right).$$

[$\because$ each term on the R.H.S. of s_{n+1} is $\geq$ the corresponding term on the R.H.S. of s_n and moreover the number of terms in the expansion of s_{n+1} is $n+2$ *i.e.*, one more than the number of terms $n+1$ in the expansion of s_n]

$\therefore \quad s_{n+1} > s_n$ for all $n \in \mathbf{N}$.

Hence, the sequence $< s_n >$ is monotonically increasing.

$\therefore \quad s_n \geq s_1 = 2, \forall \; n \in \mathbf{N}$.

From (1), we see that

$$s_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} \qquad \text{[See Ex. 18]}$$

$$= 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} < 1 + \frac{1}{1 - \frac{1}{2}} = 3, \text{ for all } n.$$

Thus $2 \leq s_n < 3$, for all n.

Hence the sequence $< s_n >$ is bounded.

Since $< s_n >$ is a bounded, monotonically increasing sequence, consequently, it converges *i.e.*, $\lim\limits_{n \to \infty} s_n$ exists and $\lim s_n = \sup < s_n >$.

Now $2 \leq s_n < 3$ for all $n \Rightarrow 2 \leq \lim\limits_{n \to \infty} s_n \leq 3$, which shows that the limit lies between 2 and 3.

Note: The actual value of the $\lim \left(1 + \frac{1}{n}\right)^n$ is defined to be equal to e. Hence e lies between 2 and 3. Taking limit of both sides of (1) as $n \to \infty$, we see that

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots \infty.$$

Example 20: *Prove that $< s_n >$ is convergent where $s_n = 2 - \frac{1}{2^{n-1}}$.*

Solution: We have

$$s_{n+1} - s_n = \left(2 - \frac{1}{2^n}\right) - \left(2 - \frac{1}{2^{n-1}}\right) = \frac{1}{2^{n-1}} - \frac{1}{2^n} > 0 \text{ for all } n.$$

Hence, the sequence $< s_n >$ is monotonically increasing.

Also, $s_n = 2 - \frac{1}{2^{n-1}} < 2$ for all n. $\left[\because \frac{1}{2^{n-1}} > 0\right]$

$\therefore$ $< s_n >$ is bounded above.

Since $< s_n >$ is a bounded above, monotonically increasing sequence, hence it converges.

We have $\lim\limits_{n \to \infty} s_n = \lim\limits_{n \to \infty} \left(2 - \frac{1}{2^{n-1}}\right) = 2.$

Example 21: *Show that the sequence $< s_n >$ defined by*

$$s_1 = \frac{1}{2},\ s_{n+1} = \frac{2\,s_n + 1}{3}\ \forall\ n \in \mathbf{N} \text{ \textit{is convergent. Also find its limit.}}$$

Solution: We have $s_1 = \frac{1}{2}$ and $s_{n+1} = \frac{2\,s_n + 1}{3}$ $\forall\ n \in \mathbf{N}$.

First applying mathematical induction we shall show that

$$s_{n+1} > s_n\ \forall\ n \in \mathbf{N}.$$

We have $$s_2 = \frac{2 \,.\, s_1 + 1}{3} = \frac{2.\left(\frac{1}{2}\right) + 1}{3} = \frac{2}{3}.$$

$\therefore$ $s_2 > s_1$.

Now assume as our induction hypothesis that $s_{n+1} > s_n$ for some positive integer n.

Then $\quad s_{n+1} > s_n \Rightarrow 2\,s_{n+1} > 2\,s_n \Rightarrow 2\,s_{n+1} + 1 > 2\,s_n + 1$

$\Rightarrow \quad \dfrac{2\,s_{n+1} + 1}{3} > \dfrac{2\,s_n + 1}{3} \quad \Rightarrow \quad s_{n+2} > s_{n+1}.$

Thus $s_2 > s_1$ and if $s_{n+1} > s_n$, then $s_{n+2} > s_{n+1}$.

$\therefore$ by induction $s_{n+1} > s_n \;\forall\; n \in \mathbf{N}$.

Thus the sequence $< s_n >$ is monotonic increasing.

Now we shall show that the sequence $< s_n >$ is also bounded above.

We have $\; s_{n+1} > s_n \;\forall\; n \in \mathbf{N}$

$\Rightarrow \quad \dfrac{2\,s_n + 1}{3} > s_n \;\forall\; n \in \mathbf{N} \quad \Rightarrow \quad 2\,s_n + 1 > 3\,s_n \;\forall\; n \in \mathbf{N}$

$\Rightarrow \quad s_n < 1 \;\forall\; n \in \mathbf{N}$.

$\therefore$ the sequence $< s_n >$ is bounded above by 1.

Since the sequence $< s_n >$ is monotonic increasing and bounded above, therefore by monotone convergence theorem $< s_n >$ converges to its supremum.

Let $\lim\limits_{n \to \infty} s_n = l$. Then $\lim\limits_{n \to \infty} s_{n+1} = l$.

Now $\quad s_{n+1} = \dfrac{2\,s_n + 1}{3} \Rightarrow \lim s_{n+1} = \lim \dfrac{2\,s_n + 1}{3}$

$\Rightarrow \quad l = \dfrac{2 \lim s_n + 1}{3} \Rightarrow l = \dfrac{2l + 1}{3} \Rightarrow l = 1.$

Hence $s_n \to 1$. We have $\inf < s_n > = s_1 = \dfrac{1}{2}$ and $\sup < s_n > = \lim s_n = 1$.

Example 22: *Show that the sequence $< s_n >$ defined by*

$$s_1 = \sqrt{2}\,, s_{n+1} = \sqrt{(2\,s_n)}$$

converges to 2.

Solution: We have $s_2 = \sqrt{(2\sqrt{2})}$.

Since $\quad 1 < \sqrt{2} \Rightarrow 2 < 2\sqrt{2} \quad \Rightarrow \quad \sqrt{2} < \sqrt{(2\sqrt{2})}$,

$\therefore \quad s_1 < s_2$.

Let us suppose that $s_m < s_{m+1}$.

Then $\quad \sqrt{(2s_m)} < \sqrt{(2s_{m+1})} \quad \Rightarrow \quad s_{m+1} < s_{m+2}$.

Hence, by mathematical induction, we have

$$s_n < s_{n+1} \text{ for all } n \in \mathbf{N}$$

i.e., $< s_n >$ is monotonically increasing.

Also, we have $s_1 = \sqrt{2} < 2$.

Let us suppose that $s_m < 2$. Then $\sqrt{(2s_m)} < \sqrt{(2.2)} = 2 \Rightarrow s_{m+1} < 2$.

Hence, by mathematical induction, we have $s_n < 2$ for all $n \in \mathbf{N}$

i.e., $< s_n >$ is bounded above by 2.

Thus $< s_n >$ is a monotonically increasing sequence bounded above by 2, hence, it converges.

Let $\lim_{n \to \infty} s_n = l$. Then $\lim_{n \to \infty} s_{n+1} = l$.

Now $\quad s_{n+1} = \sqrt{(2s_n)} \Rightarrow \lim s_{n+1} = \lim \sqrt{(2s_n)}$

$\Rightarrow \quad l = \sqrt{(2l)} \Rightarrow l(l-2) = 0 \Rightarrow l = 0, 2.$

But $\quad s_n \geq s_1 = \sqrt{2}\ \forall\ n \in \mathbf{N} \Rightarrow s_n - \sqrt{2} \geq 0\ \forall\ n \in \mathbf{N}.$

$\therefore \quad \lim (s_n - \sqrt{2}) \geq 0$ *i.e.*, $\lim s_n \geq \sqrt{2}$.

Hence l cannot be zero. Therefore $l = 2$.

Example 23: *Show that the sequence* $<s_n>$ *defined by* $s_1 = 1$ *and* $s_{n+1} = \sqrt{(2 + s_n)}, \forall\ n \in \mathbf{N}$ *is monotonically increasing and bounded. Also find its limit.*

Solution: We have $s_1 = 1$ and $(s_{n+1})^2 = 2 + s_n, \forall\ n \in \mathbf{N}$.

$\therefore \quad s_2 = \sqrt{3}, s_3 = \sqrt{(2 + \sqrt{3})}, \ldots\ldots$

Now $\quad 1 < \sqrt{3} \quad \Rightarrow \quad s_1 < s_2$.

Let us suppose that $s_m < s_{m+1}$. Then $\sqrt{(2 + s_m)} < \sqrt{(2 + s_{m+1})}$

$\Rightarrow \quad s_{m+1} < s_{m+2}$.

Hence, by mathematical induction, we have

$$s_n < s_{n+1} \text{ for all } n \in \mathbf{N}$$

i.e., $<s_n>$ is monotonically increasing.

Again, $\quad s_{n+1} > s_n \Rightarrow \sqrt{(2 + s_n)} > s_n$

$\Rightarrow \quad 2 + s_n - s_n^2 > 0 \quad \Rightarrow \quad (2 - s_n)(1 + s_n) > 0$

$\Rightarrow \quad (2 - s_n) > 0 \qquad [\because\ 1 + s_n > 0]$

or $\quad s_n < 2\ \forall\ n \in \mathbf{N}$.

Hence $<s_n>$ is bounded.

Thus $<s_n>$ is a monotonically increasing sequence bounded above by 2; consequently it converges.

Let $\lim s_n = l$. Then $\lim s_{n+1} = l$.

Now $\quad s_{n+1} = \sqrt{(2 + s_n)} \quad \Rightarrow \quad \lim s_{n+1} = \lim \sqrt{(2 + s_n)}$

$\Rightarrow \quad l = \sqrt{(2 + l)} \quad \Rightarrow \quad l^2 - l - 2 = 0 \quad \Rightarrow \quad (l + 1)(l - 2) = 0$

$\Rightarrow \quad l = -1, 2.$

But l cannot be -1 since all terms of the sequence are positive. Hence $l = 2$.

Note: If a sequence $<s_n>$ is monotonically increasing then there is no need to show that $<s_n>$ has a lower bound because s_1 is always its lower bound.

Similarly, for a monotonically decreasing sequence there is no need to find an upper bound, because s_1 will always be its upper bound.

Example 24: *Show that the sequence* $<s_n>$ *defined by*

$$s_1 = 1, s_{n+1} = \frac{4 + 3s_n}{3 + 2s_n}, n \in \mathbf{N}$$

is convergent and find its limit.

Solution: We observe that all the terms of the given sequence are positive.

First by mathematical induction we shall show that

$$s_{n+1} > s_n \;\forall\; n \in \mathbf{N}.$$

We have $\quad s_1 = 1, s_2 = \dfrac{4+3s_1}{3+2s_1} = \dfrac{4+3\cdot 1}{3+2\cdot 1} = \dfrac{7}{5}\cdot$

$\therefore \quad s_2 > s_1.$

Now assume as our induction hypothesis that for some positive integer n,

$$s_{n+1} > s_n. \qquad \ldots(1)$$

Then

$$s_{n+2} - s_{n+1} = \frac{4+3\,s_{n+1}}{3+2\,s_{n+1}} - \frac{4+3s_n}{3+2s_n}$$

$$= \frac{(4+3s_{n+1})(3+2s_n) - (4+3s_n)(3+2\,s_{n+1})}{(3+2\,s_{n+1})(3+2\,s_n)}$$

$$= \frac{s_{n+1} - s_n}{(3+2\,s_{n+1})(3+2s_n)} > 0, \text{ by (1).}$$

$\therefore \quad s_{n+2} > s_{n+1}.$

Thus $s_2 > s_1$ and if $s_{n+1} > s_n$, then we have also $s_{n+2} > s_{n+1}$.

$\therefore$ by mathematical induction $s_{n+1} > s_n, \forall\; n \in \mathbf{N}$.

Thus the sequence $< s_n >$ is monotonic increasing.

Now we shall show that the sequence $< s_n >$ is also bounded above.

We have

$$s_{n+1} = \frac{3s_n + 4}{2\,s_n + 3} = \frac{\frac{3}{2}(2\,s_n + 3) - \frac{1}{2}}{2\,s_n + 3} = \frac{3}{2} - \frac{1}{2\,(2\,s_n + 3)},$$

showing that $\quad s_{n+1} < \dfrac{3}{2}, \forall\; n \in \mathbf{N}.$

Also $\quad s_1 = 1 < \dfrac{3}{2}\cdot$

Thus $s_n < \dfrac{3}{2}$, $\forall\, n \in \mathbf{N}$. Therefore the sequence $< s_n >$ is bounded above by $\dfrac{3}{2}\cdot$

Since the sequence $< s_n >$ is monotonic increasing and bounded above, therefore by monotone convergence theorem it converges to its supremum.

Let $\lim s_n = l$. Then $\lim s_{n+1} = l$.

Now $\quad s_{n+1} = \dfrac{4+3s_n}{3+2s_n} \;\Rightarrow\; \lim s_{n+1} = \dfrac{4+3\lim s_n}{3+2\lim s_n}$

$\Rightarrow \quad l = \dfrac{4+3l}{3+2l} \;\Rightarrow\; l^2 = 2 \;\Rightarrow\; l = \pm\sqrt{2}.$

Since all terms of the sequence are positive so l cannot be negative. Hence $l = \sqrt{2}$. We have $\inf < s_n > = s_1 = 1$ and $\sup < s_n > = \lim s_n = \sqrt{2}$.

Example 25: *A sequence $< s_n >$ of positive terms is defined by*

$$s_1 = k > 0\,;\, s_{n+1} = \frac{3+2s_n}{2+s_n}, \forall\, n \in \mathbf{N}.$$

Show that the sequence converges to a limit independent of k and find the limit.

Solution: We have $s_1 = k > 0$, and $s_{n+1} = \dfrac{3 + 2\, s_n}{2 + s_n}$, $\forall\ n \in \mathbf{N}$.

Then $s_2 > 0, s_3 > 0$ and so on.

Therefore the terms of the sequence are all positive.

Now first by mathematical induction we shall show that

$$s_{n+1} > s_n\ \forall\ n \in \mathbf{N}.$$

We have $\quad s_2 - s_1 = \dfrac{3 + 2s_1}{2 + s_1} - s_1 = \dfrac{3 + 2k}{2 + k} - k = \dfrac{3 - k^2}{2 + k} > 0$ if $0 < k < \sqrt{3}$.

Thus $\quad s_2 > s_1$ if $0 < k < \sqrt{3}$.

Now assume as our induction hypothesis that for some positive integer n,

$$s_{n+1} > s_n. \qquad \ldots(1)$$

Then $\quad s_{n+2} - s_{n+1} = \dfrac{3 + 2\, s_{n+1}}{2 + s_{n+1}} - \dfrac{3 + 2\, s_n}{2 + s_n}$

$$= \frac{s_{n+1} - s_n}{(2 + s_n)(2 + s_{n+1})} > 0, \text{ by (1).}$$

$\therefore \quad s_{n+2} > s_{n+1}.$

Thus $s_2 > s_1$ and if $s_{n+1} > s_n$, then we have also $s_{n+2} > s_{n+1}$.

$\therefore$ by induction $s_{n+1} > s_n$, $\forall\ n \in \mathbf{N}$.

Thus the sequence $< s_n >$ is monotonic increasing.

Now we shall show that the sequence $< s_n >$ is also bounded above.

We have $\quad s_{n+1} > s_n\ \forall\ n \in \mathbf{N}$

$\Rightarrow \quad \dfrac{3 + 2s_n}{2 + s_n} > s_n \Rightarrow \dfrac{3 + 2s_n}{2 + s_n} - s_n > 0$

$\Rightarrow \quad \dfrac{3 - s_n^2}{2 + s_n} > 0 \Rightarrow 3 - s_n^2 > 0 \Rightarrow s_n^2 < 3$

$\Rightarrow \quad s_n < \sqrt{3}, \forall\ n \in \mathbf{N}.$

$\therefore$ $< s_n >$ is bounded above by $\sqrt{3}$.

Thus $< s_n >$ is a bounded monotonically increasing sequence. Hence it converges.

Let $\lim s_n = l$.

Then $\lim s_{n+1} = l$.

Now $\quad s_{n+1} = \dfrac{3 + 2s_n}{2 + s_n} \Rightarrow \lim s_{n+1} = \dfrac{3 + 2 \lim s_n}{2 + \lim s_n}$

$\Rightarrow \quad l = \dfrac{3 + 2l}{2 + l} \Rightarrow l^2 = 3 \Rightarrow l = \pm\sqrt{3}.$

But l cannot be negative because the terms of the sequence $< s_n >$ are all positive.

Hence $l = \sqrt{3}$ which is independent of k.

Example 26: *If u_1, v_1 are given unequal numbers and*

$$u_n = \frac{1}{2}(u_{n-1} + v_{n-1}),\ v_n = \sqrt{(u_{n-1}\, v_{n-1})},\ \text{where } n \geq 2\ ,$$

prove that (*i*) *u_n decreases, and v_n increases as n increases,*
(*ii*) *$<u_n>, <v_n>$ are both convergent and have the same limit , where $u > v > 0$, and $u_1 = \frac{1}{2}(u+v)$ and $v_1 = \sqrt{(uv)}$.*

Solution: Since $u > v > 0$, $u_1 = \frac{1}{2}(u+v)$ and $v_1 = \sqrt{(uv)}$, therefore, u_1 and v_1 are positive and $u_1 > v_1$ since A.M. > G.M.

$\therefore \quad v_1 < u_2 < u_1 \qquad [\because u_2 = \frac{1}{2}(u_1 + v_1)$ so that v_1, u_2, u_1 are in A.P.]

and $\quad v_1 < v_2 < u_1. \qquad [\because v_2 = \sqrt{(u_1 v_1)}$ so that v_1, v_2, u_1 are in G.P.]

Since u_2 is the A.M. and v_2 is the G.M. of u_1 and v_1, therefore, we have $u_2 > v_2$.
Hence as above, we get

$$v_2 < u_3 < u_2 \text{ and } v_2 < v_3 < u_2\ , \text{ and so on.}$$

Thus $v_1 < v_2 < v_3 < \ldots < \ldots < u_3 < u_2 < u_1$.
Therefore the sequences $<u_n>$ and $<v_n>$ are monotonically decreasing and monotonically increasing respectively. Obviously both are bounded so that they are convergent.
Now we have to show that $<u_n>$ and $<v_n>$ have the same limits.

Let $\quad \lim u_n = A$ and $\lim v_n = B$.

Now $\quad u_n = \frac{1}{2}(u_{n-1} + v_{n-1}) \Rightarrow \lim u_n = \lim \frac{1}{2}(u_{n-1} + v_{n-1})$

or $\quad A = \frac{1}{2}(A + B) \quad$ *i.e.,* $A = B$.

Example 27: *If x_1, x_2 are + ive and $x_{n+2} = \sqrt{(x_{n+1} \cdot x_n)}$, prove that the sequences $x_1, x_3, x_5, \ldots$ and $x_2, x_4, x_6, \ldots$ are one an increasing and the other a decreasing sequence and show that their common limit is $(x_1 x_2^2)^{1/3}$.*

Solution: Let $x_1 > x_2$. Then since $x_3 = \sqrt{(x_2 x_1)}$, we have

$$x_1 > x_3 > x_2\ ,$$

and since $x_3 > x_2$, we have $x_3 > x_4 > x_2$. $\qquad [\because x_4 = \sqrt{(x_3 x_2)}]$

Similarly, we have

$$x_3 > x_5 > x_4\ ;\ x_5 > x_6 > x_4\ ;$$
$$x_5 > x_7 > x_6\ ;\ x_7 > x_8 > x_6\ ;$$

and so on.

Thus $\quad x_2 < x_4 < x_6 < \ldots < \ldots < x_5 < x_3 < x_1$.

Hence $<x_1, x_3, x_5, \ldots>$ is monotonic decreasing and

$$<x_2, x_4, x_6, \ldots>$$

is monotonic increasing and both being bounded, are convergent.

Let $\lim x_n = A$, if n is even and $\lim x_n = B$, if n is odd.

Now $\quad x_{n+2} = \sqrt{(x_{n+1} . x_n)} \Rightarrow \lim x_{n+2} = \lim \sqrt{(x_{n+1} . x_n)}$.

$\therefore \quad B = \sqrt{(A . B)}$ or $A = B$, if n is odd

and $\quad A = \sqrt{(B . A)}$ or $A = B$, if n is even.

Hence in either case $A = B$.

Now
$$\frac{x_3}{x_2} = \frac{1}{x_2}\sqrt{(x_1\ x_2)} = \sqrt{\left(\frac{x_1}{x_2}\right)},$$
$$\frac{x_4}{x_2} = \frac{x_4}{x_3}\cdot\frac{x_3}{x_2} = \sqrt{\left(\frac{x_2}{x_3}\right)}\sqrt{\left(\frac{x_1}{x_2}\right)} = \sqrt{\left(\frac{x_1}{x_3}\right)},$$
$$\frac{x_5}{x_2} = \frac{x_5}{x_4}\cdot\frac{x_4}{x_3}\cdot\frac{x_3}{x_2} = \sqrt{\left(\frac{x_3}{x_4}\right)}\sqrt{\left(\frac{x_2}{x_3}\right)}\sqrt{\left(\frac{x_1}{x_2}\right)} = \sqrt{\left(\frac{x_1}{x_4}\right)}.$$

In a like manner, $\dfrac{x_n}{x_2} = \sqrt{\left(\dfrac{x_1}{x_{n-1}}\right)}$

or $\quad \lim (x_n \sqrt{x_{n-1}}) = \sqrt{(x_1 x_2^2)}$

i.e., $\quad A . A^{1/2} = \sqrt{(x_1 x_2^2)}$ or $A = (x_1 x_2^2)^{1/3}$.

Example 28: *If k is positive and $\alpha, -\beta$ are the positive and negative roots of $x^2 - x - k = 0$, prove that if $u_n = \sqrt{(k + u_{n-1})}$ and $u_1 > 0$, then $u_n \to \alpha$.*

Solution: Since u_1 is positive, hence by virtue of the relation

$$u_n = \sqrt{(k + u_{n-1})},\ u_2, u_3, \ldots, u_n, \ldots \text{ are all positive.}$$

Thus $\quad u_n > 0, \forall\ n \in \mathbf{N}$.

We have $u_n^2 - u_{n-1}^2 = (k + u_{n-1}) - (k + u_{n-2}) = u_{n-1} - u_{n-2}$ so that $u_n >$ or $< u_{n-1}$ according as $u_{n-1} >$ or $< u_{n-2}$ and hence $< u_n >$ is a monotonic sequence, it is an increasing or a decreasing sequence according as $u_2 >$ or $< u_1$.

Now $\quad x^2 - x - k \equiv (x - \alpha)(x + \beta) \quad \ldots(1)$

$\Rightarrow \quad u_1^2 - u_1 - k = (u_1 - \alpha)(u_1 + \beta). \quad \ldots(2)$

Let $u_1 > \alpha$. Then from (2), we have
$$u_1^2 - u_1 - k > 0$$
$\Rightarrow \quad u_1 + k < u_1^2 \Rightarrow \sqrt{(u_1 + k)} < u_1 \Rightarrow u_2 < u_1$.

Hence in this case $< u_n >$ is a decreasing sequence.

Since $u_n > 0, \forall\ n \in \mathbf{N}$, therefore $< u_n >$ is bounded below by 0.

Thus $< u_n >$ is a monotonically decreasing sequence and is bounded below and hence $< u_n >$ is convergent.

Again let $u_1 < \alpha$. Then from (2), we have
$$u_1^2 - u_1 - k < 0 \Rightarrow u_1 + k > u_1^2 \Rightarrow \sqrt{(u_1 + k)} > u_1 \Rightarrow u_2 > u_1.$$

Hence is this case $< u_n >$ is an increasing sequence.

Now $\quad u_n^2 = u_{n-1} + k < u_n + k, \quad [\because u_{n-1} < u_n]$

i.e., $u_n^2 - u_n - k < 0$ or $(u_n - \alpha)(u_n + \beta) < 0$, using (1).

$\therefore$ $u_n < \alpha$. $[\because\ u_n > 0$ and $\beta > 0 \Rightarrow u_n + \beta > 0]$

Thus in this case $u_n < \alpha, \forall\ n \in \mathbf{N}$.

Hence in this case $< u_n >$ is a monotonically increasing sequence and is bounded above by α and so $< u_n >$ is convergent.

Thus $< u_n >$ is convergent whether $u_1 >$ or $< \alpha$.

Let $\lim u_n = l$.

Now $(u_n - \alpha)(u_n + \beta) = u_n^2 - u_n - k = (u_{n-1} + k) - u_n - k = u_{n-1} - u_n$.

Taking limits, we get $(l - \alpha)(l + \beta) = l - l = 0$.

This gives $l = \alpha$ or $l = -\beta$.

Since the terms of the sequence $< u_n >$ are all positive, so its limit cannot be negative. Therefore we cannot have $l = -\beta$. Therefore we must have $l = \alpha$.

In case $u_1 = \alpha$, then from (2), we have

$$u_1^2 - u_1 - k = 0 \Rightarrow u_2 = \sqrt{(u_1 + k)} = u_1.$$

Now repeatedly using the relation

$$u_n^2 - u_{n-1}^2 = u_{n-1} - u_{n-2},$$

we observe that $u_3 = u_2$, $u_4 = u_3$, $u_5 = u_4$, and so on.

Thus in this case $u_n = u_1 = \alpha, \forall\ n \in \mathbf{N}$.

So in this case also the sequence $< u_n >$ converges to α.

Hence in all cases $< u_n >$ converges to α which is the positive root of the equation $x^2 - x - k = 0$.

Example 29: *Show that the sequence $< s_n >$ defined by*

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \text{ does not converge.}$$

Solution: We shall show that the given sequence is not a Cauchy sequence. For this we shall show that if we take $\varepsilon = \frac{1}{2} > 0$, then there exists no positive integer m such that

$$|s_n - s_m| < \varepsilon\ \forall\ n \geq m.$$

Whatever positive integer m may be, if we take $n = 2m$, then $n > m$ and we have

$$|s_n - s_m| = |s_{2m} - s_m|$$

$$= \left|\left(1 + \frac{1}{2} + \ldots. + \frac{1}{m} + \frac{1}{m+1} + \ldots. + \frac{1}{2m}\right) - \left(1 + \frac{1}{2} + \ldots. + \frac{1}{m}\right)\right|$$

$$= \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{2m}$$

$$> \frac{1}{2m} + \frac{1}{2m} + \frac{1}{2m} + \ldots. \text{ upto } m \text{ terms}$$

$$= m \cdot \frac{1}{2m} = \frac{1}{2}.$$

Thus if we take $\varepsilon = \dfrac{1}{2}$, then whatever positive integer m we take, we have $n = 2m > m$ and

$$|s_n - s_m| = |s_{2m} - s_m| > \frac{1}{2} \quad i.e., \ |s_n - s_m| > \varepsilon.$$

In this way for $\varepsilon = \dfrac{1}{2} > 0$, there exists no positive integer m such that

$$|s_n - s_m| < \varepsilon \ \forall \ n \geq m.$$

$\therefore$ the given sequence is not a Cauchy sequence.

Hence by Cauchy convergence criterion $< s_n >$ is not convergent.

Example 30: *If $< s_n >$ be a sequence of positive numbers such that*

$$s_n = \frac{1}{2}(s_{n-1} + s_{n-2}), \text{ for all } n > 2,$$

then show that $< s_n >$ converges and find $\lim s_n$.

Solution: In case $s_1 = s_2$, it can be easily seen that $s_n = s_1$ for all n, therefore $< s_n >$ converges to s_1. Now we consider the case $s_1 \neq s_2$.

We first find that

$$\begin{aligned} |s_n - s_{n-1}| &= \left|\frac{1}{2}(s_{n-1} + s_{n-2}) - s_{n-1}\right| = \frac{1}{2}|s_{n-1} - s_{n-2}| \\ &= \frac{1}{2} \cdot \frac{1}{2}|s_{n-2} - s_{n-3}| = \frac{1}{2^2}|s_{n-2} - s_{n-3}| \\ &= \frac{1}{2^{n-2}}|s_2 - s_1|, \ \text{ for } n \geq 2. \qquad \text{...(1)} \end{aligned}$$

Now for $n \geq m$, we have

$$\begin{aligned} |s_n - s_m| &= |(s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \ldots + (s_{m+1} - s_m)| \\ &\leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \ldots + |s_{m+1} - s_m| \\ &= \left(\frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \ldots + \frac{1}{2^{m-1}}\right)|s_2 - s_1| \ \text{ by (1)} \\ &= \frac{1}{2^{m-1}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-m-1}}\right)|s_2 - s_1| \\ &< \frac{1}{2^{m-2}}|s_2 - s_1|. \qquad \text{...(2)} \end{aligned}$$

$$\left[\because \ 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots + \left(\frac{1}{2}\right)^{n-m-1} < 2\right]$$

Let $\varepsilon > 0$ be given. We can choose a positive integer m such that $\dfrac{1}{2^{m-2}}|s_2 - s_1| < \varepsilon$. For this value of m, we have from (2)

$$|s_n - s_m| < \varepsilon \text{ for } n \geq m.$$

Hence $< s_n >$ is a Cauchy sequence and therefore by Cauchy's convergence criterion it converges.

Let $\lim s_n = l$. Putting $n = 3, 4, \ldots, k$ in the relation $s_n = \dfrac{1}{2}(s_{n-1} + s_{n-2})$, we get

$$\left.\begin{aligned} s_3 &= \frac{1}{2}(s_2 + s_1) \\ s_4 &= \frac{1}{2}(s_3 + s_2) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ s_{k-1} &= \frac{1}{2}(s_{k-2} + s_{k-3}) \\ s_k &= \frac{1}{2}(s_{k-1} + s_{k-2}). \end{aligned}\right\} \qquad \ldots(3)$$

Adding the corresponding sides of the relations in (3), we get

$$s_k + \frac{1}{2} s_{k-1} = \frac{1}{2}(s_1 + 2s_2).$$

Proceeding to the limit as $k \to \infty$, we get

$$\frac{3}{2} l = \frac{1}{2}(s_1 + 2s_2) \quad i.e., \quad l = \frac{1}{3}(s_1 + 2s_2).$$

Example 31: *Let $< u_n >$ be a sequence and $s_n = u_1 + u_2 + \ldots + u_n$.*

If $t_n = |u_1| + |u_2| + \ldots + |u_n|$ for each $n \in \mathbf{N}$ and $< t_n >$ is a Cauchy sequence, then $< s_n >$ is also a Cauchy sequence.

Solution: Let $\varepsilon > 0$ be given. Since $< t_n >$ is a Cauchy sequence, therefore, for given $\varepsilon > 0$ there exists $m \in \mathbf{N}$ such that

$$|t_n - t_m| < \varepsilon \ \forall \ n \geq m$$

$\Rightarrow$ $\quad |u_{m+1}| + |u_{m+2}| + \ldots + |u_n| < \varepsilon \ \forall \ n \geq m.$

But $\quad |u_{m+1}| + |u_{m+2}| + \ldots + |u_n| \geq |u_{m+1} + u_{m+2} + \ldots + u_n|.$

$\therefore$ $\quad |u_{m+1} + u_{m+2} + \ldots + u_n| < \varepsilon \ \forall \ n \geq m$

or $\quad |s_n - s_m| < \varepsilon \ \forall \ n \geq m.$

Hence $< s_n >$ is a Cauchy sequence.

Example 32: *Find* (*i*) $\lim \sqrt{\left(\frac{n+1}{n}\right)}$ $\qquad$ (*ii*) $\lim \frac{\sin(n\pi/3)}{\sqrt{n}}$.

Solution: (i) We have $\sqrt{\left(\frac{n+1}{n}\right)} = \sqrt{\left(1 + \frac{1}{n}\right)} < 1 + \frac{1}{2n}$.

$$\left[\because \left(1 + \frac{1}{2n}\right)^2 = 1 + \frac{1}{n} + \frac{1}{4n^2} > 1 + \frac{1}{n}\right].$$

Since $1 + \frac{1}{n} > 1$, hence $\sqrt{\left(1 + \frac{1}{n}\right)} > 1$.

$\therefore \quad 1 < \sqrt{\left(\frac{n+1}{n}\right)} < 1 + \frac{1}{2n}$ for all $n \in \mathbf{N}$.

But $\lim 1 = 1$ and $\lim \left(1 + \frac{1}{2n}\right) = 1$.

Hence, by Sandwich theorem, $\lim \sqrt{\left(\frac{n+1}{n}\right)} = 1$.

(ii) Let $s_n = \sin(n\pi/3)$, $t_n = 1/\sqrt{n}$.

We have $-1 \le \sin \dfrac{n\pi}{3} \le 1$ for all $n \in \mathbf{N}$.

$\therefore$ $< s_n >$ is a bounded sequence.

Also $\qquad \lim t_n = \lim \dfrac{1}{\sqrt{n}} = 0.$

Hence by theorem 4 of article 9, $\lim (s_n t_n) = 0$ *i.e.,* $\lim \dfrac{\sin (n\pi / 3)}{\sqrt{n}} = 0.$

Example 33: *Show that lim* $\left[\dfrac{1}{(n+1)^2} + \dfrac{1}{(n+2)^2} + \ldots + \dfrac{1}{(n+n)^2}\right] = 0.$

Solution: Let $\quad s_n = \dfrac{1}{(n+1)^2} + \dfrac{1}{(n+2)^2} + \ldots + \dfrac{1}{(n+n)^2}.$

For $1 \le m \le n$, we have $(n+1)^2 \le (n+m)^2 \le (n+n)^2$

or
$$\frac{1}{(n+1)^2} \ge \frac{1}{(n+m)^2} \ge \frac{1}{(n+n)^2}.$$

Putting $m = 1, 2, \ldots, n$ and adding the corresponding sides of the n inequalities thus obtained, we get

$$\frac{n}{(n+1)^2} \ge s_n \ge \frac{n}{(n+n)^2}$$

i.e.,
$$\frac{n}{4n^2} \le s_n \le \frac{n}{(n+1)^2} < \frac{n}{n^2} \quad \text{for all } n \in \mathbf{N}$$

i.e.,
$$\frac{1}{4n} \le s_n < \frac{1}{n} \text{ for all } n \in \mathbf{N}.$$

But $\lim \dfrac{1}{4n} = 0$ and $\lim \dfrac{1}{n} = 0.$

Hence, by Sandwich theorem, $\lim s_n = 0.$

Example 34: *If* $r > 0$, *show that* $\lim r^{1/n} = 1.$

Solution: **Case 1:** When $r > 1.$

Let $\quad s_n = r^{1/n} - 1.$ Then $s_n > 0$, for all n.

Now
$$s_n = r^{1/n} - 1 \in r^{1/n} = 1 + s_n \Rightarrow r = (1 + s_n)^n$$

$\Rightarrow$
$$r = 1 + n\, s_n + \ldots + s_n{}^n \ge 1 + n\, s_n \text{ for all } n \in \mathbf{N}$$

$\Rightarrow$
$$\frac{r-1}{n} \le s_n \ \forall\, n \in \mathbf{N}.$$

$\therefore$
$$0 < s_n \le \frac{r-1}{n} \ \forall\, n \in \mathbf{N}.$$

Hence, by Sandwich theorem, $\lim s_n = 0$ *i.e.,* $\lim (r^{1/n} - 1) = 0$ or $\lim r^{1/n} = 1.$

Case 2: When $r = 1.$

In this case $r^{1/n} = 1 \ \forall\, n$ and hence $< r^{1/n} >$ converges to 1.

Case 3: When $0 < r < 1.$

Since $\dfrac{1}{r} > 1$, therefore, by Case 1, $\lim \left(\dfrac{1}{r}\right)^{1/n} = 1$ *i.e.,* $\lim \dfrac{1}{r^{1/n}} = 1.$

$\therefore$ By theorem 7 of article 9, $\lim r^{1/n} = 1$.

Example 35: *Show that* $\lim \dfrac{1}{n}\left(1+\dfrac{1}{3}+\ldots+\dfrac{1}{2n-1}\right)=0.$

Solution: Let $s_n = \dfrac{1}{2n-1}\cdot$ Then $\lim s_n = \lim \dfrac{1}{2n-1} = 0.$

$\therefore$ By Cauchy's first theorem on limits, $\lim \dfrac{s_1+s_2+\ldots+s_n}{n} = 0\cdot$

Since $\qquad s_1 = 1,\ s_2 = \dfrac{1}{3},\ldots, s_n = \dfrac{1}{2n-1},$

$$\therefore \qquad \lim \frac{1}{n}\left\{1+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{2n-1}\right\}=0\cdot$$

Example 36: *Prove that* $\lim \dfrac{1}{\text{n}}(1+2^{1/2}+3^{1/3}+\ldots+n^{1/n})=1\cdot$

Solution: Let $s_n = n^{1/n}$. Then we know that $\lim n^{1/n} = 1$.

Hence, by Cauchy's first theorem on limits,

$$\lim \frac{1}{n}(s_1+s_2+\ldots+s_n)=1$$

or $\qquad \lim \dfrac{1}{n}(1+2^{1/2}+3^{1/3}+\ldots+n^{1/n})=1.$

Example 37: *Prove that lim* $\left[\left(\dfrac{2}{1}\right)\left(\dfrac{3}{2}\right)^2\left(\dfrac{4}{3}\right)^3\ldots\left(\dfrac{n+1}{n}\right)^n\right]^{1/n} = e.$

Solution: Let $s_n = \left(\dfrac{n+1}{n}\right)^n = \left(1+\dfrac{1}{n}\right)^n$. Then $\lim s_n = e$.

Clearly $s_n > 0$ for all n.

Hence, by theorem 11 of article 9, $\lim (s_1 s_2 \ldots s_n)^{1/n} = e$.

Since $\qquad s_1 = \dfrac{2}{1}, s_2 = \left(\dfrac{3}{2}\right)^2, s_3 = \left(\dfrac{4}{3}\right)^3,\ldots s_n = \left(\dfrac{n+1}{n}\right)^n,$

$$\therefore \qquad \lim\left[\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\ldots\left(\frac{n+1}{n}\right)^n\right]^{1/n} = e.$$

Example 38: *Show that*

(i) $\lim \dfrac{n}{(n\,!)^{1/n}} = e,$

(ii) *lim* $[\{(n+1)(n+2)\ldots(n+n)\}^{1/n}/n] = 4/e.$ **(Gorakhpur 2012)**

Solution: (i) Let $s_n = \dfrac{n^n}{n\,!}$, then $s_{n+1} = \dfrac{(n+1)^{n+1}}{(n+1)\,!}\cdot$

$$\therefore \qquad \frac{s_{n+1}}{s_n} = \frac{(n+1)^{n+1}}{n+1}\cdot\frac{1}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n\cdot$$

Also $s_n > 0$ for all $n \in \mathbf{N}$.

Hence by theorem 12 of article 9, we have

$$\lim s_n^{1/n} = \lim \frac{s_{n+1}}{s_n} = \lim \left(1 + \frac{1}{n}\right)^n = e$$

i.e., $$\lim \frac{n}{(n\,!)^{1/n}} = e.$$

(ii) Let $$s_n = (n+1)\,(n+2)\ldots(n+n)\,/\,n^n.$$

Then $$\frac{s_{n+1}}{s_n} = \frac{2\,(2\,n+1)}{(n+1)}\left(\frac{n}{n+1}\right)^n,$$

so that $$\lim \frac{s_{n+1}}{s_n} = \lim \left[\frac{2\,(2\,n+1)}{n+1} \cdot \frac{1}{\left(1 + \frac{1}{n}\right)^n}\right] = 4 \cdot \frac{1}{e} = \frac{4}{e}.$$

Also $s_n > 0$ for all $n \in \mathbf{N}$.

Hence by theorem 12 of article 9, we have

$$\lim s_n^{1/n} = \lim \left[\{(n+1)\,(n+2)\ldots(n+n)\}^{1/n}\,/\,n\right] = 4\,/\,e.$$

Example 39: *If $p > 0$ and c is real, then find the* $\lim \dfrac{n^c}{(1+p)^n}$.

Solution: Let k be an integer such that $k > c, k > 0$.

We have, for $n > 2k$

$$(1+p)^n > {}^nC_k\, p^k = \frac{n\,(n-1)\ldots(n-k+1)}{k\,!}\,p^k > \frac{n^k}{2^k} \cdot \frac{p^k}{k\,!}$$

$$\left[\because\ n > 2\,k \Rightarrow (n-r+1) > \frac{n}{2} \text{ for } r = 1, 2, \ldots, k\right]$$

or $$0 < \frac{1}{(1+p)^n} < \frac{2^k\,k\,!}{n^k\,p^k}, \text{ for } n > 2k.$$

$\therefore$ $$0 < \frac{n^c}{(1+p)^n} < \frac{2^k\,k\,!}{p^k} \cdot \frac{1}{n^{k-c}} \quad \text{for } n > 2k. \qquad \ldots(1)$$

Since $k - c > 0$, therefore, $\dfrac{1}{n^{k-c}} \to 0$ as $n \to \infty$.

Hence from (1), we get $$\lim_{n \to \infty} \frac{n^c}{(1+p)^n} = 0.$$

Example 40: *The usual definition of e is given by* $e = \sum_{n=0}^{\infty} \dfrac{1}{n\,!}$. *Show that e is irrational.*

Solution: Let $s_n = 1 + \dfrac{1}{1!} + \dfrac{1}{2\,!} + \dfrac{1}{3\,!} + \ldots + \dfrac{1}{n\,!}$.

Then $$e - s_n = \frac{1}{(n+1)\,!} + \frac{1}{(n+2)\,!} + \frac{1}{(n+3)\,!} + \ldots\ldots$$

$$< \frac{1}{(n+1)!}\left[1+\frac{1}{n+1}+\frac{1}{(n+1)^2}+\ldots\right]$$

$$= \frac{1}{(n+1)!}\cdot\frac{1}{1-\frac{1}{n+1}} = \frac{1}{n(n!)}\cdot$$

Thus $$0 < e - s_n < \frac{1}{n(n!)}\cdot \qquad \ldots(1)$$

Let, if possible, e be rational. Then e can be put in the form p/q where p and q are positive integers.

From (1), we have $0 < e - s_q < \dfrac{1}{q(q!)}$

or $$0 < (q!)(e - s_q) < \frac{1}{q} \qquad \ldots(2)$$

Now $$(q!)\, s_q = (q!)\left\{1+1+\frac{1}{2!}+\ldots+\frac{1}{q!}\right\}$$

is an integer. Also, by our assumption $e(q!)$ is an integer. It follows that $e(q!) - s_q(q!) = q!(e - s_q)$ is an integer. Since $q \geq 1$, therefore, (2) shows the existence of an integer between 0 and 1 which is absurd. Hence our initial assumption is wrong. So e is irrational.

Comprehensive Exercise 1

1. Write a formula or formulae for the nth term s_n for each of the following sequences :

(a) $1, -4, 9, -16, 25, -36, \ldots$ (b) 1, 0, 1, 0, 1, 0, ...

(c) 1, 3, 6, 10, 15, ...

2. Which of the sequences (a), (b), (c) in the above problem are subsequences of the sequence $< s_n >$ defined by $s_n = n$?

3. Find whether the following sequences are bounded above or below :

(i) $< \dfrac{(-1)^n}{n} >$ (ii) $< 2^n >$ (iii) $< n! >$.

4. Are the sequences $< s_n >$ defined as follows, bounded ?

(i) $s_n = 1 + \dfrac{(-1)^n}{n}$ (ii) $s_n = \left(1+\dfrac{1}{n}\right)^n$

(iii) $s_n = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \ldots + \dfrac{1}{n!}\cdot$

(iv) $s_n = 1$, if n is divisible by 3 and $s_n = 0$, otherwise.

5. Use the definition of the limit of a sequence to show that the limit of the sequence $< s_n >$ where $s_n = 2n/(n+3)$, is 2.

6. Show that the sequence $< s_n >$ where $s_n = n / (n+1)$ converges to 1.
7. If the sequence $< s_n >$ converges to l, then prove that the sequence $| s_n |$ converges to $| l |$.
8. Show by considering the sequence $< s_n = (-1)^n >$ that $<| s_n |>$ may converge but $< s_n >$ may not.
9. If $\lim s_n = l$ and $s_n \leq m$ for all $n \in \mathbf{N}$, prove that $l \leq m$.
10. Let $< s_n >$ be a sequence such that $< s_n^2 >$ converges to zero. Is it necessary that $< s_n >$ should converge to zero ?
11. If $s_n = \dfrac{2n}{n + 4n^{1/2}}$, prove that $< s_n >$ is convergent.
12. If $s_n = \dfrac{n}{2^n}$, prove that $< s_n > \to 0$.
13. If $< s_n >$ converges to $l \ (\neq 0)$, prove that $< (-1)^n s_n >$ oscillates.
14. If $< s_n >$ diverges and $c \ (\neq 0) \in \mathbf{R}$, prove that $< cs_n >$ diverges.
15. Show that the sequence $< n + (-1)^n n >$ oscillates infinitely.
16. Show that $< s_n >$ converges to e, where s_n is

    (i) $\left(1 + \dfrac{1}{n}\right)^{n+1}$ (ii) $\left(1 + \dfrac{1}{n+1}\right)^{n}$ (iii) $\left(1 - \dfrac{1}{n}\right)^{-n}$.
17. Show that the sequence $< s_n >$, where $s_n = \left(1 + \dfrac{2}{n}\right)^n$, converges to e^2.
18. If $s_n = \dfrac{n^3 - 2n + 1}{n^3 + 2n^2 - 1}$, prove that $\lim s_n = 1$.
19. If $s_n = \dfrac{(3n-1)(n^4 - n)}{(n^2+2)(n^3+1)}$, prove that $\lim s_n = 3$.
20. Prove that the sequence $< \dfrac{n^2 + 3n + 5}{2n^2 + 5n + 7} >$ converges to $\dfrac{1}{2}$.
21. Prove that the sequence $< s_n >$ where $s_n = \dfrac{n}{n^2 + 1}$ is convergent.
22. Show that the sequence $< s_n >$ defined by
$$s_n = \frac{1}{\sqrt{(n^2+1)}} + \frac{1}{\sqrt{(n^2+2)}} + \ldots + \frac{1}{\sqrt{(n^2+n)}} \text{ converges to } 1.$$
23. Prove that $\lim \left[\dfrac{1}{\sqrt{(2n^2+1)}} + \dfrac{1}{\sqrt{(2n^2+2)}} + \ldots + \dfrac{1}{\sqrt{(2n^2+n)}}\right] = \dfrac{1}{\sqrt{2}}$.
24. Show that $\lim \left[\dfrac{1}{(n+1)^{\lambda}} + \dfrac{1}{(n+2)^{\lambda}} + \ldots + \dfrac{1}{(2n)^{\lambda}}\right] = 0, \lambda > 1$.
25. A sequence $< s_n >$ is defined as follows :
$$s_1 = a > 0, s_{n+1} = \sqrt{\{(ab^2 + s_n^2) / (a+1)\}}, b > a, n \geq 1.$$
    Show that $< s_n >$ is a bounded monotonically increasing sequence and $\lim s_n = b$.

26. Prove that $\lim\limits_{n \to \infty} \left\{ \dfrac{(3n)!}{(n!)^3} \right\}^{1/n} = 27.$

27. Prove that $\lim \left[\dfrac{(n!)^{1/n}}{n} \right] = \dfrac{1}{e}$.

28. (i) Let $< s_n >$ be a sequence defined as follows :

$$s_1 = \frac{3}{2}; s_{n+1} = 2 - \frac{1}{s_n}, n \geq 1.$$

Show that $< s_n >$ is monotonic and bounded. Find the limit of the sequence.

(ii) Show that the sequence $< s_n >$ defined by the formula $s_1 = 1$, $s_{n+1} = \sqrt{(3s_n)}$ converges to 3.

29. If $< s_n >$ is a sequence such that $s_n > 0$ and $s_{n+1} \leq k\, s_n$ for all $n \geq m$ and $0 < k < 1, m$ being a fixed positive integer, then $\lim s_n = 0$.

30. Prove that if x be any real number, then $\lim\limits_{n \to \infty} \dfrac{x^n}{n!} = 0.$

31. If $s_n = 1 + \dfrac{1}{3} + \dfrac{1}{3^2} + \ldots + \dfrac{1}{3^n}$, prove that $< s_n >$ converges. Also find $\sup < s_n >$ and $\inf < s_n >$.

32. What do you understand by a monotonic sequence ? Prove that the sequence $< s_n >$, where $s_n = \dfrac{2n-7}{3n+1}$, is :

(i) monotonic increasing (ii) bounded above.

(iii) bounded below.

Also show that the sequence $< s_n >$ is convergent and find $\sup < s_n >$ and $\inf < s_n >$.

33. If $s_n = \dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \ldots + \dfrac{1}{n(n+1)}$, prove that $< s_n >$ is increasing and convergent. Find also $\sup < s_n >$ and $\inf < s_n >$.

34. If $s_n = 1 + \dfrac{1}{2} + \ldots + \dfrac{1}{n} - \log n$, prove that $< s_n >$ is decreasing and bounded.

35. Test the sequence $< s_n >$, defined by $s_n = (-1)^n\, n$, for limit points.

36. Find the limit points of the sequence $< s_n >$ defined by

$$s_n = (-1)^n \left(1 + \frac{1}{n}\right).$$

37. State and prove Cauchy's general principle of convergence for real sequences. Hence prove that the sequence $< s_n >$, where $s_n = \dfrac{n+1}{n}$ converges.

38. Show, by applying Cauchy's convergence criterion, that the sequence $< s_n >$ defined by

$$s_n = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} \text{ does not converge.}$$

39. If the sequence $< s_n >$ converges and if $< t_n >$ is a sequence such that

$$|t_n - t_m| \leq |s_n - s_m|,$$

for all positive integers m and n, then prove that $< t_n >$ converges.

40. If $< s_n >$ is a Cauchy sequence of real numbers which has a subsequence converging to l, prove that $< s_n >$ itself converges to l.

41. Show that $\lim [(n!)(a/n)^n] = 0$ or $+\infty$ according as $a < e$ or $a > e$, where a is any non-negative real number.

42. If $0 < u_1 < u_2$ and $u_n = \dfrac{2u_{n-1}\, u_{n-2}}{u_{n-1} + u_{n-2}}$

(*i.e.*, u_n is the harmonic mean of u_{n-1} and u_{n-2}), show that

$$\lim u_n = 3u_1 u_2 / (2u_1 + u_2).$$

43. If the sequences $< s_n >$ and $< t_n >$ converge to zero and if $< t_n >$ is a strictly decreasing sequence so that $t_{n+1} < t_n \ \forall\, n \in \mathbf{N}$, then

$$lim\, \frac{s_n}{t_n} = lim\, \frac{s_n - s_{n+1}}{t_n - t_{n+1}}$$

provided that the limit on the right exists, whether finite or infinite.

44. If $s_n = \dfrac{a}{1 + s_{n-1}}$, where a, s_n are positive, show that the sequence $< s_n >$ tends to a definite limit l, the positive root of the equation $x^2 + x = a$.

45. Prove that the set of limit points of every sequence is a closed set.

Answers 1

1. (a) $s_n = (-1)^{n-1} n^2$ (b) $s_n = 1$ if n is odd, $s_n = 0$ if n is even,

(c) $s_n = \dfrac{n(n+1)}{2}$

2. (c).

3. (i) Bounded above as well as bounded below

(ii) Bounded below but not above

(iii) Bounded below but not above

4. (i) Yes (ii) Yes (iii) Yes (iv) Yes

10. Yes **28.** (i) 1

35. No limit point **36.** 1 and -1

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Every bounded monotonically increasing sequence converges to
   (a) its supremum (b) its infimum
   (c) 0 (d) 1

2. If $< s_n >$ is a sequence of non-negative numbers such that $\lim s_n = l$, then
   (a) $l < 0$ (b) $l > 0$
   (c) $l \geq 0$ (d) $l = 1$

3. Every subsequence of a convergent sequence is
   (a) divergent (b) convergent
   (c) may be convergent or divergent (d) oscillatory

4. The sequence $< s_n >$ where $s_n = \dfrac{5n}{n + 3n^{1/2}}$ has the limit
   (a) 3 (b) 1
   (c) $\dfrac{1}{3}$ (d) 5

5. The sequence $< 1, -1, 1, -1, 1, -1, \ldots >$ has
   (a) no limit point
   (b) only one limit point
   (c) two limit points
   (d) an infinite number of limit points

6. The sequence $< s_n >$, where $s_n = \left(1 + \dfrac{2}{n}\right)^{n+3}$, converges to
   (a) e (b) e^2
   (c) $e + 3$ (d) $e^2 + 3$

7. If $s_n = 1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \ldots + \dfrac{1}{2^n}$, then the sequence $< s_n >$ is
   (a) unbounded (b) convergent
   (c) divergent (d) oscillatory

8. The sequence $< s_n >$, where $s_n = \dfrac{3\,n^2 + 1}{3\,n^2 - 1}$, converges to

(a) 1 (b) 3
(c) −1 (d) 0

9. Every Cauchy sequence is
(a) oscillatory (b) divergent
(c) unbounded (d) convergent

10. The value of the limit $\lim\limits_{n \to \infty} \dfrac{1}{n}(1 + 2^{1/2} + 3^{1/3} + \ldots + n^{1/n})$ is equal to

(a) 1 (b) 0
(c) 2 (d) 3

11. Every convergent sequence is :
(a) oscillatory (b) unbounded
(c) bounded (d) oscillates finitely

(Rohilkhand 2011)

Fill In The Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. The range of the sequence $< (-1)^n >$ is the set … .

2. The sequence $< 1/n >$ converges to … .

3. If $s_n = \dfrac{1}{2^n}$ for all $n \in \mathbf{N}$, then

(i) sup $< s_n >$ = ... and (ii) inf $< s_n > = \ldots .$

4. The supremum of the sequence $< \dfrac{n}{n+1} >$ is … .

(Rohilkhand 2012)

5. The nth term of the sequence $< 1, -1, 1, -1, \ldots >$ is … .

6. If the subsequences $< s_{2n-1} >$ and $< s_{2n} >$ of the sequence $< s_n >$ converge to the same limit l, then $\lim\limits_{n \to \infty} s_n = \ldots .$

7. If $s_n = n \sqrt{n}$, then $\lim\limits_{n \to \infty} s_n = \ldots .$

8. The sequence $< -2, -4, -6, \ldots, -2n, \ldots >$ diverges to … .

9. If $\lim s_n = l$ and $\lim t_n = l'$, then $\lim (s_n\, t_n) = \ldots .$

10. If $\lim_{n \to \infty} s_n = l$, then $\lim_{n \to \infty} \dfrac{s_1 + s_2 + \ldots + s_n}{n} = \ldots.$

11. Every bounded monotonic sequence is … .

12. The sequence $< \dfrac{n^2 + 4n + 7}{3n^2 + 5n - 9} >$ converges to … .

13. If $s_n = \left(1 + \dfrac{1}{n}\right)^{n+2}$, then $\lim_{n \to \infty} s_n = \ldots.$

14. If $s_n = \sqrt{(n+1)} - \sqrt{n}$, then $\lim_{n \to \infty} s_n = \ldots.$

15. If $s_n = \left(1 + \dfrac{1}{n}\right)^{n}$, then $\lim_{n \to \infty} (s_1 s_2 \ldots s_n)^{1/n} = \ldots.$

True or False

Write 'T' for true and 'F' for false statement.

1. The sequence $< 2, -2, 2, -2, 2, -2, \ldots >$ is bounded.
2. The sequence $< 1, -1, 1, -1, 1, -1, \ldots >$ is convergent.
3. The sequence $< 3, -3, 3, -3, 3, -3, \ldots >$ is a Cauchy sequence.
4. Every Cauchy sequence is always convergent.
5. Every subsequence of a divergent sequence is always divergent.
6. A sequence $< s_n >$ is said to converge to a number l, if for any given $\varepsilon > 0$ there exists a positive integer m such that $|s_n - l| > \varepsilon$ for all $n \geq m$.
7. If a sequence $< s_n >$ is convergent, then $\lim_{n \to \infty} s_n$ is unique.
8. Every convergent sequence is always bounded.
9. Every bounded sequence is always convergent.
10. The sequence $< 1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \ldots, \dfrac{1}{n}, \ldots >$ is a Cauchy sequence.
11. If $\lim s_n = l$ and $\lim t_n = l'$, then $\lim (s_n + t_n) = l + l'$.
12. Every bounded monotonically decreasing sequence converges to its infimum.
13. A monotonically decreasing sequence which is not bounded below diverges to minus infinity.

14. A monotonically increasing sequence which is not bounded above is a convergent sequence.

15. Every bounded sequence has at least one limit point.

16. The sequence $< 1, -1, 1, -1, \ldots >$ has no limit points.

17. If a sequence $< s_n >$ is convergent, then it may or may not be a Cauchy sequence.

18. The sequence $< s_n >$, where $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$, is a Cauchy sequence.

19. The sequence $< s_n >$, where $s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{(n-1)!}$, is a convergent sequence.

20. If a sequence $< s_n >$ is convergent, then the sequence $< |s_n| >$ is also convergent.

21. If a sequence $< |s_n| >$ is convergent, then the sequence $< s_n >$ is also convergent.

22. The sequence $< s_n >$, where $s_n = \frac{n}{n^3 + 1}$, converges to 0.

23. The sequence $< s_n >$, where $s_n = \left(1 + \frac{1}{n}\right)^n$, is a bounded sequence.

24. The sequence $< 1, -1, 1, -1, 1, -1, \ldots >$ is a monotonic sequence.

25. The sequence $< s_n >$, where $s_n = 3 - \frac{1}{3^{n-1}}$, converges to 2 .

Answers

Multiple Choice Questions

1. (a)	2. (c)	3. (b)	4. (d)	5. (c)
6. (b)	7. (b)	8. (a)	9. (d)	10. (a)
11. (c)				

Fill in the Blank(s)

1. $\{1, -1\}$	2. 0	3. (i) $\frac{1}{2}$ (ii) 0	4. 1	5. $(-1)^{n-1}$
6. l	7. 1	8. $-\infty$	9. ll'	10. l
11. convergent	12. $\frac{1}{3}$	13. e	14. 0	15. e

True or False

1. *T*	2. *F*	3. *F*	4. *T*	5. *T*
6. *F*	7. *T*	8. *T*	9. *F*	10. *T*
11. *T*	12. *T*	13. *T*	14. *F*	15. *T*
16. *F*	17. *F*	18. *F*	19. *T*	20. *T*
21. *F*	22. *T*	23. *T*	24. *F*	25. *F*

□

Chapter

3

Infinite Series

1 Infinite Series

A*n expression of the form* $u_1 + u_2 + \ldots + u_n + \ldots$ *in which every term is followed by another according to some definite law is called a* **series.**

The series is called a **finite series**, if the number of terms is *finite*. Symbolically, the finite series $u_1 + u_2 + \ldots + u_n$ having n terms is denoted by $\sum\limits_{r=1}^{n} u_r$.

The series is called an infinite series, if the number of terms is infinite. Symbolically, the infinite series $u_1 + u_2 + \ldots + u_n + \ldots$ is denoted by $\sum\limits_{n=1}^{\infty} u_n$ or simply by $\Sigma\, u_n$.

Since we are going to deal with infinite series only, therefore we shall simply use the term '**series**' *to denote an infinite series.*

2 Convergence and Divergence of Series

Convergent Series : **(Gorakhpur 2011; Kashi 14)**

A series $\Sigma\, u_n$ is said to be convergent if S_n, the sum of its first n terms, tends to a definite finite limit S as n tends to infinity.

We write $S = \lim_{n \to \infty} S_n$.

The finite limit S to which S_n tends is called the sum of the series.

Divergent Series: *A series $\Sigma\, u_n$ is said to be divergent if S_n, the sum of its first n terms, tends to either $+\infty$ or $-\infty$ as n tends to infinity,*

i.e., if $\lim_{n \to \infty} S_n = \infty$ *or* $-\infty$.

Oscillatory Series: *A series $\Sigma\, u_n$ is said to be an oscillatory series if S_n, the sum of its first n terms, neither tends to a definite finite limit nor to $+\infty$ or $-\infty$ as n tends to ∞.*

The series is said to *oscillate finitely,* if the value of S_n as $n \to \infty$ fluctuates within a finite range. It is said to *oscillate infinitely,* if S_n tends to infinity and its sign is alternately positive and negative.

Sequence of Partial Sums of a Series :

If S_n denotes the sum of the first n terms of the series $\Sigma\, u_n$, so that

$$S_n = u_1 + u_2 + \ldots + u_n,$$

then S_n is called the **partial sum** of the first n terms of the series and the sequence $< S_n > = < S_1, S_2, \ldots, S_n, \ldots >$ is called the **sequence of partial sums** of the given series. We can define the convergent, divergent and oscillatory series in terms of the sequence of partial sums.

Definition: *A series $\Sigma\, u_n$ is said to be convergent, divergent or oscillatory according as the sequence $< S_n >$ of its partial sums is convergent, divergent or oscillatory.*

If the sequence $< S_n >$ of partial sums of a series $\Sigma\, u_n$ converges to S then S is said to be the sum of the series $\Sigma\, u_n$.

Note: Since the limits for infinite series will be taken as $n \to \infty$, so throughout this chapter we shall write $\lim_{n \to \infty}$ as 'lim' only.

Illustration 1:

The series $1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots + \left(\frac{2}{3}\right)^{n-1} + \ldots$ is convergent.

Here the given series is a geometric series with common ratio $2/3 < 1$.

$$\therefore \quad S_n = \frac{1 . \{1 - (2/3)^n\}}{1 - (2/3)} = 3\, \{1 - (2/3)^n\}.$$

Now, $\lim S_n = \lim 3\, \{1 - (2/3)^n\} = 3\,(1 - 0)$ $\quad [\because\ 2/3 < 1]$

$= 3$, a definite finite number.

Consequently the given series is convergent.

Illustration 2:

The series $1 + 2 + 3 + \ldots + n + \ldots$ is divergent.

Here, $S_n = 1 + 2 + 3 + \ldots + n = \frac{1}{2} n(n+1).$

$\therefore$ $\lim S_n = \lim \frac{1}{2} n(n+1) = \infty.$

Consequently the given series is divergent.

Illustration 3:

The series $2 - 2 + 2 - 2 + \ldots$ is oscillatory.

Here, $S_n = 0$ if n is even,

$= 2$, if n is odd.

Therefore, the sequence $< S_n >$ of partial sums of the series, and consequently the given series, is oscillatory.

Below we give some results which will be found useful and can be easily proved.

1. The nature of a series remains unaltered if
   (i) the signs of all the terms are changed;
   (ii) a finite number of terms are added or omitted;
   (iii) each term of the series is multiplied or divided by the same fixed number c which is not zero.
2. If Σu_n converges to A and Σv_n converges to B, then $\Sigma (u_n + v_n)$ converges to $A + B$.
3. If Σu_n converges to A and $c \in \mathbf{R}$, then $\Sigma c u_n$ converges to cA.
4. If Σu_n converges to A and Σv_n converges to B and $p, q \in \mathbf{R}$, then $\Sigma (pu_n + qv_n)$ converges to $pA + qB$.
5. If Σu_n diverges and $c \in \mathbf{R}, c \neq 0$, then Σcu_n diverges.
6. If Σu_n and Σv_n are two divergent series having all terms positive, then $\Sigma (u_n + v_n)$ also diverges.

3 A Necessary Condition for Convergence

For a series Σu_n to be convergent, it is necessary that $\lim u_n = 0$.

Or *For every convergent series Σu_n, we must have $\lim u_n = 0$.* **(Gorakhpur 2010, 13)**

Let the series Σu_n be convergent. Let S_n denote the sum of n terms of the series Σu_n.

Then $S_n = u_1 + u_2 + \ldots + u_n$ and $S_{n-1} = u_1 + u_2 + \ldots + u_{n-1}$.

$\therefore$ $u_n = S_n - S_{n-1}$. ...(1)

Since the series Σu_n is convergent, therefore, S_n and S_{n-1} both will tend to the same finite limit, say S, as $n \to \infty$.

Taking limits of both sides of (1), we get

$$\lim u_n = \lim S_n - \lim S_{n-1} = S - S = 0.$$

Hence for a convergent series, it is necessary that $\lim u_n = 0$.

Note: It is to be noted that the above condition is only necessary but not sufficient for a series to be convergent *i.e.,* if $\lim u_n = 0$, then the series Σu_n may or may not be convergent. **(Gorakhpur 2010, 13)**

For example, consider the series

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\ldots+\frac{1}{\sqrt{n}}+\ldots$$

Here $u_n=\dfrac{1}{\sqrt{n}}$, so that $\lim u_n=\lim\dfrac{1}{\sqrt{n}}=0$. But the series does not converge as shown below.

We have $S_n=1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\ldots+\dfrac{1}{\sqrt{n}}>\dfrac{1}{\sqrt{n}}+\dfrac{1}{\sqrt{n}}+\dfrac{1}{\sqrt{n}}+\ldots+\dfrac{1}{\sqrt{n}}=\dfrac{n}{\sqrt{n}}=\sqrt{n}$,

i.e., $S_n>\sqrt{n}$, which tends to infinity as n tends to infinity. Hence the series is divergent.

Again consider the geometric series $\dfrac{1}{2}+\left(\dfrac{1}{2}\right)^2+\left(\dfrac{1}{2}\right)^3+\ldots$, for which $\lim u_n=\lim\left(\dfrac{1}{2}\right)^n=0$ and the series is convergent.

Thus if $u_n\to 0$, we cannot say anything about the behaviour of the series but if u_n does not tend to zero, the series definitely does not converge. The more useful form of the above test is as follows:

If a series $\Sigma\, u_n$ be such that u_n does not tend to zero as n tends to infinity, then the series does not converge.

4 Cauchy's General Principle of Convergence for Series

Sometimes it is either impossible or difficult to find the sequence of partial sums of a given series and yet we want to know whether the series converges or not. Now we shall establish a fundamental principle, for dealing with the convergence of such series, known as *Cauchy's general principle of convergence.*

Theorem: *A necessary and sufficient condition for a series $\Sigma\, u_n$ to converge is that for each $\varepsilon>0$, there exists a positive integer m, such that*

$$|u_{m+1}+u_{m+2}+\ldots+u_n|<\varepsilon \text{ for all } n>m$$

Or $$|u_{p+1}+u_{p+2}+\ldots+u_q|<\varepsilon \text{ for all } q\ge p\ge m$$

Or $$|u_{n+1}+u_{n+2}+\ldots+u_{n+p}|<\varepsilon \text{ for all } n\ge m,\ p>0.$$

Proof: Let $<S_n>$ be the sequence of partial sums of the series $\Sigma\, u_n$. The series $\Sigma\, u_n$ will converge, iff the sequence $<S_n>$ of its partial sums converges. By Cauchy's general principle of convergence for sequences, we know that a necessary and sufficient condition for the convergence of $<S_n>$ is that for each $\varepsilon>0$, there exists $m\in\mathbf{N}$ such that

$$|S_n-S_m|<\varepsilon \text{ for all } n>m$$

i.e., $$|u_{m+1}+u_{m+2}+\ldots+u_n|<\varepsilon \text{ for all } n>m.$$

Hence the result.

Illustrative Examples

Example 1: *Discuss the convergence of a geometric series.* **(Gorakhpur 2015)**

Solution: Consider the geometric series

$$a + ax + ax^2 + ax^3 + \ldots + ax^{n-1} + \ldots. \qquad \ldots(1)$$

Let S_n be the sum of first n terms of the series (1).

$\therefore \qquad S_n = \dfrac{a(1-x^n)}{1-x}$ if $x < 1$ and $S_n = \dfrac{a(x^n-1)}{x-1}$ if $x > 1$.

Case I: When $|x| < 1$ *i.e.*, $-1 < x < 1$.

If $|x| < 1$, then $x^n \to 0$ as $n \to \infty$.

$$\therefore \qquad \lim S_n = \lim \frac{a(1-x^n)}{1-x} = \frac{a(1-0)}{1-x} = \frac{a}{1-x},$$

which is a definite finite number and therefore the series is convergent.

Case II: When $x = 1$.

If $x = 1$, then each term of the series (1) is a.

$\therefore \qquad S_n = a + a + \ldots$ to n terms $= na$.

$\therefore \quad \lim S_n = \infty$ or $-\infty$ according as a is positive or negative. Hence the series is divergent.

Case III: When $x > 1$.

If $x > 1$, then $x^n \to \infty$ as $n \to \infty$.

$$\therefore \qquad \lim S_n = \lim \frac{a(x^n-1)}{x-1} = \infty \text{ or } -\infty \text{ according as } a > \text{ or } < 0.$$

Hence the series is divergent.

Case IV: When $x = -1$.

If $x = -1$, then the series (1) becomes $\quad a - a + a - a + \ldots..$

The sum of n terms of the series is a or 0 according as n is odd or even.

Hence the series is an oscillatory series, the oscillation being finite.

Case V: When $x < -1$.

If $x < -1$, then $-x > 1$.

Let $r = -x$, then $r > 1$ and so $r^n \to \infty$ as $n \to \infty$.

$$\text{Now} \qquad S_n = \frac{a(1-x^n)}{1-x} = \frac{a\{1-(-r)^n\}}{1-(-r)} \qquad [\because x = -r]$$

$$= \frac{a(1+r^n)}{1+r} \quad \text{or} \quad \frac{a(1-r^n)}{1+r}, \text{ according as } n \text{ is odd or even.}$$

$\therefore \quad$ in this case $\lim S_n$ is ∞ or $-\infty$ according as n is odd or even, provided $a > 0$ and if $a < 0$ the results are reversed.

Therefore in this case the series is an oscillatory series, the oscillation being infinite.

Hence a geometric series whose common ratio is x is convergent if $|x| < 1$, *divergent if* $x \geq 1$ *and oscillatory if* $x \leq -1$.

Example 2: *Prove that the series* $\Sigma \dfrac{1}{4^n}$ *converges to* $\dfrac{1}{3}$.

Solution: Here $S_n = \dfrac{1}{4} + \dfrac{1}{4^2} + \dfrac{1}{4^3} + \ldots + \dfrac{1}{4^n} = \dfrac{\dfrac{1}{4}\left\{1 - \left(\dfrac{1}{4}\right)^n\right\}}{1 - \dfrac{1}{4}} = \dfrac{1}{3}\left(1 - \dfrac{1}{4^n}\right)$.

$$\therefore \qquad \lim S_n = \lim \frac{1}{3}\left(1 - \frac{1}{4^n}\right) = \frac{1}{3}. \qquad \left[\because \ \lim \frac{1}{4^n} = 0\right]$$

$\therefore$ the sequence $< S_n >$ converges to $\dfrac{1}{3}$ and hence $\Sigma\, u_n$ converges to $\dfrac{1}{3}$.

Example 3: *Test the convergence of the series*

$$\log_e 2 + \log_e \frac{3}{2} + \log_e \frac{4}{3} + \log_e \frac{5}{4} + \ldots.$$

Solution: Here, $S_n = \log_e 2 + \log_e \dfrac{3}{2} + \log_e \dfrac{4}{3} + \ldots + \log_e \left(\dfrac{n+1}{n}\right)$

$$= \log_e \left\{2 \cdot \frac{3}{2} \cdot \frac{4}{3} \ldots \frac{n+1}{n}\right\} = \log_e (n+1).$$

$$\therefore \qquad \lim S_n = \lim \log (n+1) = \log \infty = \infty.$$

Hence the given series is divergent.

Example 4: *Show that the series*

$$\sqrt{\left(\frac{1}{4}\right)} + \sqrt{\left(\frac{2}{6}\right)} + \ldots + \sqrt{\left[\frac{n}{2(n+1)}\right]} + \ldots$$

does not converge.

Solution: Here,

$$u_n = \sqrt{\left[\frac{n}{2(n+1)}\right]} = \frac{1}{\sqrt{2}} \sqrt{\left(\frac{n}{n+1}\right)} = \frac{1}{\sqrt{2}} \cdot \left[\frac{1}{1 + (1/n)}\right]^{1/2}.$$

$$\therefore \qquad \lim_{n \to \infty} u_n = \frac{1}{\sqrt{2}} \neq 0.$$

Hence the given series does not converge.

Example 5: *Show that the series* $\Sigma \dfrac{1}{n}$ *does not converge.*

Solution: Let the given series converge. Then for $\varepsilon = \dfrac{1}{4}$, by Cauchy's general principle of convergence, we can find a positive integer m such that

$$\frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{n} < \frac{1}{4} \text{ for all } n > m.$$

Taking $n = 2m$, we see that

$$\frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{n}=\frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{2m}$$

$$> m\,.\,\frac{1}{2m}=\frac{1}{2}\,.$$

Thus we get a contradiction. Hence the given series does not converge.

5 Series of Positive Terms (or Series of Non-negative Terms)

If $\Sigma\, u_n$ is a series of positive terms then $u_n > 0$ for all $n \in \mathbf{N}$.

The important aspect of this series is that its sequence of partial sums is increasing.

We have $S_n = u_1 + u_2 + \ldots + u_n$, then $S_n - S_{n-1} = u_n$.

Since $u_n > 0$ for all n, therefore we get $S_n - S_{n-1} > 0$ for all n, *i.e.,* $S_n > S_{n-1}$ for all n *i.e.,* the sequence $< S_n >$ is a monotonically increasing sequence.

Now a monotonic sequence can either converge or diverge but cannot oscillate. Hence, we have only two possibilities for a series of positive terms, either the series converges or it diverges.

We give some fundamental results for series of positive terms.

Theorem 1: *A series $\Sigma\, u_n$ of positive terms converges iff there exists a number K such that $u_1 + u_2 + \ldots + u_n < K$ for all n.*

Proof: First, suppose that there exists a number K such that

$$u_1 + u_2 + \ldots + u_n < K,\ \forall\, n \quad i.e.,\quad S_n < K,\ \forall\ n.$$

This shows that the sequence $< S_n >$ of partial sums of the series $\Sigma\, u_n$ is bounded above. Also, the sequence $< S_n >$ is an increasing sequence, since the series $\Sigma\, u_n$ is of positive terms. We know that every bounded monotonic sequence converges. Therefore $< S_n >$ converges and hence $\Sigma\, u_n$ converges.

Conversely, we assume that $\Sigma\, u_n$ converges. Then, the sequence $< S_n >$ of partial sums of the series converges. We know that every convergent sequence is bounded. Therefore $< S_n >$ is bounded and hence there exist real numbers k and K such that $k < S_n < K$, for all n.

It gives $S_n < K$ *i.e.,* $u_1 + u_2 + \ldots + u_n < K$, for all n.

Note: In the light of the above theorem, we conclude that to show that a series of positive terms converges, it is sufficient to show that the sequence of its partial sums is bounded. On the other hand, to show that a series of positive terms diverges, we have to show that the sequence of its partial sums is not bounded, *i.e.,* for any real number A, there exists a positive integer m such that $S_m > A$.

Theorem 2: *A series of positive terms is divergent if each term after a fixed stage is greater than some fixed positive number.*

Proof: Let each term of the series be greater than a fixed positive number. We can assume so because the convergence or divergence of the series is not affected by omitting a finite number of terms.

So let $\Sigma\, u_n$ be the given series of positive terms and let $u_n > k$ (a fixed positive number) for all n.

Now $\quad S_n = u_1 + u_2 + \ldots + u_n > nk$.

But $\quad \lim nk = \infty$.

$\therefore \quad \lim S_n = \infty$.

Hence the series $\Sigma\, u_n$ is divergent.

Corollary: *A series of positive terms is divergent if* $\lim u_n > 0$.

Proof: Let $\lim u_n = l$, where $l > 0$. Then for a given $\varepsilon > 0$, there exists a positive integer m such that

$$|u_n - l| < \varepsilon, \text{ for all } n \geq m$$

i.e., $$l - \varepsilon < u_n < l + \varepsilon, \text{ for all } n \geq m.$$

Let $l - \varepsilon = a$. Then a is a fixed positive number because ε can be taken as small as we please. For example take $\varepsilon = \frac{1}{2} l$.

Thus $u_n > a$ for all $n \geq m$. Hence the given series is divergent.

Theorem 3: *If each term of a series $\Sigma\, u_n$ of positive terms, does not exceed the corresponding term of a convergent series $\Sigma\, v_n$ of positive terms, then $\Sigma\, u_n$ is convergent.*

While, if each term of $\Sigma\, u_n$ exceeds (or equals) the corresponding term of a divergent series of positive terms, then $\Sigma\, u_n$ is divergent.

Proof: Let $u_n \leq v_n$ for all n.

Let S_n and S_n' be the sums of first n terms of the two series $\Sigma\, u_n$ and $\Sigma\, v_n$ respectively.

Then $S_n = u_1 + u_2 + \ldots + u_n$ and $S_n' = v_1 + v_2 + \ldots + v_n$.

Since $u_n \leq v_n \;\forall\; n$, therefore, $S_n \leq S_n'$.

But $\Sigma\, v_n$ is convergent, therefore $S_n' \to S'$ (a finite quantity) as $n \to \infty$.

$\therefore \quad \lim S_n \leq S'$ (a finite quantity).

$\therefore \quad S_n$ itself tends to a finite limit as $n \to \infty$.

Hence the series $\Sigma\, u_n$ is convergent.

Now if $u_n \geq v_n$, for all n, then $S_n \geq S_n'$.

But $\Sigma\, v_n$ is divergent, therefore $S_n' \to \infty$ as $n \to \infty$ and hence $S_n \to \infty$ as $n \to \infty$. Consequently $\Sigma\, u_n$ is divergent.

6 The Auxiliary Series $\Sigma\, 1/n^p$

The infinite series

$$\Sigma \frac{1}{n^p} \quad i.e., \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{n^p} + \ldots$$

is convergent if $p > 1$ and divergent if $p \leq 1$. **(Kumaun 2001; Avadh 05; Kanpur 07; Kashi 13; Rohilkhand 14; Agra 14)**

Proof: Case I: Let $p > 1$. Since the terms of the given series are all positive, we can group them as we like. Hence we write the given series

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots=\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)$$
$$+\left(\frac{1}{8^p}+\frac{1}{9^p}+\ldots+\frac{1}{15^p}\right)+\ldots \qquad \ldots(1)$$

Now since $p > 1$,

$\therefore \qquad 3 > 2 \Rightarrow 3^p > 2^p \Rightarrow 1/3^p < 1/2^p.$

$\therefore \qquad \frac{1}{2^p}+\frac{1}{3^p}<\frac{1}{2^p}+\frac{1}{2^p}$

or $\qquad \frac{1}{2^p}+\frac{1}{3^p}<\frac{2}{2^p}.$

Similarly $\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}=\frac{4}{4^p},$

$\qquad \frac{1}{8^p}+\frac{1}{9^p}+\ldots+\frac{1}{15^p}<\frac{8}{8^p},$

and so on.

Thus we observe that on being grouped as mentioned in (1), the given series is term by term

$$<\frac{1}{1^p}+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\ldots$$

But the series on the R.H.S. of the above inequality is a geometric series and is convergent since its common ratio is $2/2^p = 1/2^{p-1}$ which is less than 1 as $p > 1$. Thus the given series on being grouped as in (1) is term by term less than a convergent series. Consequently the given series is convergent when $p > 1$.

Case II: Let $p = 1$. Then we group terms of the given series as

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$$
$$=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\ldots+\frac{1}{16}\right)+\ldots \qquad \ldots(2)$$

Now as $3 < 4$, so $\frac{1}{3}>\frac{1}{4}$ or $\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}$

or $\qquad \frac{1}{3}+\frac{1}{4}>\frac{2}{4}$ *i.e.,* $\frac{1}{3}+\frac{1}{4}>\frac{1}{2}.$

Similarly, $\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2},$

$\qquad \frac{1}{9}+\frac{1}{10}+\ldots+\frac{1}{16}>\frac{1}{2}$, and so on.

Thus we observe that on being grouped as in (2), the given series is term by term

$$>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots. \qquad \ldots(3)$$

The series on the R.H.S. of (3) is divergent as the sum of the first n terms of the series

$$= 1 + (n-1) \cdot \frac{1}{2} = \frac{1}{2}(n+1), \text{ which tends to infinity as } n \to \infty.$$

Thus the given series on being grouped as in (2) is term by term greater than a divergent series.

Consequently the given series is divergent when $p = 1$.

Case III: Let $p < 1$. Then

$$\frac{1}{n^p} > \frac{1}{n} \text{ for } n = 2, 3, 4, \ldots .$$

In this case the given series

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$$

is term by term greater than the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots .,$$

which is a divergent series, as proved in case II.

Consequently the given series is divergent when $p < 1$.

Hence the proof is complete.

Now we shall give some tests to know whether the given series of positive terms is convergent or divergent without actually finding out the sum of its n terms.

7 Comparison Test

Theorem: First form: *Let Σu_n and Σv_n be two series of positive terms such that $u_n < Kv_n$ for all n, where K is a fixed positive number. Then if Σv_n converges, so does Σu_n, and if Σu_n diverges, then Σv_n also diverges.* **(Gorakhpur 2010)**

Proof: Since $u_n < Kv_n$ for all n,

$$\therefore \quad u_1 + u_2 + \ldots + u_n < K(v_1 + v_2 + \ldots + v_n), \ \forall \ n. \quad \ldots(1)$$

Now if Σv_n converges, then there must exist a positive real number A, such that

$$v_1 + v_2 + \ldots + v_n < A, \ \forall \ n. \quad \ldots(2)$$

From (1) and (2), we get

$$u_1 + u_2 + \ldots + u_n < K A, \ \forall \ n.$$

Thus the sequence of partial sums of the series Σu_n is bounded above and hence Σu_n converges.

To prove the other result, we assume that Σu_n diverges. Then for any positive real number B, there must exist a positive integer m such that

$$u_1 + u_2 + \ldots + u_n > BK, \text{ for all } n > m. \quad \ldots(3)$$

From (1) and (3), we get

$$v_1 + v_2 + \ldots + v_n > B, \text{ for all } n > m.$$

Hence the series Σv_n diverges.

Second form: *Let Σu_n and Σv_n be two series of positive terms and let k and K be positive real numbers such that*

$$kv_n < u_n < Kv_n, \text{ for all } n.$$

Then the series Σu_n and Σv_n converge or diverge together.

Proof: From $kv_n < u_n < Kv_n$, for all n, we get

$$kv_n < u_n \quad \text{or} \quad v_n < \left(\frac{1}{k}\right) u_n, \text{ for all } n.$$

Now applying the result proved in the first form of the comparison test, we conclude that

(i) if Σu_n converges, then Σv_n also converges.

(ii) if Σv_n diverges, then Σu_n also diverges.

Again, applying the result of the first form of the comparison test for the inequality $u_n < Kv_n$, we conclude that

(iii) if Σv_n converges, then Σu_n also converges.

(iv) if Σu_n diverges, then Σv_n also diverges.

The desired result now follows from (i), (ii), (iii) and (iv).

Third form: *Let Σu_n and Σv_n be two series of positive terms and let K be a positive number such that $u_n < Kv_n$ for all $n > m$, m being a fixed positive integer. Then if the series Σv_n be convergent, then the series Σu_n is also convergent and if the series Σu_n is divergent, then the series, Σv_n is also divergent.*

Proof: The above result follows from the result of the first form of the comparison test because the convergence or the divergence of a series remains unaffected by omitting a finite number of terms of the series.

Fourth form: *Let Σu_n and Σv_n be two series of positive terms and let k and K be positive real numbers such that $kv_n < u_n < Kv_n$ for all $n > m$, m being a fixed positive integer. Then the series Σu_n and Σv_n converge or diverge together.*

Proof: Since the omission of a finite number of terms of a series has no effect on its convergence or divergence, therefore,

(i) the series $u_1 + u_2 + \ldots$ and the series $u_{m+1} + u_{m+2} + \ldots$ converge or diverge together ; and

(ii) the series $v_1 + v_2 + \ldots$ and the series $v_{m+1} + v_{m+2} + \ldots$ converge or diverge together.

Again, $kv_n < u_n < Kv_n$ for all $n > m \quad \Rightarrow \quad kv_{m+p} < u_{m+p} < Kv_{m+p}$ for all $p \in \mathbf{N}$,

therefore, by the result of the second form of the comparison test, we have

(iii) the series $u_{m+1} + u_{m+2} + \ldots$ and the series $v_{m+1} + v_{m+2} + \ldots$ converge or diverge together.

Hence from (i), (ii) and (iii), we conclude that the series Σu_n and Σv_n converge or diverge together.

Fifth form: (Important from the point of view of application to the solution of problems): *Let Σu_n and Σv_n be two series of positive terms such that*

$$\lim_{n \to \infty} \frac{u_n}{v_n} = l \text{ **(finite and non-zero)**};$$

then both the series converge or diverge together i.e., the two series $\Sigma\, u_n$ and $\Sigma\, v_n$ are either both convergent or both divergent.

Proof: We have $\dfrac{u_n}{v_n} > 0$ for all n, therefore

$$\lim_{n \to \infty} \frac{u_n}{v_n} \geq 0 \quad i.e., \quad l \geq 0.$$

Since $l \neq 0$ (given), therefore, $l > 0$.

Choose $\varepsilon > 0$ in such a way that $l - \varepsilon > 0$.

Now $\displaystyle\lim_{n \to \infty} \frac{u_n}{v_n} = l \Rightarrow$ there exists $m \in \mathbf{N}$ such that

$$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon, \text{ for all } n > m. \qquad \ldots(1)$$

Since $v_n > 0 \;\; \forall \;\; n$, hence multiplying (1) throughout by v_n, we get

$$(l - \varepsilon)\, v_n < u_n < (l + \varepsilon)\, v_n, \text{ for all } n > m. \qquad \ldots(2)$$

Now if $\Sigma\, v_n$ is convergent then $\Sigma\, (l + \varepsilon)\, v_n$ is also convergent. In this case from (2), we see that $\Sigma\, u_n$ is term by term less than a convergent series $\Sigma\, (l + \varepsilon)\, v_n$ except possibly for a finite number of terms. Therefore the series $\Sigma\, u_n$ is also convergent.

Again if $\Sigma\, v_n$ is divergent then $\Sigma\, (l - \varepsilon)\, v_n$ is also divergent. In this case from (2), we see that $\Sigma\, u_n$ is term by term greater than a divergent series $\Sigma\, (l - \varepsilon)\, v_n$ except possibly for a finite number of terms. Therefore the series $\Sigma\, u_n$ is also divergent.

Hence the series $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Sixth form: *Let $\Sigma\, u_n$ and $\Sigma\, v_n$ be two series of positive terms such that*

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}, \textit{ for all } n \geq m.$$

Then $\Sigma\, v_n$ converges $\Rightarrow \Sigma\, u_n$ converges and $\Sigma\, u_n$ diverges $\Rightarrow \Sigma\, v_n$ diverges.

Proof: We have $\dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}}$, for all $n \geq m$.

Putting $n = m + 1, m + 2, \ldots, n - 1$ in the above inequality, we get

$$\frac{u_{m+1}}{u_{m+2}} > \frac{v_{m+1}}{v_{m+2}}, \quad \frac{u_{m+2}}{u_{m+3}} > \frac{v_{m+2}}{v_{m+3}}, \; \ldots, \; \frac{u_{n-1}}{u_n} > \frac{v_{n-1}}{v_n}.$$

Multiplying the corresponding sides of these inequalities, we get

$$\frac{u_{m+1}}{u_n} > \frac{v_{m+1}}{v_n}, \text{ for all } n > m,$$

i.e.,

$$u_n < \left(\frac{u_{m+1}}{v_{m+1}}\right) v_n, \text{ for all } n > m.$$

Now the result follows from the third form.

Note 1: From the point of view of applications, the third and the fifth forms of the comparison test are the most useful.

Note 2: The geometric series $\Sigma \dfrac{1}{r^n}$ and the auxiliary series $\Sigma \dfrac{1}{n^p}$ will play a prominent role for comparison.

Working rule for applying comparison test:

The v_n-method: Comparison test is usually applied when the nth term u_n of the given series $\Sigma\, u_n$ contains the powers of n only which may be positive or negative, integral or fractional. The auxiliary series $\Sigma\,(1/n^p)$ is chosen as the series $\Sigma\, v_n$. From article 6, we know that $\Sigma\,(1/n^p)$ is convergent if $p > 1$ and divergent if $p \le 1$.

Now the question arises that how to choose v_n ? For applying comparison test, it is necessary that $\lim \dfrac{u_n}{v_n}$ should be finite and non-zero. It will be so if we take $v_n = \dfrac{1}{n^{p-q}}$, where p and q are respectively the highest indices of n in the denominator and numerator of u_n when it is in the form of a fraction. If u_n, can be expanded in ascending powers of $1/n$, then to get v_n, we should retain only the lowest power of $1/n$. After making a proper choice of v_n, we find $\lim\,(u_n/v_n)$ which should come out to be finite and non-zero. Then the series $\Sigma\, u_n$ and $\Sigma\, v_n$ are either both convergent or both divergent. The whole procedure will be clear from the examples that follow article 8.

Illustrative Examples

Example 6: *Test for convergence the series*

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \ldots + \frac{1}{n^n} + \ldots\ldots$$

Solution: Since $n^n > 2^n$ for all $n > 2$, therefore, $\dfrac{1}{n^n} < \dfrac{1}{2^n}\cdot$

Here $\qquad u_n = \dfrac{1}{n^n}\cdot$ Let $v_n = \dfrac{1}{2^n}\cdot$

Since $u_n < v_n$ for all $n > 2$ and $\Sigma\, v_n$ is a convergent series (a geometric series with common ratio $\frac{1}{2}$), therefore, by the comparison test, the given series converges.

Example 7: *Test for convergence the series whose nth terms are*

(i) $\dfrac{\sqrt{n}}{n^2+1}$ **(Kumaun 2002; Kanpur 06; Meerut 13B; Agra 14)**

(ii) $\dfrac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}}$ **(Kanpur 2009; Meerut 13)**

(iii) $\dfrac{n^p}{(1+n)^q}\cdot$

Solution: (i) Here $u_n = \dfrac{\sqrt{n}}{n^2+1}\cdot$

Take $$v_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}},$$

i.e., the auxiliary series is $\Sigma\, v_n = \Sigma\, \dfrac{1}{n^{3/2}}\cdot$

Now $$\lim \frac{u_n}{v_n} = \lim \left\{\frac{\sqrt{n}}{n^2+1} \cdot n^{3/2}\right\} = \lim \frac{n^2}{n^2+1}$$

$$= \lim \frac{1}{1+(1/n^2)} = 1, \text{ which is finite and non-zero.}$$

Since the auxiliary series $\Sigma\, v_n = \Sigma\,(1/n^{3/2})$ is convergent $\left(p = \dfrac{3}{2} > 1\right)$, therefore, by comparison test the given series $\Sigma\, u_n$ is also convergent.

(ii) Here $$u_n = \frac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}} = \frac{n^{2/3}\,(2-1/n^2)^{1/3}}{n^{3/4}\,(3+2/n^2+5/n^3)^{1/4}}$$

$$= \frac{1}{n^{1/12}} \cdot \frac{(2-1/n^2)^{1/3}}{(3+2/n^2+5/n^3)^{1/4}}\cdot$$

Take $$v_n = \frac{1}{n^{1/12}}\cdot$$

Then $$\frac{u_n}{v_n} = \frac{(2-1/n^2)^{1/3}}{(3+2/n^2+5/n^3)^{1/4}}\cdot$$

$\therefore$ $$\lim \frac{u_n}{v_n} = \frac{2^{1/3}}{3^{1/4}}, \text{ which is finite and non-zero.}$$

Hence, by comparison test, $\Sigma\, u_n$ and $\Sigma\, v_n$ are either both convergent or both divergent. But the auxiliary series $\Sigma\, v_n$ is divergent because $p = 1/12 < 1$. Hence $\Sigma\, u_n$ is also divergent.

(iii) Here $$u_n = \frac{n^p}{(n+1)^q}\cdot$$

Take $$v_n = \frac{n^p}{n^q} = \frac{1}{n^{q-p}}\cdot$$

Now $$\lim \frac{u_n}{v_n} = \lim \left\{\frac{n^p}{(n+1)^q} \cdot n^{q-p}\right\} = \lim \frac{1}{(1+1/n)^q} = 1, \text{ which is finite and}$$ non-zero.

Therefore, by comparison test, $\Sigma\, u_n$ and $\Sigma\, v_n$ are either both convergent or both divergent.

But the auxiliary series $\Sigma\, v_n = \Sigma\, \dfrac{1}{n^{q-p}}$ is convergent if $q - p > 1$ *i.e.* if $p - q + 1 < 0$ and divergent if $q - p \leq 1$ *i.e.* if $p - q + 1 \geq 0$.

Hence by comparison test the given series Σu_n is convergent if $p-q+1<0$ and divergent if $p-q+1\geq 0$.

Example 8: *Test for convergence the series whose nth terms are*

(*i*) $\dfrac{1}{1+1/n}$ **(Avadh 2012)**

(*ii*) $\sin\dfrac{1}{n}$ **(Kanpur 2012; Gorakhpur 14)**

(*iii*) $\tan^{-1}\dfrac{1}{n}$. **(Kanpur 2008)**

Solution: (i) Here $u_n=\dfrac{1}{1+1/n}$.

We have, $\lim\limits_{n\to\infty} u_n=\lim\limits_{n\to\infty}\dfrac{1}{1+(1/n)}=1$, which is >0.

$\therefore$ the given series is divergent.

(ii) Here, $u_n=\sin\dfrac{1}{n}=\dfrac{1}{n}-\dfrac{1}{3\,!}\dfrac{1}{n^3}+\dfrac{1}{5\,!}\dfrac{1}{n^5}-\ldots\ldots$

Take $v_n=1/n$, since the lowest power of $1/n$ in u_n is $1/n$. The auxiliary series $\Sigma\, v_n=\Sigma\,(1/n)$ is divergent as here $p=1$.

Now $$\lim\frac{u_n}{v_n}=\lim\left(1-\frac{1}{3\,!}\cdot\frac{1}{n^2}+\frac{1}{5\,!}\cdot\frac{1}{n^4}-\ldots\right)=1,$$

which is finite and non-zero.

Hence by comparison test the given series is divergent.

(iii) Here, $u_n=\tan^{-1}\left(\dfrac{1}{n}\right)=\dfrac{1}{n}-\dfrac{1}{3n^3}+\dfrac{1}{5n^5}-\ldots$ $\left[\because \tan^{-1}x=x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\ldots\right]$

The lowest power of $1/n$ in u_n is $1/n$. Therefore, to apply the comparison test, the auxiliary series is taken as $\Sigma\, v_n=\Sigma\,(1/n)$.

Now, $\lim\dfrac{u_n}{v_n}=\lim\left(1-\dfrac{1}{3n^2}+\dfrac{1}{5n^4}-\ldots\right)=1$, which is finite and non-zero.

But the auxiliary series $\Sigma\, v_n=\Sigma\,(1/n)$ is divergent as here $p=1$.

Hence by comparison test the given series is divergent.

Example 9: *Test the convergence of the series*

$$\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\ldots\ldots$$ **(Kumaun 2000; Avadh 10)**

Solution: Here $u_n=\dfrac{n+1}{n^p}$. Take $v_n=\dfrac{n}{n^p}=\dfrac{1}{n^{p-1}}$.

Now $$\lim\frac{u_n}{v_n}=\lim\left(\frac{n+1}{n^p}\,.\,n^{p-1}\right)=\lim\left(1+\frac{1}{n}\right)=1,$$

which is finite and non-zero.

Hence by comparison test $\Sigma\, u_n$ and $\Sigma\, v_n$ are either both convergent or both divergent.

But the auxiliary series $\Sigma\, v_n = \Sigma\, \dfrac{1}{n^{p-1}}$ is convergent if $p - 1 > 1$ *i.e.*, $p > 2$, and divergent if $p - 1 \le 1$ *i.e.* if $p \le 2$.

Hence the given series $\Sigma\, u_n$ is convergent if $p > 2$ and divergent if $p \le 2$.

Example 10: *Test the convergence of the following series*

(i) $\quad 1 + \dfrac{1}{2^2} + \dfrac{2^2}{3^3} + \dfrac{3^3}{4^4} + \dfrac{4^4}{5^5} + \ldots$ **(Avadh 2014)**

(ii) $\quad \dfrac{1}{1 + \sqrt{2}} + \dfrac{2}{1 + 2\sqrt{3}} + \dfrac{3}{1 + 3\sqrt{4}} + \ldots$

Solution: (i) Omitting the first term, if the given series is denoted by $\Sigma\, u_n$, then

$$\Sigma\, u_n = \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + \ldots = \Sigma\, \frac{n^n}{(n+1)^{n+1}}\cdot$$

Here, $\quad u_n = \dfrac{n^n}{(n+1)^{n+1}}\cdot$ Take $v_n = \dfrac{n^n}{n^{n+1}} = \dfrac{1}{n}\cdot$

Now

$$\lim \frac{u_n}{v_n} = \lim \left\{ \frac{n^n}{(n+1)^{n+1}} \cdot n \right\}$$

$$= \lim \left\{ \frac{1}{(1 + 1/n)^n \cdot (1 + 1/n)} \right\} = \frac{1}{e}, \qquad \left[\because \lim \left(1 + \frac{1}{n}\right)^n = e \right]$$

which is finite and non-zero.

But the auxiliary series $\Sigma\, v_n = \Sigma\,(1/n)$ is divergent as here $p = 1$. Hence by comparison test the given series is divergent.

(ii) Here, $\quad u_n = \dfrac{n}{1 + n\sqrt{(n+1)}}\cdot$

Take $\quad v_n = \dfrac{n}{n\sqrt{n}} = \dfrac{1}{n^{1/2}}\cdot$

Now

$$\lim \frac{u_n}{v_n} = \lim \left\{ \frac{n}{1 + n\sqrt{(n+1)}} \cdot n^{1/2} \right\}$$

$$= \lim \left\{ \frac{1}{1/n^{3/2} + \sqrt{(1 + 1/n)}} \right\} = 1, \text{ which is finite and non-zero.}$$

Since the auxiliary series $\Sigma\, v_n = \Sigma\,(1/n^{1/2})$ is divergent as here $p = 1/2 < 1$, therefore, by comparison test the given series is divergent.

Example 11: *Test the following series for convergence whose nth terms are given by*

(i) $\quad (n^3 + 1)^{1/3} - n$ **(Meerut 2013)**

(ii) $\sqrt{(n^4 + 1)} - \sqrt{(n^4 - 1)}$. **(Kanpur 2006; Avadh 06, 14; Meerut 13B; Kashi 14)**

Solution: (i) Here, $u_n = (n^3 + 1)^{1/3} - n = (n^3)^{1/3}\ (1 + 1/n^3)^{1/3} - n$

$$= n\left[\left(1+\frac{1}{n^3}\right)^{1/3} - 1\right] = n\left[1 + \frac{1}{3}\cdot\frac{1}{n^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2\,!}\cdot\frac{1}{n^6} + \ldots - 1\right]$$

$$= \frac{1}{3n^2} - \frac{1}{9n^5} + \ldots.$$

Taking the lowest power of $1/n$ in u_n, the auxiliary series is given by

$$\Sigma\, v_n = \Sigma\,(1/n^2).$$

Now $$\lim \frac{u_n}{v_n} = \lim \left\{\left(\frac{1}{3n^2} - \frac{1}{9n^5} + \ldots\right).n^2\right\} = \lim\left(\frac{1}{3} - \frac{1}{9n^3} + \ldots\right) = \frac{1}{3},$$

which is finite and non-zero.

Since the auxiliary series $\Sigma\, v_n = \Sigma\,(1/n^2)$ is convergent as here $p = 2 > 1$, therefore by comparison test the given series $\Sigma\, u_n$ is also convergent.

(ii) Here $u_n = \sqrt{(n^4 + 1)} - \sqrt{(n^4 - 1)}$

$$= n^2\ [(1 + 1/n^4)^{1/2} - (1 - 1/n^4)^{1/2}]$$

$$= n^2\left[\left\{1 + \frac{1}{2}\cdot\frac{1}{n^4} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2\,!}\cdot\frac{1}{n^8} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3\,!}\cdot\frac{1}{n^{12}} + \ldots\right\}\right.$$
$$\left. - \left\{1 - \frac{1}{2}\cdot\frac{1}{n^4} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2\,!}\cdot\frac{1}{n^8} - \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3\,!}\cdot\frac{1}{n^{12}} + \ldots\right\}\right]$$

$$= n^2\left[2\left\{\frac{1}{2n^4} + \frac{1}{16n^{12}} + \ldots\right\}\right] = \frac{1}{n^2} + \frac{1}{8n^{10}} + \ldots.$$

The lowest power of $1/n$ in u_n is $1/n^2$. Therefore to apply the comparison test we take the auxiliary series as $\Sigma\, v_n = \Sigma\, 1/n^2$, which is convergent as $p = 2 > 1$.

Now $$\lim \frac{u_n}{v_n} = \lim\left[\left\{\frac{1}{n^2} + \frac{1}{8n^{10}} + \ldots\right\}.n^2\right]$$

$$= \lim\left\{1 + \frac{1}{8n^8} + \ldots\right\} = 1, \text{ which is finite and non-zero.}$$

Therefore, by comparison test, $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together. Since $\Sigma\, v_n$ is convergent, therefore, $\Sigma\, u_n$ is also convergent.

Alternate solution: We have $u_n = \sqrt{(n^4 + 1)} - \sqrt{(n^4 - 1)}$

$$= \frac{[\sqrt{(n^4+1)} - \sqrt{(n^4-1)}]\,[\sqrt{(n^4+1)} + \sqrt{(n^4-1)}]}{\sqrt{(n^4+1)} + \sqrt{(n^4-1)}}$$

$$= \frac{(n^4+1)-(n^4-1)}{\sqrt{(n^4+1)} + \sqrt{(n^4-1)}} = \frac{2}{\sqrt{(n^4+1)} + \sqrt{(n^4-1)}}$$

$$= \frac{1}{n^2} \cdot \frac{2}{\sqrt{[1+(1/n^4)]} + \sqrt{[1-(1/n^4)]}}.$$

Take $\quad v_n = \dfrac{1}{n^2}.$

Then $\quad \dfrac{u_n}{v_n} = \dfrac{2}{\sqrt{[1+(1/n^4)]} + \sqrt{[1-(1/n^4)]}}.$

$\therefore \quad \lim\limits_{n \to \infty} \dfrac{u_n}{v_n} = 1$ which is finite and non-zero.

Hence by comparison test $\Sigma\, u_n$ and $\Sigma\, v_n$ are either both convergent or both divergent.

But for $v_n = \dfrac{1}{n^2} = \dfrac{1}{n^p}$, $p = 2 > 1$.

$\therefore \quad \Sigma\, v_n$ is convergent and hence $\Sigma\, u_n$ is also convergent.

Example 12: *Test for convergence of the following series :*

(i) $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{(a+b/n)}}$. *(ii)* $\sum\limits_{n=1}^{\infty} \dfrac{1}{2^n + 3^n}$. *(iii)* $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}\left(\dfrac{n+2}{n+3}\right)^n$.

Solution: (i) Here, $u_n = \dfrac{1}{n^{(a+b/n)}} = \dfrac{1}{n^a \,.\, n^{b/n}}$. Let $v_n = \dfrac{1}{n^a}$.

Now $\quad \lim \dfrac{u_n}{v_n} = \lim \left[\dfrac{1}{n^a \,.\, n^{b/n}} \,.\, n^a\right] = \lim \left(\dfrac{1}{n^{b/n}}\right)$

$$= \lim \frac{1}{(n^{1/n})^b} = \frac{1}{(1)^b} \qquad \left[\because \lim_{n \to \infty} n^{1/n} = 1\right]$$

$= 1$, which is finite and non-zero.

We know that $\Sigma\, v_n = \Sigma\, (1/n^a)$ is convergent if $a > 1$ and divergent if $a \le 1$.

Hence by comparison test the given series $\Sigma\, u_n$ is convergent if $a > 1$ and divergent if $a \le 1$.

(ii) Here, $\quad u_n = \dfrac{1}{2^n + 3^n} = \dfrac{1}{3^n\left[1 + \left(\dfrac{2}{3}\right)^n\right]}.$

Take $\quad v_n = \dfrac{1}{3^n}.$

We know that $\Sigma\, v_n = \Sigma\, (1/3^n)$ is a geometric series with common ratio $1/3 < 1$, hence it is convergent.

Now $\quad \lim \dfrac{u_n}{v_n} = \lim \dfrac{1}{1 + \left(\dfrac{2}{3}\right)^n} = 1, \qquad [\because \lim r^n = 0, 0 < r < 1]$

which is finite and non-zero.

Hence by comparison test the given series $\Sigma\, u_n$ is convergent.

(iii) Here, $u_n = \dfrac{1}{n^3}\left(\dfrac{n+2}{n+3}\right)^n$.

Take $v_n = \dfrac{1}{n^3}$. Then $\Sigma\, v_n = \Sigma\, \dfrac{1}{n^3}$ is convergent as $p = 3 > 1$.

Now
$$\frac{u_n}{v_n} = \frac{1}{n^3}\left(\frac{n+2}{n+3}\right)^n \cdot n^3 = \left(\frac{n+2}{n+3}\right)^n = \frac{n^n\left(1+\frac{2}{n}\right)^n}{n^n\left(1+\frac{3}{n}\right)^n} = \frac{\left(1+\frac{2}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n}.$$

We know that $\lim \left(1+\dfrac{x}{n}\right)^n = e^x$.

$\therefore$
$$\lim \frac{u_n}{v_n} = \frac{e^2}{e^3} = \frac{1}{e},$$
which is finite and non-zero.

Hence by comparison test, $\Sigma\, u_n$ is convergent.

Example 13: *Test for convergence the series*

$$\frac{1}{1+2^{-1}} + \frac{2}{1+2^{-2}} + \frac{3}{1+2^{-3}} + \frac{4}{1+2^{-4}} + \ldots$$

Solution: Here, $u_n = \dfrac{n}{1+2^{-n}}$.

$\therefore$
$$\lim u_n = \lim \frac{n}{1+\left(\frac{1}{2}\right)^n} = \infty,$$

which is > 0. Also $\Sigma\, u_n$ is a series of positive terms.

Hence the given series $\Sigma\, u_n$ is divergent.

Comprehensive Exercise 1

Test for convergence the following series :

1. (i) $\dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \dfrac{1}{4\cdot 5} + \dfrac{1}{5\cdot 6} + \ldots\ldots$

(ii) $\dfrac{1\cdot 2}{3^2\cdot 4^2} + \dfrac{3\cdot 4}{5^2\cdot 6^2} + \dfrac{5\cdot 6}{7^2\cdot 8^2} + \ldots\ldots$

(Kumaun 2002; Meerut 12B)

(iii) $\dfrac{1}{1\cdot 2\cdot 3} + \dfrac{3}{2\cdot 3\cdot 4} + \dfrac{5}{3\cdot 4\cdot 5} + \ldots\ldots$

(Avadh 2011; Meerut 12)

(iv) $\dfrac{(1+a)(1+b)}{1.2.3} + \dfrac{(2+a)(2+b)}{2.3.4} + \dfrac{(3+a)(3+b)}{3.4.5} + \ldots\ldots$

2. (i) $\dfrac{1}{\sqrt{1}+\sqrt{2}}+\dfrac{1}{\sqrt{2}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{4}}+\ldots..$

(ii) $\dfrac{\sqrt{2}-1}{3^3-1}+\dfrac{\sqrt{3}-1}{4^3-1}+\dfrac{\sqrt{4}-1}{5^3-1}+\dfrac{\sqrt{5}-1}{6^3-1}+\ldots..$

(iii) $1+\dfrac{1}{2\cdot 2^{1/100}}+\dfrac{1}{3\cdot 3^{1/100}}+\dfrac{1}{4\cdot 4^{1/100}}+\ldots..$

3. (i) $\sum\sqrt{\left(\dfrac{n}{n^5+2}\right)}$. (ii) $\sum\dfrac{1}{(2n-1)^p}$.

(iii) $\sum\left(\dfrac{1}{\sqrt{n}}\sin\dfrac{1}{n}\right)$. (iv) $\sum\cos\dfrac{1}{n}$. **(Kanpur 2007)**

4. (i) $\Sigma\left[\sqrt{(n+1)}-\sqrt{n}\right]$. (ii) $\Sigma\left[\sqrt{(n^2+1)}-n\right]$.

(iii) $\Sigma\left[\sqrt{(n^3+1)}-\sqrt{n^3}\right]$. (iv) $\Sigma\left[\sqrt{(n^4+1)}-n^2\right]$.

5. (i) $\sqrt{\left(\dfrac{1}{2^3}\right)}+\sqrt{\left(\dfrac{2}{3^3}\right)}+\sqrt{\left(\dfrac{3}{4^3}\right)}+\ldots.$

(ii) The series whose nth term is $\dfrac{1}{n}\sin\dfrac{1}{n}$. **(Kanpur 2005)**

Answers 1

1. (i) Convergent (ii) Convergent
(iii) Convergent (iv) Divergent

2. (i) Divergent (ii) Convergent
(iii) Convergent

3. (i) Convergent
(ii) Convergent if $p>1$, divergent if $p\leq 1$
(iii) Convergent (iv) Divergent

4. (i) Divergent (ii) Divergent
(iii) Convergent (iv) Convergent

5. (i) Divergent (ii) Convergent

8 Cauchy's Root Test

Theorem 1: *Let Σu_n be a series of positive terms such that $\lim u_n^{1/n}=l$. Then*

(i) Σu_n converges, if $l<1$; *(ii) Σu_n diverges, if $l>1$;*

(iii) the test fails and the series may either converge or diverge, if $l=1$.

(Here $u_n^{1/n}$ stands for positive nth root of u_n).

(Kumaun 2001; Kanpur 04, 07; Avadh 06; Meerut 12; Gorakhpur 13)

Proof: Since $u_n > 0$, for all n, and $(u_n)^{1/n}$ stands for positive nth root of u_n, $\lim u_n^{1/n} = l \geq 0$.

Since $\lim u_n^{1/n} = l$, therefore for $\varepsilon > 0$ there exists a positive integer m, such that

$$|u_n^{1/n} - l| < \varepsilon, \text{ for all } n > m,$$

i.e., $$l - \varepsilon < u_n^{1/n} < l + \varepsilon, \text{ for all } n > m,$$

i.e., $$(l - \varepsilon)^n < u_n < (l + \varepsilon)^n, \text{ for all } n > m. \quad \ldots(1)$$

(i) Let $l < 1$.

Choose $\varepsilon > 0$, such that $r = l + \varepsilon < 1$.

Then $0 \leq l < r < 1$.

From (1), we get $u_n < (l + \varepsilon)^n$ for all $n > m$ *i.e.,* $u_n < r^n$ for all $n > m$.

Since $\Sigma\, r^n$ is a geometric series with common ratio r less than unity, $\Sigma\, r^n$ is convergent.

Therefore, by comparison test, $\Sigma\, u_n$ is convergent.

(ii) Let $l > 1$.

Choose $\varepsilon > 0$, such that $r = l - \varepsilon > 1$.

From (1), we get $(l - \varepsilon)^n < u_n$ for all $n > m$ *i.e.,* $u_n > r^n$ for all $n > m$.

Since $\Sigma\, r^n$ is a geometric series with common ratio greater than unity, $\Sigma\, r^n$ is divergent.

Therefore, by comparison test, $\Sigma\, u_n$ is divergent.

(iii) Let $l = 1$.

Consider the series $\Sigma\, u_n$, where $u_n = 1/n$.

Then $$u_n^{1/n} = \left(\frac{1}{n}\right)^{1/n}, \text{ so that } \lim u_n^{1/n} = 1. \quad [\text{Note that } \lim n^{1/n} = 1].$$

Since $\Sigma\,(1/n)$ diverges, hence, we observe that if

$$\lim u_n^{1/n} = 1, \text{ the series } \Sigma\, u_n \text{ may diverge.}$$

Now, consider the series $\Sigma\, u_n$, where $u_n = 1/n^2$.

In this case also, $\lim u_n^{1/n} = 1$.

Since $\Sigma\,(1/n^2)$ converges, hence, we observe that if $\lim u_n^{1/n} = 1$, the series $\Sigma\, u_n$ may converge.

Thus the above two examples show that Cauchy's root test fails to decide the nature of the series when $l = 1$.

Note 1: In general the Root test is used when powers are involved.

Another form of Cauchy's Root Test: The root test can also be stated in the form given below :

A series $\Sigma\, u_n$ of positive terms is convergent if for every value of $n \geq m$, m being finite, $(u_n)^{1/n}$ is less than a fixed number which is less than unity.

The series is divergent if $(u_n)^{1/n} \geq 1$ for every value of $n \geq m$.

Proof: **Case 1:** Given $(u_n)^{1/n} < r, \ \forall \, n \geq m$ where r is a fixed positive number such that $r < 1$.

$\therefore \qquad u_n < r^n$, for all $n \geq m$.

Since $\Sigma\, r^n$ is a geometric series with common ratio r less than unity, $\Sigma\, r^n$ is convergent.

Therefore, by comparison test, $\Sigma\, u_n$ is convergent.

Case 2: Given $(u_n)^{1/n} \geq 1, \forall \ n \geq m$.

$\therefore \qquad u_n \geq 1, \forall \, n \geq m$.

Omitting the first $m - 1$ terms of the series because it will not affect the convergence or divergence of the series, we have

$$u_n \geq 1, \forall \, n \in \mathbf{N}$$

$\Rightarrow \qquad S_n = u_1 + \ldots + u_n \geq n \quad \Rightarrow \quad \lim S_n = \infty$

$\Rightarrow \qquad$ the series is divergent.

Theorem 2: *Let $\Sigma\, u_n$ be a series of positive terms such that $u_n^{1/n} \to \infty$ as $n \to \infty$. Then $\Sigma\, u_n$ diverges.*

Proof: Let $r > 1$. Since $u_n^{1/n} \to \infty$ as $n \to \infty$, therefore, there exists a positive integer m such that $u_n^{1/n} > r$ for all $n \geq m \quad \Rightarrow \quad u_n > r^n$ for all $n \geq m$.

For $r > 1$, the geometric series $\Sigma\, r^n$ is divergent.

Hence, by comparison test, $\Sigma\, u_n$ is divergent.

Some important limits to be remembered:

1. $\lim\limits_{n \to \infty} n^{1/n} = 1$.

2. $\lim\limits_{n \to \infty} \dfrac{\log n}{n} = 0$.

3. $\lim\limits_{n \to \infty} \left(1 + \dfrac{x}{n}\right)^n = e^x$.

4. $\lim\limits_{n \to \infty} \left(1 + \dfrac{x}{n}\right)^p = 1$, if p is finite *i.e.*, if p is a fixed real number.

5. $\lim\limits_{n \to \infty} \left(1 + \dfrac{x}{n}\right)^{n+p} = e^x$, if p is finite.

6. $$\lim_{n \to \infty} \frac{a_0\, n^p + a_1 n^{p-1} + a_2 n^{p-2} + \ldots + a_{p-1}\, n + a_p}{b_0\, n^q + b_1 n^{q-1} + b_2\, n^{q-2} + \ldots + b_{q-1}\, n + b_q}$$

$$= \lim_{n \to \infty} \frac{n^p\, [a_0 + a_1\, (1/n) + a_2\, (1/n)^2 + \ldots.]}{n^q\, [b_0 + b_1\, (1/n) + b_2\, (1/n)^2 + \ldots]}$$

$$= \begin{cases} a_0 / b_0, \text{ if } p = q \\ 0, \text{ if } q > p \\ \infty, \text{ if } p > q \text{ and } a_0 > 0, b_0 > 0. \end{cases}$$

Illustrative Examples

Example 14: *Assuming that $n^{1/n} \to 1$ as $n \to \infty$, show by applying Cauchy's nth root test that the series $\sum_{n=1}^{\infty} (n^{1/n} - 1)^n$ converges.*

Solution: Here, $u_n = (n^{1/n} - 1)^n$.

$\therefore \quad u_n^{1/n} = n^{1/n} - 1.$

$\therefore \quad \lim_{n \to \infty} u_n^{1/n} = \lim_{n \to \infty} (n^{1/n} - 1) = 0 < 1.$

Hence, by Cauchy's root test, the given series converges.

Example 15: *Test the convergence of the following series*

(*i*) $\Sigma \left(1 + \frac{1}{n}\right)^{-n^2}$ (*ii*) $\Sigma \frac{x^n}{n!}$

(*iii*) $\frac{1^3}{3} + \frac{2^3}{3^2} + \frac{3^3}{3^3} + \frac{4^3}{3^4} + \ldots + \frac{n^3}{3^n} + \ldots$

Solution: (i) Here $u_n = \left(1 + \frac{1}{n}\right)^{-n^2}$.

$\therefore \quad u_n^{1/n} = \left(1 + \frac{1}{n}\right)^{-n}.$

$\therefore \quad \lim u_n^{1/n} = \lim \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e} < 1.$ $\quad [\because 2 < e < 3]$

Hence by Cauchy's root test the given series is convergent.

(ii) Here $u_n = \frac{x^n}{n!}$.

$\therefore \quad u_n^{1/n} = \frac{x}{(n!)^{1/n}}.$

$$\therefore \quad \lim u_n^{1/n} = \lim \frac{x}{(n!)^{1/n}} = \lim \left[\frac{n}{(n!)^{1/n}} \cdot \frac{x}{n}\right] = \lim \left[\frac{(n^n)^{1/n}}{(n!)^{1/n}} \cdot \frac{x}{n}\right]$$

$$= \lim \left[\left(\frac{n^n}{n!}\right)^{1/n} \cdot \frac{x}{n}\right] = e \,.\, \lim \frac{x}{n} \qquad \left[\because \lim \left(\frac{n^n}{n!}\right)^{1/n} = e\right]$$

$$= e \,.\, 0 = 0 < 1.$$

Hence by Cauchy's root test, the given series is convergent.

(iii) Here $u_n = \dfrac{n^3}{3^n}$.

$\therefore$ $u_n^{1/n} = \dfrac{n^{3/n}}{3}$.

$\therefore$ $\lim u_n^{1/n} = \lim \dfrac{1}{3} n^{3/n} = \dfrac{1}{3} \lim (n^{1/n})^3 = \dfrac{1}{3} \cdot 1 < 1.$

Hence by Cauchy's root test the given series is convergent.

Example 16: *Test the convergence of the series*

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \ldots$$

(Kumaun 2001; Meerut 13B)

Solution: Here $u_n = \left[\dfrac{(n+1)^{n+1}}{n^{n+1}} - \dfrac{n+1}{n}\right]^{-n}$.

$\therefore$ $$u_n^{1/n} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-1} = \left(\frac{n+1}{n}\right)^{-1} \left[\left(\frac{n+1}{n}\right)^{n} - 1\right]^{-1}$$

$$= \left(1 + \frac{1}{n}\right)^{-1} \left[\left(1 + \frac{1}{n}\right)^{n} - 1\right]^{-1}.$$

$\therefore$ $\lim u_n^{1/n} = (1+0)^{-1} (e-1)^{-1} = \dfrac{1}{e-1} < 1.$ $[\because 2 < e < 3]$

Hence by Cauchy's root test the given series is convergent.

Example 17: *Test for convergence* $\Sigma\, 3^{-n-(-1)^n}$.

Solution: Here $u_n = 3^{-n-(-1)^n} = \begin{cases} 3^{-n} . 3^{-1}, & \text{if } n \text{ is even} \\ 3^{-n} . 3, & \text{if } n \text{ is odd.} \end{cases}$

$\therefore$ $$u_n^{1/n} = \begin{cases} 3^{-1} . 3^{-1/n} = \dfrac{1}{3} \cdot \dfrac{1}{3^{1/n}}, & \text{if } n \text{ is even} \\ 3^{-1} . 3^{1/n} = \dfrac{1}{3} . 3^{1/n}, & \text{if } n \text{ is odd.} \end{cases}$$

$\therefore$ $\lim u_n^{1/n} = \dfrac{1}{3} < 1.$ $[\because \lim a^{1/n} = 1 \text{ if } a > 0]$

Hence by Cauchy's root test the given series is convergent.

Example 18: *Test for convergence* $\Sigma \left(\dfrac{n+1}{n+2}\right)^n . x^n, \ (x > 0).$ **(Meerut 2013)**

Solution: Here $u_n = \left(\dfrac{n+1}{n+2}\right)^n x^n.$

$\therefore \qquad u_n^{1/n} = \dfrac{n+1}{n+2} \cdot x.$

$$\therefore \qquad \lim u_n^{1/n} = \lim \left[\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)} \cdot x \right] = x.$$

$\therefore$ By Cauchy's root test, $\Sigma\, u_n$ converges if $x < 1$ and $\Sigma\, u_n$ diverges if $x > 1$.

For $x = 1$, the test fails. When $x = 1$, $u_n = \left(\dfrac{n+1}{n+2}\right)^n$.

$$\therefore \qquad \lim u_n = \lim \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} = \frac{e}{e^2} = \frac{1}{e} > 0.$$

$\therefore$ The series $\Sigma\, u_n$ diverges when $x = 1$.

Hence the given series converges if $x < 1$ and diverges if $x \geq 1$.

Example 19: *Test for convergence* $\Sigma \dfrac{1}{(\log n)^n}$.

Solution: Here $u_n = \dfrac{1}{(\log n)^n}$.

$\therefore \qquad u_n^{1/n} = \dfrac{1}{\log n}$.

$\therefore \qquad \lim u_n^{1/n} = \lim \dfrac{1}{\log n} = 0$, which is < 1.

Hence by Cauchy's root test the given series is convergent.

Comprehensive Exercise 2

Test for convergence the following series :

1. (i) $\sum \dfrac{1}{n^{1+(1/n)}}$. **(Kanpur 2008; Avadh 12)**

(ii) $\sum \left(1+\dfrac{1}{n}\right)^{n^2}$.

2. (i) $\sum \left(\dfrac{n}{n+1}\right)^{n^2}$. **(Avadh 2013; Kashi 14)**

(ii) $\sum \left(1+\dfrac{1}{\sqrt{n}}\right)^{-n^{3/2}}$.

3. (i) $2 + \dfrac{3}{2}x + \dfrac{4}{3}x^2 + \dfrac{5}{4}x^3 + \ldots$, where $x > 0$.

(ii) $\sum_{n=1}^{\infty} \frac{3n+1}{4n+3} x^n, x > 0.$

4. $\frac{2}{1^2} x + \frac{3^2}{2^3} x^2 + \frac{4^3}{3^4} x^3 + \ldots . + \frac{(n+1)^n}{n^{n+1}} x^n + \ldots ..$

5. (i) $1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^3}{4^3} + \ldots \infty, \ x > 0.$

(ii) $x + 2x^2 + 3x^3 + 4x^4 + \ldots .$

(iii) $\frac{1}{2} + \left(\frac{2}{3}\right) x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \ldots . \infty, x > 0.$

Answers 2

1. (i) Divergent (ii) Divergent
2. (i) Convergent (ii) Convergent
3. (i) Convergent if $x < 1$ and divergent if $x \geq 1$
   (ii) Convergent if $x < 1$ and divergent if $x \geq 1$
4. Convergent if $x < 1$ and divergent if $x \geq 1$
5. (i) Convergent
   (ii) Convergent if $x < 1$ and divergent if $x \geq 1$
   (iii) Convergent if $x < 1$ and divergent if $x \geq 1$

9 D' Alembert's Ratio Test

(Avadh 2003, 05; Kanpur 05; Gorakhpur 11; Meerut 12B; Kashi 14)

Theorem 1: *Let Σu_n be a series of positive terms such that* $\lim \frac{u_{n+1}}{u_n} = l$. *Then*

(i) Σu_n *converges if* $l < 1$ *(ii)* Σu_n *diverges if* $l > 1$

and (iii) the test fails to decide the nature of the series if $l = 1$.

Proof: Since $u_n > 0$, for all n, therefore

$$\frac{u_{n+1}}{u_n} > 0 \Rightarrow \lim \frac{u_{n+1}}{u_n} = l \geq 0.$$

Since $\lim \frac{u_{n+1}}{u_n} = l$, therefore, for $\varepsilon > 0$, there exists a positive integer m such that

$$\left| \frac{u_{n+1}}{u_n} - l \right| < \varepsilon, \text{ for all } n \geq m$$

i.e.,
$$l - \varepsilon < \frac{u_{n+1}}{u_n} < l + \varepsilon, \text{ for all } n \geq m.$$

Putting $n = m, m+1, \ldots, n-1$ in succession in the above inequality and multiplying the corresponding sides of the $(n-m)$ inequalities thus obtained, we get

$$(l-\varepsilon)^{n-m} < \frac{u_n}{u_m} < (l+\varepsilon)^{n-m} \text{ for all } n > m$$

i.e.,
$$(l-\varepsilon)^n \frac{u_m}{(l-\varepsilon)^m} < u_n < (l+\varepsilon)^n \frac{u_m}{(l+\varepsilon)^m} \text{ for all } n > m. \quad \ldots(1)$$

(i) Let $l < 1$.

Choose $\varepsilon > 0$ such that $r = l + \varepsilon < 1$.

Then $0 \le l < r < 1$.

From (1), we get $u_n < \left(\frac{u_m}{r^m}\right) r^n$ for all $n > m$

i.e., $u_n < \alpha\, r^n$ for all $n > m$ where $\alpha = \frac{u_m}{r^m} \in \mathbf{R}^+$.

Since $\Sigma\, r^n$ is a geometric series with common ratio less than unity, $\Sigma\, r^n$ is convergent.

Hence by comparison test, $\Sigma\, u_n$ is convergent.

(ii) Let $l > 1$.

Choose $\varepsilon > 0$ such that $r = l - \varepsilon > 1$.

From (1), we get $\frac{u_m}{r^m} r^n < u_n$, for all $n > m$

i.e., $u_n > \beta r^n$, for all $n > m$ where $\beta = \frac{u_m}{r^m} \in \mathbf{R}^+$.

Since $\Sigma\, r^n$ is a geometric series with common ratio greater than unity, $\Sigma\, r^n$ is divergent.

Therefore, by comparison test, $\Sigma\, u_n$ is divergent.

(iii) Let $l = 1$.

Consider the series $\Sigma\, u_n$ where $u_n = 1/n^2$.

Here
$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{n^2}{(n+1)^2} = \lim \frac{1}{\left(1+\frac{1}{n}\right)^2} = 1.$$

Since the series $\Sigma\,(1/n^2)$ converges, we observe that if $l = 1$, the series may be convergent.

Now, consider the series $\Sigma\, u_n$, where $u_n = 1/n$.

Here
$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{n}{n+1} = \lim \frac{1}{1+\frac{1}{n}} = 1\cdot$$

Since the series $\Sigma\,(1/n)$ diverges, we observe that if $l = 1$, the series may be divergent.

Thus the above two examples show that the test fails to decide the nature when $l = 1$.

Note 1: Taking the reciprocals, the ratio test can also be stated in the form given below.

The series $\Sigma\, u_n$ *of positive terms is convergent if* $\lim \dfrac{u_n}{u_{n+1}} > 1$ *and divergent if* $\lim \dfrac{u_n}{u_{n+1}} < 1$.

If $\lim \dfrac{u_n}{u_{n+1}} = 1$, the test fails.

We shall usually apply the ratio test in this form which will in the later part of this chapter be more convenient for further investigation in case the ratio test fails.

The ratio test is generally applied when the *n*th term of the series involves factorials, products of several factors, or combinations of powers and factorials.

Another form of D' Alembert's Ratio Test: The ratio test can also be stated in the form given below :

An infinite series of positive terms is convergent if from and after some term the ratio of each term to the preceding term is less than a fixed number which is less than unity.

The series is divergent if the above ratio is greater than or equal to unity.

Proof: Case 1: It is given that

$$\frac{u_{n+1}}{u_n} < r \text{ for all } n \geq m, \qquad \ldots(1)$$

where r is a fixed positive number such that $r < 1$.

To prove $\Sigma\, u_n$ is convergent.

Putting $n = m, m+1, \ldots, n-1$ in succession in (1) and multiplying the corresponding sides of the $n - m$ inequalities thus obtained, we get

$$\frac{u_{m+1}}{u_m} \cdot \frac{u_{m+2}}{u_{m+1}} \cdot \frac{u_{m+3}}{u_{m+2}} \cdot \ldots \cdot \frac{u_n}{u_{n-1}} < r^{n-m}$$

$$\Rightarrow \qquad \frac{u_n}{u_m} < r^{n-m} \quad \Rightarrow \quad u_n < \frac{u_m}{r^m}\, r^n$$

$$\Rightarrow \qquad u_n < \alpha\, r^n, \text{ for all } n > m \text{ where } \alpha = \frac{u_m}{r^m} \in \mathbf{R}^+.$$

Since $\Sigma\, r^n$ is a geometric series with common ratio less than unity, $\Sigma\, r^n$ is convergent. Hence by comparison test, $\Sigma\, u_n$ is also convergent.

Case 2: It is given that

$$\frac{u_{n+1}}{u_n} \geq 1 \text{ for all } n \geq m. \qquad \ldots(2)$$

Putting $n = m, m+1, \ldots, n-1$ in succession in (2) and multiplying the corresponding sides of the $n - m$ inequalities thus obtained, we get

$$\frac{u_n}{u_m} \geq 1 \quad \Rightarrow \quad u_n \geq u_m \text{ for all } n > m.$$

Omitting the first m terms of the series because it will not affect the convergence or divergence of the series, we have

$$u_n \geq u_m \text{ for all } n \in \mathbf{N}$$

$\Rightarrow$ $S_n = u_1 + \ldots + u_n \geq n\, u_m$

$\Rightarrow$ $\lim S_n = \infty$

$\Rightarrow$ the series is divergent.

Theorem 2: *Let $\Sigma\, u_n$ be a series of positive terms such that $\dfrac{u_{n+1}}{u_n} \to \infty$ as $n \to \infty$. Then $\Sigma\, u_n$ diverges.*

Proof: Since $\dfrac{u_{n+1}}{u_n} \to \infty$ as $n \to \infty$, therefore, there exists a positive integer m such that

$$\frac{u_{n+1}}{u_n} > 2 \text{ for all } n \geq m \quad i.e., \quad u_{n+1} > 2\, u_n \text{ for all } n \geq m.$$

Replacing n by $m, m+1, m+2, \ldots, n-1$ and multiplying the $(n-m)$ inequalities, we get

$$u_n > 2^{n-m} \,.\, u_m \text{ for all } n > m$$

i.e.,
$$u_n > \left(\frac{u_m}{2^m}\right) 2^n \text{ for all } n > m.$$

Since the geometric series $\Sigma\, 2^n$ diverges, hence, by comparison test $\Sigma\, u_n$ diverges.

Note: In a similar manner it can be proved that $\Sigma\, u_n$ is convergent if

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \infty.$$

10 Remarks on the Ratio Test

It should be noted that D' Alembert's ratio test does not tell us anything about the convergence of the series $\Sigma\, u_n$ if we only know that $\dfrac{u_n}{u_{n+1}} > 1\; \forall\; n$.

If $u_n = \dfrac{1}{n}$, then $\dfrac{u_n}{u_{n+1}} = 1 + \dfrac{1}{n} > 1$ for all n while the series $\Sigma\, u_n$ is divergent. Also, for the convergence of the series $\Sigma\, u_n$ it is not necessary that $\dfrac{u_n}{u_{n+1}}$ should have a definite limit.

For a change in the order of the terms of a series of positive terms may affect the value of $\lim \dfrac{u_n}{u_{n+1}}$ but it does not affect the convergence of the series.

For example, let us consider the series

$$1 + x + x^2 + x^3 + \ldots \text{ where } 0 < x < 1. \qquad \ldots(1)$$

Changing the order of terms, the series becomes

$$x + 1 + x^3 + x^2 + x^5 + x^4 + \ldots \qquad \ldots(2)$$

Since the series (1) is convergent, therefore, the series (2) is also convergent. But in the series (2), the ratio $u_n \,/\, u_{n+1}$ is alternately x and $1 \,/\, x^3$ and consequently $\lim\, (u_n/u_{n+1})$ is not definite.

In comparison with Cauchy's root test, D'Alembert's ratio test is more useful since it is easier to apply than the root test because generally u_n / u_{n+1} is a simpler fraction than u_n. However **the root test is stronger than the ratio test**. To be more precise, whenever the ratio test indicates the nature of the series, the root test does too. But sometimes the ratio test does not apply while the root test succeeds.

Illustrative Examples

Example 20: *Test for convergence the following series :*

(*i*) $1+\dfrac{2^p}{2\,!}+\dfrac{3^p}{3\,!}+\dfrac{4^p}{4\,!}+\ldots$ **(Bundelkhand 2006)**

(*ii*) $\dfrac{1}{1+2}+\dfrac{2}{1+2^2}+\dfrac{3}{1+2^3}+\ldots$

Solution: (i) Here $u_n=\dfrac{n^p}{n\,!}\cdot$

$$\therefore \quad u_{n+1}=\frac{(n+1)^p}{(n+1)\,!}\cdot$$

Now
$$\frac{u_n}{u_{n+1}}=\frac{n^p}{n\,!}\cdot\frac{(n+1)\,!}{(n+1)^p}=\frac{(n+1)\,n^p}{(n+1)^p}=\frac{n+1}{(1+1/n)^p}\cdot$$

$$\therefore \quad \lim \frac{u_n}{u_{n+1}}=\lim\frac{n+1}{(1+1/n)^p}=\infty,$$

which is >1 for all values of p.

Hence by ratio test the series $\Sigma\, u_n$ is convergent.

(ii) Here $u_n=\dfrac{n}{1+2^n}\cdot$

$$\therefore \quad u_{n+1}=\frac{n+1}{1+2^{n+1}}\cdot$$

Now
$$\frac{u_n}{u_{n+1}}=\frac{n}{1+2^n}\cdot\frac{1+2^{n+1}}{n+1}=\frac{n\,.\,2^{n+1}\,(1+1/2^{n+1})}{2^n\,(1+1/2^n)\,.\,n\,(1+1/n)}$$

$$=\frac{2\,(1+1/2^{n+1})}{(1+1/2^n)\,(1+1/n)}\cdot$$

$$\therefore \quad \lim\frac{u_n}{u_{n+1}}=2\cdot\frac{(1+0)}{(1+0)\,(1+0)}=2, \text{ which is} > 1.$$

Therefore, by ratio test, the given series converges.

Example 21: *Test for convergence the series whose nth term is*

(*i*) $\dfrac{n^3+a}{2^n+a}$, (*ii*) $\dfrac{n\,!}{n^n}$, **(Purvanchal 2014)** (*iii*) $\sqrt{\left\{\dfrac{2^n-1}{3^n-1}\right\}}$.

Solution: (i) Here $u_n = \dfrac{n^3 + a}{2^n + a}$, $u_{n+1} = \dfrac{(n+1)^3 + a}{2^{n+1} + a}$.

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \frac{n^3 + a}{2^n + a} \cdot \frac{2^{n+1} + a}{(n+1)^3 + a}$$

$$= \frac{n^3 (1 + a/n^3) . 2^{n+1} (1 + a/2^{n+1})}{2^n (1 + a/2^n) . n^3 \{(1 + 1/n)^3 + a/n^3\}}$$

$$= 2 \cdot \frac{(1 + a/n^3)(1 + a/2^{n+1})}{(1 + a/2^n)\{(1 + 1/n)^3 + a/n^3\}}.$$

Now $$\lim \frac{u_n}{u_{n+1}} = 2 \cdot \frac{(1+0)(1+0)}{(1+0)\{(1+0)^3 + 0\}} = 2, \text{ which is } > 1.$$

Therefore, by ratio test the given series converges.

(ii) Here $u_n = \dfrac{n!}{n^n}$ so that $u_{n+1} = \dfrac{(n+1)!}{(n+1)^{n+1}}$.

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!} = \frac{(n+1)^{n+1}}{n^n . (n+1)} = \left(1 + \frac{1}{n}\right)^n.$$

$$\therefore \qquad \lim \frac{u_n}{u_{n+1}} = \lim \left(1 + \frac{1}{n}\right)^n = e, \text{ which is} > 1.$$

Therefore, by ratio test the given series converges.

(iii) Here $$u_n = \sqrt{\left(\frac{2^n - 1}{3^n - 1}\right)}, \; u_{n+1} = \sqrt{\left(\frac{2^{n+1} - 1}{3^{n+1} - 1}\right)}.$$

Now $$\frac{u_n}{u_{n+1}} = \sqrt{\left(\frac{2^n - 1}{3^n - 1} \cdot \frac{3^{n+1} - 1}{2^{n+1} - 1}\right)} = \sqrt{\left\{\frac{2^n (1 - 1/2^n) . 3^n (3 - 1/3^n)}{3^n (1 - 1/3^n) . 2^n (2 - 1/2^n)}\right\}}$$

$$= \sqrt{\left\{\frac{(1 - 1/2^n)(3 - 1/3^n)}{(1 - 1/3^n)(2 - 1/2^n)}\right\}}.$$

$$\therefore \qquad \lim \frac{u_n}{u_{n+1}} = \sqrt{\left(\frac{3}{2}\right)}, \text{ which is } > 1.$$

Therefore, by ratio test the given series converges.

Example 22: *Show that the series*

$$1 + \frac{\alpha + 1}{\beta + 1} + \frac{(\alpha + 1)(2\alpha + 1)}{(\beta + 1)(2\beta + 1)} + \frac{(\alpha + 1)(2\alpha + 1)(3\alpha + 1)}{(\beta + 1)(2\beta + 1)(3\beta + 1)} + \ldots..$$

converges if $\beta > \alpha > 0$ *and diverges if* $\alpha \geq \beta > 0$ $[\alpha > 0, \beta > 0]$.

Solution: Here,

$$u_n = \frac{(\alpha+1)(2\alpha+1)\ldots\ldots[(n-1)\alpha+1]}{(\beta+1)(2\beta+1)\ldots\ldots[(n-1)\beta+1]},$$

so that $$u_{n+1} = \frac{(\alpha+1)(2\alpha+1)\ldots\ldots[(n-1)\alpha+1](n\alpha+1)}{(\beta+1)(2\beta+1)\ldots\ldots[(n-1)\beta+1](n\beta+1)}.$$

Now $$\frac{u_n}{u_{n+1}} = \frac{n\beta+1}{n\alpha+1} = \frac{\beta+1/n}{\alpha+1/n}.$$

$\therefore$ $$\lim \frac{u_n}{u_{n+1}} = \lim \frac{\beta+1/n}{\alpha+1/n} = \frac{\beta}{\alpha}.$$

Hence by ratio test the series is convergent if $\frac{\beta}{\alpha} > 1$ *i.e.,* if $\beta > \alpha > 0$, divergent if $\frac{\beta}{\alpha} < 1$, *i.e.,* if $\alpha > \beta > 0$, and the test fails if $\frac{\beta}{\alpha} = 1$ *i.e.,* if $\beta = \alpha$.

When $\beta = \alpha$, then the given series becomes

$$1 + 1 + 1 + \ldots\ldots$$

S_n = the sum of n terms of this series = n.

Since $\lim S_n = \infty$, hence the series is divergent.

Thus the given series is convergent if $\beta > \alpha > 0$ and divergent if $\alpha \geq \beta > 0$.

Example 23: *Test for convergence the following series* :

(*i*) $1 + 3x + 5x^2 + 7x^3 + \ldots$ (*ii*) $1 + \frac{x}{2^2} + \frac{x^2}{3^2} + \frac{x^3}{4^2} + \ldots.$

Solution: (i) Here $u_n = (2n-1)\,x^{n-1}$, $u_{n+1} = (2n+1)\,x^n$.

$\therefore$ $$\frac{u_n}{u_{n+1}} = \frac{(2n-1)\,x^{n-1}}{(2n+1)\,x^n} = \frac{(2-1/n)}{(2+1/n)} \cdot \frac{1}{x}.$$

$\therefore$ $$\lim \frac{u_n}{u_{n+1}} = \frac{2}{2} \cdot \frac{1}{x} = \frac{1}{x}.$$

Hence by ratio test the series is convergent if $1/x > 1$ *i.e.* if

$$1 > x \quad \text{or} \quad x < 1,$$

the series is divergent if $1/x < 1$, *i.e.* if $x > 1$ and the test fails if $1/x = 1$ *i.e.* if $x = 1$.

When $x = 1$, then the given series becomes

$$1 + 3 + 5 + 7 + \ldots$$

S_n = sum of n terms of this series $= \frac{n}{2}(1 + 2n - 1) = n^2$.

Since $\lim S_n = \infty$, hence this series is divergent.

Thus the given series converges if $x < 1$ and diverges if $x \geq 1$.

(ii) Here $u_n = \frac{x^{n-1}}{n^2}$, so that $u_{n+1} = \frac{x^n}{(n+1)^2}$.

$\therefore$ $$\frac{u_n}{u_{n+1}} = \frac{x^{n-1}}{n^2} \cdot \frac{(n+1)^2}{x^n} = \left(1 + \frac{1}{n}\right)^2 \cdot \frac{1}{x}.$$

$\therefore \qquad \lim \dfrac{u_n}{u_{n+1}} = \lim \left(1+\dfrac{1}{n}\right)^2 \cdot \dfrac{1}{x} = \dfrac{1}{x}.$

Hence by ratio test the series converges if $1/x > 1$ *i.e.* if $x < 1$, diverges if $1/x < 1$ *i.e.* if $x > 1$ and the test fails if $1/x = 1$ *i.e.* if $x = 1$.

When $x = 1$, then $u_n = 1/n^2$. We know that $\Sigma\,(1/n^2)$ is convergent because here $p = 2 > 1$.

Thus the given series converges if $x \le 1$ and diverges if $x > 1$.

Example 24: *Test for convergence the series whose nth term is*

(*i*) $\dfrac{1}{x^n + x^{-n}}$, $\qquad$ (*ii*) $\dfrac{a^n}{x^n + a^n}$.

Solution: Here $u_n = \dfrac{1}{x^n + x^{-n}} = \dfrac{x^n}{x^{2n}+1}$, $u_{n+1} = \dfrac{x^{n+1}}{x^{2(n+1)}+1}$.

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \frac{x^n}{x^{2n}+1} \cdot \frac{x^{2(n+1)}+1}{x^{n+1}} = \frac{x^{2n+2}+1}{x^{2n}+1} \cdot \frac{1}{x}.$$

Now (u_n / u_{n+1}) can be found only if we know that

$x < 1 \quad$ or $\quad x > 1$.

Let $\qquad x < 1$.

Then
$$\lim \frac{u_n}{u_{n+1}} = \lim \left[\frac{x^{2n+2}+1}{x^{2n}+1} \cdot \frac{1}{x}\right]$$
$$= \frac{1}{x}. \qquad [\because \lim x^{2n+2} = 0 = \lim x^{2n} \text{ if } x < 1]$$

But if $x < 1$, then $1/x > 1$.

$\therefore$ if $x < 1$, we have $\lim\,(u_n / u_{n+1}) > 1$ and hence by ratio test the series converges in this case.

Now let $\quad x > 1$.

Then
$$\lim \frac{u_n}{u_{n+1}} = \lim \left[\frac{x^{2n+2}+1}{x^{2n}+1} \cdot \frac{1}{x}\right] = \lim \left[\frac{x^{2n+2}\,(1+1/x^{2n+2})}{x^{2n}\,(1+1/x^{2n})} \cdot \frac{1}{x}\right]$$
$$= \lim \left[x\,\frac{(1+1/x^{2n+2})}{(1+1/x^{2n})}\right]$$
$$= x \qquad [\because \lim 1/x^{2n+2} = 0 \text{ if } x > 1]$$

$\therefore$ if $x > 1$, we have $\lim\,(u_n / u_{n+1}) = x$ *i.e.* > 1 and hence by ratio test the series is convergent in this case also.

Again, if $x = 1$, then $u_n = \dfrac{1}{1+1} = \dfrac{1}{2}$,

i.e., $\quad$ the series becomes $\dfrac{1}{2} + \dfrac{1}{2} + \dfrac{1}{2} + \ldots$

$S_n =$ sum of its n terms $= \frac{1}{2} \cdot n.$

Since $\lim S_n = \infty$, hence, the series is divergent if $x = 1$.

Thus the given series is convergent if $x > 1$ or $x < 1$ and divergent if $x = 1$.

(ii) Here $u_n = \dfrac{a^n}{x^n + a^n}, u_{n+1} = \dfrac{a^{n+1}}{x^{n+1} + a^{n+1}}$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{a^n}{x^n + a^n} \cdot \frac{x^{n+1} + a^{n+1}}{a^{n+1}} = \frac{x^{n+1} + a^{n+1}}{a\,(x^n + a^n)}.$$

Let $x > a$.

Then $$\lim \frac{u_n}{u_{n+1}} = \lim \frac{x^{n+1} + a^{n+1}}{a\,(x^n + a^n)} = \lim \frac{x^{n+1}\,[1 + (a/x)^{n+1}]}{ax^n\,[1 + (a/x)^n]}$$

$$= \lim \frac{x}{a} \frac{[1 + (a/x)^{n+1}]}{[1 + (a/x)^n]} = \frac{x}{a}, \text{ which is} > 1 \text{ as } x > a.$$

Hence by ratio test the given series converges if $x > a$.

Let $x < a$.

Then $$\lim \frac{u_n}{u_{n+1}} = \lim \frac{a^{n+1}\,[1 + (x/a)^{n+1}]}{a\,.\,a^n\,[1 + (x/a)^n]} = \lim \frac{[1 + (x/a)^{n+1}]}{[1 + (x/a)^n]} = 1.$$

$\therefore$ the ratio test fails in this case.

But in this case, $\lim u_n = \lim \dfrac{a^n}{x^n + a^n} = \lim \dfrac{a^n}{a^n\,[1 + (x/a)^n]} = 1$, which is > 0.

$\therefore$ the given series diverges if $x < a$.

Now, if $x = a$, the series is $\frac{1}{2} + \frac{1}{2} + \ldots$, which diverges.

Hence the given series is convergent if $x > a$ and divergent if $x \le a$.

Example 25: *Test for convergence the following series*

(i) $\dfrac{1}{2\sqrt{1}} + \dfrac{x^2}{3\sqrt{2}} + \dfrac{x^4}{4\sqrt{3}} + \dfrac{x^6}{5\sqrt{4}} + \ldots$ **(Gorakhpur 2013)**

(ii) $x + \dfrac{3}{5}x^2 + \dfrac{8}{10}x^3 + \dfrac{15}{17}x^4 + \ldots + \dfrac{n^2 - 1}{n^2 + 1}x^n + \ldots.$ **(Avadh 2012)**

Solution: (i) Here $u_n = \dfrac{x^{2n-2}}{(n+1)\sqrt{n}}, u_{n+1} = \dfrac{x^{2n}}{(n+2)\sqrt{(n+1)}}$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \cdot \frac{(n+2)\sqrt{(n+1)}}{x^{2n}} = \frac{(1 + 2/n)}{(1 + 1/n)} \sqrt{\left(1 + \frac{1}{n}\right)} \cdot \frac{1}{x^2}.$$

$\therefore \qquad \lim \dfrac{u_n}{u_{n+1}} = \dfrac{1}{1} \cdot \sqrt{1} \cdot \dfrac{1}{x^2} = \dfrac{1}{x^2}.$

$\therefore$ by ratio test the given series is convergent if $1/x^2 > 1$ *i.e.*, if $x^2 < 1$, divergent if $1/x^2 < 1$ *i.e.*, if $x^2 > 1$ and the test fails if $x^2 = 1$.

When $x^2 = 1$, we have $u_n = \dfrac{1}{(n+1)\sqrt{n}}$.

Take $\qquad v_n = \dfrac{1}{n\sqrt{n}}$.

$\therefore \qquad \lim \dfrac{u_n}{v_n} = \lim \dfrac{n\sqrt{n}}{(n+1)\sqrt{n}} = \lim \dfrac{1}{(1+1/n)} = 1,$

which is finite and non-zero. Hence by comparison test $\Sigma\, u_n$ and $\Sigma\, v_n$ are either both convergent or both divergent.

Since $\Sigma\, v_n = \Sigma\,(1/n^{3/2})$ is convergent as $p = 3/2 > 1$, therefore the given series is also convergent if $x^2 = 1$.

Thus the given series is convergent if $x^2 \le 1$ and divergent if $x^2 > 1$.

(ii) Here $\qquad u_n = \dfrac{n^2 - 1}{n^2 + 1} x^n,\ u_{n+1} = \dfrac{(n+1)^2 - 1}{(n+1)^2 + 1} x^{n+1}.$

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \frac{n^2-1}{n^2+1} x^n \cdot \frac{(n+1)^2+1}{(n+1)^2-1} \cdot \frac{1}{x^{n+1}}$$

$$= \frac{1 - 1/n^2}{1 + 1/n^2} \cdot \frac{1 + 2/n + 2/n^2}{1 + 2/n} \cdot \frac{1}{x}.$$

$$\therefore \qquad \lim \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

$\therefore$ by ratio test the given series is convergent if $1/x > 1$ *i.e.*, if $x < 1$, divergent if $1/x < 1$ *i.e.*, if $x > 1$ and the test fails if $x = 1$.

When $x = 1$, $u_n = \dfrac{n^2 - 1}{n^2 + 1} = \dfrac{1 - 1/n^2}{1 + 1/n^2}$.

$\therefore \qquad \lim u_n = \lim \dfrac{1 - 1/n^2}{1 + 1/n^2} = 1$, which is > 0.

$\therefore \qquad$ the given series is divergent if $x = 1$.

Thus the given series is convergent if $x < 1$ and divergent if $x \ge 1$.

Comprehensive Exercise 3

Test for convergence the following series :

1. (i) $2x+\frac{3x^2}{8}+\frac{4x^3}{27}+\ldots.+\frac{(n+1)\,x^n}{n^3}+\ldots.$

(Kumaun 2003; Kanpur 11; Meerut 12,12B)

(ii) $\frac{x^2}{2\sqrt{1}}+\frac{x^3}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\ldots..,\, x>0.$

(iii) $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots,\, x>0.$

2. $\frac{1^2\cdot 2^2}{1!}+\frac{2^2\cdot 3^2}{2!}+\frac{3^2\cdot 4^2}{3!}+\ldots.$

3. (i) $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\ldots+\frac{x^n}{n^2+1}+\ldots..$

(ii) $1+\frac{2}{5}x+\frac{6}{9}x^2+\frac{14}{17}x^3+\ldots+\frac{2^n-2}{2^n+1}x^{n-1}+\ldots..$

4. $\frac{x}{1\cdot 2}+\frac{x^2}{2\cdot 3}+\frac{x^3}{3\cdot 4}+\ldots...$

(Gorakhpur 2012)

5. $\frac{1}{3}+\frac{1.2}{3.5}+\frac{1.2.3}{3.5.7}+\frac{1.2.3.4}{3.5.7.9}+\ldots$

6. (i) $\sum \frac{n!\,3^n}{n^n}.$ (ii) $\sum \frac{n^3}{(n-1)!}.$

7. (i) $\sum \left(\frac{3n-1}{2^n}\right)$ (ii) $\sum \left(\frac{x^n}{x+n}\right).$

8. (i) $\sum \frac{x^n}{a+\sqrt{n}}$ (ii) $\sum \frac{n^n}{n!}.$

9. Test for convergence the series whose n th term is

(i) $\frac{n^2(n+1)^2}{n!}$ (ii) $\frac{2^n-1}{3^n+1}$

(iii) $\frac{\sqrt{n}}{\sqrt{(n^2+1)}}x^n,\,(x>0)$ (iv) $\frac{n^3-1}{n^3+1}x^n,\,(x>0)$

(v) $\sqrt{\left(\dfrac{n-1}{n^3+1}\right)}\, x^n, (x>0)$ (vi) $\dfrac{3^n-2}{3^n+1}\, x^{n-1}, (x>0).$

(vii) $\dfrac{x^n}{x^n+a^n}, x>0, a>0.$

10. Examine the convergence of the series

$$\frac{1}{1^p}+\frac{x}{3^p}+\frac{x^2}{5^p}+\frac{x^3}{7^p}+\dots$$

Answers 3

1. (i) Convergent if $x \le 1$ and divergent if $x > 1$
   (ii) Convergent if $x \le 1$ and divergent if $x > 1$
   (iii) Convergent for all real values of x
2. Convergent
3. (i) Convergent if $x \le 1$ and divergent if $x > 1$
   (ii) Convergent if $x < 1$ and divergent if $x \ge 1$
4. Convergent if $x \le 1$ and divergent if $x > 1$
5. Convergent
6. (i) Divergent (ii) Convergent
7. (i) Convergent (ii) Convergent if $x < 1$ and divergent if $x \ge 1$
8. (i) Convergent if $x < 1$ and divergent if $x \ge 1$ (ii) Divergent
9. (i) Convergent (ii) Convergent
   (iii) Convergent if $x < 1$ and divergent if $x \ge 1$
   (iv) Convergent if $x < 1$ and divergent if $x \ge 1$
   (v) Convergent if $x < 1$ and divergent if $x \ge 1$
   (vi) Convergent if $x < 1$ and divergent if $x \ge 1$
   (vii) Convergent if $x < a$ and divergent if $x \ge a$
10. Convergent if $x < 1$ and divergent if $x > 1$
    In case $x = 1$, then convergent if $p > 1$ and divergent if $p \le 1$.

11 Cauchy's Condensation Test

(Avadh 2012)

Theorem: *If the function $f(n)$ is positive for all positive integral values of n and continually decreases as n increases, then the two infinite series*

$$f(1)+f(2)+f(3)+\dots+f(n)+\dots$$

and $$a\,f(a)+a^2 f(a^2)+a^3 f(a^3)+\dots+a^n f(a^n)+\dots$$

are either both convergent or both divergent, a being a positive integer greater than unity.

Proof: The terms in the series $\Sigma f(n)$ can be arranged as

$$\{f(1)+f(2)+f(3)+\ldots+f(a)\}$$
$$+\{f(a+1)+f(a+2)+\ldots+f(a^2)\}$$
$$+\{f(a^2+1)+f(a^2+2)+\ldots+f(a^3)\}+\ldots$$
$$\ldots+\{f(a^m+1)+f(a^m+2)+\ldots+f(a^{m+1})\}+\ldots. \qquad \ldots(1)$$

The terms in the $(m+1)$ th group are

$$f(a^m+1)+f(a^m+2)+\ldots+f(a^{m+1}). \qquad \ldots(2)$$

The number of terms in this group is $(a^{m+1}-a^m)$ *i.e.,* $a^m(a-1)$. Also $f(a^{m+1})$ is the smallest term in this group since the terms go on decreasing.

$\therefore$
$$f(a^m+1)+f(a^m+2)+\ldots+f(a^{m+1})>a^m(a-1)f(a^{m+1})$$

or
$$f(a^m+1)+f(a^m+2)+\ldots+f(a^{m+1})>\frac{a-1}{a}\{a^{m+1}f(a^{m+1})\}. \qquad \ldots(3)$$

Putting $m=0,1,2,3,\ldots$ successively in (3), we have

$$f(2)+f(3)+\ldots+f(a)>\frac{a-1}{a}\{a\,f(a)\}$$
$$f(a+1)+f(a+2)+\ldots+f(a^2)>\frac{a-1}{a}\{a^2 f(a^2)\}$$
$$f(a^2+1)+f(a^2+2)+\ldots+f(a^3)>\frac{a-1}{a}\{a^3 f(a^3)\}$$

... ...

... ...

Adding all the above inequalities, we get

$$\Sigma f(n)-f(1)>\frac{a-1}{a}\Sigma[a^n f(a^n)].$$

This shows that if the series $\Sigma a^n f(a^n)$ is divergent, so also is the series $\Sigma f(n)$.

Again, each term of the $(m+1)$ th group given by (2) is less than $f(a^m)$. Hence, we have

$$f(a^m+1)+f(a^m+2)+\ldots+f(a^{m+1})$$
$$<f(a^m)+f(a^m)+\ldots+f(a^m)=a^m(a-1)f(a^m)$$

i.e.
$$f(a^m+1)+f(a^m+2)+\ldots+f(a^{m+1})<(a-1)\{a^m f(a^m)\}. \qquad \ldots(4)$$

Putting $m=0,1,2,3,\ldots$ successively in (4), we have

$$f(2)+f(3)+\ldots+f(a)<(a-1)\{f(1)\}$$
$$f(a+1)+f(a+2)+\ldots+f(a^2)<(a-1)\{a\,f(a)\}$$
$$f(a^2+1)+f(a^2+2)+\ldots+f(a^3)<(a-1)\{a^2 f(a^2)\}$$

... ...

... ...

Adding all these inequalities, we get

$$\Sigma f(n)-f(1)<(a-1)f(1)+(a-1)\Sigma a^n f(a^n).$$

This shows that if $\Sigma a^n f(a^n)$ is convergent, so also is $\Sigma f(n)$.

Note: For the validity of the above theorem it is sufficient if $f(n)$ be positive and constantly decreases for values of n greater than a fixed positive integer r.

12 The Auxiliary Series $\Sigma \frac{1}{n(\log n)^p}$

Theorem: *The series*

$$1+\frac{1}{2(\log 2)^p}+\frac{1}{3(\log 3)^p}+\ldots+\frac{1}{n(\log n)^p}+\ldots$$

is convergent if $p>1$ *and divergent if* $p\leq 1$.

Proof: Case 1: Let $p\leq 0$.

Then $\frac{1}{n(\log n)^p}\geq\frac{1}{n}$ for all $n\geq 2$.

Since the series $\Sigma(1/n)$ is divergent, therefore by comparison test $\Sigma\frac{1}{n(\log n)^p}$ is also divergent.

Case 2: Let $p>0$. Let $f(n)=\frac{1}{n(\log n)^p}$.

Obviously $f(n)>0$ for all $n\geq 2$.

Now the given series $\Sigma\frac{1}{n(\log n)^p}=\Sigma f(n)$.

Since $<n(\log n)^p>$ is an increasing sequence, therefore $<f(n)>$ is a decreasing sequence. Hence by Cauchy's condensation test given in article 11, the series $\Sigma f(n)$ is convergent or divergent according as the series $\Sigma a^n f(a^n)$ is convergent or divergent.

Now $$a^n f(a^n)=\frac{a^n}{a^n(\log a^n)^p}=\frac{1}{(n\log a)^p}=\frac{1}{(\log a)^p}\cdot\frac{1}{n^p}.$$

Since $\frac{1}{(\log a)^p}$ is a constant, hence the series $\Sigma a^n f(a^n)$ is convergent or divergent according as the series $\Sigma(1/n^p)$ is convergent or divergent. But the series $\Sigma 1/n^p$ is convergent if $p>1$ and divergent if $p\leq 1$.

Hence by Cauchy's condensation test the given series is also convergent if $p>1$ and divergent if $p\leq 1$.

Illustrative Examples

Example 26: *Test the convergence of the following series :*

(*i*) $\frac{1}{\log 2}+\frac{1}{\log 3}+\frac{1}{\log 4}+\ldots$ (*ii*) $\frac{\log 2}{2}+\frac{\log 3}{3}+\frac{\log 4}{4}+\ldots$

(*iii*) $\dfrac{1}{(\log 2)^p}+\dfrac{1}{(\log 3)^p}+\ldots+\dfrac{1}{(\log n)^p}+\ldots$

Solution: (i) Here $f(n)=\dfrac{1}{\log n}>0$ for all $n\geq 2$. Also $f(n)$ decreases continually as n increases.

Now $a^n f(a^n)=\dfrac{a^n}{\log(a^n)}=\dfrac{a^n}{n\log a}$, a being taken as some positive integer >1.

Consider the series $\Sigma\, a^n f(a^n)=\Sigma\,\{a^n/(n\log a)\}=\Sigma\, v_n$, (say).

Here $v_n=\dfrac{a^n}{n\log a}$, so that $v_{n+1}=\dfrac{a^{n+1}}{(n+1)\log a}$.

$\therefore$ $$\frac{v_n}{v_{n+1}}=\frac{n+1}{n}\cdot\frac{1}{a}=\left(1+\frac{1}{n}\right)\cdot\frac{1}{a}.$$

$\therefore$ $$\lim\frac{v_n}{v_{n+1}}=\frac{1}{a} \text{ which is} <1 \text{ as by our choice } a>1.$$

$\therefore$ by ratio test the series $\Sigma\, v_n=\Sigma\, a^n f(a^n)$ is divergent.

Consequently by Cauchy's condensation test the given series

$$\Sigma f(n)=\frac{1}{\log 2}+\frac{1}{\log 3}+\ldots, \text{ is also divergent.}$$

(ii) Here $f(n)=\dfrac{\log n}{n}>0$ for all $n\geq 2$.

Also $f(n)$ decreases continually as n increases.

Now $a^n f(a^n)=a^n\left(\dfrac{\log a^n}{a^n}\right)=n\log a$, a being taken as some + ive integer >1.

Now the series $\Sigma\, a^n f(a^n)=\Sigma\,(n\log a)=\log a\,.\,\Sigma\, n$

is divergent because the series $\Sigma\, n$ is divergent.

Hence by Cauchy's condensation test the given series $\Sigma f(n)=\Sigma\dfrac{\log n}{n}$ is also divergent.

(iii) If $p\leq 0$, the given series is obviously divergent. So let us consider the case when $p>0$. Here $f(n)=\dfrac{1}{(\log n)^p}>0$ for all $n\geq 2$.

Also $f(n)$ decreases continually as n increases.

Now $a^n f(a^n)=\dfrac{a^n}{(\log a^n)^p}=\dfrac{a^n}{n^p(\log a)^p}$, a being taken >1.

Consider the series $\Sigma\, a^n f(a^n)=\Sigma\dfrac{a^n}{n^p(\log a)^p}=\Sigma\, v_n$, say.

Here $v_n=\dfrac{a^n}{n^p(\log a)^p}$, so that $v_{n+1}=\dfrac{a^{n+1}}{(n+1)^p(\log a)^p}$.

$$\therefore \qquad \frac{v_n}{v_{n+1}} = \frac{a^n}{n^p (\log a)^p} \cdot \frac{(n+1)^p (\log a)^p}{a^{n+1}} = \left(1 + \frac{1}{n}\right)^p \cdot \frac{1}{a}.$$

$$\therefore \qquad \lim \frac{v_n}{v_{n+1}} = \frac{1}{a} \text{ which is} < 1 \text{ as } a > 1.$$

$\therefore$ by ratio test the series $\Sigma\, v_n = \Sigma\, a^n\, f\,(a^n)$ is divergent.

Therefore by Cauchy's condensation test the given series $\Sigma\, f\,(n)$ is also divergent.

Example 27: *Test for convergence the following series*

(*i*) $\dfrac{(\log 2)^2}{2^2} + \dfrac{(\log 3)^2}{3^2} + \dfrac{(\log 4)^2}{4^2} + \ldots + \dfrac{(\log n)^2}{n^2} + \ldots$

(*ii*) $\dfrac{1}{(2 \log 2)^p} + \dfrac{1}{(3 \log 3)^p} + \ldots + \dfrac{1}{(n \log n)^p} + \ldots$

Solution: (i) Here we can take the first term of the series as

$$\frac{(\log 1)^2}{1^2} \text{ because } \log 1 = 0.$$

$$\therefore \qquad u_n = n\text{th term of the series} = \frac{(\log n)^2}{n^2} = f\,(n), \text{ say.}$$

It is positive for all $n \geq 2$ and decreases continually as n increases.

Now $$a^n\, f\,(a^n) = \frac{a^n\, (\log a^n)^2}{(a^n)^2} = \frac{a^n\, n^2\, (\log a)^2}{(a^n)^2} = \frac{n^2\, (\log a)^2}{a^n},$$

a being taken to be a +ive integer > 1.

Consider the series $\Sigma\, a^n\, f\,(a^n) = \Sigma\, \{ n^2\, (\log a)^2 \,/\, a^n \} = \Sigma v_n$, (say).

Here $$v_n = \frac{n^2\, (\log a)^2}{a^n},\; v_{n+1} = \frac{(n+1)^2\, (\log a)^2}{a^{n+1}}.$$

$$\therefore \qquad \frac{v_n}{v_{n+1}} = \frac{n^2\, (\log a)^2}{a^n} \cdot \frac{a^{n+1}}{(n+1)^2\, (\log a)^2} = \frac{a}{(1 + 1/n)^2}.$$

$$\therefore \qquad \lim \frac{v_n}{v_{n+1}} = \lim \frac{a}{(1 + 1/n)^2} = a > 1 \text{ since by our choice } a > 1.$$

$\therefore$ by ratio test the series $\Sigma\, v_n$ is convergent.

Hence by Cauchy's condensation test the given series $\Sigma\, f\,(n)$ is also convergent.

(ii) If $p \leq 0$, obviously the given series is divergent. So it remains to discuss the case when $p > 0$.

When $p > 0$, we have $f\,(n) = \dfrac{1}{(n \log n)^p} > 0$ for all $n \geq 2$ and it decreases continually as n increases.

Now $$a^n\, f\,(a^n) = \frac{a^n}{(a^n \log a^n)^p} = \frac{1}{a^{n\,(p-1)} \,.\, n^p\, (\log a)^p}, \; a \text{ to be taken} > 1.$$

Case I: Let $p > 1$. Then $a^{n(p-1)} > 1$ as $a > 1$.

$$\therefore \quad a^n f(a^n) = \frac{1}{a^{n(p-1)} \cdot n^p (\log a)^p} < \frac{1}{(\log a)^p} \cdot \frac{1}{n^p} \qquad \ldots(1)$$

Now $1/(\log a)^p$ is a fixed positive real number and the series $\Sigma (1/n^p)$ is convergent because $p > 1$.

Hence from (1), by comparison test (second form) given in article 7, the series $\Sigma\, a^n f(a^n)$ is convergent.

Now by Cauchy's condensation test it follows that the given series $\Sigma f(n)$ is also convergent.

Case II: Let $p \le 1$. Then $a^{n(p-1)} \le 1$ as $a > 1$.

$$\therefore \quad a^n f(a^n) \ge \frac{1}{(\log a)^p} \cdot \frac{1}{n^p}. \qquad \ldots(2)$$

Now $1/(\log a)^p$ is a fixed +ive real number and the series $\Sigma (1/n^p)$ is divergent, p being ≤ 1.

Hence from (2), by comparison test the series $\Sigma\, a^n f(a^n)$ is divergent.

Now by Cauchy's condensation test it follows that the given series $\Sigma f(n)$ is also divergent.

Hence the given series is convergent if $p > 1$ and divergent if $p \le 1$.

13 Raabe's Test

Theorem: *The series $\Sigma\, u_n$ of positive terms is convergent or divergent according as*

$$\lim \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right\} > 1 \text{ or } < 1.$$

(Gorakhpur 2014)

Proof: Case I: Let $\lim \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right\} = k$, where $k > 1$.

Choose a number p such that $k > p > 1$.

Compare the series $\Sigma\, u_n$ with the auxiliary series $\Sigma\, v_n = \Sigma \frac{1}{n^p}$, which is convergent since $p > 1$.

By article 7, sixth form of comparison test, $\Sigma\, u_n$ is convergent if after some particular term

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$$

or
$$\frac{u_n}{u_{n+1}} > \frac{1/n^p}{1/(n+1)^p} = \left(\frac{n+1}{n} \right)^p = \left(1 + \frac{1}{n} \right)^p$$

$$= 1 + p \cdot \frac{1}{n} + \frac{p(p-1)}{2!} \cdot \frac{1}{n^2} + \ldots$$

or $$n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + \frac{p(p-1)}{2!} \cdot \frac{1}{n} + \ldots \quad \ldots(1)$$

If n be taken sufficiently large the L.H.S and R.H.S. of (1) respectively approach k and p. Also k is greater than p. Therefore (1) is satisfied for sufficiently large values of n. Hence $\Sigma\, u_n$ is convergent if

$$\lim \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} > 1.$$

Case II: Let $\lim \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = l$, where $l < 1$.

Choose a number p such that $l < p < 1$.

Compare the series $\Sigma\, u_n$ with the auxiliary series $\Sigma\, v_n = \Sigma\,(1/n^p)$ which is divergent since $p < 1$.

The series $\Sigma\, u_n$ is divergent if after some particular term

$$\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}},$$ [By article 7, sixth form of comparison test]

or $$n\left(\frac{u_n}{u_{n+1}} - 1\right) < p + \frac{p(p-1)}{2!} \cdot \frac{1}{n} + \ldots \quad \ldots(2)$$

(Proceeding as in case I)

If n be taken sufficiently large the L.H.S. and R.H.S. of (2) respectively approach l and p. Also $l < p$. Thus (2) is satisfied for sufficiently large values of n. Hence $\Sigma\, u_n$ is divergent if

$$\lim \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} < 1.$$

However, if $\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = 1$, the Raabe's test fails.

Note: Rabbe's test is to be applied when D'Alembert's ratio test fails.

Illustrative Examples

Example 28: *Test the convergence of the series*

(*i*) $\Sigma \dfrac{n!\,x^n}{3.5.7\ldots(2n+1)}$ (*ii*) $\sum_{n=1}^{\infty} \dfrac{1}{1+\log n}$.

Solution: (i) Here $u_n = \dfrac{n\,!\,x^n}{3\,.5\,.7\ldots(2n+1)}$

so that $$u_{n+1} = \frac{(n+1)\,!\,x^{n+1}}{3\,.5\,.7\ldots.(2n+1)\,(2n+3)}\cdot$$

Now $$\frac{u_n}{u_{n+1}} = \frac{(2n+3)}{(n+1)\,!}\cdot\frac{n\,!}{x} = \frac{2n+3}{n+1}\cdot\frac{1}{x} = \left(\frac{2+3/n}{1+1/n}\right)\cdot\frac{1}{x}\cdot$$

$\therefore$ $$\lim\frac{u_n}{u_{n+1}} = \lim\left(\frac{2+3/n}{1+1/n}\right)\cdot\frac{1}{x} = \frac{2}{x}\cdot$$

Hence by D' Alembert's ratio test the series converges if $\dfrac{2}{x} > 1$ *i.e.,* if $x < 2$ and diverges if $2/x < 1$ *i.e.,* if $x > 2$ and the test fails when $2/x = 1$ *i.e.*, when $x = 2$.

In case $x = 2$, we apply Raabe's test.

When $$x = 2,\ \frac{u_n}{u_{n+1}} = \frac{2\,n+3}{2\,(n+1)}\cdot$$

$\therefore$ $$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{2n+3}{2n+2} - 1\right) = \frac{n}{2\,(n+1)} = \frac{1}{2\,(1+1/n)}\cdot$$

$\therefore$ $$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim\frac{1}{2\,(1+1/n)} = \frac{1}{2} < 1.$$

Hence by Raabe's test $\Sigma\, u_n$ is divergent if $x = 2$.

Thus the given series $\Sigma\, u_n$ is convergent if $x < 2$ and divergent if $x \geq 2$.

(ii) Here $$u_n = \frac{1}{1+\log n}\ ;\ u_{n+1} = \frac{1}{1+\log\,(n+1)}\cdot$$

Now $$\frac{u_n}{u_{n+1}} = \frac{1+\log\,(n+1)}{1+\log n}$$

$$= \frac{1+\log\,\{n\,(1+1/n)\}}{1+\log n} = \frac{1+\log n+\log\,(1+1/n)}{1+\log n}$$

$$= \frac{\log\,(en)+\log\,(1+1/n)}{\log\,(en)} = 1+\frac{1}{\log\,(en)}\log\,(1+1/n)$$

$$= 1+\frac{1}{\log\,(en)}\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\right)$$

$$= 1+\frac{1}{n\log\,(en)} - \frac{1}{2n^2\log\,(en)} + \ldots.$$

$\therefore$ $\lim\dfrac{u_n}{u_{n+1}} = 1$, and so the ratio test fails.

Now we apply Raabe's test. We have

$$\lim n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim n\left[\frac{1}{n\log(en)}-\frac{1}{2n^2\log(en)}+\ldots\right]$$

$$=\lim\left[\frac{1}{\log(en)}-\frac{1}{2n\log(en)}+\ldots\right]=0<1.$$

Hence by Raabe's test the given series is divergent.

Example 29: *Test the convergence of the series*

(i) $\dfrac{1^2}{4^2}+\dfrac{1^2 .5^2}{4^2 .8^2}+\dfrac{1^2 .5^2 .9^2}{4^2 .8^2 .12^2}+\dfrac{1^2 .5^2 .9^2 .13^2}{4^2 .8^2 .12^2 .16^2}+\ldots$ **(Meerut 2013)**

(*ii*) $1+\dfrac{3}{7}x+\dfrac{3.6}{7.10}x^2+\dfrac{3.6.9}{7.10.13}x^3+\dfrac{3.6.9.12}{7.10.13.16}x^4+\ldots$ **(Meerut 2013B)**

Solution: (i) Here $u_n=\dfrac{1^2 .5^2 .9^2 \ldots\ldots(4n-3)^2}{4^2 .8^2 .12^2 \ldots\ldots(4n)^2}$.

[Note that the nth term of the sequence $1^2, 5^2, 9^2, \ldots$ is

$$\{1+(n-1)4\}^2 \text{ i.e., } (4n-3)^2$$

and the nth term of the sequence

$$4^2, 8^2, 12^2, \ldots \text{ is } \{4+(n-1)4\}^2 \text{ i.e., } (4n)^2 \text{].}$$

Then $\quad u_{n+1}=\dfrac{1^2 .5^2 .9^2\ldots\ldots(4n-3)^2 .(4n+1)^2}{4^2 .8^2 .12^2\ldots\ldots(4n)^2 .(4n+4)^2}$.

Now $\quad \dfrac{u_n}{u_{n+1}}=\dfrac{(4n+4)^2}{(4n+1)^2}=\dfrac{(4+4/n)^2}{(4+1/n)^2}$.

$\therefore \quad \lim \dfrac{u_n}{u_{n+1}}=1$, so that the ratio test fails.

Now we apply Raabe's test. We have

$$\lim n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim n\left[\frac{(4n+4)^2}{(4n+1)^2}-1\right]=\lim n\left[\frac{24n+15}{(4n+1)^2}\right]$$

$$=\lim\left\{\frac{24+15/n}{(4+1/n)^2}\right\}=\frac{24}{4^2}=\frac{3}{2}, \text{ which is } >1.$$

Hence by Raabe's test the series is convergent.

(ii) Omitting the first term of the series, we have
nth term of the sequence 3, 6, 9, … is $3+(n-1)3=3n$

and nth term of the sequence 7, 10, 13, … is $7+(n-1)3=3n+4$.

$\therefore \quad u_n=\dfrac{3.6.9\ldots\ldots 3n}{7.10.13\ldots\ldots(3n+4)}x^n,$

and $\quad u_{n+1}=\dfrac{3.6.9\ldots\ldots 3n.(3n+3)}{7.10.13\ldots\ldots(3n+4)(3n+7)}x^{n+1}.$

$\therefore \qquad \dfrac{u_n}{u_{n+1}} = \left(\dfrac{3n+7}{3n+3}\right) \cdot \dfrac{1}{x} = \left(\dfrac{3+7/n}{3+3/n}\right) \cdot \dfrac{1}{x}.$

$\therefore \qquad \lim \dfrac{u_n}{u_{n+1}} = \lim \left(\dfrac{3+7/n}{3+3/n}\right) \cdot \dfrac{1}{x} = \dfrac{3}{3} \cdot \dfrac{1}{x} = \dfrac{1}{x}.$

Hence by ratio test, the series is convergent if $1/x > 1$ *i.e.,* if $x < 1$, divergent if $1/x < 1$ *i.e.,* if $x > 1$ and the test fails if $1/x = 1$ *i.e.,* if $x = 1$.

If $x = 1$, then $\dfrac{u_n}{u_{n+1}} = \dfrac{3n+7}{3n+3}.$

Now we apply Raabe's test. We have

$$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim n\left(\frac{3n+7}{3n+3} - 1\right) = \lim \frac{4n}{3n+3}$$

$$= \lim \frac{4}{3+3/n} = \frac{4}{3}, \text{ which is } > 1.$$

Hence the series is convergent when $x = 1$.

Thus the given series is convergent if $x \leq 1$ and divergent if $x > 1$.

Example 30: *Test for convergence the following series*

$$1 + a + \frac{a(a+1)}{1.2} + \frac{a(a+1)(a+2)}{1.2.3} + \ldots$$

Solution: Leaving the first term, we have

$$u_n = \frac{a(a+1)(a+2)\ldots(a+n-1)}{1.2.3\ldots n},$$

and then $\qquad u_{n+1} = \dfrac{a(a+1)(a+2)\ldots(a+n-1)(a+n)}{1.2.3\ldots n(n+1)}.$

Now $\qquad \dfrac{u_n}{u_{n+1}} = \dfrac{n+1}{a+n} = \dfrac{1+1/n}{a/n+1}.$

$\therefore \qquad \lim \dfrac{u_n}{u_{n+1}} = 1$, so that the ratio test fails.

Now we apply Raabe's test. We have

$$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim n\left(\frac{n+1}{a+n} - 1\right)$$

$$= \lim \frac{n(1-a)}{a+n} = \lim \frac{(1-a)}{1+a/n} = 1 - a.$$

Hence by Raabe's test, the given series is convergent if $1 - a > 1$ *i.e.*, if $a < 0$, divergent if $1 - a < 1$ *i.e.*, if $a > 0$ and the test fails if $1 - a = 1$ *i.e.*, if $a = 0$.

In case $a = 0$, the given series becomes $1 + 0 + 0 + 0 + \ldots$

The sum of n terms of this series is always 1. Therefore the series is convergent if $a = 0$.

Thus the given series is convergent if $a \leq 0$ and divergent if $a > 0$.

Comprehensive Exercise 4

Test for convergence the following series :

1. $1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\dots$

2. $1+\frac{1}{2}\cdot\frac{x^2}{4}+\frac{1.3.5}{2.4.6}\cdot\frac{x^4}{8}+\frac{1.3.5.7.9}{2.4.6.8.10}\cdot\frac{x^6}{12}+\dots$

3. $x^2+\frac{2^2}{3.4}x^4+\frac{2^2.4^2}{3.4.5.6}x^6+\frac{2^2.4^2.6^2}{3.4.5.6.7.8}x^8+\dots$ **(Kanpur 2014)**

4. (i) $1+\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\frac{x^7}{7}+\dots$ **(Kashi 2014)**

(ii) $\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\frac{x^7}{7}+\dots$ **(Gorakhpur 2012, 14)**

(iii) $\frac{x}{1}+\frac{1}{2}\cdot\frac{x^2}{3}+\frac{1.3}{2.4}\cdot\frac{x^3}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^4}{7}+\dots,\ x>0.$

5. $\frac{a}{a+3}+\frac{a(a+2)}{(a+3)(a+5)}x+\frac{a(a+2)(a+4)}{(a+3)(a+5)(a+7)}x^2+\dots$

6. $\sum\frac{4.7.\dots.(3n+1)}{1.2.\dots.n}x^n.$

7. Apply Cauchy's condensation test to discuss the convergence of the series

$$\sum_{n=2}^{\infty}\frac{1}{n\log n(\log\log n)^p}.$$

Answers 4

1. Convergent if $x<1$ and divergent if $x\geq 1$
2. Convergent if $x^2\leq 1$ and divergent if $x^2>1$
3. Convergent if $x^2\leq 1$ and divergent if $x^2>1$
4. (i) Convergent if $x^2\leq 1$ and divergent if $x^2>1$

   (ii) Convergent if $x^2\leq 1$ and divergent if $x^2>1$

   (iii) Convergent if $x\leq 1$ and divergent if $x>1$
5. Convergent if $x\leq 1$ and divergent if $x>1$
6. Convergent if $x<1/3$ and divergent if $x\geq 1/3$
7. Convergent if $p>1$ and divergent if $p\leq 1$

14 Logarithmic Test

Theorem: *The series $\Sigma\, u_n$ of positive terms is convergent or divergent according as*

$$\lim \left\{ n \log \frac{u_n}{u_{n+1}} \right\} > 1 \text{ or } < 1.$$

Proof: First suppose that

$$\lim \left\{ n \log \frac{u_n}{u_{n+1}} \right\} = k, \text{ where } k > 1.$$

Choose a number p such that $k > p > 1$.

Compare the given series with the auxiliary series $\Sigma\, v_n$ where $v_n = 1/n^p$, which is convergent as $p > 1$.

The series $\Sigma\, u_n$ is convergent if after some particular term

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} \qquad \text{[By article 7, sixth form of comparison test.]}$$

or $$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} = \left(1 + \frac{1}{n}\right)^p$$

or $$\log \frac{u_n}{u_{n+1}} > \log \left(1 + \frac{1}{n}\right)^p = p \log \left(1 + \frac{1}{n}\right) = p \left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots\right]$$

or $$n \log \frac{u_n}{u_{n+1}} > p - \frac{p}{2n} + \frac{p}{3n^2} - \dots \qquad \dots(1)$$

If n is taken sufficiently large the L.H.S. and R.H.S. of (1) respectively approach k and p. Also $k > p$.

Thus (1) is satisfied for sufficiently large values of n. Hence the series $\Sigma\, u_n$ is convergent if

$$\lim \left\{ n \log \frac{u_n}{u_{n+1}} \right\} > 1 \cdot$$

Similarly, it can be proved that $\Sigma\, u_n$ is divergent if

$$\lim \left\{ n \log \frac{u_n}{u_{n+1}} \right\} < 1.$$

[The procedure of proof will be the same as given in the proof of Raabe's test]

However, if $\lim \left\{ n \log \dfrac{u_n}{u_{n+1}} \right\} = 1$, the test fails.

Note: This test is an alternative to Raabe's test and is to be applied when D'Alembert's ratio test fails and when either

(i) n occurs as an exponent in $\dfrac{u_n}{u_{n+1}}$ so that it is not convenient to find $\dfrac{u_n}{u_{n+1}} - 1$

(ii) taking logarithm of $\dfrac{u_n}{u_{n+1}}$ makes the evaluation of limits easier.

Illustrative Examples

Example 31: *Test for convergence the series*

$$1+\frac{2\,x}{2\,!}+\frac{3^2\,x^2}{3\,!}+\frac{4^3\,x^3}{4\,!}+\frac{5^4\,x^4}{5\,!}+\ldots\ldots$$

(Kashi 2013; Avadh14)

Solution: Here $u_n=\dfrac{n^{n-1}}{n\,!}\,x^{n-1},\ u_{n+1}=\dfrac{(n+1)^n}{(n+1)\,!}\,x^n.$

Now
$$\frac{u_n}{u_{n+1}}=\frac{n^{n-1}}{n\,!}\,\frac{(n+1)\,!}{(n+1)^n}\cdot\frac{1}{x}=\frac{n^{n-1}\,(n+1)}{(n+1)^n}\cdot\frac{1}{x}$$
$$=\left(\frac{n}{n+1}\right)^{n-1}\cdot\frac{1}{x}=\frac{1}{(1+1/n)^{n-1}}\,\frac{1}{x}$$
$$=\frac{1}{(1+1/n)^n}\cdot(1+1/n)\cdot\frac{1}{x}\cdot$$

$\therefore$
$$\lim\frac{u_n}{u_{n+1}}=\lim\left\{\frac{(1+1/n)}{(1+1/n)^n}\cdot\frac{1}{x}\right\}=\frac{1}{ex}\cdot \qquad [\because \lim\,(1+1/n)^n=e]$$

$\therefore$ by ratio test the series $\Sigma\,u_n$ converges if $1/ex>1$ *i.e.*, if $x<1/e$, diverges if $1/ex<1$ *i.e.*, if $x>1/e$ and the test fails if $1/ex=1$ *i.e.* if $x=1/e$.

Now if $x=1/e$, $\dfrac{u_n}{u_{n+1}}=\dfrac{e\,(1+1/n)}{(1+1/n)^n}\cdot$ Applying log test, we get

$$\lim\left(n\log\frac{u_n}{u_{n+1}}\right)=\lim\left[n\log\left\{\frac{e\,(1+1/n)}{(1+1/n)^n}\right\}\right]$$
$$=\lim n\,[\log e+\log\,(1+1/n)-n\log\,(1+1/n)]$$
$$=\lim n\left[1+\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\cdots\right)-n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\cdots\right)\right]$$
$$=\lim n\left[\left(1+\frac{1}{2}\right)\cdot\frac{1}{n}+\left(-\frac{1}{2}-\frac{1}{3}\right)\frac{1}{n^2}+\ldots\right]$$
$$=\lim\left[\frac{3}{2}-\frac{5}{6n}+\ldots\right]=\frac{3}{2}\quad i.e.,\ >1.$$

Therefore the series $\Sigma\,u_n$ converges when $x=1/e$.

Hence the given series is convergent if $x\le 1/e$ and divergent if $x>1/e$.

Example 32: *Test for convergence the series*

$$\frac{(a+x)}{1!}+\frac{(a+2x)^2}{2\,!}+\frac{(a+3x)^3}{3\,!}+\ldots$$

Solution: Here $u_n=\dfrac{(a+nx)^n}{n\,!},\ u_{n+1}=\dfrac{[a+(n+1)\,x]^{n+1}}{(n+1)\,!}\cdot$

Now $$\frac{u_n}{u_{n+1}} = \frac{(a+nx)^n}{n!} \cdot \frac{(n+1)!}{[a+(n+1)x]^{n+1}} = \frac{(a+nx)^n (n+1)}{[a+(n+1)x]^{n+1}}$$

$$= \frac{(nx)^n (a/nx+1)^n n(1+1/n)}{(n+1)^{n+1} x^{n+1} [a/(n+1)x+1]^{n+1}}$$

$$= \frac{1}{x} \cdot \frac{n^{n+1} (1+a/nx)^n (1+1/n)}{n^{n+1} (1+1/n)^{n+1} [1+a/(n+1)x]^{n+1}}$$

$$= \frac{1}{x} \cdot \frac{\left[1+\frac{(a/x)}{n}\right]^n}{\left[1+\frac{1}{n}\right]^n \left[1+\frac{(a/x)}{n+1}\right]^{n+1}}.$$

$\therefore$ $$\lim \frac{u_n}{u_{n+1}} = \frac{1}{x} \frac{e^{a/x}}{e.\, e^{a/x}} \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x\right]$$

$$= \frac{1}{ex}.$$

Hence by ratio test the given series is convergent if $1/e\,x > 1$ *i.e.*, if $x < 1/e$, divergent if $1/ex < 1$ *i.e.*, if $x > 1/e$ and the test fails if $1/ex = 1$ *i.e.*, if $x = 1/e$.

If $x = 1/e$, $$\frac{u_n}{u_{n+1}} = \frac{e\left(1+\frac{ea}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n \left[1+\frac{ae}{n+1}\right]^{n+1}}.$$

Applying logarithmic test, we get

$$\lim\left(n \log \frac{u_n}{u_{n+1}}\right) = \lim n \log \left[\frac{e\left(1+\frac{ea}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n \left\{1+\frac{ae}{n+1}\right\}^{n+1}}\right]$$

$$= \lim n\left[\log e + n\log\left(1+\frac{ea}{n}\right) - n\log\left(1+\frac{1}{n}\right) - (n+1)\log\left(1+\frac{ae}{n+1}\right)\right]$$

$$= \lim n\left[1 + n\left(\frac{ea}{n} - \frac{e^2a^2}{2n^2} + \frac{e^3a^3}{3n^3} - \ldots\right) - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\right)\right.$$
$$\left. - (n+1)\left\{\frac{ea}{n+1} - \frac{e^2a^2}{2(n+1)^2} + \ldots\right\}\right]$$

$$= \lim n\left[\left(-\frac{e^2a^2}{2} + \frac{1}{2}\right)\frac{1}{n} + \frac{e^2a^2}{2(n+1)} + \left(\frac{e^3a^3}{3} - \frac{1}{3}\right)\frac{1}{n^2} + \ldots\right]$$

$$= \lim\left[\left(-\frac{e^2a^2}{2} + \frac{1}{2}\right) + \frac{e^2a^2}{2(1+1/n)} + \left(\frac{e^3a^3-1}{3n}\right) + \ldots\right]$$

$$= -\frac{e^2 a^2}{2} + \frac{1}{2} + \frac{e^2 a^2}{2} = \frac{1}{2}, \text{ which is } < 1.$$

$\therefore$ the series is divergent if $x = 1/e$.

Thus the given series is convergent if $x < 1/e$ and divergent if $x \geq 1/e$.

Comprehensive Exercise 5

Test for convergence the following series :

1. $x^2 (\log 2)^q + x^3 (\log 3)^q + x^4 (\log 4)^q + \ldots$

2. $\dfrac{1}{(\log 2)^p} + \dfrac{1}{(\log 3)^p} + \ldots + \dfrac{1}{(\log n)^p} + \ldots$

3. (i) $x + \dfrac{2^2 x^2}{2!} + \dfrac{3^3 x^3}{3!} + \dfrac{4^4 x^4}{4!} + \dfrac{5^5 x^5}{5!} + \ldots$ **(Gorakhpur 2013)**

   (ii) $1 + \dfrac{2^2 x^2}{2!} + \dfrac{3^3 x^3}{3!} + \dfrac{4^4 x^4}{4!} + \dfrac{5^5 x^5}{5!} + \ldots$

4. $1 + \dfrac{1!}{2} x + \dfrac{2!}{3^2} x^2 + \dfrac{3!}{4^3} x^3 + \dfrac{4!}{5^4} x^4 + \ldots$ **(Kanpur 2014)**

Answers 5

1. Convergent if $x < 1$ and divergent if $x \geq 1$
2. Divergent for all values of p
3. (i) Convergent if $x < 1/e$ and divergent if $x \geq 1/e$

   (ii) Convergent if $x < 1/e$ and divergent if $x \geq 1/e$
4. Convergent if $x < e$ and divergent if $x \geq e$

15 De Morgan's and Bertrand's Test

Theorem: *The series Σu_n of positive terms is convergent or divergent according as*

$$\lim \left[\left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) - 1 \right\} \log n \right] > 1 \; or < 1.$$

Proof: Let $\lim \left[\left\{ n \left(\dfrac{u_n}{u_{n+1}} - 1 \right) - 1 \right\} \log n \right] = k$, where $k > 1$.

Take a number p such that $k > p > 1$.

Compare the series $\Sigma\, u_n$ with the auxiliary series $\Sigma\, v_n$, where $v_n = \dfrac{1}{n\,(\log n)^p}$, which is convergent as $p > 1$.

The series $\Sigma\, u_n$ is convergent if after some particular term

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}, \qquad \text{[By article 7, sixth from of comparison test]}$$

i.e.,
$$\frac{u_n}{u_{n+1}} > \frac{1}{n\,(\log n)^p} \cdot (n+1)\,\{\log\,(n+1)\}^p, \qquad \left[\because\ v_n = \frac{1}{n\,(\log n)^p}\right]$$

i.e.,
$$\frac{u_n}{u_{n+1}} > \left(\frac{n+1}{n}\right)\left[\frac{\log\,\{n\,(1+1/n)\}}{\log n}\right]^p$$

i.e.,
$$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left[\frac{\log n + \log\,(1+1/n)}{\log n}\right]^p$$

i.e.,
$$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left[\frac{\log n + \dfrac{1}{n} - \dfrac{1}{2n^2} + \ldots}{\log n}\right]^p$$

i.e.,
$$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left[1 + \frac{1}{n\log n} - \frac{1}{2n^2\log n} + \ldots\right]^p$$

i.e.,
$$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left[1 + \frac{p}{n\log n} + \ldots\right]$$

i.e.,
$$\frac{u_n}{u_{n+1}} > 1 + \frac{1}{n} + \frac{p}{n\log n} + \ldots$$

i.e.,
$$n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1 + \frac{p}{\log n} + \ldots$$

i.e.,
$$n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 > \frac{p}{\log n} + \ldots$$

i.e.,
$$\left[n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right]\log n > p + \text{terms containing } n \text{ or } \log n$$

in the denominator. ...(1)

Now as n becomes sufficiently large the L.H.S. and R.H.S. of (1) respectively approach k and p. Also $k > p$.

Thus (1) is satisfied for sufficiently large values of n.

Hence the series $\Sigma\, u_n$ is convergent if

$$\lim\left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n\right] > 1.$$

Similarly, it can be proved as in the case of Raabe's test that $\Sigma\, u_n$ is divergent if

$$\lim\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]<1.$$

Note: This test is to be applied when both D' Alembert's ratio test and Raabe's test fail.

16 An Alternative to Bertrand's Test

Theorem: *The series* $\Sigma\, u_n$ *of positive terms is convergent or divergent according as*

$$\lim\left[\left(n\log\frac{u_n}{u_{n+1}}-1\right)\log n\right]>1 \text{ or } <1.$$

Proof: Let $\lim\left[\left(n\log\frac{u_n}{u_{n+1}}-1\right)\log n\right]=k$, where $k>1$.

Take a number p such that $k>p>1$.

Compare the given series $\Sigma\, u_n$ with the auxiliary series $\Sigma\, v_n$ where $v_n=\dfrac{1}{n\,(\log n)^p}$,which is convergent since $p>1$. The series $\Sigma\, u_n$ is convergent if after some particular term

$$\frac{u_n}{u_{n+1}}>\frac{v_n}{v_{n+1}}, \text{ by article 7, sixth form of comparison test}$$

i.e. $$\frac{u_n}{u_{n+1}}>1+\frac{1}{n}+\frac{p}{n\log n}+\dots \qquad \text{[Proceeding as in article 15]}$$

i.e. $$\log\frac{u_n}{u_{n+1}}>\log\left\{1+\left(\frac{1}{n}+\frac{p}{n\log n}+\dots\right)\right\}$$

i.e. $$\log\frac{u_n}{u_{n+1}}>\left(\frac{1}{n}+\frac{p}{n\log n}+\dots\right)-\frac{1}{2}\left(\frac{1}{n}+\frac{p}{n\log n}+\dots\right)^2+\dots$$

i.e. $$n\log\frac{u_n}{u_{n+1}}>n\left[\frac{1}{n}+\frac{p}{n\log n}-\frac{1}{2n^2}+\dots\right]$$

i.e. $$n\log\frac{u_n}{u_{n+1}}>1+\frac{p}{\log n}-\frac{1}{2n}+\dots$$

i.e., $$n\log\frac{u_n}{u_{n+1}}-1>\frac{p}{\log n}-\frac{1}{2n}+\dots$$

i.e., $$\left(n\log\frac{u_n}{u_{n+1}}-1\right)\log n>p-\frac{1}{2}\left\{\frac{\log n}{n}\right\}+\dots \qquad \dots(1)$$

Now as n becomes sufficiently large the L.H.S. and R.H.S. of (1) respectively approach k and p. Also $k>p$. Thus (1) is satisfied for sufficiently large values of n. Hence the series $\Sigma\, u_n$ is convergent if

$$\lim \left[\left\{ n \log \frac{u_n}{u_{n+1}} - 1 \right\} \log n \right] > 1.$$

Similarly, it can be proved that Σu_n is divergent if

$$\lim \left[\left\{ n \log \frac{u_n}{u_{n+1}} - 1 \right\} \log n \right] < 1.$$

Note: This test is to be applied when the log test of article 14 fails *i.e.*, when $\lim \dfrac{u_n}{u_{n+1}} = 1$ and also $\lim n \log \dfrac{u_n}{u_{n+1}} = 1$.

Illustrative Examples

Example 33: *Test for convergence the following series :*

(*i*) $\dfrac{1^2}{2^2} + \dfrac{1^2 . 3^2}{2^2 . 4^2} x + \dfrac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} x^2 + \dots$

(Bundelkhand 2014)

(*ii*) $1 + \dfrac{\alpha . \beta}{1 . \gamma} x + \dfrac{\alpha (\alpha + 1) \beta (\beta + 1)}{1 . 2 . \gamma (\gamma + 1)} x^2 + \dfrac{\alpha (\alpha + 1)(\alpha + 2) \beta (\beta + 1)(\beta + 2)}{1 . 2 . 3 . \gamma (\gamma + 1)(\gamma + 2)} x^3 + \dots$

Solution: (i) Here $u_n = \dfrac{1^2 . 3^2 \dots (2n-1)^2}{2^2 . 4^2 \dots (2n)^2} x^{n-1}$,

and $$u_{n+1} = \frac{1^2 . 3^2 \dots . (2n-1)^2 . (2n+1)^2}{2^2 . 4^2 \dots . (2n)^2 (2n+2)^2} x^n.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} \cdot \frac{1}{x} = \left\{ \frac{2 + 2/n}{2 + 1/n} \right\}^2 \cdot \frac{1}{x}.$$

$$\therefore \quad \lim \frac{u_n}{u_{n+1}} = \lim \left[\left\{ \frac{2 + 2/n}{2 + 1/n} \right\}^2 \cdot \frac{1}{x} \right] = \frac{2^2}{2^2} \cdot \frac{1}{x} = \frac{1}{x}.$$

$\therefore$ by ratio test the given series Σu_n is convergent if $1/x > 1$ *i.e.*, $x < 1$, divergent if $1/x < 1$ *i.e.*, $x > 1$ and the test fails if $1/x = 1$ *i.e.*, $x = 1$.

When $x = 1$, we have

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2}.$$

$$\therefore \quad n \left\{ \frac{u_n}{u_{n+1}} - 1 \right\} = n \left\{ \frac{(2n+2)^2}{(2n+1)^2} - 1 \right\} = \frac{n(4n+3)}{(2n+1)^2} = \frac{4 + 3/n}{(2 + 1/n)^2}.$$

$$\therefore \quad \lim n \left\{ \frac{u_n}{u_{n+1}} - 1 \right\} = \lim \frac{4 + 3/n}{(2 + 1/n)^2} = \frac{4}{2^2} = 1.$$

$\therefore$ Raabe's test also fails when $x = 1$ and so we shall now apply De Morgan's test.

Now $$n\left\{\frac{u_n}{u_{n+1}}-1\right\}-1=\frac{n(4n+3)}{(2n+1)^2}-1=\frac{-n-1}{(2n+1)^2}.$$

$\therefore$ $$\lim\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]$$

$$=\lim\left[\left\{\frac{-n-1}{(2n+1)^2}\right\}\log n\right]=\lim\left[\frac{-1-1/n}{(2+1/n)^2}\cdot\frac{\log n}{n}\right]$$

$$=\frac{-1}{2^2}.0=0<1. \qquad \left[\textbf{Note} \text{ that } \lim\frac{\log n}{n}=0\right]$$

$\therefore$ by De Morgan's test $\Sigma\, u_n$ is divergent when $x=1$.

Hence the given series $\Sigma\, u_n$ is convergent if $x<1$ and divergent if $x\geq 1$.

(ii) Omitting the first term, we have

$$u_n=\frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)\,\beta(\beta+1)(\beta+2)\ldots\ldots(\beta+n-1)}{1.2\ldots n.\gamma(\gamma+1)(\gamma+2)\ldots(\gamma+n-1)}x^n,$$

$$u_{n+1}=\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)(\alpha+n)\,\beta(\beta+1)\ldots(\beta+n-1)(\beta+n)}{1.2\ldots n(n+1).\gamma(\gamma+1)\ldots(\gamma+n-1)(\gamma+n)}x^{n+1}.$$

Now $$\frac{u_n}{u_{n+1}}=\frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}\cdot\frac{1}{x}=\frac{(1+1/n)(\gamma/n+1)}{(\alpha/n+1)(\beta/n+1)}\cdot\frac{1}{x}.$$

$\therefore$ $\lim\dfrac{u_n}{u_{n+1}}=\dfrac{1.1}{1.1}\cdot\dfrac{1}{x}=\dfrac{1}{x}$ so that by ratio test the series is convergent if $1/x>1$ *i.e.*, $x<1$ and divergent if $1/x<1$ *i.e.*, $x>1$ and the test fails if $1/x=1$ *i.e.*, $x=1$.

When $x=1$, we have

$$\frac{u_n}{u_{n+1}}=\frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}=\frac{n^2+(\gamma+1)n+\gamma}{n^2+(\alpha+\beta)n+\alpha\beta}.$$

$\therefore$ $$n\left(\frac{u_n}{u_{n+1}}-1\right)=n\left[\frac{n^2+(\gamma+1)n+\gamma}{n^2+(\alpha+\beta)n+\alpha\beta}-1\right]$$

$$=\frac{n\{(\gamma+1-\alpha-\beta)n+(\gamma-\alpha\beta)\}}{n^2+(\alpha+\beta)n+\alpha\beta}$$

$$=\frac{(\gamma+1-\alpha-\beta)+(\gamma-\alpha\beta)/n}{1+(\alpha+\beta)/n+\alpha\beta/n^2}.$$

$\therefore$ $$\lim n\left(\frac{u_n}{u_{n+1}}-1\right)=\frac{\gamma+1-\alpha-\beta}{1}=\gamma+1-\alpha-\beta.$$

$\therefore$ if $x=1$, then by Raabe's test, the series is convergent if $\gamma+1-\alpha-\beta>1$ *i.e.*, if $\gamma>\alpha+\beta$, divergent if $\gamma+1-\alpha-\beta<1$ *i.e.*, if $\gamma<\alpha+\beta$, and the test fails if $\gamma+1-\alpha-\beta=1$ *i.e.*, if $\gamma=\alpha+\beta$.

When $\gamma=\alpha+\beta$, we have

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=\frac{n\{n+\alpha+\beta-\alpha\beta\}}{n^2+(\alpha+\beta)n+\alpha\beta}.$$

Now we shall apply De Morgan's test.

We have

$$\lim\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]=\lim\left[\left\{\frac{n(n+\alpha+\beta-\alpha\beta)}{n^2+(\alpha+\beta)n+\alpha\beta}-1\right\}\log n\right]$$

$$=\lim\left[\frac{-\alpha\beta n-\alpha\beta}{n^2+(\alpha+\beta)n+\alpha\beta}\cdot\log n\right]$$

$$=\lim\left[\frac{-\alpha\beta(1+1/n)}{1+(\alpha+\beta)/n+\alpha\beta/n^2}\cdot\frac{\log n}{n}\right]$$

$$=\frac{-\alpha\beta}{1}\cdot 0=0, \text{ which is} <1. \qquad \left[\textbf{Note} \text{ that } \lim\frac{\log n}{n}=0\right]$$

$\therefore$ by De-Morgan's test the series is divergent if $\gamma=\alpha+\beta$.

Thus the given series is convergent if $x<1$, divergent if $x>1$ and for $x=1$, the series is convergent if $\gamma>\alpha+\beta$ and divergent if $\gamma\le\alpha+\beta$.

Example 34: *Test for convergence the series*

$$1^p+\left(\frac{1}{2}\right)^p+\left(\frac{1.3}{2.4}\right)^p+\left(\frac{1.3.5}{2.4.6}\right)^p+\ldots$$

Solution: Omitting the first term 1^p, we have

$$u_n=\left[\frac{1.3.5.....(2n-1)}{2.4.6.....(2n)}\right]^p,$$

and then $$u_{n+1}=\left[\frac{1.3.5...(2n-1)(2n+1)}{2.4.6...(2n)(2n+2)}\right]^p.$$

Now $$\frac{u_n}{u_{n+1}}=\left(\frac{2n+2}{2n+1}\right)^p=\left(\frac{1+1/n}{1+1/2n}\right)^p.$$

$\therefore$ $$\lim\frac{u_n}{u_{n+1}}=\left(\frac{1}{1}\right)^p=1 \; i.e., \text{ the ratio test fails.}$$

Now we apply logarithmic test.

We have $$\log\frac{u_n}{u_{n+1}}=\log\left(\frac{2n+2}{2n+1}\right)^p=\log\left(\frac{1+1/n}{1+1/2n}\right)^p$$

$$=p[\log(1+1/n)-\log(1+1/2n)]$$

$$=p\left[\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right)-\left(\frac{1}{2n}-\frac{1}{2.2^2n^2}+\frac{1}{3.2^3n^3}-\ldots\right)\right]$$

$$=p\left[\left\{1-\frac{1}{2}\right\}\frac{1}{n}-\frac{1}{2}\cdot\left\{1-\frac{1}{4}\right\}\frac{1}{n^2}+\frac{1}{3}\left\{1-\frac{1}{8}\right\}\frac{1}{n^3}-\ldots\right]$$

$$= p\left[\frac{1}{2n} - \frac{3}{8n^2} + \frac{7}{24n^3} - \ldots\right].$$

$$\therefore \qquad n \log \frac{u_n}{u_{n+1}} = p\left[\frac{1}{2} - \frac{3}{8n} + \frac{7}{24n^2} - \ldots\right].$$

$\therefore\quad \lim n \log \dfrac{u_n}{u_{n+1}} = \dfrac{p}{2}$, so that the series is convergent if $p/2 > 1$ *i.e.*, if $p > 2$, divergent if $p/2 < 1$ *i.e.*, if $p < 2$ and the test fails if $p/2 = 1$ *i.e.*, if $p = 2$.

If $p = 2$, we have

$$n \log \frac{u_n}{u_{n+1}} = 2\left[\frac{1}{2} - \frac{3}{8n} + \frac{7}{24n^2} - \ldots\right] = 1 - \frac{3}{4n} + \frac{7}{12n^2} - \ldots.$$

$$\therefore \qquad \lim\left[\left(n \log \frac{u_n}{u_{n+1}} - 1\right)\log n\right] = \lim\left[\left\{-\frac{3}{4n} + \frac{7}{12n^2} - \ldots\right\}.\log n\right]$$

$$= \lim\left[\left\{-\frac{3}{4} + \frac{7}{12n} - \ldots\right\}\cdot\frac{\log n}{n}\right] = \left\{-\frac{3}{4}\right\}.0 = 0, \text{ which is } < 1.$$

Hence by Alternative to Bertrand's test given in article 16, the series is divergent when $p = 2$.

Thus the given series is convergent if $p > 2$ and divergent if $p \le 2$.

Comprehensive Exercise 6

Test for convergence the following series :

1. $1 + \dfrac{2^2}{3^2} + \dfrac{2^2.4^2}{3^2.5^2} + \dfrac{2^2.4^2.6^2}{3^2.5^2.7^2} + \ldots$ **(Kashi 2013; Meerut 13)**

2. $\dfrac{1^2}{2^2} + \dfrac{1^2.3^2}{2^2.4^2} + \dfrac{1^2.3^2.5^2}{2^2.4^2.6^2} + \ldots$ **(Kumaun 2003)**

3. $\dfrac{a}{b} + \dfrac{a(a+1)}{b(b+1)} + \dfrac{a(a+1)(a+2)}{b(b+1)(b+2)} + \ldots$

4. $1 + \dfrac{a(1-a)}{1^2} + \dfrac{(1+a)a(1-a)(2-a)}{1^2.2^2} + \dfrac{(2+a)(1+a)a(1-a)(2-a)(3-a)}{1^2.2^2.3^2} + \ldots$

5. $1 + \dfrac{\alpha}{1.\beta}x + \dfrac{\alpha(\alpha+1)^2}{1.2\beta(\beta+1)}x^2 + \dfrac{\alpha(\alpha+1)^2(\alpha+2)^2}{1.2.3\beta(\beta+1)(\beta+2)}x^3 + \ldots$

6. $\left\{\dfrac{1}{2.4}\right\}^{2/3} + \left\{\dfrac{1.3}{2.4.6}\right\}^{2/3} + \left\{\dfrac{1.3.5}{2.4.6.8}\right\}^{2/3} + \ldots$

7. $x + x^{1+1/2} + x^{1+1/2+1/3} + x^{1+1/2+1/3+1/4} + \ldots$

Answers 6

1. Divergent
2. Divergent
3. Convergent if $b - a > 1$ and divergent if $b - a \leq 1$
4. Divergent
5. Convergent if $x < 1$, divergent if $x > 1$ and when $x = 1$ then convergent if $\beta > 2\alpha$ and divergent if $\beta \leq 2\alpha$
6. Divergent
7. Convergent if $x < 1/e$ and divergent if $x \geq 1/e$

17 Summary of Tests

Let the given series of positive terms be Σu_n. Then to test the series for convergence we proceed as follows :

1. **Find lim u_n:** (a) If $\lim u_n > 0$, the series is divergent.

   (b) If $\lim u_n = 0$, then the series may or may not be convergent. In this case we apply further tests to decide the nature of the series.

2. **If lim $u_n = 0$** and u_n can be arranged as an algebraic fraction in n, then usually **comparison test** should be applied.

3. **If n occurs as an exponent in** u_n and $\lim (u_n)^{1/n}$ can be easily evaluated, then **Cauchy's root test** should be applied.

4. **Cauchy's condensation test** is generally applied when u_n involves $\log n$.

   In case all the above tests are not applicable then we adopt the following scheme of testing in the given order.

5. **D' Alembert's ratio test:** For this we find $\lim \frac{u_n}{u_{n+1}}$. The series is convergent or divergent according as this limit is > 1 or < 1. In case this limit is equal to 1 (unity), this test fails. Then we proceed to apply either test 6 (a) or 6 (b) or 6(c) given below depending upon the nature of u_n and u_n / u_{n+1}.

6. **(a) Comparison test:** In some cases when D' Alembert's ratio test fails, the convergence of the series may be decided by comparison test.

   (b) Raabe's test: For this we find $\lim n\left(\frac{u_n}{u_{n+1}} - 1\right)$. The series is convergent or divergent according as this limit is > 1 or < 1. In case the limit is equal to 1, this test fails and we apply test 7 (a).

(c) Logarithmic test: If $\frac{u_n}{u_{n+1}} - 1$ cannot be evaluated easily while $\log \frac{u_n}{u_{n+1}}$ can be easily evaluated then we apply logarithmic test. Here we find $\lim \left(n \log \frac{u_n}{u_{n+1}} \right)$. If this limit > 1, the series is convergent and if this limit < 1, the series is divergent. In case the limit = 1, this test fails and we apply test 7 (b).

7. **(a) De Morgan's and Bertrand's test:**

Find $\lim \left[\left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) - 1 \right\} \log n \right]$.

The series is convergent or divergent according as this limit is > 1 or < 1.

Note: When this test is applied, we shall generally find that the limit comes out to be equal to zero and since 0 < 1, the series is divergent.

(b) Alternative to Bertrand's test:

To apply this test we find $\lim \left[\left(n \log \frac{u_n}{u_{n+1}} - 1 \right) \log n \right]$.

The series is convergent or divergent according as this limit is > 1 or < 1.

18 Kummer's Test

Theorem: *Let* $\Sigma\, u_n$ *and* $\Sigma\,(1/d_n)$ *be two series of positive terms and let* $v_n = d_n\,(u_n / u_{n+1}) - d_{n+1}$. *Then*

(*i*) *if a fixed positive number* k *can be found so that after a certain stage, say for* $n \geq m$, $v_n \geq k$, *the series,* $\Sigma\, u_n$ *is convergent;*

(*ii*) *if* $v_n \leq 0$ *for* $n \geq m$ and $\Sigma\,(1/d_n)$ *is divergent,* $\Sigma\, u_n$ *is divergent.*

Proof: (i) From the condition given in the statement of the theorem for $\Sigma\, u_n$ to be convergent, we have for $n \geq m$, where m is a fixed +ive integer,

$$v_n \geq k > 0 \quad i.e., \quad d_n \left(\frac{u_n}{u_{n+1}} \right) - d_{n+1} \geq k$$

i.e.,
$$d_n\, u_n - d_{n+1}\, u_{n+1} \geq k\, u_{n+1} \qquad \ldots(1)$$

[**Note** that u_{n+1} is positive]

Replacing n by $m, m+1, m+2, \ldots, n-1$ in succession in (1), we get

$$d_m\, u_m - d_{m+1}\, u_{m+1} \geq k\, u_{m+1},$$
$$d_{m+1}\, u_{m+1} - d_{m+2}\, u_{m+2} \geq k\, u_{m+2},$$
$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$
$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$
$$d_{n-1}\, u_{n-1} - d_n\, u_n \geq k\, u_n.$$

Adding the corresponding sides of these inequalities, we get

$$d_m u_m - d_n u_n \geq k\,(u_{m+1} + u_{m+2} + \ldots + u_n)$$

or $$u_{m+1} + u_{m+2} + \ldots + u_n \leq \frac{1}{k}(d_m u_m - d_n u_n) \qquad [\because\ k > 0]$$

or $$u_{m+1} + u_{m+2} + \ldots + u_n < \frac{1}{k} d_m u_m \qquad \ldots(2)$$

or $$S_n - S_m < \frac{1}{k} d_m u_m,$$

where $S_n = u_1 + \ldots + u_m + u_{m+1} + \ldots + u_n = S_m + u_{m+1} + \ldots + u_n$

or $$S_n < S_m + \frac{1}{k} d_m u_m, \text{ using (2).}$$

Since S_n is less than a fixed number, hence the series $\Sigma\, u_n$ is convergent.

(ii) We have $v_n \leq 0$ for $n \geq m$ (given)

i.e., $$d_n \left(\frac{u_n}{u_{n+1}} \right) - d_{n+1} \leq 0 \text{ for } n \geq m$$

i.e., $$d_n u_n \leq d_{n+1}\, u_{n+1}, \text{ for } n \geq m.$$

Putting $n = m, m+1, m+2, \ldots, n-1$ in succession, we have

$$d_m u_m \leq d_{m+1} u_{m+1} \leq d_{m+2}\, u_{m+2} \leq \ldots\ldots \leq d_n u_n$$

i.e., $$d_m u_m \leq d_n u_n$$

or $$u_n \geq (d_m u_m) / d_n.$$

Now $d_m u_m$ is a fixed number and the series $\Sigma\,(1/d_n)$ is divergent (given), hence, by comparison test the series $\Sigma\, u_n$ is also divergent.

19 Gauss's Test

Theorem: *Let $\Sigma\, u_n$ be a series of positive terms and u_n / u_{n+1} can be expressed in the form*

$$\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + \frac{b_n}{n^p},$$

where $p > 1$ and $|b_n| <$ a fixed number k or (in particular) b_n tends to a finite limit as $n \to \infty$, then $\Sigma\, u_n$ converges if $a > 1$ and diverges if $a \leq 1$.

Proof: It is given that

$$\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + \frac{b_n}{n^p} \quad i.e., \quad n\left(\frac{u_n}{u_{n+1}} - 1 \right) = a + \frac{b_n}{n^{p-1}}.$$

$\therefore$ $$\lim n \left(\frac{u_n}{u_{n+1}} - 1 \right) = a. \qquad [\because\ p > 1 \text{ and } |b_n| < k]$$

Hence by Raabe's test $\Sigma\, u_n$ converges if $a > 1$ and diverges if $a < 1$. The test fails if $a = 1$ and then we apply Kummer's test to find the convergence of $\Sigma\, u_n$.

When $a = 1$, we have

$$\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{b_n}{n^p}.$$

Since the series $\frac{1}{2\log 2} + \frac{1}{3\log 3} + \ldots + \frac{1}{n\log n} + \ldots$ is divergent, we take $d_n = n\log n$ for all $n \geq 2$ in Kummer's test. Then we have

$$v_n = d_n \frac{u_n}{u_{n+1}} - d_{n+1}$$

$$= (n\log n).\left(1 + \frac{1}{n} + \frac{b_n}{n^p}\right) - (n+1)\log(n+1)$$

$$= n\left(1 + \frac{1}{n}\right)\log n + \frac{b_n}{n^{p-1}}\log n - (n+1)\log(n+1)$$

$$= (n+1)\{\log n - \log(n+1)\} + \frac{b_n}{n^{p-1}}\log n$$

$$= (n+1)\log\left(\frac{n}{n+1}\right) + \frac{b_n}{n^{p-1}}\log n$$

$$= (n+1)\log\left(1 - \frac{1}{n+1}\right) + \frac{\log n}{n^{p-1}}\cdot b_n$$

$$= (n+1)\left[-\frac{1}{n+1} - \frac{1}{2(n+1)^2} - \ldots.\right] + \frac{\log n}{n^{p-1}}\cdot b_n$$

$$= \left[-1 - \frac{1}{2(n+1)} - \ldots\right] + \frac{\log n}{n^{p-1}}\cdot b_n \qquad \ldots(1)$$

Now $\lim \frac{\log n}{n^{p-1}} = 0$ as $p > 1$ and $< b_n >$ is a bounded sequence because $|b_n| < k$.

$$\therefore \qquad \lim\left[\frac{\log n}{n^{p-1}}\cdot b_n\right] = 0 \cdot \qquad \ldots(2)$$

Hence taking limit when $n \to \infty$, we get from (1) with the help of (2) $\lim v_n = -1 + 0 = -1$, which shows that after a certain stage $v_n < 0$. Also $\Sigma\,(1/d_n)$ *i.e.,* $\Sigma\,(1/n\log n)$ is divergent. Hence by Kummer's test the series $\Sigma\, u_n$ is divergent. Thus the series $\Sigma\, u_n$ is convergent if $a > 1$ and divergent if $a \leq 1$.

20 Cauchy-Maclaurin's Integral Test

Improper integrals: Integrals of the form $\int_a^\infty f(x)\,dx$ where $a \in \mathbf{R}$ are called improper integrals.

Let $F(t) = \int_a^t f(x)\,dx$ for $a \leq t < \infty$.

If $\lim_{t \to \infty} F(t)$ exists and is equal to $l \in \mathbf{R}$, the improper integral $\int_a^\infty f(x)\,dx$ is said to **converge** to l, otherwise it is called a **divergent integral**.

Theorem: *Let $f(x)$ be a non-negative monotonically decreasing integrable function on* $[1, \infty[$. *Then the series* $\sum_{n=1}^{\infty} f(n)$ *and the improper integral* $\int_1^{\infty} f(x)\,dx$ *converge or diverge together i.e., the series* $\Sigma f(n)$ *converges or diverges according as the integral* $\int_1^{\infty} f(x)\,dx$ *tends to a finite limit or diverges to* ∞ *as* $n \to \infty$.

Proof: Since $f(x)$ is non-negative on $[1, \infty[$, therefore

$$f(x) \geq 0 \quad \forall\, x \geq 1$$

i.e., the series $\sum_{n=1}^{\infty} f(n)$ is of non-negative terms.

For any $x \in [1, \infty[$, we can find $n \in \mathbf{N}$ such that $n \leq x \leq n+1$.

Since f is monotonically decreasing on $[1, \infty[$, therefore, we have

$$f(n) \geq f(x) \geq f(n+1) \text{ if } n \leq x \leq n+1.$$

$$\therefore \quad \int_n^{n+1} f(n)\,dx \geq \int_n^{n+1} f(x)\,dx \geq \int_n^{n+1} f(n+1)\,dx$$

or $$f(n) \geq \int_n^{n+1} f(x)\,dx \geq f(n+1) \cdot \qquad \ldots(1)$$

Putting $n = 1, 2, \ldots, (n-1)$ in (1) in succession and then adding all the results, we get

$$f(1) + f(2) + \ldots + f(n-1) \geq \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \ldots + \int_{n-1}^{n} f(x)\,dx$$

$$\geq f(2) + f(3) + \ldots + f(n). \qquad \ldots(2)$$

Let $$s_n = f(1) + f(2) + \ldots + f(n)$$

and $$I_n = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \ldots + \int_{n-1}^{n} f(x)\,dx = \int_1^n f(x)\,dx.$$

Then (2) can be written as

$$s_n - f(n) \geq I_n \geq s_n - f(1)$$

or $$-f(n) \geq I_n - s_n \geq -f(1)$$

or $$f(n) \leq s_n - I_n \leq f(1). \qquad \ldots(3)$$

The result (3) is true for all $n \in \mathbf{N}$.

Let $u_n = s_n - I_n$ for all $n \in \mathbf{N}$.

Now $$u_{n+1} - u_n = (s_{n+1} - I_{n+1}) - (s_n - I_n) = (s_{n+1} - s_n) - (I_{n+1} - I_n)$$

$$= f(n+1) - \int_n^{n+1} f(x)\,dx \leq 0, \text{ using (1).}$$

$\therefore \quad u_{n+1} \leq u_n$ for all $n \in \mathbf{N}$

i.e., $< u_n >$ is a monotonically decreasing sequence.

Also by (3), $u_n \geq f(n) \geq 0$ for all n and hence $< u_n >$ is bounded below. Thus the sequence $< u_n >$ is convergent *i.e.,* $< u_n >$ tends to a finite limit as $n \to \infty$.

Since $s_n = u_n + I_n$ and $< u_n >$ is convergent, it follows that the sequences $< s_n >$ and $< I_n >$ converge or diverge together. Consequently the series $\Sigma\, f(n)$ and the integral $\int_1^\infty f(x)\,dx$ converge or diverge together.

Note: The series $\Sigma\, f(n)$ and the integral $\int_a^\infty f(x)\,dx$ converge or diverge together for $a \geq 1$.

Illustrative Examples

Example 35: *Test the convergence of the series*

$$\frac{1^2}{2^2} + \frac{1^2 . 3^2}{2^2 . 4^2} + \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} + \ldots\ldots$$

Solution: Here, $u_n = \dfrac{1^2 . 3^2 . 5^2 \ldots (2n-1)^2}{2^2 . 4^2 . 6^2 \ldots (2n)^2}$

and $$u_{n+1} = \frac{1^2 . 3^2 . 5^2 \ldots (2n-1)^2\ (2n+1)^2}{2^2 . 4^2 . 6^2 \ldots (2n)^2\ (2n+2)^2}.$$

Now $$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2}.$$

$\therefore$ $$\lim \frac{u_n}{u_{n+1}} = 1.$$

Hence D'Alembert's ratio test fails. Now we apply Raabe's test. We have

$$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim n\left\{\frac{(2n+2)^2}{(2n+1)^2} - 1\right\} = \lim \frac{n\,(4n+3)}{(2n+1)^2} = 1$$

i.e. Raabe's test also fails. Now we apply Gauss test. We can write

$$\frac{u_n}{u_{n+1}} = \left(1 + \frac{1}{n}\right)^2 \left(1 + \frac{1}{2n}\right)^{-2}$$

$$= \left(1 + \frac{2}{n} + \frac{1}{n^2}\right)\left(1 - 2 . \frac{1}{2n} + 3 . \frac{1}{4n^2} - \ldots\right)$$

$$= 1 + \frac{1}{n} - \frac{1}{4n^2} + \ldots = 1 + \frac{1}{n} + \frac{1}{n^2}\left(-\frac{1}{4} + \ldots\right)$$

$$= 1 + \frac{a}{n} + \frac{b_n}{n^2} \text{ where } b_n \to -\frac{1}{4} \text{ as } n \to \infty.$$

Since $a = 1$, hence by Gauss's test, the series $\Sigma\, u_n$ is divergent.

Example 36: *Test for convergence the series*

$$1 + \frac{2^2}{3^2} + \frac{2^2 . 4^2}{3^2 . 5^2} + \frac{2^2 . 4^2 . 6^2}{3^2 . 5^2 . 7^2} + \ldots\ldots$$

Solution: Here, $u_n = \dfrac{2^2.4^2.6^2......(2n-2)^2}{3^2.5^2.7^2......(2n-1)^2}$

and $$u_{n+1} = \frac{2^2.4^2.6^2......(2n-2)^2\,(2n)^2}{3^2.5^2.7^2......(2n-1)^2\,(2\,n+1)^2}.$$

$\therefore$ $$\lim \frac{u_n}{u_{n+1}} = \lim \frac{(2n+1)^2}{(2n)^2} = 1 \text{ so that ratio test fails.}$$

Now we apply Raabe's test. We have
$$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim n\left\{\frac{(2n+1)^2}{4n^2} - 1\right\} = \lim \frac{n\,(4n+1)}{4n^2} = 1.$$

Hence Raabe's test also fails. Now we apply Gauss test. We can write
$$\frac{u_n}{u_{n+1}} = \frac{4n^2+4n+1}{4n^2} = 1 + \frac{1}{n} + \frac{1}{4n^2} = 1 + \frac{a}{n} + \frac{b_n}{n^p}.$$

Here $a = 1$, $b_n = \frac{1}{4}$, $p = 2 > 1$. Consequently the series is divergent by Gauss's test.

Example 37: *Show by Cauchy's integral test that the series*
$$\sum_{n=2}^{\infty} \frac{1}{n\,(\log n)^p}$$
converges if $p > 1$ *and diverges if* $0 < p \le 1$.

Solution: Let $f(x) = \dfrac{1}{x\,(\log x)^p}$, $p > 0$ and $x \in [2, \infty[$. Then $f(x) > 0$ and is monotonically decreasing for $2 \le x < \infty$.

Let $$I_n = \int_2^n \frac{1}{x\,(\log x)^p}\,dx.$$

Then $$I_n = \left[\frac{(\log x)^{1-p}}{1-p}\right]_2^n, \text{ when } p \ne 1$$
$$= \frac{1}{1-p}\left[(\log n)^{1-p} - (\log 2)^{1-p}\right]$$

and when $p = 1$, we have $I_n = [\log\log x]_2^n = \log\log n - \log\log 2$.

Hence when $n \to \infty$, we have
$$\lim_{n\to\infty} \int_2^n f(x)\,dx = \infty, \text{ if } p \le 1, \quad \text{and} \quad = -\frac{1}{1-p}(\log 2)^{1-p} \text{ if }.$$

Thus the integral $\int_2^\infty f(x)\,dx$ converges if $p > 1$ and diverges if $0 < p \le 1$.

Hence by Cauchy's integral test the series
$$\sum_{n=2}^{\infty} f(n) = \sum_{n=2}^{\infty} \frac{1}{n\,(\log n)^p}$$
converges if $p > 1$ and diverges if $0 < p \le 1$.

Example 38: *Show by integral test that the series* $\Sigma \frac{1}{n^p}$ $(p > 0)$ *converges if* $p > 1$ *and diverges if* $0 < p \le 1$. **(Agra 2014)**

Solution: Let $f(x) = \frac{1}{x^p}$, $p > 0$ and $x \in [1, \infty[$.

Then $f(x) > 0$ and is monotonically decreasing on $[1, \infty[$.

Let
$$I_n = \int_1^n \frac{1}{x^p}\, dx = \int_1^n x^{-p}\, dx$$
$$= \begin{cases} \frac{n^{1-p}}{1-p} - \frac{1}{1-p}, & \text{if } p \ne 1 \\ \log n, & \text{if } p = 1. \end{cases}$$

Now, when $n \to \infty$, $n^{1-p} = \frac{1}{n^{p-1}} \to 0$ if $p > 1$, $n^{1-p} \to \infty$ if $p < 1$ and $\log n \to \infty$.

$$\therefore \qquad \lim_{n \to \infty} I_n = -\frac{1}{1-p} = \frac{1}{p-1} \text{ if } p > 1 \text{ and } \lim I_n = \infty \text{ if } p \le 1.$$

Thus the integral $\int_1^\infty f(x)\, dx$ converges if $p > 1$ and diverges if $p \le 1$ and hence by Cauchy's integral test the series $\Sigma \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$.

Example 39: *Show that* $\lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n\right)$ *exists.*

Solution: Let $f(x) = \frac{1}{x}$ where $1 \le x < \infty$.

Then $f(x) > 0$ and monotonically decreasing on $[1, \infty[$.

Take $s_n = f(1) + f(2) + \ldots + f(n) = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$,

and
$$I_n = \int_1^n f(x)\, dx = \int_1^n \frac{1}{x}\, dx = \log n.$$

Then proceeding as in article 20 and thus here using condition (3) of article 20, we get

$$f(n) \le s_n - I_n \le f(1) \text{ for all } n, \quad \text{or} \quad 0 < \frac{1}{n} \le s_n - I_n \le 1 \text{ for all } n.$$

This shows that the sequence $< s_n - I_n >$ is bounded below.

Also as shown in article 20, $< s_n - I_n >$ is a monotonically decreasing sequence and hence it converges.

We call the limit of this sequence **Euler's constant** and denote it by γ which is $0 \cdot 577$ approximately.

Hence $\lim \left\{1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n\right\}$ exists and is equal to γ, where γ is called Euler's constant.

Comprehensive Exercise 7

1. Apply integral test to check the convergence of the following series :

   (i) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, (ii) $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$, (iii) $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$.

2. Apply Cauchy's integral test to prove the convergence of the series

   (i) $\Sigma \frac{1}{n^2+1}$, (ii) $\Sigma \frac{1}{n(n+1)}$, (iii) $\Sigma \frac{1}{n\sqrt{(n^2-1)}}$.

3. If $f(x)$ is positive and monotonically decreasing when $x \geq 1$, then prove that the sequence whose nth term is

$$f(1) + f(2) + \ldots + f(n) - \int_1^n f(x)\, dx$$

   converges to a finite limit.

Answers 7

1. (i) Divergent (ii) Convergent (iii) Convergent

21 Alternating Series

So far we have mainly dealt with series of positive terms. We have seen that a series of positive terms either converges or diverges and cannot oscillate. But a series which contains an infinite number of positive and an infinite number of negative terms may either converge or diverge or oscillate.

Alternating Series: Definition: *A series whose terms are alternately positive and negative is called an* **alternating series.** Thus an alternating series is of the form

$$u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1} u_n + \ldots$$

where $u_n > 0$ for all n. It is denoted as

$$\sum_{n=1}^{\infty} (-1)^{n-1} u_n .$$

The following are some examples of an alternating series.

(i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots$

(ii) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \ldots + \frac{(-1)^{n-1}(n+1)}{2n} + \ldots$

(iii) $1 - \frac{2}{\log 2} + \frac{3}{\log 3} - \frac{4}{\log 4} + \ldots$

Theorem: Alternating Series Test (Leibnitz's Test): *An infinite series* $\Sigma\,(-1)^{n-1}\,u_n$ *in which the terms are alternately positive and negative is convergent if each term is numerically less than the preceding term and* $\lim u_n = 0$.
Symbolically, the alternating series

$$u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1}\,u_n + \ldots,\ (u_n > 0\ \textit{for all } n)$$

converges if

(*i*) $u_{n+1} \le u_n$ *for all n i.e.,* $u_1 \ge u_2 \ge u_3 \ge u_4 \ge \ldots$

and (*ii*) $\lim u_n = 0$ *i.e.,* $u_n \to 0$ *as* $n \to \infty$.

Proof: Let $S_n = u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1}\,u_n$ so that $< S_n >$ is the sequence of partial sums of the given series.

We shall prove the theorem in **two steps**.

(i) First we shall prove that the subsequences $< S_{2n} >$ and $< S_{2n+1} >$ of the sequence $< S_n >$ converge to the same limit, say S.

We have $\quad S_{2n} = u_1 - u_2 + \ldots + u_{2n-1} - u_{2n}$

and $\quad S_{2n+2} = u_1 - u_2 + \ldots + u_{2n-1} - u_{2n} + u_{2n+1} - u_{2n+2}$.

$\therefore\quad S_{2n+2} - S_{2n} = u_{2n+1} - u_{2n+2} \ge 0$ for all n because it is given that $u_{n+1} \le u_n$ for all n.

$\therefore\quad S_{2n+2} \ge S_{2n}$ for all n and so the sequence $< S_{2n} >$ is monotonically increasing.

Again for all n,

$$S_{2n} = u_1 - [(u_2 - u_3) + (u_4 - u_5) + \ldots + (u_{2n-2} - u_{2n-1}) + u_{2n}]$$

$= u_1 -$ some positive number because $u_2 - u_3, \ldots, u_{2n-2} - u_{2n-1}, u_{2n}$ are all positive

$\le u_1$.

Thus $S_{2n} \le u_1$ for all n and so the sequence $< S_{2n} >$ is bounded above.

Since the sequence $< S_{2n} >$ is monotonically increasing and bounded above, therefore it converges. Let $\lim S_{2n} = S$.

Now $\quad S_{2n+1} = S_{2n} + u_{2n+1}$.

$$\therefore\quad \lim S_{2n+1} = \lim S_{2n} + \lim u_{2n+1}$$
$$= S + 0 \qquad [\because\ \lim u_n = 0]$$
$$= S.$$

$\therefore\quad$ the sequence $< S_{2n+1} >$ also converges to S.

Thus the subsequences $< S_{2n} >$ and $< S_{2n+1} >$ of the sequence $< S_n >$ converge to the same limit S.

(ii) Now we shall show that the sequence $< S_n >$ also converges to S.

Let $\varepsilon > 0$ be given. Since the sequences $< S_{2n} >$ and $< S_{2n+1} >$ both converge to S, therefore there exist +ive integers m_1 and m_2 such that

$$|\,S_{2n} - S\,| < \varepsilon \text{ for all } n \ge m_1$$

and
$$|\,S_{2n+1} - S\,| < \varepsilon \text{ for all } n \ge m_2.$$

Let $\quad m = \max\,(m_1, m_2)$.

Then $\quad |\,S_n - S\,| < \varepsilon$ for all $n \ge 2m$.

$\therefore$ the sequence $< S_n >$ converges to S.

Hence the given series $\Sigma\, (-1)^{n-1}\, u_n$ converges.

Note 1: The above test is equally applicable to the series $\Sigma\,(-1)^n\, u_n, u_n > 0$ for all n, provided both the conditions (i) and (ii) are satisfied.

Note 2: If in the case of an alternating series

$$u_1 - u_2 + u_3 - u_4 + \ldots\ldots\ (u_n > 0 \text{ for all } n),$$

the terms continually decrease, we cannot say that the series is convergent unless $\lim u_n = 0$. Because if $\lim u_n \neq 0$, then $\lim S_{2n}$ and $\lim S_{2n+1}$ will differ and so the series will not be convergent. Such a series is an oscillatory series.

For example, consider the series

$$2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \ldots .$$

Here the terms are alternately positive and negative and each term is numerically less than the preceding term because

$$2 > \frac{3}{2} > \frac{4}{3} > \frac{5}{4} > \ldots\ldots$$

But here $\lim u_n = \lim \dfrac{n+1}{n} = \lim \left(1 + \dfrac{1}{n}\right) = 1 \neq 0$. Hence the given series is not convergent. As a matter of fact it is an oscillatory series.

Illustrative Examples

Example 40: *Show that the series* $1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \ldots$ *converges.* **(Avadh 2012)**

Solution: The given series is an alternating series

$$u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1} u_n + \ldots, (u_n > 0 \text{ for all } n).$$

Here $u_n = 1/n > 0$ for all n.

We have $\quad u_{n+1} - u_n = \dfrac{1}{n+1} - \dfrac{1}{n} = \dfrac{n-n-1}{n\,(n+1)} = \dfrac{-1}{n\,(n+1)} < 0$ for all n.

Thus $u_{n+1} < u_n$ for all n *i.e.*, each term is numerically less than the preceding term.

Also $\quad \lim u_n = \lim \dfrac{1}{n} = 0.$

Hence by Leibnitz's test for alternating series, the given series is convergent.

Example 41: *Show that the following series are convergent.*

(i) $1^{-p} - 2^{-p} + 3^{-p} - \ldots$ *when* $p > 0$.

(ii) $\dfrac{1}{x+1} - \dfrac{1}{x+2} + \dfrac{1}{x+3} - \ldots$ *except when x is a negative integer.*

Solution: (i) The given series is an alternating series

$$u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1} u_n + \ldots, (u_n > 0 \text{ for all } n).$$

Here $u_n = 1 / n^p > 0$ for all n.

Also since $p > 0$, we have $\frac{1}{1^p} > \frac{1}{2^p} > \frac{1}{3^p} > \ldots.$

Thus $u_{n+1} < u_n$ for all n.

Also $$\lim u_n = \lim \frac{1}{n^p} = 0, \text{ since } p > 0.$$

Hence by alternating series test the given series is convergent for $p > 0$.

(ii) The given series is

$$\frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3} - \ldots, x \text{ is not a –ive integer.}$$

If $x > -1$, then the terms are alternately positive and negative from the beginning. If $x < -1$, excluding –ive integers, then the terms are *ultimately* alternating in sign.

Since the removal of a finite number of terms does not affect the convergence of the series, therefore we may assume the series to be alternating in sign in both the cases.

Obviously $u_1 > u_2 > u_3 > u_4 > \ldots$ *i.e.*, each term of the series is numerically less than the preceding term.

Also $$\lim u_n = \lim \frac{1}{x+n} = 0.$$

Hence by alternating series test, the given series is convergent.

Comprehensive Exercise 8

1. Examine the convergence of the series

$$\frac{1}{1.2} - \frac{1}{3.4} + \frac{1}{5.6} - \frac{1}{7.8} + \ldots.$$

2. Show that the series

$$\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} - \ldots \text{ converges.}$$

3. Examine the convergence of the series

$$\sum_{n=1}^{\infty} (-1)^n \left[\frac{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}}{n} \right].$$

4. Test the convergence of the series

$$\sum_{n=1}^{\infty} \left[\frac{1}{n} + \frac{(-1)^{n+1}}{\sqrt{n}} \right].$$

Answers 8

1. Convergent **3.** Convergent **4.** Divergent

22 Absolute Convergence and Conditional Convergence

(Meerut 2012B)

Absolutely Convergent Series:

Definition: *A series Σu_n is said to be* **absolutely convergent** *if the series $\Sigma |u_n|$ is convergent.*

If Σu_n is a series of positive terms, then Σu_n and $\Sigma |u_n|$ are the same series and so if Σu_n is convergent, it is also absolutely convergent. Hence for a series of positive terms the concepts of convergence and absolute convergence are the same.

But if a series Σu_n contains an *infinite* number of positive and an infinite number of negative terms, then Σu_n is absolutely convergent only if the series $\Sigma |u_n|$ obtained from Σu_n by making all its terms positive is convergent.

For example the series

$$\Sigma u_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \dots$$

is absolutely convergent. Here we see that the series

$$\Sigma |u_n| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$$

is an infinite geometric series of positive terms with common ratio $\frac{1}{2}$ which is < 1 and so it is convergent. Hence the given series Σu_n is absolutely convergent.

Non-absolutely convergent or semi-convergent or conditionally convergent series:

Definition: *A series Σu_n is said to be* **semi-convergent** *or* **conditionally convergent** *or* **non-absolutely convergent** *if Σu_n is convergent but $\Sigma |u_n|$ is divergent.*

For example, consider the series

$$\Sigma u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots.$$

It is an alternating series in which each term is numerically less than the preceding term and $\lim u_n = \lim (1/n) = 0$. Hence by alternating series test, Σu_n is a convergent series.

But the series $\Sigma |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is the series $\Sigma (1/n^p)$, for $p = 1$, and we know that it is divergent. Thus here Σu_n is convergent while $\Sigma |u_n|$ is divergent.

Hence Σu_n is a semi-convergent or conditionally convergent or non-absolutely convergent series.

Tests for absolute convergence: To test the absolute convergence of the series $\Sigma\, u_n$, we have to simply test the convergence of the series $\Sigma \,|\, u_n \,|$ which is a series of positive terms. Hence the various tests given for the series of positive terms are precisely the tests which we are to apply to check the absolute convergence of the series $\Sigma\, u_n$. We have to simply replace u_n by $|\, u_n \,|$ in these tests. For example by Cauchy's root test, the series $\Sigma\, u_n$ is absolutely convergent if $\lim |\, u_n \,|^{1/n} < 1 \cdot$ Similarly by D'Alembert's ratio test the series $\Sigma\, u_n$ is absolutely convergent if

$$\lim \frac{|\, u_n|}{|\, u_{n+1}|} = \lim \left| \frac{u_n}{u_{n+1}} \right| > 1.$$

Similarly comparison test or other tests may be used.

However these tests cannot give any information about the conditional convergence of the series.

23 Some Important Theorems on Absolutely Convergent Series

Theorem 1: *Every absolutely convergent series is convergent but the converse is not necessarily true i.e., convergence need not imply absolute convergence.*

Theorem 2: *In an absolutely convergent series, the series formed by its positive terms alone is convergent and the series formed by its negative terms alone is convergent.*

Theorem 3: Re-arrangement of terms of an absolutely convergent series:

If the terms of an absolutely convergent series are re-arranged the series remains convergent and its sum unaltered.

Or The sum of an absolutely convergent series is independent of the order of terms.

Illustrative Examples

Example 42: *Show that the series*

$$\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$$ **(Meerut 2012)**

is conditionally convergent.

Solution: The given series is an alternating series

$$u_1 - u_2 + u_3 - \dots + (-1)^{n-1}\, u_n + \dots, (u_n > 0 \text{ for all } n).$$

Here $u_n = \dfrac{1}{\sqrt{n}} > 0$ for all n.

Also for all n, $\sqrt{(n+1)} > \sqrt{n}$

$$\Rightarrow \qquad \frac{1}{\sqrt{(n+1)}} < \frac{1}{\sqrt{n}} \quad \Rightarrow \quad u_{n+1} < u_n \text{ for all } n.$$

Again $$\lim u_n = \lim \frac{1}{\sqrt{n}} = 0.$$

Hence by Leibnitz's test, the given series $\Sigma\, [(-1)^{n-1} / \sqrt{n}]$ is convergent.

Now $\Sigma \left| \dfrac{(-1)^{n-1}}{\sqrt{n}} \right| = \Sigma\, \dfrac{1}{\sqrt{n}}$ is divergent because $\Sigma\, (1/n^p)$ is divergent if $p \leq 1$ and here $p = \dfrac{1}{2}$.

Hence the given series is semi-convergent or conditionally convergent.

Example 43: *Examine the convergence and absolute convergence of the series*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, n}{n^2 + 1}.$$ **(Kashi 2013)**

Solution: Obviously the given series is an alternating series

$$u_1 - u_2 + u_3 - u_4 + \ldots, u_n > 0 \text{ for all } n.$$

Here $u_n = \dfrac{n}{n^2 + 1} > 0$ for all n.

Also $$u_{n+1} - u_n = \frac{n+1}{(n+1)^2 + 1} - \frac{n}{n^2 + 1} = \frac{-n^2 - n + 1}{(n^2 + 1)\,[(n+1)^2 + 1]} < 0 \text{ for all } n.$$

Thus $u_{n+1} < u_n$ for all n.

Again $$\lim u_n = \lim \frac{n}{n^2 + 1} = \lim \frac{1}{n\,[1 + (1/n^2)]} = 0.$$

Hence by Leibnitz's test, the given series converges.

Now we shall test the given series for absolute convergence.

Consider the series $\Sigma\, u_n'$ of positive terms, where

$$u_n' = \left| \frac{(-1)^{n+1}\, n}{n^2 + 1} \right| = \frac{n}{n^2 + 1} = \frac{1}{n\,[1 + (1/n^2)]}.$$

Take $v_n = \dfrac{1}{n}$. Then $\lim \dfrac{u_n'}{v_n} = \lim \dfrac{1}{1 + (1/n^2)} = 1$ which is finite and non-zero. Hence by comparison test $\Sigma u_n'$ and $\Sigma\, v_n$ are either both convergent or both divergent. But for v_n, $p = 1$ so that $\Sigma\, v_n$ is divergent. Hence $\Sigma\, u_n'$ is divergent.

Hence the given series is not absolutely convergent *i.e.*, it is conditionally convergent.

Example 44: *Show that the series* $\Sigma\, (-1)^n\, [\sqrt{(n^2 + 1)} - n]$ *is conditionally convergent.*

Solution: The given series is an alternating series $\Sigma\, (-1)^n\, u_n, u_n > 0$ for all n.

Here $$u_n = \sqrt{(n^2 + 1)} - n = \frac{[\sqrt{(n^2 + 1)} - n]\,[\sqrt{(n^2 + 1)} + n]}{\sqrt{(n^2 + 1)} + n}$$

$$= \frac{n^2 + 1 - n^2}{\sqrt{(n^2 + 1)} + n} = \frac{1}{\sqrt{(n^2 + 1)} + n}.$$

Obviously $u_n > 0$ for all n.

Also $u_{n+1} < u_n$ for all n.

Again $\qquad \lim u_n = \lim \dfrac{1}{\sqrt{(n^2+1)}+n} = \lim \dfrac{1}{n\,[(1+1/n^2)^{1/2}+1]} = 0 \cdot$

Hence by Leibnitz's test, the given series is convergent.

Now let $\Sigma\, u_n'$ denote the series obtained from the given series by making all its terms positive *i.e.*,

$$u_n' = |(-1)^n \{\, |\sqrt{(n^2+1)} - n\,\}| = \sqrt{(n^2+1)} - n.$$

We shall apply comparison test to check the convergence of $\Sigma\, u_n'$.

We have $\qquad u_n' = \sqrt{(n^2+1)} - n = \dfrac{1}{\sqrt{(n^2+1)}+n} = \dfrac{1}{n\,[(1+1/n^2)^{1/2}+1]} \cdot$

Take $v_n = \dfrac{1}{n} \cdot$ Then $\dfrac{u_n'}{v_n} = \dfrac{1}{[(1+1/n^2)^{1/2}+1]} \cdot$

$\therefore \qquad \lim \dfrac{u_n'}{v_n} = \lim \dfrac{1}{[(1+1/n^2)^{1/2}+1]} = 1$ which is finite and non-zero.

Hence by comparison test $\Sigma\, u_n'$ and $\Sigma\, v_n$ are either both convergent or both divergent. But for v_n, $p = 1$ and so $\Sigma\, v_n$ is divergent. Therefore $\Sigma\, u_n'$ is also divergent.

Hence the given series converges conditionally.

Example 45: *Show that the exponential series*

$$1 + \frac{x}{1\,!} + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \ldots$$

converges absolutely for all values of x.

Solution: Let the given series be $\Sigma\, u_n$, so that

$$u_n = \frac{x^{\,n-1}}{(n-1)\,!} \text{ and } u_{n+1} = \frac{x^{\,n}}{n\,!} \cdot$$

Now
$$\frac{|u_n|}{|u_{n+1}|} = \left|\frac{u_n}{u_{n+1}}\right| = \left|\frac{x^{\,n-1}}{(n-1)\,!} \cdot \frac{n\,!}{x^{\,n}}\right|$$
$$= \left|\frac{n}{x}\right| = \frac{n}{|x|} \cdot$$

$\therefore \qquad \lim \dfrac{|u_n|}{|u_{n+1}|} = \lim \dfrac{n}{|x|} = +\infty$, whatever $(x \neq 0)$ may be.

Therefore, by D'Alembert's ratio test, the series $\Sigma\,|u_n|$ converges for all x (except for $x = 0$) .

For $x = 0$, the series $\Sigma\,|u_n|$ obviously converges.

Hence the given series $\Sigma\, u_n$ converges absolutely for all values of x *i.e.*, for all $x \in \mathbf{R}$.

Note: The sum of this series is denoted by e^x.

Also since $\Sigma\, u_n$ converges, we must have

$$\lim u_n = 0 \quad \textit{i.e.}, \quad \lim_{n\to\infty} (x^n / n\,!) = 0.$$

Example 46: *Show that the series* $x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$ *converges if and only if* $-1 \le x \le 1$.

Solution: Let the given series be Σu_n. Then

$$u_n = (-1)^{n-1} \frac{x^{2n-1}}{2n-1} \quad \text{and} \quad u_{n+1} = (-1)^n \frac{x^{2n+1}}{2n+1}.$$

$$\therefore \quad \left|\frac{u_n}{u_{n+1}}\right| = \frac{2n+1}{2n-1} \cdot \frac{1}{x^2}.$$

Now $$\lim_{n \to \infty} \left|\frac{u_n}{u_{n+1}}\right| = \lim \left[\frac{(2+1/n)}{(2-1/n)} \cdot \frac{1}{x^2}\right] = \frac{1}{x^2}.$$

$\therefore$ by D'Alembert's ratio test, $\Sigma |u_n|$ converges if $1/x^2 > 1$ *i.e.,* $x^2 < 1$ *i.e.,* $|x| < 1$ and diverges if $|x| > 1$. Since every absolutely convergent series is convergent, therefore the given series Σu_n converges when $|x| < 1$ *i.e.,* $-1 < x < 1$.

When $x = 1$, the series Σu_n becomes

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

which converges by Leibnitz's test for alternating series.

When $x = -1$, the series Σu_n becomes

$$-\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right)$$

which is again convergent by Leibnitz's test.

When $x > 1$ or when $x < -1$, obviously u_n does not tend to zero as $n \to \infty$. So the series Σu_n does not converge when

$$|x| > 1.$$

Hence the given series converges iff $-1 \le x \le 1$.

Example 47: *Show that the binomial series*

$$1 + nx + \frac{n(n-1)}{2!} x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!} x^r + \ldots$$

is absolutely convergent when $|x| < 1$.

Solution: Omitting the first term of the given series, let u_r denote the r th term of the resulting series.

Then $$\frac{u_r}{u_{r+1}} = \frac{r+1}{n-r} \cdot \frac{1}{x} = \frac{(1+1/r)}{(n/r-1)} \cdot \frac{1}{x}.$$

$$\therefore \quad \left|\frac{u_r}{u_{r+1}}\right| = \frac{1+1/r}{|n/r-1|} \cdot \frac{1}{|x|}.$$

$$\therefore \quad \lim_{n \to \infty} \left|\frac{u_r}{u_{r+1}}\right| = \frac{1}{|x|} \text{ which is} > 1 \text{ if } |x| < 1.$$

$\therefore$ The series $\Sigma|u_r|$ converges if $|x|<1$ *i.e.*, the given series is absolutely convergent when $|x|<1$.

Example 48: *Show that a series of positive terms, if convergent, is absolutely convergent. Prove that the series*

$$2\ sin\ \frac{x}{3}+4\ sin\ \frac{x}{9}+8\ sin\ \frac{x}{27}+\ldots$$

converges absolutely for all finite values of x.

Solution: **First part:** Let $\Sigma\, u_n$ be a convergent series of positive terms. Since $u_n>0 \Rightarrow |u_n| = u_n$, therefore the series $\Sigma|u_n| = \Sigma\, u_n$ is also convergent and hence the series $\Sigma\, u_n$ is absolutely convergent.

Second part: Let the given series be denoted by $\Sigma\, u_n$. Then

$$u_n = 2^n \sin(x/3^n) \text{ and } u_{n+1} = 2^{n+1}\sin(x/3^{n+1}).$$

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \frac{1}{2}\cdot\sin(x/3^n)\cdot\frac{1}{\sin(x/3^{n+1})}$$

$$= \frac{1}{2}\cdot\frac{\sin(x/3^n)}{x/3^n}\cdot\frac{x/3^{n+1}}{\sin(x/3^{n+1})}\cdot 3.$$

To test the convergence of the series $\Sigma|u_n|$, we have

$$\left|\frac{u_n}{u_{n+1}}\right| = \frac{3}{2}\cdot\left|\frac{\sin(x/3^n)}{x/3^n}\right|\cdot\left|\frac{x/3^{n+1}}{\sin(x/3^{n+1})}\right|.$$

$$\therefore \qquad \lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \frac{3}{2} \text{ for all finite values of } x, \text{ because } \lim_{n\to\infty}\frac{\sin(x/3^n)}{x/3^n} = 1.$$

Since $\lim\left|\dfrac{u_n}{u_{n+1}}\right| > 1$ for all finite values of x, therefore by ratio test, the series $\Sigma|u_n|$ converges for all finite values of x. Hence the series $\Sigma\, u_n$ converges absolutely for all finite values of x.

Example 49: *Discuss the convergence of the logarithmic series*

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots$$

Solution: Let $\quad \Sigma\, u_n = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots\cdot$

The series $\Sigma\, u_n$ is absolutely convergent if the series $\Sigma\,|\,u_n\,|$ is convergent. To discuss the convergence of $\Sigma\,|\,u_n\,|$ we shall apply ratio test.

We have $\qquad \left|\dfrac{u_n}{u_{n+1}}\right| = \left|\dfrac{x^n}{n}\cdot\dfrac{n+1}{x^{n+1}}\right| = \dfrac{n+1}{n}\cdot\dfrac{1}{|x|} = \left(1+\dfrac{1}{n}\right)\cdot\dfrac{1}{|x|}\cdot$

$$\therefore \qquad \lim\left|\frac{u_n}{u_{n+1}}\right| = \lim\left[\left(1+\frac{1}{n}\right)\cdot\frac{1}{|x|}\right] = \frac{1}{|x|}\cdot$$

So by ratio test, the series $\Sigma|u_n|$ is convergent if $1/|x|>1$ *i.e.*, $|x|<1$ *i.e.* $-1<x<1$.

$\therefore$ the given series is absolutely convergent and hence also convergent if $-1 < x < 1$ *i.e.*, if $|x| < 1$.

When $x = 1$, the given series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

which converges by Leibnitz's test but converges conditionally.

When $x = -1$, the given series is $-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right)$ which diverges to $-\infty$.

When $x > 1$ or $x < -1$ *i.e.*, $|x| > 1$, obviously $\lim u_n \neq 0$ and so the series $\Sigma\, u_n$ does not converge.

Hence the given series is convergent if $-1 < x \leq 1$. For $|x| < 1$ *i.e.*, $-1 < x < 1$, it converges absolutely.

Comprehensive Exercise 9

1. Test for convergence the following series :

(i) $1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots$ (ii) $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \ldots$

(iii) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$ (iv) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{[n(n+1)(n+2)]}}$.

(v) $\Sigma \frac{(-1)^{n-1}(n+1)}{2n}$.

(vi) $\frac{1}{x} - \frac{1}{x+a} + \frac{1}{x+2a} - \frac{1}{x+3a} + \ldots, x > 0, a > 0$.

(vii) $\log\left(\frac{1}{2}\right) - \log\left(\frac{2}{3}\right) + \log\left(\frac{3}{4}\right) - \log\left(\frac{4}{5}\right) + \ldots$

(viii) $\log\left(\frac{2}{1}\right) - \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) - \log\left(\frac{5}{4}\right) + \ldots$.

2. Test the absolute convergence or conditional convergence of the following series :

(i) $1 - x + x^2 - x^3 + \ldots$. $(0 < x < 1)$

(ii) $\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots$ $(p > 0)$

(iii) $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \ldots$

(iv) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

(v) $\frac{1}{1.2} - \frac{1}{3.4} + \frac{1}{5.6} - \frac{1}{7.8} + \ldots$

(vi) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\dots$ **(Meerut 2012B)**

(vii) $\Sigma\,(-1)^n \frac{\sin n\alpha}{n^3}$, $\alpha \in \mathbf{R}$.

(viii) $\Sigma\,(-1)^n \frac{\cos n\alpha}{n\sqrt{n}}$, $\alpha \in \mathbf{R}$.

(ix) $\Sigma\, \frac{(-1)^{n-1}}{2n+3}$.

(x) $\Sigma\,(-1)^{n-1} \frac{n^2}{(n+1)!}$.

(xi) $\sum_{n=1}^{\infty} (-1)^n \sin \frac{1}{n}$.

(xii) $\sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{1}{n^2}+\frac{1}{(n+1)^2}\right]$.

3. Show that the series $\sum_{n=1}^{\infty} (-1)^{n+1} [\sqrt{(n+1)}-\sqrt{n}]$ is semi-convergent.

4. Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(\sqrt{n}+\sqrt{a})^2}$ is semi-convergent.

5. Show that the series $\left(\frac{1}{2}\right)^2-\left(\frac{1.3}{2.4}\right)^2+\left(\frac{1.3.5}{2.4.6}\right)^2-\dots$

is conditionally convergent.

6. Test the series

$$\frac{1}{2(\log 2)^p}-\frac{1}{3(\log 3)^p}+\frac{1}{4(\log 4)^p}-\dots,\ p>0$$

for convergence and absolute convergence.

7. Test for convergence the following series :

(i) $\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{2^n+5}$.

(ii) $\sum_{n=1}^{\infty} (-1)^{n-1} \left[\frac{1}{\sqrt{n^5}}+\frac{1}{\sqrt{(n+1)^5}}\right]$.

8. Show that the series $1-2+3-4+5-6+\dots$ oscillates infinitely.

9. Discuss the convergence including absolute convergence of the series

$$1-2x+3x^2-4x^3+\dots$$

10. Show that the series $\frac{2}{1^2}-\frac{3}{2^2}+\frac{4}{3^2}-\frac{5}{4^2}+\dots$ converges conditionally.

11. Show that the series $\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}$ is absolutely convergent.

12. Define absolute convergence. Show that the series

$$1-\frac{1}{2^3}-\frac{1}{4^3}+\frac{1}{3^3}-\frac{1}{6^3}-\frac{1}{8^3}+\dots+\frac{1}{(2n-1)^3}-\frac{1}{(4n-2)^3}-\frac{1}{(4n)^3}+\dots$$

is absolutely convergent.

13. Prove that the series

$$z+\frac{a-b}{2!}z^2+\frac{(a-b)(a-2b)}{3!}z^3+\frac{(a-b)(a-2b)(a-3b)}{4!}z^4+\dots$$

is absolutely convergent if $|z|<\frac{1}{|b|}$.

Answers 9

1. (i) Convergent (ii) Convergent (iii) Convergent
   (iv) Convergent (v) Oscillate (vi) Convergent
   (vii) Convergent (viii) Convergent
2. (i) Absolutely convergent
   (ii) Absolutely convergent if $p > 1$ and conditionally convergent if $0 < p \leq 1$
   (iii) Absolutely convergent
   (iv) Absolutely convergent for all $x \in \mathbf{R}$
   (v) Absolutely convergent
   (vi) Absolutely convergent
   (vii) Absolutely convergent
   (viii) Absolutely convergent
   (ix) Conditionally convergent
   (x) Absolutely convergent
   (xi) Semi-convergent
   (xii) Absolutely convergent
6. Absolutely convergent if $p > 1$ and semi-convergent if $0 < p \leq 1$
7. (i) Absolutely convergent (ii) Absolutely convergent
9. Absolutely convergent if $|x| < 1$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The series $\Sigma \dfrac{1}{n^p}$ is convergent if

   (a) $p < 1$ (b) $p = 1$

   (c) $p > 1$ (d) none of these

2. The series $\Sigma\, u_n$ of positive terms is convergent if

   (a) $\lim\limits_{n \to \infty} u_n^{1/n} < 1$ (b) $\lim\limits_{n \to \infty} u_n^{1/n} > 1$

   (c) $\lim\limits_{n \to \infty} \dfrac{u_n}{u_{n+1}} < 1$ (d) $\lim\limits_{n \to \infty} \left\{ n \left(\dfrac{u_n}{u_{n+1}} - 1 \right) \right\} < 1$

3. If u_n denotes the nth term of the series $\frac{1^2}{2^2} + \frac{1^2 . 3^2}{2^2 . 4^2} x + \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} x^2 + \ldots$, then

(a) $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = x$ (b) $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$

(c) $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = x^2$ (d) $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$

4. The series $\Sigma \frac{1}{n^{2/3}}$ is

(a) convergent (b) divergent
(c) oscillatory (d) none of these

5. The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$ is

(a) divergent (b) oscillatory
(c) convergent (d) absolutely convergent

6. The series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$ is

(a) absolutely convergent (b) oscillatory
(c) divergent (d) semi-convergent

7. The series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$ is

(a) divergent (b) absolutely convergent
(c) semi-convergent (d) oscillatory

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. For every convergent series $\Sigma\, u_n$, we must have $\lim_{n \to \infty} u_n = \ldots.$
2. The infinite geometric series $a + ax + ax^2 + ax^3 + \ldots$ is convergent if and only if $|x| < \ldots.$
3. The series $\Sigma \frac{1}{n^p}$ is divergent if $p \le \ldots.$
4. The nth term of the series $\frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + \ldots$, is ... .
5. Let $\Sigma\, u_n$ be a series of positive terms such that $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = l.$

   Then $\Sigma\, u_n$ converges if ... .
6. Let $\Sigma\, u_n$ be a series of positive terms such that $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = l.$

   Then $\Sigma\, u_n$ diverges if ... .

7. If u_n denotes the nth term of the series

$$\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+\ldots, \text{ then } u_n=\ldots.$$

8. If u_n denotes the nth term of the series

$$1+3x+5x^2+7x^3+\ldots\infty, \text{ then } \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\ldots.$$

9. The nth term of the series

$$\frac{1}{3}+\frac{1.2}{3.5}+\frac{1.2.3}{3.5.7}+\frac{1.2.3.4}{3.5.7.9}+\ldots, \text{ is } \ldots .$$

10. If u_n denotes the nth term of the series

$$\frac{1^2}{2^2}+\frac{1^2.3^2}{2^2.4^2}+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}+\ldots, \text{ then } \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\ldots.$$

11. A series $\Sigma\, u_n$ is said to be semi-convergent if … .
12. A series $\Sigma\, u_n$ is said to be absolutely convergent if … .
13. A series in which the terms are alternately positive and negative is called an … .
14. An infinite series $\Sigma\,(-1)^{n-1}\,u_n$ in which the terms are alternately positive and negative is convergent if each term is numerically less than the preceding term and $\lim_{n\to\infty} u_n=\ldots.$
15. The series $\Sigma\, u_n$ of positive terms is divergent if $\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}<\ldots.$

True or False

Write 'T' for true and 'F' for false statement.

1. A series $\Sigma\, u_n$ of positive terms is divergent if $\lim_{n\to\infty} u_n>0.$
2. A series $\Sigma\, u_n$ is convergent if $\lim_{n\to\infty} u_n=0.$
3. A series $\Sigma\, u_n$ of positive terms is convergent if $\lim_{n\to\infty} u_n^{1/n}=1.$
4. A series $\Sigma\, u_n$ of positive terms is divergent if $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}<1.$
5. The series $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$ is divergent.
6. The series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$ is divergent.
7. The series $\sum_{n=1}^{\infty}\cos\frac{1}{n}$ is divergent.
8. The series $\sum_{n=1}^{\infty}\sin\frac{1}{n}$ is convergent.

9. The series whose nth term is $\dfrac{\sqrt{n}}{n^2+1}$ is convergent.

10. The series whose nth term is $\dfrac{1}{1+(1/n)}$ is convergent.

11. The series $\sum_{n=1}^{\infty} \dfrac{x^{n-1}}{(n-1)!}$ is absolutely convergent.

12. The series $\dfrac{1}{\sqrt{1}} - \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{4}} + \dots$ is absolutely convergent.

13. The series $\sum_{n=1}^{\infty} (-1)^n [\sqrt{(n^2+1)} - n]$ is semi-convergent.

14. The series $\dfrac{1}{1.2} - \dfrac{1}{3.4} + \dfrac{1}{5.6} - \dfrac{1}{7.8} + \dots$ is divergent.

(Purvanchal 2014)

15. The series $1 - \dfrac{1}{1!} + \dfrac{1}{2!} - \dfrac{1}{3!} + \dots$ is semi-convergent.

16. The series $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{(\sqrt{n} + \sqrt{a})^2}$ is absolutely convergent.

17. Every absolutely convergent series is convergent.

18. Every convergent series is absolutely convergent.

19. The series $\sum_{n=1}^{\infty} (-1)^n \sin \dfrac{1}{n}$ is convergent but is not absolutely convergent.

20. The series $\dfrac{1}{\sqrt{1}+\sqrt{2}} + \dfrac{1}{\sqrt{2}+\sqrt{3}} + \dfrac{1}{\sqrt{3}+\sqrt{4}} + \dots$ is divergent.

21. The series $\Sigma [\sqrt{(n+1)} - \sqrt{n}]$ is convergent.

22. The series $\Sigma [\sqrt{(n^4+1)} - n^2]$ is convergent.

23. The series $\Sigma [\sqrt{(n^3+1)} - \sqrt{n^3}]$ is divergent.

24. The series $1 - \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} - \dfrac{1}{4\sqrt{4}} + \dots$ is absolutely convergent.

25. The series $\left(1+\dfrac{1}{1}\right)^1 + \left(1+\dfrac{1}{2}\right)^2 + \dots + \left(1+\dfrac{1}{n}\right)^n + \dots$ is convergent.

26. The infinite series of positive terms is always convergent or divergent and is never an oscillatory series.

27. The series $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{4}} + \dots + \dfrac{1}{\sqrt{n}} + \dots$ is divergent.

28. Let $\Sigma\, u_n$ be an infinite series having all terms positive and let $\lim_{n \to \infty} \dfrac{u_n}{u_{n+1}} = l$.

If $l > 1$, then $\Sigma\, u_n$ is divergent.

Answers

Multiple Choice Questions

1.	(c)	**2.**	(a)	**3.**	(d)	**4.**	(b)	**5.**	(c)
6.	(a)	**7.**	(c)						

Fill in the Blank(s)

1. 0 **2.** 1 **3.** 1 **4.** $\dfrac{n^n}{(n+1)^{n+1}}$ **5.** $l < 1$

6. $l < 1$ **7.** $\dfrac{x^{2n-2}}{(n+1)\sqrt{n}}$ **8.** $\dfrac{1}{x}$ **9.** $\dfrac{1 \,.\, 2 \,.\, 3 \,.\, 4 \ldots n}{3 \,.\, 5 \,.\, 7 \,.\, 9 \ldots (2n+1)}$

10. 1 **11.** $\Sigma\, u_n$ is convergent but $\Sigma\, |u_n|$ is divergent

12. $\Sigma\, |u_n|$ is convergent **13.** alternating series

14. 0 **15.** 1

True or False

1.	*T*	**2.**	*F*	**3.**	*F*	**4.**	*T*	**5.**	*F*
6.	*T*	**7.**	*T*	**8.**	*F*	**9.**	*T*	**10.**	*F*
11.	*T*	**12.**	*F*	**13.**	*T*	**14.**	*F*	**15.**	*F*
16.	*F*	**17.**	*T*	**18.**	*F*	**19.**	*T*	**20.**	*T*
21.	*F*	**22.**	*T*	**23.**	*F*	**24.**	*T*	**25.**	*F*
26.	*T*	**27.**	*T*	**28.**	*F*				

□

Chapter 4

Limits And Continuity

1 Definitions

Constant: A symbol which retains the same value throughout a set of mathematical operations is called a **constant**.

Variable: A **variable** is a quantity, or a symbol representing a number, which is capable of assuming different values.

Continuous Variable: A **continuous variable** is one which can take all the numerical values between two given numbers.

Independent Variable: **An independent variable** is one which may take up any arbitrary value that may be assigned to it.

Dependent Variable: **A dependent variable** is a symbol which can assume its value as a result of some other variable taking some assigned value.

Domain of A Variable: If we give the independent variable x only those values which lie between $x = a$ and $x = b$, then all these numerical values taken collectively will be called **domain** or **interval** of the variable. The domain is said to be **closed** if a and b are included in it and is denoted by the symbol $[a, b]$. An **open** domain is denoted by $]a, b[$ or by (a, b). Similarly the symbols $[a, b[$ and $]a, b]$ stand for semi-open domains. These semi-open domains are also denoted by $[a, b)$ and $(a, b]$ respectively.

Function: If y depends upon x in such a manner that for every value of x in its domain of variation there corresponds a definite (*i.e.*, a unique) value of y, then y is said to be a single-valued function of x and is denoted by $y = f(x)$, f denoting the kind of dependence or relationship that exists between x and y.

This relationship is often called functional relation and $f(x_1), f(x_2), \ldots, f(x_r)$ are called functional values of $f(x)$ for $x = x_1, x_2, \ldots, x_r$ respectively.

Note: The essential thing about the definition of a function is that for each value of x there must correspond a definite value of $f(x)$. We must be in possession of a set of rules which determine for each value of x in a certain interval, a definite value of the function. These rules may take the shape of a single compact formula such as $f(x) = \sin x$ or a number of such formulae that apply to different parts of the domain of x, for example

$$\left.\begin{array}{lll} f(x) = \sin x & \text{for} & 0 \le x \le \pi/2 \\ f(x) = x & \text{for} & \pi/2 < x < \pi \\ f(x) = \cos x & \text{for} & x \ge \pi. \end{array}\right\} \cdot \qquad \ldots(1)$$

In the first case $f(x) = \sin x$ is defined for values of x in any interval. In the second case $f(x)$ given by (1) is defined in the interval $[0, \infty[$.

The above definition of a function of x brings about

(1) idea of the dependence of the function on x

(2) idea of definiteness of the values of the function for each value of x

(3) idea of single valuedness of the function

(4) idea of the domain of the variable x.

We are accustomed to think that every function is capable of graphical representation. Majority of functions are certainly capable of graphical representation but there are some functions which cannot be represented by a graph. The function defined as follows is such a function :

$$f(x) = 0 \text{ when } x \text{ is rational}, \quad f(x) = 1 \text{ when } x \text{ is irrational}.$$

Set-theoretic definition of a function: Let A and B be two given sets. Suppose there exists a correspondence denoted by f, which associates to **each** member of A a **unique** member of B. Then f is called a **function** or a **mapping** from A to B.

The mapping f of A to B is denoted by $f : A \to B$. The set A is called the **domain** of the function f, and B is called the **co-domain** of f. The element $y \in B$ which the mapping f associates to an element $x \in A$ is denoted by $f(x)$ and is called the **f-image of** x or the **value** of the function f for x. Each element of A has a unique image and each element of B need not appear as the image of an element in A. We define the **range** of f to consist of those elements in B which appear as the image of at least one element in A.

Equality of two functions: Two functions f and g of $A \to B$ are said to be *equal* if and only if $f(x) = g(x)\ \forall\ x \in A$ and we write $f = g$. For two unequal mappings from A to B, there must exist at least one element $x \in A$ such that $f(x) \ne g(x)$.

Constant function: A function $f : A \to B$ is called a **constant function** if the same element $b \in B$ is assigned to every element in A.

Real valued function: If both A and B are the sets of real numbers, then $f : A \to B$ is called a real valued function of a real variable.

Single-valued and multiple-valued functions: If y has only one definite value when a definite value is given to x then y is called a **single-valued function of x**. When y has more than one value for a value of x, it is called a **multiple-valued function of x.**

Odd and Even functions: A function is said to be **odd** if it changes sign when the sign of the variable is changed *i.e.*, if $f(-x) = -f(x)$.

A function is said to be **even** if its sign does not change when the sign of the variable is changed *i.e.*, if $f(-x) = f(x)$.

Bounded and unbounded functions: If for all values of x in a given interval, $f(x)$ is never greater than some fixed number M, the number M is said to be an **upper bound** for f in that interval, whereas if $f(x)$ is never less than some number m then m is called a **lower bound** for f in that interval. If both upper and lower bounds of a function are finite, the function is said to be bounded otherwise it is said to be unbounded.

By a supremum of f in an interval we mean the least of all the upper bounds of f in that interval. Similarly an **infimum** of f is the greatest of all the lower bounds of f in the interval.

A rational integral function, or a **polynomial**, is a function of the form

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$$

where $a_0, a_1, \ldots, a_n$ are constants and n is a positive integer or zero.

A rational function is defined as the quotient of one polynomial by another. For example,

$$\frac{7x+4}{2x^2+3x+6} \text{ is a rational function.}$$

An Algebraical Function: **An algebraical function** is a function which can be expressed as the root of an equation of the form

$$y^n + A_1 y^{n-1} + A_2 y^{n-2} + \ldots + A_{n-1} y + A_n = 0$$

where $A_1, A_2, \ldots, A_n$ are rational functions of x. In particular a rational function is also algebraical.

A Transcendental Function: **A transcendental function** is a function which is not algebraical. Trigonometrical, exponential and logarithmic functions are examples of transcendental functions.

Monotonic functions: The function $y = f(x)$ is said to be **monotonically increasing** if corresponding to an increase in the value of x in a certain interval I in which the function $f(x)$ is defined, the value of y never decreases *i.e.,*

$$x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2) \quad \forall \quad x_1, x_2 \in I.$$

Similarly the function $f(x)$ is **monotonically decreasing** if

$$x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2) \quad \forall\, x_1, x_2 \in I.$$

Also f is said to be **strictly increasing** iff $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$ and **strictly decreasing** iff $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$.

The function f defined by $f(x) = \sin x$ is monotonically increasing in the interval $0 \le x \le \frac{1}{2}\pi$ and monotonically decreasing in the interval $\frac{1}{2}\pi \le x \le \pi$.

Explicit and implicit functions: A function is said to be *explicit* when expressed directly in terms of the independent variable or variables *e.g.*, $y = \sin^{-1} x + \log x$.

If the function cannot be expressed directly in terms of the independent variable or variables, the function is said to be *implicit e.g.*, the equation $x^y + y^x = a^b$ expresses y as an implicit function of x.

Sum, Difference, Product and Quotient of two functions. Let f, g be two functions with domains D_1 and D_2. If $D = D_1 \cap D_2$, then D is common to the domains of f and g.

The *sum function* $f + g$ is defined as $(f + g)(x) = f(x) + g(x) \ \forall x \in D$.
If $c \in \mathbf{R}$, the function cf is defined as $(cf)(x) = c f(x) \ \forall x \in D_1$.
The *difference function* $f - g$ is defined as $(f - g)(x) = f(x) - g(x) \ \forall x \in D$.
The *product function* fg is defined as $(fg)(x) = f(x)\, g(x) \ \forall x \in D$.
The *reciprocal function* $1/g$ of the function g is defined as

$$\left(\frac{1}{g}\right)(x) = \frac{1}{g(x)} \ \forall x \in D_2 \text{ and } g(x) \ne 0.$$

The *quotient function* f/g is defined as $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} \ \forall x \in D$ and $g(x) \ne 0$.

2 Limits

Consider the function $y = (x^2 - 1)/(x - 1)$. The value of this function at $x = 1$ is of the form $0/0$ which is meaningless. In this case we cannot divide the numerator by the denominator since $x - 1$ is zero. Now suppose x is not actually equal to 1 but very nearly equal to 1. Then $x - 1$ is not equal to zero. Hence in this case we can divide the numerator by the denominator.

$$\therefore \qquad \frac{x^2 - 1}{x - 1} = x + 1.$$

If x is little greater than 1, then the value of y will be greater than 2 and as x gets nearer to 1, y comes nearer to 2. Now the difference between y and 2 is

$$\frac{x^2 - 1}{x - 1} - 2 = \frac{x^2 - 2x + 1}{x - 1} = \frac{(x - 1)^2}{x - 1} = x - 1.$$

This difference $(x - 1)$ can be made as small as we please by letting x tend to 1.

Thus we see that when x has a fixed value 1, the value of y is meaningless but when x tends to 1, y tends to 2 and we say that the limit of y is 2 when x tends to 1. Thus we write as

$$\lim_{x \to 1} [(x^2 - 1)/(x - 1)] = 2.$$

Definition of limit: **(Bundelkhand 2006; Purvanchal 10; Kashi 14)**

Let f be a function defined on some neighbourhood of a point a except possibly at a itself. Then a real number l is said to be the **limit** *of f as x approaches a if for any arbitrarily chosen positive number ε, however small but not zero, there exists a corresponding number δ greater than zero such that*

$$|f(x) - l| < \varepsilon$$

for all values of x for which $0 < |x - a| < \delta$, where $|x - a|$ means the absolute value of $x - a$ without any regard to sign.

In symbols, we then write $\lim_{x \to a} f(x) = l.$

We have to negate the above definition in order to show that f does not approach l as x approaches a.

If it is not true that for every $\varepsilon > 0$, there is some $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - l| < \varepsilon,$$

then there must exist an $\varepsilon > 0$, such that for every $\delta > 0$, there is some x for which

$$0 < |x - a| < \delta \text{ but } |f(x) - l| \nless \varepsilon.$$

This means that in order to show that f does not approach l as x approaches a, it is sufficient to produce an $\varepsilon > 0$ such that for each $\delta > 0$ there is some x satisfying $0 < |x - a| < \delta$ and $|f(x) - l| \geq \varepsilon$.

Note 1: It is not at all necessary for $\lim_{x \to a} f(x)$ to exist that f be defined at $x = a$.

It is enough that for some $\delta > 0$, f be defined whenever $0 < |x - a| < \delta$.

Note 2: If N be a neighbourhood of a, then $N \sim \{a\}$ is called a *deleted neighbourhood* of a.

Note 3: If a function f has a finite limit at a point a, then by the definition of the limit of a function a deleted neighbourhood of a exists on which f is bounded.

Now we shall prove a theorem which is the foundation on which the definition of limit rests. If this theorem were not true, the definition of limit would have been useless.

Theorem: *If $\lim_{x \to a} f(x) = l$, and $\lim_{x \to a} f(x) = m$, then $l = m$ i.e., if $\lim_{x \to a} f(x)$ exists, then it is unique.*

Proof: Suppose, if possible, $l \neq m$. Let us take $\varepsilon = \frac{1}{2}|l - m|$. Then $\varepsilon > 0$.

Since $\lim_{x \to a} f(x) = l$, for a given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$|f(x) - l| < \varepsilon \text{ whenever } 0 < |x - a| < \delta_1. \qquad \ldots(1)$$

Again since $\lim_{x \to a} f(x) = m$, for a given $\varepsilon > 0$, there exists $\delta_2 > 0$ such that

$$|f(x) - m| < \varepsilon \text{ whenever } 0 < |x - a| < \delta_2. \qquad \ldots(2)$$

If we choose $\delta = \min. \{\delta_1, \delta_2\}$, then $0 < |x - a| < \delta$ implies that both $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$ hold, and hence, we have

$$|f(x) - l| < \varepsilon \text{ and } |f(x) - m| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

This implies that if $0 < |x - a| < \delta$, then

$$|l - m| = |\{f(x) - m\} - \{f(x) - l\}| \leq |f(x) - m| + |f(x) - l|$$
$$< \varepsilon + \varepsilon = 2\varepsilon = |l - m|$$

i.e., $|l - m| < |l - m|$, which is absurd and so our assumption is wrong.

Hence, $l = m$ *i.e.,* $\lim_{x \to a} f(x)$ is unique.

3 Algebra Of Limits

Now we shall give some theorems on limits of functions which are similar to those of limits of sequences.

Theorem 1: *If* $\lim_{x \to a} f(x) = l \neq 0$, *then there exist numbers* $k > 0$ *and* $\delta > 0$ *such that*

$$|f(x)| > k \text{ whenever } 0 < |x - a| < \delta.$$

Also then $\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{l}$.

Proof: Let $\varepsilon = \frac{1}{2}|l|$. Then $\varepsilon > 0$, because $l \neq 0$.

Since $\lim_{x \to a} f(x) = l$, therefore, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - l| < \varepsilon, \text{ whenever } 0 < |x - a| < \delta. \qquad \ldots(1)$$

Now $|l| = |l - f(x) + f(x)| \leq |l - f(x)| + |f(x)|$

$$< \varepsilon + |f(x)|, \text{ whenever } 0 < |x - a| < \delta, \text{ from (1).}$$

$\therefore$ Whenever $0 < |x - a| < \delta$, we have

$$|f(x)| > |l| - \varepsilon = |l| - \frac{1}{2}|l| = \frac{1}{2}|l| > 0. \qquad \ldots(2)$$

Thus taking $k = \frac{1}{2} \infty l \infty > 0$, we get $|f(x)| > k$ whenever $0 < |x - a| < \delta$.

This proves the first part of the theorem.

Second part: Now to prove that $\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{l}$.

We have $$\left|\frac{1}{f(x)} - \frac{1}{l}\right| = \left|\frac{l - f(x)}{l \cdot f(x)}\right| = \frac{|l - f(x)|}{|l| \cdot |f(x)|}. \qquad \ldots(3)$$

By first part of this theorem there exist numbers $k > 0$ and $\delta_1 > 0$ such that

$$|f(x)| > k \text{ i.e., } \frac{1}{|f(x)|} < \frac{1}{k} \quad \text{whenever} \quad 0 < |x - a| < \delta_1. \qquad \ldots(4)$$

Let $\varepsilon' > 0$ be given.

Since $\lim\limits_{x \to a} f(x) = l$, therefore, given $\varepsilon' > 0$, there exists $\delta_2 > 0$ such that

$$|f(x) - l| < k\,|l|\,\varepsilon' \text{ whenever } 0 < |x - a| < \delta_2. \qquad \ldots(5)$$

Let $\delta = \min. \{\delta_1, \delta_2\}$. Then from (3), (4) and (5), we have

$$\left|\frac{1}{f(x)} - \frac{1}{l}\right| < \frac{1}{|l|} \cdot k\,|l|\,\varepsilon' \cdot \frac{1}{k} \text{ whenever } 0 < |x - a| < \delta$$

$$= \varepsilon'.$$

Thus for given $\varepsilon' > 0$, there exists $\delta > 0$ such that

$$\left|\frac{1}{f(x)} - \frac{1}{l}\right| < \varepsilon' \text{ whenever } 0 < |x - a| < \delta.$$

Hence, $\quad \lim\limits_{x \to 0} \dfrac{1}{f(x)} = \dfrac{1}{l}.$

Theorem 2: *The limit of a sum is equal to the sum of the limits.*

Proof: Let $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m.$

We have to show that $\lim\limits_{x \to a} \{(f + g)(x)\} = l + m.$

Let $\varepsilon > 0$ be given. Since $\lim\limits_{x \to a} f(x) = l$, therefore, there exists $\delta_1 > 0$ such that

$$|f(x) - l| < \frac{1}{2}\varepsilon \text{ whenever } 0 < |x - a| < \delta_1.$$

Again since $\lim\limits_{x \to a} g(x) = m$, therefore, there exists $\delta_2 > 0$ such that

$$|g(x) - m| < \frac{1}{2}\varepsilon \text{ whenever } 0 < |x - a| < \delta_2.$$

If we take $\delta = \min. \{\delta_1, \delta_2\}$, then $0 < |x - a| < \delta$

$\Rightarrow$ both $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$ hold,

and consequently if $0 < \infty x - a \infty < \delta$, then both $|f(x) - l| < \frac{1}{2}\varepsilon$ and $|g(x) - m| < \frac{1}{2}\varepsilon$ are true.

Now if $0 < |x - a| < \delta$, then

$$|(f + g)(x) - (l + m)| = |f(x) - l + g(x) - m|$$
$$\leq |f(x) - l| + |g(x) - m| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

Thus $|(f + g)(x) - (l + m)| < \varepsilon$ whenever $0 < |x - a| < \delta.$

$\therefore \quad \lim_{x \to a} (f+g)(x)$ exists and $\lim_{x \to a} (f+g)(x) = l + m$.

The above result can be extended to any finite number of functions.

In the same way, we can prove that $\lim_{x \to a} (f - g)(x) = l - m$.

Theorem 3: *The limit of a product is equal to the product of the limits.*

(Gorakhpur 2015)

Proof: Using the notations of theorem 2, we have to prove that

$$\lim_{x \to a} (fg)(x) = lm.$$

Let $\varepsilon > 0$ be given.

Now
$$\begin{aligned} |(fg)(x) - lm| &= |f(x)\, g(x) - lg(x) + lg(x) - lm| \\ &\le |f(x)\, g(x) - lg(x)| + |lg(x) - lm| \\ &= |g(x)|\,|f(x) - l| + |l|\,|g(x) - m|. \qquad \ldots(1) \end{aligned}$$

Since $\lim_{x \to a} g(x) = m$, therefore $g(x)$ is bounded in some deleted neighbourhood of $x = a$. Hence there exists $k > 0$ and $\delta_1 > 0$ such that $|g(x)| \le k$ whenever $0 < |x - a| < \delta_1$.

Since $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, therefore, corresponding to any given $\varepsilon > 0$, we can find positive numbers δ_2 and δ_3 such that $|f(x) - l| < \dfrac{\varepsilon}{2k}$ whenever $0 < |x - a| < \delta_2$

and $\quad |g(x) - m| < \dfrac{\varepsilon}{2(|l| + 1)}$ whenever $0 < |x - a| < \delta_3$.

If we take $\delta = \min. \{\delta_1, \delta_2, \delta_3\}$, then from (1), we get

$$|(fg)(x) - lm| < k \cdot \frac{\varepsilon}{2k} + |l| \cdot \frac{\varepsilon}{2(|l| + 1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \left[\because \frac{|l|}{|l| + 1} < 1\right]$$
$$= \varepsilon \quad \text{whenever } 0 < |x - a| < \delta.$$

Thus for $\varepsilon > 0$, we have $\delta > 0$ such that $|(fg)(x) - lm| < \varepsilon$ whenever $0 < |x - a| < \delta$.

$\therefore \quad \lim_{x \to a} (fg)(x) = \lim_{x \to a} f(x)\, g(x)$ exists and $\lim_{x \to a} (fg)(x) = lm$.

The above theorem can evidently be extended to any finite number of functions.

Theorem 4: *The limit of a quotient is equal to the quotient of the limits provided the limit of the denominator is not zero.*

Proof: Let $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m \ne 0$.

Now $$\left|\frac{f(x)}{g(x)} - \frac{l}{m}\right| = \left|\left\{\frac{f(x)}{g(x)} - \frac{f(x)}{m}\right\} + \left\{\frac{f(x)}{m} - \frac{l}{m}\right\}\right|$$

$$= \left| \frac{f(x)}{m\, g(x)} \{m - g(x)\} + \frac{1}{m} \{f(x) - l\} \right|$$

$$\le \frac{|f(x)|}{|m||g(x)|} |m - g(x)| + \frac{1}{|m|} |f(x) - l| \qquad ...(1)$$

Since $\lim_{x \to a} f(x) = l$, therefore there exists a deleted neighbourhood $]a - \delta_1, a + \delta_1[- \{a\}$ of the point $x = a$ in which the function f is bounded. Let $K > 0$ be such that

$$|f(x)| \le K \text{ whenever } 0 < |x - a| < \delta_1.$$

Again since $g(x) \ne 0$ for all x in the domain of g and $\lim_{x \to a} g(x) = m \ne 0$, therefore there exist numbers $k > 0$ and $\delta_2 > 0$ such that

$$|g(x)| > k \ \text{ i.e., } \frac{1}{|g(x)|} < \frac{1}{k} \text{ whenever } 0 < |x - a| < \delta_2.$$

[See theorem 1 of article 3]

Let $\delta' = \min(\delta_1, \delta_2)$.

The inequality (1) can then be written as

$$\left| \frac{f(x)}{g(x)} - \frac{l}{m} \right| \le \frac{K}{k|m|} |m - g(x)| + \frac{1}{|m|} |f(x) - l|, \qquad ...(2)$$

for all x such that $0 < |x - a| < \delta'$.

Now take any given $\varepsilon > 0$.

Since $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, we can find positive numbers δ_3 and δ_4 such that

$$|f(x) - l| < |m| \cdot \frac{\varepsilon}{2} \text{ whenever } 0 < |x - a| < \delta_3$$

and
$$|g(x) - m| < \frac{k|m|}{K} \cdot \frac{\varepsilon}{2} \text{ whenever } 0 < |x - a| < \delta_4.$$

Take $\delta = \min\{\delta', \delta_3, \delta_4\}$. Then from (2), we get

$$\left| \frac{f(x)}{g(x)} - \frac{l}{m} \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ whenever } 0 < |x - a| < \delta.$$

$\therefore$
$$\lim_{x \to a} \frac{f(x)}{g(x)} \text{ exists}$$

and
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \left(\frac{f}{g} \right)(x) = \frac{l}{m}, \text{ if } m \ne 0.$$

Alternative Proof:

Since $m \ne 0$, therefore, by theorem 1 of article 3, $\lim_{x \to a} \frac{1}{g(x)}$ exists and equals $\frac{1}{m}$.

Now
$$\lim_{x \to a} \left(\frac{f}{g} \right)(x) = \lim_{x \to a} \left\{ f(x) \cdot \frac{1}{g(x)} \right\}$$

$$= \left\{ \lim_{x \to a} f(x) \right\} \left\{ \lim_{x \to a} \frac{1}{g(x)} \right\}$$ [By theorem 3 of article 3]

$$= l \cdot \frac{1}{m} = \frac{l}{m} \cdot$$

Theorem 5: *Let f be defined on D and let $f(x) \geq 0$ for all $x \in D$.*

If $\lim_{x \to a} f(x)$ *exists, then* $\lim_{x \to a} f(x) \geq 0$.

Proof: Suppose that $\lim_{x \to a} f(x) = l$ and l is negative.

Taking $\varepsilon = -\frac{1}{2} l$, we can find a positive number $\delta > 0$ such that

$$|f(x) - l| < -\frac{1}{2} l \text{ whenever } 0 < |x - a| < \delta.$$

It gives that $\frac{3l}{2} < f(x) < \frac{l}{2} < 0$ whenever $0 < |x - a| < \delta$.

This is a contradiction since we are given that $f(x) \geq 0$ for all $x \in D$. Hence l cannot be negative.

Consequently $\lim_{x \to a} f(x) \geq 0$.

Corollary: *Let f be defined on D and let $f(x) > 0$ for all $x \in D$.*

If $\lim_{x \to a} f(x)$ *exists, then* $\lim_{x \to a} f(x) \geq 0$.

Proof: Since $f(x) > 0 \to f(x) \geq 0$, therefore now we can apply theorem 5, article 3.

Theorem 6: *Let f and g be defined on D and let $f(x) \geq g(x)$ for all $x \in D$. Then*

$$\lim_{x \to a} f(x) \geq \lim_{x \to a} g(x), \textit{ provided these limits exist.}$$

Proof: Let $\lim_{x \to a} f(x) = l$, $\lim_{x \to a} g(x) = m$.

Let us define a function h by $h(x) = f(x) - g(x) \ \forall x \in D$. Then, we have

(i) $h(x) \geq 0 \ \forall x \in D$.

(ii) $\lim_{x \to a} h(x)$ exists and $\lim_{x \to a} h(x) = l - m$.

(iii) $\lim_{x \to a} h(x) \geq 0$, by theorem 5 of article 3.

Thus, from (ii) and (iii), we get

$$l - m \geq 0 \text{ i.e., } l \geq m, \quad \text{i.e., } \lim_{x \to a} f(x) \geq \lim_{x \to a} g(x).$$

Corollary: *Let $f(x) > g(x)$ for all $x \in D$. Then* $\lim_{x \to a} f(x) \geq \lim_{x \to a} g(x)$, *provided these limits exist.*

Theorem 7: *Let f, g and h be defined on D and let* $f(x) \geq g(x) \geq h(x)$ *for all x.*

Let $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$.

Then $\lim_{x \to a} g(x)$ *exists, and* $\lim_{x \to a} g(x) = \lim_{x \to a} f(x) = \lim_{x \to a} h(x)$.

Proof: Let $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = l$.

Then corresponding to any given $\varepsilon > 0$, we can find positive numbers δ_1 and δ_2 such that

$$|f(x) - l| < \varepsilon \text{ whenever } 0 < |x - a| < \delta_1$$

i.e., $l - \varepsilon < f(x) < l + \varepsilon$ whenever $0 < |x - a| < \delta_1$ ...(1)

and $l - \varepsilon < h(x) < l + \varepsilon$ whenever $0 < |x - a| < \delta_2$. ...(2)

Choosing δ to be smaller than δ_1 and δ_2, we see from (1) and (2) that

$$l - \varepsilon < h(x) \leq g(x) \leq f(x) < l + \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

Thus $l - \varepsilon < g(x) < l + \varepsilon$ whenever $0 < |x - a| < \delta$

or $|g(x) - l| < \varepsilon$ whenever $0 < |x - a| < \delta$.

Hence $\lim_{x \to a} g(x)$ exists and $\lim_{x \to a} g(x) = l$.

Theorem 8: *If* $\lim_{x \to a} f(x) = l$, *then* $\lim_{x \to a} |f(x)| = |l|$.

(Gorakhpur 2010)

Proof: We have $|f(x) - l| \geq ||f(x)| - |l||$, for all x. ...(1)

$[\because |p - q| \geq ||p| - |q||]$

Let $\varepsilon > 0$ be given.

Since $\lim_{x \to a} f(x) = l$, therefore, given $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x) - l| < \varepsilon \text{ whenever } 0 < |x - a| < \delta. \quad ...(2)$$

From (1) and (2), we get $||f(x)| - |l|| < \varepsilon$ whenever $0 < |x - a| < \delta$.

Consequently $\lim_{x \to a} |f(x)|$ exists and $\lim_{x \to a} |f(x)| = |l|$.

Theorem 9: *If there is a number* $\delta > 0$ *such that* $h(x) = 0$ *whenever* $0 < |x - a| < \delta$, *then*

$$\lim_{x \to a} h(x) = 0.$$

Proof: For any $\varepsilon > 0$, the number $\delta > 0$, given in the hypothesis of the theorem is such that

$$h(x) = 0 \text{ whenever } 0 < |x - a| < \delta$$

or $|h(x) - 0| = 0 < \varepsilon$ whenever $0 < |x - a| < \delta$.

Hence $\lim_{x \to a} h(x) = 0$.

Corollary: *If there is a number* $\delta > 0$ *such that* $f(x) = g(x)$ *whenever* $0 < |x - a| < \delta$, *then*

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x).$$

Proof: Let us define a function h by setting $h(x) = f(x) - g(x)$ for all x.

Then $h(x) = 0$ whenever $0 < |x - a| < \delta$.

Now, apply theorem 9 of article 3.

Note: The above corollary has deep implications. It asserts that the concept of limit is a 'local' one. If two functions agree on some neighbourhood of a point a, then they cannot approach different limits as x approaches a.

Theorem 10: *If* $\lim_{x \to a} f(x) = 0$ *and* $g(x)$ *is bounded in some deleted neighbourhood of* a, *then* $\lim_{x \to a} f(x)\, g(x) = 0$.

Proof: Since $g(x)$ is bounded in some deleted neighbourhood of a, therefore there exist numbers $k > 0$ and $\delta_1 > 0$ such that

$$|g(x)| \leq k \text{ whenever } 0 < |x - a| < \delta_1. \qquad \ldots(1)$$

Now take any given $\varepsilon > 0$.

Since $\lim_{x \to a} f(x) = 0$, therefore there exists $\delta_2 > 0$ such that

$$|f(x) - 0| = |f(x)| < \frac{\varepsilon}{k} \text{ whenever } 0 < |x - a| < \delta_2 \qquad \ldots(2)$$

Now take $\delta = \min(\delta_1, \delta_2)$. Then for all x such that $0 < |x - a| < \delta$, we have

$$|f(x)\, g(x) - 0| = |f(x)\, g(x)|$$

$$= |f(x)| . |g(x)| < \frac{\varepsilon}{k} \cdot k = \varepsilon, \text{ using (1) and (2).}$$

Hence $\lim_{x \to a} f(x)\, g(x) = 0$.

Illustration: We have $\lim_{x \to 0} x \sin(1/x) = 0$

because $\lim_{x \to 0} x = 0$ and $|\sin(1/x)| \leq 1$ for all $x \neq 0$ *i.e.*, $\sin(1/x)$ is bounded in some deleted neighbourhood of zero.

4 Right Hand and Left Hand Limits

(Bundelkhand 2008)

Definition: **(Right-hand limit):** *A function f is said to approach l as x approaches a from right (or from above) if corresponding to an arbitrary positive number* ε, *there exists a positive number* δ *such that* $|f(x) - l| < \varepsilon$ *whenever* $a < x < a + \delta$.

It is written as $\lim_{x \to a+0} f(x) = l$ or $f(a + 0) = l$.

Working Rule for Finding the Right Hand Limit (R.H.L.):

"Put $a + h$ for x in $f(x)$ where h is + ive and very very small and make h approach zero".

In short, we have $f(a+0) = \lim_{h \to 0} f(a+h)$.

Definition: (Left-hand limit): *A function f is said to approach l as x approaches a from the left (or from below) if corresponding to an arbitrary positive number* ε*, there exists a positive number* δ *such that* $|f(x) - l| < \varepsilon$ *whenever* $a - \delta < x < a$.

It is written as $\lim_{x \to a-0} f(x) = l$ or $f(a-0) = l$.

Working Rule for Finding the Left Hand Limit (L.H.L.):

"Put $a - h$ for x in $f(x)$ where h is + ive and very very small and make h approach zero."

In this case, we have $f(a-0) = \lim_{h \to 0} f(a-h)$.

Note: If both right hand limit and left hand limit of f as $x \to a$, exist and are equal in value, their common value, evidently, will be the limit of f as $x \to a$. If, however, either or both of these limits do not exist, the limit of f as $x \to a$ does not exist. Even if both these limits exist but are not equal in value then also the limit of f as $x \to a$ does not exist.

5 Limits As $x \to +\infty(-\infty)$

Definition: *A function f is said to approach l as x becomes positively infinite, if corresponding to each* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $|f(x) - l| < \varepsilon$ *whenever* $x \geq \delta$.

Then we write $\lim_{x \to \infty} f(x) = l$ *or* $f(x) \to l$ *as* $x \to \infty$.

Definition: *A function f is said to approach l as x becomes negatively infinite, if corresponding to each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $|f(x) - l| < \varepsilon$ *whenever* $x \leq -\delta$.

Then we write $\lim_{x \to -\infty} f(x) = l$ *or* $f(x) \to l$ *as* $x \to -\infty$.

Note 1: The results on the limits of sum, product and quotient of functions also hold good here provided that in these cases $l + m$, lm, l / m are defined.

Note 2: If $\lim_{x \to \infty} f(x) = l$ exists, $\lim_{x \to \infty} g(x)$ does not exist (as a finite real number), even then $\lim_{x \to \infty} f(x)\, g(x)$ can exist. Similar is the case as $x \to -\infty$.

6 Infinite Limits

Definition: *A function f is said to approach* $+\infty$ *as x approaches a, if corresponding to any* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $f(x) > \varepsilon$ *whenever* $0 < |x - a| < \delta$.

Then we write $\lim_{x \to a} f(x) = \infty$ *or* $f(x)$ *tends to* ∞ *as x tends to a.*

Definition: *A function f is said to approach* $-\infty$ *as x approaches a, if corresponding to any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$f(x) < -\varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

Then we write $\lim_{x \to a} f(x) = -\infty$ *or* $f(x)$ *tends to* $-\infty$ *as x tends to a.*

Illustrative Examples

Example 1: *Let f be the function given by* $f(x) = \dfrac{x^2 - a^2}{x - a}$, $x \neq a$.

Using (ε, δ) *definition show that* $\lim_{x \to a} f(x) = 2a$.

Solution: Let $\varepsilon > 0$ be given. In order to show that

$$\lim_{x \to a} f(x) = 2a,$$

we have to show that for any given $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x) - 2a| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

If $x \neq a$, then $|f(x) - 2a| = \left| \dfrac{x^2 - a^2}{x - a} - 2a \right| = |(x + a) - 2a|$ $[\because x \neq a]$

$$= |x - a|.$$

$\therefore$ $|f(x) - 2a| < \varepsilon$, if $|x - a| < \varepsilon$.

Choosing a number δ such that $0 < \delta \leq \varepsilon$, we have

$$|f(x) - 2a| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$

Hence $\lim_{x \to a} f(x) = 2a$.

Example 2: *Using* (ε, δ) *definition show that* $\lim_{x \to 0} \left(x \sin \dfrac{1}{x} \right) = 0 \cdot$

(Gorakhpur 2011; Meerut 12, 13; Rohilkhand 13B)

Solution: Let $\varepsilon > 0$ be given. In order to show that $\lim_{x \to 0} \left(x \sin \dfrac{1}{x} \right) = 0$,

we have to show that for any given $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$\left| x \sin \frac{1}{x} - 0 \right| < \varepsilon \text{ whenever } 0 < |x - 0| < \delta.$$

Now $\left| x \sin \dfrac{1}{x} - 0 \right| = |x| \left| \sin \dfrac{1}{x} \right| \leq |x|$, because $\left| \sin \dfrac{1}{x} \right| \leq 1.$

$\therefore$ $\left| x \sin \dfrac{1}{x} - 0 \right| < \varepsilon$ whenever $|x| < \varepsilon$.

Choosing a number δ such that $0 < \delta \leq \varepsilon$, we have

$$\left| x \sin \frac{1}{x} - 0 \right| < \varepsilon \text{ whenever } 0 < |x| < \delta.$$

Hence $\lim_{x \to 0} x \sin \frac{1}{x} = 0.$

Example 3: *Show by* (ε, δ) *method that the function f, defined on* $\mathbf{R} \sim \{0\}$ *by* $f(x) = \sin(1/x)$ *whenever* $x \neq 0$, *does not tend to* 0 *as x tends to* 0. **(Meerut 2013B)**

Solution: In order to show that $\sin(1/x)$ does not tend to 0 as x tends to 0, take $\varepsilon = \frac{1}{2}$.

By Archimedean property of real numbers for any $\delta > 0$ there exists a positive integer n such that

$$n > \frac{1}{\pi\delta} \qquad \text{i.e.,} \qquad \delta > \frac{1}{n\pi}.$$

$$\therefore \qquad 0 < \frac{2}{(4n+1)\pi} < \frac{1}{2n\pi} < \frac{1}{n\pi} < \delta.$$

Take $x = \frac{2}{(4n+1)\pi}$. Then $0 < |x - 0| < \delta$.

Also, $|\sin(1/x) - 0| = |\sin(2n\pi + \frac{1}{2}\pi)| = 1 > \varepsilon$.

Thus we have shown that there exists an $\varepsilon > 0$, namely $\frac{1}{2}$, such that for every $\delta > 0$ there is an $x \left[= \frac{2}{(4n+1)\pi}, \text{where } n \text{ is a positive integer such that } \frac{2}{(4n+1)\pi} < \delta \right]$ such that

$$0 < |x - 0| < \delta \text{ and } |\sin(1/x) - 0| > \varepsilon.$$

Hence $\sin(1/x)$ does not tend to 0 as x tends to 0.

Example 4: *Show that* $\lim_{x \to 2} \frac{|x-2|}{x-2}$ *does not exist.*

Solution: Let $f(x) = |x - 2|/(x - 2)$.

We have the right hand limit *i.e.*,

$$f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} \frac{|2+h-2|}{(2+h-2)}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1;$$

and the left hand limit *i.e.*,

$$f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} \frac{|2-h-2|}{(2-h-2)}$$

$$= \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h}$$

$$= \lim_{h \to 0} -1 = -1.$$

Since $f(2+0) \neq f(2-0)$, hence $\lim_{x \to 2} \frac{|x-2|}{x-2}$ does not exist.

Example 5: *Evaluate the following limits if they exist :*

(a) $\lim\limits_{x \to 2} \dfrac{x^2 + 3x + 2}{x - 2}$.

Solution: Here the right hand limit *i.e.*,

$$f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} \frac{(2+h)^2 + 3(2+h) + 2}{2 + h - 2}$$

$$= \lim_{h \to 0} \frac{12 + 7h + h^2}{h} = \lim_{h \to 0} \left(\frac{12}{h} + 7 + h\right) = \infty;$$

and the left hand limit *i.e.,*

$$f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} \frac{(2-h)^2 + 3(2-h) + 2}{2 - h - 2}$$

$$= \lim_{h \to 0} \frac{12 - 7h + h^2}{-h} = \lim_{h \to 0} \left(-\frac{12}{h} + 7 - h\right) = -\infty.$$

Since $f(2+0) \neq f(2-0)$, hence $\lim\limits_{x \to 2} f(x)$ does not exist.

(b) $\lim\limits_{x \to 0} (1+x)^{1/x}$.

Solution: Here the right hand limit *i.e.,*

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} (1+h)^{1/h}$$

$$= \lim_{h \to 0} \left[1 + \frac{1}{h} \cdot h + \frac{\frac{1}{h}\left(\frac{1}{h} - 1\right)}{1.2} h^2 + \frac{\frac{1}{h}\left(\frac{1}{h} - 1\right)\left(\frac{1}{h} - 2\right)}{1.2.3} h^3 + \ldots\right]$$

$$= \lim_{h \to 0} \left[1 + \frac{1}{1!} + \frac{1.(1-h)}{2!} + \frac{1.(1-h)(1-2h)}{3!} + \ldots\right]$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots \infty = e.$$

Similarly, the left hand limit *i.e.,*

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} (1-h)^{-1/h} = e.$$

Thus both $f(0+0)$ and $f(0-0)$ exist and are equal to e.

Hence $\lim\limits_{x \to 0} (1+x)^{1/x} = e.$

(c) $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$.

(Bundelkhand 2008; Kanpur 09)

Solution: Let $f(x) = \dfrac{\sin x}{x}$.

Here $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \dfrac{\sin h}{h}$

$$= \lim_{h \to 0} \frac{h - \dfrac{h^3}{3!} + \dfrac{h^5}{5!} - \dots}{h} = \lim_{h \to 0} \left(1 - \frac{h^2}{3!} + \frac{h^4}{5!} - \dots\right) = 1.$$

Similarly $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$

$$= \lim_{h \to 0} \frac{\sin(-h)}{-h} = \lim_{h \to 0} \frac{\sin h}{h} = 1.$$

Since $f(0+0) = f(0-0) = 1$, hence $\lim_{x \to 0} \dfrac{\sin x}{x} = 1.$

(d) $\lim_{x \to \infty} \dfrac{\sin x}{x}$.

(Bundelkhand 2008)

Solution: $\lim_{x \to \infty} \dfrac{\sin x}{x} = \lim_{y \to 0} y \sin(1/y)$, putting $x = 1/y$.

Let $f(y) = y \sin(1/y) \cdot$

$$f(0+0) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} h \sin(1/h)$$

$= 0 \times$ a finite quantity lying between -1 and 1

$= 0.$

Similarly, $f(0-0) = \lim_{h \to 0} f(0-h)$

$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} (-h)\sin(-1/h)$$

$$= \lim_{h \to 0} h \sin(1/h) = 0.$$

Since $f(0+0) = f(0-0) = 0$, therefore

$\lim_{y \to 0} y \sin(1/y) = 0$ *i.e.,* $\lim_{x \to \infty} \dfrac{\sin x}{x} = 0.$

(e) $\lim_{x \to 0} \sin \dfrac{1}{x}$.

(Kanpur 2010)

Solution: Let $f(x) = \sin(1/x)$.

Here $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \sin \dfrac{1}{h}$.

As $h \to 0$, the value of $\sin(1/h)$ oscillates between $+1$ and -1, passing through zero and intermediate values an infinite number of times. Hence there is no definite number l to which sin $(1/h)$ tends as h tends to zero. Therefore the right hand limit $f(0+0)$ does not exist.

Similarly the left hand limit $f(0-0)$ also does not exist. Thus $\lim_{x \to 0} \sin(1/x)$ does not exist.

(f) $\lim_{x \to 0} \dfrac{a^x - 1}{x}$. **(Meerut 2003; Lucknow 08; Kanpur 10)**

Solution: Let $f(x) = \dfrac{a^x - 1}{x}$

$$= \frac{1 + x \log a + (x^2/2!)(\log a)^2 + \ldots - 1}{x}$$

$$= \frac{x\left[\log a + \frac{1}{2} x (\log a)^2 + \ldots\right]}{x}.$$

Here $\quad f(0+0) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \dfrac{h\left[\log a + \frac{h}{2}(\log a)^2 + \ldots\right]}{h} = \log a.$

Also $\quad f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \log a.$

Since $\quad f(0+0) = f(0-0) = \log a$, therefore $\lim_{x \to 0} \dfrac{a^x - 1}{x} = \log a.$

(g) $\lim_{x \to 0} \dfrac{1}{x} \cdot e^{1/x}$.

Solution: Let $f(x) = \dfrac{1}{x} \cdot e^{1/x}$.

Then $\quad f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \dfrac{1}{h} e^{1/h}$

$= \infty \quad$, since both $1/h$ and $e^{1/h}$ tend to ∞ as $h \to 0$.

Also $\quad f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} -\dfrac{1}{h} e^{-1/h}$

$$= \lim_{h \to 0} \frac{-1}{he^{1/h}} = \lim_{h \to 0} \frac{-1}{h\left(1 + \frac{1}{h} + \frac{1}{2!} \cdot \frac{1}{h^2} + \ldots\right)}$$

$$= \lim_{h \to 0} \frac{-1}{h + 1 + (1/2h) + \ldots} = 0.$$

Since $\quad f(0+0) \neq f(0-0)$, therefore $\lim_{x \to 0} \dfrac{1}{x} e^{1/x}$ does not exist.

(h) $\lim_{x \to 0} \dfrac{(1+x)^n - 1}{x}$.

Let $f(x) = \dfrac{(1+x)^n - 1}{x}$.

Then $f(0+0) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \dfrac{(1+h)^n - 1}{h}$

$$= \lim_{h \to 0} \frac{1 + nh + \dfrac{n(n-1)}{2!} h^2 + \ldots - 1}{h}$$

$$= \lim_{h \to 0} \frac{h\left[n + \dfrac{n(n-1)}{2!} h + \ldots\right]}{h}$$

$$= \lim_{h \to 0} \left[n + \frac{n(n-1)}{2!} h + \ldots\right] = n.$$

Also $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \dfrac{(1-h)^n - 1}{-h}$

$$= \lim_{h \to 0} \frac{1 + n(-h) + \dfrac{n(n-1)}{2!}(-h)^2 + \ldots - 1}{-h} = n.$$

Since $f(0+0) = f(0-0) = n$, therefore $\lim_{x \to 0} f(x) = n$.

(i) $\lim_{x \to a} \dfrac{x^m - a^m}{x - a}$.

Solution: Let $f(x) = \dfrac{x^m - a^m}{x - a}$.

Then $f(a+0) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} \dfrac{(a+h)^m - a^m}{a + h - a}$

$$= \lim_{h \to 0} \frac{a^m\left[\left(1 + \dfrac{h}{a}\right)^m - 1\right]}{h}$$

$$= \lim_{h \to 0} \frac{a^m}{h}\left[1 + m \cdot \frac{h}{a} + \frac{m(m-1)}{2!}\frac{h^2}{a^2} + \ldots - 1\right]$$

$$= \lim_{h \to 0} a^m \left[\frac{m}{a} + \frac{m(m-1)}{2} \cdot \frac{h}{a^2} + \ldots\right] = a^m \cdot \frac{m}{a} = ma^{m-1}.$$

Also $f(a-0) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} \dfrac{(a-h)^m - a^m}{a - h - a} = ma^{m-1}$.

Since $f(a+0) = f(a-0) = ma^{m-1}$, hence $\lim_{x \to a} f(x) = ma^{m-1}$.

Example 6: *Find the right hand and the left hand limits in the following cases and discuss the existence of the limit in each case :*

(i) $\lim_{x \to 2} \dfrac{2x^2 - 8}{x - 2}$; (ii) $\lim_{x \to 0} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$;

(Meerut 2003; Kanpur 11; Gorakhpur 12; Rohilkhand 14)

(iii) $\lim_{x \to 0} f(x)$ *where* $f(x)$ *is defined as*

$$f(x) = x, \text{ when } x > 0 \text{ ; } f(x) = 0, \text{ when } x = 0 \text{ ; } f(x) = -x, \text{ when } x < 0.$$

(Purvanchal 2008)

Solution: **(i)** Let $f(x) = \dfrac{2x^2 - 8}{x - 2}$.

We have $f(2 + 0) = \lim_{h \to 0} f(2 + h) = \lim_{h \to 0} \dfrac{2(2 + h)^2 - 8}{2 + h - 2}$

$$= \lim_{h \to 0} \frac{2(4 + 4h + h^2) - 8}{h} = \lim_{h \to 0} \frac{8h + 2h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(8 + 2h)}{h} = \lim_{h \to 0} (8 + 2h) = 8.$$

Again $f(2 - 0) = \lim_{h \to 0} f(2 - h) = \lim_{h \to 0} \dfrac{2(2 - h)^2 - 8}{2 - h - 2}$

$$= \lim_{h \to 0} \frac{2(4 - 4h + h^2) - 8}{-h} = \lim_{h \to 0} \frac{-8h + 2h^2}{-h}$$

$$= \lim_{h \to 0} \frac{-h(8 - 2h)}{-h} = \lim_{h \to 0} (8 - 2h) = 8.$$

Since $f(2 + 0) = f(2 - 0) = 8$, therefore $\lim_{x \to 2} \dfrac{2x^2 - 8}{x - 2}$ exists and is equal to 8.

(ii) Let $f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$.

Here the right hand limit, *i.e.,*

$$f(0 + 0) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1}$$

$$= \lim_{h \to 0} \frac{e^{1/h}[1 - (1/e^{1/h})]}{e^{1/h}[1 + (1/e^{1/h})]} = 1.$$

Again the left hand limit, *i.e.,*

$$f(0 - 0) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1}$$

$$= \lim_{h \to 0} \frac{(1/e^{1/h}) - 1}{(1/e^{1/h}) + 1} = \frac{0 - 1}{0 + 1} = -1.$$

Since $\quad f(0+0) \neq f(0-0)$, hence $\lim\limits_{x \to 0} \dfrac{e^{1/x}-1}{e^{1/x}+1}$ does not exist.

(iii) We have the right hand limit *i.e.,* $f(0+0)$

$$= \lim_{h \to 0} f(0+h), \text{ where } h \text{ is + ive but sufficiently small}$$

$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} h, \qquad [\because h > 0 \text{ and } f(x) = x \text{ if } x > 0]$$

$$= 0.$$

Also, the left hand limit, *i.e.,* $f(0-0)$

$$= \lim_{h \to 0} f(0-h), \text{ where } h \text{ is + ive but sufficiently small}$$

$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} -(-h), \qquad [\because -h < 0 \text{ and } f(x) = -x \text{ if } x < 0]$$

$$= \lim_{h \to 0} h = 0.$$

Thus both the limits $f(0+0)$ and $f(0-0)$ exist and are equal to zero.

Hence $\lim\limits_{x \to 0} f(x)$ exists and is equal to zero.

Example 7: *Let* $\quad f(x) = \begin{cases} x & \text{if } x \textit{ is rational} \\ -x & \text{if } x \textit{ is irrational.} \end{cases}$

Show that $\lim\limits_{x \to a} f(x)$ *exists only when* $a = 0$.

(Purvanchal 2007)

Solution: **Case I. If a is a non-zero rational number.**

In this case $f(a-0) = \lim\limits_{h \to 0} f(a-h)$

$$= \lim_{h \to 0} (a-h) \quad \text{or} \quad \lim_{h \to 0} -(a-h),$$

according as $(a-h)$ is rational or irrational

$= a \quad$ or $\quad -a$ *i.e.,* is not unique.

$\therefore \quad f(a-0)$ does not exist. $\quad \therefore \quad \lim\limits_{x \to a} f(x)$ does not exist.

Case II: If $a = 0$. In this case $f(0-0) = \lim\limits_{h \to 0} f(0-h) = \lim\limits_{h \to 0} f(-h)$

$$= \lim_{h \to 0} (-h) \quad \text{or} \quad \lim_{h \to 0} h, \text{ according as } -h \text{ is rational or irrational}$$

$$= 0.$$

Again $f(0+0) = \lim\limits_{h \to 0} f(0+h) = \lim\limits_{h \to 0} f(h)$

$$= \lim_{h \to 0} h \quad \text{or} \quad \lim_{h \to 0} (-h), \text{ according as } h \text{ is rational or irrational}$$

$$= 0.$$

Since $f(0+0) = f(0-0) = 0$, hence $\lim_{x \to 0} f(x)$ exists and is equal to zero.

Case III. If *a* is an irrational number.

In this case $f(a-0) = \lim_{h \to 0} f(a-h)$

$$= \lim_{h \to 0} (a-h) \text{ or } \lim_{h \to 0} -(a-h),$$

according as $(a-h)$ is rational or irrational

$= a$ or $-a$ *i.e.,* is not unique.

$\therefore$ $f(a-0)$ does not exist. $\therefore$ $\lim_{x \to a} f(x)$ does not exist.

Thus we see that $\lim_{x \to a} f(x)$ exists only when $a = 0$.

Example 8: *Discuss the existence of the limit of the function f defined as*

$$f(x) = 1, \text{ if } x < 1; \quad f(x) = 2 - x, \text{ if } 1 < x < 2; \quad f(x) = 2, \text{ if } x \geq 2$$

at $x = 1$ *and* $x = 2$.

Solution: At $\boldsymbol{x = 1}$. We have

$$f(1+0) = \lim_{h \to 0} f(1+h), \quad \text{where } h \text{ is + ive and sufficiently small}$$

$$= \lim_{h \to 0} [2 - (1+h)] = \lim_{h \to 0} (1-h) = 1;$$

and $f(1-0) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} (1) = 1.$

Since $f(1+0) = f(1-0) = 1$, hence $\lim_{x \to 1} f(x)$ exists and is equal to 1.

At $\boldsymbol{x = 2}$. We have $f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} (2) = 2;$

and $f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} [2 - (2-h)] = \lim_{h \to 0} h = 0.$

Since $f(2+0) \neq f(2-0)$, hence $\lim_{x \to 2} f(x)$ does not exist.

Example 9: *If* $\lim_{x \to a} f(x) = \pm \infty$, *then prove that* $\lim_{x \to a} \dfrac{1}{f(x)} = 0.$

Solution: Let $\lim_{x \to a} f(x) = +\infty.$

Let $\varepsilon > 0$ be given. If $\varepsilon_1 = 1/\varepsilon$, then $\varepsilon_1 > 0$.

Since $\lim_{x \to a} f(x) = \infty$, therefore for $\varepsilon_1 > 0$, there exists $\delta > 0$ such that

$$f(x) > \varepsilon_1 \text{ whenever } 0 < |x-a| < \delta \ \textit{i.e.,} \ \frac{1}{f(x)} < \frac{1}{\varepsilon_1} \text{ whenever } 0 < |x-a| < \delta$$

i.e., $0 < \dfrac{1}{f(x)} < \varepsilon$ whenever $0 < |x - a| < \delta$ $\qquad [\because \ \varepsilon = 1/\varepsilon_1]$

i.e., $-\varepsilon < \dfrac{1}{f(x)} < \varepsilon$ whenever $0 < |x - a| < \delta$

i.e, $\left|\dfrac{1}{f(x)} - 0\right| < \varepsilon$ whenever $0 < |x - a| < \delta$.

$\therefore \quad \lim\limits_{x \to a} \dfrac{1}{f(x)} = 0 \cdot$

Similarly it can be proved that $\lim\limits_{x \to a} \dfrac{1}{f(x)} = 0$ when $\lim\limits_{x \to a} f(x) = -\infty$.

Example 10: *If* $f(x) = \dfrac{\sin [x]}{[x]}$, $[x] \neq 0$ *and* $f(x) = 0$, $[x] = 0$,

where $[x]$ *denotes the greatest integer less than or equal to x, then find* $\lim\limits_{x \to 0} f(x)$.

(Kanpur 2010)

Solution: Here $f(0+0) = \lim\limits_{h \to 0} f(0+h) = \lim\limits_{h \to 0} f(h)$

$$= \lim_{h \to 0} 0 \qquad [\because \ [h] = 0]$$

$$= 0.$$

Also $\qquad f(0-0) = \lim\limits_{h \to 0} f(0-h) = \lim\limits_{h \to 0} f(-h)$

$$= \lim_{h \to 0} \frac{\sin [-h]}{[-h]}, \qquad [\because \ [-h] = -1 \neq 0]$$

$$= \lim_{h \to 0} \frac{\sin (-1)}{(-1)} = \frac{\sin (-1)}{(-1)} = \sin 1 \neq 0.$$

Since $f(0+0) \neq f(0-0)$, therefore $\lim\limits_{x \to 0} f(x)$ does not exist.

Comprehensive Problems 1

1. If $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ do not exist, can

$\lim\limits_{x \to a} [f(x) + g(x)]$ or $\lim\limits_{x \to a} [f(x)\, g(x)]$ exist ?

(Kumaun 2008)

2. If $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} [f(x) + g(x)]$ both exist, must $\lim\limits_{x \to a} g(x)$ exist ?

3. If $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} [f(x)\, g(x)]$ both exist, must $\lim\limits_{x \to a} g(x)$ exist ?

4. Show that $\lim_{x \to 0} f(x) = \lim_{x \to a} f(x - a)$.

5. Using definition of limit, show that $\lim_{x \to 0} f(x) = 1$ where $f(x) = \begin{cases} 1 + x^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

6. If f is defined on **R** as $f(x) = \begin{cases} 2, & \text{if } x \text{ is irrational} \\ 1, & \text{if } x \text{ is rational,} \end{cases}$

prove that $\lim_{x \to a} f(x)$ does not exist for any $a \in \mathbf{R}$.

7. If f is defined on **R** as $f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1, & \text{if } x \text{ is rational,} \end{cases}$

prove that $\lim_{x \to a} f(x)$ does not exist for any $a \in \mathbf{R}$.

8. If $x \to 0$, then does the limit of the following function f exist or not ?
$f(x) = x$, when $x < 0$; $f(x) = 1$, when $x = 0$; $f(x) = x^2$, when $x > 0$.

9. Use the formula $\lim_{x \to 0} \frac{a^x - 1}{x} = \log a$ to find $\lim_{x \to 0} \frac{2^x - 1}{(1 + x)^{1/2} - 1}$.

10. If $f(x) = e^{-1/x}$, show that at $x = 0$, the right hand limit is zero while the left hand limit is $+\infty$, and thus there is no limit of the function at $x = 0$.

11. Prove that $\lim_{x \to 0+0} f(x) = \lim_{x \to 0-0} f(-x)$.

12. Give an example to show that $\lim_{x \to a} f(x)$ may exist even when the function is not defined for $x = a$.

13. Let $f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 3 - x, & 1 \leq x \leq 2. \end{cases}$

Show that $\lim_{x \to 1+} f(x) = 2$. Does the limit of $f(x)$ at $x = 1$ exist ? Give reasons for your answer.

14. Evaluate : $\lim_{x \to 0} \frac{x - |x|}{x}$. **(Meerut 2001)**

15. Evaluate : $\lim_{x \to 0} \frac{|\sin x|}{x}$.

16. Evaluate : $\lim_{x \to 0} \frac{e^{1/x}}{e^{1/x} + 1}$. **(Meerut 2006; Rohilkhand 05, 08; Avadh 10)**

17. If $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n$ then prove that $\lim_{x \to c} f(x) = f(c)$. **(Garhwal 2011)**

Answers 1

1. Yes. If we take $f(x) = \sin(1/x)$, $g(x) = -\sin(1/x)$ whenever $x \neq 0$, then $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ do not exist but $\lim_{x \to 0} [f(x) + g(x)]$ exists.

   Again if we take $f(x) = g(x) = 1$ for all rational x and $f(x) = g(x) = -1$ for all irrational x, then both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ do not exist for any real number a but $\lim_{x \to a} [f(x)\, g(x)]$ exists for every real number a.

2. Yes.
3. No. If we take $f(x) = x \ \forall\, x \in \mathbf{R}$ and $g(x) = \sin(1/x)$, if $x \neq 0$, $g(0) = 0$, then $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} [f(x)\, g(x)]$ both exist but $\lim_{x \to 0} g(x)$ does not exist.

8. Yes; $\lim_{x \to 0} f(x) = 0$.
9. 2 log 2.
13. Does not exist.
14. Right hand limit is 0 and left hand limit is 2 and so the limit does not exist.
15. Does not exist because the right hand limit is 1 and the left hand limit is -1.
16. The limit does not exist because the right hand limit is 1 and the left hand limit is 0.

7 Continuity

(Purvanchal 2010, 11; Gorakhpur 11; Avadh 14)

The intuitive concept of continuity is derived from geometrical considerations. If the graph of the function $y = f(x)$ is a continuous curve, it is natural to call the function continuous. This requires that there should be no sudden changes in the value of the function. A small change in x should produce only a small change in y. Moreover for the graph to be a continuous running curve, it should possess a definite direction at each point.

But the continuity as defined in pure analysis is quite distinct from the intuitive or the geometrical concept of the term. Sometimes drawing a graph is difficult. We now give the arithmetical definition of continuity given by Cauchy.

Cauchy's definition of continuity.

A real valued function f defined on an open interval I is said to be continuous at $a \in I$ iff for any arbitrarily chosen positive number ε, however small, we can find a corresponding number $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \text{ whenever } |x - a| < \delta. \qquad \ldots(1)$$

(Bundelkhand 2010; Kanpur 11)

We say that f is a **continuous function** if it is continuous at every $x \in I$.

In other words, f is continuous at a if for any given $\varepsilon > 0$, we can find a $\delta > 0$ such that

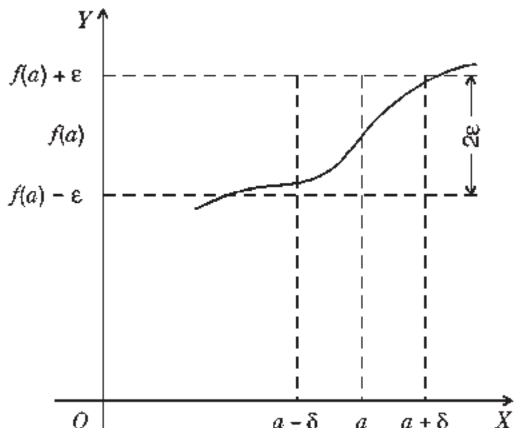
$$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

This means that the function f will be continuous at $x = a$ if the difference between $f(a)$ and the value of $f(x)$ at any point in the interval $]a-\delta, a+\delta[$ can be made less than a pre-assigned positive number ε. Note that we choose δ after we have chosen ε.

A geometrical interpretation of the above definition is immediate. Corresponding to any pre-assigned positive number ε, we can determine an interval of width 2δ about the point $x = a$ (see the figure) such that for any point x lying in the interval $]a-\delta, a+\delta[$, $f(x)$ is confined to lie between

$$f(a) - \varepsilon \text{ and } f(a) + \varepsilon.$$

The inequality (1) may be written in the form of an equality as $f(x) = f(a) + \eta$, where $|\eta| < \varepsilon$.



Note 1: For a function $f(x)$ to be continuous at $x = a$, it is necessary that $\lim\limits_{x \to a} f(x)$ must exist.

Note 2: $|f(x) - f(x)| \Rightarrow f(a) - \in < f(x) < f(a) + \in$

and $|x - a| < \delta \Rightarrow a - \delta < x < a + \delta$

Note 3: The function must be defined at the point of continuity.

Note 4: The value of δ depends upon the values of ε and a.

Note 5: The interval I may be of any one of the forms :

$$]a, b[\,,]-\infty, b\,[\,,]a, \infty[\,,]-\infty, \infty[.$$

An alternative definition of continuity of a Function At a Point: *A function f is said to be continuous at $a \in I$ iff $\lim\limits_{x \to a} f(x)$ exists, is finite and is equal to $f(a)$ otherwise the function is discontinuous at $x = a$.*

This definition of continuity follows immediately from the definition of limit and the definition of continuity. Thus a function f is said to be continuous at a, if $f\ (a+0) = f\ (a-0) = f\ (a)$. This is a working formula for testing the continuity of a function at a given point. **(Bundelkhand 2008, 10; Kashi 12)**

Polynomial Function:

Theorem 1: *A polynomial function is always a continuous function.*

Proof: If $f\ (x) = a_0\ x^n + a_1\ x^{n-1} + \ldots + a_{n-1}\ x + a_n$ is a polynomial in x of degree n, then by the above definition it can be easily seen that $f\ (x)$ is continuous for all $x \in \mathbf{R}$.

If c be any real number, then

$$\lim_{x \to c} f\ (x) = \lim_{x \to c} \{a_0\ x^n + a_1\ x^{n-1} + \ldots + a_{n-1}\ x + a_n\}$$

$$= a_0 \lim_{x \to c} x^n + a_1 \lim_{x \to c} x^{n-1} + \ldots + a_{n-1} \lim_{x \to c} x + \lim_{x \to c} a_n$$

$$= a_0\ c^n + a_1\ c^{n-1} + \ldots + a_{n-1}\ c + a_n \qquad \left[\because \lim_{x \to c} x = c\right]$$

$$= f\ (c).$$

Since $\lim_{x \to c} f\ (x) = f\ (c)$, therefore $f\ (x)$ is continuous at $x = c$.

Thus $f\ (x)$ is continuous at every real number c and so $f\ (x)$ is continuous for all $x \in \mathbf{R}$.

Thus remember that a polynomial function $f\ (x)$ is always continuous at each point of its domain.

Continuity from left and continuity from right:

Let f be a function defined on an open interval I and let $a \in I$. We say that f is continuous from the left at a if $\lim_{x \to a-0} f\ (x)$ *exists and is equal to* $f\ (a)$. *Similarly f is said to be continuous from the right at a if* $\lim_{x \to a+0} f\ (x)$ *exists and is equal to* $f\ (a)$.

From the above definitions it is clear that for a function f to be continuous at a, it is necessary as well as sufficient that f be continuous from the left as well as from the right at a.

Continuous function: *A function f is said to be a continuous function if it is continuous at each point of its domain.*

Continuity in an open interval: A function f is said to be continuous in the open interval $]a, b[$ if it is continuous at each point of the interval. **(Bundelkhand 2009)**

Continuity in a closed interval: Let f be a function defined on the closed interval $[a, b]$. We say that f is continuous at a if it is continuous from the right at a and also that f is continuous at b if it is continuous from the left at b. Further, f is said to be continuous on the closed interval $[a, b]$, if (i) it is continuous from the right at a, (ii) continuous from the left at b and (iii) continuous on the open interval $]a, b[$.

Thus if a function f is defined on the closed interval $[a, b]$, then

(i) it is continuous at the left end point a if $f(a) = f(a + 0)$

i.e., $$f(a) = \lim_{x \to a+0} f(x)$$

(ii) it is continuous at the right end point b if $f(b) = f(b - 0)$

i.e., $$f(b) = \lim_{x \to b-0} f(x)$$

and (iii) it is continuous at an interior point c of $[a, b]$ *i.e.*, at $c \in]\, a, b\, [$

if $$f(c - 0) = f(c) = f(c + 0)$$

i.e., if $$\lim_{x \to c-0} f(x) = f(c) = \lim_{x \to c+0} f(x).$$

8 Discontinuity

Definition: *If a function is not continuous at a point, then it is said to be discontinuous at that point and the point is called a point of discontinuity of this function.*

Types of Discontinuity:

(i) Removable discontinuity: **(Meerut 2011; Avadh 14)**

A function f is said to have a *removable discontinuity* at a point a if $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$ *i.e.*, if

$$f(a + 0) = f(a - 0) \neq f(a).$$

The function can be made continuous by defining it in such a way that $\lim_{x \to a} f(x) = f(a)$.

(ii) Discontinuity of the first kind or ordinary discontinuity: **(Meerut 2010B)**

A function f is said to have a *discontinuity of the first kind* or *ordinary discontinuity* at a if $f(a + 0)$ and $f(a - 0)$ both exist but are not equal. The point a is said to be a point of discontinuity from the left or right according as $f(a - 0) \neq f(a) = f(a + 0)$ or $f(a - 0) = f(a) \neq f(a + 0)$.

(iii) Discontinuity of the second kind: **(Meerut 2003, 10B)**

A function f is said to have a *discontinuity of the second kind*, at a if none of the limits $f(a + 0)$ and $f(a - 0)$ exist. The point a is said to be a point of discontinuity of the second kind from the left or right according as $f(a - 0)$ or $f(a + 0)$ does not exist.

(iv) Mixed discontinuity: **(Meerut 2012B)**

A function f is said to have a *mixed discontinuity* at a, if f has a discontinuity of second kind on one side of a and on the other side a discontinuity of first kind or may be continuous.

(v) Infinite discontinuity:

A function f is said to have an *infinite discontinuity* at a if $f\ (a+0)$ or $f\ (a-0)$ is $+\infty$ or $-\infty$. Obviously, if f has a discontinuity at a and is unbounded in every neighbourhood of a, then f is said to have an infinite discontinuity at a.

9 Jump of a Function at a Point

If both $f\ (a+0)$ and $f\ (a-0)$ exist, then the **jump** in the function at a is defined as the non-negative difference $f\ (a+0) \sim f\ (a-0)$. A function having a finite number of jumps in a given interval is **called piecewise continuous.**

10 Algebra of Continuous Functions

Theorem 1: *Let f and g be defined on an interval I. If f and g are continuous at $a \in I$, then $f + g$ is also continuous at a.*

Theorem 2: *Let f and g be defined on an interval I. If f and g are continuous at $a \in I$, then fg is continuous at a.*

Theorem 3: *If f is continuous at a point a and $c \in \mathbf{R}$, then cf is continuous at a.*

Theorem 4: *Let f and g be defined on an interval I, and let $g\ (a) \neq 0$. If f and g are continuous at $a \in I$, then $f\ /\ g$ is continuous at a.*

Theorem 5: *If f is continuous at a then $|\,f\,|$ is also continuous at a.*

Note: The converse is not true. For example, if

$f\ (x) = -1$, for $x < a$ and $f\ (x) = 1$ for $x \geq a$ then

$\lim_{x \to a} |\,f\ (x)\,| = 1 = |\,f\ (a)\,|$, but $\lim_{x \to a} f\ (x)$ does not exist.

Thus $|\,f\,|$ is continuous at a while f is not continuous at a.

Illustrative Examples

Example 11: *Test the following functions for continuity :*

(i) $f\ (x) = x \sin (1/x),\ x \neq 0,\ f\ (0) = 0$ *at* $x = 0$. **(Kanpur 2005; Avadh 08; Meerut 09B; Purvanchal 09; Kashi 12; Rohilkhand 14; Gorakhpur 12, 14)**

Also draw the graph of the function. **(Lucknow 2007)**

(ii) $f\ (x) = 2^{1/x}$ *when* $x \neq 0,\ f\ (0) = 0$ *at* $x = 0$.

(iii) $f\ (x) = 1/(1 - e^{-1/x}),\ x \neq 0,\ f\ (0) = 0$ *at* $x = 0$.

Solution: (i) Here $f(0+0) = \lim_{h \to 0} f(0+h), h > 0$

$$= \lim_{h \to 0} f(h) = \lim_{h \to 0} h \sin \frac{1}{h} = 0.$$

$$\left[\because \lim_{h \to 0} h = 0 \text{ and } \left|\sin \frac{1}{h}\right| \leq 1 \text{ for all } h \neq 0 \text{ i.e., } \sin(1/h) \text{ is bounded in some deleted neighbourhood of zero}\right]$$

Similarly $f(0-0) = \lim_{h \to 0} f(0-h), h > 0$

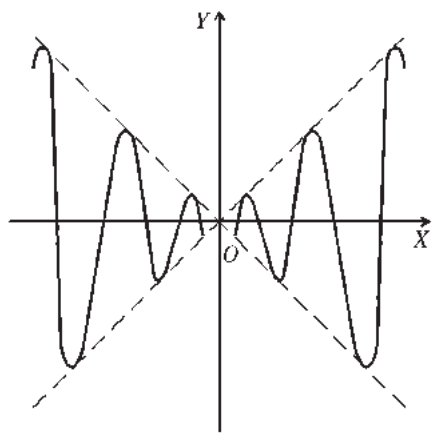
$$= \lim_{h \to 0} f(-h) = \lim_{h \to 0} (-h) \sin\left(\frac{1}{-h}\right)$$

$$= \lim_{h \to 0} h \sin \frac{1}{h} = 0, \text{ as before.}$$

Also $f(0) = 0$. Thus $f(0-0) = f(0) = f(0+0)$.

$\therefore$ the function $f(x)$ is continuous at $x = 0$. To draw the graph of the function we put $y = f(x)$.

So the graph of the function is the curve $y = x \sin(1/x), x \neq 0$ and $y = 0$ when $x = 0$.



If we put $-x$ in place of x, the equation of this curve does not change and so this curve is symmetrical about the y-axis and it is sufficient to draw the graph when $x > 0$.

Also

$|f(x)| = |x \sin(1/x)| = |x| \cdot |\sin(1/x)|$

$\leq |x|$. $\quad [\because |\sin(1/x)| \leq 1]$

$\therefore$ for all x the curve $y = x \sin(1/x)$ lies between the lines $y = x$ and $y = -x$.

Excluding origin the curve meets the y-axis at the points where

$$\sin \frac{1}{x} = 0 \text{ i.e., where } \frac{1}{x} = \pi, 2\pi, 3\pi, \ldots \text{ i.e., where } x = \frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \ldots.$$

Also $\quad y = x$ at the points where $\sin \frac{1}{x} = 1$ i.e., $\frac{1}{x} = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \ldots$

$$\text{i.e., } x = \frac{2}{\pi}, \frac{2}{5\pi}, \frac{2}{9\pi}, \ldots$$

and $\quad y = -x$ at the points where $\sin \frac{1}{x} = -1$ i.e., $\frac{1}{x} = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$

$$\text{i.e., } x = \frac{2}{3\pi}, \frac{2}{7\pi}, \ldots.$$

We have $\dfrac{dy}{dx} = \sin\dfrac{1}{x} + x\left(\cos\dfrac{1}{x}\right)\left(-\dfrac{1}{x^2}\right) = \sin\dfrac{1}{x} - \dfrac{1}{x}\cos\dfrac{1}{x}\cdot$

So at the points where $\sin(1/x) = 1$, we have $\cos(1/x) = 0$ and $dy/dx = 1$ *i.e.*, at these points the curve touches the straight line $y = x$. Similarly at the points where $\sin(1/x) = -1$, the curve touches the straight line $y = -x$.

Also
$$\lim_{x\to\infty} x\sin\frac{1}{x} \qquad [\text{Form } \infty\times 0]$$
$$= \lim_{x\to\infty}\frac{\sin(1/x)}{1/x} \qquad \left[\text{Form } \frac{0}{0}\right]$$
$$= \lim_{\theta\to 0}\frac{\sin\theta}{\theta}, \text{ putting } \frac{1}{x} = \theta \text{ so that } \theta\to 0 \text{ as } x\to\infty$$
$$= 1.$$

Thus $y \to 1$ as $x \to \infty$ and so the straight line $y = 1$ is an asymptote of the curve.

Although the function is continuous at the origin, yet the graph of the function in the vicinity of the origin cannot be drawn, since the function oscillates infinitely often in any interval containing the origin.

From the graph it is clear that the function makes an infinite number of oscillations in the neighbourhood of $x = 0$. The oscillations, however, go on diminishing in length as $x \to 0$.

Note 1: If we are to check the continuity of $f(x)$ at any point $x = c$, where $c \neq 0$, then we see that

$$\lim_{x\to c} f(x) = \lim_{x\to c} x\sin\frac{1}{x} = c\sin\frac{1}{c} = f(c)$$

and so $f(x)$ is continuous at $x = c$.

Thus $f(x)$ is continuous for all $x \in \mathbf{R}$ *i.e.*, $f(x)$ is continuous on the whole real line.

Note 2: If we take $f(0) = 2$, the function becomes discontinuous at $x = 0$ and has a **removable discontinuity** at $x = 0$.

(ii) Here $f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} 2^{1/h} = 2^{\infty} = \infty$,

$$f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} 2^{-1/h} = 2^{-\infty} = 0 \quad \text{and} \quad f(0) = 0.$$

Since $f(0+0) \neq f(0-0)$, therefore the function is discontinuous at the origin.

It has an **infinite discontinuity** there.

(iii) Here $f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\dfrac{1}{1-e^{-1/h}} = 1$,

$$f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0}\frac{1}{1-e^{1/h}} = 0.$$

Since $f(0+0) \neq f(0-0)$, hence $f(x)$ is discontinuous at $x = 0$ and has discontinuity of the first kind. This function has a jump of one unit at 0 since $f(0+0) - f(0-0) = 1$.

Example 12: *Consider the function f defined by $f(x) = x - [x]$, where x is a positive variable and [x] denotes the integral part of x and show that it is discontinuous for integral values of x and continuous for all others. Draw its graph.*

Solution: From the definition of the function $f(x)$, we have

$$f(x) = x - (n-1) \qquad \text{for } n-1 < x < n,$$
$$f(x) = 0 \qquad \text{for } x = n,$$
$$f(x) = x - n \qquad \text{for } n < x < n+1, \text{ where } n \text{ is an integer.}$$

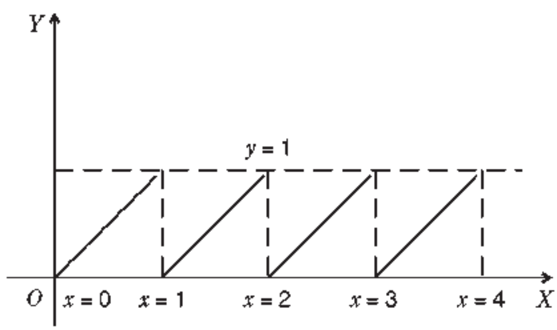
We shall test the function $f(x)$ for continuity at $x = n$.

We have $\quad f(n) = 0;$

$$f(n+0) = \lim_{h \to 0} f(n+h) = \lim_{h \to 0} \{(n+h) - n\} \qquad [\because n < n+h < n+1]$$
$$= \lim_{h \to 0} h = 0;$$

and
$$f(n-0) = \lim_{h \to 0} f(n-h) = \lim_{h \to 0} \{(n-h) - (n-1)\} \qquad [\because n-1 < n-h < n]$$
$$= \lim_{h \to 0} (1-h) = 1.$$

Since $f(n+0) \neq f(n-0)$, the function $f(x)$ is discontinuous at $x = n$. Thus $f(x)$ is discontinuous for all integral values of x. It is obviously continuous for all other values of x.



Since x is a positive variable, putting $n = 1, 2, 3, 4, 5, \ldots$ we see that the graph of $f(x)$ consists of the following straight lines :

$y = x$	when $0 < x < 1$,	$y = 0$ when $x = 1$
$y = x - 1$	when $1 < x < 2$,	$y = 0$ when $x = 2$
$y = x - 2$	when $2 < x < 3$,	$y = 0$ when $x = 3$
$y = x - 3$	when $3 < x < 4$,	$y = 0$ when $x = 4$ and so on.

The graph of the function thus obtained is shown by thick lines from $x = 0$ to $x = 4$.
From the graph it is evident that :

(i) The function is discontinuous for all integral values of x but continuous for other values of x.

(ii) The function is bounded between 0 and 1 in every domain which includes an integer.

(iii) The lower bound 0 is attained but the upper bound 1 is not attained since $f(x) \neq 1$ for any value of x.

Example 13: *Show that the function* $f(x) = [x] + [-x]$ *has removable discontinuity for integral values of* x. **(Kanpur 2009)**

Solution: We observe that $f(x) = 0$, when x is an integer and $f(x) = -1$, when x is not an integer. Hence if n is any integer, we have $f(n-0) = f(n+0) = -1$ and $f(n) = 0$. So the function $f(x)$ has a removable discontinuity at $x = n$, where n is an integer.

Example 14: *Let* $y = E(x)$, *where* $E(x)$ *denotes the integral part of* x. *Prove that the function is discontinuous where* x *has an integral value. Also draw the graph.*

Solution: From the definition of $E(x)$, we have

$$E(x) = n - 1 \quad \text{for} \quad n - 1 \leq x < n,$$
$$E(x) = n \quad \text{for} \quad n \leq x < n + 1$$
$$E(x) = n + 1 \quad \text{for} \quad n + 1 \leq x < n + 2,$$

and so on where n is an integer.

We consider $x = n$.

Then $E(n) = n$, $E(n-0) = n - 1$ and $E(n+0) = n$.

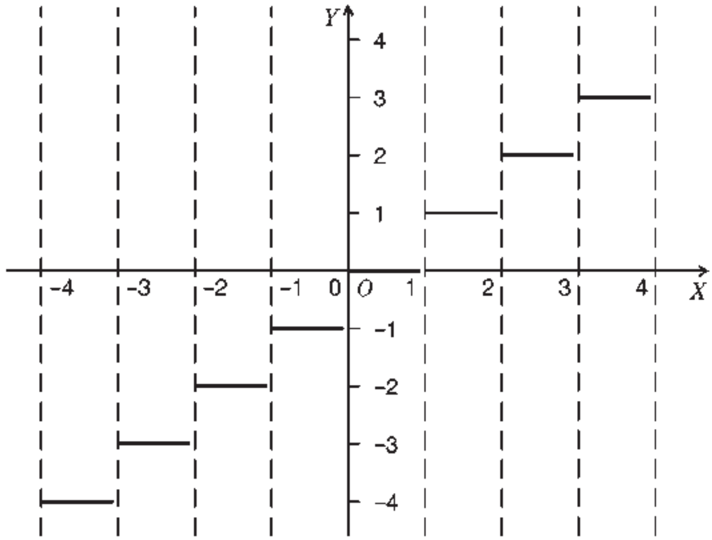
Since $E(n+0) \neq E(n-0)$, the function $E(x)$ is discontinuous at $x = n$ *i.e.,* when x has an integral value.

Evidently it is continuous for all other values of x.

To draw the graph, we put $n = \ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$, so that

$$y = -4, \quad \text{when} \quad -4 \leq x < -3,$$
$$y = -3, \quad \text{when} \quad -3 \leq x < -2,$$
$$y = -2, \quad \text{when} \quad -2 \leq x < -1,$$
$$y = -1, \quad \text{when} \quad -1 \leq x < 0,$$
$$y = 0, \quad \text{when} \quad 0 \leq x < 1$$
$$y = 1, \quad \text{when} \quad 1 \leq x < 2$$
$$y = 2, \quad \text{when} \quad 2 \leq x < 3$$
$$y = 3, \quad \text{when} \quad 3 \leq x < 4$$
$$y = 4, \quad \text{when} \quad 4 \leq x < 5 \text{ and so on.}$$

The graph is shown by thick lines.



Example 15: *Show that the function ϕ defined as*

$$\phi(x)=\begin{cases}0 & \text{for } x=0\\ \frac{1}{2}-x & \text{for } 0<x<\frac{1}{2}\\ \frac{1}{2} & \text{for } x=\frac{1}{2}\\ \frac{3}{2} & \text{for } \frac{1}{2}<x<1\\ 1 & \text{for } x=1\end{cases}$$

has three points of discontinuity which you are required to find. Also draw the graph of the function.

(Rohilkhand 2009; Avadh 10, 13)

Solution: Here the domain of the function $\phi(x)$ is the closed interval $[0,1]$.

When $0<x<\frac{1}{2}$, $\phi(x)=\frac{1}{2}-x$ which is a polynomial in x of degree 1. We know that a polynomial function is continuous at each point of its domain and so $\phi(x)$ is continuous at each point of the open interval $0<x<\frac{1}{2}\cdot$

Again when $\frac{1}{2}<x<1$, $\phi(x)=\frac{3}{2}-x$ which is also a polynomial in x and so $\phi(x)$ is also continuous at each point of the open interval $\frac{1}{2}<x<1$.

Now it remains to test the function $\phi(x)$ for continuity at $x=0,\frac{1}{2}$ and 1.

(i) For $x=0$, we have $\phi(0)=0$,

$$\phi(0+0)=\lim_{h\to 0}\phi(0+h)=\lim_{h\to 0}\phi(h)=\lim_{h\to 0}\left(\frac{1}{2}-h\right)=\frac{1}{2}\cdot$$

Since $\phi(0)\neq\phi(0+0)$, the function $\phi(x)$ is discontinuous at $x=0$ and the discontinuity is ordinary.

(ii) For $x = \frac{1}{2}$, we have $\phi\left(\frac{1}{2}\right) = \frac{1}{2}$,

$$\phi\left(\frac{1}{2} - 0\right) = \lim_{h \to 0} \phi\left(\frac{1}{2} - h\right) = \lim_{h \to 0} \left[\frac{1}{2} - \left(\frac{1}{2} - h\right)\right],$$

$$\left[\text{Note that } 0 < \frac{1}{2} - h < \frac{1}{2}\right]$$

$$= \lim_{h \to 0} h = 0.$$

Since $\phi\left(\frac{1}{2} - 0\right) \neq \phi\left(\frac{1}{2}\right)$, the function $\phi(x)$ is discontinuous from the left at $x = 1/2$.

Again $\phi\left(\frac{1}{2} + 0\right) = \lim_{h \to 0} \phi\left(\frac{1}{2} + h\right), h > 0$

$$= \lim_{h \to 0} \left[\frac{3}{2} - \left(\frac{1}{2} + h\right)\right] \qquad \left[\because \frac{1}{2} < \frac{1}{2} + h < 1\right]$$

$$= \lim_{h \to 0} (1 - h) = 1 \neq \phi\left(\frac{1}{2}\right) = \frac{1}{2}.$$

Thus the function $\phi(x)$ is discontinuous from the right also at $x = \frac{1}{2}$.

In this way $\phi(x)$ has discontinuity of the first kind *i.e.*, ordinary discontinuity at $x = \frac{1}{2}$ and the jump of the function at $x = 1/2$ is $\phi\left(\frac{1}{2} + 0\right) - \phi\left(\frac{1}{2} - 0\right)$ *i.e.*, $1 - 0$ *i.e.*, 1.

(iii) For $x = 1$, we have $\phi(1) = 1$,

$$\phi(1 - 0) = \lim_{h \to 0} \phi(1 - h)$$

$$= \lim_{h \to 0} [(3/2) - (1 - h)], \qquad \left[\text{Note that } \frac{1}{2} < 1 - h < 1\right]$$

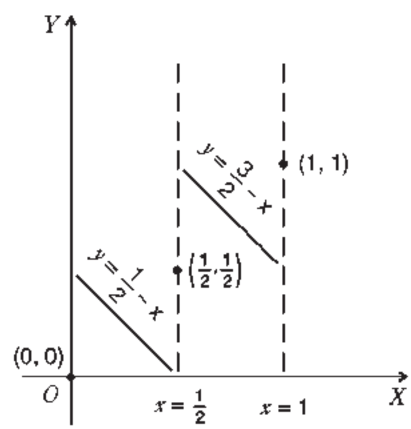
$$= \lim_{h \to 0} \left(\frac{1}{2} + h\right) = \frac{1}{2}.$$

Since $\phi(1) \neq \phi(1 - 0)$, $\phi(x)$ is discontinuous at $x = 1$ and the discontinuity is ordinary.

Hence the function $\phi(x)$ has three points of discontinuity at $x = 0, \frac{1}{2}$ and 1.

The graph of the function consists of the point $(0, 0)$; the segment of the line $y = \frac{1}{2} - x$, $0 < x < \frac{1}{2}$; the point $\left(\frac{1}{2}, \frac{1}{2}\right)$; the segment of the line $y = \frac{3}{2} - x$, $\frac{1}{2} < x < 1$; and the point $(1, 1)$.

Thus the graph is as shown in the figure. From the graph we observe that the function is discontinuous at $x = 0, \frac{1}{2}$ and 1.



Example 16: *Determine the values of a, b, c for which the function*

$$f(x) = \begin{cases} \dfrac{\sin (a+1) x + \sin x}{x} & \text{for } x < 0 \\ c & \text{for } x = 0 \\ \dfrac{(x + bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} & \text{for } x > 0 \end{cases}$$

is continuous at $x = 0$.

Solution: Here $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \dfrac{(h + bh^2)^{1/2} - h^{1/2}}{bh^{3/2}}$

$$= \lim_{h \to 0} \frac{(1 + bh)^{1/2} - 1}{bh} = \lim_{h \to 0} \frac{\{1 + \frac{1}{2} bh + \ldots\} - 1}{bh} = \frac{1}{2},$$

which is independent of b and so b may have any real value except 0.

Again $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \dfrac{\sin (a+1)(-h) + \sin (-h)}{(-h)}$

$$= \lim_{h \to 0} \frac{\sin (a+1) h + \sin h}{h}$$

$$= \lim_{h \to 0} \frac{2 \sin \left(\frac{1}{2} a + 1 \right) h \cos (ah/2)}{h}$$

$$= \lim_{h \to 0} \frac{\sin \{(a+2)/2\} h}{\{(a+2)/2\} h} (a+2) \cos (ah/2) = a + 2.$$

For continuity at $x = 0$, we have $f(0+0) = f(0-0) = f(0)$

i.e., $\dfrac{1}{2} = a + 2 = c$. $\therefore$ $c = \dfrac{1}{2}$ and $a = -\dfrac{3}{2}$.

Example 17: *A function* $f(x)$ *is defined as follows :*

$$f(x) = \begin{cases} (x^2/a) - a, & \text{when } x < a \\ 0, & \text{when } x = a \\ a - (a^2/x), & \text{when } x > a. \end{cases}$$

Prove that the function $f(x)$ *is continuous at* $x = a$.

(Bundelkhand 2007; Avadh 09; Rohilkhand 13)

Solution: We have

$$f(a+0) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} \left[a - \frac{a^2}{(a+h)} \right],$$

$[\because f(x) = a - (a^2/x) \text{ for } x > a]$

$$= [a - (a^2 / a)] = a - a = 0 \; ;$$

$$f(a-0) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} \left[\frac{(a-h)^2}{a} - a \right],$$

$$[\because \; f(x) = (x^2 / a) - a \text{ for } x < a]$$

$$= [(a^2 / a) - a] = a - a = 0.$$

Also, we have $f(a) = 0$.

Since $f(a+0) = f(a-0) = f(a)$, therefore $f(x)$ is continuous at $x = a$.

Example 18: *Examine the function defined below for continuity at* $x = a$:

$$f(x) = \frac{1}{x-a} \, cosec \left(\frac{1}{x-a} \right), x \neq a$$

and $\qquad f(x) = 0, \; x = a.$ **(Avadh 2004; Lucknow 08)**

Solution: We have $f(a+0) = \lim_{h \to 0} f(a+h)$

$$= \lim_{h \to 0} \frac{1}{a+h-a} \operatorname{cosec} \frac{1}{a+h-a} = \lim_{h \to 0} \frac{1}{h \sin (1/h)}$$

$= + \infty$, since $h \sin (1/h) \to 0$ as $h \to 0$.

$$f(a-0) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} \frac{1}{a-h-a} \operatorname{cosec} \left(\frac{1}{a-h-a} \right)$$

$$= \lim_{h \to 0} - \left[\frac{1}{h} \cdot \frac{1}{\sin \{-(1/h)\}} \right] = \lim_{h \to 0} \frac{1}{h \sin (1/h)}$$

$= + \infty$, since $h \sin (1/h) \to 0$ as $h \to 0$.

Also, we have $f(a) = 0$.

Since $f(a+0) = f(a-0) \neq f(a)$, the function $f(x)$ is discontinuous at $x = a$, having an infinite discontinuity of the second kind.

Example 19: *Examine the function defined below for continuity at* $x = 0$:

$$f(x) = \frac{sin^2 \, ax}{x^2} \; for \; x \neq 0, \; f(x) = 1 \, for \; x = 0.$$

(Lucknow 2006, 07; Meerut 10)

Solution: We have $f(0) = 1$;

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} \frac{\sin^2 ah}{h^2}$$

$$= \lim_{h \to 0} \left(\frac{\sin ah}{ah} \right)^2 . \, a^2 = 1 \, . \, a^2 = a^2 \; ;$$

and $$f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)=\lim_{h\to 0}\frac{\sin^2(-ah)}{(-h)^2}$$

$$=\lim_{h\to 0}\frac{\sin^2 ah}{h^2}=a^2.$$

Since $f(0+0)=f(0-0)\neq f(0)$.

Hence $f(x)$ is discontinuous at $x=0$.

Example 20: *A function $f(x)$ is defined as follows :*

$$f(x)=1+x \text{ if } x\leq 2 \text{ and } f(x)=5-x \text{ if } x\geq 2.$$

Is the function continuous at $x=2$? **(Meerut 2002, 06; Lucknow 09)**

Solution: Here $f(2)=1+2$ or $5-2=3$;

$$f(2+0)=\lim_{h\to 0} f(2+h), \text{ where } h \text{ is + ive and sufficiently small}$$

$$=\lim_{h\to 0}[5-(2+h)], \qquad [\because 2+h>2 \text{ and } f(x)=5-x \text{ if } x>2]$$

$$=\lim_{h\to 0}(3-h)=3;$$

and $$f(2-0)=\lim_{h\to 0} f(2-h), \text{ where } h \text{ is + ive and sufficiently small}$$

$$=\lim_{h\to 0}[1+(2-h)], \qquad [\because 2-h<2 \text{ and } f(x)=1+x \text{ if } x<2]$$

$$=\lim_{h\to 0}(3-h)=3.$$

Thus $f(2+0)=f(2-0)=f(2)$. Hence the function $f(x)$ is continuous at $x=2$.

Example 21: *Discuss the continuity of the function $f(x)$ defined as follows:*

$$f(x)=x^2 \text{ for } x<-2,\quad f(x)=4 \text{ for } -2\leq x\leq 2,\quad f(x)=x^2 \text{ for } x>2.$$

Solution: We shall test the continuity of $f(x)$ only at the points $x=-2$ and 2. Obviously it is continuous at all other points.

At $x=-2$. We have $f(-2)=4$;

$$f(-2+0)=\lim_{h\to 0} f(-2+h)=\lim_{h\to 0} 4=4;$$

$$f(-2-0)=\lim_{h\to 0} f(-2-h)=\lim_{h\to 0}(-2-h)^2, \qquad [\because -2-h<-2]$$

$$=4.$$

Since $f(-2+0)=f(-2-0)=f(-2)$, the function is continuous at $x=-2$.

At $x=2$. We have $f(2)=4$;

$$f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} (2+h)^2 = 4;$$

$$f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} 4 = 4.$$

Since $f(2+0) = f(2-0) = f(2)$, the function is continuous at $x = 2$.

Comprehensive Exercise 2

1. Discuss the continuity and discontinuity of the following functions :

(i) $f(x) = x^3 - 3x$

(ii) $f(x) = x + x^{-1}$

(iii) $f(x) = e^{-1/x}$

(iv) $f(x) = \sin x$.

(v) $f(x) = \cos(1/x)$ when $x \neq 0$, $f(0) = 0$. **(Lucknow 2003)**

(vi) $f(x) = \sin(1/x)$ when $x \neq 0$, $f(0) = 0$. **(Lucknow 2011)**

(vii) $f(x) = \dfrac{\sin x}{x}$ when $x \neq 0$ and $f(0) = 1$. **(Kanpur 2007; Avadh 08)**

(viii) $f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ when $x \neq 0$ and $f(0) = 1$. **(Meerut 2004B; Kumaun 10)**

(ix) $f(x) = \dfrac{e^{1/x}}{1 + e^{1/x}}$ when $x \neq 0$, $f(0) = 0$. **(Lucknow 2011; Bundelkhand 11)**

(x) $f(x) = \dfrac{x e^{1/x}}{1 + e^{1/x}} + \sin(1/x)$ when $x \neq 0$, $f(0) = 0$.

(xi) $f(x) = \sin x \cos(1/x)$ when $x \neq 0$, $f(0) = 0$.

2. (i) Examine at $x = 0$, the continuity of $f(x) = \begin{cases} \dfrac{e^{1/x^2}}{1 - e^{1/x^2}}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0. \end{cases}$

(Meerut 2008)

(ii) If $f(x) = \dfrac{1}{x-a} \sin \dfrac{1}{x-a}$, find $f(a+0)$ and $f(a-0)$.

Is the function continuous at $x = a$?

3. Find out the points of discontinuity of the following functions :

(i) $f(x) = (2 + e^{1/x})^{-1} + \cos e^{1/x}$ for $x \neq 0, f(0) = 0$.

(ii) $f(x) = 1/2^n$ for $1/2^{n+1} < x \leq 1/2^n, n = 0, 1, 2, \ldots$ and $f(0) = 0$.

4. If $f(x) = \dfrac{1}{x} \sin \dfrac{1}{x}$ for $x \neq 0$ and $f(0) = 0$, show that $f(x)$ is finite for every value of x in the interval $[-1, 1]$ but is not bounded. Determine the points of discontinuity of the function if any.

5. A function f defined on $[0, 1]$ is given by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f takes every value between 0 and 1 (both inclusive), but it is continuous only at the point $x = \dfrac{1}{2}$.

(Rohilkhand 2012B)

6. Prove that the function f defined by $f(x) = \begin{cases} \dfrac{1}{2}, & \text{if } x \text{ is rational} \\ \dfrac{1}{3}, & \text{if } x \text{ is irrational} \end{cases}$

is discontinuous everywhere.

7. (i) Show that the function f defined by $f(x) = \dfrac{x\, e^{1/x}}{1 + e^{1/x}}, x \neq 0, f(0) = 1$ is not continuous at $x = 0$ and also show how the discontinuity can be removed. **(Rohilkhand 2006; Lucknow 08; Meerut 11)**

(ii) Show that the function $f(x) = 3x^2 + 2x - 1$ is continuous for $x = 2$.

(iii) Show that the function $f(x) = (1 + 2x)^{1/x}, x \neq 0,$ and $f(x) = e^2, x = 0$ is continuous at $x = 0$.

8. Examine the continuity of the function $f(x) = \begin{cases} -x^2 & \text{if } x \leq 0 \\ 5x - 4 & \text{if } 0 < x \leq 1 \\ 4x^2 - 3x & \text{if } 1 < x < 2 \\ 3x + 4 & \text{if } x \geq 2 \end{cases}$

at $x = 0, 1$ and 2. **(Meerut 2004, 06B, 07B; Lucknow 06; Avadh 06; Purvanchal 06, 10; Gorakhpur 15)**

9. (i) Show that the function

$f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}, x \neq 0$ and $f(0) = 0$ is discontinuous at $x = 0$.

(ii) Show that the following function is continuous at $x = 0$:

$f(x) = \dfrac{\sin^{-1} x}{x}, x \neq 0, f(0) = 1.$

10. Discuss the continuity of the function $f(x)=\dfrac{1}{1-e^{1/x}}$ when $x \neq 0$ and $f(0)=0$ for all values of x. **(Lucknow 2010)**

11. Prove that the function $f(x)=\dfrac{|x|}{x}$ for $x \neq 0, f(0)=0$ is continuous at all points except $x=0$. **(Kanpur 2008; Meerut 09; Gorakhpur 11)**

12. Test the continuity of the function $f(x)$ at $x=0$ if

$$f(x)=\frac{e^{1/x}\sin(1/x)}{1+e^{1/x}},\ x \neq 0 \text{ and } f(x)=0,\ x=0.$$

(Meerut 2005)

13. Examine the following function for continuity at $x=0$ and at $x=1$:

$$f(x)=\begin{cases} x^2 & \text{for } x \leq 0 \\ 1 & \text{for } 0<x\leq 1 \\ 1/x & \text{for } x>1. \end{cases}$$

(Meerut 2001,03, 04B, 05)

14. Discuss the continuity of the following function at $x=0$:

$$f(x)=\begin{cases} \cos x, & x \geq 0 \\ -\cos x, & x<0. \end{cases}$$

15. Test the continuity of the following functions at $x=0$:

(i) $f(x)=x\cos(1/x)$, when $x \neq 0, f(0)=0$. **(Meerut 2007)**

(ii) $f(x)=x\log x$, for $x>0, f(0)=0$.

16. Discuss the nature of discontinuity at $x=0$ of the function $f(x)=[x]-[-x]$ where $[x]$ denotes the integral part of x.

17. Discuss the continuity of $f(x)=(1/x)\cos(1/x)$.

18. Give an example of each of the following types of functions :

(i) The function which possesses a limit at $x=1$ but is not defined at $x=1$.

(ii) The function which is neither defined at $x=1$ nor has a limit at $x=1$.

(iii) The function which is defined at two points but is nevertheless discontinuous at both the points.

19. In the closed interval $[-1, 1]$ let f be defined by

$$f(x)=x^2\sin(1/x^2) \text{ for } x \neq 0 \text{ and } f(0)=0.$$

In the given interval (i) Is the function bounded ? (ii) Is it continuous ?

Answers 2

1. (i) Continuous for all x (ii) Discontinuous at $x=0$
(iii) Discontinuous at $x=0$ (iv) Continuous for all x
(v) Discontinuous at $x=0$ (vi) Discontinuous at 0
(vii) Continuous for all x (viii) Discontinuous at 0

(ix) Discontinuous at 0 (x) Discontinuous at 0

(xi) Continuous for all x

2. No, it has a discontinuity of second kind. Here both $f(a+0)$ and $f(a-0)$ do not exit

3. (i) Discontinuous at $x=0$

(ii) Discontinuous at $x = 1/2^n, = 1,2,3,...$

4. Discontinuous at 0

8. Continuous at $x = 1, 2$ and discontinuous at $x=0$

10. Discontinuous only at $x=0$ and the discontinuity is ordinary

12. Discontinuity of the second kind at $x=0$

13. Discontinuous at $x=0$ and continuous at $x=1$

14. Discontinuous at $x=0$

15. (i) Continuous (ii) Continuous

16. Discontinuity of the first kind

17. Continuous for all x, except at $x=0$ where it has discontinuity of the second kind

18. (i) $f(x) = x^2$ for $x > 1$, $f(x) = x^3$ for $x < 1$

(ii) $f(x) = -x^2$ for $x < 1$, $f(x) = x^2$ for $x > 1$

(iii) $f(x) = 0$ for $x \leq 0$, $f(x) = \frac{3}{2} - x$ for $0 < x \leq \frac{1}{2}$, $f(x) = \frac{3}{2} + x$ for $x > \frac{1}{2}$

19. (i) Yes (ii) Yes

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. $\lim\limits_{x \to 0} \frac{|x|}{x}$ is equal to

(a) 1 (b) –1

(c) 2 (d) The limit does not exist

2. $\lim\limits_{x \to 0} \frac{e^x - 1}{x}$ is equal to

(a) 0 (b) 1

(c) –1 (d) 2

3. $\lim\limits_{x \to 2+} \frac{|x-2|}{x-2}$ is equal to

(a) –1 (b) 1

(c) 2 (d) –2

4. $\lim_{x \to 3-} \dfrac{|x-3|}{x-3}$ is equal to

(a) -1 (b) 3

(c) -3 (d) 1 **(Meerut 2003)**

5. $\lim_{x \to 0+} \dfrac{e^{1/x}-1}{e^{1/x}+1}$ is equal to

(a) -1 (b) 1

(c) 0 (d) 2

6. The value of K for which $f(x) = \begin{cases} \dfrac{\sin 5x}{3x}, & \text{if } x \neq 0 \\ K, & \text{if } x = 0 \end{cases}$

is continuous at $x = 0$, shall be

(a) 1/3 (b) 3/5

(c) 0 (d) 5/3 **(Kumaun 2008)**

7. The value of $\lim_{n \to 0} \sin \dfrac{1}{n}$ shall be

(a) 1 (b) -1

(c) 0 (d) non-existent **(Kumaun 2009)**

8. The value of $\lim_{x \to a} \dfrac{x^m - a^m}{x-a}$ shall be

(a) mx^{m-1} (b) a^{m-1}

(c) ma^{m-1} (d) x^{m-1} **(Kumaun 2011)**

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. A function $f(x)$ is continuous at a point $x = a$ if $\lim_{x \to a} f(x) = \ldots\ldots$

(Bundelkhand 2008; Kumaun 14)

2. $\lim_{x \to 1} \dfrac{x^2-1}{x-1} = \ldots\ldots$

3. $\lim_{x \to 0} \dfrac{\sin(x/4)}{x} = \ldots\ldots$

4. $\lim_{x \to 1} \dfrac{x^3-1}{x^2-1} = \ldots\ldots$

5. $\lim_{x \to 0} (1+x)^{1/x} = \ldots\ldots$

6. $\lim_{x \to 0} \dfrac{a^x-1}{x} = \ldots\ldots$

7. If $f(x) = x - [x]$, where $[x]$ denotes the greatest integer less than or equal to x, then $f(x) = \ldots\ldots$, for $3 < x < 4$.

8. Let $f(x) = \begin{cases} x & , \ 0 \le x < 1 \\ 3 - x, & 1 \le x \le 2. \end{cases}$

   Then $\lim_{x \to 1-} f(x) = \ldots\ldots$

9. Let $f(x) = \begin{cases} 1 & , \quad x < 1 \\ 2 - x, & 1 \le x < 2 \\ 2 & , \quad x \ge 2. \end{cases}$

   Then (i) $f\left(\frac{3}{2}\right) = \ldots\ldots$

   (ii) $\lim_{x \to 1+} f(x) = \ldots\ldots$ and

   (iii) $\lim_{x \to 2-} f(x) = \ldots\ldots$

(Meerut 2003)

10. $\lim_{x \to 0-} \frac{|\sin x|}{x} = \ldots\ldots$

11. A function $f(x)$ has a removable discontinuity at $x = a$ if $\lim_{x \to a} f(x)$ exists but is not equal to …… .

12. The domain of the function $f(x) = \frac{\sin x}{x}$ is …… .

13. The domain of the function $f(x) = \begin{cases} \frac{\sin x}{x}, & x \ne 0 \\ 1 & , \ x = 0 \end{cases}$

    is …… .

True or False

Write 'T' for true and 'F' for false statement.

1. If $f(x) = \begin{cases} x, & \text{when } x < 0 \\ 1, & \text{when } x = 0 \\ x^2, & \text{when } x > 0, \end{cases}$

   then $\lim_{x \to 0} f(x) = 0$.

2. The function $f(x) = \begin{cases} \frac{\sin x}{x}, & x \ne 0 \\ 2 & , \ x = 0 \end{cases}$

   is continuous at $x = 0$.

3. The function $f(x)=\begin{cases}\sin x, & x\geq 0\\ -\sin x, & x<0\end{cases}$

is continuous at $x=0$.

4. For $\lim\limits_{x\to a} f(x)$ to exist, the function $f(x)$ must be defined at $x=a$.

5. The function $f(x)=\begin{cases}x\cos(1/x), & x\neq 0\\ 0, & x=0\end{cases}$

is discontinuous at $x=0$.

6. If a function f is continuous at a, then $|f|$ is also continuous at a.

7. The function $f(x)=\begin{cases}\dfrac{\sin x}{x}, & x\neq 0\\ 1, & x=0\end{cases}$

is continuous at $x=0$.

8. The function $f(x)=\begin{cases}1, & x<1\\ 2-x, & 1\leq x<2\\ 2, & x\geq 2\end{cases}$

is discontinuous at $x=1$.

9. $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=1.$

10. $\lim\limits_{x\to 0}\dfrac{\sin 3x}{x}=1.$

11. $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}=2.$

12. $\lim\limits_{x\to 0}\dfrac{\sin x}{2x}=\dfrac{1}{2}.$

Answers

Multiple Choice Questions

1.	(d)	2.	(b)	3.	(b)	4.	(a)	5.	(b)
6.	(d)	7.	(d)	8.	(c)				

Fill in the Blank(s)

1.	$f(a)$	2.	2	3.	$\frac{1}{4}$	4.	$\frac{3}{2}$	5.	e		
6.	$\log_e a$	7.	$x-3$	8.	1	9.	(i) $\frac{1}{2}$		(ii) 1		(iii) 0
10.	-1	11.	$f(a)$	12.	$\mathbf{R}-\{0\}$	13.	$\mathbf{R}$				

True or False

1.	*T*	**2.**	*F*	**3.**	*T*	**4.**	*F*	**5.**	*F*
6.	*T*	**7.**	*T*	**8.**	*F*	**9.**	*F*	**10.**	*F*
11.	*T*	**12.**	*T*						

□

Chapter 5

Differentiability

1 Definitions

Derivative at a Point. **(Bundelkhand 2010)**

Let I denote the open interval $]a, b[$ in $\mathbf{R}$ *and let $x_0 \in I$. Then a function $f : I \to \mathbf{R}$ is said to be* **differentiable** (*or* **derivable**) *at x_0 iff*

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \text{ or equivalently } \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists finitely and this limit, if it exists finitely, is called the **differential coefficient** *or* **derivative** *of f with respect to x at $x = x_0$.*

It is denoted by $f'(x_0)$ or by $Df(x_0)$.

Progressive and regressive derivatives.

The *progressive derivative* of f at $x = x_0$ is given by

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}, h > 0.$$

It is also called the **right hand differential coefficient** of f at $x = x_0$ and is denoted by $R\, f'(x_0)$ or by $f'(x_0 + 0)$.

The *regressive derivative* of f at $x = x_0$ is given by

$$\lim_{h \to 0} \frac{f(x_0 - h) - f(x_0)}{-h}, h > 0.$$

It is also called the **left hand differential coefficient** of f at $x = x_0$ and is denoted by $L\, f'(x_0)$ or by $f'(x_0 - 0)$.

It is obvious that *f is derivable at x_0 iff $L\, f'(x_0)$ and $R\, f'(x_0)$ both exist and are equal.*

Remark: If $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$ is a polynomial in x of degree n, then $f(x)$ is differentiable at every point a of $\mathbf{R}$.

Differentiability in an interval: **(Meerut 2003)**

Open interval $]a, b[$. *A function $f:]a, b[\to \mathbf{R}$ is said to be differentiable in $]a, b[$ iff it is differentiable at every point of $]a, b[$.*

Closed interval $]a, b[$. *A function $f: [a, b] \to \mathbf{R}$ is said to be differentiable in $[a, b]$ iff $R\, f'(a)$ exists, $L\, f'(b)$ exists and f is differentiable at every point of $]a, b[$.*

Derivative of a function: **(Gorakhpur 2010)**

Let f be a function whose domain is an interval I. If I_1 be the set of all those points x of I at which f is differentiable *i.e.,* $f'(x)$ exists and if $I_1 \neq \varnothing$, we get another function f' with domain I_1. It is called the *first derivative* of f (or simply the derivative of f). Similarly 2nd, 3rd, ..., nth derivatives of f are defined and are denoted by $f'', f''', \ldots, f^{(n)}$ respectively.

Note: The derivative of a function at a point and the derivative of a function are two different but related concepts. The derivative of f at a point a is a number while the derivative of f is a function. However, very often the term derivative of f is used to denote both number and function and it is left to the context to distinguish what is intended.

An alternate definition of differentiability:

Let f be a function defined on an interval I and let a be an interior point of I. Then, by the definition of $f'(a)$, assuming it to exist, we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

i.e., $f'(a)$ exists if for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < \varepsilon \text{ whenever } 0 < |x - a| < \delta$$

or equivalently

$$x \in]a - \delta, a + \delta[\Rightarrow f'(a) - \varepsilon < \frac{f(x) - f(a)}{x - a} < f'(a) + \varepsilon.$$

2 Geometrical Meaning of a Derivative

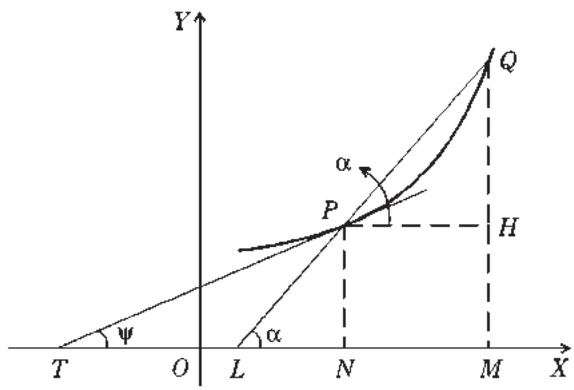
We take two neighbouring points $P\,[a, f\,(a)]$ and $Q\,[a+h, f\,(a+h)]$ on the curve $y = f\,(x)$.

Let the chord PQ and the tangent at P meet the x-axis in L and T respectively. Let $\angle Q\,LX = \alpha$ and $\angle PTX = \psi$. Draw PN and $QM \perp$ to OX and $PH \perp$ to QM.

Then $\quad PH = NM = OM - ON = a + h - a = h,$

and $\quad QH = QM - MH = QM - PN = f\,(a+h) - f\,(a).$

$\therefore \quad \tan\alpha = \dfrac{QH}{PH} = \dfrac{f\,(a+h) - f\,(a)}{h}.$ ...(1)



As $h \to 0$, the point Q moving along the curve approaches the point P, the chord PQ approaches the tangent line TP as its limiting position and the angle α approaches the angle ψ.

Hence taking limits as $h \to 0$, the equation (1) gives

$$\tan\psi = f\,'(a).$$

Hence $f\,'(a)$ is the tangent of the angle which the tangent line to the curve $y = f\,(x)$ at the point $P\,[a, f\,(a)]$ makes with x-axis.

3 Meaning of the Sign of Derivative

Let $f\,'(c) > 0$ where c is an interior point of the domain of the function f ; then

$$\lim_{x \to c} \frac{f\,(x) - f\,(c)}{x - c} = f\,'(c) > 0.$$

If $\varepsilon > 0$ be any number $< f\,'(c)$, there exists $\delta > 0$ such that

$$|x - c| \le \delta \Rightarrow \left| \frac{f\,(x) - f\,(c)}{x - c} - f'\,(c) \right| < \varepsilon$$

i.e., $\quad x \in [c - \delta, c + \delta], x \neq c$

$\Rightarrow \quad \dfrac{f\,(x) - f\,(c)}{x - c} \in \,]\, f\,'(c) - \varepsilon, f\,'(c) + \varepsilon\,[\cdot$ ...(1)

Since ε was chosen smaller than $f\,'(c)$, we conclude from (1) that

$$\frac{f\,(x) - f\,(c)}{x - c} > 0 \text{ when } x \in [c - \delta, c + \delta], x \neq c.$$

We then have, $f\,(x) - f\,(c) > 0$ when $c < x \le c + \delta$

and $\qquad f(x) - f(c) < 0$ when $c - \delta \le x < c$.

Thus we have shown that if $f'(c) > 0$, there exists a neighbourhood $[c - \delta, c + \delta]$ of c such that

$$f(x) > f(c) \quad \forall\, x \in \,]c, c + \delta] \text{ and } f(x) < f(c) \quad \forall\, x \in [c - \delta, c[.$$

If $f'(c) < 0$, it can be similarly shown that there exists a neighbourhood $[c - \delta, c + \delta]$ of c such that

$$f(x) > f(c) \quad \forall\, x \in [c - \delta, c[\text{ and } f(x) < f(c) \quad \forall\, x \in \,]c, c + \delta].$$

Similarly, it can be shown for the end points a and b that there exist intervals $]a, a + \delta]$ and $[b - \delta, b[$ such that

$$f'(a) > 0 \Rightarrow f(x) > f(a) \;\; \forall\;\; x \in \,]a, a + \delta]$$

$$f'(a) < 0 \Rightarrow f(x) < f(a) \;\; \forall\;\; x \in \,]a, a + \delta]$$

$$f'(b) > 0 \Rightarrow f(x) < f(b) \;\; \forall\;\; x \in [b - \delta, b[$$

and $\qquad f'(b) < 0 \Rightarrow f(x) > f(b) \;\; \forall\;\; x \in [b - \delta, b[\,.$

4 A Necessary Condition for the Existence of a Finite Derivative

Theorem 1. *Continuity is a necessary but not a sufficient condition for the existence of a finite derivative.* **(Kanpur 2007, 12; Meerut 10, 10B, 11; Avadh 10; Kashi 14; Gorakhpur 13, 14)**

Proof: Let f be differentiable at x_0. Then $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ exists and equals $f'(x_0)$. Now, we can write $\quad f(x) - f(x_0) = \dfrac{f(x) - f(x_0)}{x - x_0}(x - x_0)$, if $x \ne x_0$.

Taking limits as $x \to x_0$, we get

$$\lim_{x \to x_0} [f(x) - f(x_0)] = \lim_{x \to x_0} \left\{ \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \right\}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0) = f'(x_0) \,.\, 0 = 0,$$

so that $\qquad \lim\limits_{x \to x_0} f(x) = f(x_0)$.

Hence f is continuous at x_0. Thus continuity is a necessary condition for differentiability but it is not a sufficient condition for the existence of a finite derivative. The following example illustrates this fact :

Let $f(x) = x \sin(1/x), x \ne 0$ and $f(0) = 0$. **(Gorakhpur 2014)**

This function is continuous at $x = 0$ but not differentiable at $x = 0$.

Since $\lim_{x \to 0} f(x) = \lim_{x \to 0} x \sin \frac{1}{x} = 0 = f(0)$, therefore the function $f(x)$ is continuous at $x = 0$.

Now $R f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{h \sin (1/h) - 0}{h} = \lim_{h \to 0} \sin \frac{1}{h},$$

which does not exist. Similarly $L f'(0)$ does not exist.

Thus $f(x)$ is not differentiable at $x = 0$, though it is continuous there.

5 Algebra of Derivatives

Now we shall establish some fundamental theorems regarding the differentiability of the sum, product and quotient of differentiable functions.

Theorem 1: *If a function f is differentiable at a point x_0 and c is any real number, then the function $c\,f$ is also differentiable at x_0 and $(c\,f)'(x_0) = c\,f'(x_0)$.*

Proof: By the definition of $f'(x_0)$, we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).$$

Now $\lim_{x \to x_0} \frac{(c\,f)(x) - (c\,f)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{c\,f(x) - c\,f(x_0)}{x - x_0}$

$$= \lim_{x \to x_0} \left\{ c \cdot \frac{f(x) - f(x_0)}{x - x_0} \right\} = c \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = c\,f'(x_0).$$

Hence cf is differentiable at x_0 and $(c\,f)'(x_0) = c\,f'(x_0)$.

Theorem 2: *Let f and g be defined on an interval I. If f and g are differentiable at $x_0 \in I$, then so also is $f + g$ and*

$$(f + g)'(x_0) = f'(x_0) + g'(x_0).$$

Proof: Since f and g are differentiable at x_0, therefore

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0), \quad \ldots(1)$$

and $$\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = g'(x_0). \quad \ldots(2)$$

Now $\lim_{x \to x_0} \frac{(f + g)(x) - (f + g)(x_0)}{x - x_0}$

$$= \lim_{x \to x_0} \frac{[f(x) + g(x)] - [f(x_0) + g(x_0)]}{x - x_0}$$

$$= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} \right]$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0},$$

as the limit of a sum is equal to the sum of the limits

$= f'(x_0) + g'(x_0)$, using (1) and (2).

Hence $f + g$ is differentiable at x_0 and

$$(f + g)'(x_0) = f'(x_0) + g'(x_0).$$

Theorem 3: *Let f and g be defined on an interval I. If f and g are differentiable at $x_0 \in I$, then so also is fg and* $(fg)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$.

Proof: Since f and g are differentiable at x_0, we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \qquad \ldots(1)$$

and
$$\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = g'(x_0) \qquad \ldots(2)$$

Now
$$\lim_{x \to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x)\, g(x) - f(x_0)\, g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x)\, g(x) - f(x_0)\, g(x) + f(x_0)\, g(x) - f(x_0)\, g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} \cdot g(x) + f(x_0) \frac{g(x) - g(x_0)}{x - x_0} \right]$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} g(x) + f(x_0) \cdot \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$

$= f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$,

using (1), (2) and the fact that $\lim_{x \to x_0} g(x) = g(x_0)$.

Note that $g(x)$ is differentiable at $x = x_0$ implies that $g(x)$ is continuous at x_0 and so

$$\lim_{x \to x_0} g(x) = g(x_0).$$

Hence fg is differentiable at x_0 and $(fg)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$.

Theorem 4: *If f is differentiable at x_0 and $f(x_0) \neq 0$, then the function $1/f$ is differentiable at x_0 and* $(1/f)'(x_0) = -f'(x_0) / \{f(x_0)\}^2$.

Proof: Since f is differentiable at x_0, therefore

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0). \qquad \ldots(1)$$

Since f is differentiable at x_0, it is continuous at x_0, therefore

$$\lim_{x \to x_0} f(x) = f(x_0) \neq 0. \qquad \ldots(2)$$

Also, since $f(x_0) \neq 0$, hence, $f(x_0) \neq 0$ in some neighbourhood N of x_0. Now, we have for $x \in N$

$$\lim_{x \to x_0} \frac{\dfrac{1}{f(x)} - \dfrac{1}{f(x_0)}}{x - x_0} = \lim_{x \to x_0} \left\{ -\frac{f(x) - f(x_0)}{x - x_0} \cdot \frac{1}{f(x)} \cdot \frac{1}{f(x_0)} \right\}$$

$$= -\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} \frac{1}{f(x)} \cdot \frac{1}{f(x_0)}$$

$$= -f'(x_0) \cdot \frac{1}{f(x_0)} \cdot \frac{1}{f(x_0)}, \text{ using (1) and (2)}$$

$$= -f'(x_0) / \{f(x_0)\}^2.$$

Hence $1/f$ is differentiable at x_0 and $(1/f)'(x_0) = -f'(x_0) / \{f(x_0)\}^2$.

Theorem 5: *Let f and g be defined on an interval I. If f and g are differentiable at $x_0 \in I$, and $g(x_0) \neq 0$, then the function f/g is differentiable at x_0 and*

$$\left(\frac{f}{g}\right)'(x_0) = \frac{[g(x_0) f'(x_0) - f(x_0) g'(x_0)]}{[g(x_0)]^2}.$$

Proof: Use theorems 3 and 4 of article 5.

6 The Chain Rule of Differentiability

Theorem 1. *Let f and g be functions such that the range of f is contained in the domain of g. If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then $g \circ f$ is differentiable at x_0, and*

$$(g \circ f)'(x_0) = g'(f(x_0)) . f'(x_0).$$

Proof: Let $y = f(x)$ and $y_0 = f(x_0)$.

Since f is differentiable at x_0, we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

or $$f(x) - f(x_0) = (x - x_0)[f'(x_0) + \lambda(x)] \qquad \ldots(1)$$

where $\lambda(x) \to 0$ as $x \to x_0$.

Further since g is differentiable at y_0, we have

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = g'(y_0)$$

or $$g(y) - g(y_0) = (y - y_0)[g'(y_0) + \mu(y)] \qquad \ldots(2)$$

where $\mu(y) \to 0$ as $y \to y_0$.

Now $$(g \circ f)(x) - (g \circ f)(x_0) = g(f(x)) - g(f(x_0)) = g(y) - g(y_0)$$

$$= (y - y_0)[g'(y_0) + \mu(y)], \text{ by (2)}$$
$$= [f(x) - f(x_0)][g'(y_0) + \mu(y)]$$
$$= (x - x_0)[f'(x_0) + \lambda(x)][g'(y_0) + \mu(y)], \text{ by (1)}.$$

Thus if $x \neq x_0$, then

$$\frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = [g'(y_0) + \mu(y)].[f'(x_0) + \lambda(x)]. \qquad \ldots(3)$$

Also f being differentiable at x_0, is continuous at x_0 and hence as $x \to x_0$, $f(x) \to f(x_0)$ *i.e.*, $y \to y_0$.

Consequently $\mu(y) \to 0$ as $x \to x_0$ and $\lambda(x) \to 0$ as $x \to x_0$.

Taking the limits as $x \to x_0$, we get from (3)

$$\lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = g'(y_0) . f'(x_0).$$

Hence the function gof is differentiable at x_0 and $(gof)'(x_0) = g'(f(x_0)) f'(x_0)$.

7 Derivative of the Inverse Function

Theorem: *If f be a continuous one-to-one function defined on an interval and let f be differentiable at x_0, with $f'(x_0) \neq 0$, then the inverse of the function f is differentiable at $f(x_0)$ and its derivative at $f(x_0)$ is $1/f'(x_0)$.*

Proof: Before proving the theorem we remind that if the domain of f be X and its range be Y, then the inverse function g of f usually denoted by f^{-1} is the function with domain Y and range X such that $f(x) = y \Leftrightarrow g(y) = x$. Also g exists if f is one-one.

Let $\qquad y = f(x) \quad$ and $\quad y_0 = f(x_0)$.

Since f is differentiable at x_0, we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

or $\qquad f(x) - f(x_0) = (x - x_0)[f'(x_0) + \lambda(x)] \qquad \ldots(1)$

where $\lambda(x) \to 0$ as $x \to x_0$. Further, we have

$$g(y) - g(y_0) = x - x_0, \text{ by definition of } g.$$

$$\therefore \quad \frac{g(y) - g(y_0)}{y - y_0} = \frac{x - x_0}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0) + \lambda(x)}, \text{ by (1)}.$$

It can be easily seen that if $y \to y_0$, then $x \to x_0$.

In fact, f is continuous at x_0 implies that $g = f^{-1}$ is continuous at $f(x_0) = y_0$ and consequently

$$g(y) \to g(y_0) \text{ as } y \to y_0 \text{ i.e., } x \to x_0 \text{ as } y \to y_0,$$

so that $\lambda(x) \to 0$ as $y \to y_0$.

$$\therefore \quad \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{1}{f'(x_0) + \lambda(x)} = \frac{1}{f'(x_0)}$$

or $\quad g'(y_0) = \dfrac{1}{f'(x_0)}$ or $g'(f(x_0)) = \dfrac{1}{f'(x_0)}$.

Illustrative Examples

Example 1: *Prove that the function $f(x) = |x|$ is continuous at $x = 0$, but not differentiable at $x = 0$ where $|x|$ means the numerical value or the absolute value of x.*

(Rohilkhand 2007; Bundelkhand 08; Meerut 13B; Avadh 11)

Also draw the graph of the function.

Solution: We have $f(0) = |0| = 0$,

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} |h| = \lim_{h \to 0} h = 0$$

and

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} |-h| = \lim_{h \to 0} h = 0.$$

$\therefore \quad f(0) = f(0+0) = f(0-0).$

Hence $f(x)$ is continuous at $x = 0$.

Also, we have $Rf'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \dfrac{f(h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{|h| - 0}{h} = \lim_{h \to 0} \frac{h}{h}, \text{ (h being positive)}$$

$$= \lim_{h \to 0} 1 = 1,$$

and

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{|-h| - 0}{-h} = \lim_{h \to 0} \frac{h}{-h}, \text{ (h being positive)}$$

$$= \lim_{h \to 0} -1 = -1.$$

Since $Rf'(0) \neq Lf'(0)$, the function $f(x)$ is not differentiable at $x = 0$.

To draw the graph of the function $f(x) = |x|$.

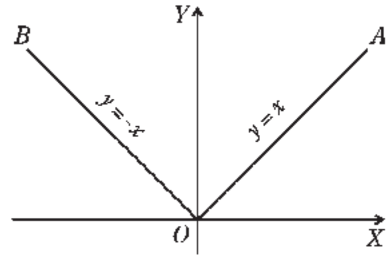
We have $f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0. \end{cases}$

Let $y = f(x)$. Then the graph of the function consists of the following straight lines :

$$y = x, \quad x \geq 0$$
$$y = -x, \quad x \leq 0.$$

The graph is as shown in the figure. From the graph we observe that the function is continuous at the point O *i.e.*, at the point $x = 0$ but it is not differentiable at this point.

The tangent to the curve at the point O from the right is the straight line OA and from the left is the straight line OB. Thus the tangent to the curve at O does not exist and so the function is not differentiable at O.



Example 2: *Show that the function* $f(x) = |x| + |x - 1|$ *is not differentiable at* $x = 0$ *and* $x = 1$. **(Meerut 2005B, 08; Kashi 14)**

Solution: We first observe that if $x < 0$, then

$$|x| = -x \text{ and } |x - 1| = |1 - x| = 1 - x \;;$$

if $0 \le x \le 1$, then $|x| = x$ and $|x - 1| = |1 - x| = 1 - x$;

and if $x > 1$, then $|x| = x$ and $|x - 1| = x - 1$.

$\therefore$ the function $f(x)$ is given by

$$f(x) = \begin{cases} 1 - 2x, & \text{if } x < 0 \\ 1, & \text{if } 0 \le x \le 1 \\ 2x - 1, & \text{if } x > 1. \end{cases}$$

At $x = 0$. We have $Rf'(0) = \lim_{h \to 0} \dfrac{f(0 + h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{1 - 1}{h}, \text{ as } f(x) = 1 \text{ if } 0 \le x \le 1$$

$$= \lim_{h \to 0} 0 = 0,$$

and $$Lf'(0) = \lim_{h \to 0} \frac{f(0 - h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{f(-h) - f(0)}{-h} = \lim_{h \to 0} \frac{[1 - 2(-h)] - 1}{-h}$$

$$[\because f(x) = 1 - 2x, \text{ if } x < 0]$$

$$= \lim_{h \to 0} \frac{2h}{-h} = \lim_{h \to 0} -2 = -2 \cdot$$

$\because$ $Rf'(0) \ne Lf'(0)$, so the given function is not differentiable at $x = 0$.

At $x = 1$. We have

$$Rf'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{[2(1 + h) - 1] - 1}{h}$$

$$= \lim_{h \to 0} \frac{2 + 2h - 1 - 1}{h} = \lim_{h \to 0} \frac{2h}{h} = \lim_{h \to 0} 2 = 2,$$

and $$Lf'(1) = \lim_{h \to 0} \frac{f(1 - h) - f(1)}{-h} = \lim_{h \to 0} \frac{1 - 1}{-h} = \lim_{h \to 0} 0 = 0.$$

$\because$ $Rf'(1) \ne Lf'(1)$, so the given function $f(x)$ is not differentiable at $x = 1$.

Example 3: *Let $f(x)$ be an even function. If $f'(0)$ exists, find its value.* **(Kanpur 2010)**

Solution: $f(x)$ is an even function, so $f(-x) = f(x) \ \forall x$.

$f'(0)$ exists $\Rightarrow Rf'(0) = Lf'(0) = f'(0)$.

Now $$f'(0) = Rf'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}, h > 0$$

$$= \lim_{h \to 0} \frac{f(-h) - f(0)}{h} \qquad [\because f(-x) = f(x)]$$

$$= -\lim_{h \to 0} \frac{f(-h) - f(0)}{-h} = -Lf'(0) = -f'(0).$$

$\therefore$ $2f'(0) = 0 \Rightarrow f'(0) = 0$.

Example 4: *Let* $f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ x - 1, & 0 < x \le 2, \end{cases}$ *and* $g(x) = f(|x|) + |f(x)|$. *Test the differentiability of* $g(x)$ *in* $]-2, 2[$.

Solution: When $-2 \le x \le 0, |x| = -x$ and when $0 < x \le 2, |x| = x$.

Now $-2 \le x \le 0 \Rightarrow |x| = -x$

$\Rightarrow$ $f(|x|) = f(-x) = -x - 1$. $\qquad [\because 0 < -x \le 2]$

So we have $$f(|x|) = \begin{cases} x - 1, & 0 < x \le 2 \\ -x - 1, & -2 \le x \le 0 \end{cases}$$

and $$|f(x)| = \begin{cases} 1, & -2 \le x \le 0 \\ -x + 1, & 0 < x \le 1 \\ x - 1, & 1 < x \le 2. \end{cases}$$

$\therefore$ $$g(x) = f(|x|) + |f(x)| = \begin{cases} -x, & -2 \le x \le 0 \\ 0, & 0 < x \le 1 \\ 2x - 2, & 1 < x \le 2. \end{cases}$$

We see that $g(x)$ is differentiable $\forall x \in]-2, 2[$, except possibly at $x = 0$ and 1.

$$Lg'(0) = \lim_{h \to 0} \frac{g(0 - h) - g(0)}{-h} = \lim_{h \to 0} \frac{g(-h) - g(0)}{-h}$$

$$= \lim_{h \to 0} \frac{h - 0}{-h} = -1,$$

$$Rg'(0) = \lim_{h \to 0} \frac{g(0 + h) - g(0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0.$$

Since $Lg'(0) \ne Rg'(0)$, $g(x)$ is not differentiable at $x = 0$.

Again $$Rg'(1) = \lim_{h \to 0} \frac{g(1 + h) - g(1)}{h} = \lim_{h \to 0} \frac{2(1 + h) - 2 - 0}{h} = 2,$$

$$L\,g'(1) = \lim_{h \to 0} \frac{g(1-h) - g(1)}{-h} = \lim_{h \to 0} \frac{0-0}{-h} = 0 \neq R\,g'(1).$$

$\therefore$ g is not differentiable at $x = 1$.

Example 5: *Suppose the function f satisfies the conditions :*

(*i*) $f(x+y) = f(x)\,f(y)\ \forall\ x, y$

(*ii*) $f(x) = 1 + x\,g(x)$ *where* $\lim_{x \to 0} g(x) = 1$.

Show that the derivative $f'(x)$ *exists and* $f'(x) = f(x)$ *for all x.*

Solution: Putting δx for y in the first condition, we have

$$f(x + \delta x) = f(x)\,f(\delta x).$$

Then $$f(x + \delta x) - f(x) = f(x)\,f(\delta x) - f(x)$$

or $$\frac{f(x+\delta x) - f(x)}{\delta x} = \frac{f(x)\,[f(\delta x) - 1]}{\delta x}$$

$$= \frac{f(x)\,\delta x\,g(\delta x)}{\delta x}, \text{ by given condition (ii)}$$

$$= f(x)\,g(\delta x).$$

$$\therefore \quad \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} f(x)\,g(\delta x) = f(x)\,.\,1.$$

$$\left[\because \lim_{\delta x \to 0} g(\delta x) = 1\right]$$

$\therefore$ $f'(x) = f(x)$. Since $f(x)$ exists, $f'(x)$ also exists.

Example 6: *Show that the function f given by* $f(x) = x \tan^{-1}(1/x)$ *for* $x \neq 0$ *and* $f(0) = 0$ *is continuous but not differentiable at* $x = 0$. **(Purvanchal 2008; Lucknow 11; Bundelkhand 12; Meerut 13)**

Solution: Since $\lim_{x \to 0} f(x) = \lim_{x \to 0} x \tan^{-1} \frac{1}{x} = 0 = f(0)$, therefore the function f is continuous at $x = 0$.

Now $$R\,f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h \tan^{-1}(1/h) - 0}{h} = \lim_{h \to 0} \tan^{-1}\left(\frac{1}{h}\right)$$

$$= \tan^{-1} \infty = \frac{\pi}{2}$$

and $$L\,f'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-h \tan^{-1}(-1/h) - 0}{-h}$$

$$= \lim_{h \to 0} \tan^{-1}\left(-\frac{1}{h}\right) = -\tan^{-1}\infty = -\frac{\pi}{2}.$$

Since $R f'(0) \neq L f'(0)$, f is not differentiable at $x = 0$.

Example 7: *Investigate the following function from the point of view of its differentiability. Does the differential coefficient of the function exist at $x = 0$ and $x = 1$?*

$$f(x) = \begin{cases} -x & \text{if} \quad x < 0 \\ x^2 & \text{if} \quad 0 \leq x \leq 1 \\ x^3 - x + 1 & \text{if} \quad x > 1. \end{cases}$$

(Meerut 2006)

Solution: We check the function $f(x)$ for differentiability at $x = 0$ and $x = 1$ only. For other values of x, obviously $f(x)$ is differentiable because it is a polynomial function. It can be seen that $f(x)$ is continuous at $x = 0$ and $x = 1$.

Now $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{-(0-h) - 0}{-h} = -1$$

and $$R f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{(0+h)^2 - 0}{h} = 0.$$

$\because$ $L f'(0) \neq R f'(0)$, the function is not differentiable at $x = 0$.

Again $$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{(1-h)^2 - 1}{-h}$$

$$= \lim_{h \to 0} \frac{-2h + h^2}{-h} = \lim_{h \to 0} (2 - h) = 2$$

and $$R f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^3 - (1+h) + 1 - 1}{h}$$

$$= \lim_{h \to 0} \frac{2h + 3h^2 + h^3}{h} = \lim_{h \to 0} (2 + 3h + h^2) = 2 = L f'(1).$$

Hence $f'(1)$ exists *i.e.*, the function is differentiable at $x = 1$.

Example 8: *Find $f'(1)$ if* $f(x) = \begin{cases} \dfrac{x-1}{2x^2 - 7x + 5}, & \textit{when} \quad x \neq 1 \\ -1/3, & \textit{when} \quad x = 1. \end{cases}$

Solution: We have $$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \left[\frac{1 + h - 1}{2(1+h)^2 - 7(1+h) + 5} - \left(\frac{-1}{3}\right)\right] \Big/ h$$

$$= \lim_{h \to 0} \frac{3h + 2(1+h)^2 - 7(1+h) + 5}{3h\,[2(1+h)^2 - 7(1+h) + 5]}$$

$$= \lim_{h \to 0} \frac{2h^2}{3h(-3h + 2h^2)} = \lim_{h \to 0} \frac{2}{-9 + 6h} = -\frac{2}{9}.$$

Example 9: *Test the continuity and differentiability in $-\infty < x < \infty$, of the following function :*

$$
\begin{aligned}
f(x) &= 1 && \text{in} \quad -\infty < x < 0 \\
&= 1 + \sin x && \text{in} \quad 0 \le x < \tfrac{1}{2}\pi \\
&= 2 + \left(x - \tfrac{1}{2}\pi\right)^2 && \text{in} \quad \tfrac{1}{2}\pi \le x < \infty.
\end{aligned}
$$

(Avadh 2009)

Solution: We shall test $f(x)$ for continuity and differentiability at $x = 0$ and $\pi/2$. It is obviously continuous as well as differentiable at all other points.

(i) Continuity and differentiability of $f(x)$ at $x = 0$

We have $f(0) = 1 + \sin 0 = 1;$

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} (1 + \sin h) = 1;$$

and $$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} 1 = 1.$$

Since $f(0) = f(0+0) = f(0-0)$, $f(x)$ is continuous at $x = 0$.

Now $$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{(1 + \sin h) - (1 + \sin 0)}{h} = \lim_{h \to 0} \frac{\sin h}{h} = 1;$$

and $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{1 - (1 + \sin 0)}{-h} = \lim_{h \to 0} \frac{0}{-h} = \lim_{h \to 0} 0 = 0.$$

Since $Rf'(0) \ne Lf'(0)$, $f(x)$ is not differentiable at $x = 0$.

(ii) Continuity and differentiability of $f(x)$ at $x = \frac{1}{2}\pi$.

We have $$f\left(\frac{1}{2}\pi\right) = 2 + \left(\frac{1}{2}\pi - \frac{1}{2}\pi\right)^2 = 2;$$

$$f\left(\frac{1}{2}\pi + 0\right) = \lim_{h \to 0} f\left(\frac{1}{2}\pi + h\right) = \lim_{h \to 0} \left[2 + \left\{\left(\frac{1}{2}\pi + h\right) - \frac{1}{2}\pi\right\}^2\right]$$

$$= \lim_{h \to 0} (2 + h^2) = 2;$$

and $$f\left(\frac{1}{2}\pi - 0\right) = \lim_{h \to 0} f\left(\frac{1}{2}\pi - h\right) = \lim_{h \to 0} \left[1 + \sin\left(\frac{1}{2}\pi - h\right)\right]$$

$$= \lim_{h \to 0} (1 + \cos h) = 1 + 1 = 2.$$

Since $f\left(\frac{1}{2}\pi\right)=f\left(\frac{1}{2}\pi+0\right)=f\left(\frac{1}{2}\pi-0\right)$, f is continuous at $x=\frac{1}{2}\pi$.

Now $$R f'\left(\frac{1}{2}\pi\right)=\lim_{h\to 0}\frac{f\left(\frac{1}{2}\pi+h\right)-f\left(\frac{1}{2}\pi\right)}{h}$$

$$=\lim_{h\to 0}\frac{\left[2+\left\{\frac{1}{2}\pi+h-\frac{1}{2}\pi\right\}^2\right]-\left[2+\left(\frac{1}{2}\pi-\frac{1}{2}\pi\right)^2\right]}{h}$$

$$=\lim_{h\to 0}\frac{2+h^2-2}{h}=\lim_{h\to 0}h=0\,;$$

and $$L f'\left(\frac{1}{2}\pi\right)=\lim_{h\to 0}\frac{f\left(\frac{1}{2}\pi-h\right)-f\left(\frac{1}{2}\pi\right)}{-h}$$

$$=\lim_{h\to 0}\frac{1+\sin\left(\frac{1}{2}\pi-h\right)-2}{-h}=\lim_{h\to 0}\frac{-1+\cos h}{-h}$$

$$=\lim_{h\to 0}\frac{1-\cos h}{h}=\lim_{h\to 0}\frac{2\sin^2(h/2)}{h}$$

$$=\lim_{h\to 0}\left[\frac{\sin(h/2)}{h/2}\,.\sin(h/2)\right]=1\times 0=0.$$

Since $R f'(0)=L f'(0)$, $f(x)$ is differentiable at $x=\frac{1}{2}\pi$.

Example 10: *If $f(x)=x^2\sin(1/x)$, for $x\neq 0$ and $f(0)=0$, then show that $f(x)$ is continuous and differentiable everywhere and that $f'(0)=0$. Also show that the function $f'(x)$ has a discontinuity of second kind at the origin.* **(Meerut 2006B; Avadh 06; Kumaun 12; Kanpur 14)**

Solution: We have $f(0+0)=\lim_{h\to 0}(0+h)^2\sin\frac{1}{0+h}=\lim_{h\to 0}h^2\sin\frac{1}{h}=0;$

$$f(0-0)=\lim_{h\to 0}f(0-h)=\lim_{h\to 0}f(-h)$$

$$=\lim_{h\to 0}(-h)^2\sin(-1/h)=-\lim_{h\to 0}h^2\sin\frac{1}{h}=0.$$

$\because$ $f(0+0)=f(0-0)=f(0)$, so the function is continuous at $x=0$.

Now $$R f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{h^2\sin(1/h)-0}{h}=\lim_{h\to 0}h\sin\frac{1}{h}=0;$$

and $$L f'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{(-h)^2 \sin(-1/h) - 0}{-h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0.$$

Thus $R f'(0) = L f'(0)$ implies that $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$.

For all other values of x, $f(x)$ is easily seen to be continuous and differentiable.

Now $$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x} \text{ at } x \neq 0 \text{ and } f'(0) = 0.$$

$\therefore$ $$f'(0+0) = \lim_{h \to 0} f'(0+h) = \lim_{h \to 0} f'(h)$$

$$= \lim_{h \to 0} \left(2h \sin \frac{1}{h} - \cos \frac{1}{h}\right), \text{ which does not exist.}$$

Similarly it can be shown that $f'(0-0)$ does not exist.

Hence f' is discontinuous at the origin. Since both the limits $f'(0-0)$ and $f'(0+0)$ do not exist, therefore the discontinuity is of the second kind.

Example 11: *A function f is defined by* $f(x) = x^p \cos(1/x), x \neq 0; f(0) = 0.$

What conditions should be imposed on p so that f may be

(*i*) *continuous at* $x = 0$ (*ii*) *differentiable at* $x = 0$?

Solution: We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} [(0+h)^p \cos\{1/(0+h)\}]$$

$$= \lim_{h \to 0} h^p \cos(1/h) \qquad ...(1)$$

and $$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} [(0-h)^p \cos\{1/(0-h)\}]$$

$$= \lim_{h \to 0} (-h)^p \cos(1/h). \qquad ...(2)$$

Now if the function $f(x)$ is to be continuous at $x = 0$, then

$$f(0+0) = f(0) = 0 = f(0-0)$$

i.e., the limits given in (1) and (2) must both tend to zero.

This is possible only if $p > 0$, which is the required condition.

Now $$R f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^p \cos(1/h) - 0}{h} = \lim_{h \to 0} h^{p-1} \cos \frac{1}{h} \qquad ...(3)$$

and $$L f'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{(-h)^p \cos(-1/h) - 0}{-h}$$

$$= \lim_{h \to 0} -(-1)^p \, h^{p-1} \cos(1/h). \qquad \ldots(4)$$

Now if $f'(x)$ exists at $x = 0$ then we must have $Rf'(0) = Lf'(0)$ and this is possible only if $p - 1 > 0$ *i.e.,* $p > 1$ which gives $Rf'(0) = 0 = Lf'(0)$. Hence in order that f is differentiable at $x = 0$, p must be greater than 1.

Example 12: *For a real number y, let [y] denote the greatest integer less than or equal to y. Then if* $f(x) = \dfrac{\tan(\pi[x - \pi])}{1 + [x]^2}$, *show that* $f'(x)$ *exists for all x.*

Solution: From the definition of $[y]$, we see that $[x - \pi]$ is an integer for all values of x. Then $\pi(x - \pi)$ is an integral multiple of π and so $\tan(\pi[x - \pi]) = 0 \;\; \forall x$. Since $[x]$ is an integer so $1 + [x]^2 \neq 0$ for any x. Thus $f(x) = 0$ for all x *i.e.,* $f(x)$ is a constant function and so it is continuous and differentiable *i.e.,* $f'(x)$ exists $\forall x$ and is equal to zero.

Example 13: *Determine the set of all points where the function* $f(x) = x/(1 + |x|)$ *is differentiable.*

Solution: Since $|x| = x,\ x > 0,\ |x| = -x,\ x < 0,\ |x| = 0,\ x = 0,$

$$\therefore \qquad f(x) = \frac{x}{1 - x}, x < 0;\ f(x) = 0,\ x = 0;\ f(x) = \frac{x}{1 + x}, x > 0.$$

We have $\quad f(0 + 0) = \lim\limits_{h \to 0} f(0 + h) = \lim\limits_{h \to 0} f(h) = \lim\limits_{h \to 0} \dfrac{h}{1 + h} = 0;$

$$f(0 - 0) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{-h}{1 + h} = 0.$$

Since $\quad f(0 + 0) = f(0) = f(0 - 0) = 0$ so the function is continuous at $x = 0$.

Further $\quad Lf'(0) = \lim\limits_{h \to 0} \dfrac{f(0 - h) - f(0)}{-h} = \lim\limits_{h \to 0} \dfrac{f(-h) - f(0)}{-h}$

$$= \lim_{h \to 0} \frac{[-h/(1 + h)] - 0}{-h} = \lim_{h \to 0} \frac{1}{1 + h} = 1;$$

$$Rf'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{[h/(1 + h)] - 0}{h} = 1.$$

Since $Lf'(0) = Rf'(0)$, so the function is differentiable at $x = 0$. It is obviously differentiable for all other real values of x. Hence it is differentiable in the interval $]-\infty, \infty[$.

Example 14: *Let* $f(x) = \sqrt{(x)}\,\{1 + x \sin(1/x)\}$ *for* $x > 0$, $f(0) = 0$,

$$f(x) = -\sqrt{(-x)}\,\{1 + x \sin(1/x)\} \text{ for } x < 0.$$

Show that $f'(x)$ *exists everywhere and is finite except at* $x = 0$ *where its value is* $+\infty$.

Solution: We have

$$R\, f\,'(0) = \lim_{h \to 0} \frac{f\,(0+h) - f\,(0)}{h} = \lim_{h \to 0} \frac{f\,(h) - f\,(0)}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{h})\,\{1 + h \sin (1/h)\} - 0}{h}$$

$$= \lim_{h \to 0} \left[\frac{1}{\sqrt{h}} + (\sqrt{h}) \sin (1/h)\right] = \infty + 0 = \infty$$

and $$L\, f\,'(0) = \lim_{h \to 0} \frac{f\,(0-h) - f\,(0)}{-h} = \lim_{h \to 0} \frac{f\,(-h) - f\,(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-\sqrt{[-(-h)]}\left[1 + (-h) \sin \dfrac{1}{-h}\right] - 0}{-h}$$

$$= \lim_{h \to 0} \left[\frac{1}{\sqrt{h}} + (\sqrt{h}) \sin \frac{1}{h}\right] = \infty + 0 = \infty.$$

Since $R\, f\,'(0) = L\, f\,'(0) = \infty$, $\therefore f\,'(0) = \infty$.

We have $$f\,'(x) = \frac{1}{2\sqrt{x}} + \frac{3}{2}\sqrt{x} \sin \frac{1}{x} - \frac{1}{\sqrt{x}} \cos \frac{1}{x} \text{ for } x > 0$$

and $$f\,'(x) = \frac{1}{2\sqrt{(-x)}} + \frac{3}{2}\sqrt{(-x)} \sin \frac{1}{x} - \frac{1}{\sqrt{(-x)}} \cos \frac{1}{x} \text{ for } x < 0.$$

Hence $f\,'(a)$ is finite for all $a \neq 0$.

Example 15: *Draw the graph of the function* $y = |x-1| + |x-2|$ *in the interval* $[0, 3]$ *and discuss the continuity and differentiability of the function in this interval.*

(Garhwal 2008; Meerut 07B, 09; Gorakhpur 12)

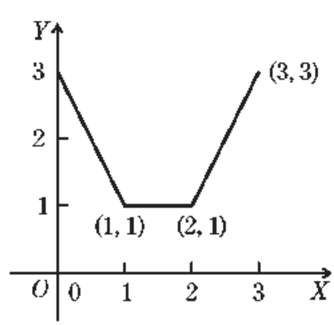
Solution: From the given definition of the function, we have

$y = 1 - x + 2 - x = 3 - 2x$ when $x \le 1$

$y = x - 1 + 2 - x = 1$ when $1 \le x \le 2$

$y = x - 1 + x - 2 = 2x - 3$ when $x \ge 2$.

Thus the graph consists of the segments of the three straight lines $y = 3 - 2x$, $y = 1$ and $y = 2x - 3$ corresponding to the intervals $[0, 1]$, $[1, 2]$, $[2, 3]$ respectively. The graph of the function for the interval $[0, 3]$ is as given in the figure.



The graph shows that the function is continuous throughout the interval but is not differentiable at $x = 1, 2$ because the slopes at these points are different on the left and right hand sides.

To test it analytically, we write $y = f\,(x)$. Then

$f\,(x) = 3 - 2x$ when $x \le 1$

$= 1$ when $1 \le x \le 2$

$= 2x - 3$ when $x \ge 2$.

This function is obviously continuous and differentiable at all points of the interval $[0,3]$ except possibly at $x=1$ and at $x=2$.

At $x = 1$, we have $f(1)=1$;

$$f(1-0)=\lim_{h\to 0}[3-2(1-h)]=1;\ f(1+0)=\lim_{h\to 0}(1)=1.$$

Since $\quad f(1-0)=f(1+0)=f(1)$, f is continuous at $x=1$.

Again $\quad Rf'(1)=\lim_{h\to 0}\dfrac{f(1+h)-f(1)}{h}=\lim_{h\to 0}\dfrac{1-1}{h}=0$

and $\quad Lf'(1)=\lim_{h\to 0}\dfrac{f(1-h)-f(1)}{-h}=\lim_{h\to 0}\dfrac{3-2(1-h)-1}{-h}=-2.$

Since $\quad Rf'(1)\neq Lf'(1)$, f is not differentiable at $x=1$.

At $x = 2$, we have $f(2)=1$;

$$f(2-0)=\lim_{h\to 0}(1)=1;\ f(2+0)=\lim_{h\to 0}[2(2+h)-3]=1.$$

Since $\quad f(2-0)=f(2+0)=f(2)$, f is continuous at $x=2$.

Again $\quad Rf'(2)=\lim_{h\to 0}\dfrac{f(2+h)-f(2)}{h}=\lim_{h\to 0}\dfrac{2(2+h)-3-1}{h}=2$

and $\quad Lf'(2)=\lim_{h\to 0}\dfrac{f(2-h)-f(2)}{-h}=\lim_{h\to 0}\dfrac{1-1}{-h}=0\cdot$

Since $\quad Rf'(2)\neq Lf'(2)$, f is not differentiable at $x=2$.

Example 16: *Show that the function $f:\mathbf{R}\to\mathbf{R}$ defined by*

$$f(x)=x\left[1+\frac{1}{3}\sin\log x^2\right],\ x\neq 0 \text{ and } f(0)=0$$

is everywhere continuous but has no differential coefficient at the origin. **(Garhwal 2009)**

Solution: Obviously the function $f(x)$ is continuous at every point of $\mathbf{R}$ except possibly at $x=0$. We test at $x=0$. Given $f(0)=0$.

$$f(0+0)=\lim_{h\to 0}f(0+h)=\lim_{h\to 0}\left[(0+h)\left\{1+\frac{1}{3}\sin\log(0+h)^2\right\}\right]$$

$$=\lim_{h\to 0}[h+(h/3)\sin\log h^2]=0+0\times\text{ a finite quantity}=0.$$

[$\because$ $\sin\log h^2$ oscillates between -1 and $+1$ as $h\to 0$]

Similarly we can show that $f(0-0)=0$.

Hence f is continuous at $x=0$.

Now $\quad Rf'(0)=\lim_{h\to 0}\dfrac{(0+h)\left\{1+\frac{1}{3}\sin\log(0+h)^2\right\}-0}{h}$

$$=\lim_{h\to 0}\left\{1+\frac{1}{3}\sin\log h^2\right\},$$ which does not exist since $\sin\log h^2$ oscillates between -1 and 1 as $h\to 0$.

$$L\,f'(0)=\lim_{h\to 0}\frac{(0-h)\left\{1+\frac{1}{3}\sin\log(0-h)^2\right\}-0}{-h}$$

$$=\lim_{h\to 0}\left[1+\frac{1}{3}\sin\log h^2\right],\text{ which does not exist as above.}$$

Hence f has no differential coefficient at $x=0$.

Example 17: *Let* $f(x)=x\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}},\ x\neq 0;\ f(0)=0.$

Show that $f(x)$ *is continuous but not derivable at* $x=0$. **(Meerut 2005; Purvanchal 07; Kanpur 08; Lucknow 09; Gorakhpur 10; Bundelkhand 14)**

Solution: We have $f(0)=0$;

$$f(0+0)=\lim_{h\to 0}f(0+h)=\lim_{h\to 0}f(h)=\lim_{h\to 0}h\frac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}}$$

$$=\lim_{h\to 0}h\frac{1-e^{-2/h}}{1+e^{-2/h}},\quad\text{dividing the Nr. and Dr. by } e^{1/h}$$

$$=0\times\frac{1-0}{1+0}=0\times 1=0\,;$$

and
$$f(0-0)=\lim_{h\to 0}f(0-h)=\lim_{h\to 0}f(-h)$$

$$=\lim_{h\to 0}-h\frac{e^{1/-h}-e^{-1/-h}}{e^{1/-h}+e^{-1/-h}}=\lim_{h\to 0}-h\frac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}}$$

$$=\lim_{h\to 0}-h\frac{e^{-2/h}-1}{e^{-2/h}+1}=0\times\frac{0-1}{0+1}=0.$$

Since $f(0+0)=f(0-0)=f(0)$, the function is continuous at $x=0$.

Now
$$R\,f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$=\lim_{h\to 0}\left[h\frac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}}-0\right]\Big/ h=\lim_{h\to 0}\frac{1-e^{-2/h}}{1+e^{-2/h}}=\frac{1-0}{1+0}=1.$$

and
$$L\,f'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\left[(-h)\frac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}}-0\right]\Big/(-h)$$

$$=\lim_{h\to 0}\frac{e^{-2/h}-1}{e^{-2/h}+1}=\frac{0-1}{0+1}=-1.$$

Since $R\,f'(0)\neq L\,f'(0)$, the function is not derivable at $x=0$.

Example 18: *Let $f(x)=e^{-1/x^2}\sin(1/x)$ when $x\neq 0$ and $f(0)=0$. Show that at every point f has a differential coefficient and this is continuous at $x=0$.* **(Avadh 2006)**

Solution: We test differentiability at $x=0$.

$$Rf'(0)=\lim_{h\to 0}\frac{e^{-1/h^2}\sin(1/h)-0}{h}=\lim_{h\to 0}\frac{\sin(1/h)}{he^{1/h^2}}$$

$$=\lim_{h\to 0}\frac{\sin(1/h)}{h\left\{1+\dfrac{1}{h^2}+\dfrac{1}{2!h^4}+\ldots\right\}}=\lim_{h\to 0}\frac{\sin(1/h)}{h+\dfrac{1}{h}+\dfrac{1}{2!}\cdot\dfrac{1}{h^3}+\ldots}$$

$$=\frac{\text{a finite quantity lying between }-1\text{ and }+1}{\infty}=0.$$

Similarly $Lf'(0)=0$.

Since $Rf'(0)=Lf'(0)=0$, hence the function $f(x)$ is differentiable at $x=0$ and $f'(0)=0$.

If x is any point other than zero, then

$$f'(x)=(2/x^3)\,e^{-1/x^2}\sin(1/x)-(1/x^2)\,e^{-1/x^2}\cos(1/x)$$

$$=\{(2/x)\sin(1/x)-\cos(1/x)\}(1/x^2)(1/e^{1/x^2}) \qquad \ldots(1)$$

Now $$f'(0+0)=\lim_{h\to 0}f'(0+h)=\lim_{h\to 0}\left(\frac{2}{h}\sin\frac{1}{h}-\cos\frac{1}{h}\right)\cdot\frac{1}{h^2e^{1/h^2}}$$

$$=\lim_{h\to 0}\left(\frac{2\sin(1/h)}{h^3e^{1/h^2}}-\frac{\cos(1/h)}{h^2e^{1/h^2}}\right)$$

$$=\lim_{h\to 0}\left[\frac{2\sin(1/h)}{h^3\left(1+\dfrac{1}{h^2}+\dfrac{1}{2!.h^4}+\ldots\right)}-\frac{\cos(1/h)}{h^2\left(1+\dfrac{1}{h^2}+\dfrac{1}{2!.h^4}+\ldots\right)}\right]$$

$$=\frac{\text{some finite quantity}}{\infty}-\frac{\text{some finite quantity}}{\infty}=0\cdot$$

Similarly $f'(0-0)=0$. Hence f' is continuous at $x=0$.

Comprehensive Exercise 1

1. Show that $f(x)=|x-1|, 0\le x\le 2$ is not derivable at $x=1$. Is it continuous in $[0,2]$?

2. (a) If $f(x)=\dfrac{x}{1+e^{1/x}}, x\neq 0, f(0)=0,$ show that f is continuous at $x=0$, but $f'(0)$ does not exist. **(Lucknow 2005, 10; Gorakhpur 13)**

(b) If $f(x) = \dfrac{x e^{1/x}}{1 + e^{1/x}}$ for $x \neq 0$ and $f(0) = 0$, show that $f(x)$ is continuous at $x = 0$, but $f'(0)$ does not exist. **(Lucknow 2006)**

3. A function ϕ is defined as follows :

$$\phi(x) = -x \text{ for } x \leq 0, \phi(x) = x \text{ for } x \geq 0.$$

Test the character of the function at $x = 0$ as regards continuity and differentiability.

4. Show that the function f defined on **R** by

$$f(x) = |x-1| + 2|x-2| + 3|x-3|$$

is continuous but not differentiable at the points 1, 2, and 3. **(Bundelkhand 2009)**

5. Show that the function $f(x) = x, 0 < x \leq 1$
$= x - 1, 1 < x \leq 2$

has no derivative at $x = 1$.

6. Show that the function $f(x) = x^2 - 1, x \geq 1$
$= 1 - x, x < 1$

has no derivative at $x = 1$.

7. The following limits are derivatives of certain functions at a certain point. Determine these functions and the points.

(i) $\lim\limits_{x \to 2} \dfrac{\log x - \log 2}{x - 2}$ (ii) $\lim\limits_{h \to 0} \dfrac{\sqrt{(a+h)} - \sqrt{a}}{h}$.

8. Let $f(x) = x^2 \sin(x^{-4/3})$ except when $x = 0$ and $f(0) = 0$. Prove that $f(x)$ has zero as a derivative at $x = 0$.

9. A function ϕ is defined as : $\phi(x) = 1 + x$ if $x \leq 2$, $\phi(x) = 5 - x$ if $x > 2$.
Test the character of the function at $x = 2$ as regards its continuity and differentiability. **(Avadh 2007)**

10. Examine the following curve for continuity and differentiability :

$$\begin{aligned} y &= x^2 && \text{for} \quad x \leq 0 \\ y &= 1 && \text{for} \quad 0 < x \leq 1 \\ y &= 1/x && \text{for} \quad x > 1. \end{aligned}$$

Also draw the graph of the function. **(Meerut 2003, 04B, 09B)**

11. A function $f(x)$ is defined as follows :

$$\begin{aligned} f(x) &= 1 + x && \text{for} \quad x \leq 0, \\ f(x) &= x && \text{for} \quad 0 < x < 1, \\ f(x) &= 2 - x && \text{for} \quad 1 \leq x \leq 2, \\ f(x) &= 3x - x^2 && \text{for} \quad x > 2. \end{aligned}$$

Discuss the continuity of $f(x)$ and the existence of $f'(x)$ at $x = 0, 1$ and 2.

12. Discuss the continuity and differentiability of the following function :

$$f(x) = x^2 \quad \text{for} \quad x < -2$$
$$f(x) = 4 \quad \text{for} \quad -2 \le x \le 2$$
$$f(x) = x^2 \quad \text{for} \quad x > 2.$$

Also draw the graph. **(Meerut 2007, 10B)**

13. A function $f(x)$ is defined as follows :

$$f(x) = x \text{ for } 0 \le x \le 1, \; f(x) = 2 - x \text{ for } x \ge 1.$$

Test the character of the function at $x = 1$ as regards the continuity and differentiability. **(Meerut 2003)**

14. Examine the function defined by $f(x) = x^2 \cos(e^{1/x}), x \ne 0, f(0) = 0$ with regard to (i) continuity (ii) differentiability in the interval $]-1, 1[$.

15. (a) Define continuity and differentiability of a function at a given point. If a function possesses a finite differential coefficient at a point, show that it is continuous at this point. Is the converse true ? Give example in support of your answer.

(b) What do you understand by the derivative of a real valued function at the point $b \in \mathbf{R}$? Apply your definition to discuss the derivative of $f(x) = |x|, x \in \mathbf{R}$ *at* $x = 0$.

(c) Prove that if a function $f(x)$ possesses a finite derivative in a closed interval $[a, b]$, then $f(x)$ is continuous in $[a, b]$.

Answers 1

1. Yes
3. Continuous at $x = 0$ but not differentiable at $x = 0$
7. (i) The function is $\log x$ and the point is $x = 2$

   (ii) The function is $\sqrt{x}$ and the point is $x = a$
9. Continuous but not differentiable at $x = 2$
10. Discontinuous and non-differentiable at $x = 0$, continuous and non-differentiable at $x = 1$
11. Discontinuous and non-differentiable at $x = 0, 2$ and continuous but not differentiable at $x = 1$
12. Continuous but not differentiable at $x = -2, 2$
13. Continuous but non-differentiable at $x = 1$
14. Continuous and differentiable throughout $\mathbf{R}$

8 Rolle's Theorem

If a function $f(x)$ is such that

(i) *$f(x)$ is continuous in the closed interval $[a, b]$,*

(ii) *$f'(x)$ exists for every point in the open interval $]a, b[$,*

(iii) *$f(a) = f(b)$, then there exists at least one value of x, say c, where $a < c < b$, such that $f'(c) = 0$.* **(Lucknow 2007; Purvanchal 07; Kanpur 08, 12; Meerut 12B; Kashi 13, 14; Gorakhpur 12, 13, 14; Avadh 08, 11, 14)**

Proof: Since f is continuous on $[a, b]$, it is bounded on $[a, b]$. Let M and m be the supremum and infimum of f respectively in the closed interval $[a, b]$.

Now either $M = m$ or $M \neq m$.

If $M = m$, then f is a constant function over $[a, b]$ and consequently $f'(x) = 0$ for all x in $[a, b]$. Hence the theorem is proved in this case.

If $M \neq m$, then at least one of the numbers M and m must be different from the equal values $f(a)$ and $f(b)$. For the sake of definiteness, let $M \neq f(a)$.

Since every continuous function on a closed interval attains its supremum, therefore, there exists a real number c in $[a, b]$ such that $f(c) = M$. Also, since $f(a) \neq M \neq f(b)$, therefore, c is different from both a and b. This implies that $c \in]a, b[$.

Now $f(c)$ is the supremum of f on $[a, b]$, therefore,

$$f(x) \leq f(c) \quad \forall \quad x \text{ in } [a, b]. \qquad \ldots(1)$$

In particular, for all positive real numbers h such that $c - h$ lies in $[a, b]$,

$$f(c - h) \leq f(c)$$

$$\Rightarrow \qquad \frac{f(c - h) - f(c)}{-h} \geq 0. \qquad \ldots(2)$$

Since $f'(x)$ exists at each point of $]a, b[$, and hence, in particular $f'(c)$ exists, so taking limit as $h \to 0$, (2) gives $\quad Lf'(c) \geq 0. \qquad \ldots(3)$

Similarly, from (1), for all positive real numbers h such that $c + h$ lies in $[a, b]$, we have

$$f(c + h) \leq f(c).$$

By the same argument as above, we get

$$Rf'(c) \leq 0. \qquad \ldots(4)$$

Since $f'(c)$ exists, hence, $Lf'(c) = f'(c) = Rf'(c)$. $\qquad \ldots(5)$

From (3), (4) and (5) we conclude that $f'(c) = 0$.

In the same manner we can consider the case $M = f(a) \neq m$.

Note 1: There may be more than one point like c at which $f'(x)$ vanishes.

Note 2: Rolle's theorem will not hold good

(i) if $f(x)$ is discontinuous at some point in the interval $a \leq x \leq b$,

or (ii) if $f'(x)$ does not exist at some point in the interval $a < x < b$,

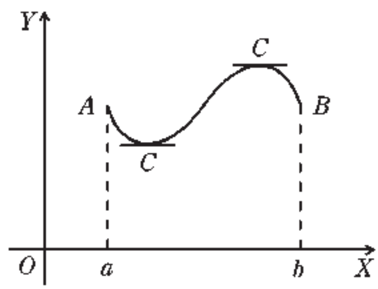
or (iii) if $f(a) \neq f(b)$.

Note 3: It can be seen that the conditions of Rolle's theorem are not necessary for $f'(x)$ to vanish at some point in $]a, b[$. For example, $f(x) = \cos(1/x)$ is discontinuous at $x = 0$ in the interval $[-1, 2]$ but $f'(x)$ vanishes at an infinite number of points in the interval.

Geometrical interpretation of Rolle's Theorem: **(Gorakhpur 2015)**

Suppose the function $f(x)$ is not constant and satisfies the conditions of Rolle's theorem in the interval $[a, b]$. Then its geometrical interpretation is that the curve representing the graph of the function f must have a tangent parallel to x-axis, at least at one point between a and b.

Algebraical interpretation of Rolle's Theorem:



Rolle's theorem leads to a very important result in the theory of equations, when $f(a) = f(b) = 0$ and $f: [a, b] \to \mathbf{R}$ is a polynomial function $f(x)$. Here a and b are the roots of the equation $f(x) = 0$. Since a polynomial function $f(x)$ is continuous and differentiable at every point of its domain and we have taken $f(a) = f(b)$, therefore, all the three conditions of Rolle's theorem are satisfied and consequently there exists a point $c \in]a, b[$ such that $f'(c) = 0$ *i.e., if a and b are any two roots of the polynomial equation* $f(x) = 0$, *then there exists at least one root of the equation* $f'(x) = 0$ *which lies between a and b.*

Illustrative Examples

Example 19: *Discuss the applicability of Rolle's theorem for* $f(x) = 2 + (x-1)^{2/3}$ *in the interval* $[0, 2]$. **(Meerut 2012)**

Solution: We have $f(x) = 2 + (x-1)^{2/3}$. Here $f(0) = 3 = f(2)$, which shows that the third condition of Rolle's theorem is satisfied.

Since $f(x)$ is an algebraic function of x, it is continuous in the closed interval $[0, 2]$. Thus the first condition of Rolle's theorem is satisfied.

Now $f'(x) = \dfrac{2}{3} \cdot [1/(x-1)^{1/3}]$. We see that for $x = 1$, $f'(x)$ is not finite while $x = 1$ is a point of the open interval $0 < x < 2$. Thus the second condition of Rolle's theorem is not satisfied.

Hence the Rolle's theorem is not applicable for the function $f(x) = 2 + (x-1)^{2/3}$ in the interval $[0, 2]$.

Example 20: *Discuss the applicability of Rolle's theorem in the interval* $[-1, 1]$ *to the function* $f(x) = |x|$.

Solution: Given $f(x) = |x|$. Here $f(-1) = |-1| = 1$, $f(1) = |1| = 1$, so that $f(-1) = f(1)$.

Further the function $f(x)$ is continuous throughout the closed interval $[-1, 1]$ but it is not differentiable at $x = 0$ which is a point of the open interval $]-1, 1[$. Thus the second condition of Rolle's theorem is not satisfied. Hence the Rolle's theorem is not applicable here.

Example 21: *Are the conditions of Rolle's theorem satisfied in the case of the following functions ?*

(i) $f(x) = x^2$ *in* $2 \le x \le 3$, *(ii)* $f(x) = \cos(1/x)$ *in* $-1 \le x \le 1$,

(iii) $f(x) = \tan x$ *in* $0 \le x \le \pi$.

Solution: **(i)** The function $f(x) = x^2$ is continuous and differentiable in the interval $[2, 3]$. Also $f(2) = 4$ and $f(3) = 9$, so that $f(2) \ne f(3)$.

Thus the first two conditions of Rolle's theorem are satisfied and the third condition is not satisfied.

(ii) The function $f(x) = \cos(1/x)$ is discontinuous at $x = 0$ and consequently is not differentiable there. Thus the first two conditions of Rolle's theorem are not satisfied.

Here $f(-1) = \cos(-1) = \cos 1$ and $f(1) = \cos 1$. Thus $f(-1) = f(1)$ *i.e.*, the third condition is satisfied.

(iii) The function $f(x) = \tan x$ is not continuous at $x = \pi/2$ and consequently is not differentiable there. Thus the first two conditions of Rolle's theorem are not satisfied here.

Further $f(0) = \tan 0 = 0$ and $f(\pi) = \tan \pi = 0$. Thus $f(0) = f(\pi)$ *i.e.*, the third condition is satisfied.

Example 22: *Discuss the applicability of Rolle's theorem to* $f(x) = \log\left[\dfrac{x^2 + ab}{(a+b)x}\right]$, *in the interval* $[a, b]$, $0 < a < b$. **(Lucknow 2008, 11; Rohilkhand 14)**

Solution: Here $f(a) = \log\left[\dfrac{a^2 + ab}{(a+b)a}\right] = \log 1 = 0$,

and $$f(b) = \log\left[\frac{b^2 + ab}{(a+b)b}\right] = \log 1 = 0.$$

Thus $$f(a) = f(b) = 0.$$

Also $$Rf'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log\left\{\frac{(x+h)^2 + ab}{(a+b)(x+h)}\right\} - \log\left\{\frac{x^2 + ab}{(a+b)x}\right\}\right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\log \frac{(x^2 + 2xh + h^2 + ab)(a+b)x}{(a+b)(x+h)(x^2+ab)} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\log \left\{ \frac{(x^2 + 2xh + h^2 + ab)}{x^2 + ab} \times \frac{x}{x+h} \right\} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\log \left\{ 1 + \frac{2xh + h^2}{x^2 + ab} \right\} - \log \left\{ 1 + \frac{h}{x} \right\} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{2xh + h^2}{x^2 + ab} - \frac{h}{x} + \ldots \right], \qquad \ldots(1)$$

$$\left[\because \log(1+y) = y - \frac{1}{2} y^2 + \ldots \right]$$

$$= \frac{2x}{x^2 + ab} - \frac{1}{x}.$$

Again $\quad L f'(x) = \lim_{h \to 0} \left[\frac{f(x-h) - f(x)}{-h} \right]$

$$= \lim_{h \to 0} \frac{1}{(-h)} \left[\frac{-2hx + h^2}{x^2 + ab} - \frac{(-h)}{x} + \ldots \right],$$

replacing h by $-h$ in (1)

$$= \frac{2x}{x^2 + ab} - \frac{1}{x}.$$

Since $R f'(x) = L f'(x)$, $f(x)$ is differentiable for all values of x in $[a, b]$. This implies that $f(x)$ is also continuous for all values of x in $[a, b]$. Thus all the three conditions of Rolle's theorem are satisfied. Hence $f'(x) = 0$ for at least one value of x in the open interval $]a, b[$.

Now $\quad f'(x) = 0 \Rightarrow \frac{2x}{x^2 + ab} - \frac{1}{x} = 0$

or $\quad 2x^2 - (x^2 + ab) = 0$

or $\quad x^2 = ab \quad$ or $\quad x = \sqrt{(ab)}$,

which being the geometric mean of a and b lies in the open interval $]a, b[$. Hence the Rolle's theorem is verified.

Remark: In this question to find $f'(x)$, we can also proceed as follows :

We have

$$f(x) = \log(x^2 + ab) - \log(a+b) - \log x.$$

$$\therefore \quad f'(x) = \frac{2x}{x^2 + ab} - \frac{1}{x}.$$

Obviously $f'(x)$ exists for all values of x in $[a, b]$.

Example 23: *Verify Rolle's theorem in the case of the functions*

(i) $f(x) = 2x^3 + x^2 - 4x - 2,$ **(Lucknow 2009)**

(ii) $f(x) = \sin x$ *in* $[0, \pi]$,

(iii) $f(x) = (x - a)^m (x - b)^n$, *where m and n are +ive integers, and x lies in the interval* $[a, b]$.

Solution: **(i)** Since $f(x)$ is a rational integral function of x, therefore, it is continuous and differentiable for all real values of x. Thus the first two conditions of Rolle's theorem are satisfied in any interval.

Here $\quad f(x) = 0$ gives $2x^3 + x^2 - 4x - 2 = 0$

or $\quad (x^2 - 2)(2x + 1) = 0$ *i.e.,* $x = \pm\sqrt{2}, -\dfrac{1}{2}$.

Thus $\quad f(\sqrt{2}) = f(-\sqrt{2}) = f\left(-\dfrac{1}{2}\right) = 0.$

If we take the interval $[-\sqrt{2}, \sqrt{2}]$, then all the three conditions of Rolle's theorem are satisfied in this interval. Consequently there is at least one value of x in the open interval $]-\sqrt{2}, \sqrt{2}[$ for which $f'(x) = 0$.

Now $\quad f'(x) = 0 \Rightarrow 6x^2 + 2x - 4 = 0 \Rightarrow 3x^2 + x - 2 = 0$

or $\quad (3x - 2)(x + 1) = 0 \quad$ or $\quad x = -1, 2/3 \quad$ *i.e.,* $\quad f'(-1) = f'(2/3) = 0.$

Since both the points $x = -1$ and $x = 2/3$ lie in the open interval $]-\sqrt{2}, \sqrt{2}[$, Rolle's theorem is verified.

(ii) The function $f(x) = \sin x$ is continuous and differentiable in $[0, \pi]$.

Also $f(0) = 0 = f(\pi)$. Thus all the three conditions of Rolle's theorem are satisfied. Hence $f'(x) = 0$ for at least one value of x in the open interval $]0, \pi[$.

Now $\quad f'(x) = 0 \Rightarrow \cos x = 0 \Rightarrow x = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, \ldots$.

Since $x = \pi/2$ lies in the open interval $]0, \pi[$, the Rolle's theorem is verified.

(iii) We have $f(x) = (x - a)^m (x - b)^n$.

As m and n are positive integers, $(x - a)^m$ and $(x - b)^n$ are polynomials in x on being expanded by binomial theorem. Hence $f(x)$ is also a polynomial in x. Consequently $f(x)$ is continuous and differentiable in the closed interval $[a, b]$. Also $f(a) = f(b) = 0$.

Thus all the three conditions of Rolle's theorem are satisfied so that there is at least one value of x in the open interval $]a, b[$ where $f'(x) = 0$.

Now $\quad f'(x) = (x - a)^m . n(x - b)^{n-1} + m(x - a)^{m-1}(x - b)^n.$

Solving the equation $f'(x) = 0$, we get $x = a, b, (na + mb)/(m + n)$

Out of these values the value $(na + mb)/(m + n)$ is a point which lies in the open interval $]a, b[$, since it divides the interval $]a, b[$ internally in the ratio $m : n$. Hence the Rolle's theorem is verified.

Example 24: *Verify Rolle's theorem for*

$$f(x) = x(x+3)\,e^{-x/2} \text{ in } [-3, 0]. \qquad \textbf{(Gorakhpur 2015)}$$

Solution: We have $f(x) = x(x+3)\,e^{-x/2}$.

$$\therefore \qquad f'(x) = (2x+3)\,e^{-x/2} + (x^2+3x)\,e^{-x/2}.\left(-\frac{1}{2}\right)$$

$$= e^{-x/2}\left[2x+3-\frac{1}{2}(x^2+3x)\right] = -\frac{1}{2}(x^2-x-6)\,e^{-x/2},$$

which exists for every value of x in the interval $[-3, 0]$. Hence $f(x)$ is differentiable and so also continuous in the interval $[-3, 0]$. Also $f(-3) = f(0) = 0$.

Thus all the three conditions of Rolle's theorem are satisfied. So $f'(x) = 0$ for at least one value of x lying in the open interval $]-3, 0[$.

Now $\qquad f'(x) = 0 \;\Rightarrow\; -\frac{1}{2}(x^2-x-6)\,e^{-x/2} = 0 \;\text{ or }\; x^2 - x - 6 = 0$

or $\qquad (x-3)(x+2) = 0 \;\text{ or }\; x = 3, -2$.

Since the value $x = -2$ lies in the open interval $]-3, 0[$, the Rolle's theorem is verified.

Example 25: *If the functions f, g, h are continuous on $[a, b]$ and twice differentiable on $]a, b[$, prove that there exist $\xi, \eta \in\,]a, b[$ such that*

$$\begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix} = \frac{1}{2}(b-c)(c-a)(a-b)\begin{vmatrix} f(a) & f'(\xi) & f''(\eta) \\ g(a) & g'(\xi) & g''(\eta) \\ h(a) & h'(\xi) & h''(\eta) \end{vmatrix}$$

where $a < c < b$.

Solution: Consider the function F defined by

$$F(x) = \begin{vmatrix} f(a) & f(b) & f(x) \\ g(a) & g(b) & g(x) \\ h(a) & h(b) & h(x) \end{vmatrix} - \frac{(x-a)(x-b)}{(c-a)(c-b)}\begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix}.$$

We have $F(a) = F(b) = F(c) = 0$. The function F satisfies the conditions of Rolle's theorem on $[a, c]$ and $[c, b]$. Hence, we get

$$F'(x_1) = 0 = F'(x_2) \text{ where } a < x_1 < c \text{ and } c < x_2 < b.$$

Again, applying Rolle's theorem for $F'(x)$ on $[x_1, x_2]$, we get

$$F''(\eta) = 0, \text{ where } x_1 < \eta < x_2 .$$

Now $F''(\eta) = 0$

$$\Rightarrow \quad \begin{vmatrix} f(a) & f(b) & f''(\eta) \\ g(a) & g(b) & g''(\eta) \\ h(a) & h(b) & h''(\eta) \end{vmatrix} = \frac{2}{(c-a)(c-b)}\begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix} \qquad \ldots(1)$$

Again, let ϕ be the function defined by

$$\phi(x) = \begin{vmatrix} f(a) & f(x) & f''(\eta) \\ g(a) & g(x) & g''(\eta) \\ h(a) & h(x) & h''(\eta) \end{vmatrix} - \frac{2(x-a)}{(c-a)(c-b)(b-a)} \begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix}.$$

Then $\phi(a) = 0$ and using (1), we have $\phi(b) = 0$.

Therefore applying Rolle's theorem for ϕ on $[a, b]$, we have $\phi'(\xi) = 0$ where $a < \xi < b$

i.e., $$\begin{vmatrix} f(a) & f'(\xi) & f''(\eta) \\ g(a) & g'(\xi) & g''(\eta) \\ h(a) & h'(\xi) & h''(\eta) \end{vmatrix} = \frac{2}{(c-a)(c-b)(b-a)} \begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix}.$$

Comprehensive Exercise 2

1. (i) State Rolle's Theorem. **(Kanpur 2005; Lucknow 07)**

(ii) Verify Rolle's theorem when $f(x) = e^x \sin x, a = 0, b = \pi$. **(Gorakhpur 2012)**

2. Verify Rolle's theorem for the following functions :

(i) $f(x) = (x-4)^5 (x-3)^4$ in the interval $[3, 4]$.

(ii) $f(x) = x^3 - 6x^2 + 11x - 6$.

(iii) $f(x) = x^3 - 4x$ in $[-2, 2]$.

(iv) $f(x) = e^x (\sin x - \cos x)$ in $[\pi/4, 5\pi/4]$. **(Meerut 2013B)**

(v) $f(x) = 10x - x^2$ in $[0, 10]$. **(Kanpur 2006)**

3. Discuss the applicability of Rolle's theorem to the function

$$f(x) = x^2 + 1, \text{ when } 0 \le x \le 1$$
$$= 3 - x, \text{ when } 1 < x \le 2.$$

4. Show that between any two roots of $e^x \cos x = 1$ there exists at least one root of $e^x \sin x - 1 = 0$.

5. State and prove Rolle's theorem. Interpret it geometrically. Verify Rolle's theorem for the function $f(x) = x^2$ in $[-1, 1]$. **(Lucknow 2010)**

6. Verify the truth of Rolle's theorem for the function $f(x) = x^2 - 3x + 2$ on the interval $[1, 2]$.

7. Does the function $f(x) = |x - 2|$ satisfy the conditions of Rolle's theorem in the interval $[1, 3]$? Justify your answer with correct reasoning.

8. The function f is defined in $[0,1]$ as follows :

$$f(x) = 1 \qquad \text{for } 0 \le x < \frac{1}{2}$$
$$= 2 \qquad \text{for } \frac{1}{2} \le x \le 1.$$

Show that $f(x)$ satisfies none of the conditions of Rolle's theorem, yet $f'(x) = 0$ for many points in $[0,1]$.

9. If $a + b + c = 0$, then show that the quadratic equation $3ax^2 + 2bx + c = 0$ has at least one root in $]0,1[$.

10. Let $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots + \frac{a_{n-1}}{2} + a_n = 0$. Show that there exists at least one real x between 0 and 1 such that $a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0$.

(Lucknow 2009)

11. If $f(x) = \begin{vmatrix} \sin x & \sin \alpha & \sin \beta \\ \cos x & \cos \alpha & \cos \beta \\ \tan x & \tan \alpha & \tan \beta \end{vmatrix}$ where $0 < \alpha < \beta < \pi / 2$,

show that $f'(\xi) = 0$, where $\alpha < \xi < \beta$.

12. Show that there is no real number k for which the equation $x^3 - 3x + k = 0$, has two distinct roots in $]0, 1[$.

Answers 2

3. The given function is not differentiable at $x = 1$ and so Rolle's theorem is not applicable to the given function in the interval $[0, 2]$.
7. The function does not satisfy the third condition that $f(x)$ must be differentiable in the open interval $]1, 3[$.

9 Lagrange's Mean Value Theorem

Theorem: *If a function $f(x)$ is* **(Lucknow 2006, 09; Avadh 07, 12, 14; Meerut 12; Kanpur 11; Rohilkhand 12, 12B; Gorakhpur 10, 12, 14; Kashi 14)**

(i) continuous in a closed interval $[a, b]$,

and (ii) differentiable in the open interval $]a, b[$ i.e., $a < x < b$, then there exists at least one value 'c' of x lying in the open interval $]a, b[$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof: Consider the function $\phi(x)$ defined by $\phi(x) = f(x) + Ax$, ...(1)

where A is a constant to be chosen such that $\phi(a) = \phi(b)$

i.e., $f(a) + Aa = f(b) + Ab$

or $A = -\dfrac{f(b) - f(a)}{b - a}$. ...(2)

(i) Now the function f is given to be continuous on $[a, b]$ and the mapping $x \to Ax$ is continuous on $[a, b]$, therefore ϕ is continuous on $[a, b]$.

(ii) Also, since f is given to be differentiable on $]a, b[$ and the mapping $x \to Ax$ is differentiable on $]a, b[$, therefore, ϕ is differentiable on $]a, b[$.

(iii) By our choice of A, we have $\phi(a) = \phi(b)$.

From (i), (ii) and (iii), we find that ϕ satisfies all the conditions of Rolle's theorem on $[a, b]$. Hence there exists at least one point, say $x = c$, of the open interval $]a, b[$, such that $\phi'(c) = 0$.

But $\phi'(x) = f'(x) + A$, from (1).

$\therefore$ $\phi'(c) = 0 \Rightarrow f'(c) + A = 0$

or $f'(c) = -A = \dfrac{f(b) - f(a)}{b - a}$, from (2).

This proves the theorem. It is usually known as the 'First Mean Value Theorem of Differential Calculus'.

Another form of Lagrange's mean value theorem:

If in the above theorem, we take $b = a + h$, then a number c, lying between a and b can be written as $c = a + \theta h$, where θ is some real number such that $0 < \theta < 1$.

Now Lagrange's theorem can be stated as follows :

If f be defined and continuous on $[a, a + h]$ and differentiable on $]a, a + h[$, then there exists a point $c = a + \theta h$ $(0 < \theta < 1)$ in the open interval $]a, a + h[$ such that

$$\frac{f(a + h) - f(a)}{h} = f'(a + \theta h)$$

or $$\boldsymbol{f(a + h) - f(a) = hf'(a + \theta h).}$$

Geometrical interpretation of the mean value theorem:

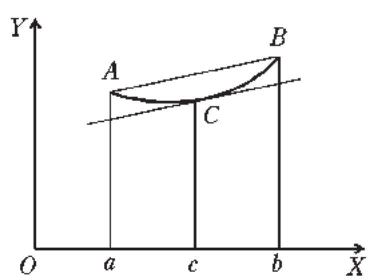
(Gorakhpur 2012, 14)

Let $y = f(x)$ and let ACB be the graph of $y = f(x)$ in $[a, b]$. The coordinates of the point A are $(a, f(a))$ and those of B are $(b, f(b))$. If the chord AB makes an angle α with the x-axis, then

$$\tan \alpha = \frac{f(b) - f(a)}{b - a}$$

$$= f'(c),$$

by Lagrange's mean value theorem where $a < c < b$.



Thus Lagrange's mean value theorem says that there is some point c in $]a, b[$ such that the tangent to the curve at this point is parallel to the chord joining the points on the graph with abscissae a and b.

10 Some Important Deductions from the Mean Value Theorem

Theorem 1: *If a function f is continuous on* $[a, b]$, *differentiable on* $]a, b[$ *and if* $f'(x) = 0$ *for all x in* $]a, b[$, *then* $f(x)$ *has a constant value throughout* $[a, b]$.

Proof: Let c be any point of $]a, b]$. Then the function f is continuous on $[a, c]$ and differentiable on $]a, c[$. Thus f satisfies all the conditions of Lagrange's mean value theorem on $[a, c]$. Consequently there exists a real number d between a and c *i.e.*, $a < d < c$ such that

$$f(c) - f(a) = (c - a) f'(d).$$

But by hypothesis $f'(x) = 0$ throughout the interval $]a, b[$, therefore, in particular $f'(d) = 0$ and hence $f(c) - f(a) = 0$ or $f(c) = f(a)$. Since c is any point of $]a, b]$, therefore, it gives that $f(x) = f(a)\ \forall\ x$ in $]a, b]$. Thus $f(x)$ has a constant value throughout $[a, b]$.

Theorem 2: *If* $f(x)$ *and* $\phi(x)$ *are functions continuous on* $[a, b]$ *and differentiable on* $]a, b[$ *and if* $f'(x) = \phi'(x)$ *throughout the interval* $]a, b[$, *then* $f(x)$ *and* $\phi(x)$ *differ only by a constant.*

Proof: Consider the function $F(x) = f(x) - \phi(x)$. Throughout the interval $]a, b[$, we have

$$F'(x) = f'(x) - \phi'(x) = 0, \text{ because } f'(x) = \phi'(x).$$

Consequently, from theorem 1, we get

$$F(x) = \text{constant or } f(x) - \phi(x) = \text{constant}.$$

Theorem 3: *If* $f'(x) = k$ *for each point x of* $[a, b]$, *k being a constant, then*

$$f(x) = k\,x + C\ \forall\ x \in [a, b], \text{ where } C \text{ is a constant.}$$

Proof: Consider the interval $[a, x]$ such that $[a, x]$ lies in the interval $[a, b]$ *i.e.*, $[a, x] \subset [a, b]$. Since $f'(x)$ exists $\forall\ x \in [a, b]$, f is differentiable on $[a, b]$ and hence on $[a, x]$ and consequently continuous on $[a, x]$. Thus f satisfies all the conditions of Lagrange's mean value theorem on $[a, x]$ and hence there is a point $c \in\ ]a, x[$ such that

$$f(x) - f(a) = (x - a) f'(c).$$

But by hypothesis $f'(x) = k\ \ \forall\ \ x \in [a, b]$, therefore, in particular $f'(c) = k$ as $a < c < x < b$ *i.e.*, $a < c < b$.

Hence $f(x) - f(a) = (x - a)\,k$ or $f(x) = k\,x + f(a) - ak$

or $$f(x) = k\,x + C \text{ where } C = f(a) - ak \text{ is a constant.}$$

Theorem 4: *If f is continuous on* $[a, b]$ *and* $f'(x) \geq 0$ *in* $]a, b[$, *then f is increasing in* $[a, b]$.

Proof: Let x_1 and x_2 be any two distinct points of $[a, b]$ such that $x_1 < x_2$. Then f satisfies the conditions of the Lagrange's mean value theorem in $[x_1, x_2]$. Consequently there exists a number c such that $x_1 < c < x_2$, and

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(c).$$

Now $x_2 - x_1 > 0$ and $f'(c) \geq 0$ (as $f'(x) \geq 0 \;\; \forall \;\; x \in]a, b[$ and c is a point of $]a, b[$), therefore

$$f(x_2) - f(x_1) \geq 0 \text{ i.e., } f(x_1) \leq f(x_2).$$

Thus $\quad x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \;\; \forall \;\; x_1, x_2 \in [a, b].$

Hence f is an increasing function in the interval $[a, b]$.

Similarly, we can prove that if $f'(x) \leq 0$ in $]a, b[$, then f is decreasing in $[a, b]$.

Corollary: *If f is continuous on* $[a, b]$*, then f is strictly increasing or strictly decreasing on* $[a, b]$ *according as*

$$f'(x) > 0 \text{ or } < 0 \;\; \text{in }]a, b[\cdot$$

11 Cauchy's Mean Value Theorem

(Kanpur 2007; Lucknow 10; Avadh 12; Rohilkhand 14)

Theorem: *If two functions* $f(x)$ *and* $g(x)$ *are*

(i) *continuous in a closed interval* $[a, b]$,

(ii) *differentiable in the open interval* $]a, b[$,

(iii) $g'(x) \neq 0$ *for any point of the open interval* $]a, b[$, *then there exists at least one value c of x in the open interval* $]a, b[$, *such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}, \; a < c < b.$$

Proof: First we observe that as a consequence of condition (iii), $g(b) - g(a) \neq 0$. For if $g(b) - g(a) = 0$ *i.e.,* $g(b) = g(a)$, then the function $g(x)$ satisfies all the conditions of Rolle's theorem in $[a, b]$ and consequently there is some x in $]a, b[$ for which $g'(x) = 0$, thus contradicting the hypothesis that $g'(x) \neq 0$ for any point of $]a, b[$.

Now consider the function $F(x)$ defined on $[a, b]$, by setting

$$F(x) = f(x) + A g(x), \qquad \ldots(1)$$

where A is a constant to be chosen such that $F(a) = F(b)$

i.e., $\quad f(a) + A g(a) = f(b) + A g(b)$

or $$-A = \frac{f(b) - f(a)}{g(b) - g(a)} \cdot \qquad \ldots(2)$$

Since $g(b) - g(a) \neq 0$, therefore A is a definite real number.

(i) Now f and g are continuous on $[a, b]$, therefore, F is also continuous on $[a, b]$.

(ii) Again, since f and g are differentiable on $]a, b[$, therefore F is also differentiable on $]a, b[$.

(iii) By our choice of A, $F(a) = F(b)$.

Thus the function $F(x)$ satisfies the conditions of Rolle's theorem in the interval $[a, b]$. Consequently there exists, at least one value, say c, of x in the open interval $]a, b[$ such that $F'(c) = 0$.

But $\qquad F'(x) = f'(x) + Ag'(x)$, from (1).

$\therefore \qquad F'(c) = 0 \Rightarrow f'(c) + Ag'(c) = 0$

or $\qquad -A = \dfrac{f'(c)}{g'(c)}\cdot$ ...(3)

From (2) and (3), we get

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}\cdot$$

Another form: If $b = a + h$, then $a + \theta h = a$ when $\theta = 0$ and $a + \theta h = b$ when $\theta = 1$. Therefore, if $0 < \theta < 1$, then $a + \theta h$ means some value between a and b. So putting $b = a + h$ and $c = a + \theta h$, the result of the above theorem can be written as

$$\frac{f(a+h) - f(a)}{g(a+h) - g(a)} = \frac{f'(a + \theta h)}{g'(a + \theta h)}, \; 0 < \theta < 1.$$

Note 1: If we take $g(x) = x$ for all x in $[a, b]$, then Cauchy's mean value theorem gives Lagrange's mean value theorem as a particular case. For $g(x) = x$ means $g(b) = b$, $g(a) = a$, $g'(x) = 1$ and so $g'(c) = 1$. Putting these values in Cauchy's mean value theorem, we get Lagrange's mean value theorem. Thus Cauchy's mean value theorem is more general than Lagrange's mean value theorem.

Note 2: Cauchy's mean value theorem cannot be obtained by applying Lagrange's mean value theorem to the functions f and g.

For applying Lagrange's mean value theorem to $f(x)$ and $g(x)$ separately, we get

$$f(b) - f(a) = (b - a) f'(c_1), \text{ where } a < c_1 < b$$

and $\qquad g(b) - g(a) = (b - a) g'(c_2)$, where $a < c_2 < b$.

Dividing, we have $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c_1)}{g'(c_2)}\cdot$

Note that here c_1 is not necessarily equal to c_2.

Illustrative Examples

Example 26: *If $f(x) = (x - 1)(x - 2)(x - 3)$ and $a = 0$, $b = 4$, find 'c' using Lagrange's mean value theorem.* **(Lucknow 2006, 07, 11; Rohilkhand 14)**

Solution: We have

$$f(x) = (x - 1)(x - 2)(x - 3) = x^3 - 6x^2 + 11x - 6.$$

$\therefore \qquad f(a) = f(0) = -6$ and $f(b) = f(4) = 6$.

$\therefore \qquad \dfrac{f(b) - f(a)}{b - a} = \dfrac{6 - (-6)}{4 - 0} = \dfrac{12}{4} = 3\cdot$

Also $f'(x) = 3x^2 - 12x + 11$ gives $f'(c) = 3c^2 - 12c + 11$.

Putting these values in Lagrange's mean value theorem

$$\frac{f(b)-f(a)}{b-a} = f'(c), (a < c < b), \text{ we get}$$

$$3 = 3c^2 - 12c + 11 \quad \text{or} \quad 3c^2 - 12c + 8 = 0$$

or $$c = \frac{12 \pm \sqrt{(144-96)}}{6} = 2 \pm \frac{2\sqrt{3}}{3}.$$

As both of these values of c lie in the open interval $]0, 4[$, hence both of these are the required values of c.

Example 27: *Let $f : [0, 1] \to \mathbf{R}$ be defined by*

$$f(x) = (x-1)^2 + 2 \;\; \forall \, x \in [0, 1].$$

Find the equation of the tangent to the graph of this curve which is parallel to the chord joining the points $(0, 3)$ and $(1, 2)$ of the curve.

Solution: Since $f(x)$ is a polynomial function, therefore it is continuous on $[0, 1]$ and differentiable in $]0, 1[$. Hence, by Lagrange's mean value theorem, there is some $c \in \,]0, 1[$ such that

$$\frac{f(1)-f(0)}{1-0} = f'(c)$$

or $$\frac{2-3}{1} = f'(c) \quad \text{or} \quad -1 = f'(c).$$

Now $f'(x) = 2(x-1)$ gives $f'(c) = 2(c-1)$.

Thus $2(c-1) = -1$ *i.e.*, $c = \frac{1}{2}$.

$\therefore f(c) = \frac{9}{4}$, so that the point of contact of the tangent is $\left(\frac{1}{2}, \frac{9}{4}\right)$ and its slope is $f'(c) = -1$. Hence the equation of the required tangent is

$$y - \frac{9}{4} = -1\left(x - \frac{1}{2}\right)$$

or $$4x + 4y = 11.$$

Example 28: *Compute the value of θ in the first mean value theorem*

$$f(x+h) = f(x) + hf'(x+\theta h), \text{ if } f(x) = ax^2 + bx + c.$$

Solution: Here $f(x) = ax^2 + bx + c$.

$\therefore$ $$f(x+h) = a(x+h)^2 + b(x+h) + c,$$

$$f'(x) = 2ax + b, f'(x+\theta h) = 2a(x+\theta h) + b.$$

Substituting all these values in the Lagrange's mean value theorem, we get

$$a(x+h)^2 + b(x+h) + c = ax^2 + bx + c + h[2a(x+\theta h) + b] \qquad \ldots(1)$$

The relation (1) being identically true for all values of x, hence when $x \to 0$, we have

$$ah^2 + bh + c = c + h[2a\theta h + b]$$

or $$ah^2 = 2a\theta h^2 \quad \text{or} \quad \theta = 1/2.$$

Example 29: *A function* $f(x)$ *is continuous in the closed interval* $[0, 1]$ *and differentiable in the open interval* $]0, 1[$ *, prove that*

$$f'(x_1) = f(1) - f(0), \text{ where } 0 < x_1 < 1.$$

Solution: Here $a = 0$, $b = 1$ so that

$$\frac{f(b) - f(a)}{b - a} = \frac{f(1) - f(0)}{1 - 0} = f(1) - f(0).$$

If we take $c = x_1$, and substitute these values in the result of Lagrange's mean value theorem, we get

$$f(1) - f(0) = f'(x_1) \text{ where } 0 < x_1 < 1.$$

This is a particular case of Lagrange's mean value theorem. Students can give an independent proof of this.

Example 30: *Separate the intervals in which the polynomial*

$2x^3 - 15x^2 + 36x + 1$ *is increasing or decreasing.*

Solution: We have $f(x) = 2x^3 - 15x^2 + 36x + 1.$

$\therefore$ $$f'(x) = 6x^2 - 30x + 36 = 6(x - 2)(x - 3).$$

Now $f'(x) > 0$ for $x < 2$ or for $x > 3$,

$f'(x) < 0$ for $2 < x < 3$, and $f'(x) = 0$ for $x = 2, 3$.

Thus $f'(x)$ is +ive in the intervals $]-\infty, 2[$ and $]3, \infty[$ and negative in the interval $]2, 3[$. Hence f is monotonically increasing in the intervals $]-\infty, 2]$, $[3, \infty[$ and monotonically decreasing in the interval $[2, 3]$.

Example 31: *Show that*

$$\frac{x}{1 + x} < \log(1 + x) < x \text{ for } x > 0.$$

(Bundelkhand 2011)

Solution: Let $f(x) = \log(1 + x) - \dfrac{x}{1 + x}\cdot$

$\therefore$ $$f(0) = 0.$$

Then $$f'(x) = \frac{1}{1 + x} - \frac{1.(1 + x) - x.1}{(1 + x)^2} = \frac{1}{1 + x} - \frac{1}{(1 + x)^2} = \frac{x}{(1 + x)^2}\cdot$$

We observe that $f'(x) > 0$ for $x > 0$. Hence $f(x)$ is monotonically increasing in the interval $[0, \infty[$. Therefore

$$f(x) > f(0) \text{ for } x > 0 \text{ i.e., } \left[\log(1 + x) - \frac{x}{1 + x}\right] > 0 \text{ for } x > 0$$

i.e., $$\log(1 + x) > \frac{x}{1 + x} \text{ for } x > 0. \qquad \ldots(1)$$

Again, let $\phi(x) = x - \log(1 + x).$

$\therefore$ $\phi(0) = 0.$

Then $\qquad \phi'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x}\cdot$

We observe that $\phi'(x) > 0$ for $x > 0$. Hence $\phi(x)$ is monotonically increasing in the interval $[0, \infty[$. Therefore

$$\phi(x) > \phi(0) \text{ for } x > 0 \text{ i.e., } [x - \log(1+x)] > 0 \text{ for } x > 0$$

i.e., $\qquad x > \log(1+x)$ for $x > 0$. $\qquad$ ...(2)

From (1) and (2), we get

$$\frac{x}{1+x} < \log(1+x) < x \text{ for } x > 0.$$

Example 32: *Verify Cauchy's mean value theorem for the functions x^2 and x^3 in the interval* $[1, 2]$. **(Avadh 2013)**

Solution: Let $f(x) = x^2$ and $g(x) = x^3$. Then $f(x)$ and $g(x)$ are continuous in the closed interval $[1, 2]$ and differentiable in the open interval $]1, 2[$. Also $g'(x) = 3x^2 \neq 0$ for any point in the open interval $]1, 2[$. Hence by Cauchy's mean value theorem there exists at least one real number c in the open interval $]1, 2[$, such that

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)}\cdot \qquad \text{...(1)}$$

Now $\qquad \dfrac{f(2) - f(1)}{g(2) - g(1)} = \dfrac{4-1}{8-1} = \dfrac{3}{7}\cdot$

Also $f'(x) = 2x$, $g'(x) = 3x^2$.

$\therefore \quad \dfrac{f'(c)}{g'(c)} = \dfrac{2c}{3c^2} = \dfrac{2}{3c}\cdot$ Putting these values in (1), we get $\dfrac{3}{7} = \dfrac{2}{3c}$ or $c = \dfrac{14}{9}$ which lies in the open interval $]1, 2[$. Hence Cauchy's mean value theorem is verified.

Example 33: *If in the Cauchy's mean value theorem, we write $f(x) = e^x$ and $g(x) = e^{-x}$, show that 'c' is the arithmetic mean between a and b.*

Solution: Here $\quad \dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{e^b - e^a}{e^{-b} - e^{-a}} = -e^a e^b = -e^{a+b}$,

and $\qquad \dfrac{f'(x)}{g'(x)} = \dfrac{e^x}{-e^{-x}}$ so that $\dfrac{f'(c)}{g'(c)} = \dfrac{e^c}{-e^{-c}} = -e^{2c}$.

Putting these values in Cauchy's mean value theorem, we get

$$-e^{a+b} = -e^{2c} \qquad \text{or} \qquad 2c = a + b \qquad \text{or} \qquad c = \frac{1}{2}(a+b).$$

Thus c is the arithmetic mean between a and b.

Example 34: *If in the Cauchy's mean value theorem, we write*

(i) $\quad f(x) = \sqrt{x}$ *and* $g(x) = 1/\sqrt{x}$, *then c is the geometric mean between a and b,* **(Rohilkhand 2014)**

and if

(ii) $\quad f(x) = 1/x^2$ *and* $g(x) = 1/x$, *then c is the harmonic mean between a and b.* **(Rohilkhand 2005)**

Solution: **(i)** Here $\dfrac{f(b)-f(a)}{g(b)-g(a)}=\dfrac{\sqrt{b}-\sqrt{a}}{(1/\sqrt{b})-(1/\sqrt{a})}=-\sqrt{(ab)}$,

and $$\frac{f'(x)}{g'(x)}=\frac{\frac{1}{2}x^{-1/2}}{-\frac{1}{2}x^{-3/2}} \text{ so that } \frac{f'(c)}{g'(c)}=-\frac{c^{-1/2}}{c^{-3/2}}=-c.$$

Putting these values in Cauchy's mean value theorem, we get

$$-\sqrt{(ab)}=-c \quad \text{or} \quad c=\sqrt{(ab)}$$

i.e., c is the geometric mean between a and b.

(ii) Here $$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{(1/b^2)-(1/a^2)}{(1/b)-(1/a)}=\frac{a+b}{ab}$$

and $$\frac{f'(x)}{g'(x)}=\frac{-2x^{-3}}{-x^{-2}} \text{ so that } \frac{f'(c)}{g'(c)}=\frac{-2c^{-3}}{-c^{-2}}=\frac{2}{c}\cdot$$

Putting these values in Cauchy's mean value theorem, we get

$$\frac{a+b}{ab}=\frac{2}{c} \quad \text{or} \quad c=\frac{2ab}{a+b}$$

i.e., c is the harmonic mean between a and b.

Comprehensive Exercise 3

1. State Lagrange's mean value theorem. Test if Lagrange's mean value theorem holds for the function $f(x)=|x|$ in the interval $[-1,1]$.
(Kanpur 2010; Rohilkhand 13B)

2. If $f(x)=1/x$ in $[-1,1]$, will the Lagrange's mean value theorem be applicable to $f(x)$? **(Meerut 2012B)**

3. Verify Lagrange's mean value theorem for the function
$$f:[-1,1]\to \mathbf{R} \text{ given by } f(x)=x^3.$$

4. Find 'c' of the mean value theorem, if
$$f(x)=x(x-1)(x-2);\ a=0, b=\frac{1}{2}\cdot$$

5. Find 'c' of Mean value theorem when
   (i) $f(x)=x^3-3x-2$ in $[-2,3]$
   (ii) $f(x)=2x^2+3x+4$ in $[1,2]$
   (iii) $f(x)=x(x-1)$ in $[1,2]$ **(Meerut 2013B)**
   (iv) $f(x)=x^2-3x-1$ in $\left[-\dfrac{11}{7},\dfrac{13}{7}\right]\cdot$

6. Show that any chord of the parabola $y = Ax^2 + Bx + C$ is parallel to the tangent at the point whose abscissa is same as that of the middle point of the chord.

7. If $f''(c)$ exists for all points in $[a, b]$ and $\dfrac{f(c) - f(a)}{c - a} = \dfrac{f(b) - f(c)}{b - c}$ where $a < c < b$, then there is a number ξ such that $a < \xi < b$ and $f''(\xi) = 0$.

8. State the conditions for the validity of the formula

$$f(x + h) = f(x) + h f'(x + \theta h)$$

and investigate how far these conditions are satisfied and whether the result is true, when $f(x) = x \sin(1/x)$ (being defined to be zero at $x = 0$) and $x < 0 < x + h$.

9. (a) Show that $x^3 - 3x^2 + 3x + 2$ is monotonically increasing in every interval.

(b) Show that $\log(1 + x) - \dfrac{2x}{2 + x}$ is increasing when $x > 0$.

10. Determine the intervals in which the function

$$(x^4 + 6x^3 + 17x^2 + 32x + 32) e^{-x}$$

is increasing or decreasing.

11. Use the function $f(x) = x^{1/x}$, $x > 0$ to determine the bigger of the two numbers e^{π} and π^{e}.

12. If $a = -1, b \geq 1$ and $f(x) = 1/|x|$, show that the conditions of Lagrange's mean value theorem are not satisfied in the interval $[a, b]$, but the conclusion of the theorem is true if and only if $b > 1 + \sqrt{2}$.

13. State Cauchy's mean value theorem. **(Kanpur 2007)**

Verify Cauchy's mean value theorem for $f(x) = \sin x, g(x) = \cos x$ in $[-\pi/2, 0]$. **(Lucknow 2007)**

14. If $f(x) = x^2, g(x) = \cos x$, then find the point $c \in]0, \pi/2[$ which gives the result of Cauchy's mean value theorem in the interval $[0, \pi/2]$ for the functions $f(x)$ and $g(x)$.

15. Show that $\dfrac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta$, where $0 < \alpha < \theta < \beta < \dfrac{\pi}{2}\cdot$

16. Use Cauchy's mean value theorem to evaluate $\lim_{x \to 1} \left[\dfrac{\cos \frac{1}{2} \pi x}{\log(1/x)} \right]$.

Answers 3

1. The mean value theorem does not hold since the given function is not differentiable at $x = 0$

2. not applicable

4. $1 - \dfrac{\sqrt{21}}{6}$

5. (i) $\pm\sqrt{(7/3)}$ (ii) 3/2 (iii) 3/2 (iv) 1/7
8. Condition of differentiability is not satisfied in $x < 0 < x + h$ since $f(x)$ is non-differentiable at $x = 0$.
10. Increasing in the intervals $[-2, -1]$ and $[0, 1]$ and decreasing in the intervals $]-\infty, -2[$, $[-1, 0]$ and $[1, \infty[$.
11. e^{π} is bigger than π^{e}.
14. Root of the equation $\sin c - (8c/\pi^2) = 0$ in the open interval $]\pi/6, \pi/2[$.
16. $\pi/2$.

12 Taylor's Theorem with Lagrange's form of Remainder After n Terms

Theorem: *If $f(x)$ is a single-valued function of x such that*

(i) all the derivatives of $f(x)$ upto $(n-1)$ th are continuous in $a \le x \le a+h$,

(ii) $f^{(n)}(x)$ exists in $a < x < a+h$, then

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a+\theta h), \text{ where } 0 < \theta < 1.$$

Proof: Consider the function ϕ defined by

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) + \frac{A}{n!}(a+h-x)^n,$$

where A is a constant to be suitably chosen.

We choose A such that $\phi(a) = \phi(a+h)$.

Now $\phi(a) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{A}{n!} h^n,$

and $\phi(a+h) = f(a+h)$.

Hence A is given by

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} A. \quad \ldots(1)$$

Now, by hypothesis, all the functions

$$f(x), f'(x), f''(x), \ldots\ldots, f^{(n-1)}(x)$$

are continuous in the closed interval $[a, a+h]$ and differentiable in the open interval $]a, a+h[$.

Further $(a+h-x), (a+h-x)^2/2!, \ldots, (a+h-x)^n/n!$, all being polynomials, are continuous in the closed interval $[a, a+h]$ and differentiable in the open interval $]a, a+h[$. Also A is a constant.

$\therefore$ $\phi(x)$ is continuous in the closed interval $[a, a+h]$ and differentiable in the open interval $]a, a+h[$.

By our choice of A, $\phi(a) = \phi(a+h)$. Hence $\phi(x)$ satisfies all the conditions of Rolle's theorem.

Consequently $\phi'(a+\theta h) = 0$, where $0 < \theta < 1$.

Now
$$\phi'(x) = f'(x) - f'(x) + (a+h-x) f''(x) - (a+h-x) f''(x)$$
$$+ \ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(x) - \frac{A}{(n-1)!}(a+h-x)^{n-1}$$
$$= \frac{(a+h-x)^{n-1}}{(n-1)!}[f^{(n)}(x) - A],$$

since other terms cancel in pairs.

$\therefore$ $\phi'(a+\theta h) = 0$ gives
$$\frac{[a+h-(a+\theta h)]^{n-1}}{(n-1)!}[f^{(n)}(a+\theta h) - A] = 0$$

or
$$\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}[f^{(n)}(a+\theta h) - A] = 0$$

or
$$f^{(n)}(a+\theta h) - A = 0 \quad \text{or} \quad A = f^{(n)}(a+\theta h).$$

$[\because h \neq 0, (1-\theta) \neq 0 \text{ as } 0 < \theta < 1]$

Putting this value of A in (1), we get
$$\boldsymbol{f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots}$$
$$\boldsymbol{+ \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a+\theta h).}$$

This is **Taylor's development** of $f(a+h)$ in ascending integral powers of h. The $(n+1)$th term $\boldsymbol{\frac{h^n}{n!} f^{(n)}(a+\theta h)}$ is called **Lagrange's form of remainder** after n terms in Taylor's expansion of $f(a+h)$.

Note: If we take $n = 1$, we see that Lagrange's mean value theorem is a particular case of the above theorem.

Corollary. (Maclaurin's development):

If we take the interval $[0, x]$ instead of $[a, a+h]$, so that changing a to 0 and h to x in Taylor's theorem, we get

$$f(x) = f(0) + xf'(0 + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n}{n!} f^{(n)}(\theta x)$$

which is known as **Maclaurin's theorem** or **Maclaurin's development** of $f(x)$ in the interval $[0, x]$ with **Lagrange's form of remainder** $\frac{x^n}{n!} f^{(n)}(\theta x)$ after n terms.

13 Taylor's Theorem with Cauchy's Form of Remainder

Theorem: *If $f(x)$ is a single-valued function of x such that*

(i) all the derivatives of $f(x)$ upto $(n-1)$th are continuous in $a \le x \le a+h$,

(ii) $f^{(n)}(x)$ exists in $a < x < a+h$, then

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a)$$
$$+ \frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(a+\theta h), \text{ where } 0 < \theta < 1.$$

Proof: Consider the function ϕ defined by

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots$$
$$+ \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) + (a+h-x) A,$$

where A is a constant to be suitably chosen. We choose A such that $\phi(a) = \phi(a+h)$.

Now $\quad \phi(a) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + h A,$

and $\quad \phi(a+h) = f(a+h).$

Hence A is given by

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + hA. \quad \ldots(1)$$

As explained earlier in article 12, it can be easily seen that $\phi(x)$ satisfies all the conditions of Rolle's theorem. Consequently

$$\phi'(a+\theta h) = 0, \text{ where } 0 < \theta < 1.$$

Now $\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(x) - A$, since other terms cancel in pairs.

$\therefore$ $\quad \phi'(a+\theta h)=0$ gives $\dfrac{[a+h-(a+\theta h)]^{n-1}}{(n-1)!} f^{(n)}(a+\theta h)-A=0$

or $\quad A=\dfrac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(a+\theta h).$

Putting this value of A in (1), we get

$$\boldsymbol{f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)}$$
$$\boldsymbol{+\frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(a+\theta h).}$$

The $(n+1)$th term $\dfrac{h^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(a+\theta h)$ is called **Cauchy's form of remainder** after n terms in the **Taylor's expansion** of $f(a+h)$ in ascending integral powers of h.

Corollary. (Maclaurin's development with Cauchy's form of remainder): If we change a to 0 and h to x in the above result, we get

$$\boldsymbol{f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)}$$
$$\boldsymbol{+\frac{x^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(\theta x),}$$

which is **Maclaurin's theorem with Cauchy's form of remainder**. The $(n+1)$th term $\dfrac{x^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(\theta x)$ is known as Cauchy's form of remainder after n terms in Maclaurin's development of $f(x)$ in the interval $[0, x]$.

14 Expansions of Some Basic Functions

(i) Expansion of e^x.

Let $\quad f(x)=e^x.$

Then $\quad f^{(n)}(x)=e^x \;\; \forall\, n\in \mathbf{N}, \forall\, x\in \mathbf{R}$ so that

$\quad f^{(n)}(0)=e^0=1 \;\; \forall\; n\in \mathbf{N}.$

Now Maclaurin's expansion of $f(x)$ with Lagrange's form of remainder is

$$f(x)=f(0)+x f'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+R_n$$

where $\quad R_n=\dfrac{x^n}{n!}f^{(n)}(\theta x),\; 0<\theta<1.$

Now $$\lim_{n \to \infty} R_n = \lim_{n \to \infty} \frac{x^n}{n!} f^{(n)}(\theta x) = \lim_{n \to \infty} \frac{x^n}{n!} e^{\theta x}$$

$$= e^{\theta x} \lim_{n \to \infty} \frac{x^n}{n!} = e^{\theta x} \times 0 = 0. \qquad \left[\because \lim_{n \to \infty} \frac{x^n}{n!} = 0\right]$$

Thus $f^{(n)}(x)$ exists in $[0, x]$ for each $n \in \mathbf{N}$ and $R_n \to 0$ as $n \to \infty$ *i.e.*, all the conditions of Maclaurin's series expansion are satisfied.

Hence $\forall\, x \in \mathbf{R}$ the expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

(ii) Expansion of sin *x*.

Let $f(x) = \sin x.$

Then $f^{(n)}(x) = \sin\left(x + \frac{1}{2} n\pi\right) \; \forall\, \boldsymbol{n} \in \mathbf{N}, \forall\, \boldsymbol{x} \in \mathbf{R}$

so that $f^{(n)}(0) = \sin\left(\frac{1}{2} n\pi\right) \; \forall\; n \in \mathbf{N}$

or $$f^{(n)}(0) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ (-1)^{(n-1)/2}, & \text{if } n \text{ is odd.} \end{cases}$$

The Maclaurin's expansion of $f(x)$ with Lagrange's form of remainder is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n$$

where $$R_n = \frac{x^n}{n!} f^{(n)}(\theta x) = \frac{x^n}{n!} \sin\left(\theta x + \frac{n\pi}{2}\right), \; 0 < \theta < 1.$$

Now $$|R_n| = \left|\frac{x^n}{n!} \sin\left(\theta x + \frac{n\pi}{2}\right)\right| = \left|\frac{x^n}{n!}\right| \left|\sin\left(\theta x + \frac{n\pi}{2}\right)\right| \le \left|\frac{x^n}{n!}\right|.$$

$\therefore$ $$\lim_{n \to \infty} |R_n| \le \lim_{n \to \infty} \left|\frac{x^n}{n!}\right| = 0 \qquad \left[\because \lim_{n \to \infty} \frac{x^n}{n!} = 0\right]$$

or $$\lim_{n \to \infty} R_n = 0.$$

Thus all the conditions of Maclaurin's series expansion are satisfied. Hence the expansion of $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \ldots$$

(iii) Expansion of cos *x*.

Proceed as above. In this case we get

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

(iv) Expansion of $\log_e (1 + x)$.

Let $\quad f(x) = \log_e (1 + x), \quad (-1 < x \le 1).$

Then $\quad f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n} \ \forall\ n \in \mathbf{N},$

so that $\quad f^{(n)}(0) = (-1)^{n-1}(n-1)! \ \forall\ n \in \mathbf{N}.$

The Maclaurin's expansion of $f(x)$ with Lagrange's form of remainder is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n$$

where $\quad R_n = \dfrac{x^n}{n!} f^{(n)}(\theta x), \ 0 < \theta < 1$

$$= \frac{x^n}{n!} \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n} = (-1)^{n-1} \cdot \frac{1}{n}\left(\frac{x}{1+\theta x}\right)^n.$$

In order to show that $R_n \to 0$ as $n \to \infty$, we consider two cases :

Case I: $\quad 0 \le x \le 1.$

Since $0 \le x \le 1$ and $0 < \theta < 1$, therefore $x < 1 + \theta x$

and hence $\displaystyle\lim_{n \to \infty} \left(\frac{x}{1+\theta x}\right)^n = 0.$ Also $\displaystyle\lim_{n \to \infty} \frac{(-1)^{n-1}}{n} = 0.$

Thus, in this case, $\displaystyle\lim_{n \to \infty} R_n = 0.$

Case II: $\quad -1 < x < 0.$

In this case it will not be convenient to show that Lagrange's form of remainder $R_n \to 0$ as $n \to \infty$ because $x / (1 + \theta x)$ may not be numerically less than unity. Therefore we use the Cauchy's form of remainder. We have

$$R_n = \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(\theta x)$$

$$= \frac{x^n}{(n-1)!}(1-\theta)^{n-1}(-1)^{n-1}(n-1)!(1+\theta x)^{-n}$$

$$= (-1)^{n-1} x^n \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \cdot \frac{1}{1+\theta x}, \qquad (0 < \theta < 1).$$

Now as $0 < \theta < 1$ and $-1 < x < 0$, we have

$$0 < \frac{1-\theta}{1+\theta x} < 1, \text{ so that } \lim_{h \to \infty} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} = 0.$$

Also, $\displaystyle\lim_{h \to \infty} x^n = 0$ as $-1 < x < 0.$

Thus, in this case also $R_n \to 0$ as $n \to \infty$.

Hence, $f(x)$ satisfies all the conditions of Maclaurin's series expansion for $-1 < x \le 1$.

Therefore, for $-1 < x \le 1$, we get

$$\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1} \frac{x^n}{n} + \ldots$$

(v) Expansion of $(1 + x)^m$.

Let $\quad f(x) = (1+x)^m, \ \forall \ x \in \mathbf{R}$.

Then $\quad f^{(n)}(x) = m(m-1)(m-2)\ldots(m-n+1)(1+x)^{m-n} \quad \forall \ n \in \mathbf{N}$.

Now we consider two cases :

Case I: If m is a positive integer.

In this case, we notice that for $n > m$, $f^{(n)}(x) = 0$. So all the terms after the $(m + 1)$th term vanish and so the expansion consists of finite number of terms in the form

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^m}{m!} f^{(m)}(0).$$

Case II: If m is a fraction or a negative integer.

In this case, let $|x| < 1$.

We have

$$f^{(n)}(x) = m(m-1)(m-2)\ldots(m-n+1)(1+x)^{m-n}, \quad x \ne -1.$$

Here, we use Maclaurin's expansion with Cauchy's form of remainder. Thus, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n$$

where $\quad R_n = \dfrac{x^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(\theta x), 0 < \theta < 1$

$$= \frac{x^n}{(n-1)!}(1-\theta)^{n-1} . m(m-1)\ldots(m-n+1)(1+\theta x)^{m-n}$$

$$= \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} . (1+\theta x)^{m-1} \cdot \frac{m(m-1)\ldots(m-n+1)}{(n-1)!} \cdot x^n.$$

If $\quad a_n = \dfrac{m(m-1)\ldots(m-n+1)}{(n-1)!} \cdot x^n$, then $\dfrac{a_{n+1}}{a_n} = \dfrac{m-n}{n} \cdot x$

(on simplification).

$\therefore \quad \lim\limits_{n \to \infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n \to \infty} \left(\dfrac{m}{n} - 1\right) x = (0-1)x = -x.$

This gives $\lim\limits_{n \to \infty} a_n = 0$, since $|-x| = |x| < 1$.

Further $0 < \theta < 1 \Rightarrow 0 < 1 - \theta < 1 + \theta x$

so that $\dfrac{1-\theta}{1+\theta x} < 1$ which gives $\lim\limits_{n \to \infty} \left(\dfrac{1-\theta}{1+\theta x}\right)^{n-1} = 0.$

Hence in this case $\lim\limits_{n \to \infty} R_n = 0.$ Thus $f(x)$ satisfies the conditions of Maclaurin's series expansion.

Therefore for $-1 < x < 1$, we have

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!} \cdot x^2 + \frac{m(m-1)(m-2)}{3!} \cdot x^3 + \ldots$$

Illustrative Examples

Example 35: *Prove that*

$$\sin ax = ax - \frac{a^3 x^3}{3!} + \frac{a^5 x^5}{5!} - \ldots + \frac{a^{n-1} x^{n-1}}{(n-1)!} \sin\left(\frac{n-1}{2}\pi\right)$$
$$+ \frac{a^n x^n}{n!} \sin\left(a\theta x + \frac{n\pi}{2}\right).$$

Solution: Let $f(x) = \sin ax$.

Then $$f'(x) = a \cos ax = a \sin\left(ax + \frac{1}{2}\pi\right)$$

$$f''(x) = a^2 \cos\left(ax + \frac{1}{2}\pi\right) = a^2 \sin\left(ax + 2 \cdot \frac{1}{2}\pi\right)$$

$$f'''(x) = a^3 \cos\left(ax + 2 \cdot \frac{1}{2}\pi\right) = a^3 \sin\left(ax + 3 \cdot \frac{1}{2}\pi\right)$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$f^{(n-1)}(x) = a^{n-1} \sin\left(ax + \frac{n-1}{2}\pi\right)$$

and $$f^{(n)}(x) = a^n \sin\left(ax + \frac{n\pi}{2}\right)$$

so that $$f^{(n)}(\theta x) = a^n \sin\left(a\theta x + \frac{n\pi}{2}\right).$$

$\therefore$ $$f(0) = \sin 0 = 0,\ f'(0) = a \sin\frac{1}{2}\pi = a,\ f''(0) = a^2 \sin \pi = 0,$$

$$f'''(0) = a^3 \sin\frac{3\pi}{2} = -a^3,\ f^{iv}(0) = a^4 \sin 2\pi = 0,$$

$$f^{v}(0) = a^5 \sin\frac{5\pi}{2} = a^5, \ldots\ldots\ldots,$$

$$f^{(n-1)}(0) = a^{n-1} \sin\frac{n-1}{2}\pi.$$

Putting these values in

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$$

$$+ \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n}{n!} f^{(n)}(\theta x), \text{ we get}$$

$$\sin ax = 0 + xa + 0 - \frac{x^3}{3!} a^3 + 0 + \frac{x^5}{5!} a^5 + \ldots$$

$$+ \frac{x^{n-1}}{(n-1)!} a^{n-1} \sin \frac{n-1}{2} \pi + \frac{x^n}{n!} a^n \sin \left(a\theta x + \frac{n\pi}{2} \right)$$

or $$\sin ax = xa - \frac{x^3 a^3}{3!} + \frac{x^5 a^5}{5!} - \ldots + \frac{x^{n-1} a^{n-1}}{(n-1)!} \sin \left(\frac{n-1}{2} \pi \right)$$

$$+ \frac{a^n x^n}{n!} \sin \left(a\theta x + \frac{n\pi}{2} \right).$$

Example 36: *If* $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta x)$, ...(1)

find the value of θ *as* $x \to 1$, $f(x)$ *being* $(1-x)^{5/2}$. **(Lucknow 2009)**

Solution: Here $f(x) = (1-x)^{5/2}$.

$\therefore$ $$f'(x) = -\frac{5}{2}(1-x)^{3/2}$$

and $$f''(x) = \frac{15}{4}(1-x)^{1/2}.$$

Thus $$f(0) = 1, \; f'(0) = -\frac{5}{2}, \; f''(\theta x) = \frac{15}{4}(1-\theta x)^{1/2}.$$

Putting these values in (1), we get

$$(1-x)^{5/2} = 1 - \frac{5}{2} x + \frac{x^2}{2!} \times \frac{15}{4} (1-\theta x)^{1/2}.$$

Therefore as $x \to 1$, we have

$$0 = 1 - \frac{5}{2} + \frac{1}{2!} \cdot \frac{15}{4} (1-\theta)^{1/2}$$

or $$(1-\theta)^{1/2} = \frac{4}{5} \quad \text{or} \quad (1-\theta) = \frac{16}{25} \quad \text{or} \quad \theta = \frac{9}{25}.$$

Example 37: *Assuming the derivatives which occur are continuous, apply the mean value theorem to prove that*

$$\phi'(x) = F'\{f(x)\} f'(x) \text{ where } \phi(x) = F\{f(x)\}.$$

Solution: Let $f(x) = t$ so that $\phi(x) = F(t)$.

Now $$\phi'(x) = \lim_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h} = \lim_{h \to 0} \frac{F\{f(x+h)\} - F\{f(x)\}}{h}$$

$$= \lim_{h \to 0} \frac{F\{f(x) + h f'(x+\theta_1 h)\} - F\{f(x)\}}{h}, \ (0 < \theta_1 < 1)$$

[$\because f(x+h) = f(x) + h f'(x + \theta_1 h)$, by mean value theorem]

$$= \lim_{h \to 0} \frac{F(t+H) - F(t)}{h}, \text{ where } H = h f'(x+\theta_1 h)$$

$$= \lim_{h \to 0} \frac{HF'(t+\theta_2 H)}{h}$$

[$\because F(t+H) = F(t) + HF'(t+\theta_2 H)$, by mean value theorem]

$$= \lim_{h \to 0} \frac{h f'(x+\theta_1 h) F'[t + \theta_2 h f'(x+\theta_1 h)]}{h}$$

$$= f'(x) F'(t) = F'\{f(x)\} f'(x).$$

Note: This example gives an alternative proof of the chain rule.

Comprehensive Exercise 4

1. If $f(x+h) = f(x) + h f'(x) + \dfrac{h^2}{2!} f''(x+\theta h)$

find the value of θ as $x \to a$, $f(x)$ being $(x-a)^{5/2}$. **(Lucknow 2010)**

2. Find θ, if

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x+\theta h), 0 < \theta < 1, \textit{ and}$$

(i) $f(x) = ax^3 + bx^2 + cx + d$ (ii) $f(x) = x^3$.

3. Show that 'θ' (which occurs in the Lagrange's mean value theorem) approaches the limit $\dfrac{1}{2}$ as 'h' approaches zero provided that $f''(a)$ is not zero. It is assumed that $f''(x)$ is continuous.

4. Show that the number θ which occurs in the Taylor's theorem with Lagrange's form of remainder after n terms approaches the limit $1/(n+1)$ as $h \to 0$ provided that $f^{(n+1)}(x)$ is continuous and different from zero at $x = a$.

Answers 4

1. $\dfrac{64}{225}$

2. (i) $\dfrac{1}{3}$ (ii) $\dfrac{1}{3}$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The function $f(x) = |x - 1|$ is not differentiable at
   (a) $x = 0$ (b) $x = -1$
   (c) $x = 1$ (d) $x = 2$
2. The function $f(x) = |x + 3|$ is not differentiable at
   (a) $x = 3$ (b) $x = -3$
   (c) $x = 0$ (d) $x = 1$
3. A function $f(x)$ is differentiable at $x = a$ if
   (a) $R f'(a) = Lf'(a)$ (b) $R f'(a) = 0$
   (c) $L f'(a) = 0$ (d) $R f'(a) \neq Lf'(a)$
4. A function $\phi(x)$ is defined as follows :
   $$\phi(x) = 1 + x \text{ if } x \le 2$$
   $$\phi(x) = 5 - x \text{ if } x > 2.$$
   Then
   (a) $\phi(x)$ is continuous but not differentiable at $x = 2$
   (b) $\phi(x)$ is differentiable at every point of **R**
   (c) $\phi(x)$ is neither continuous nor differentiable at $x = 2$
   (d) $\phi(x)$ is differentiable at $x = 2$ but is not continuous at $x = 2$.
5. Out of the following four functions tell the function for which the conditions of Rolle's theorem are satisfied.
   (a) $f(x) = |x|$ in $[-1, 1]$ (b) $f(x) = x^2$ in $2 \le x \le 3$
   (c) $f(x) = \sin x$ in $[0, \pi]$ (d) $f(x) = \tan x$ in $0 \le x \le \pi$
6. The function $f(x) = \sin x$ is increasing in the interval
   (a) $[0, \pi]$ (b) $\left[0, \frac{\pi}{2}\right]$
   (c) $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ (d) $\left[\frac{\pi}{2}, \pi\right]$ **(Kumaun 2014)**
7. The value of 'c' of Lagrange's mean value theorem for $f(x) = x(x - 1)$ in $[1, 2]$ is given by
   (a) $c = \frac{5}{4}$ (b) $c = \frac{3}{2}$
   (c) $c = \frac{7}{4}$ (d) $c = \frac{11}{6}$

8. The value of 'c' of Rolle's theorem for the function $f(x) = e^{x} \sin x$ in $[0, \pi]$ is given by

(a) $c = \dfrac{3\pi}{4}$ (b) $c = \dfrac{\pi}{4}$

(c) $c = \dfrac{\pi}{2}$ (d) $c = \dfrac{5\pi}{6}$

9. The function $f(x) = |x|$ at $x = 0$ shall be

(a) differentiable

(b) continuous but not differentiable

(c) discontinuous

(d) none of these **(Kumaun 2009)**

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. A function $f(x)$ is said to be differentiable at $x = a$ if

$$\lim_{x \to a} \frac{f(x) - \ldots}{x - a} \text{ exists.}$$

2. The right hand derivative of $f(x)$ at $x = a$ is given by

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{\ldots}, h > 0,$$

provided the limit exists.

3. The left hand derivative of $f(x)$ at $x = a$ is given by

$$\lim_{h \to 0} \frac{f(a-h) - f(a)}{\ldots}, h > 0,$$

provided the limit exists.

4. A function $f :]a, b[\to \mathbf{R}$ is said to be differentiable in $]a, b[$ if and only if it is differentiable at every point in … .

5. If a function $f(x)$ is differentiable at $x = a$, then $f'(a)$ is the tangent of the angle which the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ makes with … .

6. Continuity is a necessary but not a … condition for the existence of a finite derivative.

7. The function $f(x) = |x|$ is differentiable at every point of $\mathbf{R}$ except at $x = \ldots$.

8. If a function $f(x)$ is such that

(i) $f(x)$ is continuous in the closed interval $[a, b]$,

(ii) $f'(x)$ exists for every point in the open interval $]a, b[$,

(iii) $f(a) = f(b)$, then there exists at least one value of x, say c, where $a < c < b$, such that $f'(c) = 0$.

The above theorem is known as … .

9. If a function $f(x)$ is
   (i) continuous in the closed interval $[a, b]$, and
   (ii) differentiable in the open interval $]a, b[$ *i.e.*, $a < x < b$, then there exists at least one value 'c' of x lying in the open interval $]a, b[$ such that
$$\frac{f(b) - f(a)}{b - a} = \ldots.$$

10. If two functions $f(x)$ and $g(x)$ are
   (i) continuous in a closed interval $[a, b]$
   (ii) differentiable in the open interval $]a, b[$, and
   (iii) $g'(x) \neq 0$ for any point of the open interval $]a, b[$, then there exists at least one value c of x in the open interval $]a, b[$, such that
$$\frac{f(b) - f(a)}{\ldots} = \frac{f'(c)}{g'(c)}.$$

11. If f is continuous in $[a, b]$ and $f'(x) \geq 0$ in $]a, b[$, then f is ... in $[a, b]$.
12. If $f(x) = \sin x$, then
$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \ldots.$$

True or False

Write 'T' for true and 'F' for false statement.

1. If a function $f(x)$ is continuous at $x = a$, it must also be differentiable at $x = a$.
2. If a function $f(x)$ is differentiable at $x = a$, it must be continuous at $x = a$.
3. If a function $f(x)$ is differentiable at $x = a$, it may or may not be continuous at $x = a$.
4. The function $f(x) = |x|$ is differentiable at every point of $\mathbf{R}$.
5. Rolle's theorem is applicable for $f(x) = \sin x$ in $[0, 2\pi]$.
6. Rolle's theorem is applicable for $f(x) = |x|$ in $[-1, 1]$.
7. Lagrange's mean value theorem is applicable for $f(x) = |x|$ in $[-1, 1]$.
8. The function $f(x) = \sin x$ is increasing in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
9. If $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has no root in $]0, 1[$.
10. If f is continuous on $[a, b]$ and $f'(x) \leq 0$ in $]a, b[$, then f is increasing in $[a, b]$.
11. The function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is decreasing in the interval $[2, 3]$.
12. Let $f(x) = |x| + |x - 1|$. Then $Rf'(0) = 0$.
13. Rolle's theorem is not applicable for the function $f(x) = x(x + 2)e^{-x/2}$ in $[-2, 0]$.

14. The value of 'c' of Lagrange's mean value theorem for the function $f(x) = 2x^2 + 3x + 4$ in $[1, 2]$ is given by $c = \frac{5}{4}$.

15. If $f(x) = x^n$, then $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h} = n\, x^{n-1}$.

16. If $f(x) = \cos x$, then $\lim\limits_{x \to a} \frac{f(x) - f(a)}{x - a} = -\sin a$.

17. If $f(x) = e^x$, then $\lim\limits_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = e^x$.

Answers

Multiple Choice Questions

1.	(c)	**2.**	(b)	**3.**	(a)	**4.**	(a)	**5.**	(c)
6.	(b)	**7.**	(b)	**8.**	(a)	**9.**	(b)		

Fill in the Blank(s)

1.	$f(a)$	**2.**	h	**3.**	$-h$	**4.**	$]a, b[$	**5.**	the x-axis
6.	sufficient	**7.**	0	**8.**	Rolle's theorem			**9.**	$f'(c)$
10.	$g(b) - g(a)$			**11.**	increasing	**12.**	$\cos x$		

True or False

1.	F	**2.**	T	**3.**	F	**4.**	F	**5.**	T
6.	F	**7.**	F	**8.**	T	**9.**	F	**10.**	F
11.	T	**12.**	T	**13.**	F	**14.**	F	**15.**	T
16.	T	**17.**	F						

□

Chapter 6

More About Continuity and Darboux Theorem

In the present chapter we shall study the sequential continuity, boundedness and intermediate value properties of continuous functions, uniform continuity, Meaning of sign of derivative and Darboux theorem.

1 Criteria for Continuity or Equivalent Definitions of Continuity

Theorem 1: (Heine's definition of continuity). Sequential Continuity: *The necessary and sufficient condition for a function f defined on $I \subset \mathbf{R}$ to be continuous at $a \in I$ is that for each sequence $< a_n >$ in I which converges to a, we have*

$$\lim_{n \to \infty} f(a_n) = f(a) .$$

Proof: The condition is necessary: Let f be continuous at a and let $< a_n >$ be a sequence in I such that $\lim\limits_{n \to \infty} a_n = a$.

Let ε be any positive number. Since f is continuous at a, therefore, for a given $\varepsilon > 0$, we can find a number $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \text{ whenever } |x - a| < \delta. \qquad \ldots(1)$$

Also, since $\lim\limits_{n \to \infty} a_n = a$, therefore, there exists a positive integer m such that

$$|a_n - a| < \delta \text{ whenever } n > m. \qquad \ldots(2)$$

Setting $x = a_n$ in (1), we get

$$|f(a_n) - f(a)| < \varepsilon \text{ whenever } |a_n - a| < \delta. \qquad \ldots(3)$$

From (2) and (3), we get

$$|f(a_n) - f(a)| < \varepsilon \text{ whenever } n > m.$$

$\therefore$
$$\lim_{n \to \infty} f(a_n) = f(a).$$

Hence the condition is necessary.

The condition is sufficient:

Suppose for every sequence $< a_n >$ in I converging to a, we have

$$\lim_{n \to \infty} f(a_n) = f(a).$$

Then we have to show that f is continuous at a.

Let us suppose that f is not continuous at a. Then there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there is an x such that $|x - a| < \delta$ but $|f(x) - f(a)| \geq \varepsilon$.

If we take $\delta = 1/n$, we find that for each positive integer n, there is an a_n such that

$$|a_n - a| < 1/n \text{ but } |f(a_n) - f(a)| \geq \varepsilon.$$

Then $\lim\limits_{n \to \infty} a_n = a$, but $f(a_n)$ does not converge to $f(a)$ *i.e.,*

$$\lim_{n \to \infty} f(a_n) \neq f(a).$$

But this is a contradiction. Hence f must be continuous at $x = a$.

Illustrative Examples

Example 1: *A function f defined on $[0, 1]$ is given by*

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational}. \end{cases}$$

Show that f takes every value between 0 and 1 (both inclusive), but it is continuous only at the point $x = \frac{1}{2}$.

Solution: Let $c \in [0, 1]$.

If c is rational, then $f(c) = c$.

If c is irrational, then $1 - c$ is also irrational

and $\qquad 0 < 1 - c < 1$ *i.e.,* $1 - c \in [0, 1]$.

We have $f(1 - c) = 1 - (1 - c) = c$.

Thus f takes every value c in $[0, 1]$.

Now to show that f is continuous only at the point $x = \frac{1}{2}\cdot$

Let x_0 be any point of $[0,1]$. For each positive integer n we select a rational number a_n and an irrational number b_n, both in $[0,1]$, such that

$$|a_n - x_0| < 1/n, |b_n - x_0| < 1/n.$$

$$\therefore \qquad \lim_{n \to \infty} a_n = x_0 = \lim_{n \to \infty} b_n.$$

If f is to be continuous at x_0, then we must have

$$\lim_{n \to \infty} f(a_n) = f(x_0) = \lim_{n \to \infty} f(b_n).$$

Now $f(a_n) = a_n$ for all n and $f(b_n) = 1 - b_n$ for all n.

$$\therefore \qquad \lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} a_n = x_0$$

and
$$\lim_{n \to \infty} f(b_n) = \lim_{n \to \infty} (1 - b_n) = 1 - x_0.$$

So for f to be continuous at x_0, we must have

$$x_0 = f(x_0) = 1 - x_0 \quad i.e., \quad x_0 = \frac{1}{2}\cdot$$

Thus $x = \frac{1}{2}$ is the only possible point of $[0, 1]$ where f can be continuous.

Now we shall show that f is actually continuous at the point $x = 1/2$.

We have $f(1/2) = \frac{1}{2}\cdot$

Let $\varepsilon > 0$ be given.

Take a positive real number $\delta = \frac{1}{2}\varepsilon$. Then if x is rational, we have

$$\left|x - \frac{1}{2}\right| < \delta \Rightarrow \left|f(x) - f\left(\frac{1}{2}\right)\right| = \left|x - \frac{1}{2}\right| < \delta = \frac{1}{2}\varepsilon < \varepsilon$$

and if x is irrational, we have

$$\left|x - \frac{1}{2}\right| < \delta \Rightarrow \left|f(x) - f\left(\frac{1}{2}\right)\right| = \left|(1 - x) - \frac{1}{2}\right| = \left|x - \frac{1}{2}\right| < \delta = \frac{1}{2}\varepsilon < \varepsilon.$$

Thus, we have

$$\left|x - \frac{1}{2}\right| < \delta \Rightarrow \left|f(x) - f\left(\frac{1}{2}\right)\right| < \varepsilon,$$

so that f is continuous at $x = \frac{1}{2}\cdot$

Hence f is continuous only at the point $x = 1/2$.

Example 2: *Let f be a function defined on* $]0,1[$ *by setting* $f(x) = 0$ *when x is irrational and* $f(x) = 1/q$ *when x is a rational number of the form* p/q *where p and q are positive integers having no factor in common. Show that f is continuous at each irrational point and discontinuous at each rational point.*

Solution: First, let $a = p/q$ be any rational number in $]0,1[$, where p and q are positive integers having no factor in common. For each positive integer n, we choose a positive irrational number x_n such that $|x_n - a| < 1/n$. Then the sequence $< x_n >$ converges to a. Also it is given that $f(x_n) = 0$ for each n, so that $\lim_{n \to \infty} f(x_n) = 0 \neq f(a)$, since $f(a) = f(p/q) = 1/q \neq 0$.

It follows from the preceding theorem that f is not continuous at a rational point.

Next, let b be an irrational number in $]0,1[$ so that $f(b) = 0$. Let $\varepsilon > 0$ be given. Choose a positive integer n such that $1/n < \varepsilon$. Evidently there can be only a finite number of rational numbers p/q in $]0,1[$ such that $q < n$. Hence, we can find a number $\delta > 0$ such that no rational number in $]b - \delta, b + \delta[\subset]0,1[)$ has its denominator less than n.

Thus we have shown that

$$|x - b| < \delta \Rightarrow |f(x) - f(b)| = |f(x)| = 0 < \varepsilon, \quad \ldots(1)$$

if x is irrational

and $$|x - b| < \delta \Rightarrow |f(x) - f(b)| = |f(x)| \leq 1/n < \varepsilon, \quad \ldots(2)$$

if x is rational.

From (1) and (2), it follows that $|f(x) - f(b)| < \varepsilon$ whenever $|x - b| < \delta$.

Hence f is continuous at b.

Theorem 2: *A function $f : \mathbf{R} \to \mathbf{R}$ is continuous iff for every open set G in $\mathbf{R}$, the inverse image $f^{-1}(G)$ is an open set in $\mathbf{R}$.*

Proof: The 'only if' part: Let f be continuous and let G be any open set in $\mathbf{R}$. If $f^{-1}(G)$ is empty, then it is open. If $f^{-1}(G)$ is not empty, let $a \in f^{-1}(G)$. Then $f(a) \in G$. Since G is an open set containing $f(a)$, therefore, there exists an $\varepsilon > 0$ such that

$$]f(a) - \varepsilon, f(a) + \varepsilon[\subset G.$$

Now f is continuous at a, so we can find a number $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

i.e., $$x \in]a - \delta, a + \delta[\Rightarrow f(x) \in]f(a) - \varepsilon, f(a) + \varepsilon[\subset G$$

or $$x \in]a - \delta, a + \delta[\Rightarrow f(x) \in G$$
$$\Rightarrow x \in f^{-1}(G)$$

Hence $]a - \delta, a + \delta[\subset f^{-1}(G)$. Thus $f^{-1}(G)$ is a neighbourhood of a. Since a is any point of $f^{-1}(G)$, therefore, it follows that $f^{-1}(G)$ is open.

The 'if' part: Let the inverse image $f^{-1}(G)$ of every open set G be open. To show that f is continuous. Consider any point $a \in \mathbf{R}$. Let $\varepsilon > 0$ be given. Then $]f(a) - \varepsilon, f(a) + \varepsilon[$ is an open set containing $f(a)$ and hence by hypothesis $f^{-1}(]f(a) - \varepsilon, f(a) + \varepsilon[)$ is an open set containing a. Consequently, we can find a number $\delta > 0$ such that

$$]a - \delta, a + \delta[\subset f^{-1}(]f(a) - \varepsilon, f(a) + \varepsilon[), \text{ so that}$$

$$f(]a - \delta, a + \delta[) \subset]f(a) - \varepsilon, f(a) + \varepsilon[.$$

Thus, for any given $\varepsilon > 0$ we have found a number $\delta > 0$ such that

$$x \in]\, a - \delta, a + \delta\, [\Rightarrow f(x) \in]\, f(a) - \varepsilon, f(a) + \varepsilon\, [\text{ i.e.,}$$
$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

This shows that f is continuous at a. Since a is any point of **R**, hence, f is continuous on **R**.

Theorem 3: *A function $f : \mathbf{R} \to \mathbf{R}$ is continuous on **R** iff for every closed set H in **R**, $f^{-1}(H)$ is closed in **R**.*

Proof: First, let f be continuous and let H be any closed set in **R**. Then $\mathbf{R} \sim H$ is an open set in **R**, so that by the preceding theorem, $f^{-1}(\mathbf{R} \sim H)$ is an open set in **R**. Since

$$f^{-1}(\mathbf{R} \sim H) = \mathbf{R} \sim f^{-1}(H),$$

hence it follows that $\mathbf{R} \sim f^{-1}(H)$ is an open set in **R** and consequently $f^{-1}(H)$ is a closed set in **R**.

Conversely, let $f^{-1}(H)$ be closed for every closed set H. To show that f is continuous. Let G be any open set in **R**. Then $\mathbf{R} \sim G$ is a closed set in **R** and hence by hypothesis $f^{-1}(\mathbf{R} \sim G)$ is a closed set in **R**.

Since $f^{-1}(\mathbf{R} \sim G) = \mathbf{R} \sim f^{-1}(G)$, therefore, this means that $\mathbf{R} \sim f^{-1}(G)$ is a closed set in **R** and hence $f^{-1}(G)$ is an open set in **R**. Thus $f^{-1}(G)$ is open in **R** whenever G is open in **R**. Therefore f is continuous by the preceding theorem.

2 Algebra of Continuous Functions

Theorem 1: *Let f and g be defined on an interval I. If f and g are continuous at $a \in I$, then $f + g$ is also continuous at a.*

Proof: Let $< a_n >$ be any sequence in I converging to a. Since f and g are both continuous at $a \in I$, therefore

$$\lim_{n \to \infty} f(a_n) = f(a) \text{ and } \lim_{n \to \infty} g(a_n) = g(a).$$

We have,
$$\lim_{n \to \infty} (f + g)(a_n) = \lim_{n \to \infty} \{f(a_n) + g(a_n)\}$$
$$= \lim_{n \to \infty} f(a_n) + \lim_{n \to \infty} g(a_n)$$
$$= f(a) + g(a) = (f + g)(a),$$

which shows that $f + g$ is continuous at a.

Aliter: Since both f and g are continuous at a, therefore

$$f(a) = \lim_{x \to a} f(x) \text{ and } g(a) = \lim_{x \to a} g(x)$$

Now
$$\lim_{x \to a} (f + g)(x) = \lim_{x \to a} \{f(x) + g(x)\}$$

[$\because$ by def. of the function $f + g$, we have $(f + g)(x) = f(x) + g(x)$]

$$= \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

[$\because$ limit of a sum = sum of the limits provided both the limits exist]

$$= f(a) + g(a) = (f + g)(a).$$

Since $\lim_{x \to a} (f + g)(x) = (f + g)(a)$, therefore $f + g$ is continuous at a.

Theorem 2: *Let f and g be defined on an interval I. If f and g are continuous at $a \in I$, then fg is continuous at a.*

Proof: Let $< a_n >$ be any sequence in I converging to a. Since f and g are both continuous at $a \in I$, therefore,

$$\lim_{n \to \infty} f(a_n) = f(a) \text{ and } \lim_{n \to \infty} g(a_n) = g(a).$$

We have, $\lim_{n \to \infty} (fg)(a_n) = \lim_{n \to \infty} \{f(a_n) g(a_n)\}$

$$= \lim_{n \to \infty} f(a_n) . \lim_{n \to \infty} g(a_n)$$

$$= f(a) . g(a) = (f g)(a),$$

which shows that $f g$ is continuous at a.

Alternative Proof: Since both f and g are continuous at a, therefore

$$f(a) = \lim_{x \to a} f(x) \text{ and } g(a) = \lim_{x \to a} g(x).$$

Now $\lim_{x \to a} (fg)(x) = \lim_{x \to a} f(x) . g(x)$

[$\because$ by def. of the function fg, we have $(fg)(x) = f(x) g(x)$]

$$= \lim_{x \to a} f(x) . \lim_{x \to a} g(x)$$

[$\because$ limit of a product = product of the limits provided both the limits exist]

$$= f(a) g(a) = (fg)(a).$$

Since $\lim_{x \to a} (f g)(x) = (f g)(a)$, therefore fg is continuous at a.

Theorem 3: *If f is continuous at a point a and $c \in \mathbf{R}$, then cf is continuous at a.*

Proof: Let $< a_n >$ be any sequence in I converging to a. Since f is continuous at a, therefore

$$\lim_{n \to \infty} f(a_n) = f(a).$$

We have, $\lim_{n \to \infty} (c f)(a_n) = \lim_{n \to \infty} c f(a_n) = c \lim_{n \to \infty} f(a_n)$

$$= c f(a) = (c f)(a),$$

which shows that cf is continuous at a.

Alternative Proof: Since f is continuous at a, therefore $f(a) = \lim_{x \to a} f(x)$.

Now
$$\lim_{x \to a} (cf)(x) = \lim_{x \to a} cf(x)$$

[$\because$ by def. of the function cf, we have $(cf)(x) = cf(x)$]

$$= c \lim_{x \to a} f(x) = cf(a) = (cf)(a).$$

Since $\lim_{n \to \infty} (cf)(x) = (cf)(a)$, therefore cf is continuous at a.

Theorem 4: *Let f and g be defined on an interval I, and let $g(a) \neq 0$. If f and g are continuous at $a \in I$, then f/g is continuous at a .*

Proof: Let $< a_n >$ be any sequence in I converging to a.

Since f and g are continuous at a, therefore
$$\lim_{n \to \infty} f(a_n) = f(a) \text{ and } \lim_{n \to \infty} g(a_n) = g(a).$$

Also, since $g(a) \neq 0$, therefore, there exists a positive integer m such that $g(a_n) \neq 0$, whenever $n > m$.

We have,
$$\lim_{n \to \infty} (f/g)(a_n) = \lim_{n \to \infty} \{f(a_n)/g(a_n)\}$$
$$= \lim_{n \to \infty} f(a_n) / \lim_{n \to \infty} g(a_n)$$
$$= f(a)/g(a) = (f/g)(a)$$

which shows that f/g is continuous at a.

Aliter: Since both f and g are continuous at a, therefore
$$f(a) = \lim_{x \to a} f(x) \text{ and } g(a) = \lim_{x \to a} g(x)$$

Since $g(a) \neq 0$ and $g(x)$ is continuous at $x = a$, therefore there exists a neighbourhood of a at each point of which $g(x) \neq 0$.

Now
$$\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \lim_{x \to a} \frac{f(x)}{g(x)}$$

[$\because$ by def. of the function f/g we have $(f/g)(x) = f(x)/g(x)$]

$$= \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

[$\because$ limit of a quotient = quotient of the limits provided both the limits exist and the limit of the denominator is not zero]

$$= \frac{f(a)}{g(a)} = \left(\frac{f}{g}\right)(a).$$

Since $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \left(\frac{f}{g}\right)(a)$, therefore $\frac{f}{g}$ is continuous at a.

Theorem 5: *If f is continuous at a then $|f|$ is also continuous at a.*

Proof: Let $< a_n >$ be any sequence in I converging to a. Since f is continuous at a, therefore

$$\lim_{n \to \infty} f(a_n) = f(a)$$

Now $$\lim_{n \to \infty} |f|(a_n) = \lim_{n \to \infty} |f(a_n)| = \left| \lim_{n \to \infty} f(a_n) \right|$$

$$= |f(a)| = |f|(a), \text{ showing that } |f| \text{ is continuous at } a.$$

Aliter: It is given that f is continuous at a. To prove that $|f|$ is also continuous at a.

Take any given $\varepsilon > 0$.

Since f is continuous at a, therefore there exists $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon, \text{ whenever } |x - a| < \delta. \quad \text{...(1)}$$

Now if x and y are any real numbers, we know that

$$||x| - |y|| \leq |x - y|.$$

$\therefore$ $$||f(x)| - |f(a)|| \leq |f(x) - f(a)| \quad \text{...(2)}$$

From (1) and (2), we have

$$||f(x)| - |f(a)|| < \varepsilon, \text{ whenever } |x - a| < \delta$$

$\Rightarrow$ $$||f|(x) - |f|(a)| < \varepsilon, \text{ whenever } |x - a| < \delta.$$

[$\because$ by def. of the function $|f|$, we have $|f|(x) = |f(x)|$]

Thus for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$||f|(x) - |f|(a)| < \varepsilon, \text{ whenever } |x - a| < \delta.$$

$\therefore$ by definition of continuity of a function at a point, the function $|f|$ is continuous at a.

Note: The converse is not true. For example, if

$$f(x) = -1, \text{ for } x < a \text{ and } f(x) = 1 \text{ for } x \geq a \text{ then}$$

$$\lim_{x \to a} |f(x)| = 1 = |f(a)|, \text{ but } \lim_{x \to a} f(x) \text{ does not exist.}$$

Thus $|f|$ is continuous at a while f is not continuous at a.

Theorem 6: *Let f and g be defined on an interval I. If both f and g are continuous at $a \in I$, then the functions max. $\{f, g\}$ and min. $\{f, g\}$ are both continuous at a.*

Proof: We have $\max\{f, g\} = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$,

$$\text{min. } \{f, g\} = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|.$$

The theorem now follows from theorems 1, 3, 5, of article 2.

Theorem 7: *Let f and g be defined on intervals I and J respectively and let $f(I) \subset J$. If f is continuous at $a \in I$ and g is continuous at $f(a)$, then the composite map $g \circ f$ is continuous at a.*

Proof: Let $< a_n >$ be any sequence in I converging to a. Then $< f(a_n) >$ converges to $f(a)$, f being continuous at a. Again, since $f(I) \subset J$, hence $< f(a_n) >$ is a sequence in J.

Now g being continuous at $f(a)$ and $< f(a_n) >$ is a sequence in J converging to $f(a)$, therefore $< g(f(a_n)) >$ converges to $g(f(a))$ *i.e.,* $< (g \circ f)(a_n) >$ converges to $(g \circ f)(a)$.

Thus $< a_n > \to a \Rightarrow < (g \circ f)(a_n) > \to (g \circ f)(a)$.

Hence $g \circ f$ is continuous at a.

3 Boundedness and Intermediate Value Properties of Continuous Functions

Theorem 1: (Borel's Theorem) *If f is a continuous function on the closed interval $[a, b]$, then the interval can always be divided up into a finite number of subintervals such that, given $\varepsilon > 0$, $|f(x_1) - f(x_2)| < \varepsilon$, where x_1 and x_2 are any two points in the same sub-interval.*

Proof: Let us assume that the theorem is false. Then for any mode of sub-division, there must be at least one of the subintervals for which the theorem is false. Let us divide $[a, b]$ into two equal parts. Let c be the point of division. Then the theorem must be false in at least one of the two parts. Suppose it is false in $[c, b]$. Renaming the interval $[c, b]$ as $[a_1, b_1]$, we divide $[a_1, b_1]$ into two equal parts. Again, the theorem must be false in at least one of these two parts. Continuing this process of repeated bisection indefinitely, we get a sequence of closed intervals

$[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n], \ldots$ such that

$$a \le a_1 \le a_2 \le \ldots \le a_n \le \ldots \le b_n \le \ldots \le b_2 \le b_1 \le b, \qquad \ldots(1)$$

$$b_1 - a_1 = \frac{1}{2}(b-a), b_2 - a_2 = \frac{1}{2}(b_1 - a_1) = \frac{1}{2^2}(b-a), \ldots\ldots,$$

$$b_n - a_n = \frac{1}{2^n}(b-a), \ldots \qquad \ldots(2)$$

and the theorem is false for each interval $[a_n, b_n]$. ...(3)

From (1), we find that the sequence $< a_n >$ is increasing and bounded above by b and $< b_n >$ is decreasing and bounded below by a. Hence both $< a_n >$ and $< b_n >$ are convergent. So there exist $l_1, l_2 \in \mathbf{R}$ such that $\lim a_n = l_1$ and $\lim b_n = l_2$.

From (2), $\lim (b_n - a_n) = \lim \frac{1}{2^n}(b-a) = (b-a) \lim \frac{1}{2^n} = (b-a) . 0 = 0$

i.e., $\lim b_n - \lim a_n = 0$ *i.e.,* $l_2 - l_1 = 0$ *i.e.,* $l_2 = l_1$.

Now $l_1 = \sup < a_n >$,

$< a_n >$ being an increasing and bounded above sequence

$= \inf < b_n >$,

$< b_n >$ being a decreasing and bounded below sequence.

$\therefore$ $a_n \le l_1$ and $l_1 \le b_n$ *i.e*, $a_n \le l_1 \le b_n \ \forall\, n \in \mathbf{N}$

$\Rightarrow$ $a \le l_1 \le b$ *i.e.,* either $a < l_1 < b$ or $a = l_1$ or $b = l_1$.

Case I: $a < l_1 < b$. Since f is continuous in $[a, b]$, it is also continuous at $x = l_1$. Then for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(l_1)| < \frac{1}{2}\varepsilon, \text{ whenever } |x - l_1| < \delta.$$

Now we can choose n so large that the interval $[a_n, b_n]$ lies completely inside the interval $]l_1 - \delta, l_1 + \delta[$.

Hence if x_1 and x_2 are any two points in $[a_n, b_n]$, then

$$|f(x_1) - f(l_1)| < \varepsilon/2, |f(x_2) - f(l_1)| < \varepsilon/2.$$

Now
$$\begin{aligned} |f(x_1) - f(x_2)| &= |f(x_1) - f(l_1) + f(l_1) - f(x_2)| \\ &\le |f(x_1) - f(l_1)| + |f(l_1) - f(x_2)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \Rightarrow \text{the result is true in } [a_n, b_n]. \end{aligned}$$

This is a contradiction to the fact (3) and so our initial assumption that the theorem is not true is wrong. Hence the theorem must be true.

Similarly we can prove the cases for $l_1 = a$ or $l_1 = b$.

Note: The possibility $l_1 = a$ or $l_1 = b$ shows that the interval considered in the theorem must be closed so that f is continuous at l_1.

Theorem 2: (Boundedness Theorem) *If a function $f(x)$ is continuous in a closed interval $[a, b]$, then it is bounded in that interval.*

Proof: By the above theorem, for a given $\varepsilon > 0$, we can sub-divide the interval $[a, b]$ into a finite number of sub-intervals say $[a = a_0, a_1], [a_1, a_2], \ldots, [a_{n-1}, a_n = b]$ such that

$$|f(x_1) - f(x_2)| < \varepsilon \qquad \ldots(1)$$

for any two points x_1, x_2 in the same sub-interval. Let x be any point in the first sub-interval $[a, a_1]$. Then by (1), we have, $\forall\, x \in [a, a_1]$

$$|f(x) - f(a)| < \varepsilon \text{ i.e., } f(a) - \varepsilon < f(x) < f(a) + \varepsilon. \qquad \ldots(2)$$

In particular, for $x = a_1$, $|f(a_1) - f(a)| < \varepsilon$. ...(3)

Again $\forall\, x \in [a_1, a_2], |f(x) - f(a_1)| < \varepsilon$. ...(4)

$\therefore$ $\forall\, x \in [a_1, a_2]$, we have

$$\begin{aligned} |f(x) - f(a)| &= |f(x) - f(a_1) + f(a_1) - f(a)| \\ &\le |f(x) - f(a_1)| + |f(a_1) - f(a)| \\ &< \varepsilon + \varepsilon, \text{ from (3) and (4)} \\ &= 2\varepsilon. \end{aligned}$$

Thus $\forall\, x \in [a_1, a_2]$, we have $|f(x) - f(a)| < 2\varepsilon$

i.e.,
$$f(a) - 2\varepsilon < f(x) < f(a) + 2\varepsilon. \qquad \ldots(5)$$

From (2) and (5), we see that all the values of $f(x)$ in the first two sub-intervals lie between

$$f(a) - 2\varepsilon \text{ and } f(a) + 2\varepsilon.$$

Proceeding in the same way, we can show that $\forall\, x \in [a_{n-1}, a_n = b]$, we have

$$f(a) - n\varepsilon < f(x) < f(a) + n\varepsilon.$$

Hence all the values of $f(x)$ in the interval $[a, b]$ will lie between $f(a) - n\varepsilon$ and $f(a) + n\varepsilon$.

Thus $f(x)$ is bounded in $[a, b]$.

Note: A bounded function in $[a, b]$ need not be continuous in $[a, b]$. For example, the function

$$f(x) = \sin(1/x) \text{ for } x \neq 0, \ f(0) = 0$$

is bounded in $[0, 1]$ but not continuous in $[0, 1]$ since it is discontinuous at $x = 0$.

Theorem 3: (The Mostest Theorem) *If a function $f(x)$ is continuous in $[a, b]$, then it attains its supremum and infimum at least once in $[a, b]$.*

Proof: Since $f(x)$ is continuous in $[a, b]$, $f(x)$ is bounded in $[a, b]$. Let M and m be the supremum and infimum of f in $[a, b]$ respectively. We shall show that f attains its supremum M at least once in this interval. Let, if possible, $f(x)$ does not attain M, then

$$f(x) \neq M \ \forall \ x \in [a, b] \quad i.e., \quad M - f(x) \neq 0 \ \forall \ x \in [a, b].$$

Now, M, being a constant, is always a continuous function and f is given to be continuous in $[a, b]$.

$\therefore$ $\qquad M - f(x)$ is also continuous in $[a, b]$

$\Rightarrow$ $\qquad \dfrac{1}{M - f(x)}$ is also continuous in $[a, b]$. $\qquad [\because M - f(x) \neq 0 \ \forall \ x \in [a, b]]$

Consequently $1/\{M - f(x)\}$ is bounded in $[a, b]$. Let $G \in \mathbf{R}_+$ be its upper bound in $[a, b]$ so that $\forall \ x \in [a, b]$

$$\frac{1}{M - f(x)} \leq G \ i.e., \ M - f(x) \geq \frac{1}{G} \quad i.e., \quad f(x) \leq M - (1/G) < M$$

i.e., $\qquad M - (1/G) < M$ is also an upper bound of f.

But this contradicts the fact that M is the supremum of f in $[a, b]$. Hence $f(x) = M$ for at least one value of x in $[a, b]$. Similarly we can show that f attains its infimum at least once in $[a, b]$.

Theorem 4: *If f is continuous at $x = x_0$ where $f(x_0) \neq 0$, then a positive number δ can be found such that $f(x)$ has the same sign as $f(x_0)$ for every value of x in $]x_0 - \delta, x_0 + \delta[$.*

Proof: Since f is continuous at $x = x_0$, hence for a given $\varepsilon > 0$, we can find a number $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$, whenever $|x - x_0| < \delta$

i.e., $\qquad f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$, whenever $x_0 - \delta < x < x_0 + \delta$.

Now $\qquad f(x_0) \neq 0 \Rightarrow |f(x_0)| > 0$.

If we choose ε such that $0 < \varepsilon < |f(x_0)|$, we see that $f(x_0) - \varepsilon$ and $f(x_0) + \varepsilon$ have the same sign as $f(x_0)$. It implies that $f(x)$ has the same sign as $f(x_0)$ for all x in the interval $]x_0 - \delta, x_0 + \delta[$.

Theorem 5: ((Bolzano's Theorem)) *If f is continuous in $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then there is at least one value of x for which $f(x)$ vanishes.*

Proof: For definiteness, let $f(a) < 0$ and $f(b) > 0$. Consider a subset S of $[a, b]$ defined as follows :

$$S = \{x : a \leq x \leq b \text{ and } f(x) < 0\}.$$

Then $S \neq \varnothing$, since $a \in S$, $f(a)$ being < 0. By definition, b is an upper bound of S. It follows that S has a supremum, say u, by the completeness property of real numbers. Obviously $a \leq u \leq b$. Now, we shall show that $f(u) = 0$.

Step I: First we shall show that $u \neq a$. Since $f(a) < 0$, we can find a positive number δ such that $f(x) < 0$ whenever $a \leq x < a + \delta$.

It follows that $[a, a+\delta[\; S$ and hence the supremum of S must be greater than or equal to $a + \delta$.

$\therefore$ $$u \geq a + \delta \text{ i.e., } u \neq a.$$

Step II: We shall show that $u \neq b$. In fact, since $f(b) > 0$, therefore, there exists a positive number δ_1 such that $f(x) > 0$ whenever $b - \delta_1 < x \leq b$.

It gives that $b - \delta_1$ is an upper bound of S, and hence

$$u = \sup S \leq b - \delta_1 < b \quad \text{i.e., } u < b.$$

Step III: We shall show that $f(u) \ngtr 0$. Since $a < u < b$, therefore, if $f(u) > 0$, then we can find a positive number δ_2 such that $f(x) > 0$ whenever

$$u - \delta_2 < x < u + \delta_2 .$$

Also, since $u = \sup S$, there exists $x_0 \in S$ such that $u - \delta_2 < x_0 \leq u$. This means that $f(x_0) > 0$.

Again, $x_0 \in S \Rightarrow f(x_0) < 0$, by definition of S. This contradiction implies that $f(u) \ngtr 0$.

Step IV: We shall show that $f(u) \nless 0$. Since, if $f(u) < 0$, then we can find a positive number δ_3 such that $u + \delta_3 < b$ and $f(x) < 0$ whenever $u - \delta_3 < x < u + \delta_3$.

If x_1 is any point such that $u < x_1 < u + \delta_3$, then $f(x_1) < 0$. But this is a contradiction to the fact that u is the supremum of all those points of $[a, b]$ for which f is negative. Consequently $f(u) \nless 0$.

It follows from steps III and IV that $f(u) = 0$.

Theorem 6: (The Intermediate Value Theorem) *If a function f is continuous in the closed interval $[a, b]$, then $f(x)$ must take at least once all values between $f(a)$ and $f(b)$.*

Proof: Let $f(a) < d < f(b)$. Let us define a function F such that

$$F(x) = f(x) - d.$$

Since f is continuous in the closed interval $[a, b]$, F must also be continuous in $[a, b]$. Also $F(a) < 0$ and $F(b) > 0$ *i.e.,* $F(a)$ and $F(b)$ are of opposite signs. It follows that there exists x_0 in $]a, b[$ such that $F(x_0) = 0$ *i.e.,* $f(x_0) - d = 0$ or $f(x_0) = d$.

Since d is any value between $f(a)$ and $f(b)$ it follows that f takes all values between $f(a)$ and $f(b)$ at least once.

The converse of the above theorem is not true. For example, let f be the function defined as $f(x) = \sin(1/x), x \neq 0$ and $f(0) = 0$.

In the interval $[-2/\pi, 2/\pi]$ this function takes all values between $f(-2/\pi)$ and $f(2/\pi)$ *i.e.,* between -1 and 1 an infinite number of times as x varies from $-2/\pi$ to $2/\pi$, but this function is not continuous in $[-2/\pi, 2/\pi]$ as it is discontinuous at $x = 0$.

Corollary 1: *Let f be continuous on $[a, b]$ and let $k \in [m, M]$ where $m = \inf f$ and $M = \sup f$ on $[a, b]$. Then there exists $c \in [a, b]$ such that $f(c) = k$.*

Proof: Since f is continuous on $[a, b]$ and every function defined and continuous on a closed interval attains its supremum and infimum, therefore, there exist $x_1, x_2 \in [a, b]$ such that

$$m = f(x_1) \text{ and } M = f(x_2).$$

If $x_1 = x_2$, then f is a constant function on $[a, b]$ and the result follows.
Let $x_1 < x_2$. Then $[x_1, x_2] \subset [a, b]$ and f is continuous on $[x_1, x_2]$.
Hence by the above theorem there exists $c \in [x_1, x_2] \subset [a, b]$ such that $f(c) = k$.
Similarly we can prove the result if $x_1 > x_2$.

Corollary 2: *Let f be continuous on $[a, b]$. Then $f([a, b]) = [m, M]$, where $m = \inf f$ and $M = \sup f$ on $[a, b]$ and thus $f([a, b])$ is a closed set.*

Proof: By Corollary 1, f takes all values between m and M and hence $[m, M] \subset f([a, b])$. Since every value of f on $[a, b]$ lies between m and M, hence, $f([a, b]) \subset [m, M]$. Thus $f([a, b]) = [m, M]$ which is a closed set because every closed interval is a closed set.

4 Uniform Continuity

Let a function f be continuous for every value of x in $[a, b]$. It means that if $x_0 \in [a, b]$, then, given $\varepsilon > 0$, there exists $\delta > 0$, such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$.

The number δ will depend upon x_0 as well as ε and so we may write it symbolically as $\delta(\varepsilon, x_0)$. For some functions it may happen that given $\varepsilon > 0$ the same δ serves for all $x_0 \in [a, b]$ in the condition of continuity. Such functions are called uniformly continuous on $[a, b]$.

Definition: *A function f defined on an interval I is said to be uniformly continuous on I if given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$, where x, y are in I.*

(Meerut 2001)

It should be noted carefully that uniform continuity is a property associated with an interval and not with a single point. The concept of continuity is local in character whereas the concept of uniform continuity is global in character.

Note: A function f is not uniformly continuous on I, if there exists some $\varepsilon > 0$ for which no $\delta > 0$ serves *i.e.,* for any $\delta > 0$, there exist $x, y \in I$ such that $|f(x) - f(y)| \geq \varepsilon$ and $|x - y| < \delta$.

The following two theorems express the relation between continuity and uniform continuity. The first one gives that uniform continuity always implies continuity. The second one gives a sufficient condition under which continuity implies uniform continuity.

Theorem 1: *If f is uniformly continuous on an interval I, then it is continuous on I.*

Proof: Let x_0 be any point of I and let $\varepsilon > 0$ be given. Since f is uniformly continuous on I, therefore, given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta, \ \forall\, x, y \text{ in } I.$$

Taking $y = x_0$, we have in particular,

$$|f(x) - f(x_0)| < \varepsilon \text{ whenever } |x - x_0| < \delta \ \ \forall\, x \text{ in } I.$$

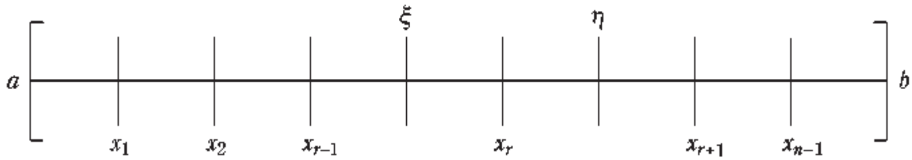
This means that f is continuous at x_0.
Since x_0 is an arbitrary point of I, therefore f is continuous at every point of I.
Hence f is continuous on I.

Theorem 2: *A function which is continuous in a closed and bounded interval $I = [a, b]$ is uniformly continuous in $[a, b]$.*

Proof: Take any given $\varepsilon > 0$. By theorem 1 of article 3, the interval $[a, b]$ can be divided up into sub-intervals $[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]$ such that for any two points α, β in the same sub-interval, we have $|f(\alpha) - f(\beta)| < \frac{1}{2}\varepsilon$. ...(1)

Let δ be a positive number which does not exceed the least of the numbers

$$x_1 - a, x_2 - x_1, \ldots, b - x_{n-1}.$$



Let ξ, η be any two points in $[a, b]$ such that $|\xi - \eta| \not> \delta$.

If these two points are in the same sub-interval, then by (1), we have

$$|f(\xi) - f(\eta)| < \frac{1}{2}\varepsilon.$$

If ξ, η do not lie in the same sub-interval, then surely they lie one in each of the two consecutive intervals. Let x_r be the point of division such that $x_{r-1} < \xi < x_r < \eta < x_{r+1}$. Then we have

$$\begin{aligned} |f(\xi) - f(\eta)| &= |f(\xi) - f(x_r) + f(x_r) - f(\eta)| \\ &\le |f(\xi) - f(x_r)| + |f(x_r) - f(\eta)| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \text{ by (1).} \end{aligned}$$

Thus we have shown that, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(\xi) - f(\eta)| < \varepsilon \text{ whenever } |\xi - \eta| < \delta, \ \forall\, \xi, \eta \text{ in } [a, b].$$

Hence f is uniformly continuous in $[a, b]$.

Illustrative Examples

Example 3: *Give an example to show that a function continuous in an open interval may fail to be uniformly continuous in the interval.*

Solution: Consider the function f defined on the open interval $]0, 1[$ as follows :

$$f(x) = 1/x, \forall\, x \in\,]0, 1[.$$

First we shall show that f is continuous in $]0, 1[$ *i.e.*, f is continuous at each point c in $]0, 1[$.

We have $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} \frac{1}{x} = \frac{1}{c} = f(c)$.

So $f(x)$ is continuous at each point c in $]0, 1[$ and hence $f(x)$ is continuous in $]0, 1[$.

Now we shall show that $f(x)$ is not uniformly continuous in $]0, 1[$.

For any $\delta > 0$, we can find a positive integer m such that $1/m < \delta$.

Let $x_1 = 1/m$ and $x_2 = 1/2m$. Then $0 < x_1 < 1$ and $0 < x_2 < 1$ so that $x_1, x_2 \in]0,1[$.

We have $|x_1 - x_2| = \left|\frac{1}{m} - \frac{1}{2m}\right| = \left|\frac{1}{2m}\right| = \frac{1}{2m} < \frac{1}{m} < \delta$

and $$|f(x_1) - f(x_2)| = \left|\frac{1}{x_1} - \frac{1}{x_2}\right| = |m - 2m| = |-m| = m > \frac{1}{2}.$$

[$\because$ m is a + ive integer]

Thus if we take $\varepsilon = \frac{1}{2} > 0$, then what ever $\delta > 0$ we try there exist $x_1, x_2 \in]0,1[$ such that

$$|x_1 - x_2| < \delta \text{ but } |f(x_1) - f(x_2)| > \varepsilon = \frac{1}{2}.$$

In this way for $\varepsilon = \frac{1}{2} > 0$, there exists no $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \varepsilon \text{ whenever } |x_1 - x_2| < \delta, x_1, x_2 \in]0,1[.$$

Hence $f(x) = 1/x$ is not uniformly continuous in $]0,1[$.

Example 4: *Prove that the function f defined on* $\mathbf{R}^+$ *as* $f(x) = \sin\frac{1}{x}$, $\forall\ x > 0$ *is continuous but not uniformly continuous on* $\mathbf{R}^+$.

Solution: Let a be any positive real number.

We have $$f(a-0) = \lim_{n \to 0} f(a-h) = \lim_{n \to 0} \sin\frac{1}{a-h} = \sin\frac{1}{a},$$

$$f(a+0) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} \sin\frac{1}{a+h} = \sin\frac{1}{a}.$$

Also $$f(a) = \sin\frac{1}{a}.$$

Since $f(a+0) = f(a-0) = f(a), f$ is continuous at a.

But a is an arbitrary point of $\mathbf{R}^+$, so f is continuous on $\mathbf{R}^+$.

It remains to show that f is not uniformly continuous on $\mathbf{R}^+$.

We shall show that no δ works for $\varepsilon = \frac{1}{2}$.

Let δ be any positive number. Take $x_1 = \frac{1}{n\pi}$, $x_2 = \frac{1}{n\pi + (\pi/2)} = \frac{2}{(2n+1)\pi}$,

where n is a positive integer such that $x_1 - x_2 = \frac{1}{n\pi} - \frac{2}{(2n+1)\pi} < \delta$.

(We can always choose such a positive integer n for each positive δ.)

Now $|x_1 - x_2| < \delta$ but $|f(x_1) - f(x_2)| = |\sin n\pi - \sin\frac{1}{2}(2n+1)\pi| = 1 > \varepsilon$.

This shows that for this choice of ε, we are unable to find $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \varepsilon \text{ whenever } |x_1 - x_2| < \delta, \quad \forall\, x_1, x_2 \in \mathbf{R}^+.$$

Hence f is not uniformly continuous on $\mathbf{R}^+$.

Example 5: *Show that the function f defined by $f(x) = x^3$, is uniformly continuous in* $[-2, 2]$.

Solution: Let $x_1, x_2 \in [-2, 2]$. We have

$$|f(x_2) - f(x_1)| = |x_2^3 - x_1^3| = |(x_2 - x_1)(x_2^2 + x_1^2 + x_1 x_2)|$$
$$\leq |x_2 - x_1| \{|x_2|^2 + |x_1|^2 + |x_1||x_2|\}$$
$$\leq 12 |x_2 - x_1|.$$
$$[\because x_1, x_2 \in [-2, 2] \Rightarrow |x_1| \leq 2 \text{ and } |x_2| \leq 2]$$

$\therefore$ $$|f(x_2) - f(x_1)| < \varepsilon \text{ whenever } |x_2 - x_1| < \varepsilon / 12.$$

Thus given $\varepsilon > 0$, there exists $\delta = \varepsilon / 12$ such that

$$|f(x_2) - f(x_1)| < \varepsilon \text{ whenvever } |x_2 - x_1| < \delta \ \forall \ x_1, x_2 \in [-2, 2].$$

Hence $f(x)$ is uniformly continuous in $[-2, 2]$.

Example 6: *Let $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$. Show that f is not uniformly continuous on* $\mathbf{R}$.

Solution: Let $\varepsilon > 0$ be given. The function $f(x)$ will be uniformly continuous on $\mathbf{R}$ if we are able to find $\delta > 0$ such that

$$x_1, x_2 \in \mathbf{R}, |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon. \quad \ldots(1)$$

The function $f(x)$ will not be uniformly continuous on $\mathbf{R}$ if we produce some $\varepsilon > 0$ for which no δ works *i.e.*, for which for every $\delta > 0$, there exist $x_1, x_2 \in \mathbf{R}$ such that $|x_1 - x_2| < \delta$ but $|f(x_1) - f(x_2)| \geq \varepsilon$.

So here we shall show that for some given $\varepsilon > 0$, there exists no $\delta > 0$ which satisfies the condition (1).

By the axiom of Archimedes, for any $\delta > 0$, there exists a positive integer n such that

$$n\delta^2 > \varepsilon. \quad \ldots(2)$$

If we take $x_1 = n\delta$ and $x_2 = n\delta + \frac{1}{2}\delta$, then $|x_1 - x_2| = \frac{1}{2}\delta < \delta$,

but $|f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = |x_1 - x_2||x_1 + x_2|$

$$= \frac{1}{2}\delta \left(2n\delta + \frac{1}{2}\delta\right) = n\delta^2 + \frac{1}{4}\delta^2 > \varepsilon, \text{ by (2)}.$$

Hence for these two points x_1, x_2, we would always have $|f(x_1) - f(x_2)| > \varepsilon$, whatever $\delta > 0$ we take. This contradicts (1). Hence f is not uniformly continuous on $\mathbf{R}$.

Example 7: *In the closed interval* $[-1, 1]$ *let f be defined by*

$$f(x) = x^2 \sin(1/x^2) \text{ for } x \neq 0 \text{ and } f(0) = 0.$$

In the given interval (i) Is the function bounded ? (ii) Is it continuous ? (iii) Is it uniformly continuous ?

Solution: (i) If $x \in [-1, 1]$ and $x \neq 0$, we have

$$|f(x)| = |x^2 \sin(1/x^2)| = |x^2| \cdot |\sin(1/x^2)|$$
$$= |x|^2 \cdot |\sin(1/x^2)| \leq 1 \cdot 1 = 1.$$
$$[\because |\sin(1/x^2)| \leq 1 \text{ and } -1 \leq x \leq 1 \Rightarrow |x| \leq 1]$$

Also $f(0)=0 \Rightarrow |f(0)|=0<1$.

Thus $|f(x)| \le 1, \forall\ x \in [-1,1]$ and so f is bounded in $[-1,1]$.

(ii) Let $c \in [-1,1]$ and $c \neq 0$.

We have $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} x^2 \sin \dfrac{1}{x^2} = c^2 \sin \dfrac{1}{c^2} = f(c)$.

$\therefore$ $f(x)$ is continuous at every point c of $[-1,1]$ if $c \neq 0$.

Now to check the continuity of $f(x)$ at $x=0$.

We have $\quad f(0-0) = \lim\limits_{h \to 0} f(0-h) = \lim\limits_{h \to 0} f(-h), h>0$

$$= \lim_{h \to 0} (-h)^2 \sin \left\{\frac{1}{(-h)^2}\right\} = \lim_{h \to 0} h^2 \sin \frac{1}{h^2} = 0.$$

$$\left[\because \lim_{h \to 0} h^2 = 0 \text{ and } \left|\sin \frac{1}{h^2}\right| \le 1 \text{ if } h \neq 0\right]$$

Again $\quad f(0+0) = \lim\limits_{h \to 0} f(0+h) = \lim\limits_{h \to 0} f(h)$

$$= \lim_{h \to 0} h^2 \sin \frac{1}{h^2} = 0.$$

Also $\quad f(0)=0$.

Since $f(0-0)=f(0)=f(0+0)$, therefore $f(x)$ is continuous at $x=0$.

Thus $f(x)$ is continuous at each point of $[-1,1]$ and so it is continuous in $[-1,1]$.

(iii) Since f is continuous in the closed interval $[-1,1]$, therefore it is also uniformly continuous in $[-1,1]$.

Example 8: *Prove that if f and g are bounded and uniformly continuous on an interval I, then the product function fg is also uniformly continuous on I. Is boundedness of each function necessary for the uniform continuity of the product ? If not so, give a counter example.*

Solution: It is given that the functions f and g are bounded and uniformly continuous on I.

To prove that fg is also uniformly continuous on I.

Since f is bounded on I, therefore there exists $k_1>0$ such that

$$|f(x)| \le k_1, \forall\ x \in I. \qquad ...(1)$$

Again g is also bounded on I and so there exists $k_2>0$ such that

$$|g(x)| \le k_2, \forall\ x \in I. \qquad ...(2)$$

Now take any given $\varepsilon>0$.

Since f is uniformly continuous on I, therefore there exists $\delta_1>0$ such that

$$|f(x)-f(y)| < \frac{\varepsilon}{2k_2},$$

whenever $|x-y|<\delta_1$, where $x, y \in I$. ...(3)

Again g is also uniformly continuous on I and so there exists $\delta_2 > 0$ such that

$$|g(x) - g(y)| < \frac{\varepsilon}{2k_1},$$

whenever $|x - y| < \delta_2$, where $x, y \in I$. ...(4)

Take $\delta = \text{mini}(\delta_1, \delta_2)$. Then from (3) and (4), we have

$$|f(x) - f(y)| < \frac{\varepsilon}{2k_2}$$

and $$|g(x) - g(y)| < \frac{\varepsilon}{2k_1}, \qquad \text{...(5)}$$

whenever $|x - y| < \delta$, where $x, y \in I$.

Now $\delta > 0$ is such that if $x, y \in I$ and $|x - y| < \delta$, then

$$|(f\,g)(x) - (f\,g)(y)| = |f(x)\,g(x) - f(y)\,g(y)|$$

$$= |f(x)\,g(x) - f(y)\,g(x) + f(y)\,g(x) - f(y)\,g(y)|$$

$$= |\{f(x) - f(y)\}\,g(x) + \{g(x) - g(y)\}\,f(y)|$$

$$\leq |f(x) - f(y)|.|g(x)| + |g(x) - g(y)|.|f(y)|$$

$$< \frac{\varepsilon}{2k_2}\cdot k_2 + \frac{\varepsilon}{2k_1}\cdot k_1, \text{ using (1), (2) and (5)}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$|(f\,g)(x) - (f\,g)(y)| < \varepsilon$, whenever $|x - y| < \delta$ where $x, y \in I$.

Hence $f\,g$ is uniformly continuous on I.

Second part: Boundedness of each function is not necessary for the uniform continuity of the product as is obvious from the following examples.

Example Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = x, \forall\, x \in \mathbf{R} \text{ and } g(x) = 1, \forall\, x \in \mathbf{R}.$$

Here both the functions f and g are uniformly continuous on $\mathbf{R}$. The function f is not bounded. But the product function $f\,g = f$ is uniformly continuous on $\mathbf{R}$.

Example Consider the functions f and g defined on $[0, \infty[$ by

$$f(x) = g(x) = \sqrt{x}, \forall\, x \in [0, \infty[.$$

Both the functions f and g are not bounded. The product function $f\,g$ is given by

$$(f\,g)(x) = x, \forall\, x \in [0, \infty[$$

which is obviously uniformly continuous on $[0, \infty[$.

Example 9: *Find the points of discontinuity of the function* $f : \mathbf{R} \to \mathbf{R}$ *defined by*

$$f(x) = \lim_{m \to \infty} \left\{ \lim_{n \to \infty} (\cos m!\,\pi\, x)^{2n} \right\}.$$

Solution: Let x be a rational number, say p / q, where p, q are integers prime to each other. Choosing m sufficiently large, $m ! \pi x$ can be made an integral multiple of π so that $\cos(m ! \pi x) = \pm 1$.

$$\therefore \quad \lim_{n \to \infty} (\cos m ! \pi x)^{2n} = \lim_{n \to \infty} (\pm 1)^{2n} = 1.$$

Hence $f(x) = 1$, when x is rational.

If x is irrational, then for any integral value of m, $\cos m ! \pi x$ will always lie between -1 and $+1$.

$$\therefore \quad (\cos m ! \pi x)^{2n} = (r_m)^{2n} \text{ where } |r_m| < 1, \text{ for a fixed value of } m.$$

Hence $f(x) = \lim_{m \to \infty} \lim_{n \to \infty} (r_m)^{2n} = 0$, when x is irrational.

Since $f(x)$ is 1 for rational values of x and 0 for irrational values of x, f is totally discontinuous *i.e.,* discontinuous for every value of x. This is so because at any point a (rational or irrational) the limits $f(a+0)$ as well as $f(a-0)$ do not exist. Note that there are infinite number of rational and infinite number of irrational points in any neighbourhood $]a - h, a + h[$ of a, however small h may be and at these points the functional values differ widely.

Here $\overline{f(a+0)} = 1$, $\underline{f(a+0)} = 0$, $\overline{f(a-0)} = 1$, $\underline{f(a-0)} = 0$.

Hence there is a discontinuity of second kind at every point.

Example 10: *Show that the function $f : \mathbf{R} \to \mathbf{R}$ defined by*

$$f(x) = \lim_{n \to \infty} \left[\lim_{t \to 0} \frac{\sin^2 (n ! \pi x)}{\sin^2 (n ! \pi x) + t^2} \right]$$

is equal to 0 when x is rational and to 1 when x is irrational. Hence show that the function is totally discontinuous.

Solution: Let x be a rational number say p / q, where p, q are integers prime to each other. By taking n sufficiently large, $n ! \pi x$ can be made an integral multiple of π so that $\sin(n ! \pi x) = 0$.

Hence $f(x) = \lim_{t \to 0} \dfrac{0}{0 + t^2} = 0$, when x is rational.

If x is irrational, then $0 < \sin^2 (n ! \pi x) < 1$.

$$\text{Hence} \quad f(x) = \lim_{n \to \infty} \lim_{t \to 0} \frac{1}{1 + t^2 / \sin^2 (n ! \pi x)}$$

$$= \frac{1}{1+0} = 1, \text{ when } x \text{ is irrational.}$$

Thus $f(x) = 0$ when x is rational and 1 when x is irrational. Hence f is totally discontinuous.

Example 11: *Show that the function* $f : \mathbf{R} \to \mathbf{R}$ *defined by*

$$f(x) = \lim_{t \to \infty} \frac{(1 + \sin \pi x)^t - 1}{(1 + \sin \pi x)^t + 1}$$

is discontinuous at the points $x = 0, 1, 2, \ldots, n, \ldots$

Solution: For $x = 0, 1, 2, 3, \ldots, n, \ldots$ we have $\sin \pi x = 0$, so that at these values of x, $f(x) = \lim_{n \to \infty} \dfrac{(1+0)^t - 1}{(1+0)^t + 1} = 0.$

Now if $2m < x < 2m + 1$ (m being an integer), then $\sin \pi x$ is positive. Hence for such values of x, we have

$$f(x) = \lim_{t \to \infty} \frac{1 - \dfrac{1}{(1 + \sin \pi x)^t}}{1 + \dfrac{1}{(1 + \sin \pi x)^t}} = \frac{1 - \dfrac{1}{\infty}}{1 + \dfrac{1}{\infty}} = 1. \qquad \left[\because \frac{1}{\infty} = 0\right]$$

Again if $2m + 1 < x < 2m + 2$, $\sin \pi x$ is negative and so

$$\lim_{t \to \infty} (1 + \sin \pi x)^t = 0.$$

Hence for such values of x, $f(x) = \dfrac{0 - 1}{0 + 1} = -1.$

From the values of $f(x)$ mentioned above, we observe that

(i) if x is an even integer, then

$$f(x) = 0,\ f(x + 0) = 1 \text{ and } f(x - 0) = -1$$

and (ii) if x is an odd integer, then

$$f(x) = 0,\ f(x + 0) = -1 \text{ and } f(x - 0) = 1.$$

Hence f has discontinuities of the first kind at

$$x = 0, 1, 2, \ldots, n, \ldots$$

Example 12: *Let a function* $f : \mathbf{R} \to \mathbf{R}$ *satisfy the equation*

$$f(x + y) = f(x) + f(y),\ \forall\ x, y \in \mathbf{R}.$$ *Show that*

(i) If f is continuous at the point $x = a$, *then it is continuous for all* $x \in \mathbf{R}$.

(ii) If f is continuous then $f(x) = kx$, *for some constant k.*

Solution: **(i)** Let f be continuous at the point $x = a$.

We have $f(a + 0) = \lim_{h \to 0} f(a + h) = \lim_{h \to 0} [f(a) + f(h)]$, by definition of f

$$= \lim_{h \to 0} f(a) + \lim_{h \to 0} f(h) = f(a) + \lim_{h \to 0} f(h).$$

Since f is continuous at a, we have $f(a) = f(a + 0)$

$$\Rightarrow \quad f(a) = f(a) + \lim_{h \to 0} f(h) \Rightarrow \lim_{h \to 0} f(h) = 0. \qquad \ldots(1)$$

Similarly $\quad f(a) = f(a - 0) \Rightarrow \lim_{h \to 0} f(-h) = 0. \qquad \ldots(2)$

Now let α be any real number. We shall show that f is continuous at α.

We have $$f(\alpha+0) = \lim_{h\to 0} f(\alpha+h) = \lim_{h\to 0} [f(\alpha)+f(h)]$$

$$= \lim_{h\to 0} f(\alpha) + \lim_{h\to 0} f(h) = f(\alpha), \text{ using (1).}$$

Similarly, using (2) we have $f(\alpha-0) = f(\alpha)$.

Thus $$f(\alpha+0) = f(\alpha-0) = f(\alpha).$$

Hence f is continuous at $x=\alpha$. Since α is arbitrary so f is continuous for all $x \in \mathbf{R}$.

(ii) Here we consider the following cases :

Case I: Let $x=0$. Since $f(x+x) = f(x)+f(x)$, we have

$$f(0) = f(0+0) = f(0)+f(0) \text{ and so } f(0) = 0.$$

Hence $f(x) = kx$ for every constant k in this case.

Case II: Let x be any positive integer. Then

$$f(x) = f(1+1+\dots x \text{ times}) = f(1)+f(1)+\dots x \text{ times}$$
$$= x f(1) = kx, \text{ where } k = f(1) \text{ i.e., a constant.}$$

Case III: Let x be any negative integer. Put $x = -y$ so that y is a positive integer.

Now $$0 = f(0), \text{ by Case I}$$
$$= f(y-y) = f(y) + f(-y).$$

$\therefore$ $$f(-y) = -f(y).$$

Hence $$f(x) = -f(y) = -y\, f(1), \text{ by Case II}$$
$$= kx, \text{ where } k = f(1) \text{ i.e., a constant.}$$

Case IV: Let x be any rational number. Put $x = p/q$ where q is a positive integer and p is any integer, positive, negative or zero.

Now $$f\left(q \cdot \frac{p}{q}\right) = f\left(\frac{p}{q}+\frac{p}{q}+\dots q \text{ times}\right)$$

$$= f\left(\frac{p}{q}\right) + f\left(\frac{p}{q}\right) + \dots q \text{ times.}$$

$\therefore$ $$f(p) = q f\left(\frac{p}{q}\right).$$

But $f(p) = kp$, by previous cases.

Hence $$q f\left(\frac{p}{q}\right) = kp \quad \text{or} \quad f\left(\frac{p}{q}\right) = k\frac{p}{q}$$

or $f(x) = kx$, in this case also.

Case V: Finally let x be any real number. Let $< x_n >$ be a sequence of rational numbers converging to x. Since f is continuous at x, the sequence $< f(x_n) >$ converges to $f(x)$. Thus, we have

$$\lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} f(x_n) = f(x)$$

Since x_n is a rational number, we have by case IV,

$$f(x_n) = kx_n.$$

$\therefore \quad \lim\limits_{n \to \infty} f(x_n) = \lim\limits_{n \to \infty} kx_n = k \lim\limits_{n \to \infty} x_n = kx$

or $\quad f(x) = kx.$

Hence $\quad f(x) = kx, \ \forall \ x \in \mathbf{R}.$

5 Meaning of the Sign of Derivative

Let $f'(c) > 0$ where c is an interior point of the domain of the function f; then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) > 0.$$

If $\varepsilon > 0$ be any number $< f'(c)$, there exists $\delta > 0$ such that

$$|x - c| \leq \delta \Rightarrow \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon$$

i.e., $\quad x \in [c - \delta, c + \delta], x \neq c$

$\Rightarrow \quad \dfrac{f(x) - f(c)}{x - c} \in \,]\, f'(c) - \varepsilon, f'(c) + \varepsilon\, [\qquad \ldots(1)$

Since ε was chosen smaller than $f'(c)$ we conclude from (1) that

$$\frac{f(x) - f(c)}{x - c} > 0 \text{ when } x \in [c - \delta, c + \delta], x \neq c.$$

We then have, $f(x) - f(c) > 0$ when $c < x \leq c + \delta$

and $\quad f(x) - f(c) < 0$ when $c - \delta \leq x < c.$

Thus we have shown that if $f'(c) > 0$, there exists a neighbourhood $[c - \delta, c + \delta]$ of c such that

$$f(x) > f(c) \ \forall \ x \in \,]\, c, c + \delta\, [\text{ and } f(x) < f(c) \ \forall \ x \in [c - \delta, c\,].$$

If $f'(c) < 0$, it can be similarly shown that there exists a neighbourhood $]\, c - \delta, c + \delta\, [$ of c such that

$$f(x) > f(c) \ \forall \ x \in [c - \delta, c\, [\text{ and } f(x) < f(c) \ \forall \ x \in \,]\, c, c + \delta\, [.$$

Similarly, it can be shown for the end points a and b that there exist intervals, $]\, a, a + \delta]$ and $[\, b - \delta, b\, [$ such that

$$f'(a) > 0 \Rightarrow f(x) > f(a) \ \forall \ x \in \,]\, a, a + \delta\,]$$

$$f'(a) < 0 \Rightarrow f(x) < f(a) \ \forall \ x \in \,]\, a, a + \delta\,]$$

$$f'(b) > 0 \Rightarrow f(x) < f(b) \ \forall \ x \in [\, b - \delta, b\, [$$

and $\quad f'(b) < 0 \Rightarrow f(x) > f(b) \ \forall \ x \in [\, b - \delta, b\, [$

6 Intermediate value Theorem for Derivatives or Darboux Theorem

Theorem: *If f is finitely differentiable in a closed interval $[a, b]$ and $f'(a)$, $f'(b)$ are of opposite signs, then there exists at least one point $c \in \,]\, a, b\, [$ such that $f'(c) = 0$.* **(Gorakhpur 2011)**

Proof : For definiteness, suppose that $f'(a)>0$ and $f'(b)<0$. Then there exist intervals $]a, a+h]$ and $[b-h, b[$, h being positive such that

$$f(x)>f(a)\ \forall\, x\in\,]a, a+h]$$
$$f(x)>f(b)\ \forall\, x\in[b-h, b[\,.$$

Further f being finitely differentiable, is continuous in $[a,b]$ and hence it is bounded on $[a,b]$ and attains its supremum and infimum at least once in $[a,b]$.

Thus if M be the least upper bound of f in $[a,b]$, then there exists $c\in[a,b]$ such that $f(c)=M$. It is clear from (1) and (2) that the upper bound is not attained at the end points a and b so that $c\in\,]a,b[$. We shall prove that $f'(c)=0$.

If $f'(c)>0$, then there exists an interval $]c, c+h], h>0$, such that

$$f(x)>f(c)=M\ \forall\, x\in\,]c, c+h\,],$$

which is not possible since M is the least upper bound of the function $f(x)$ in $[a,b]$.

If $f'(c)<0$, then there exists an interval $[c-h, c\,[, h>0$ such that

$$f(x)>f(c)=M\ \ \forall\ \ x\in[c-h, c\,[,$$

which is again not possible for the same reason as mentioned earlier.

Hence we conclude that $f'(c)=0$.

Corollary 1: *If f is finitely differentiable on $[a,b]$ and $f'(a)\neq f'(b)$ and k is any number lying between $f'(a)$ and $f'(b)$, then there exists at least one point $c\in\,]a,b[$ such that $f'(c)=k$. In other words $f'(x)$ takes every value intermediate between $f'(a)$ and $f'(b)$.*

Proof : Let k be any real number lying between $f'(a)$ and $f'(b)$. Define a function ϕ by $\phi(x)=f(x)-kx$. Since f and kx both are finitely differentiable in $[a,b]$, ϕ is also finitely differentiable in $[a,b]$.

We have $\phi'(x)=f'(x)-k\ \forall\, x\in[a,b]$.

Hence $\phi'(a)=f'(a)-k$ and $\phi'(b)=f'(b)-k$.

Since k lies between $f'(a)$ and $f'(b)$, $\phi'(a)$ and $\phi'(b)$ are of opposite signs. Hence by the above theorem, there exists at least one point c of $]a,b[$ such that $\phi'(c)=0$

or $\qquad f'(c)-k=0$ or $f'(c)=k$

Corollary 2: *If f is finitely differentiable on $[a,b]$ and $f'(x)\neq 0$ for any $x\in\,]a,b[$, then $f'(x)$ retains the same sign, positive or negative in $]a,b[$ [i.e., $f'(x)$ is either positive or negative for all values of $x\in\,]a,b[$.*

Proof : If possible, let x_1 and x_2 be two distinct elements of $]a,b[$ and let $f'(x_1)$ and $f'(x_2)$ be of opposite signs. Then by the above theorem, there exists $c\in\,]a,b[$ such that $f'(c)=0$ which contradicts the fact that $f'(x)\neq 0\ \forall\, x\in\,]a,b[$. Hence $f'(x)$ must retain the same sign in $]a,b[$.

Corollary 3: *If f is finitely differentiable on $I=[a,b]$, then the range $f'(I)$ of f' on I is either an interval or a singleton.*

Proof: Let $f'(I)=J$ and let p_1, p_2 be two distinct elements of J. Then there exist two distinct elements x_1, x_2 of I such that $f'(x_1)=p_1$ and $f'(x_2)=p_2$. Suppose that $x_1<x_2$. Then

$$[x_1, x_2]\subset[a,b].$$

Let p be any real number lying between p_1 and p_2. Then, by the above corollary 1, there exists $c \in]\, x_1, x_2 [\, \subset [a, b]$ such that $f'(c) = p$. Hence $p \in J$. This shows that every number lying between p_1 and p_2 belongs to J. Hence J is an interval.

If J does not contain at least two distinct elements, then obviously it is a singleton.

Comprehensive Exercise 1

1. Let f be the function defined on $[-1, 1]$ by
   $f(x) = x$, if x is irrational, $f(x) = 0$, if x is rational.
   Show that f is continuous only at $x = 0$.
2. Let $f : \mathbf{R} \to \mathbf{R}$ be such that
   $$\begin{aligned} f(x) &= x \quad \text{when } x \text{ is irrational} \\ &= -x \quad \text{when } x \text{ is rational.} \end{aligned}$$
   Show that $f(x)$ is continuous only at $x = 0$.
3. Show that the function f defined on R by
   $f(x) = 1$ when x is rational, $f(x) = -1$ when x is irrational
   is discontinuous at every point of R.
4. Show that $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$ is continuous but not uniformly continuous on $\mathbf{R}$.
5. Explain fully uniform continuity and discuss the uniform continuity of the function $f(x) = x^2, \forall\, x \in \mathbf{R}$ in $]\,0, 1\,[$.
6. Let $f :]-1, 1[\;\to [0, 1]$ be a function defined by $f(x) = x^2$. Using definition show that f is uniformly continuous on its domain.
7. Show that the function $f(x) = 1/x, x > 0$ is continuous in $(0, 1)$ but not uniformly continuous.
8. Define uniform continuity and show that the function
   $f(x) = x^2 + 3x, x \in [-1, 1]$ is uniformly continuous in $[-1, 1]$.
9. Show that a function which is continuous in a closed interval is bounded in that interval. Verify the theorem for
   $$f(x) = \cos x \;\; in \; [-\frac{1}{2}\pi, \frac{1}{2}\pi] \; .$$
10. Give an example to show that a function continuous on an open interval need not be bounded on that interval.
11. If the function f is continuous in the closed interval $[a, b]$, prove that it attains its least upper bound and greatest lower bound in $[a, b]$. Verify the theorem for the function $f(x) = \sin x$ in $[0, 2\pi]$.
12. If f is continuous in $[a, b]$ and $f(a) . f(b) < 0$, show that $f(c) = 0$ for at least one $c \in [a, b]$.

13. Let f be continuous on $[a, b]$ and suppose that $f(x) = 0$ for every rational x in $[a, b]$. Prove that $f(x) = 0$ for all x in $[a, b]$.

14. If a function is continuous on a closed interval $[a, b]$, then it attains its bounds at least once in $[a, b]$. Give an example of a function which is continuous and bounded, and attains its supremum but does not attain its infimum.

15. Prove that the identity mapping of any interval I is uniformly continuous on I.

16. Show that the function $f: \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \frac{1}{1 + e^{1/\sin(n!\,\pi x)}}$$

can be made discontinuous at any rational point in the interval $[0, 1]$ by a proper choice of n.

17. If $f: \mathbf{R} \to \mathbf{R}$ is a continuous function and satisfies the relation $f(x + y) = f(x) f(y), \forall\ x, y \in \mathbf{R}$, then either $f(x) = 0 \quad \forall\ x \in \mathbf{R}$ or there exists an $a > 0$ such that

$$f(x) = a^x, \forall\ x \in \mathbf{R}.$$

18. Show that a function f is continuous at a iff for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x_1, x_2 \in\ ]a - \delta, a + \delta[\ \Rightarrow |f(x_1) - f(x_2)| < \varepsilon.$$

19. Let f and g be continuous on $[a, b]$ and let $f(a) < g(a)$ but $f(b) > g(b)$. Prove that $f(c) = g(c)$ for some $c \in\ ]a, b[$.

20. Let $f: \mathbf{R} \to \mathbf{R}$ be continuous and let f be zero on a dense set (*i.e.*, a set whose intersection with every interval is non-empty). Then f is identically zero.

21. Determine the discontinuities of the function $f: \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \lim_{n \to \infty} \frac{\{1 + \sin(\pi / x)\}^n - 1}{\{1 + \sin(\pi / x)\}^n + 1}, 0 < x \le 1.$$

22. Find the type of discontinuity at $x = 1$ for the function

$$f(x) = \lim_{n \to \infty} \frac{x^n}{1 + x^n e^x}.$$

23. Show that the function

$$\phi(x) = \lim_{n \to \infty} \frac{x^{2n+2} - \cos x}{x^{2n} + 1}$$

does not vanish anywhere in the interval $[0, 2]$ though $\phi(0)$ and $\phi(2)$ differ in sign. Discuss the continuity of the function at $x = 1$.

24. Examine for continuity the function f defined by

$$f(x) = \lim_{n \to \infty} \frac{e^x - x^n \sin x}{1 + x^n} \left(0 \le x \le \frac{\pi}{2}\right)$$

at $x = 1$. Explain why the function f does not vanish anywhere in $[0, \pi / 2]$ although $f(0) \cdot f(\pi / 2) < 0$.

Answers 1

5. Uniformly continuous in $]0,1[$
19. Discontinuity of the first kind at $x=1$
21. f has a discontinuity of second kind at $x=0$ and ordinary discontinuous at $x=1, \frac{1}{2}, \frac{1}{3}, \ldots$
22. Discontinuous of the first kind
24. Discontinuous at $x=1$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The necessary and sufficient condition for a function f defined on $I \subset R$ to be continuous at $a \in I$ is that for each sequence $< a_n >$ in **I** which converges to a, we have $\lim\limits_{n \to \infty} f(a_n) =$

(a) $f'(a)$ (b) $f^2(a)$

(c) $f(a)$ (d) none of these

2. The function f defined on $[-1, 1]$ by

$$f(x) = x, \text{ if } x \text{ is irrational}$$
$$f(x) = 0, \text{ if } x \text{ is rational.}$$

is continuous at $x =$

(a) 0 (b) 1

(c) $-1, 1$ (d) none of these

3. At $x = 1$, the function $f(x) = \lim\limits_{n \to \infty} \dfrac{x^n}{1 + x^n e^x}$

(a) is continuous
(b) has discontinuity of first kind
(c) has discontinuity of second kind
(d) is uniformly continuous.

4. For all real values of x, the function $f(x) = x^2$ is

(a) continuous (b) discontinuous

(c) uniformly continuous (d) not uniformly continuous.

5. In the interval $[-1, 1]$ the function f defined by $f(x) = x^2 \sin(1/x^2)$ for $x \neq 0$ and $f(0) = 0$ is

(a) bounded
(b) uniformly continuous
(c) unbounded
(d) none of these

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. If f and g are continuous at $a \in I$ then $f + g$ is ……… at a.
2. If a function $f(x)$ is continuous in a closed interval $[a, b]$, it is ……… in that interval.
3. If f is continuous in a closed and bounded interval I, it is ……… on I.
4. The function $f : R \to R$ defined by

$$f(x) = \lim_{t \to \infty} \frac{(1 + \sin \pi x)^t - 1}{(1 + \sin \pi x)^t + 1}$$

is ……… at the points $x = 0, 1, 2, \ldots$

5. The function $f(x) = x^3$ is ……… continuous in $[-2, 2]$.
6. The function f defined on R^+ as $f(x) = \sin \frac{1}{x}$, $\forall\ x > 0$ is ……… but not ……… on R^+.
7. The function $f(x) = \lim_{n \to \infty} \frac{e^x - x^n \sin x}{1 + x^n} \left(0 \le x \le \frac{\pi}{2}\right)$ is ……… at $x = 1$.

True or False

Write 'T' for true and 'F' for false statement.

1. If a function $f(x)$ is continuous in a closed interval $[a, b]$, then it is bounded in $[a, b]$.
2. If a function $f(x)$ is uniformly continuous in an interval I, then it is also continuous in I.
3. If a function $f(x)$ is continuous in an open interval I, then it is also uniformly continuous in I.
4. If a function $f(x)$ is continuous in a closed interval $[a, b]$, then it is also uniformly continuous in $[a, b]$.
5. If a function f is continuous at a, then $|f|$ is also continuous at a.
6. A function continuous on an open interval is bounded on that interval.
7. The function $f(x) = x^2 + 3x$, $x \in [-1, 1]$ is uniformly continuous in this interval.
8. The function $f(x) = x^2$ is uniformly continuous on $]0, 1[$.
9. A function defined on $[0, 1]$ and given by $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational} \end{cases}$,

is discontinuous at $x = \frac{1}{2}$.

Answers

Multiple Choice Questions

1. (c) **2.** (a) **3.** (b) **4.** (a), (d) **5.** (a), (b)

Fill in the Blank(s)

1. continuous **2.** bounded

3. uniformly continuous

4. discontinuous **5.** uniformly

6. continuous, uniformly continuous **7.** discontinuous

True or False

1. *T* **2.** *T* **3.** *F* **4.** *T* **5.** *T*

6. *F* **7.** *T* **8.** *T* **9.** *F*

□

Chapter

7

The Riemann Integral

1 Introduction

In elementary treatments, the process of integration is generally introduced as the inverse of differentiation. If $F'(x) = f(x)$ for all x belonging to the domain of the function f, F is called an integral of the given function f.

Historically, however, the subject of integral arose in connection with the problem of finding areas of plane regions in which the area of a plane region is calculated as the limit of a sum. This notion of integral as summation is based on geometrical concepts.

A German mathematician G.F.B. Riemann gave the first rigorous arithmetic treatment of definite integral free from geometrical concepts. Riemann's definition covered only bounded functions. It was Cauchy who extended this definition to unbounded functions. Later on early in the twentieth century Lebesgue introduced the integral on a firm foundation with many refinements and generalisations.

In the present chapter we shall study the Riemann integral of real valued, bounded functions defined on some closed interval.

2 Partitions and Riemann Sums

Definition 1: *Let $I = [a, b]$ be a closed and bounded interval. Then by a* **partition** *(or a* **dissection** *or a* **net***) of I we mean a finite set of real numbers $P = \{x_0, x_1, \dots, x_{n-1}, x_n\}$ having the property that $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$.*

The closed sub-intervals

$$I_1 = [x_0, x_1], I_2 = [x_1, x_2], \dots, I_n = [x_{n-1}, x_n]$$

determined by P constitute the **segments** *of the partition.*

By changing the set of points, the partition can be changed. Thus there can be an infinite number of partitions of the interval $[a, b]$. The family of all partitions of $[a, b]$ shall be denoted by $\mathbf{P}[a, b]$.

We write $\Delta x_r = x_r - x_{r-1}$ for $r = 1, 2, \dots, n$ so that Δx_r is the length of the segment $[x_{r-1}, x_r]$. The **norm** (or **mesh**) of a partition P is the greatest of the lengths of the segments of a partition P and it is denoted by $\|P\|$. Thus

$$\|P\| = \text{max.} (\Delta x_r : r = 1, 2, \dots, n).$$

Sometimes the norm of a partition P is also denoted by $\mu(P)$.

Definition 2: *A partition P^* is called a* **refinement** *of another partition P or we say that P^* is* **finer than** *P iff $P^* \supset P$, i.e., if every point of P is used in the construction of P^*.*

If $P^* = P_1 \cup P_2$, then P^* is called the **common refinement** *of the given two partitions P_1 and P_2.*

Definition 3: *Let f be a bounded function defined on a bounded interval $[a, b]$. Also let $P = \{a = x_0, x_1, \dots, x_n = b\}$ be any partition of $[a, b]$. Also let m_r and M_r be the infimum and supremum respectively of the function f on I_r, for $r = 1, 2, \dots, n$ i.e.,*

$$m_r = \inf \{ f(x) : x_{r-1} \le x \le x_r \}$$

and

$$M_r = \sup \{ f(x) : x_{r-1} \le x \le x_r \}.$$

Let us now form two sums

$$L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r \text{ and } U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r .$$

Then $L(P, f)$ is called the **lower Riemann sum** *(or* **lower Darboux sum***) of f on $[a, b]$ with respect to the partition P and $U(P, f)$ is called the* **upper Riemann sum** *(or* **upper Darboux sum***) of f on $[a, b]$ with respect to the partition P.*

In brief, we shall refer to these sums as the *lower and upper R-sums* of f with respect to P. Obviously $L(P, f) \le U(P, f)$.

Theorem 1: *Let f be a bounded function defined on $[a, b]$ and let m and M be the infimum and supremum of $f(x)$ in $[a, b]$. Then for any partition $\mathbf{P}$ of $[a, b]$, we have*

$$m(b - a) \le L(P, f) \le U(P, f) \le M(b - a).$$ **(Garhwal 2008)**

Proof: Let $P = \{a = x_0, x_1, \dots, x_n = b\}$ be any partition of $[a, b]$. Then $I_r = [x_{r-1}, x_r], r = 1, 2, \dots, n$ are the subintervals of $[a, b]$. Let m_r and M_r be the infimum and supremum of $f(x)$ in $[x_{r-1}, x_r]$. Then for every value of r, we have

$$m \le m_r \le M_r \le M$$

$\Rightarrow \qquad m\,\Delta x_r \le m_r\,\Delta x_r \le M_r\,\Delta x_r \le M\,\Delta x_r \qquad [\because \Delta x_r > 0]$

$\Rightarrow \qquad \sum_{r=1}^{n} m\,\Delta x_r \le \sum_{r=1}^{n} m_r\,\Delta x_r \le \sum_{r=1}^{n} M_r\,\Delta x_r \le \sum_{r=1}^{n} M\,\Delta x_r\,. \qquad ...(1)$

Now $\quad \sum_{r=1}^{n} m\,\Delta x_r = m \sum_{r=1}^{n} \Delta x_r = m \sum_{r=1}^{n} (x_r - x_{r-1})$

$$= m\,(x_1 - x_0 + x_2 - x_1 + \ldots + x_n - x_{n-1})$$
$$= m\,(x_n - x_0) = m\,(b-a).$$

Similarly $\quad \sum_{r=1}^{n} M\Delta\,x_r = M\,(b-a).$

Also $\quad \sum_{r=1}^{n} m_r\,\Delta x_r = L\,(P, f)$ and $\sum_{r=1}^{n} M_r\,\Delta x_r = U\,(P, f).$

Hence, from (1), we conclude that

$$m\,(b-a) \le L\,(P, f) \le U\,(P, f) \le M\,(b-a) \;\; \forall\, P \in \boldsymbol{P}\,[a, b].$$

It follows that the sets of upper sums and lower sums are bounded.

Theorem 2: *If* $f : [a, b] \to \mathrm{R}$ *is a bounded function, then*

$$U\,(P, -f) = -L\,(P, f) \;\; \textit{and} \;\; L\,(P, -f) = -U\,(P, f). \qquad \textbf{(Meerut 2012)}$$

Proof: Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be any partition of $[a, b]$. Let M_r and m_r be the supremum and infimum of f in I_r.

Now f is bounded on $[a, b] \Rightarrow -f$ is bounded on $[a, b]$.

Again M_r, m_r are supremum and infimum of f in I_r

$\Rightarrow \qquad -m_r, -M_r$ are supremum and infimum of $-f$ in I_r.

We have $\quad U\,(P, -f) = \sum_{r=1}^{n} (-m_r)\,\Delta x_r$, by definition of upper R-sum

$$= -\sum_{r=1}^{n} m_r\,\Delta x_r = -L\,(P, f).$$

Also $\quad L\,(P, -f) = \sum_{r=1}^{n} (-M_r)\,\Delta x_r$, by definition of lower R-sum

$$= -\sum_{r=1}^{n} M_r\,\Delta x_r = -U\,(P, f).$$

Theorem 3: *Let f be a bounded function defined on $[a, b]$ and let P be a partition of $[a, b]$. If P^* is a refinement of P, then*

$$L\,(P^*, f) \ge L\,(P, f) \;\textit{and}\; U\,(P^*, f) \le U\,(P, f).$$

Proof: Let $\quad P = \{a = x_0, x_1, x_2, \ldots, x_{r-1}, x_r, \ldots, x_n = b\}$

and $\quad P^* = \{a = x_0, x_1, x_2, \ldots, x_{r-1}, y_1, x_r, \ldots, x_n = b\}$,

so that P^* has one more partition point y_1 than P.

Let m_r and M_r be the infimum and supremum of f in $[x_{r-1}, x_r]$. Let M_r', M_r'' be the suprema of f in $[x_{r-1}, y_1], [y_1, x_r]$ and m_r', m_r'' be the infima of f in $[x_{r-1}, y_1], [y_1, x_r]$ respectively.

Then $M_r \ge M_r', M_r''$ and $m_r \le m_r', m_r''$.

Since the rth subinterval only of P is split into two more subintervals of P^* and the remaining subintervals are identical in P and P^*, therefore, we have

$$U(P, f) - U(P^*, f)$$
$$= M_r(x_r - x_{r-1}) - \{M_r'(y_1 - x_{r-1}) + M_r''(x_r - y_1)\}.$$

But $\quad M_r'(y_1 - x_{r-1}) + M_r''(x_r - y_1) \le M_r(y_1 - x_{r-1}) + M_r(x_r - y_1)$
$$= M_r(x_r - x_{r-1}).$$

It gives $\quad U(P, f) - U(P^*, f) \ge M_r(x_r - x_{r-1}) - M_r(x_r - x_{r-1}) = 0$

i.e., $\quad U(P, f) \ge U(P^*, f)$ or $U(P^*, f) \le U(P, f)$.

If P^* contains p more partition points than P then repeating p times the above argument we can show that

$$U(P^*, f) \le U(P, f). \quad ...(1)$$

In a similar manner we can show that

$$L(P^*, f) \ge L(P, f). \quad ...(2)$$

Note: We know that for any partition P^*,

$$L(P^*, f) \le U(P^*, f). \quad ...(3)$$

Thus from (1), (2) and (3), we get

$$L(P, f) \le L(P^*, f) \le U(P^*, f) \le U(P, f).$$

Theorem 4: *If P_1 and P_2 be any two partitions of $[a, b]$ then*

$$U(P_1, f) \ge L(P_2, f).$$

Proof: Let $P = P_1 \cup P_2$. Then P is the common refinement of both the partitions P_1 and P_2. Therefore by the theorem 3 above, we get

$$L(P_2, f) \le L(P, f) \text{ and } U(P, f) \le U(P_1, f).$$

But we have $\quad L(P, f) \le U(P, f)$.

Thus $\quad L(P_2, f) \le L(P, f) \le U(P, f) \le U(P_1, f)$.

It follows that $L(P_2, f) \le U(P_1, f)$ or $U(P_1, f) \ge L(P_2, f)$, *that is, every upper sum for f is greater than or equal to every lower sum for f.*

Theorem 5: *Let f, g be bounded functions defined on $[a, b]$ and let P be any partition of $[a, b]$. Then*

$$L(P, f + g) \ge L(P, f) + L(P, g)$$

and $$U(P, f + g) \le U(P, f) + U(P, g).$$

Proof: Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a, b]$. Since f and g are bounded functions on $[a, b]$, $f + g$ is also bounded on $[a, b]$.

Let m_r', M_r' be the infimum and supremum of f on I_r,

m_r'', M_r'' be the infimum and supremum of g on I_r

and m_r, M_r be the infimum and supremum of $f + g$ on I_r.

By definition of infimum we find that

$$f(x) \ge m_r', \ g(x) \ge m_r'' \ \forall\, x \in I_r$$

$$\Rightarrow \quad f(x) + g(x) \ge m_r' + m_r'' \ \forall\, x \in I_r$$

$\Rightarrow \qquad (f+g)(x) \geq m_r' + m_r'' \;\forall\; x \in I_r$

$\Rightarrow \qquad m_r' + m_r''$ is a lower bound of $f+g$ on I_r.

But m_r is the greatest lower bound of $f+g$ on I_r.

$\therefore \qquad m_r \geq m_r' + m_r'' \Rightarrow m_r\,\Delta x_r \geq m_r'\,\Delta x_r + m_r''\,\Delta x_r$

$\Rightarrow \qquad \sum_{r=1}^{n} m_r\,\Delta x_r \geq \sum_{r=1}^{n} m_r'\,\Delta x_r + \sum_{r=1}^{n} m_r''\,\Delta x_r$

$\Rightarrow \qquad L(P, f+g) \geq L(P, f) + L(P, g).$

Similarly we can prove the other result.

3 Lower and Upper Riemann Integrals

(Purvanchal 2009, 12)

Let f be a real valued bounded function defined on $[a, b]$. We know that the set of all numbers $L(P, f)$ with respect to all possible partitions P of $[a, b]$ is bounded above by $M(b-a)$ and hence there exists a supremum of $L(P, f)$. The **lower Riemann integral (lower *R*-integral)** of f over $[a, b]$ is the supremum of $L(P, f)$ over all partitions $P \in \mathbf{P}[a, b]$. It is denoted by

$$\underline{\int_a^b} f(x)\,dx.$$

Similarly the set of numbers $U(P, f)$ is bounded below by $m(b-a)$ and so it possesses an infimum. The **upper Riemann integral** (**upper *R*-integral**) of f over $[a, b]$ is the infimum of $U(P, f)$ over all partitions $P \in \mathbf{P}[a, b]$. It is denoted by

$$\overline{\int_a^b} f(x)\,dx.$$

Thus $\qquad \underline{\int_a^b} f(x)\,dx = \sup\{L(P, f) : P \text{ is a partition of } [a, b]\}$

and $\qquad \overline{\int_a^b} f(x)\,dx = \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}.$

We denote the lower and upper integrals of f simply by

$$\underline{\int_a^b} f \quad \text{and} \quad \overline{\int_a^b} f.$$

Since $L(P, -f) = -U(P, f)$ and $U(P, -f) = -L(P, f)$, it gives that

$$\underline{\int_a^b}(-f) = -\overline{\int_a^b} f \quad \text{and} \quad \overline{\int_a^b}(-f) = -\underline{\int_a^b} f.$$

Theorem 1: *The lower R-integral cannot exceed the upper R-integral, i.e.,*

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

(Purvanchal 2007, 09; Rohilkhand 11)

Proof: If P_1 and P_2 are any two partitions of $[a, b]$, then by theorem 4 of article 2, we have

$$L(P_1, f) \leq U(P_2, f). \qquad ...(1)$$

First, keeping P_2 fixed and taking the supremum over all partitions P_1, (1) gives

$$\underline{\int_a^b} f \leq U(P_2, f). \qquad \ldots(2)$$

Now taking infimum over all partitions P_2, (2) gives

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Theorem 2: (Darboux Theorem): *Let f be a bounded function defined on $[a, b]$. Then to every $\varepsilon > 0$, there corresponds $\delta > 0$ such that*

$$U(P, f) < \overline{\int_a^b} f + \varepsilon \quad \textit{and} \quad L(P, f) > \underline{\int_a^b} f - \varepsilon$$

for all partitions P with $\|P\| \leq \delta$.

Proof: Let $\varepsilon > 0$ be given. Since $\overline{\int_a^b} f$ is the infimum of $U(P, f)$ and $\underline{\int_a^b} f$ is the supremum of $L(P, f)$ for all partitions P, therefore, for given $\varepsilon > 0$ there exist partitions P_1 and P_2 such that

$$U(P_1, f) < \overline{\int_a^b} f + \varepsilon \qquad \ldots(1)$$

and
$$L(P_2, f) > \underline{\int_a^b} f - \varepsilon \quad . \qquad \ldots(2)$$

Let P_3 be the common refinement of P_1 and P_2. Then by theorem 3 of article 7.2, we get

$$U(P_3, f) \leq U(P_1, f) \text{ and } L(P_3, f) \geq L(P_2, f). \qquad \ldots(3)$$

Therefore, from (1), (2) and (3), we get

$$U(P, f) < \overline{\int_a^b} f + \varepsilon \quad \text{and} \quad L(P, f) > \underline{\int_a^b} f - \varepsilon$$

for all partitions P of $[a, b]$ with $\|P\| \leq \delta$, where $\delta = \|P_3\| > 0$.

Corollary: *If f is bounded on $[a, b]$ and P is a partition of $[a, b]$, then*

(i) $\displaystyle \lim_{\|P\| \to 0} L(P, f) = \underline{\int_a^b} f$

(ii) $\displaystyle \lim_{\|P\| \to 0} U(P, f) = \overline{\int_a^b} f.$

Proof: (i) Since $\underline{\int_a^b} f$ is the supremum of $L(P, f)$ for all partitions P, therefore we have

$$L(P, f) \leq \underline{\int_a^b} f. \qquad \ldots(1)$$

Using the above theorem, we see that for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$L(P, f) > \underline{\int_a^b} f - \varepsilon \qquad \ldots(2)$$

for all partitions P with $\|P\| \leq \delta$.

From (1) and (2), we get

$$\underline{\int_a^b} f - \varepsilon < L(P, f) \leq \underline{\int_a^b} f < \underline{\int_a^b} f + \varepsilon$$

or $$\underline{\int_a^b} f - \varepsilon < L(P, f) < \underline{\int_a^b} f + \varepsilon.$$

By definition of limit, this implies that

$$\lim_{\|P\| \to 0} L(P, f) = \underline{\int_a^b} f.$$

(ii) Since $\overline{\int_a^b} f$ is the infimum of $U(P, f)$ for all partitions P, therefore we have

$$U(P, f) \geq \overline{\int_a^b} f. \qquad \text{...(3)}$$

Using the above theorem, we see that for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$U(P, f) < \overline{\int_a^b} f + \varepsilon \qquad \text{...(4)}$$

for all partitions P with $\|P\| \leq \delta$.

From (3) and (4), we get

$$\overline{\int_a^b} f \leq U(P, f) < \overline{\int_a^b} f + \varepsilon$$

or $$\overline{\int_a^b} f - \varepsilon < U(P, f) < \overline{\int_a^b} f + \varepsilon.$$

By definition of limit, this implies that

$$\lim_{\|P\| \to 0} U(P, f) = \overline{\int_a^b} f.$$

4 *R*-Integrability

(Purvanchal 2012)

Definition: *Let f be a bounded function defined on the bounded interval $[a, b]$, then f is called* **Riemann integrable** *(or simply R-integrable) on $[a, b]$ iff*

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

and their common value is called the R-integral of f on $[a, b]$ and is denoted by $\int_a^b f$.

The class of all bounded functions f which are Riemann integrable on $[a, b]$ is denoted by $\mathbf{R}[a, b]$. The numbers a and b are called the **lower and upper limits of integration** respectively.

If $\underline{\int_a^b} f \neq \overline{\int_a^b} f$ then f is not Riemann integrable on $[a, b]$.

Note 1: The concept of integrability of a function over an interval as introduced here is subject to two very important limitations, *viz*.

(i) The function is bounded.

(ii) The interval of integration is finite *i.e.*, neither of the end points is infinite.

Note 2: It is not necessary that every bounded function is integrable *i.e.* there may exist a bounded function f for which

$$\underline{\int_a^b} f \neq \overline{\int_a^b} f .$$

Note 3: The statement that $\int_a^b f$ exists indicates that the function f is bounded and integrable over $[a, b]$.

5 Another Definition of Riemann Integral

A function f defined on $[a, b]$ is said to be Riemann integrable over $[a, b]$ iff for every $\varepsilon > 0$ there exists a $\delta > 0$ and a number I such that for every partition

$$P = \{a = x_0 , x_1, \ldots , x_n = b\}$$

with $\| P \| \leq \delta$ and for every choice of $\xi_r \in [x_{r-1}, x_r]$,

$$\left| \sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1}) - I \right| < \varepsilon .$$

In such a case I is said to be *Riemann integral* of f over $[a, b]$ *i.e.*,

$$I = \int_a^b f(x)\, dx.$$

Theorem: *Definitions of article 4 and article 5 are equivalent.*

Proof: (i) Let f be integrable according to definition of article 4. Then f is bounded and we have

$$\underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f . \qquad \ldots(1)$$

Let $\varepsilon > 0$ be given. Then by Darboux theorem, there exists $\delta > 0$ such that for every partition P with $\| P \| \leq \delta$,

$$L(P, f) > \underline{\int_a^b} f - \varepsilon = \int_a^b f - \varepsilon. \qquad \ldots(2)$$

and
$$U(P, f) < \overline{\int_a^b} f + \varepsilon = \int_a^b f + \varepsilon. \qquad \ldots(3)$$

If ξ_r is any point of the interval $[x_{r-1}, x_r]$, then

$$L(P, f) \leq \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r \leq U(P, f). \qquad \ldots(4)$$

From (2), (3) and (4), we conclude that for every partition P with $\| P \| \leq \delta$,

$$\int_a^b f - \varepsilon < \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r < \int_a^b f + \varepsilon$$

i.e.,
$$\left| \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r - \int_a^b f \right| < \varepsilon$$

or
$$\left| \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r - I \right| < \varepsilon, \text{ where } I = \int_a^b f .$$

$\therefore f$ is Riemann integrable according to the definition of article 5.

Thus the definition of article 4 $\Rightarrow$ the definition of article 5.

(ii) Let f be integrable according to the definition of article 5.

Then for $\varepsilon = 1 > 0$ there exists $\delta > 0$ and a number I such that for every partition P with $\|P\| \le \delta$ and for every choice of $\xi_r \in [x_{r-1}, x_r]$,

$$\left| \sum_{r=1}^{n} f(\xi_r)\,\Delta x_r - I \right| < 1 \quad i.e., \quad \left| \sum_{r=1}^{n} f(\xi_r)\,\Delta x_r \right| < |I| + 1. \qquad ...(1)$$

We have to show that f is bounded on $[a, b]$ and the lower and upper integrals of f over $[a, b]$ are equal.

Suppose f is not bounded on $[a, b]$, then f must not be bounded in at least one subinterval of P, say, $[x_{m-1}, x_m]$. Hence there exists a point $\xi_m \in [x_{m-1}, x_m]$ such that $f(\xi_m)$ is infinite.

Now, to form the sum $\sum_{r=1}^{n} f(\xi_r)\,\Delta x_r$ we have to choose points ξ_r in each subinterval $[x_{r-1}, x_r]$. Choose that ξ_m in the interval $[x_{m-1}, x_m]$ for which $f(\xi_m)$ is infinite.

In that case $\left| \sum_{r=1}^{n} f(\xi_r)\,\Delta x_r \right| > |I| + 1,$

which is a contradiction to (1). Hence f is bounded on $[a, b]$.

Again, by definition of article 7.5, for any $\varepsilon > 0$ there exists $\delta > 0$ and a number $I \in R$ such that for every partition P of $[a, b]$ with $\|P\| \le \delta$ and for every choice of $\xi_r \in [x_{r-1}, x_r]$,

$$\left| \sum_{r=1}^{n} f(\xi_r)\,\Delta x_r - I \right| < \frac{1}{2}\varepsilon$$

i.e.,
$$I - \frac{1}{2}\varepsilon < \sum_{r=1}^{n} f(\xi_r)\,\Delta x_r < I + \frac{1}{2}\varepsilon. \qquad ...(2)$$

Let m_r, M_r be the infimum and supremum of f on $[x_{r-1}, x_r]$, so that there exist points $\alpha_r, \beta_r \in [x_{r-1}, x_r]$ such that

$$f(\alpha_r) > M_r - \frac{\varepsilon}{2(b-a)} \quad \text{and} \quad f(\beta_r) < m_r + \frac{\varepsilon}{2(b-a)}.$$

$\therefore$
$$\sum_{r=1}^{n} f(\alpha_r)\,\Delta x_r > \sum_{r=1}^{n} M_r\,\Delta x_r - \sum_{r=1}^{n} \frac{\varepsilon}{2(b-a)}\,\Delta x_r$$

and
$$\sum_{r=1}^{n} f(\beta_r)\,\Delta x_r < \sum_{r=1}^{n} m_r\,\Delta x_r + \sum_{r=1}^{n} \frac{\varepsilon}{2(b-a)}\,\Delta x_r$$

i.e.
$$\sum_{r=1}^{n} f(\alpha_r)\,\Delta x_r > U(P, f) - \frac{\varepsilon}{2} \qquad ...(3)$$

and
$$\sum_{r=1}^{n} f(\beta_r)\,\Delta x_r < L(P, f) + \frac{\varepsilon}{2} \qquad ...(4)$$

Now from (2) and (3), we get

$$I + \frac{1}{2}\varepsilon > U(P, f) - \frac{1}{2}\varepsilon \qquad ...(5)$$

and from (2) and (4), we get

$$I - \frac{1}{2}\varepsilon < L(P, f) + \frac{1}{2}\varepsilon \qquad ...(6)$$

i.e., $\quad I + \varepsilon > U(P, f)$ and $I - \varepsilon < L(P, f) \qquad ...(7)$

But $\quad L(P, f) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(P, f). \qquad ...(8)$

From (7) and (8), we get

$$I - \varepsilon < \underline{\int_a^b} f \leq \overline{\int_a^b} f < I + \varepsilon \qquad ...(9)$$

or $\quad 0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f < 2\varepsilon$ or $0 \leq \overline{\int_a^b} f - \underline{\int_a^b} f \leq 0,$

as $\varepsilon > 0$ is arbitrary

or $\quad \overline{\int_a^b} f - \underline{\int_a^b} f = 0$ *i.e.,* $\overline{\int_a^b} f = \underline{\int_a^b} f. \qquad ...(10)$

$\therefore f$ is Riemann integrable according to the definition of article 4. Now from (9) and (10), we have

$$I - \varepsilon < \int_a^b f < I + \varepsilon.$$

Since ε is arbitrary, $I = \int_a^b f.$

Thus the definition of article 5 $\Rightarrow$ the definition of article 4.

Hence the two definitions of article 4 and article 5 are equivalent.

6 Riemann's Necessary and Sufficient Condition for *R*-Integrability

Oscillatory sum: With usual notations, we have

$$L(P, f) = \sum_{r=1}^{n} m_r \, \Delta x_r, \; U(P, f) = \sum_{r=1}^{n} M_r \, \Delta x_r .$$

Let $\omega_r = M_r - m_r$ so that ω_r is the oscillation of f on $[x_{r-1}, x_r]$.

$\therefore \quad U(P, f) - L(P, f) = \sum_{r=1}^{n} (M_r - m_r)\, \Delta x_r = \sum_{r=1}^{n} \omega_r \, \Delta x_r .$

The sum $\sum_{r=1}^{n} \omega_r \, \Delta x_r$ is called the **oscillatory sum** for the function f corresponding to the partition P and is denoted by $\omega(P, f)$.

Theorem: *A necessary and sufficient condition for R-integrability of a bounded function* $f : [a, b] \to \mathbf{R}$ *over* $[a, b]$ *is that for every* $\varepsilon > 0$*, there exists a partition P of* $[a, b]$ *such that for P and all its refinements*

$$0 \leq U(P, f) - L(P, f) < \varepsilon.$$ **(Rohilkhand 2009; Gorakhpur 14, 15)**

Proof: The condition is necessary. Let $f \in \boldsymbol{R}[a,b]$ so that

$$\underline{\int_a^b} f = \overline{\int_a^b} f. \quad ...(1)$$

By Darboux theorem, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all partitions P of $[a,b]$ with $\|P\| \le \delta$,

$$U(P, f) < \overline{\int_a^b} f + \frac{\varepsilon}{2} \quad ...(2)$$

and $$L(P, f) > \underline{\int_a^b} f - \frac{\varepsilon}{2}. \quad ...(3)$$

Adding the inequalities (2) and (3), we get

$$U(P, f) + \underline{\int_a^b} f - \frac{\varepsilon}{2} < L(P, f) + \overline{\int_a^b} f + \frac{\varepsilon}{2}.$$

In view of (1), this gives

$$U(P, f) < L(P, f) + \varepsilon \;\; i.e., \;\; U(P, f) - L(P, f) < \varepsilon \quad ...(4)$$

Since $U(P, f) \ge L(P, f)$, the inequality (4) can be written as

$$0 \le U(P, f) - L(P, f) < \varepsilon.$$

Hence the condition is necessary.

The condition is sufficient: Let for every $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that for P and all its refinements,

$$0 \le U(P, f) - L(P, f) < \varepsilon. \quad ...(5)$$

By the definition of upper and lower integrals, we have

$$\overline{\int_a^b} f \le U(P, f) \text{ and } \underline{\int_a^b} f \ge L(P, f)$$

or $$-\underline{\int_a^b} f \le -L(P, f).$$

$\therefore$ $$\overline{\int_a^b} f - \underline{\int_a^b} f \le U(P, f) - L(P, f) < \varepsilon \quad ...(6)$$

or $$\overline{\int_a^b} f - \overline{\int_a^b} f < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\overline{\int_a^b} f - \underline{\int_a^b} f \le 0. \quad ...(7)$$

Also we know that the lower Riemann integral can never exceed the upper Riemann integral

i.e., $$\overline{\int_a^b} f - \underline{\int_a^b} f \ge 0. \quad ...(8)$$

From (7) and (8), we get

$$\overline{\int_a^b} f - \underline{\int_a^b} f = 0 \quad \text{or} \quad \overline{\int_a^b} f = \underline{\int_a^b} f$$

i.e., the function f is Riemann integrable over $[a,b]$.

Note: **Another statement of the above theorem:** *A necessary and sufficient condition for a bounded function f to be integrable over $[a, b]$ is that for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that the oscillatory sum $\omega(P, f) < \varepsilon$*

or $\lim \omega(P, f) = 0$ *as $\|P\|$ tends to zero.*

Illustrative Examples

Example 1: *Show that if f is defined on $[a, b]$ by*

$$f(x) = k \;\; \forall \;\; x \in [a, b]$$

where k is a constant, then $f \in \mathbf{R}[a, b]$ and $\int_a^b k = k(b - a)$. **(Gorakhpur 2015)**

Solution: Obviously the given function is bounded over $[a, b]$.

Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a, b]$. Then for any subinterval $[x_{r-1}, x_r]$, we have $m_r = k, M_r = k$.

Now,
$$U(P, f) = \sum_{r=1}^{n} M_r \,\Delta x_r = \sum_{r=1}^{n} k \,\Delta x_r = k \sum_{r=1}^{n} \Delta x_r$$
$$= k[\Delta x_1 + \Delta x_2 + \ldots + \Delta x_n]$$
$$= k[(x_1 - x_0) + (x_2 - x_1) + \ldots + (x_n - x_{n-1})]$$
$$= k(x_n - x_0) = k(b - a)$$

and
$$L(P, f) = \sum_{r=1}^{n} m_r \,\Delta x_r = \sum_{r=1}^{n} k \,\Delta x_r = k(b - a).$$

Hence
$$\overline{\int_a^b} f = \inf U(P, f) = \inf\{k(b - a)\} = k(b - a)$$

and
$$\underline{\int_a^b} f = \sup L(P, f) = \sup\{k(b - a)\} = k(b - a).$$

Thus
$$\overline{\int_a^b} f = \underline{\int_a^b} f = k(b - a).$$

Hence
$$f \in R[a, b] \text{ and } \int_a^b f = k(b - a).$$

Example 2: *Let $f(x) = x$ on $[0, 1]$. Calculate $\underline{\int_0^1} x\,dx$ and $\overline{\int_0^1} x\,dx$ by dissecting $[0, 1]$ into n equal parts and hence show that $f \in \mathbf{R}[0, 1]$.*

(Purvanchal 2011; Garhwal 12; Meerut 12)

Solution: Let $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{r-1}{n}, \frac{r}{n}, \ldots, \frac{n}{n} = 1\right\}$. Here

$$m_r = \frac{r-1}{n}, M_r = \frac{r}{n} \text{ and } \Delta x_r = \frac{1}{n} \text{ for } r = 1, 2, \ldots, n.$$

Now, we have

$$L(P, f) = \sum_{r=1}^{n} m_r \,\Delta x_r = \sum_{r=1}^{n} \frac{r-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{r=1}^{n} (r - 1)$$

$$= \frac{1}{n^2}[1+2+3+\ldots+(n-1)] = \frac{(n-1).n}{2n^2} = \frac{n-1}{2n}$$

and $$U(P,f) = \sum_{r=1}^{n} M_r \Delta x_r = \sum_{r=1}^{n} \frac{r}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{r=1}^{n} r$$

$$= \frac{1}{n^2}[1+2+3+\ldots+n] = \frac{n(n+1)}{2n^2} = \frac{n+1}{2n}.$$

Hence by Corollary 1, theorem 2 of article 3, we get

$$\underline{\int_0^1} x\,dx = \lim_{||P|| \to 0} L(P,f) = \lim_{n \to \infty} \frac{n-1}{2n} = \frac{1}{2}$$

and $$\overline{\int_0^1} x\,dx = \lim_{||P|| \to 0} U(P,f) = \lim_{n \to \infty} \frac{n+1}{2n} = \frac{1}{2}.$$

Since $\underline{\int_0^1} f = \overline{\int_0^1} f, f \in \mathbf{R}[0,1]$ and $\int_0^1 x\,dx = \frac{1}{2}$.

Example 3: *Let $f(x) = x^2$ on $[0,a], a > 0$. Show that $f \in \mathbf{R}[0,a]$ and find $\int_0^a f$.*

(Rohilkhand 2010; Gorakhpur 14)

Solution: Let $P = \left\{0, \frac{a}{n}, \frac{2a}{n}, \ldots, \frac{(n-1)a}{n}, \frac{na}{n} = a\right\}$ be the partition of $[0,a]$ obtained by dissecting $[0,a]$ into n equal parts. Then

$$\Delta x_r = a/n, r = 1, 2, \ldots, n.$$

Also $I_r = r$th subinterval $= \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$.

Since $f(x) = x^2$ is an increasing function in $[0,a]$,

$$m_r = \frac{(r-1)^2 a^2}{n^2} \quad \text{and} \quad M_r = \frac{r^2 a^2}{n^2}, r = 1, 2, \ldots, n.$$

Now $$L(P,f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n} \frac{(r-1)^2 a^2}{n^2} \cdot \frac{a}{n}$$

$$= \frac{a^3}{n^3} \sum_{r=1}^{n} (r-1)^2 = \frac{a^3}{n^3} \cdot \frac{(n-1)n(2n-1)}{6}.$$

$\therefore$ $$\underline{\int_0^a} x^2\,dx = \lim_{n \to \infty} L(P,f) = \lim_{n \to \infty} \frac{a^3}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) = \frac{a^3}{3}.$$

Again $$U(P,f) = \sum_{r=1}^{n} M_r \Delta x_r = \sum_{r=1}^{n} \frac{r^2 a^2}{n^2} \cdot \frac{a}{n}$$

$$= \frac{a^3}{n^3} \sum_{r=1}^{n} r^2 = \frac{a^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}.$$

$\therefore$ $$\overline{\int_0^a} x^2\,dx = \lim_{n \to \infty} U(P,f) = \lim_{n \to \infty} \frac{a^3}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) = \frac{a^3}{3}.$$

Since $\overline{\int_0^a} f = \underline{\int_0^a} f, f \in \mathbf{R}[0,a]$ and $\int_0^a x^2\,dx = \frac{a^3}{3}$.

Example 4: *If a function f is defined on* $[0, a]$ *,* $a > 0$ *by* $f(x) = x^3$*, then show that f is Riemann integrable on* $[0, a]$ *and* $\int_0^a f(x)\,dx = \frac{a^4}{4}\cdot$

Solution: Let $P = \left\{0, \frac{a}{n}, \frac{2a}{n}, \ldots, \frac{(n-1)a}{n}, \frac{na}{n} = a\right\}$ be the partition of $[0, a]$ obtained by dissecting $[0, a]$ into n equal parts. Then

$$I_r = r \text{ th sub-interval} = \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$$

and $$\Delta x_r = \text{length of } I_r = \frac{a}{n}, r = 1, 2, \ldots, n.$$

Let m_r and M_r be respectively the infimum and supremum of f in I_r.
Since $f(x) = x^3$ is an increasing function in $[0, a]$, therefore

$$m_r = \frac{(r-1)^3 a^3}{n^3} \quad \text{and} \quad M_r = \frac{r^3 a^3}{n^3}, r = 1, 2, \ldots, n.$$

Now $$L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n}\left[\frac{(r-1)^3 a^3}{n^3}\cdot\frac{a}{n}\right] = \frac{a^4}{n^4}\sum_{r=1}^{n}(r-1)^3$$

$$= \frac{a^4}{n^4}[1^3 + 2^3 + \ldots + (n-1)^3] = \frac{a^4}{n^4}\cdot\left[\frac{(n-1)n}{2}\right]^2 = \frac{a^4}{4}\left(1 - \frac{1}{n}\right)^2.$$

$\therefore$ $$\underline{\int_0^a} f(x)\,dx = \lim_{n\to\infty} L(P, f) = \lim_{n\to\infty}\frac{a^4}{4}\left(1 - \frac{1}{n}\right)^2 = \frac{a^4}{4}.$$

Again $$U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r = \sum_{r=1}^{n}\left[\frac{r^3 a^3}{n^3}\cdot\frac{a}{n}\right] = \frac{a^4}{n^4}\sum_{r=1}^{n} r^3$$

$$= \frac{a^4}{n^4}(1^3 + 2^3 + \ldots + n^3) = \frac{a^4}{n^4}\cdot\left[\frac{n(n+1)}{2}\right]^2 = \frac{a^4}{4}\left(1 + \frac{1}{n}\right)^2.$$

$\therefore$ $$\overline{\int_0^a} f(x)\,dx = \lim_{n\to\infty} U(P, f) = \lim_{n\to\infty}\frac{a^4}{4}\left(1 + \frac{1}{n}\right)^2 = \frac{a^4}{4}.$$

Since $\underline{\int_0^a} f = \overline{\int_0^a} f$, f is Riemann integrable on $[0, a]$ and

$$\int_0^a f(x)\,dx = \int_0^a x^3\,dx = \frac{a^4}{4}\cdot$$

Example 5: *Let f be the function defined on* $[0, 1]$ *by*

$$f(x) = \begin{cases} 0 & \textit{when x is irrational,} \\ 1 & \textit{when x is rational.}\end{cases}$$

Calculate $\underline{\int_0^1} f$ *and* $\overline{\int_0^1} f$ *and hence show that* $f \notin \boldsymbol{R}[0, 1]$.

(Purvanchal 2008, 09; Garhwal 11; Rohilkhand 11)

Solution: First, we observe that f is bounded, for evidently

$$0 \le f(x) \le 1 \;\forall\; x \in [0, 1].$$

Let P be any partition of $[0,1]$. Then for any subinterval $[x_{r-1}, x_r]$ of P, we have $m_r = 0$ and $M_r = 1$, because every subinterval will contain rational as well as irrational numbers. Note that rational as well as irrational points are everywhere dense.

It follows that

$$L(P, f) = \sum_{r=1}^{n} m_r \, \Delta x_r = \sum_{r=1}^{n} 0 \, . \, \Delta x_r = 0$$

and $$U(P, f) = \sum_{r=1}^{n} M_r \, \Delta x_r = \sum_{r=1}^{n} 1 \, . \, \Delta x_r = \sum_{r=1}^{n} \Delta x_r = 1.$$

$\therefore$ $$\underline{\int_0^1} f = \lim_{n \to \infty} L(P, f) = 0 \text{ and } \overline{\int_0^1} f = \lim_{n \to \infty} U(P, f) = 1.$$

Since $\underline{\int_0^1} f \neq \overline{\int_0^a} f$, we have $f \notin \boldsymbol{R}[0,1]$.

Example 6: *Give an example of a bounded function which is not R-integrable over* $[0,1]$.

(Garhwal 2006, 09)

Solution: See example 5 above. As another example, consider the function f defined on $[0,1]$ as follows :

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational.} \end{cases}$$

Evidently f is bounded on $[0,1]$.

If P is any partition of $[0,1]$, then for any subinterval $[x_{r-1}, x_r]$ of P, we have $m_r = -1$ and $M_r = 1, r = 1, 2, \ldots, n$.

$\therefore$ $$L(P, f) = \sum_{r=1}^{n} m_r \, \Delta x_r = \sum_{r=1}^{n} -1 \, . \, \Delta x_r = -1 \, . \sum_{r=1}^{n} \Delta x_r$$

$$= -1 \, . \, (1 - 0) = -1 \, . \, 1 = -1,$$

$$U(P, f) = \sum_{r=1}^{n} M_r \, \Delta x_r = 1. \sum_{r=1}^{n} \Delta x_r = 1 \, . \, (1 - 0) = 1.$$

Hence $$\underline{\int_0^1} f = \lim_{n \to \infty} L(P, f) = -1, \; \overline{\int_0^1} f = \lim_{n \to \infty} U(P, f) = 1.$$

Thus $\underline{\int_0^1} f \neq \overline{\int_0^1} f$ and consequently $f \notin \boldsymbol{R}[0,1]$.

Example 7: *Show that* $f(x) = \sin x$ *is integrable on* $\left[0, \frac{1}{2}\pi\right]$ *and* $\int_0^{\pi/2} \sin x \, dx = 1$.

Solution: Let $P = \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \ldots, \frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}, \ldots, \frac{n\pi}{2n} = \frac{\pi}{2}\right\}$

be the partition of $[0, \pi/2]$ obtained by dissecting $[0, \frac{1}{2}\pi]$ into n equal parts. The length of each subinterval $= \pi / 2n$ and the r th subinterval $I_r = \left[\frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}\right]$.

Since $f(x) = \sin x$ is increasing in $[0, \frac{1}{2}\pi]$, we have

$$m_r = \sin \frac{(r-1)\pi}{2n} \text{ and } M_r = \sin \frac{r\pi}{2n}, r = 1, 2, \ldots, n.$$

Now $$U(P, f) = \sum_{r=1}^{n} M_r \, \Delta x_r = \sum_{r=1}^{n} \left(\sin \frac{r\pi}{2n} \right) \cdot \frac{\pi}{2n}$$

$$= \frac{\pi}{2n} \left[\sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin \frac{n\pi}{2n} \right].$$

We know that

$$\sin a + \sin (a+d) + \ldots + \sin \{a + (n-1) d\} = \frac{\sin \left(a + \frac{n-1}{2} d \right) \sin \frac{nd}{2}}{\sin (d/2)}.$$

$\therefore$ $$U(P, f) = \frac{\pi}{2n} \left[\frac{\sin \left(\frac{\pi}{2n} + \frac{n-1}{2} \cdot \frac{\pi}{2n} \right) \sin \frac{n\pi}{4n}}{\sin \frac{\pi}{4n}} \right] = \frac{\frac{\pi}{2n} \sin \frac{(n+1)\pi}{4n} \cdot \sin \frac{\pi}{4}}{\sin \frac{\pi}{4n}}$$

$$= \frac{\frac{\pi}{2n} \cdot \sin \left(\frac{\pi}{4} + \frac{\pi}{4n} \right) \cdot \frac{1}{\sqrt{2}}}{\sin \frac{\pi}{4n}} = \frac{\frac{\pi}{2\sqrt{2}n} \left\{ \sin \frac{\pi}{4} \cos \frac{\pi}{4n} + \cos \frac{\pi}{4} \sin \frac{\pi}{4n} \right\}}{\sin \left(\frac{\pi}{4n} \right)}$$

$$= \frac{\pi}{2\sqrt{2}n} \cdot \frac{1}{\sqrt{2}} \left(\cot \frac{\pi}{4n} + 1 \right) = \frac{\pi}{4n} \left(\cot \frac{\pi}{4n} + 1 \right).$$

Similarly, we can find that

$$L(P, f) = \frac{\pi}{4n} \left(\cot \frac{\pi}{4n} - 1 \right).$$

Now $$\underline{\int_0^{\pi/2}} f = \lim_{n \to \infty} L(P, f) = \lim_{n \to \infty} \frac{\pi}{4n} \left(\cot \frac{\pi}{4n} - 1 \right)$$

$$= \lim_{n \to \infty} \frac{(\pi/4n)}{\tan (\pi/4n)} - \lim_{n \to \infty} \frac{\pi}{4n} = 1 - 0 = 1$$

$$\left[\because \lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1 \right]$$

and $$\overline{\int_0^{\pi/2}} f = \lim_{n \to \infty} U(P, f) = \lim_{n \to \infty} \frac{\pi}{4n} \left(\cot \frac{\pi}{4n} + 1 \right) = 1.$$

Since $$\underline{\int_0^{\pi/2}} f = \overline{\int_0^{\pi/2}} f, f \in \mathbf{R} \left[0, \frac{\pi}{2} \right] \text{ and } \int_0^{\pi/2} f = 1.$$

Example 8: *Let $f(x)$ be a function defined on $[0, \frac{1}{4}\pi]$ by*

$$f(x) = \begin{cases} \cos x, & \text{if } x \text{ is rational} \\ \sin x, & \text{if } x \text{ is irrational}. \end{cases}$$

Show that f is not Riemann integrable over $\left[0, \frac{1}{4}\pi \right]$.

Solution: Consider the partition

$$P = \left\{ 0, \frac{\pi}{4n}, \frac{2\pi}{4n}, \ldots, \frac{(r-1)\pi}{4n}, \frac{r\pi}{4n}, \ldots, \frac{n\pi}{4n} = \frac{\pi}{4} \right\}$$

obtained by dissecting $\left[0, \frac{1}{4}\pi\right]$ into n equal parts.

Then for any subinterval $\left[\frac{(r-1)\pi}{4n}, \frac{r\pi}{4n}\right], r = 1, 2, \ldots, n,$ we have

$$m_r = \sin \frac{(r-1)\pi}{4n} \text{ and } M_r = \cos \frac{(r-1)\pi}{4n}, \Delta x_r = \frac{\pi}{4n}.$$

Hence
$$L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n} \sin \frac{(r-1)\pi}{4n} \cdot \frac{\pi}{4n}$$

$$= \frac{\pi}{4n} \left[\sin \frac{\pi}{4n} + \ldots + \sin \frac{(n-1)\pi}{4n} \right]$$

$$= \frac{\pi}{4n} \cdot \frac{\sin\left(\frac{\pi}{4n} + \frac{n-2}{2} \cdot \frac{\pi}{4n}\right) \cdot \sin \frac{n\pi}{8n}}{\sin \frac{\pi}{8n}}$$

[summing up the series]

$$= \frac{(\pi/8n)}{\sin(\pi/8n)} \cdot 2\sin^2 \frac{\pi}{8}$$

and
$$U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r = \sum_{r=1}^{n} \cos \frac{(r-1)\pi}{4n} \cdot \frac{\pi}{4n}$$

$$= \frac{\pi}{4n} \left[\cos 0 + \cos \frac{\pi}{4n} + \ldots + \cos \frac{(n-1)\pi}{4n} \right]$$

$$= \frac{\pi}{4n} \cdot \frac{\cos\left(\frac{n-1}{2} \cdot \frac{\pi}{4n}\right) \sin \frac{n\pi}{8n}}{\sin \frac{\pi}{8n}}$$

[summing up the series]

$$= \frac{(\pi/8n)}{\sin(\pi/8n)} \cdot 2\cos \frac{(n-1)\pi}{8n} \cdot \sin \frac{\pi}{8}.$$

Hence
$$\underline{\int_0^{\pi/4}} f = \lim_{||P|| \to 0} L(P, f) = \lim_{n \to \infty} L(P, f)$$

$$= \lim_{n \to \infty} \frac{(\pi/8n)}{\sin(\pi/8n)} \cdot 2\sin^2 \frac{\pi}{8} = 2\sin^2 \frac{\pi}{8} = \left(1 - \cos \frac{\pi}{4}\right)$$

$$= 1 - \frac{1}{\sqrt{2}}$$

and
$$\overline{\int_0^{\pi/4}} f = \lim_{n \to \infty} U(P, f)$$

$$= \lim_{n \to \infty} \frac{(\pi/8n)}{\sin(\pi/8n)} \cdot 2\cos \frac{(n-1)\pi}{8n} \cdot \sin \frac{\pi}{8}$$

$$= \lim_{n \to \infty} \frac{(\pi/8n)}{\sin(\pi/8n)} \cdot 2\cos \frac{\pi}{8}\left(1 - \frac{1}{n}\right) \sin \frac{\pi}{8}$$

$$= 2\cos \frac{\pi}{8} \sin \frac{\pi}{8} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Since $\underline{\int}_0^{\pi/4} f \neq \overline{\int}_0^{\pi/4} f$, f is not Riemann integrable over $\left[0, \frac{\pi}{4}\right]$.

Example 9: *Let f be a function on $[0,1]$ defined by*

$$f(x) = \begin{cases} 1 & \text{if } x \neq \frac{1}{2} \\ 0 & \text{if } x = \frac{1}{2} \end{cases}.$$

Show that $f \in \mathbf{R}[0,1]$ and find $\int_0^1 f$.

Solution: Obviously $f(x)$ is bounded in $[0,1]$ since

$$0 \le f(x) \le 1 \; \forall \; x \in [0,1].$$

Let P be a partition of $[0,1]$ such that the point $\frac{1}{2}$ belongs to the open interval $]x_{s-1}, x_s[$.

Then we have (with usual notations)

$$m_r = M_r = 1 \text{ for } r = 1, 2, \ldots, n$$

and $\quad r \neq s, m_s = 0, M_s = 1.$

Now $\quad U(P, f) - L(P, f)$

$$= \sum_{\substack{r=1 \\ r \neq s}}^{n} (M_r - m_r)\,\Delta x_r + (M_s - m_s)(x_s - x_{s-1})$$

$$= \sum_{r=1}^{n} (1-1)(x_r - x_{r-1}) + (1-0)(x_s - x_{s-1})$$

$$= x_s - x_{s-1}. \qquad \ldots(1)$$

Let $\varepsilon > 0$ be given. Then we choose a partition P such that the point $\frac{1}{2}$ is an interior point of one of the subintervals whose length is less than ε. Then, it follows from (1) that

$$U(P, f) - L(P, f) < \varepsilon.$$

Hence, by theorem of article 7.6, $f \in \mathbf{R}[0,1]$.

Now to find $\int_0^1 f$, it is enough to find $\underline{\int}_0^1 f$ or $\overline{\int}_0^1 f$.

We calculate $\overline{\int}_0^1 f$. For any partition P, we have

$$U(P, f) = \sum_{r=1}^{n} M_r\,\Delta x_r = \sum_{r=1}^{n} 1 . \Delta x_r = \sum_{r=1}^{n} \Delta x_r$$

$$= \text{length of the interval } [0,1] = 1.$$

Hence $\quad \overline{\int}_0^1 f = \lim_{n \to \infty} U(P, f) = 1.$

Thus $\quad f \in \mathbf{R}[0,1]$ and $\int_0^1 f = \overline{\int}_0^1 f = 1.$

Example 10: *If* $f(x) = x + x^2$ *for rational values of* x *in the interval* $[0,2]$ *and* $f(x) = x^2 + x^3$ *for irrational values of x in the same interval, evaluate the upper and the lower Riemann integrals of* f *over* $[0,2]$. **(Purvanchal 2007)**

Solution: We have $(x + x^2) - (x^2 + x^3) = x - x^3 = x(1 - x^2)$

so that $\quad (x + x^2) - (x^2 + x^3) > 0 \quad \text{if } 0 < x < 1$

and $\quad < 0 \quad \text{if } 1 < x < 2.$

If P is any partition of $[0,2]$, then any subinterval of P, however small it may be, will contain rational as well as irrational points.

With usual notations, we have for all values of r

$$M_r = x + x^2, \quad \text{if} \quad 0 < x < 1$$
$$= x^2 + x^3, \quad \text{if} \quad 1 < x < 2$$

and

$$m_r = x^2 + x^3, \quad \text{if} \quad 0 < x < 1$$
$$= x + x^2, \quad \text{if} \quad 1 < x < 2.$$

Hence

$$\overline{\int_0^2} f(x)\, dx = \int_0^1 (x + x^2)\, dx + \int_1^2 (x^2 + x^3)\, dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} + \frac{x^4}{4}\right]_1^2$$

$$= \left(\frac{1}{2} + \frac{1}{3}\right) - 0 + \left(\frac{8}{3} + \frac{16}{4}\right) - \left(\frac{1}{3} + \frac{1}{4}\right) = \frac{83}{12} = 6\,\frac{11}{12}$$

and

$$\underline{\int_0^2} f(x)\, dx = \int_0^1 (x^2 + x^3)\, dx + \int_1^2 (x + x^2)\, dx$$

$$= \left[\frac{x^3}{3} + \frac{x^4}{4}\right]_0^1 + \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{3} + \frac{1}{4} + \frac{4}{2} + \frac{8}{3} - \frac{1}{2} - \frac{1}{3} = \frac{53}{12} = 4\,\frac{5}{12}.$$

Example 11: *Find the upper and lower R-integrals for the function* f *defined on* $[0,1]$ *as follows:*

$$f(x) = \begin{cases} \sqrt{(1 - x^2)}, & \textit{if } x \textit{ is rational} \\ (1 - x), & \textit{if } x \textit{ is irrational}. \end{cases}$$

Solution: Here we have

$$(1 - x^2) - (1 - x)^2 = 2x - 2x^2 = 2x(1 - x) > 0 \;\; \forall\, x \in]0,1[$$

i.e., $\quad \sqrt{(1 - x^2)} > (1 - x) \;\; \forall\, x \in]0,1[.$

With usual notations, we have for all values of r

$$M_r = \sqrt{(1 - x^2)} \text{ and } m_r = (1 - x).$$

Hence

$$\underline{\int_0^1} f = \int_0^1 (1 - x)\, dx = \left[x - \frac{x^2}{2}\right]_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$$

and $\quad \overline{\int_0^1} f = \int_0^1 \sqrt{(1-x^2)}\, dx = \left[\frac{1}{2} x \sqrt{(1-x^2)} + \frac{1}{2} \sin^{-1} x\right]_0^1$

$$= \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.$$

Thus $\quad \underline{\int_0^1} f \neq \overline{\int_0^1} f$. It follows that $f \notin \boldsymbol{R}[0,1]$.

7 Some Classes of Integrable Functions

Integrability of Continuous Functions: **(Garhwal 2007; Purvanchal 07; 08, 10; Rohilkhand 11; Gorakhpur 10, 13, 15)**

Theorem 1: *If f is continuous on $[a,b]$, then $f \in \boldsymbol{R}[a,b]$.*

Proof: Since f is continuous on $[a,b]$, f is bounded on $[a,b]$. Also since f is continuous on a closed and bounded interval $[a,b]$, f is uniformly continuous on $[a,b]$. Hence for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all points x', x'' of $[a,b]$

$$|f(x') - f(x'')| < \frac{\varepsilon}{b-a} \text{ whenever } |x' - x''| < \delta. \qquad ...(1)$$

Let $\quad P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a,b]$ with

$$\|P\| < \delta.$$

Since f is continuous on $[x_{r-1}, x_r]$ it attains its infimum m_r and supremum M_r at some points c_r and d_r of $[x_{r-1}, x_r]$ respectively so that

$$m_r = f(c_r) \text{ and } M_r = f(d_r). \qquad ...(2)$$

Since $|c_r - d_r| < \delta$, therefore from (1), we get

$$|f(c_r) - f(d_r)| < \varepsilon/(b-a).$$

But $\quad |f(c_r) - f(d_r)| = f(d_r) - f(c_r) \qquad [\because f(d_r) \geq f(c_r)]$

$$= M_r - m_r.$$

Thus $\quad M_r - m_r < \varepsilon/(b-a), r = 1, 2, \ldots, n.$

Now for the partition P of $[a,b]$, we have

$$0 \leq U(P,f) - L(P,f) = \sum_{r=1}^{n} (M_r - m_r)\, \Delta x_r$$

$$< \sum_{r=1}^{n} \{\varepsilon/(b-a)\}\, \Delta x_r$$

$$= \frac{\varepsilon}{b-a} \sum_{r=1}^{n} \Delta x_r = \frac{\varepsilon}{b-a} \cdot (b-a) = \varepsilon.$$

Hence by theorem of article 6, f is R-integrable over $[a,b]$,

i.e., $\quad f \in \boldsymbol{R}[a,b]$.

Note: There exist functions which are integrable but not continuous. So, continuity is a sufficient condition but not necessary for integrability.

Integrability of Monotonic Functions:

Theorem 2: *If f is monotonic on $[a,b]$, then $f \in \boldsymbol{R}[a,b]$.*

(Garhwal 2007; Purvanchal 08, 12; Rohilkhand 10)

Proof: Suppose f is monotonic increasing on $[a, b]$.

Then $$f(a) \leq f(x) \leq f(b) \;\; \forall \;\; x \in [a, b].$$

$\therefore$ f is bounded on $[a, b]$ and $\inf f = f(a)$, $\sup f = f(b)$.

Let $\varepsilon > 0$ be given and $P = \{ a = x_0, x_1, x_2, \ldots, x_n = b \}$ be a partition of $[a, b]$ with $\| P \| \leq \varepsilon / [f(b) - f(a) + 1]$.

If m_r and M_r be the inf and sup of f on I_r, then $m_r = f(x_{r-1})$ and $M_r = f(x_r)$ because f is monotonic increasing on $[a, b]$.

Hence
$$U(P, f) - L(P, f) = \sum_{r=1}^{n} (M_r - m_r) \Delta x_r$$
$$= \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})] \Delta x_r$$
$$\leq \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})] \cdot \frac{\varepsilon}{f(b) - f(a) + 1}$$
$$= \frac{\varepsilon}{f(b) - f(a) + 1} \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})]$$
$$= \frac{\varepsilon}{f(b) - f(a) + 1} [f(x_1) - f(x_0) + f(x_2) - f(x_1) + \ldots + f(x_n) - f(x_{n-1})]$$
$$= \frac{\varepsilon}{f(b) - f(a) + 1} [f(b) - f(a)] \qquad [\because f(x_0) = f(a), f(x_n) = f(b)]$$
$$< \varepsilon.$$

It follows from the theorem of article 7.6 that $f \in \mathbf{R}[a, b]$.

If f is monotonic decreasing on $[a, b]$, then $-f$ is monotonic increasing on $[a, b]$ and so $-f \in \mathbf{R}[a, b]$.

$\therefore$ $$\underline{\int_a^b} - f(x)\, dx = \overline{\int_a^b} - f(x)\, dx$$

i.e., $$-\underline{\int_a^b} f(x)\, dx = -\overline{\int_a^b} f(x)\, dx$$

or $$\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx$$

i.e., f is R-integrable on $[a, b]$.

Hence if f is monotonic on $[a, b]$, $f \in \mathbf{R}[a, b]$.

Note: If we had taken $f(b) - f(a)$ instead of $f(b) - f(a) + 1$, the proof would not have been valid when $f(b) - f(a) = 0$ *i.e.*, when f is a constant function. It is to include this case that we have used this artifice.

Theorem 3: *If the set of points of discontinuity of a bounded function f defined on $[a, b]$ is finite, then $f \in \mathbf{R}[a, b]$.* **(Garhwal 2012)**

Proof: Since f is discontinuous on $[a, b]$, the supremum M and the infimum m of f in $[a, b]$ are not equal *i.e.*, $M - m \neq 0$.

Let $\{c_1, c_2, \ldots, c_p\}$ be the finite set of points of discontinuity of f in $[a, b]$, where $c_1 < c_2 < \ldots < c_p$. Let $\varepsilon > 0$ be given. We enclose the points $c_1, c_2, \ldots, c_p$ respectively in p mutually disjoint intervals

$$[a_1, b_1], [a_2, b_2], \ldots, [a_p, b_p] \qquad \ldots(1)$$

such that the sum of their lengths is $< \varepsilon / 2(M - m)$.

Now f is continuous on each of the subintervals

$$[a, a_1], [b_1, a_2], [b_2, a_3], \ldots, [b_p, b]. \qquad \ldots(2)$$

Consequently, there exist partitions $P_r, r = 1, 2, \ldots, p+1$ respectively, of the subintervals in (2) such that, using theorem 1 of article 7,

$$\omega(P_r, f) < \frac{\varepsilon}{2(p+1)} \text{ for } r = 1, 2, \ldots, p+1.$$

Now consider the partition P of $[a, b]$ defined by

$$P = \cup \{P_r : r = 1, 2, \ldots, p+1\}.$$

The subintervals of P can be divided into two groups :

(i) all the subintervals of $P_r, r = 1, 2, \ldots, p+1$

(ii) all the subintervals given in (1).

Corresponding to these two groups of subintervals we have two contributions to the oscillatory sum $\omega(P, f)$.

The total contribution to the oscillatory sum $\omega(P, f)$ of the subintervals in (i) is

$$< \frac{\varepsilon}{2(p+1)}(p+1) = \frac{\varepsilon}{2}.$$

Also since the oscillation of f in each of the subintervals in (ii) is $\leq M - m$, the total contribution to the oscillatory sum $\omega(P, f)$ of the subintervals in (ii) is

$$< \frac{\varepsilon}{2(M-m)} \cdot (M - m) = \frac{\varepsilon}{2}.$$

$\therefore$ for the partition P of $[a, b]$, we have

$$\omega(P, f) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus for each $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that

$$0 \leq \omega(P, f) < \varepsilon.$$

Hence $f \in \boldsymbol{R}[a, b]$, by theorem of article 6.

Theorem 4: *If the set of points of discontinuity of a bounded function f defined on $[a, b]$ has only a finite number of limit points then $f \in \boldsymbol{R}[a, b]$.*

Proof: Since f is discontinuous on $[a, b]$, the supremum M and the infimum m of f in $[a, b]$ are not equal *i.e.,* $M - m \neq 0$.

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ be the finite set of limit points of the set of the points of discontinuity of f in $[a, b]$, where $\alpha_1 < \alpha_2 < \ldots < \alpha_p$. Let $\varepsilon > 0$ be given. We enclose the points $\alpha_1, \alpha_2, \ldots, \alpha_p$ respectively in p mutually disjoint intervals

$$[a_1, b_1], [a_2, b_2], \ldots, [a_p, b_p] \qquad \ldots(1)$$

such that the sum of their lengths is $< \varepsilon / 2(M - m)$.

In each of the remaining $p+1$ subintervals

$$[a, a_1], [b_1, a_2], [b_2, a_3], \ldots, [b_p, b] \qquad \ldots(2)$$

f has only a finite number of points of discontinuity because none of these $p+1$ sub-intervals contains a limit point of the set of points of discontinuity of f.

Hence, by the previous theorem, there exist partitions $P_r, r = 1, 2, \ldots, p+1$ respectively of the subintervals in (2) such that

$$\omega(P_r, f) < \frac{\varepsilon}{2(p+1)} \text{ for } r = 1, 2, \ldots, p+1.$$

Now consider the partition P of $[a, b]$ determined by

$$P = \cup \{P_r : r = 1, 2, \ldots, p+1\}.$$

The subintervals of P can be divided into two groups :

(i) all the subintervals of $P_r, r = 1, 2, \ldots, p+1$.

(ii) all the subintervals given in (1).

The total contribution to the oscillatory sum $\omega(P, f)$ of the subintervals in (i) is

$$< \{\varepsilon / 2(p+1)\}.(p+1) = \varepsilon / 2.$$

Also since the oscillation of f in each of the subintervals in (ii) is $\leq M - m$, the total contribution to the oscillatory sum $\omega(P, f)$ of the subintervals in (ii) is

$$< \{\varepsilon / 2(M-m)\}.(M-m) = \varepsilon / 2.$$

Thus for any $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$\omega(P, f) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence, by theorem of article 6, $f \in \boldsymbol{R}[a, b]$.

Illustrative Examples

Example 12: *A function f is defined on $[0, 1]$ by*

$$f(x) = 1/2^n \text{ for } 1/2^{n+1} < x \leq 1/2^n, n = 0, 1, 2, \ldots.$$

and $f(0) = 0$. Show that $f \in \boldsymbol{R}[0, 1]$ and calculate the value of

$$\int_0^x f(t)\, dt$$

where x lies between $1/2^m$ and $1/2^{m-1}$. **(Garhwal 2007)**

Solution: Here, for $n = 1, 2, 3, \ldots$, we have $f\left(\frac{1}{2^n} + 0\right) = \frac{1}{2^{n-1}}$ and $f\left(\frac{1}{2^n} - 0\right) = \frac{1}{2^n}$,

which shows that the function $f(x)$ is discontinuous at $x = 1/2^n$, $n = 1, 2, 3, \ldots$.

Also for $n = 0, f\left(\frac{1}{2^n}\right) = f(1) = 1$

and $\qquad f\left(\frac{1}{2^n} - 0\right) = 1$ so that $f(x)$ is continuous at $x = \frac{1}{2^0} = 1$.

Thus the points of discontinuity of f are

$$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots$$

Since the set of these points of discontinuity of f has only one limiting point at $x = 0$, it follows from theorem 4 of article 7 that $f \in \boldsymbol{R}[0,1]$.

Now $$\int_0^x f(t)\,dt = \int_{1/2^m}^{x} f + \int_{1/2^{m+1}}^{1/2^m} f + \int_{1/2^{m+2}}^{1/2^{m+1}} f + \dots$$

$$= \int_{1/2^m}^{x} \frac{1}{2^{m-1}} + \int_{1/2^{m+1}}^{1/2^m} \frac{1}{2^m} + \int_{1/2^{m+2}}^{1/2^{m+1}} \frac{1}{2^{m+1}} + \dots$$

$$= \frac{1}{2^{m-1}}\left[x - \frac{1}{2^m}\right] + \frac{1}{2^m}\left[\frac{1}{2^m} - \frac{1}{2^{m+1}}\right] + \frac{1}{2^{m+1}}\left[\frac{1}{2^{m+1}} - \frac{1}{2^{m+2}}\right] + \dots$$

$$= \frac{1}{2^{m-1}}\left[x - \frac{1}{2^m}\right] + \frac{1}{2^{2m+1}} + \frac{1}{2^{2m+3}} + \frac{1}{2^{2m+5}} + \dots$$

$$= \frac{x}{2^{m-1}} - \frac{1}{2^{2m-1}} + \frac{1/2^{2m+1}}{1 - \frac{1}{4}}$$

(Summing up the infinite G.P.)

$$= \frac{x}{2^{m-1}} - \frac{1}{2^{2m-1}} + \frac{1}{3 \cdot 2^{2m-1}} = \frac{x}{2^{m-1}} - \frac{1}{3 \cdot 2^{2m-2}}.$$

Example 13: *Let the function f be defined on $[0,1]$ as follows :*

$$f(x) = 2rx \quad \textit{when} \quad \frac{1}{r+1} < x \le \frac{1}{r}, \quad r = 1, 2, 3, \dots$$

Prove that f is R-integrable in $[0,1]$ and evaluate $\int_0^1 f(x)\,dx$.

(Rohilkhand 2010; Gorakhpur 13)

Solution: The given function f is not defined for $x = 0$. We may, however, define f at this point in any manner we like provided f remains bounded.

Here
$$f(x) = 2x \quad \text{when} \quad \frac{1}{2} < x \le 1$$
$$f(x) = 4x \quad \text{when} \quad \frac{1}{3} < x \le \frac{1}{2}$$
$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$
$$f(x) = 2(r-1)x \quad \text{when} \quad \frac{1}{r} < x \le \frac{1}{r-1}$$
$$f(x) = 2rx \quad \text{when} \quad \frac{1}{r+1} < x \le \frac{1}{r}$$
$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

For $r = 2, 3, 4, \dots$, we have

$$f\left(\frac{1}{r} + 0\right) = \lim_{h \to 0} 2(r-1)\left(\frac{1}{r} + h\right) = 2 - \frac{2}{r}$$

and $\quad f\left(\frac{1}{r}-0\right)=\lim_{h\to 0} 2r\left(\frac{1}{r}-h\right)=2.$

Since $\quad f\left(\frac{1}{r}+0\right)\neq f\left(\frac{1}{r}-0\right)$, f is not continuous at $x=1/r$.

Also $f(1)=2$ and $f(1-0)=\lim_{h\to 0} 2(1-h)=2$, so that f is continuous at $x=1$.

Thus the given function f is not continuous at $x=1/r$, $r=2,3,4,\ldots$ and the set of points of discontinuity $\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots$ of f has only one limit point at $x=0$. Consequently, by theorem 4 of article 7, the given function is R-integrable.

Now $\quad \int_{1/(n+1)}^{1} f(x)\,dx=\int_{1/2}^{1} f+\int_{1/3}^{1/2} f+\int_{1/4}^{1/3} f+\ldots+\int_{1/(n+1)}^{1/n} f$

$$=\sum_{r=1}^{n}\int_{1/(r+1)}^{1/r} f. \qquad \ldots(1)$$

We have $\quad \int_{1/(r+1)}^{1/r} f(x)\,dx=\int_{1/(r+1)}^{1/r} 2rx\,dx=[rx^2]_{1/(r+1)}^{1/r}$

$$=r\left[\frac{1}{r^2}-\frac{1}{(r+1)^2}\right]. \qquad \ldots(2)$$

Putting $r=1,2,\ldots,n$ in (2) and then adding the partial integrals, in view of (1), we have

$$\int_{1/(n+1)}^{1} f(x)\,dx=1\left[\frac{1}{1^2}-\frac{1}{2^2}\right]+2\left[\frac{1}{2^2}-\frac{1}{3^2}\right]+3\left[\frac{1}{3^2}-\frac{1}{4^2}\right]+\ldots$$
$$+n\left[\frac{1}{n^2}-\frac{1}{(n+1)^2}\right]$$
$$=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2}-\frac{n}{(n+1)^2}$$
$$=\sum_{r=1}^{n}\frac{1}{r^2}-\frac{1/n}{\left(1+\frac{1}{n}\right)^2}.$$

Letting n tend to infinity, we get

$$\int_0^1 f(x)\,dx=\sum_1^{\infty}\frac{1}{r^2}-0=\sum_1^{\infty}\frac{1}{r^2}=\frac{\pi^2}{6}.$$

Example 14: *A function f is defined on $[0,1]$ by*

$$f(x)=1/n \text{ for } 1/(n+1)<x\leq 1/n,\ n=1,2,3,\ldots \text{ and } f(0)=0.$$

Prove that $f\in \boldsymbol{R}[0,1]$ and evaluate $\int_0^1 f(x)\,dx$.

Solution: It can be easily seen that the points of discontinuity of f are $\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots$.

Since the set of points of discontinuity of f has only one limit point at $x=0$, it follows from theorem 4 of article 7 that $f\in \boldsymbol{R}[0,1]$.

Now as in the previous example,

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} \int_{1/(r+1)}^{1/r} \frac{1}{r}\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{r}\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$= \lim_{n\to\infty} \left[\left(1-\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2}-\frac{1}{3}\right) + \ldots + \frac{1}{n}\left(\frac{1}{n} - \frac{1}{n+1}\right)\right]$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2}\right) - \left(\frac{1}{2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n.(n+1)}\right)\right]$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2}\right) - \left(\frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n} - \frac{1}{(n+1)}\right)\right]$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2}\right) - \left(1 - \frac{1}{n+1}\right)\right] = \frac{\pi^2}{6} - 1.$$

$$\left[\because \text{ the series } 1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2} + \ldots \text{converges to } \frac{\pi^2}{6}\right]$$

Example 15: *Let a function f be defined on $[0,1]$ as follows :*

If x is irrational let $f(x) = 0$, if x is a rational number p/q in its lowest terms, let $f(x) = 1/q$, also let $f(0) = f(1) = 0$. Show that f is integrable over $[0,1]$ and $\int_0^1 f(x)\,dx = 0$.

Solution: Evidently, the function f is bounded on $[0,1]$. Let $\varepsilon > 0$ be given. Then there exist only a finite number of fractions $\frac{1}{q}$ such that $\frac{1}{q} > \frac{\varepsilon}{2}$, for $\frac{1}{q} > \frac{\varepsilon}{2}$ iff $q < \frac{2}{\varepsilon}$ and ε being given, $\frac{2}{\varepsilon}$ is finite. We enclose these exceptional points, in order, in mutually disjoint closed intervals

$$[a_1, b_1], [a_2, b_2], \ldots, [a_m, b_m] \qquad \ldots(1)$$

such that the sum of their lengths is less than $\varepsilon/2$. On the remaining sub-intervals

$$[b_0 = 0, a_1], [b_1, a_2], [b_2, a_3], \ldots, [b_m, a_{m+1} = 1] \qquad \ldots(2)$$

at each point the value of f is $< \varepsilon/2$.

It is obvious that the oscillation of f in each of the subintervals (1) cannot exceed 1 and the oscillation of f in each of the sub-intervals (2) is less than $\varepsilon/2$. Thus for the partition

$$P = \{0 = b_0, a_1, b_1, a_2, b_2, \ldots, a_m, b_m, b_{m+1} = 1\}$$

we have, $\omega(P, f) = U(P, f) - L(P, f)$

$$< \sum_{r=1}^{m} 1.(b_r - a_r) + \sum_{r=0}^{m} \frac{\varepsilon}{2}(a_{r+1} - b_r)$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

$$\left[\because \sum_{r=1}^{m}(b_r - a_r) < \frac{\varepsilon}{2}, \sum_{r=0}^{m}(a_{r+1} - b_r) < 1\right]$$

Thus for a given $\varepsilon > 0$ there exists a partition P such that $\omega(P, f) < \varepsilon$.

Hence $f \in \boldsymbol{R}[0,1]$.

Also since for every partition $P, L(P, f) = 0$, we have

$$\int_0^1 f = \underline{\int_0^1} f = 0.$$

8 Algebra of Integrable Functions

Definition: *If $b < a$, we define $\int_a^b f$ to be $-\int_b^a f$ provided that $f \in \mathbf{R}[a, b]$.*

Also by definition, we write $\int_a^a f = 0$.

Theorem 1: *Let $f \in \mathbf{R}[a, b]$ and let m, M be bounds of f on $[a, b]$. Then*

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a) \text{ if } b \geq a,$$

$$m(b-a) \geq \int_a^b f(x)\,dx \geq M(b-a) \text{ if } b \leq a.$$

(Purvanchal 2008)

Proof: If $a = b$, the result is trivial. Let $b > a$. Then by theorem 1 of article 7.2, we have for all partitions P of $[a, b]$,

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a)$$

or $\quad m(b-a) \leq L(P, f) \leq M(b-a)$

or $\quad m(b-a) \leq \sup L(P, f) \leq M(b-a)$

or $\quad m(b-a) \leq \underline{\int_a^b} f \leq M(b-a)$

or $\quad m(b-a) \leq \int_a^b f \leq M(b-a). \qquad \left[\because \underline{\int_a^b} f = \int_a^b f\right]$

Now let $b < a$ so that $a > b$. Hence, as proved above we get

$$m(a-b) \leq \int_b^a f \leq M(a-b)$$

$\Rightarrow \quad -m(b-a) \leq -\int_a^b f \leq -M(b-a)$, by the definition mentioned above

$\Rightarrow \quad m(b-a) \geq \int_a^b f \geq M(b-a).$

Corollary 1: *Let $f \in \mathbf{R}[a, b]$. Then there exists a number μ lying between the bounds m and M of f such that $\int_a^b f(x)\,dx = \mu(b-a)$.* **(Purvanchal 2011)**

This result is known as the **first mean value theorem of integral calculus.**

Corollary 2: *Let f be continuous on $[a, b]$. Then there exists a point $c \in [a, b]$ such that*

$$\int_a^b f(x)\,dx = (b-a) f(c).$$

Proof: Since f is continuous, $f \in \mathbf{R}[a, b]$. Therefore, by Corollary 1, there exists a number μ lying between m and M, such that

$$\int_a^b f(x)\,dx = \mu(b-a).$$

Now f being continuous on $[a, b]$, so it takes every value between its bounds m and M *i.e.,* in particular it takes the value μ lying between m and M. Consequently there is a point $c \in [a, b]$ such that $f(c) = \mu$ and hence

$$\int_a^b f(x)\, dx = (b - a) f(c).$$

Corollary 3: *Let $f \in \mathbf{R}[a, b]$ and let K be a number such that*

$$|f(x)| \le K \;\forall\; x \in [a, b].$$

Then $$\left| \int_a^b f(x)\, dx \right| \le K\,|b - a|.$$

(Purvanchal 2010)

Proof: The result is trivial when $a = b$. Let m, M be the bounds of f on $[a, b]$.

Let $b > a$. We have, for all $x \in [a, b]$

$$|f(x)| \le K \Rightarrow -K \le f(x) \le K \quad \Rightarrow \quad -K \le m \le f(x) \le M \le K$$

$$\Rightarrow \quad -K(b - a) \le m(b - a) \le \int_a^b f(x)\, dx \le M(b - a) \le K(b - a)$$

$$\Rightarrow \quad \left| \int_a^b f(x)\, dx \right| \le K(b - a). \qquad \text{...(1)}$$

Now let $b < a$ so that $a > b$.

Then we get from (1) that

$$\left| \int_b^a f(x)\, dx \right| \le K(a - b) \Rightarrow \left| -\int_a^b f(x)\, dx \right| \le K(a - b)$$

$$\Rightarrow \quad \left| \int_a^b f(x)\, dx \right| \le K(a - b). \qquad \text{...(2)}$$

Hence, from (1) and (2), we get

$$\left| \int_a^b f(x)\, dx \right| \le K\,|b - a|.$$

Corollary 4: *Let $f \in \mathbf{R}[a, b]$ and let $f(x) \ge 0 \;\forall\; x \in [a, b]$. Then*

$$\int_a^b f(x)\, dx \ge 0 \text{ if } b \ge a \text{ and } \int_a^b f(x)\, dx \le 0 \text{ if } b \le a.$$

Proof: Since $f(x) \ge 0 \;\forall\; x \in [a, b]$, hence, $m \ge 0$.

If $b \ge a$, then from the first result of the above theorem, we get

$$\int_a^b f(x)\, dx \ge m(b - a) \ge 0$$

and if $b \le a$, then from the second result of the above theorem, we get

$$\int_a^b f(x)\, dx \le m(b - a) \le 0. \qquad [\because m \ge 0 \text{ and } b - a \le 0]$$

Corollary 5: *Let $f, g \in \mathbf{R}[a, b]$. Then*

$$f \ge g \Rightarrow \begin{cases} \int_a^b f(x)\, dx \ge \int_a^b g(x)\, dx \text{ if } b \ge a \\ \int_a^b f(x)\, dx \le \int_a^b g(x)\, dx \text{ if } b \le a. \end{cases}$$

Proof: We have

$$f \ge g \Rightarrow [f(x) - g(x)] \ge 0 \;\forall\; x \in [a, b]$$

$$\Rightarrow \qquad \int_a^b [f(x) - g(x)]\, dx \geq 0 \text{ or } \leq 0$$

according as $b \geq a$ or $b \leq a$ by Corollary 4,

$$\Rightarrow \qquad \left[\int_a^b f(x)\, dx - \int_a^b g(x)\, dx\right] \geq 0 \quad \text{or} \quad \leq 0$$

$$\Rightarrow \qquad \int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx \qquad \text{or} \qquad \int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$$

according as $b \geq a$ or $b \leq a$.

Theorem 2: *If $f \in \mathbf{R}[a,b]$ and $K \in \mathbf{R}$, then $Kf \in \mathbf{R}[a,b]$ and*

$$\int_a^b K f(x)\, dx = K \int_a^b f(x)\, dx.$$

Proof: If $K = 0$, the theorem is obvious. Suppose that $K \neq 0$. Since $f \in \mathbf{R}[a,b]$, f is bounded on $[a,b]$ and

$$\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx = \int_a^b f(x)\, dx. \qquad ...(1)$$

We know that $|K f| = |K| \cdot |f|$, so that Kf is bounded on $[a,b]$.

Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be any partition of $[a,b]$ and m_r and M_r be the infimum and supremum of f in $[x_{r-1}, x_r]$.

Case I: Let $K > 0$. Then $K\, m_r$ and $K\, M_r$ are the inf and sup of $K\, f$ in $[x_{r-1}, x_r]$. So we have

$$\underline{\int_a^b} Kf(x)\, dx = \sup\left[\sum_{r=1}^{n} K\, m_r\, \Delta x_r\right]$$

$$= K \sup\left[\sum_{r=1}^{n} m_r\, \Delta x_r\right] = K \underline{\int_a^b} f(x)\, dx \qquad ...(2)$$

$$= K \overline{\int_a^b} f(x)\, dx, \quad \text{using (1)}$$

$$= K. \inf\left[\sum_{r=1}^{n} M_r\, \Delta x_r\right] = \inf\left[\sum_{r=1}^{n} K\, M_r\, \Delta x_r\right]$$

$$= \underline{\int_a^b} K\, f(x)\, dx.$$

Thus $$\underline{\int_a^b} K\, f(x)\, dx = \overline{\int_a^b} K\, f(x)\, dx.$$

Hence $Kf \in \mathbf{R}[a,b]$.

Also from (2), we have

$$\underline{\int_a^b} K\, f(x)\, dx = K \underline{\int_a^b} f(x)\, dx$$

$$\Rightarrow \qquad \int_a^b K\, f(x)\, dx = K \int_a^b f(x)\, dx. \qquad [\because f \in \mathbf{R}[a,b] \text{ and } K f \in \mathbf{R}[a,b]]$$

Case II: Let $K < 0$. Then $K\, M_r$ and $K\, m_r$ respectively denote the inf and sup of $K\, f$ in $[x_{r-1}, x_r]$. We have

$$\underline{\int_a^b} Kf(x)\, dx = \sup\left[\sum_{r=1}^{n} K\, M_r\, \Delta x_r\right]$$

$$= K \inf \left[\sum_{r=1}^{n} M_r \, \Delta x_r \right], \text{as } K < 0$$

$$= K \overline{\int_a^b} f(x)\, dx = K \underline{\int_a^b} f(x)\, dx \quad \ldots(3)$$

$$= K \sup \left[\sum_{r=1}^{n} m_r \, \Delta x_r \right] = \inf \left[\sum_{r=1}^{n} K m_r \, \Delta x_r \right], \text{as } K < 0$$

$$= \overline{\int_a^b} K f(x)\, dx.$$

Thus in this case, $\underline{\int_a^b} K f(x)\, dx = \overline{\int_a^b} K f(x)\, dx.$

Hence $Kf \in \boldsymbol{R}[a, b]$. Also from (3), we have

$$\underline{\int_a^b} K f(x)\, dx = K \underline{\int_a^b} f(x)\, dx$$

$$\Rightarrow \quad \int_a^b K f(x)\, dx = K \int_a^b f(x)\, dx. \qquad [\because f \in \boldsymbol{R}[a, b] \text{ and } Kf \in \boldsymbol{R}[a, b]]$$

Theorem 3: *Let $a < c < b$. Then $f \in \boldsymbol{R}[a, b]$ iff $f \in \boldsymbol{R}[a, c]$ and $f \in \boldsymbol{R}[c, b]$. In either case*

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

(Gorakhpur 2015)

Proof: Obviously f is bounded on $[a, c]$ and $[c, b]$ iff f is bounded on $[a, b]$.

Let $f \in \boldsymbol{R}[a, b]$. Then for a given $\varepsilon > 0$, there is a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon. \quad \ldots(1)$$

Let $P^* = P \cup \{c\}$. Then P^* is also a partition of $[a, b]$ and it is a refinement of P so that

$$U(P^*, f) - L(P^*, f) \le U(P, f) - L(P, f) < \varepsilon. \quad \ldots(2)$$

Let P_1 and P_2 be the partitions of $[a, c]$ and $[c, b]$ respectively such that $P^* = P_1 \cup P_2$. Then

$$U(P^*, f) - L(P^*, f) = [U(P_1, f) + U(P_2, f)] - [L(P_1, f) + L(P_2, f)]$$

$$= [U(P_1, f) - L(P_1, f)] + [U(P_2, f) - L(P_2, f)]$$

$$< \varepsilon, \text{ by (2)}.$$

Since each of $[U(P_1, f) - L(P_1, f)]$ and $[U(P_2, f) - L(P_2, f)]$ is ≥ 0, each of them is less than ε.

Hence $f \in \boldsymbol{R}[a, c]$ and $f \in \boldsymbol{R}[c, b]$.

Also $U(P^*, f) = U(P_1, f) + U(P_2, f)$

$$\Rightarrow \quad \inf U(P^*, f) = \inf U(P_1, f) + \inf U(P_2, f)$$

$$\Rightarrow \quad \overline{\int_a^b} f(x)\, dx = \overline{\int_a^a} f(x)\, dx + \overline{\int_c^b} f(x)\, dx$$

$$\Rightarrow \quad \int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

$$[\because f \in \boldsymbol{R}[a, b], f \in \boldsymbol{R}[a, c], f \in \boldsymbol{R}[c, b]]$$

Conversely, let $f \in \boldsymbol{R}[a,c]$ and $f \in \boldsymbol{R}[c,b]$.

Then given $\varepsilon > 0$, there exist partitions P_1 and P_2 of $[a,c]$ and $[c,b]$ respectively such that

$$U(P_1, f) - L(P_1, f) < \varepsilon/2$$

and $$U(P_2, f) - L(P_2, f) < \varepsilon/2.$$

Let $P = P_1 \cup P_2$, then P is partition of $[a,b]$.

Now
$$\begin{aligned} U(P,f) - L(P,f) &= [U(P_1, f) + U(P_2, f)] - [L(P_1, f) + L(P_2, f)] \\ &= [U(P_1, f) - L(P_1, f)] + [U(P_2, f) - L(P_2, f)] \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence $f \in \boldsymbol{R}[a,b]$. The remaining part of the theorem can be proved as before.

Theorem 4: *If* $f, g \in \boldsymbol{R}[a,b]$ *then* $f \pm g \in \boldsymbol{R}[a,b]$

and $$\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx.$$

Proof: Since $f, g \in \boldsymbol{R}[a,b]$, for a given $\varepsilon > 0$, there exist partitions P_1 and P_2 of $[a,b]$ such that

$$\left.\begin{aligned} &U(P_1, f) - L(P_1, f) < \varepsilon/2 \\ \text{and} \quad &U(P_2, g) - L(P_2, g) < \varepsilon/2 \end{aligned}\right\} \qquad \text{....(1)}$$

Let $P = P_1 \cup P_2$, then P is a common refinement of P_1 and P_2 and it is a partition of $[a,b]$. We have

$$\begin{aligned} &U(P, f+g) - L(P, f+g) \\ &\le [U(P,f) + U(P,g)] - [L(P,f) + L(P,g)], \end{aligned}$$

[by theorem 5, article 2]

$$\begin{aligned} &= [U(P,f) - L(P,f)] + [U(P,g) - L(P,g)] \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \text{ by (1).} \end{aligned}$$

Since $U(P, f+g) - L(P, f+g) < \varepsilon$, $f + g \in \boldsymbol{R}[a,b]$.

Again, by the second definition of Riemann integrability, $f, g \in \boldsymbol{R}[a,b] \Rightarrow$ for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every partition P of $[a,b]$ with $\|P\| \le \delta$ and for every choice of $\xi_r \in [x_{r-1}, x_r]$,

$$\left| \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r - \int_a^b f(x)\, dx \right| < \varepsilon/2 \qquad \text{...(2)}$$

and $$\left| \sum_{r=1}^{n} g(\xi_r)\, \Delta x_r - \int_a^b g(x)\, dx \right| < \varepsilon/2 \qquad \text{...(3)}$$

From (2) and (3), we get

$$\left| \sum_{r=1}^{n} [f(\xi_r) + g(\xi_r)]\, \Delta x_r - \left[\int_a^b f(x)\, dx + \int_a^b g(x)\, dx\right] \right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

Similarly we can prove the result for the difference $f - g$ of the functions f and g.

Lemma: *Let f be a bounded function with bounds m and M defined on $[a, b]$. Then the oscillation $M - m$ of f on $[a, b]$ is the supremum of the set*

$$\{|f(x) - f(y)| : x, y \in [a, b]\}$$

of numbers.

Theorem 5: *If $f \in \mathbf{R}[a, b]$, $g \in \mathbf{R}[a, b]$, then $fg \in \mathbf{R}[a, b]$.*

(Garhwal 2010; Purvanchal 10; Bundelkhand 11)

Proof: Since $f, g \in \mathbf{R}[a, b]$, they are bounded on $[a, b]$.

So there exists a positive real number M such that

$$|f(x)| \leq M \text{ and } |g(x)| \leq M \quad \forall\ x \in [a, b].$$

It follows that $|f(x)\, g(x)| \leq M^2 \ \forall\, x \in [a, b]$.

Thus fg is bounded on $[a, b]$.

Again $f, g \in \cup [a, b] \Rightarrow$ for a given $\varepsilon > 0$, there exist partitions P_1 and P_2 of $[a, b]$ such that

$$U(P_1, f) - L(P_1, f) < \varepsilon / 2M$$

and $$U(P_2, g) - L(P_2, g) < \varepsilon / 2M.$$

Let $P = P_1 \cup P_2 = \{a = x_0, x_1, \ldots, x_n = b\}$ so that P is refinement of both P_1 and P_2. It follows by theorem 3 of article 2 that

$$\left.\begin{aligned} U(P, f) - L(P, f) &< \varepsilon / 2M \\ \text{and} \quad U(P, g) - L(P, g) &< \varepsilon / 2M \end{aligned}\right\} \quad \ldots(1)$$

Let $m_r, M_r; m_r', M_r'$ and m_r'', M_r'' be respectively the bounds of fg, f and g in the r th interval $I_r = [x_{r-1}, x_r]$. Then for all

$x, y \in [x_{r-1}, x_r]$, we have

$$\begin{aligned} |(fg)(y) - (fg)(x)| &= |f(y)\, g(y) - f(x)\, g(x)|, \text{ by the def. of } fg \\ &= |g(y)[f(y) - f(x)] + f(x)[g(y) - g(x)]| \\ &\leq |g(y)||f(y) - f(x)| + |f(x)||g(y) - g(x)| \\ &\leq M|f(y) - f(x)| + M|g(y) - g(x)| \quad \ldots(2) \end{aligned}$$

Taking suprema of both sides of (2), we have by the lemma given above

$$M_r - m_r \leq M(M_r' - m_r') + M(M_r'' - m_r'')$$

$$\Rightarrow \sum_{r=1}^{n} (M_r - m_r)\, \Delta x_r \leq M \sum_{r=1}^{n} (M_r' - m_r')\, \Delta x_r + M \sum_{r=1}^{n} (M_r'' - m_r'')\, \Delta x_r$$

or
$$\begin{aligned} U(P, fg) - L(P, fg) &\leq M[U(P, f) - L(P, f)] + M[U(P, g) - L(P, g)] \\ &< M(\varepsilon / 2M) + M(\varepsilon / 2M), \text{ by (1)} \\ &= \varepsilon / 2 + \varepsilon / 2 = \varepsilon. \end{aligned}$$

Hence $fg \in \mathbf{R}[a, b]$ by theorem of article 6.

Corollary: *If $f \in \mathbf{R}[a, b]$ then $f^2 \in \mathbf{R}[a, b]$.*

Theorem 6: *If* $f \in \boldsymbol{R}[a,b]$ *then* $|f| \in \boldsymbol{R}[a,b]$. **(Purvanchal 2008)**

Show that $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.

Proof: Since $f \in \boldsymbol{R}[a,b]$, f is bounded on $[a,b]$ so that there exists a positive number k such that $|f(x)| \le k \;\forall\; x \in [a,b]$ *i.e.* $|f(x)|$ is bounded on $[a,b]$.

Again, since $f \in \boldsymbol{R}[a,b]$, for given $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that

$$U(P,f) - L(P,f) < \varepsilon \qquad \text{...(1)}$$

Let m_r, M_r and m_r', M_r' be the bounds of $|f|$ and f respectively in $[x_{r-1}, x_r]$. Then for all $x, y \in [x_{r-1}, x_r]$, we have

$$\Big|\,|f(y)| - |f(x)|\,\Big| \le |f(y) - f(x)| \qquad \text{...(2)}$$

Taking suprema of both sides of (2), we have by the lemma mentioned above

$$(M_r - m_r) \le (M_r' - m_r')$$

$$\Rightarrow \quad \sum_{r=1}^{n} (M_r - m_r)\,\Delta x_r \le \sum_{r=1}^{n} (M_r' - m_r')\,\Delta x_r$$

$$\Rightarrow \quad U(P,|f|) - L(P,|f|) \le U(P,f) - L(P,f) < \varepsilon, \text{ by (1).}$$

Hence $|f| \in \boldsymbol{R}[a,b]$.

Now, let $\quad f_1(x) = \dfrac{1}{2}\{|f(x)| + f(x)\}$

and $\quad f_2(x) = \dfrac{1}{2}\{|f(x)| - f(x)\}$.

Then $\quad |f(x)| = f_1(x) + f_2(x);\; f(x) = f_1(x) - f_2(x)$.

Since $f_1(x) \ge 0$ and $f_2(x) \ge 0 \;\;\forall\; x \in [a,b]$, we have

$$\int_a^b f_1(x)\,dx \ge 0 \text{ and } \int_a^b f_2(x)\,dx \ge 0.$$

Hence, we get

$$\begin{aligned}\left|\int_a^b f(x)\,dx\right| &= \left|\int_a^b f_1(x)\,dx - \int_a^b f_2(x)\,dx\right| \\ &\le \left|\int_a^b f_1(x)\,dx\right| + \left|\int_a^b f_2(x)\,dx\right| \\ &= \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx \\ &= \int_a^b [f_1(x) + f_2(x)]\,dx = \int_a^b |f(x)|\,dx.\end{aligned}$$

Thus $\quad \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.

Note: The converse of the above theorem need not be true. Consider the function f defined on $[0,1]$ by

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational.} \end{cases}$$

We have seen earlier that $f \notin \boldsymbol{R}[0,1]$.

But $|f| \in \boldsymbol{R}[0,1]$. For, we have $|f(x)| = 1 \;\forall\; x \in [0,1]$ and we know that every constant function is R-integrable.

9 Fundamental Theorem of Integral Calculus

(Meerut 2012)

In this section we shall establish the close relationship between the derivative and the integral in a rigorous manner. In fact, we shall prove that integration and differentiation are, in a certain sense, inverse operations.

Integral function: Let $f \in \boldsymbol{R}[a,b]$. We define a new real valued function F with domain $[a,b]$ by setting

$$F(x) = \int_a^x f(t)\,dt, a < x \le b, F(a) = 0.$$

The function F is called an **integral function** or an **indefinite integral** of the integrable function f. The function F is well defined on $[a,b]$, as $f \in \boldsymbol{R}[a,x]$ where $a < x \le b$ and the condition $F(a) = 0$ is consistent with our previous definition that $\int_a^a f = 0$.

Primitive: Definition: *A differentiable function ϕ defined on $[a,b]$ such that its derivative ϕ' equals a given function f on $[a,b]$ is called a* **primitive** *or* **anti-derivative** *of f on $[a,b]$.*

If ϕ is a primitive of f then $\phi + c$ will also be a primitive of f where c is any constant. Hence the primitive of a function is not unique.

Theorem 1: *Let $f \in \boldsymbol{R}[a,b]$. Then the function F defined on $[a,b]$ by*

$$F(x) = \int_a^x f(t)\,dt$$

is continuous on $[a,b]$.

Proof: Since $f \in \boldsymbol{R}[a,b]$, f is bounded on $[a,b]$. It follows that there exists a positive number M such that

$$|f(t)| \le M \;\forall\; t \in [a,b].$$

Let $a \le x < y \le b$. Then we have

$$\begin{aligned} |F(y) - F(x)| &= \left| \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \right| \\ &= \left| \int_a^y f(t)\,dt + \int_x^a f(t)\,dt \right| && \left[\because \int_a^b f = -\int_b^a f\right] \\ &= \left| \int_x^y f(t)\,dt \right| && \text{[by theorem 3 of article 8]} \\ &\le M\,|y - x| && \text{[by Cor. 3 of theorem 1 of article 8]} \\ &= M(y - x). && \text{...(1)} \end{aligned}$$

Let $\varepsilon > 0$ be given. Then, if $|y - x| < \varepsilon / M$, we conclude from (1) that

$$|F(y) - F(x)| < \varepsilon.$$

Thus given $\varepsilon > 0$, there exists $\delta (= \varepsilon / M) > 0$ such that

$$|F(y) - F(x)| < \varepsilon \text{ whenever } |y - x| < \delta \ \forall\, x, y \in [a, b].$$

Consequently the function F is uniformly continuous on $[a, b]$ and hence continuous on $[a, b]$.

Note: The above theorem can also be stated as follows :

The integral of an integrable function is continuous.

Theorem 2: *Let f be continuous on $[a, b]$ and let*

$$F(x) = \int_a^x f(t)\, dt \ \forall\, x \in [a, b].$$

Then $F'(x) = f(x) \ \forall\, x \in [a, b].$ **(Gorakhpur 2015)**

Proof: Let $x \in [a, b]$ be fixed. Choose $h \neq 0$ such that

$$x + h \in [a, b].$$

Then, we have

$$F(x + h) - F(x) = \int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt$$

$$= \int_a^{x+h} f(t)\, dt + \int_x^a f(t)\, dt$$

$$= \int_x^{x+h} f(t)\, dt.$$

Since f is continuous on $[a, b]$ there exists a number c in the interval $[x, x + h]$, such that

$$\int_x^{x+h} f(t)\, dt = h\, f(c). \qquad \text{[By Cor. 2 of theorem 1, article 8]}$$

We see that if $h \to 0$, then $c \to x$.

Thus $F(x + h) - F(x) = h\, f(c)$

$$\Rightarrow \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = \lim_{h \to 0} f(c)$$

$$= \lim_{c \to x} f(c) = f(x), \text{ as } f \text{ is continuous.}$$

Hence, we get $F'(x) = f(x) \ \forall\, x \in [a, b]$.

Note: The following theorem is an improvement on the above theorem for only the R-integrability of f and the continuity of f at the point x_0 is assumed.

Theorem 3: *Let $f \in \mathbf{R}[a, b]$ and let f be continuous at $x_0 \in [a, b]$.*

If $F(x) = \int_a^x f(t)\, dt, a \le x \le b$, *then*

$$F'(x_0) = f(x_0).$$

Proof: Since f is continuous at x_0, for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x_0 + h) - f(x_0)| < \varepsilon \qquad \text{...(1)}$$

provided $|h| < \delta$ and $a \le x_0 + h \le b$.

We have $F(x_0 + h) - F(x_0) = \int_a^{x_0 + h} f(t)\, dt - \int_a^{x_0} f(t)\, dt$

$$= \int_a^{x_0 + h} f(t)\, dt$$

and $\qquad \int_{x_0}^{x_0+h} f(x_0)\,dt = f(x_0)\,.\,[(x_0+h)-x_0] = h\,f(x_0).$

Now
$$\left|\frac{F(x_0+h)-F(x_0)}{h} - f(x_0)\right|$$
$$= \left|\frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,dt - \frac{1}{h}\int_{x_0}^{x_0+h} f(x_0)\,dt\right|$$
$$= \left|\frac{1}{h}\int_{x_0}^{x_0+h} [f(t)-f(x_0)]\,dt\right|. \qquad \text{...(2)}$$

By virtue of (1) and (2), we get on using Cor. 3 of theorem 1 of article 8
$$\left|\frac{F(x_0+h)-F(x_0)}{h} - f(x_0)\right| < \frac{1}{|h|}\cdot|h|\,\varepsilon = \varepsilon \text{ for all } h \text{ with } |h| < \delta.$$

Since $\varepsilon > 0$ is arbitrary, we have
$$\lim_{h \to 0} \frac{F(x_0+h)-F(x_0)}{h} = f(x_0)$$

i.e., $\qquad F'(x_0) = f(x_0).$

Theorem 4: (Fundamental theorem of Integral Calculus): *Let f be a continuous function on $[a, b]$ and let ϕ be a differentiable function on $[a, b]$ such that $\phi'(x) = f(x)$ for all $x \in [a, b]$. Then*
$$\int_a^b f(x)\,dx = \phi(b) - \phi(a).$$

Proof: Let $F(x) = \int_a^x f(t)\,dt$. Then by theorem 2 of article 7.9, we have
$$F'(x) = f(x) \;\; \forall \;\; x \in [a, b]. \qquad \text{...(1)}$$
By hypothesis, $\phi'(x) = f(x)\ \forall\ x \in [a, b]$. ...(2)

From (1) and (2), we have $\forall\ x \in [a, b]$,
$$F'(x) = \phi'(x) \quad \text{or} \quad F'(x) - \phi'(x) = 0$$
$\Rightarrow \qquad (F-\phi)'(x) = 0 \Rightarrow (F-\phi)(x) = c$ for some $c \in \mathbf{R}$

$\Rightarrow \qquad F(x) - \phi(x) = c.$

Thus $\qquad F(x) = \phi(x) + c \;\; \forall \;\; x \in [a, b].$

Now $\qquad F(b) - F(a) = [\phi(b)+c] - [\phi(a)+c] = \phi(b) - \phi(a).$ ...(3)

Also $\qquad F(a) = \int_a^a f(t)\,dt = 0$ and $F(b) = \int_a^b f(t)\,dt.$

Putting these values in (3), we get
$$\int_a^b f(t)\,dt = \phi(b) - \phi(a)$$
or
$$\int_a^b f(x)\,dx = \phi(b) - \phi(a).$$

Note: The following theorem is an improvement on the above theorem since only the R-integrability of f is assumed. In fact theorem 4 is a corollary of theorem 5.

Theorem 5: (Fundamental theorem of Integral Calculus): *Let $f \in \boldsymbol{R}[a, b]$ and let ϕ be a differentiable function on $[a, b]$ such that $\phi'(x) = f(x)$ for all x $[a, b]$. Then* $\int_a^b f(x)\, dx = \phi(b) - \phi(a)$. **(Garhwal 2006, 07, 09, 11; Rohilkhand 11, 12)**

Proof: Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a, b]$. Now ϕ is differentiable on $[a, b]$ implies that ϕ is differentiable on each subinterval $[x_{r-1}, x_r]$. Hence by the mean value theorem of differential calculus, we find that there exists ξ_r in $[x_{r-1}, x_r], r = 1, 2, \ldots, n$, such that

$$\phi(x_r) - \phi(x_{r-1}) = (x_r - x_{r-1})\, \phi'(\xi_r) = (\phi)'(\xi_r) \,.\, \Delta x_r$$

or
$$\phi(x_r) - \phi(x_{r-1}) = f(\xi_r) \,.\, \Delta x_r \qquad [\because \phi'(\xi_r) = f(\xi_r)]$$

or
$$\sum_{r=1}^{n} [\phi(x_r) - \phi(x_{r-1})] = \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r \,. \qquad \ldots(1)$$

Now
$$\sum_{r=1}^{n} [\phi(x_r) - \phi(x_{r-1})] = \phi(x_1) - \phi(x_0) + \phi(x_2) - \phi(x_1) + \ldots$$
$$\ldots + \phi(x_n) - \phi(x_{n-1})$$
$$= \phi(x_n) - \phi(x_0) = \phi(b) - \phi(a).$$

It follows from (1) that

$$\phi(b) - \phi(a) = \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r \qquad \ldots(2)$$

Taking limit as $|| P || \to 0$,

we get
$$\phi(b) - \phi(a) = \int_a^b f(x)\, dx,$$

since $\sum_{r=1}^{n} f(\xi_r)\, \Delta x_r$ tends to $\int_a^b f(x)\, dx$ as $|| P || \to 0$.

The result of the above theorem is usually written in the form

$$\int_a^b \phi'(x)\, dx = \phi(b) - \phi(a).$$

Note: Some authors call theorem 2 or theorem 3 as the **first fundamental theorem** and the theorem 4 or theorem 5 as the **second fundamental theorem** of integral calculus.

10 Mean Value Theorems of Integral Calculus

Theorem 1: (First Mean Value Theorem): *Let $f \in \boldsymbol{R}[a, b]$. Then there exists a number μ lying between the bounds m and M of f such that*

$$\int_a^b f(x)\, dx = \mu\, (b - a).$$

Moreover if f is continuous, then

$$\int_a^b f(x)\, dx = (b - a)\, f(c), a \le c \le b.$$

(Purvanchal 2012)

Proof: See Cor. 1 and Cor. 2 of theorem 1 of article 8.

Theorem 2: (Second Mean Value Theorem): *Let*

$$f \in \boldsymbol{R}[a,b] \text{ and } g \in \boldsymbol{R}[a,b],$$

and $g(x) \geq 0$ *or* $\leq 0 \ \forall \ x \in [a,b]$.

Then there exists a number μ *with* $m \leq \mu \leq M$ *such that*

$$\int_a^b f(x)\, g(x)\, dx = \mu \int_a^b g(x)\, dx$$

where m, M are the bounds of f on $[a, b]$.

Proof: First let $g(x) \geq 0 \ \forall \ x \in [a,b]$. **(Purvanchal 2012)**

Then $mg(x) \leq f(x)\, g(x) \leq Mg(x) \ \forall \ x \in [a,b]$.

If follows from Cor. 5 of theorem 1 of article 8 that

$$m \int_a^b g(x)\, dx \leq \int_a^b f(x)\, g(x)\, dx \leq M \int_a^b g(x)\, dx$$

or
$$m \int_a^b g(x)\, dx \geq \int_a^b f(x)\, g(x)\, dx \geq M \int_a^b g(x)\, dx$$

according as $a \leq b$ or $a \geq b$.

Hence there exists a number μ with $m \leq \mu \leq M$ such that

$$\int_a^b f(x)\, g(x)\, dx = \mu \int_a^b g(x)\, dx.$$

Now let $g(x) \leq 0 \ \ \forall \ x \in [a,b]$.

Then $-g(x) \geq 0 \ \forall \ x \in [a,b]$.

Hence by the above result for some $\mu \in [m, M]$, we have

$$\int_a^b f(x)\, [-g(x)]\, dx = \mu \int_a^b [-g(x)]\, dx$$

or
$$\int_a^b f(x)\, g(x)\, dx = \mu \int_a^b g(x)\, dx.$$

Corollary: *If, in addition to the conditions of the theorem, f is continuous as well, then there exists* $\xi \in [a,b]$ *such that*

$$\int_a^b f(x)\, g(x)\, dx = f(\xi) \int_a^b g(x)\, dx.$$

Proof: Since *f* is continuous on $[a, b]$, it takes every value between m and M. Hence by the above theorem there exists a point $\xi \in [a,b]$ such that

$$\int_a^b f(x)\, g(x)\, dx = f(\xi) \int_a^b g(x)\, dx.$$

Theorem 3: (Bonnet's Mean Value Theorem): *Let* $g \in \boldsymbol{R}[a,b]$ *and let f be monotonic and non-negative on* $[a,b]$. *Then for some* ξ *or* $\eta \in [a,b]$

$$\int_a^b f(x)\, g(x)\, dx = f(a) \int_a^{\xi} g(x)\, dx$$

or
$$\int_a^b f(x)\, g(x)\, dx = f(b) \int_{\eta}^b g(x)\, dx$$

according as f is monotonically non-increasing or non-decreasing on $[a,b]$.

Proof: If $a = b$, the result is trivial. Let $b > a$ and let f be non-negative and monotonically non-increasing on $[a, b]$.

Let $\quad P = \{a = x_0, x_1, \ldots\ldots, x_n = b\}$

be any partition of $[a, b]$. Let m_r, M_r be the bounds of g on $[x_{r-1}, x_r]$ and ξ_r any point on $[x_{r-1}, x_r]$. Then

$$m_r \Delta x_r \le g(\xi_r) \Delta x_r \le M_r \Delta x_r$$

and
$$m_r \Delta x_r \le \int_{x_{r-1}}^{x_r} g \le M_r \Delta x_r .$$

[Theorem 1 of article 8]

On summing for each $r = 1, 2, \ldots, p \le n$, we get

$$\sum_1^p m_r \Delta x_r \le \sum_1^p g(\xi_r) \Delta x_r \le \sum_1^p M_r \Delta x_r \qquad \ldots(1)$$

and
$$\sum_1^p m_r \Delta x_r \le \int_a^{x_p} g \le \sum_1^p M_r \Delta x_r . \qquad \ldots(2)$$

Then (1) and (2) give

$$\left| \int_a^{x_p} g - \sum_1^p g(\xi_r) \Delta x_r \right| \le \sum_1^p (M_r - m_r) \Delta x_r$$

$$\le \sum_1^n (M_r - m_r) \Delta x_r = U(P, g) - L(P, g)$$

$$= \omega(P, g). \qquad \ldots(3)$$

[Note that if $b < a$, the inequalities (1) and (2) are reversed but (3) remains the same].

Now by theorem 1 of article 9, $\int_a^x g$ is continuous on $[a, b]$ and hence is bounded on $[a, b]$. Let m, M be its bounds on $[a, b]$. Then (3) gives

$$m - \omega(P, f) \le \sum_1^p g(\xi_r) \Delta x_r \le M + \omega(P, f).$$

Using Abel's lemma*, we get

$$f(a)[m - \omega(P, f)] \le \sum_1^p f(\xi_r) g(\xi_r) \Delta x_r$$

$$\le f(a)[M - \omega(P, f)]. \qquad \ldots(4)$$

Since f is monotonic, we have $f \in \boldsymbol{R}[a, b]$.

***Abel's Lemma.** If $v_1 \ge v_2 \ge \ldots \ge v_n \ge 0$ and numbers $u_1, u_2, \ldots, u_n$ and k_1, k_2 are such that

$$k_1 \le \sum_{r=1}^n u_r \le k_2 \text{, then } k_1 v_1 \le \sum_{r=1}^n u_r v_r \le k_2 v_1.$$

In our case, $\quad u_r = g(\xi_r) \Delta x_r$ and $v_r = f(\xi_r)$.

Also $\qquad g \in \boldsymbol{R}[a, b]$.

Hence $\qquad fg \in \boldsymbol{R}[a, b]$.

Now $\qquad f \in \boldsymbol{R}[a, b] \Rightarrow \omega(P, f) \to 0$ as $\|P\| \to 0$.

Hence (4) gives $f(a)\, m \le \int_a^b fg \le f(a)\, M$.

Thus $\qquad \int_a^b fg = f(a)\, \mu$

where μ lies between the bounds m, M of the continuous function

$$\int_a^x g \text{ on } [a, b].$$

Hence $\int_a^x g$ must take the value μ at some point $\xi \in [a, b]$

so that $\qquad \mu = \int_a^\xi g(x)\, dx.$

Therefore $\qquad \int_a^b f(x)\, g(x)\, dx = f(a) \int_a^\xi g(x)\, dx.$

If f is monotonically non-decreasing on $[a, b]$, then it is monotonically non-increasing on $[b, a]$ and so as above, we get

$$\int_a^b fg = f(b) \int_a^\eta g \text{ for some } \eta \in [a, b]$$

or $\qquad \int_a^b fg = f(b) \int_\eta^b g$

i.e., $\qquad \int_a^b f(x)\, g(x)\, dx = f(b) \int_\eta^b g(x)\, dx.$

Theorem 4: Weierstrass's (Second) Mean Value Theorem.

Let $g \in \boldsymbol{R}[a, b]$ and let f be bounded and monotonic on $[a, b]$.

Then $\qquad \int_a^b fg = f(a) \int_a^\xi g + f(b) \int_\xi^b g.$

Proof: We assume that f is monotonically non-increasing on $[a, b]$. Then $f(x) - f(b)$ is monotonically non-increasing and non-negative on $[a, b]$. Hence by the Bonnet's theorem (theorem 3 of article 10), there exists some $\xi \in [a, b]$ such that

$$\int_a^b [f(x) - f(b)]\, g(x)\, dx = [f(a) - f(b)] \int_a^\xi g(x)\, dx$$

or
$$\int_a^b f(x)\, g(x)\, dx - f(b) \int_a^b g(x)\, dx$$
$$= f(a) \int_a^\xi g(x)\, dx - f(b) \int_a^\xi g(x)\, dx$$

or
$$\int_a^b f(x)\, g(x)\, dx = f(a) \int_a^\xi g(x)\, dx + f(b) \left[\int_a^b g(x)\, dx - \int_a^\xi g(x)\, dx\right]$$
$$= f(a) \int_a^\xi g(x)\, dx + f(b) \int_\xi^b g(x)\, dx.$$

Now let f be monotonically non-decreasing on $[a, b]$. Then $-f$ is monotonically non-increasing on $[a, b]$. Hence by the above result, we have

$$\int_a^b [-f(x)]\, g(x)\, dx = [-f(a)] \int_a^{\xi} g(x)\, dx + [-f(b)] \int_{\xi}^{b} g(x)\, dx$$

or $$\int_a^b f(x)\, g(x)\, dx = f(a) \int_a^{\xi} g(x)\, dx + f(b) \int_{\xi}^{b} g(x)\, dx \text{ as before.}$$

Remark: Note that the condition that f is monotone on $[a, b]$ cannot be dropped in the above theorem. Consider, for example, the functions $f(x) = \cos x$ and $g(x) = x^2$ and the interval $[-\pi/2, \pi/2]$. In this case

$$f\left(-\frac{\pi}{2}\right) \int_{-\pi/2}^{\xi} g(x)\, dx + f\left(\frac{\pi}{2}\right) \int_{\xi}^{\pi/2} g(x)\, dx$$
$$= \cos\left(-\frac{\pi}{2}\right) \int_{-\pi/2}^{\xi} x^2\, dx + \cos\frac{\pi}{2} \int_{\xi}^{\pi 2} x^2\, dx = 0.$$

But $$\int_{-\pi/2}^{\pi/2} x^2 \cos x\, dx > 0.$$

Thus the theorem does not hold.

Illustrative Examples

Example 16: *Compute* $\int_1^2 x^3\, dx.$

Solution: Let $f(x) = x^3, 1 \le x \le 2$. Then f is continuous on $[1, 2]$. Moreover, if $\phi(x) = x^4/4\ (1 \le x \le 2)$, then $\phi'(x) = x^3 = f(x), (1 \le x \le 2)$.

Hence by the fundamental theorem of integral calculus, we have

$$\int_1^2 x^3\, dx = \phi(2) - \phi(1) = \frac{2^4}{4} - \frac{1^4}{4} = \frac{15}{4}.$$

Example 17: *(i) Taking $f(x) = x$, $g(x) = e^x$, verify the second mean value theorem in* $[-1, 1]$. [*Theorem* 2 *of article* 10].

(*ii*) *Also verify Bonnet's mean value theorem in* $[-1, 1]$ *for the functions*

$$f(x) = e^x \quad \textit{and} \quad g(x) = x.$$

Solution: (i) Since f and g are continuous on $[-1, 1]$, we have

$$f, g \in \boldsymbol{R}[-1, 1].$$

Also $g(x) > 0$ for all $x \in [-1, 1]$. Hence the conditions of theorem 2 of article 10 are satisfied. Now

$$\int_{-1}^{1} f(x)\, g(x)\, dx = \int_{-1}^{1} xe^x\, dx = [xe^x - e^x]_{-1}^{1} = \frac{2}{e}. \qquad \text{...(1)}$$

and $$\int_{-1}^{1} g(x)\, dx = \int_{-1}^{1} e^x\, dx = [e^x]_{-1}^{1} = e - e^{-1} = \frac{e^2 - 1}{e}.$$

Since f is continuous on $[-1, 1]$, it takes every value between $f(-1) = -1$ and $f(1) = 1$. Let $\mu = 2/(e^2 - 1)$. Since $e > 2$, we have $e^2 > 4 \Rightarrow e^2 - 1 > 3$ so that $0 < \mu < 1$.

It follows that there is a point ξ in $[-1, 1]$ such that

$$f(\xi) = 2/(e^2 - 1).$$

Accordingly, we have

$$f(\xi) \int_{-1}^{1} g(x)\, dx = \frac{2}{e^2 - 1} \cdot \frac{e^2 - 1}{e} = \frac{2}{e} \cdot \qquad \ldots(2)$$

From (1) and (2), we have

$$\int_{-1}^{1} f(x)\, g(x)\, dx = f(\xi) \int_{-1}^{1} g(x)\, dx.$$

Thus the second mean value theorem is verified.

(ii) Since $g(x) = x$ is continuous on $[-1, 1]$, we have

$$g \in \boldsymbol{R}[-1, 1].$$

Also $f(x) = e^x$ is monotonically non-decreasing and positive on $[-1, 1]$. Hence all the conditions of the Bonnet's mean value theorem are satisfied. As in (i), we have

$$\int_{-1}^{1} f(x)\, g(x)\, dx = \int_{-1}^{1} e^x\, x\, dx = \frac{2}{e} \cdot$$

Now $\quad \int_{\eta}^{1} g(x)\, dx = \int_{\eta}^{1} x\, dx = \frac{1}{2}[1 - \eta^2].$

$\therefore \quad f(1) \int_{\eta}^{1} g(x)\, dx = \frac{e}{2}(1 - \eta^2).$

We choose η such that

$$\frac{2}{e} = \frac{e}{2}(1 - \eta^2) \quad i.e., \quad \eta^2 = \frac{e^2 - 4}{e^2} \cdot$$

Also it is easy to see that $0 < \eta < 1$, where $\eta = \dfrac{\sqrt{(e^2 - 4)}}{e} \cdot$

For this value of η, we then have

$$\int_{-1}^{1} f(x)\, g(x)\, dx = f(1) \int_{\eta}^{1} g(x)\, dx.$$

Hence Bonnet's mean value theorem is verified.

Example 18: *Show that the Bonnet's mean value theorem does not hold on* $[-1, 1]$ *for* $f(x) = g(x) = x^2$.

Solution: The function $f(x) = x^2$ is not monotonic on $[-1, 1]$ since for the interval $[-1, 0]$ it is non-increasing and for $[0, 1]$ it is non-decreasing. Thus the conditions of the Bonnet's mean value theorem are not satisfied and hence the theorem does not hold in $[-1, 1]$.

Example 19: *Prove Bonnet's form of the second mean value theorem that if* f' *is continuous and of constant sign, and* $f(b)$ *has the same sign as* $f(a) - f(b)$, *then*

$$\int_a^b f(x)\, g(x)\, dx = f(a) \int_a^{\xi} g(x)\, dx$$

where ξ lies between a and b. Show that if $q > p > 0$, then

$$\left| \int_p^q \frac{\sin x}{x}\, dx \right| < \frac{2}{p}.$$

Solution: Let $\int_a^x g(t)\, dt = F(x)$

so that $F'(x) = g(x)$.

Then
$$\int_a^b f(x)\, g(x)\, dx = \int_a^b f(x)\, F'(x)\, dx$$
$$= [f(x)\, F(x)]_a^b - \int_a^b f'(x)\, F(x)\, dx$$
[Integrating by parts]
$$= f(b)\, F(b) - \int_a^b f'(x)\, F(x)\, dx \qquad [\because\ F(a) = 0]$$
$$= f(b)\, F(b) - F(x_0) \int_a^b f'(x)\, dx \quad [a \le x_0 \le b]$$
[by Cor. of theorem article 10]
$$= f(b)\, F(b) - F(x_0)\, [f(b) - f(a)] \qquad \ldots(1)$$
$$= f(a).\mu \text{ where } \mu \text{ lies between } F(x_0) \text{ and } F(b).$$

Since F is continuous, there exists a point ξ between x_0 and b such that

$$\mu = F(\xi) = \int_a^\xi f(x)\, dx.$$

It follows that $\int_a^b f(x)\, g(x)\, dx = f(a) \int_a^\xi g(x)\, dx.$

Writing $f(x) = 1/x$, $g(x) = \sin x$, we find

$$\int_p^q \frac{\sin x}{x}\, dx = \frac{1}{p} \int_p^\xi \sin x\, dx, \ (p < \xi < q)$$
$$= \frac{1}{p} [-\cos x]_p^\xi = \frac{1}{p} [\cos p - \cos \xi].$$

Hence
$$\left| \int_p^q \frac{\sin x}{x}\, dx \right| = \left| \frac{1}{p} (\cos p - \cos \xi) \right| < \frac{2}{p}.$$

Remark: From (1), we get

$$\int_a^b f(x)\, g(x)\, dx = f(a)\, F(x_0) + f(b)\, [F(b) - F(x_0)]$$
$$= f(a) \int_a^{x_0} g(x)\, dx + f(b) \left[\int_a^b g(x)\, dx - \int_a^{x_0} g(x)\, dx \right]$$
$$= f(a) \int_a^{x_0} g(x)\, dx + f(b) \int_{x_0}^b g(x)\, dx$$

which is the theorem 4 of article 10.

Comprehensive Exercise 1

1. (i) Let $f(x) = x \ (0 \le x \le 1)$. Let P be the partition $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$ of $[0, 1]$. Compute $U(P, f)$ and $L(P, f)$.

   (ii) Let $f(x) = x \ (0 \le x \le 3)$. Let P be the partition $\{0, 1, 2, 3\}$ of $[0, 3]$. Compute $U(P, f)$ and $L(P, f)$.

2. Show by definition that $\int_0^1 x^4 \, dx = \frac{1}{5}$.

3. Let $f(x) = x^{-1/2}$ on $[1, 4]$. Consider the partition obtained by dividing $[1, 4]$ into n equal parts and hence show that $\int_1^4 x^{-1/2} \, dx = 2$.

4. If $f(x) = \cos x \ \forall \ x \in [0, \pi/2]$, show that f is integrable on $[0, \pi/2]$ and $\int_0^{\pi/2} \cos x \, dx = 1$.

5. Show that $f(x) = 3x + 1$ is integrable on $[1, 2]$ and $\int_1^2 (3x + 1) \, dx = \frac{11}{2}$.

6. Give an example to prove that a bounded function need not be R-integrable.

7. Give an example of a discontinuous function which is R-integrable on $[0, 1]$.

8. Let f be defined on $[0, 1]$ by $f(x) = \frac{n}{n+1}$, when

$$\frac{1}{n+1} < x \le \frac{1}{n}, \ n = 1, 2, 3, \ldots$$

and $\quad f(x) = 1, x = 0$.

Then, show that f is Riemann integrable on $[0, 1]$ and $\int_0^1 f(x) \, dx = \frac{\pi^2}{6} - 1$.

9. A function $f(x)$ is defined on $[0, 1]$ as follows :

$$f(x) = \frac{n}{n+2}, \text{ when } \frac{1}{n+1} \le x \le \frac{1}{n} \ (n = 1, 2, 3, \ldots)$$

and $\quad f(x) = 1$, when $x = 0$.

Show that $f(x)$ is R-integrable on $[0, 1]$ and $\int_0^1 f(x) \, dx = \frac{1}{2}$. **(Garhwal 2008)**

10. Let a function $f(x)$ be defined on $[0, 1]$ as follows :

$$f(x) = \frac{1}{a^{r-1}}, \text{ when } \frac{1}{a^r} < x < \frac{1}{a^{r-1}},$$

for $r = 1, 2, 3, \ldots$ where a is an integer greater than 1. Show that $\int_0^1 f(x) \, dx$ exists and is equal to $\frac{a}{a+1}$.

11. Calculate the values of upper and lower integrals for the function f defined on $[0, 2]$ as follows :

$$f(x) = x^2 \quad \text{when } x \text{ is rational}$$

and $$f(x) = x^3 \quad \text{when } x \text{ is irrational.}$$

12. Let $f(x)$ be a function bounded on $[a, b]$ and let P_1 and P_2 be two partitions of $[a, b]$ such that $P_1 \subset P_2$. Then prove that

$$U(P_1, f) - L(P_1, f) \geq U(P_2, f) - L(P_2, f).$$

Answers 1

1. (i) $U(P, f) = \dfrac{2}{3}$, $L(P, f) = \dfrac{1}{3}$

(ii) $U(P, f) = 6$, $L(P, f) = 3$

11. Upper integral $= \dfrac{49}{12}$, lower integral $= \dfrac{31}{12}$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Let f be a bounded function defined on $[a, b]$ and let P be a partition of $[a, b]$. If P^* is a refinement of P, then

(a) $L(P^*, f) \leq L(P, f)$ (b) $U(P^*, f) \leq U(P, f)$

(c) $U(P^*, f) \geq U(P, f)$ (d) $L(P^*, f) = U(P, f)$.

2. Let f, g be bounded functions defined on $[a, b]$ and let P be any partition of $[a, b]$. Then

(a) $U(P, f + g) \leq U(P, f) + U(P, g)$

(b) $U(P, f + g) \geq U(P, f) + U(P, g)$

(c) $U(P, f + g) = U(P, f) + U(P, g)$

(d) $L(P, f + g) \leq L(P, f) + L(P, g)$.

3. Let f be a bounded function defined on the bounded interval $[a, b]$. Then f is Riemann integrable on $[a, b]$ if and only if

(a) $\underline{\int}_a^b f \leq \overline{\int}_a^b f$ (b) $\underline{\int}_a^b f \geq \overline{\int}_a^b f$

(c) $\underline{\int}_a^b f = \overline{\int}_a^b f$ (d) $\overline{\int}_a^b f + \underline{\int}_a^b f = 0$.

4. If f is Riemann integrable on $[a,b]$, then

(a) $\left|\int_a^b f(x)\,dx\right| = \int_a^b |f(x)|\,dx$

(b) $\left|\int_a^b f(x)\,dx\right| \geq \int_a^b |f(x)|\,dx$

(c) $\left|\int_a^b f(x)\,dx\right| = -\int_a^b |f(x)|\,dx$

(d) $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. Let $I = [a,b]$ be a closed and bounded interval. Then by a partition of I we mean a set of real numbers $P = \{x_0, x_1, \dots, x_{n-1}, x_n\}$ having the property that ....

2. Let P and P^* be two partitions of a closed and bounded interval $[a,b]$. Then P^* is called a refinement of P if ....

3. Let P_1 and P_2 be two partitions of a closed and bounded interval $[a,b]$. If $P^* = P_1 \cup P_2$, then P^* is called the ...... of P_1 and P_2.

4. Let f be a bounded function defined on a bounded interval $[a,b]$. If corresponding to any partition P of $[a,b]$, $L(P,f)$ is the lower Riemann sum of f on $[a,b]$ and $U(P,f)$ is the upper Riemann sum of f on $[a,b]$, then $L(P,f)$ ...... $U(P,f)$.

5. Let f be a bounded function defined on $[a,b]$ and let P be a partition of $[a,b]$. If P^* is a refinement of P, then $L(P^*,f)$ ...... $L(P,f)$.

6. If P_1 and P_2 be any two partitions of $[a,b]$, then $U(P_1,f)$ ...... $L(P_2,f)$.

7. Let f be a real valued bounded function defined on $[a,b]$. The lower Riemann integral of f over $[a,b]$ is the ...... of $L(P,f)$ over all partitions $P \in \boldsymbol{P}[a,b]$.

8. Let f be a real valued bounded function defined on $[a,b]$. The upper Riemann integral of f over $[a,b]$ is the ...... of $U(P,f)$ over all partitions $P \in \boldsymbol{P}[a,b]$.

9. Let f be a real valued bounded function defined on $[a,b]$. Then the lower Riemann integral of f over $[a,b]$ cannot ...... the upper Riemann integral of f over $[a,b]$.

10. Let f be a bounded function defined on the bounded interval $[a,b]$. Then f is called Riemann integrable on $[a,b]$ if $\underline{\int_a^b} f = \dots\dots$

11. A necessary and sufficient condition for Riemann integrability of a bounded function $f : [a,b] \to \mathbf{R}$ over $[a,b]$ is that for every $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that for P and all its refinements

$$0 \leq U(P,f) - L(P,f) < \dots\dots$$

12. Let f be Riemann integrable on $[a,b]$ and let ϕ be a differentiable function on $[a,b]$ such that $\phi'(x) = f(x)$ for all $x \in [a,b]$. Then $\int_a^b f(x)\,dx = \dots\dots$

True or False

Write 'T' for true and 'F' for false statement.

1. Every continuous function defined on $[a, b]$ is Riemann integrable on $[a, b]$.
2. Every bounded function f defined on $[a, b]$ is Riemann integrable on $[a, b]$.
3. If a function f is discontinuous on $[a, b]$, then f cannot be Riemann integrable on $[a, b]$.
4. If a function f is monotonic on $[a, b]$, then f is Riemann integrable on $[a, b]$.
5. If f is Riemann integrable on $[a, b]$, then $|f|$ may or may not be Riemann integrable on $[a, b]$.
6. If a function f is Riemann integrable on $[a, b]$, then the function F defined on $[a, b]$ by
$$F(x) = \int_a^x f(t)\, dt$$
is uniformly continuous on $[a, b]$.
7. Let f be continuous on $[a, b]$ and let $F(x) = \int_a^x f(t)\, dt \ \forall\, x \in [a, b]$.
Then $\quad F'(x) = f(x) \ \forall\, x \in [a, b]$.
8. Let f be a continuous function on $[a, b]$ and let ϕ be a differentiable function on $[a, b]$ such that $\phi'(x) = f(x)$ for all $x \in [a, b]$. Then
$$\int_a^b f(x)\, dx = \phi(a) - \phi(b).$$
9. Let f be a bounded function defined on $[a, b]$, where $b \geq a$, and let m be the infimum of $f(x)$ in $[a, b]$. Then for any partition P of $[a, b]$, we have
$$m(b - a) \geq L(P, f).$$
10. If $f : [a, b] \to \mathbf{R}$ is a bounded function, then
$$L(P, -f) = -U(P, f).$$
11. Let f be a bounded function defined on $[a, b]$ and let P be a partition of $[a, b]$. If P^* is a refinement of P, then
$$U(P^*, f) \geq U(P, f).$$
12. Let f be a bounded function defined on $[a, b]$. Then
$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$
13. Let f be a continuous function defined on $[a, b]$. Then
$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$
14. Let f be a bounded function defined on $[a, b]$. If the set of points of discontinuity of f on $[a, b]$ is finite, then
$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

Answers

Multiple Choice Questions

1.	(b)	**2.**	(a)	**3.**	(c)	**4.**	(d)

Fill in the Blank(s)

1. $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ **2.** $P^* \supset P$

3. common refinement **4.** $\leq$ **5.** $\geq$

6. $\geq$ **7.** supremum **8.** infimum

9. exceed **10.** $\overline{\int_a^b} f$ **11.** ε

12. $\phi(b) - \phi(a)$

True or False

1.	T	**2.**	F	**3.**	F	**4.**	T	**5.**	F
6.	T	**7.**	T	**8.**	F	**9.**	F	**10.**	T
11.	F	**12.**	F	**13.**	T	**14.**	T		

□

Chapter 8

The Riemann-Stieltjes Integral

1 A Generalization of the Riemann Integral

In the preceding chapter we have seen that the definitions of Riemann integral are founded on finite sums of the form

$$\sum_{r=1}^{n} m_r \, \Delta x_r \,, \quad \sum_{r=1}^{n} M_r \, \Delta x_r \quad \text{or} \quad \sum_{r=1}^{n} f(\xi_r) \, \Delta x_r .$$

If in the above sums we introduce $\Delta g_r = g(x_r) - g(x_{r-1})$ in place of Δx_r, where g is a more or less an arbitrary function, then a generalization of the Riemann integral is achieved which reduces to the Riemann case in the particular case $g(x) = x$. It was Thomas Jan Stieltjes who gave a generalization of Riemann integral. In the present chapter we shall not discuss the Stieltjes integrals as such but only a sort of special case thereof known as Riemann-Stieltjes integral in some detail.

2 Some Definitions

For convenience we repeat some definitions again here although we have given them in the preceding chapter.

Partition: Definition. *By a partition of a closed interval* $I = [a, b]$ *we mean a finite set of real numbers*

$$P = \{x_0, x_1, \ldots, x_{n-1}, x_n\} \text{ having the property that}$$
$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b.$$

The closed sub-intervals

$$I_1 = [x_0, x_1], I_2 = [x_1, x_2], \ldots, I_n = [x_{n-1}, x_n]$$

determined by P constitute the **segments** of the partition.

We write $\Delta x_r = x_r - x_{r-1}$ for $r = 1, 2, \ldots, n$, so that Δx_r is the **length** of the segment $[x_{r-1}, x_r]$. The **norm** of a partition P is the greatest of the lengths of the segments of a partition P and it is denoted by $\|P\|$. Thus

$$\|P\| = \max. (\Delta x_r : r = 1, 2, \ldots, n).$$

A partition P^* *is called a* ***refinement*** *of another partition* P *or we say that* P^* *is* ***finer than*** P *iff* $P^* \supset P$, *i.e., if every point of* P *is used in the construction of* P^*.

If $P^* = P_1 \cup P_2$, *then* P^* *is called the* ***common refinement*** *of the given two partitions* P_1 *and* P_2.

By an ***intermediate partition*** *of* P, *we mean a set* $Q = \{\xi_1, \xi_2, \ldots, \xi_n\}$ *of at most* n *points having the property that* $x_{r-1} \leq \xi_r \leq x_r$ *for* $r = 1, 2, \ldots, n$.

According to this definition, two consecutive points of Q can be identical. Thus, it may happen that $\xi_{r-1} = \xi_r = x_r$ for some value of r. But three or more points cannot coincide.

3 Lower and Upper Riemann-Stieltjes Sums

Let f be a bounded real valued function defined on $[a, b]$ and let g be a monotonically non-decreasing (real valued) function on $[a, b]$. We shall write

$$\Delta g_r = g(x_r) - g(x_{r-1})$$

corresponding to each partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$.

Note that $\Delta g_r \geq 0$ since g is monotonically non-decreasing on $[a, b]$. Also, we define

$$m_r = \inf \{f(x) : x_{r-1} \leq x \leq x_r\},$$
$$M_r = \sup \{f(x) : x_{r-1} \leq x \leq x_r\},$$
$$m = \inf \{f(x) : a \leq x \leq b\}, M = \sup \{f(x) : a \leq x \leq b\}.$$

Let us now form the two sums

$$L(P, f, g) = \sum_{r=1}^{n} m_r \Delta g_r \text{ and } U(P, f, g) = \sum_{r=1}^{n} M_r \Delta g_r.$$

These two numbers are called the **lower** and **upper Riemann Stieltjes Sums** (or in brief lower and upper RS sums) respectively, of f with respect to g and corresponding to the partition P.

In case $g(x) = x$, these reduce to the lower and upper Riemann sums defined in the preceding chapter.

Obviously $\quad L(P, f, g) \le U(P, f, g)$, for any partition P.

If P^* is a refinement of P, then

$$L(P, f, g) \le L(P^*, f, g) \qquad \text{...(1)}$$

and

$$U(P^*, f, g) \le U(P, f, g). \qquad \text{...(2)}$$

For any partitions P_1, P_2 of $[a, b]$, we have

$$L(P_1, f, g) \le U(P_2, f, g). \qquad \text{...(3)}$$

The results (1), (2) and (3) can be proved similarly as proved in the preceding chapter with x's replaced by $g(x)$'s and Δx_r by Δg_r etc.

4 The Lower and Upper Riemann-Stieltjes Integrals

(Gorakhpur 2010)

Definition: *Let f be a bounded function and g a non-decreasing function defined on $[a, b]$. The **lower Riemann-Stieltjes integral** of f relative to g over $[a, b]$ is the supremum of $L(P, f, g)$ over all partitions P of $[a, b]$.* It is denoted by

$$\underline{\int_a^b} f\, dg \quad \text{or} \quad \underline{\int_a^b} f(x)\, dg(x).$$

*The **upper Riemann-Stieltjes integral** of f relative to g over $[a, b]$ is the infimum of $U(P, f, g)$ over all partitions P of $[a, b]$.* It is denoted by

$$\overline{\int_a^b} f\, dg \quad \text{or} \quad \overline{\int_a^b} f(x)\, dg(x).$$

Theorem: *Let f be a bounded function and g a non-decreasing function on $[a, b]$. Then the lower RS-integral of f relative to g cannot exceed the upper RS-integral*

i.e.,
$$\underline{\int_a^b} f\, dg \le \overline{\int_a^b} f\, dg.$$

(Gorakhpur 2010)

Proof: If P_1 and P_2 are any two partitions of $[a, b]$ then by (3) of article 3, we get

$$L(P_1, f, g) \le U(P_2, f, g). \qquad \text{...(1)}$$

First, keeping P_2 fixed and taking the supremum over all partitions P_1, (1) gives

$$\underline{\int_a^b} f\, dg \le U(P_2, f, g). \qquad \text{...(2)}$$

Now taking infimum over all partitions P_2, (2) gives

$$\underline{\int_a^b} f\, dg \le \overline{\int_a^b} f\, dg.$$

5 The Riemann-Stieltjes Integral

Definition: *Let f be a bounded function and g a non-decreasing function on $[a, b]$. Then f is called **Riemann-Stieltjes integrable** (or simply RS-integrable) on $[a, b]$ relative to g iff*

$$\underline{\int_a^b} f\, dg = \overline{\int_a^b} f\, dg$$

and their common value is called the RS-integral of f on $[a, b]$ relative to g and is denoted by

$$\int_a^b f\, dg \quad or \quad \int_a^b f(x)\, dg(x).$$

*f is called the **Integrand** and g the **Integrator**.* **(Gorakhpur 2012)**

The class of all RS-integrable functions relative to g over $[a, b]$ is denoted by $RS([a, b], g)$ or simply by $RS(g)$.

From the above definition, it is obvious that quite apart from conditions which might be necessary to the existence of an RS-integral, the integrand f must be bounded and the integrator g non-decreasing.

As in the case of Riemann integral, we define

$$\int_a^b f\, dg = -\int_b^a f\, dg \text{ for } b < a \text{ and } \int_a^a f\, dg = 0.$$

If $g(x) = x$, then the lower and upper RS-integrals reduce respectively to the lower and upper R-integrals. Hence the RS-integral of f relative to x is the same as the R-integral of f.

Note: The integral depends on f, g, a and b and not on the so-called variable of integration

i.e.,
$$\int_a^b f(x)\, dg(x) = \int_a^b f(y)\, dg(y).$$

So omitting the variable of integration we shall prefer to write

$$\int_a^b f\, dg \text{ in place of } \int_a^b f(x)\, dg(x).$$

6 Necessary and Sufficient Condition for RS-Integrability

Theorem: *Let f be a bounded and g a non-decreasing function on $[a, b]$. Then $f \in RS(g)$ if and only if for every $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that*

$$U(P, f, g) - L(P, f, g) < \varepsilon.$$ **(Gorakhpur 2012)**

Proof: The condition is necessary (Only if part).

Let $f \in RS(g)$ so that

$$\underline{\int_a^b} f\,dg = \overline{\int_a^b} f\,dg. \quad ...(1)$$

Since $\underline{\int_a^b} f\,dg$ is the supremum of $L(P, f, g)$ over all partitions P, there exists a partition P_1 such that

$$\underline{\int_a^b} f\,dg < L(P_1, f, g) + \frac{\varepsilon}{2}.$$

Similarly since $\overline{\int_a^b} f\,dg$ is the infimum of $U(P, f, g)$ over all partitions P, there exists a partition P_2 such that

$$U(P_2, f, g) < \overline{\int_a^b} f\,dg + \frac{\varepsilon}{2}.$$

If $P = P_1 \cup P_2$, then P is the common refinement of P_1 and P_2. It follows from (1) and (2) of article 3 that

$$\underline{\int_a^b} f\,dg < L(P, f, g) + \frac{\varepsilon}{2} \quad ...(2)$$

and $$U(P, f, g) < \int_a^b f\,dg + \frac{\varepsilon}{2}. \quad ...(3)$$

Adding the inequalities (2) and (3), we get

$$\underline{\int_a^b} f\,dg + U(P, f, g) < L(P, f, g) + \overline{\int_a^b} f\,dg + \varepsilon.$$

In view of (1), this gives

$$U(P, f, g) < L(P, f, g) + \varepsilon \text{ i.e., } U(P, f, g) - L(P, f, g) < \varepsilon.$$

Hence the condition is necessary.

The condition is sufficient (if part).

Let for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(P, f, g) - L(P, f, g) < \varepsilon. \quad ...(4)$$

By the definition of upper and lower RS-integrals, we have

$$\overline{\int_a^b} f\,dg \le U(P, f, g) \text{ and } \underline{\int_a^b} f\,dg \ge L(P, f, g)$$

i.e., $$L(P, f, g) \le \underline{\int_a^b} f\,dg \le \overline{\int_a^b} f\,dg \le U(P, f, g). \quad ...(5)$$

It follows from (4) and (5) that

$$\overline{\int_a^b} f\,dg - \underline{\int_a^b} f\,dg \le U(P, f, g) - L(P, f, g) < \varepsilon$$

or $$\overline{\int_a^b} f\,dg - \underline{\int_a^b} f\,dg < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\overline{\int_a^b} f\, dg = \underline{\int_a^b} f\, dg$ *i.e.*, the function f is RS-integrable with respect to g.

Note: It is worth while to observe that the details of the proof of this theorem are exactly similar to those of the corresponding theorem for R-integrals given in the preceding chapter. In fact most of the elementary properties of R-integrals remain valid for RS-integrals with their corresponding proofs exactly similar except for the notation used for the two types of integrals.

Illustrative Examples

Example 1: *Let f be a constant function on $[a, b]$ defined by $f(x) = k$ and g a monotonically non-decreasing function on $[a, b]$. Prove that $\int_a^b f\, dg$ exists and $\int_a^b f\, dg = k\,[g(b) - g(a)]$.*

(Meerut 2004)

Solution: Obviously the given function is bounded over $[a, b]$. Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a, b]$. Then for any sub-interval $[x_{r-1}, x_r]$, we have

$$m_r = k,\ M_r = k.$$

Now
$$U(P, f, g) = \sum_{r=1}^{n} k\, \Delta g_r = k \sum_{r=1}^{n} [g(x_r) - g(x_{r-1})]$$

$$= k\,[g(x_1) - g(x_0) + g(x_2) - g(x_1) + \ldots + g(x_n) - g(x_{n-1})]$$
$$= k\,[g(x_n) - g(x_0)] = k\,[g(b) - g(a)],$$

and
$$L(P, f, g) = \sum_{r=1}^{n} k\, \Delta g_r = k\,[g(b) - g(a)].$$

$\therefore$
$$\overline{\int_a^b} f\, dg = \inf U(P, f, g) = \inf k\,[g(b) - g(a)]$$
$$= k\,[g(b) - g(a)]$$

and
$$\underline{\int_a^b} f\, dg = \sup L(P, f, g) = \sup k\,[g(b) - g(a)] = k\,[g(b) - g(a)].$$

Since $\underline{\int_a^b} f\, dg = \overline{\int_a^b} f\, dg$, the integral $\int_a^b f\, dg$ exists

and
$$\int_a^b f\, dg = k\,[g(b) - g(a)].$$

Example 2: *If f is non-negative and integrable function with respect to α on $[a, b]$ and if α is an increasing function on $[a, b]$, then show that*

$$\int_a^b f\, d\alpha \geq 0.$$

Solution: Let m be the infimum on $[a, b]$.

We have $$m \geq 0 \qquad \ldots(1)$$

$\{\because f(x) \geq 0, \forall\ x \in [a, b]\}$

and $$\int_a^b f\, d\alpha \geq m\,[\alpha(b) - \alpha(a)]. \qquad \ldots(2)$$

Since α is an increasing function on $[a, b]$, therefore we have

$$\alpha(b) - \alpha(a) \geq 0. \qquad \ldots(3)$$

From (1) and (3), we have

$$m\,[\alpha(b) - \alpha(a)] \geq 0.$$

Hence, from (2), we conclude

$$\int_a^b f\, d\alpha \geq 0.$$

7 The RS-integral as a Limit of Sums

So far RS-integral $\int_a^b f\, dg$ is defined by means of the sums $L\,(P, f, g)$ and $U\,(P, f, g)$. The numbers m_r, M_r which appear in these sums are not necessarily values of f. Now we shall define $\int_a^b f\, dg$ as the limit of a sequence of sums in which m_r, M_r are replaced by values of f.

Definition: *Let f be a bounded and g a non-decreasing function on $[a, b]$. Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a, b]$ and $Q = \{\xi_1, \xi_2, \ldots, \xi_n\}$ an intermediate partition of P so that*

$$x_{r-1} \leq \xi_r \leq x_r \text{ for } r = 1, 2, \ldots, n.$$

Then the number

$$RS\,(P, Q, f, g) = \sum_{r=1}^{n} f\,(\xi_r)\,\Delta g_r$$

is defined as the ***Riemann-Stieltjes sum (or RS-sum)*** *of f relative to g on $[a, b]$ and corresponding to the partition P and the intermediate partition Q.*

We say that $\lim\limits_{\|P\| \to 0} RS\,(P, Q, f, g) = I$

iff, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|\,RS\,(P, Q, f, g) - I\,| < \varepsilon \text{ whenever } \|P\| < \delta.$$

The following theorem gives $\int_a^b f\, dg$ as the limit of sums.

Theorem 1: *If $\lim\limits_{\|P\| \to 0} RS\,(P, Q, f, g)$ exists, then $f \in RS\,(g)$ on $[a, b]$ and*

$$\lim_{\|P\| \to 0} RS\,(P, Q, f, g) = \int_a^b f\, dg.$$

Proof: Suppose that

$$\lim_{\|P\|\to 0} RS\,(P, Q, f, g) = \lim_{\|P\|\to 0} \sum_{r=1}^{n} f\,(\xi_r)\,\Delta g_r \text{ exists and is equal to } I.$$

Let $\varepsilon > 0$ be given.

Then by the above definition, there exists $\delta > 0$ such that

$$|\,RS\,(P, Q, f, g) - I\,| < \frac{1}{4}\varepsilon \text{ whenever } \|\,P\,\| < \delta$$

$$\Rightarrow \qquad I - \frac{1}{4}\varepsilon < RS\,(P, Q, f, g) < I + \frac{1}{4}\varepsilon \text{ whenever } \|\,P\,\| < \delta \qquad \text{...(1)}$$

Choosing one such P, letting the points ξ_r range over the intervals $[x_{r-1}, x_r]$ and taking the infimum and supremum of the numbers $RS\,(P, Q, f, g)$ thus obtained, we get from (1)

$$I - \frac{1}{4}\varepsilon \le L\,(P, f, g) \le U\,(P, f, g) \le I + \frac{1}{4}\varepsilon. \qquad \text{...(2)}$$

It follows that

$$U\,(P, f, g) \le I + \frac{1}{4}\varepsilon \text{ and } -L\,(P, f, g) \le -I + \frac{1}{4}\varepsilon.$$

Adding these relations, we get

$$U\,(P, f, g) - L\,(P, f, g) \le \frac{1}{2}\varepsilon < \varepsilon.$$

Consequently $f \in RS\,(g)$.

It follows that lower and upper RS-integrals are equal

i.e., $$\underline{\int_a^b} f\,dg = \overline{\int_a^b} f\,dg = \int_a^b f\,dg.$$

But $$L\,(P, f, g) \le \underline{\int_a^b} f\,dg \le \overline{\int_a^b} f\,dg \le U\,(P, f, g).$$

$$\therefore \qquad L\,(P, f, g) \le \int_a^b f\,dg \le U\,(P, f, g). \qquad \text{...(3)}$$

From (2) and (3), we get

$$I - \frac{1}{4}\varepsilon \le \int_a^b f\,dg \le I + \frac{1}{4}\varepsilon$$

$$\Rightarrow \qquad \left| I - \int_a^b f\,dg \right| \le \frac{1}{4}\varepsilon < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$I = \int_a^b f\,dg$$

or $$\lim_{\|P\|\to 0} RS\,(P, Q, f, g) = \int_a^b f\,dg.$$

Illustration 1: *Let f be an arbitrary function and $g = k$, a constant function on $[a, b]$. Then for any partition P and intermediate partition Q of P, we have*

$$\sum_{r=1}^{n} f(\xi_r)\,\Delta g_r = \sum_{r=1}^{n} f(\xi_r)\,[g(x_r) - g(x_{r-1})] = \sum_{r=1}^{n} f(\xi_r)\,[k - k] = 0.$$

$$\therefore \qquad \int_a^b f\,dk = 0.$$

Thus any function f on $[a, b]$ is RS-integrable relative to constant integrator and the value of integral is zero.

Illustration 2: *Let $f = k$ be a constant function on $[a, b]$ and let g be any monotonically increasing function on $[a, b]$. Then for any partition P and an intermediate partition Q of P, we have*

$$\sum_{r=1}^{n} f(\xi_r)\,\Delta g_r = \sum_{r=1}^{n} k\,[g(x_r) - g(x_{r-1})]$$

$$= k\,[g(x_1) - g(x_0) + g(x_2) - g(x_1) + \ldots + g(x_n) - g(x_{n-1})]$$

$$= k\,[g(x_n) - g(x_0)] = k\,[g(b) - g(a)].$$

It gives that

$$\int_a^b k\,dg = \lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\,\Delta g_r = k\,[g(g) - g(a)] \qquad \ldots(1)$$

For $k = 1$, we have $\int_a^b dg = g(b) - g(a)$.

$$\therefore \qquad k\int_a^b dg = k\,[g(b) - g(a)] \qquad \ldots(2)$$

From (1) and (2), we get the formula

$$\int_a^b k\,dg = k\int_a^b dg.$$

Note: Compare the solution of this example with that of the example of article 5.

Theorem 2: *If f is continuous and g is monotonically non-decreasing on $[a, b]$ then $f \in RS(g)$. Moreover, to every $\varepsilon > 0$, there corresponds a $\delta > 0$ such that*

$$\left| \sum_{r=1}^{n} f(\xi_r)\,\Delta g_r - \int_a^b f\,dg \right| < \varepsilon$$

for every partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ with $\|P\| < \delta$ and for every intermediate partition $Q = \{\xi_1, \xi_2, \ldots, \xi_n\}$ of P,

$$i.e., \qquad \lim_{\|P\| \to 0} RS(P, Q, f, g) = \int_a^b f\,dg.$$

(Gorakhpur 2014)

Proof: Let $\varepsilon > 0$ be given. Choose $\eta > 0$ such that

$$\eta\,|g(b) - g(a)| < \varepsilon. \qquad \ldots(1)$$

The function f is uniformly continuous on $[a, b]$ since it is continuous on the closed and bounded interval $[a, b]$. It follows that there exists a $\delta > 0$ such that

$$x, y \in [a, b], |x - y| < \delta \Rightarrow |f(x) - f(y)| < \eta. \qquad \text{...(2)}$$

Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition with $\|P\| < \delta$.

The function f is continuous on each sub-interval $[x_{r-1}, x_r]$ as it is continuous on $[a, b]$ and hence it attains the bounds m_r, M_r on $[x_{r-1}, x_r]$, *i.e*, there exist points $c, d \in [x_{r-1}, x_r]$ such that

$$f(c) = m_r, \ f(d) = M_r.$$

From (2), we have

$$|f(d) - f(c)| < \eta$$

$$\Rightarrow \quad M_r - m_r < \eta, \ r = 1, 2, \ldots, n.$$

Hence $U(P, f, g) - L(P, f, g)$

$$= \sum_{r=1}^{n} (M_r - m_r) \Delta g_r < \eta \sum_{r=1}^{n} \Delta g_r = \eta \sum_{r=1}^{n} [g(x_r) - g(x_{r-1})]$$

$$= \eta [g(x_1) - g(x_0) + g(x_2) - g(x_1) + \ldots + g(x_n) - g(x_{n-1})]$$

$$= \eta [g(x_n) - g(x_0)] = \eta [g(b) - g(a)] < \varepsilon, \text{ by } (1).$$

Thus $\quad U(P, f, g) - L(P, f, g) < \varepsilon. \qquad$ ...(3)

Hence $f \in RS(g)$. It follows that

$$\underline{\int_a^b} f\, dg = \overline{\int_a^b} f\, dg = \int_a^b f\, dg. \qquad \text{...(4)}$$

This proves the first part of the theorem.

To prove the remaining part of the theorem, we get from (3), for all P with $\|P\| < \delta$

$$U(P, f, g) - \varepsilon < L(P, f, g) \Rightarrow \overline{\int_a^b} f\, dg - \varepsilon < L(P, f, g)$$

$$\Rightarrow \quad \int_a^b f\, dg - \varepsilon < L(P, f, g), \text{ using } (4).$$

Again $\quad U(P, f, g) < L(P, f, g) + \varepsilon \Rightarrow U(P, f, g) < \underline{\int_a^b} f\, dg + \varepsilon$

$$\Rightarrow \quad U(P, f, g) < \int_a^b f\, dg + \varepsilon, \text{ using } (4).$$

Thus $\quad \int_a^b f\, dg - \varepsilon < L(P, f, g) \leq U(P, f, g) < \int_a^b f\, dg + \varepsilon. \qquad$ ...(5)

Also $\quad L(P, f, g) \leq RS(P, Q, f, g) \leq U(P, f, g). \qquad$ ...(6)

It follows from (5) and (6) that

$$\int_a^b f\, dg - \varepsilon < RS(P, Q, f, g) < \int_a^b f\, dg + \varepsilon$$

or $$\left| RS\,(P, Q, f, g) - \int_a^b f\,dg \right| < \varepsilon$$

or $$\lim_{\|P\| \to 0} RS\,(P, Q, f, g) = \int_a^b f\,dg.$$

Theorem 3: *Let $f \in RS\,(g)$.and let g be continuous on $[a, b]$. Then with the usual notations,*

$$\lim_{\|P\| \to 0} RS\,(P, Q, f, g) = \int_a^b f\,dg.$$

(Gorakhpur 2011)

Proof: Let $f \in RS\,(g)$. so that

$$\underline{\int_a^b} f\,dg = \overline{\int_a^b} f\,dg = \int_a^b f\,dg. \quad \text{...(1)}$$

Let $\varepsilon > 0$ be given.

Since $\overline{\int_a^b} f\,dg$ is the infimum of $U\,(P, f, g)$, there exists a partition P^* such that

$$U\,(P^*, f, g) < \overline{\int_a^b} f\,dg + \frac{\varepsilon}{2}$$

or $$U\,(P^*, f, g) < \int_a^b f\,dg + \frac{\varepsilon}{2}, \text{using (1)} \quad \text{...(2)}$$

Set $M = \sup\,\{|\,f(x)|: a \le x \le b\}$.

The function g is uniformly continuous on $[a, b]$ since it is continuous on the closed and bounded interval $[a, b]$. It follows that there exists a $\delta_1 > 0$ such that

$$|\,x - y\,| < \delta_1 \Rightarrow |\,g\,(x) - g\,(y)\,| < \frac{\varepsilon}{2pM} \quad \text{...(3)}$$

where p is the number of sub-intervals into which $[a, b]$ is divided by P^*. If $P = \{a = x_0, x_1, \dots, x_n = b\}$ be any partition with $\|\,P\,\| < \delta_1$ then (3) gives

$$\Delta\,g_r = g\,(x_r) - g\,(x_{r-1}) < \frac{\varepsilon}{2pM}, r = 1, 2, \dots, n. \quad \text{...(4)}$$

[Note that since g is monotonically non-decreasing,

$$|\,g\,(x_r) - g\,(x_{r-1})\,| = g\,(x_r) - g\,(x_{r-1}).]$$

Now we divide the subintervals of P into two groups :

(i) those which are contained in a subinterval of P^* and

(ii) those which contain in their interior one or more points of sub-division of P^*.

The contribution of the subintervals (i) to the sum $U\,(P, f, g)$ obviously does not exceed $U\,(P^*, f, g)$ while the number of sub-intervals (ii) in P does not exceed $p - 1$, and hence their contribution to $U\,(P, f, g)$ does not exceed $(p - 1)\,kM$ where

$$k = \max.\,\{\Delta g_r, r = 1, 2, \dots, n\}.$$

Thus $\qquad U(P, f, g) \le U(P^*, f, g) + (p-1)\,kM.$ ...(5)

Also by (4), we get $k < \dfrac{\varepsilon}{2pM}\cdot$ ...(6)

Now combining (5) with (2) and (6), we get

$$U(P, f, g) < \int_a^b f\,dg + \frac{\varepsilon}{2} + (p-1)\,M\,\frac{\varepsilon}{2pM}$$

$$< \int_a^b f\,dg + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \int_a^b f\,dg + \varepsilon.$$

Thus for all partitions P with $\|P\| < \delta_1$,

$$U(P, f, g) < \int_a^b f\,dg + \varepsilon. \qquad \text{...(7)}$$

Similarly, it can be shown that there exists a $\delta_2 > 0$ such that for all partitions P with $\|P\| < \delta_2$,

$$L(P, f, g) > \int_a^b f\,dg - \varepsilon \qquad \text{...(8)}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then (7) and (8) hold for every P with $\|P\| < \delta$. Obviously

$$RS(P, Q, f, g) < \int_a^b f\,dg + \varepsilon \text{ and } RS(P, Q, f, g) > \int_a^b f\,dg - \varepsilon$$

i.e., $$\int_a^b f\,dg - \varepsilon < RS(P, Q, f, g) < \int_a^b f\,dg + \varepsilon$$

for all partitions P with $\|P\| < \delta$.

It gives $\displaystyle\lim_{\|P\| \to 0} RS(P, Q, f, g) = \int_a^b f\,dg.$

Thus the theorem is established.

8 Some Classes of RS-integrable Functions

Theorem 1: *Let f be continuous and g monotonically non-decreasing on $[a, b]$. Then $f \in RS(g)$.*

Proof: See the first part of theorem 2 of article 7.

Theorem 2: *Let f be monotonic on $[a, b]$ and let g be continuous and non-decreasing on $[a, b]$. Then $f \in RS(g)$.*

Proof: Let $\varepsilon > 0$ be given. Since g is continuous on $[a, b]$ so it will take all values between $g(a)$ and $g(b)$. Also g is monotonically non-decreasing on $[a, b]$. Hence for any positive integer n, there exists a partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ of $[a, b]$ such that

$$\Delta g_r = g(x_r) - g(x_{r-1}) = \frac{g(b) - g(a)}{n}, r = 1, 2, \ldots, n. \qquad \ldots(1)$$

Let us assume that f be monotonically increasing on $[a, b]$.

If m_r and M_r be the inf. and sup. of f on I_r then

$$m_r = f(x_{r-1}), M_r = f(x_r). \qquad \ldots(2)$$

Hence $U(P, f, g) - L(P, f, g) = \sum_{r=1}^{n} (M_r - m_r) \Delta g_r$

$$= \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})] \frac{g(b) - g(a)}{n}, \text{ by (1) and (2)}$$

$$= \frac{g(b) - g(a)}{n} \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})] = \frac{g(b) - g(a)}{n} [f(b) - f(a)]$$

$< \varepsilon$, taking n sufficiently large.

Hence $f \in RS(g)$.

9 Algebra of RS-integrable Functions

Theorem 1: *Let $f \in RS(g)$ on $[a, b]$. Then*

$$m[g(b) - g(a)] \leq \int_a^b f\, dg \leq M[g(b) - g(a)]$$

where m, M are the bounds of f on $[a, b]$.

Proof: Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be any partition of $[a, b]$. Then $I_r = [x_{r-1}, x_r]$, $r = 1, 2, \ldots, n$ are the subintervals of $[a, b]$. Let m_r and M_r be the infimum and supremum of $f(x)$ in $[x_{r-1}, x_r]$. Then for every value of r, we have

$$m \leq m_r \leq M_r \leq M$$

$$\Rightarrow \quad m\, \Delta g_r < m_r\, \Delta g_r \leq M_r\, \Delta g_r \leq M\, \Delta g_r \qquad [\because \Delta g_r > 0]$$

$$\Rightarrow \quad \sum_{r=1}^{n} m \Delta g_r \leq \sum_{r=1}^{n} m_r\, \Delta g_r \leq \sum_{r=1}^{n} M_r\, \Delta g_r \leq \sum_{r=1}^{n} M\, \Delta g_r$$

$$\Rightarrow \quad m[g(b) - g(a)] \leq L(P, f, g) \leq U(P, f, g) \leq M[g(b) - g(a)] \qquad \ldots(1)$$

But $$L(P, f, g) \leq \underline{\int_a^b} f\, dg \leq \overline{\int_a^b} f\, dg \leq U(P, f, g) \qquad \ldots(2)$$

Then from (1) and (2), we have

$$m[g(b) - g(a)] \leq \underline{\int_a^b} f\, dg \leq \overline{\int_a^b} f\, dg \leq M[g(b) - g(a)] \qquad \ldots(3)$$

Also $f \in RS(g)$ so that

$$\underline{\int_a^b} f\, dg = \overline{\int_a^b} f\, dg = \int_a^b f\, dg \qquad \ldots(4)$$

If follows from (3) and (4) that

$$m\,[g\,(b) - g\,(a)] \le \int_a^b f\,dg \le M\,[g\,(b) - g\,(a)].$$

Corollary 1: Let $f \in RS\,(g)$ Then there exists a number μ lying between the bounds m *and* M of f such that

$$\int_a^b f\,dg = \mu\,[g\,(b) - g\,(a)].$$

Corollary 2: *Let f be continuous on $[a, b]$. Then there exists a point $c \in [a, b]$ such that*

$$\int_a^b f\,dg = f\,(c)\,[g\,(b) - g\,(a)].$$

Proof: By corollary 1, we have

$$\int_a^b f\,dg = \mu\,[g\,(b) - g\,(a)].$$

Now f being continuous on $[a, b]$, so it takes every value between its bounds m and M *i.e.*, in particular it takes the value μ lying between m and M. Consequently there is a point $c \in [a, b]$ such that $f\,(c) = \mu$ and hence

$$\int_a^b f\,dg = f\,(c)\,[g\,(b) - g\,(a)].$$

Corollary 3: *Let $f \in RS\,(g)$ and let K be a number such that $|\,f\,(x)\,| \le K\ \forall\ x \in [a, b]$.*

Then $$\left|\int_a^b f\,dg\right| \le K\,[g\,(b) - g\,(a)].$$

Proof. We have, for all $x \in [a, b]$,

$$|\,f(x) \le K \Rightarrow -K \le f\,(x) \le K$$

$\Rightarrow$ $$-K \le m \le f(x) \le M \le K$$

$\Rightarrow$ $$-K \le m \le m_r \le f(x) \le M_r \le M \le K$$

$\Rightarrow$ $$-K \le m_r \le M_r \le K.$$

Now proceeding as in the above theorem, we have

$$-\,K\,[g\,(b) - g\,(a)] \le \int_a^b f\,dg \le K\,[g\,(b) - g\,(a)]$$

$\Rightarrow$ $$\left|\int_a^b f\,dg\right| \le K\,[g\,(b) - g\,(a)].$$

Corollary 4: *Let $f \in RS\,(g)$ and let $f\,(x) \ge 0\ \forall\ x \in [a, b]$. Then*

$$\int_a^b f\,dg \ge 0.$$

Proof: Since $f\,(x) \ge 0\ \forall\ x \ge [a, b]$, hence, $m \ge 0$.

Also $$g\,(b) - g\,(a) \ge 0.$$

Hence by the above theorem, we get

$$\int_a^b f\,dg \geq m\,[g(b) - g(a)] \geq 0.$$

Corollary 5: *Let* $f_1, f_2 \in RS(g)$ *on* $[a,b]$ *and* $f_1(x) \leq f_2(x)$ *on* $[a,b]$, *then*

$$\int_a^b f_1\,dg \leq \int_a^b f_2\,dg.$$

Proof: We have $f_1(x) \leq f_2(x)\ \forall\ x \in [a,b]$

$\Rightarrow$ $f_2(x) - f_1(x) \geq 0\ \forall\ x \in [a,b]$

$\Rightarrow$ $\int_a^b [f_2(x) - f_1(x)]\,dg \geq 0$ i.e., $\int_a^b f_2(x)\,dg - \int_a^b f_1(x)\,dg \geq 0$

$\Rightarrow$ $\int_a^b f_2\,dg \geq \int_a^b f_1\,dg$, i.e., $\int_a^b f_1\,dg \leq \int_a^b f_2\,dg.$

Theorem 2: *Let* $f \in RS(g)$ *on* $[a,b]$. *Then* $cf \in RS(g)$ *on* $[a,b]$ *for every constant c and*

$$\int_a^b (cf)\,dg = c\int_a^b f\,dg.$$

Proof: If $c = 0$, the theorem is obvious. Suppose that $c \neq 0$. Since $f \in RS(g)$ on $[a,b]$, we have

$$\underline{\int_a^b} f\,dg = \overline{\int_a^b} f\,dg = \int_a^b f\,dg. \qquad ...(1)$$

Let $\varepsilon > 0$ be given. Since f is RS-integrable over $[a,b]$, there exists a partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ such that

$$U(P, f, g) - L(P, f, g) < \frac{\varepsilon}{c}\cdot \qquad ...(2)$$

If m_r and M_r be the infimum and supremum of f in $[x_{r-1}, x_r]$ and $c > 0$ then cm_r and cM_r are the inf. and sup. of cf in $[x_{r-1}, x_r]$.

Hence for the same partition P of $[a,b]$, we have

$$U(P, cf, g) = cU(P, f, g) \qquad ...(3)$$

and $$L(P, cf, g) = cL(P, f, g) \qquad ...(4)$$

Now $$U(P, cf, g) - L(P, cf, g) = c\,[U(P, f, g) - L(P, f, g)]$$

$$< c\,.\frac{\varepsilon}{c} = \varepsilon, \text{by } (2) \qquad ...(5)$$

If $c < 0$, let $c = -d$ where $d > 0$. Then we have

$$U(P, -df, g) = -dL(P, f, g) \qquad ...(6)$$

and $$L(P, -df, g) = -dU(P, f, g) \qquad ...(7)$$

Using the relations (6) and (7) and proceeding as above it can be easily seen that the inequality (5) holds for $c < 0$. Thus for a given $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that

$$U(P, cf, g) - L(P, cf, g) < \varepsilon.$$

Hence, for every constant $c, cf \in RS(g)$ on $[a, b]$.

It follows that

$$\underline{\int_a^b} cf\ dg = \overline{\int_a^b} cf\ dg = \int_a^b cf\ dg. \qquad ...(8)$$

To prove the second part, taking infimum of both sides of (3) over all partitions P of $[a, b]$, we get for $c > 0$

$$\overline{\int_a^b} (cf)\ dg = c \overline{\int_a^b} f\ dg.$$

Using (1) and (8), this gives

$$\int_a^b (cf)\ dg = c \int_a^b f\ dg.$$

If $c < 0, (c = -d)$, we have

$$\int_a^b (cf)\ dg = \overline{\int_a^b} (cf)\ dg = \overline{\int_a^b} (-df)\ dg$$

$$= \inf fU(P, -df, g) = \inf [-dL(P, f, g)]$$

$$= (-d) \sup L(P, f, g) = (-d) \underline{\int_a^b} f\ dg = c \int_a^b f\ dg.$$

Hence $\quad \int_a^b (cf)\ dg = c \int_a^b f\ dg$, for every constant c.

Corollary. *If $f \in RS(g)$, then so is $-f$ and*

$$\int_a^b (-f)\ dg = -\int_a^b f\ dg.$$

Theorem 3: *Let $f_1, f_2 \in RS(g)$ on $[a, b]$.*

Then $\quad f_1 + f_2 \in RS(g)$ *on* $[a, b]$

and $\quad \int_a^b (f_1 + f_2)\ dg = \int_a^b f_1\ dg + \int_a^b f_2\ dg.$

(Gorakhpur 2011)

Proof: Since $f_1, f_2 \in RS(g)$ on $[a, b]$, for a given $\varepsilon > 0$, there exist partitions P_1 and P_2 of $[a, b]$ such that

$$U(P_1, f_1, g) - L(P_1, f_1, g) < \frac{1}{2}\varepsilon \qquad ...(1)$$

and $$U(P_2, f_2, g) - L(P_2, f_2, g) < \frac{1}{2}\varepsilon \qquad ...(2)$$

Let $P = P_1 \cup P_2$. Then P is a common refinement of P_1 and P_2 and it is a partition of $[a, b]$. Both the inequalities (1) and (2) hold for P also. Thus

$$U(P, f_1, g) - L(P, f_1, g) < \frac{1}{2}\varepsilon, U(P, f_2, g) - L(P, f_2, g) < \frac{1}{2}\varepsilon. \qquad ...(3)$$

If $f = f_1 + f_2$ and P is a partition of $[a, b]$ then we have

$$L(P, f, g) \geq L(P, f_1, g) + L(P, f_2, g) \qquad ...(4)$$

and $$U(P, f, g) \leq U(P, f_1, g) + U(P, f_2, g) \qquad ...(5)$$

From (4) and (5), we get

$$L(P, f_1, g) + L(P, f_2, g) \leq L(P, f, g) \leq U(P, f, g)$$
$$\leq U(P, f_1, g) + U(P, f_2, g)$$

$$\Rightarrow \quad U(P, f, g) - L(P, f, g) \leq U(P, f_1, g) + U(P, f_2, g)$$
$$- L(P, f_1, g) - L(P, f_2, g)$$
$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \text{ by (3).}$$

Hence $f = f_1 + f_2 \in RS(g)$ on $[a, b]$.

Again from (3), we have

$$U(P, f_1, g) < \int_a^b f_1 \, dg + \frac{\varepsilon}{2} \qquad ...(6)$$

and $$U(P, f_2, g) < \int_a^b f_2 \, dg + \frac{\varepsilon}{2}. \qquad ...(7)$$

Thus we have

$$\int_a^b f \, dg = \overline{\int_a^b} f \, dg \leq U(P, f, g) \leq U(P, f_1, g) + U(P, f_2, g)$$
$$< \int_a^b f_1 \, dg + \frac{\varepsilon}{2} + \int_a^b f_2 \, dg + \frac{\varepsilon}{2}, \text{ by (6) and (7)}$$
$$= \int_a^b f_1 \, dg + \int_a^b f_2 \, dg + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\int_a^b f \, dg \leq \int_a^b f_1 \, dg + \int_a^b f_2 \, dg \qquad ...(8)$$

Since f_1, f_2 were any RS-integrable functions, replacing them by $-f_1$ and $-f_2$ in (8), we get

$$\int_a^b (-f) \, dg = \int_a^b (-f_1 - f_2) \, dg \leq \int_a^b (-f_1) \, dg + \int_a^b (-f_2) \, dg$$

or $$-\int_a^b f \, dg \leq -\left[\int_a^b f_1 \, dg + \int_a^b f_2 \, dg\right]$$

or $$\int_a^b f \, dg \geq \int_a^b f_1 \, dg + \int_a^b f_2 \, dg \qquad ...(9)$$

It follows from (8) and (9) that

$$\int_a^b f \, dg = \int_a^b f_1 \, dg + \int_a^b f_2 \, dg$$

or $$\int_a^b (f_1 + f_2)\, dg = \int_a^b f_1\, dg + \int_a^b f_2\, dg.$$

Corollary 1: *If $f_1, f_2 \in RS\,(g)$ on $[a,b]$ then so is $c_1 f_1 + c_2 f_2$ where c_1, c_2 are constants and*

$$\int_a^b (c_1 f_1 + c_2 f_2)\, dg = c_1 \int_a^b f_1\, dg + c_2 \int_a^b f_2\, dg.$$

The result follows from theorems 2 and 3.

In particular $$\int_a^b (f_1 - f_2)\, dg = \int_a^b f_1\, dg - \int_a^b f_2\, dg.$$

Corollary 2: *If $f_1(x) \le f_2(x)$ on $[a,b]$*

then $$\int_a^b f_1(x)\, dg \le \int_a^b f_2(x)\, dg.$$

Theorem 4: *Let $f \in RS\,(g_1)$ and $f \in RS\,(g_2)$ on $[a,b]$. Then*

$f \in RS\,(g_1 + g_2)$ on $[a,b]$

and $$\int_a^b f\, d(g_1 + g_2) = \int_a^b f\, dg_1 + \int_a^b f\, dg_2\,.$$

If $f \in RS\,(g)$ and c is a constant then $f \in RS\,(cg)$

and $$\int_a^b f\, d(cg) = c \int_a^b f\, dg.$$

Hence if f is RS-integrable relative to each of the functions g_1 and g_2 on $[a,b]$ then f is RS-integrable relative to $c_1 g_1 + c_2 g_2$, where c_1 and c_2 are constants. Moreover,

$$\int_a^b f\, d(c_1 g_1 + c_2 g_2) = c_1 \int_a^b f\, dg_1 + c_2 \int_a^b f\, dg_2.$$

Proof: It is left as an exercise for the reader. It is similar to other theorems preceding this theorem.

Theorem 5: *If $f \in RS\,(g)$ on $[a,b]$ and $a < c < b$ then $f \in RS\,(g)$ on $[a,c]$, $f \in RS\,(g)$ on $[c,b]$ and*

$$\int_a^b f\, dg = \int_a^c f\, dg + \int_c^b f\, dg.$$

Proof: Let $\varepsilon > 0$ be given. Since $f \in RS\,(g)$ on $[a,b]$, there exists a partition P of $[a,b]$ such that

$$\omega(P, f, g) = U(P, f, g) - L(P, f, g) < \varepsilon. \qquad \text{...(1)}$$

If P_1 and P_2 be the sets of those points of P which constitute the partitions of $[a,c]$ and $[c,b]$ respectively then the inequality (1) gives

$$\omega(P_1, f, g) < \varepsilon \text{ and } \omega(P_2, f, g) < \varepsilon.$$

Hence $f \in RS$ (g) on $[a,c]$ and $f \in RS\,(g)$ on $[c,b]$.

Thus $$\underline{\int_a^b} f\, dg = \overline{\int_a^b} f\, dg = \int_a^b f\, dg \qquad \text{...(2)}$$

$$\underline{\int_a^c} f\, dg = \overline{\int_a^c} f\, dg = \int_a^c f\, dg \qquad \text{...(3)}$$

and $$\underline{\int_c^b} f\, dg = \overline{\int_c^b} f\, dg = \int_c^b f\, dg \qquad \text{...(4)}$$

Now if P_1 and P_2 are any partitions of $[a, c]$ and $[c, b]$ respectively then $P_1 \cup P_2$ is a partition of $[a, b]$. It contains subintervals of P_1 together with those of P_2.

$\therefore$ $$L(P_1, f, g) + L(P_2, f, g) = L(P_1 \cup P_2, f, g) \le \underline{\int_a^b} f\, dg$$

or $$L(P_1, f, g) + L(P_2, f, g) \le \int_a^b f\, dg, \text{ using (2).}$$

Keeping P_2 fixed and taking the supremum over all P_1, we get

$$\underline{\int_a^c} f\, dg + L(P_2, f, g) \le \int_a^b f\, dg$$

or $$\int_a^c f\, dg + L(P_2, f, g) \le \int_a^b f\, dg, \text{ using (3).}$$

Now taking supremum over all P_2 and using (4), we get

$$\int_a^c f\, dg + \int_c^b f\, dg \le \int_a^b f\, dg. \qquad \text{...(5)}$$

Again $$U(P_1, f, g) + U(P_2, f, g) = U(P_1 \cup P_2, f, g) \ge \overline{\int_a^b} f\, dg$$

or $$U(P_1, f, g) + U(P_2, f, g) \ge \int_a^b f\, dg, \text{ using (2).}$$

Keeping P_2 fixed and taking the infimum over all P_1, we get

$$\overline{\int_a^c} f\, dg + U(P_2, f, g) \ge \int_a^b f\, dg$$

or $$\int_a^c f\, dg + U(P_2, f, g) \ge \int_a^b f\, dg, \text{ using (3).}$$

Now taking infimum over all P_2 and using (4), we get

$$\int_a^c f\, dg + \int_c^b f\, dg \ge \int_a^b f\, dg \qquad \text{...(6)}$$

It follows from (5) and (6) that

$$\int_a^c f\, dg + \int_c^b f\, dg = \int_a^b f\, dg.$$

Corollary: *If* $\int_a^b f\, dg$ *exists and if* $[c, d] \subset [a, b]$ *then* $\int_c^d f\, dg$ *exists also.*

Theorem 6: *Let $f \in RS(g)$ on $[a,b]$, let $m = \inf f$ and $M = \sup f$ on $[a,b]$ and let $\phi : [m, M] \to \mathbf{R}$ be continuous. Then the composite function $h = \phi \circ f \in RS(g)$ on $[a,b]$.*

Proof: Let $\varepsilon > 0$ be given. The function ϕ is uniformly continuous on $[m, M]$ since it is continuous on a closed and bounded interval $[m, M]$. Therefore there exists a $\delta_1 > 0$ that

$$x, y \in [m, M], |x - y| < \delta_1 \Rightarrow |\phi(x) - \phi(y)| < \varepsilon \qquad ...(1)$$

Let $\delta = \min(\delta_1, \varepsilon)$. Now $f \in RS(g)$ on $[a,b]$ implies that for $\delta^2 > 0$, there exists a partition

$$P = \{a = x_0, x_1, \ldots, x_n = b\}$$

of $[a,b]$ such that $U(P, f, g) - L(P, f, g) < \delta^2$ ...(2)

Let $\quad m_r = \inf f, M_r = \sup f$ on $I_r = [x_{r-1}, x_r]$

and $\quad m_r^* = \inf h, M_r^* = \sup h$ on I_r.

Now divide the numbers $1, 2, \ldots, n$ into two groups A and B

such that $\quad A = \{i : M_i - m_i < \delta\}, B = \{i : M_i - m_i \geq \delta\}$.

Then $r \in A$ and $x_{r-1} \leq x \leq y \leq x_r$

$$\Rightarrow \quad |f(x) - f(y)| \leq M_r - m_r < \delta \leq \delta_1$$

$$\Rightarrow \quad |\phi(f(x)) - \phi(f(y))| < \varepsilon, \quad \text{using (1)}$$

$$\Rightarrow \quad |(\phi \circ f)(x) - (\phi \circ f)(y)| < \varepsilon$$

$$\Rightarrow \quad |h(x) - h(y)| < \varepsilon$$

$$\Rightarrow \quad |M_r^* - m_r^*| \leq \varepsilon \qquad ...(3)$$

From this it follows that

$$\sum_{r \in A} (M_r^* - m_r^*) \Delta g_r \leq \varepsilon \sum_{r \in A} \Delta g_r$$

$$\leq \varepsilon \sum_{r=1}^{n} \Delta g_r$$

$$= \varepsilon [g(b) - g(a)]. \qquad ...(4)$$

Again $r \in B \Rightarrow M_r - m_r \geq \delta$. It follows that

$$\delta \sum_{r \in B} \Delta g_r \leq \sum_{r \in B} (M_r - m_r) \Delta g_r \leq \sum_{r=1}^{n} (M_r - m_r) \Delta g_r$$

$$= U(P, f, g) - L(P, f, g) < \delta^2, \text{ by (2)}$$

$$\Rightarrow \quad \sum_{r \in B} \Delta g_r < \delta \leq \varepsilon \qquad ...(5)$$

Take $\quad K = \sup\{|\phi(x)| : m \leq x \leq M\}$ ...(6)

We have $U(P, h, g) - L(P, h, g)$

$$= \sum_{r \in A} (M_r^* - m_r^*) \Delta g_r + \sum_{r \in B} (M_r^* - m_r^*) \Delta g_r$$

$$\le \varepsilon\,[g(b) - g(a)] + \sum_{r \in B} 2K\,\Delta g_r$$
$$< \varepsilon\,[g(b) - g(a)] + 2K\,\varepsilon = \varepsilon\,[g(b) - g(a) + 2K].$$

Since $[g(b) - g(a) + 2K]$ is constant and $\varepsilon > 0$ is arbitrary, we conclude that $h = \phi \circ f \in RS(g)$ on $[a, b]$.

Deduction 1: *If $f \in RS(g)$ on $[a,b]$ and c is a constant then $cf \in RS(g)$ on $[a,b]$ and*

$$\int_a^b cf\, dg = c\int_a^b f\, dg.$$

Proof: Define $\phi(t) = ct$ so that $\phi(f(x)) = cf(x)$. Hence by the above theorem $cf \in RS(g)$ on $[a, b]$.

Using the definition of upper and lower integrals, we get

$$\overline{\int_a^b} cf\, dg = \begin{cases} c\,\overline{\int_a^b} f\, dg, \text{ if } c \ge 0 \\ c\,\underline{\int_a^b} f\, dg, \text{ if } c < 0. \end{cases}$$

The bars may be removed since f and cf are RS-integrable. Thus for every constant c, we have

$$\int_a^b cf\, dg = c\int_a^b f\, dg.$$

Deduction 2: *If $f \in RS(g)$ on $[a,b]$ then $|f| \in RS(g)$ on $[a,b]$, and*

$$\left|\int_a^b f\, dg\right| \le \int_a^b |f|\, dg.$$

Proof: If $\int_a^b f\, dg \ge 0$, leave it as it is but if $\int_a^b f\, dg < 0$ multiply it by -1 to make it > 0. This means that we choose either $c = 1$ or -1 to make

$$c\int_a^b f\, dg \ge 0.$$

Then we have

$$\left|\int_a^b f\, dg\right| = c\int_a^b f\, dg = \int_a^b cf\, dg, \text{ by deduction 1}$$

$$\le \int_a^b |f|\, dg. \qquad [\because\ cf = \pm f \le |f|]$$

Hence $$\left|\int_a^b f\, dg\right| \le \int_a^b |f|\, dg.$$

Deduction 3: *If $f \in RS(g)$ on $[a,b]$ then $f^2 \in RS(g)$ on $[a,b]$.*

Proof: Define $\phi(t) = t^2$ so that $\phi(f(x)) = f^2(x)$.

Also $\phi(t) = t^2$ is continuous on $[m, M]$. Hence by the above theorem, $f^2 \in RS(g)$ on $[a, b]$.

Theorem 7: *If f and $\psi \in RS(g)$ on $[a, b]$ then*

$f\psi \in RS(g)$ on $[a, b]$.

Proof: It follows from the identity

$$f\psi = \frac{1}{4}[(f + \psi)^2 - (f - \psi)^2]$$

and the preceding results.

10 A Relation between R-integral and RS-integral

We have seen that Riemann integral is a special case of Riemann-Stieltjes integral. But under certain conditions a simple relationship exists between them.

Theorem: *Let f be R-integrable on $[a, b]$ and let g be a monotonically non-decreasing function on $[a, b]$ such that its derivative g' is R-integrable on $[a, b]$. Then $f \in RS(g)$ on $[a, b]$ and*

$$\int_a^b f\, dg = \int_a^b f(x)\, g'(x)\, dx \qquad \text{...(A)}$$

i.e., RS-integral of f relative to g on $[a, b]$ is equal to the R-integral of fg' on $[a, b]$.

(Gorakhpur 2012)

Proof: Since $f \in \mathbf{R}[a, b]$, $g' \in \mathbf{R}[a, b]$ so $fg' \in \mathbf{R}[a, b]$ and hence right hand side of (A) is meaningful.

Again $f \in \mathbf{R}[a, b] \Rightarrow f$ is bounded on $[a, b]$ so that for some constant $M > 0$, we have

$$|f| \le M \qquad \text{...(1)}$$

Let $\varepsilon > 0$ be given. Since fg' is R-integrable, there exists $\delta_1 > 0$ such that for all partitions P with $\|P\| < \delta_1$ and for all ξ_r with $x_{r-1} \le \xi_r \le x_r$, we have

$$\left| \Sigma f(\xi_r)\, g'(\xi_r)\, \Delta x_r - \int_a^b fg' \right| < \varepsilon \qquad \text{...(2)}$$

Again since g' is R-integrable, there exists $\delta_2 > 0$ such that for all partitions P with $\|P\| < \delta_2$ and $\xi_r \in [x_{r-1}, x_r]$, we have

$$\left| \Sigma g'(\xi_r)\, \Delta x_r - \int_a^b g' \right| < \varepsilon \qquad \text{...(3)}$$

Varying ξ_r in (3), it is easy to see that

$$\Sigma | g'(\xi_r) - g'(\eta_r) | \Delta x_r < 2\varepsilon \qquad \text{...(4)}$$

if $\|P\| < \delta_2$ and $\xi_r, \eta_r \in [x_{r-1}, x_r]$.

Choose $\delta = \min\{\delta_1, \delta_2\}$. Let P be a partition with $\|P\| < \delta$ and $\xi_r \in [x_{r-1}, x_r]$. Using mean value theorem of differential calculus, there exists point $\eta_r \in [x_{r-1}, x_r]$ such that

$$\Delta g_r = g(x_r) - g(x_{r-1}) = g'(\eta_r)(x_r - x_{r-1}) = g'(\eta_r)\,\Delta x_r.$$

Thus, we have

$$\Sigma\ f(\xi_r)\,\Delta g_r = \Sigma\ f(\xi_r)\,g'(\eta_r)\,\Delta x_r$$
$$= \Sigma\ f(\xi_r)[g'(\eta_r) - g'(\xi_r)]\,\Delta x_r + \Sigma\ f(\xi_r)\,g'(\xi_r)\,\Delta x_r\,.$$

Hence $$\left|\Sigma f(\xi_r)\,\Delta g_r - \int_a^b fg'\right|$$

$$= \left|\Sigma f(\xi_r)[g'(\eta_r) - g'(\xi_r)]\,\Delta x_r + \Sigma\ f(\xi_r)\,g'(\xi_r)\,\Delta x_r - \int_a^b fg'\right|$$

$$\leq \Sigma\ |f(\xi_r)|\,|g'(\eta_r) - g'(\xi_r)|\,\Delta x_r + \left|\Sigma f(\xi_r)\,g'(\xi_r)\,\Delta x_r - \int_a^b fg'\right|$$

$< M.2\varepsilon + \varepsilon$, using (1), (2) and (4)

$= (2M + 1)\,\varepsilon.$

It follows that $$\lim_{\|P\|\to 0} \Sigma f(\xi_r)\,\Delta g_r = \int_a^b fg' \qquad ...(5)$$

Since $\int_a^b fg'$ exists $\Rightarrow \lim \Sigma\ f(\xi_r)\,\Delta g_r$ exists, it follows from theorem 1 of article 7 that

$$\lim_{\|P\|\to 0} \Sigma f(\xi_r)\,\Delta g_r = \int_a^b f\,dg \qquad ...(6)$$

Hence from (5) and (6), we get

$$\int_a^b f\,dg = \int_a^b f(x)\,g'(x)\,dx.$$

Illustrative Examples

Example 3: *Let $f(x) = x$, $g(x) = x^2$.*

Does $\int_0^1 f\,dg$ exist ? If it exists, find its value.

(Meerut 2004, 08)

Solution: Here f is a continuous function and g is a non-decreasing function on $[0, 1]$.

Hence, by theorem 2 of article 7 $\int_0^1 f\,dg$ exists.

To find its value, consider the partition

$$P = \{0, 1/n, 2/n, \ldots, r/n, \ldots, n/n = 1\}$$

and the intermediate partition $Q = \{1/n, 2/n, \ldots, n/n = 1\}$.

Now $$RS(P, Q, f, g) = \sum_{r=1}^{n} f(\xi_r)\, \Delta g_r$$

$$= \sum_{r=1}^{n} f\left(\frac{r}{n}\right)\left[g\left(\frac{r}{n}\right) - g\left(\frac{r-1}{n}\right)\right] = \sum_{r=1}^{n} \frac{r}{n}\left[\frac{r^2}{n^2} - \frac{(r-1)^2}{n^2}\right]$$

$$= \frac{1}{n^3}\sum_{r=1}^{n}(2r^2 - r) = \frac{2}{n^3}\sum_{r=1}^{n} r^2 - \frac{1}{n^3}\sum_{r=1}^{n} r$$

$$= \frac{2}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{1}{n^3}\cdot\frac{n(n+1)}{2}$$

$$= \frac{1}{6n^2}[2(2n^2 + 3n + 1) - 3n - 3] = \frac{4n^2 + 3n - 1}{6n^2}$$

$$= \frac{1}{6}\left(4 + \frac{1}{2n} - \frac{1}{6n^2}\right).$$

$\therefore$ $$\int_0^1 f\, dg = \lim_{\|P\| \to 0} RS(P, Q, f, g)$$

$$= \lim_{n \to \infty} \frac{1}{6}\left(4 + \frac{1}{2n} - \frac{1}{6n^2}\right) = \frac{1}{6}(4 + 0 - 0) = \frac{2}{3}.$$

Example 4: *Let $I = [0, 1]$ and let $f, \alpha : I \to R$ be functions such that $f(x) = \alpha(x) = x^2$. Then find the value of $\int_0^1 x^2\, dx^2$.*

Solution: Let $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1\right\}$ be the partition of $[0, 1]$ obtained by dissecting $[0, 1]$ into n equal parts. Then

$$I_r = r\text{th sub-interval} = \left[\frac{r-1}{n}, \frac{r}{n}\right]$$

Let M_r and m_r be respectively the supremum and infimum of f in I_r.

Then $$M_r = \sup_{x \in I_r} f(x) = \frac{r^2}{n^2}$$

and $$m_r = \inf_{x \in I_r} f(x) = \frac{(r-1)^2}{n^2}$$

$\therefore$ $$\Delta\alpha_r = \alpha(x_r) - \alpha(x_{r-1})$$

$$= \frac{r^2}{n^2} - \frac{(r-1)^2}{n^2} = \frac{2r-1}{n^2}$$

Now $$U(P, f, \alpha) = \sum_{r=1}^{n} M_r\, \Delta\alpha_r = \sum_{r=1}^{n} \frac{r^2}{n^2}\cdot\frac{2r-1}{n^2}$$

$$= \frac{1}{n^4}\left\{2\sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} r^2\right\}$$

$$= \frac{1}{n^4}\left\{2 . \frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\right\} = \frac{3n^3 + 4n^2 - 1}{6n^3}.$$

$\therefore$ $$\int_0^1 f\, d\alpha = \inf U(p, f, \alpha) = \lim_{n\to\infty} \frac{3n^3 + 4n^2 - 1}{6n^3} = \frac{1}{2}.$$

Again $$L(P, f, \alpha) = \sum_{r=1}^{n} m_r \Delta\alpha_r$$

$$= \sum_{r=1}^{n} \frac{(r-1)^2}{n^2} \cdot \frac{(2r-1)}{n^2}$$

$$= \frac{1}{n^4}\left\{2 \sum_{r=1}^{n} (r-1)^3 + \sum_{r=1}^{n} (r-1)^2\right\}$$

$$= \frac{1}{n^4}\left\{2 \cdot \frac{(n-1)^2 \cdot n^2}{4} + \frac{(n-1)\cdot n \cdot (2n-1)}{6}\right\}$$

$$= \frac{3n^3 - 4n^2 + 1}{6n^3}.$$

$\therefore$ $$\underline{\int_0^1} f\, d\alpha = \sup L(P, f, \alpha) = \lim_{n\to\infty} \frac{3n^3 - 4n^2 + 1}{6n^3} = \frac{1}{2}.$$

Thus $$\underline{\int_0^1} f\, d\alpha = \overline{\int_0^1} f\, d\alpha = \frac{1}{2}.$$

Hence $\int_0^1 f\, d\alpha$ exists and the value of $\int_0^1 x^2 dx^2 = \frac{1}{2}$.

Alternative Method 1: Consider the partition

$P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{r}{n}, \ldots \frac{n}{n} = 1\right\}$ and the intermediate partition $Q = \left\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1\right\}$.

Now $$RS(P, Q, f, \alpha) = \sum_{r=1}^{n} f(\xi_r)\,\Delta\alpha_r$$

$$= \sum_{r=1}^{n} \frac{2r-1}{2n}\left[\left(\frac{r}{n}\right)^2 - \left(\frac{r-1}{n}\right)^2\right]$$

$$= \frac{1}{4n^4} \sum_{r=1}^{n} (2r-1)^3 = \frac{1}{2n^3}\left(n^3 - \frac{n}{2}\right) = \frac{1}{2}\left(1 - \frac{1}{2n^2}\right).$$

$\therefore$ $$\int_0^1 f\, \alpha = \lim_{\|P\|\to 0} RS(P, Q, f, \alpha) = \lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{2n^2}\right) = \frac{1}{2}.$$

Hence $\int_0^1 x^2 dx^2 = \frac{1}{2}$.

Alternative Method 2: By using theorem of article 10, we get

$$\int_0^1 x^2\, dx^2 = \int_0^1 x^2\,(2x)\,dx = 2\int_0^1 x^3 dx = 2\left[\frac{x^4}{4}\right]_0^1 = \frac{1}{2}.$$

Example 5: *Find the value of* $\int_{-1}^{2} x^3 d(|x|^5)$.

Solution: Using theorem of article 10, we have

$$\int_{-1}^{2} x^3 d(|x|^5) = \int_{-1}^{0} x^3\, d(-x^5) + \int_{0}^{2} x^3\, d(x^5)$$

$$= -5\int_{-1}^{0} x^7\, dx + 5\int_{0}^{2} x^7\, dx$$

$$= -5\left[\frac{x^8}{8}\right]_{-1}^{0} + 5\left[\frac{x^8}{8}\right]_{0}^{2} = \frac{5}{8} + 160.$$

Comprehensive Exercise 1

1. Prove that if g is an increasing function on $[a, b]$ and if f is non-negative and integrable with respect to g on $[a, b]$, then

$$\int_a^b f\, dg \geq 0.$$

2. Let g be increasing and f non-negative and integrable relative to g on $[a, b]$. Prove that if $a \leq c \leq d \leq b$ then

$$\int_c^d f\, dg \leq \int_a^b f\, dg.$$

3. Let $g : [a, b] \to \mathbf{R}$ be monotonic increasing on $[a, b]$ and continuous at x' where $a \leq x' \leq b$. Define $f : [a, b] \to \mathbf{R}$ as follows :

$$f(x) = \begin{cases} 0 & \text{when } x \neq x' \\ 1 & \text{when } x = x' \end{cases}.$$

Prove that $f \in RS(g)$ on $[a, b]$ and $\int_a^b f\, dg = 0$.

4. Evaluate the following integrals :

(i) $\int_0^2 x^2\, dx^2$ (ii) $\int_0^2 [x]\, dx^2$

Answers 1

4. (i) 8 (ii) 3

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Let f be a bounded function and g a non-decreasing function on $[a,b]$. Then f is Riemann-Stieltjes integrable on $[a,b]$ relative to g if and only if

   (a) $\underline{\int_a^b} f\,dg = \overline{\int_a^b} f\,dg$ (b) $\underline{\int_a^b} f\,dg + \overline{\int_a^b} f\,dg = 0$

   (c) $\underline{\int_a^b} f\,dg \geq \overline{\int_a^b} f\,dg$ (d) $\underline{\int_a^b} f\,dg \leq \overline{\int_a^b} f\,dg$

2. Let f be a bounded function and g be a monotonically non-decreasing function on $[a,b]$. If P^* is a refinement of P, then

   (a) $U(P^*, f, g) \geq U(P, f, g)$ (b) $U(P^*, f, g) = U(P, f, g)$

   (c) $L(P^*, f, g) \leq L(P, f, g)$ (d) $L(P^*, f, g) = U(P, f, g)$

3. Let f be a bounded function and g a non-decreasing function on $[a,b]$. Let $f \in RS(g)$ and $f(x) \geq 0 \;\forall\; x \in [a,b]$

   (a) $\int_a^b f\,dg = 0$ (b) $\int_a^b f\,dg < 0$

   (c) $\int_a^b f\,dg \geq 0$ (d) None of these

4. Let f be a bounded function and g a non-decreasing function on $[a,b]$. Let $f_1, f_2 \in RS(g)$ on $[a,b]$ and $f_1(x) \leq f_2(x)$ on $[a,b]$, then

   (a) $\int_a^b f_1\,dg \leq \int_a^b f_2\,dg$ (b) $\int_a^b f_1\,dg \geq \int_a^b f_2\,dg$

   (c) $\int_a^b f_1\,dg + \int_a^b f_2\,dg = 0$ (d) $\int_a^b f_1\,dg - \int_a^b f_2\,dg = 0$

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. Let f be a bounded and g a non-decreasing function on $[a,b]$. Then $f \in RS(g)$ if and only if for every $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that

   $$U(P, f, g) - L(P, f, g) \ldots\ldots \varepsilon.$$

2. Let f be a bounded and g a non-decreasing function on $[a,b]$. Let $f \in RS(g)$ on $[a,b]$. Then $cf \in RS(g)$ on $[a,b]$ for every constant c and $\int_a^b (c\,f)\,dg = \ldots\ldots$.

3. Let f be a bounded and g a non-decreasing function on $[a,b]$. If $f \in RS(g)$ on $[a,b]$ and $a < c < b$ then $f \in RS(g)$ on $[a,c]$, $f \in RS(g)$ on $[c,b]$ and $\int_a^b f\,dg = \ldots\ldots$.

4. If $f \in RS(g)$ on $[a, b]$ then $|f| \in RS(g)$ on $[a, b]$ and $\left|\int_a^b f\, dg\right|$...... $\int_a^b |f|\, dg$.

True or False

Write 'T' for true and 'F' for false statement.

1. Let f be a bounded function and g a non-decreasing function on $[a, b]$. Then the lower RS-integral of f relative to g cannot exceed the upper RS-integral.
2. Let f be continuous and g monotonically non-decreasing on $[a, b]$. Then $f \notin RS(g)$.
3. Let f be monotonic on $[a, b]$ and let g be continuous and non-decreasing on $[a, b]$. Then $f \in RS(g)$.
4. If $f \in RS(g)$, then so is $(-f)$ and $\int_a^b (-f)\, dg = \int_a^b f\, dg$.
5. Let f be R-integrable on $[a, b]$ and let g be a monotonically non-decreasing function on $[a, b]$ such that its derivative g' is R-integrable on $[a, b]$. Then $f \in RS(g)$ on $[a, b]$ and $\int_a^b f\, dg \neq \int_a^b f(x) g'(x)\, dx$.

Answers

Multiple Choice Questions

1. (a) 2. (b) 3. (c) 4. (a)

Fill in the Blank(s)

1. $<$ 2. $c\int_a^b f\, dg$ 3. $\int_a^c f\, dg + \int_c^b f\, dg$ 4. $\leq$

True or False

1. *T* 2. *F* 3. *T* 4. *F* 5. *F*

□

Chapter

9

Convergence of Improper Integrals

1 Some Definitions

1. **Infinite Interval:** The interval whose length (range) is infinite is said to be an *infinite interval*. Thus the intervals (a, ∞), $(-\infty, b)$ and $(-\infty, \infty)$ are infinite intervals.

2. **Bounded Functions:** A function $f(x)$ is said to be *bounded* over the interval I if there exist two real numbers a and b $(b > a)$ such that

$$a \leq f(x) \leq b \text{ for all } x \in I.$$

A function $f(x)$ is said to be unbounded at a point, if it becomes infinite at that point. Thus the function

$$f(x) = x / \{(x-1)(x-2)\}$$

is unbounded at each of the points $x = 1$ and $x = 2$.

3. **Monotonic functions:** A real valued function f defined on an interval I is said to be **monotonically** increasing if

$$x > y \Rightarrow f(x) > f(y) \;\forall\; x, y \in I$$

and **monotonically decreasing** if

$$x > y \Rightarrow f(x) < f(y) \;\forall\; x, y \in I.$$

A function f defined on an interval I is said to be a monotonic function if it is either monotonically decreasing or monotonically increasing on I.

For example the function f defined by $f(x) = \sin x$ is monotonically increasing in the interval $0 \le x \le \frac{1}{2}\pi$ and monotonically decreasing in the interval $\frac{1}{2}\pi \le x \le \pi$.

4. **Proper Integral:** The definite integral $\int_a^b f(x)\,dx$ is said to be a *proper integral* if the range of integration is finite and the integrand $f(x)$ is bounded. The integral $\int_0^{\pi/2} \sin x\,dx$ is a proper integral. Also $\int_0^1 \frac{\sin x}{x}\,dx$ is an example of a proper integral because $\lim_{x \to 0} \frac{\sin x}{x} = 1.$

5. **Improper Integrals:** The definite integral $\int_a^b f(x)\,dx$ is said to be an *improper integral* if (*i*) the interval (a, b) is not finite (*i.e.*, is infinite) and the function $f(x)$ is bounded over this interval; or (*ii*) the interval (a, b) is finite and $f(x)$ is not bounded over this interval; or (*iii*) neither the interval (a, b) is finite nor $f(x)$ is bounded over it.

6. **Improper integrals of the first kind or infinite integrals:** A definite integral $\int_a^b f(x)\,dx$ in which the range of integration is infinite (*i.e.*, either $b = \infty$ or $a = -\infty$ or both) and the integrand $f(x)$ is bounded, is called an improper integral of the first kind or an infinite integral. Thus $\int_0^\infty \frac{dx}{1+x^2}$ is an improper integral of the first kind since the upper limit of integration is infinite and the integrand $1/(1+x^2)$ is bounded. Similarly $\int_{-\infty}^0 e^x\,dx$ is an example of an improper integral of the first kind because here the lower limit of integration is infinite. Also $\int_{-\infty}^\infty \frac{dx}{1+x^2}$ is an improper integral of the first kind.

In case the **interval (a, b) is infinite and the integrand $f(x)$ is bounded**, we define

(i) $\int_a^\infty f(x)\,dx = \lim_{x \to \infty} \int_a^x f(x)\,dx,$

provided that the limit exists finitely *i.e.*, the limit is equal to a definite real number.

(ii) $\int_{-\infty}^b f(x)\,dx = \lim_{x \to \infty} \int_{-x}^b f(x)\,dx,$

provided that the limit exists finitely.

(iii) $\int_{-\infty}^\infty f(x)\,dx = \lim_{x_1 \to \infty} \int_{-x_1}^c f(x)\,dx + \lim_{x_2 \to \infty} \int_c^{x_2} f(x)\,dx$

provided that both these limits exist finitely.

7. **Improper integrals of the second kind:** A definite integral $\int_a^b f(x)\,dx$ in which the range of integration is finite but the integrand $f(x)$ is unbounded at one or more points of the interval $a \le x \le b$, is called an improper integral of the second kind.

Thus $\int_0^4 \frac{dx}{(x-2)(x-3)}$

and $\int_0^1 \frac{1}{x^2}\,dx$ are improper integrals of the second kind.

In the case of the definite integral

$$\int_a^b f(x)\,dx,$$

if the range of integration (a, b) is finite and the integrand $f(x)$ **is unbounded at one or more points of the given interval,** we define the value of the integral as follows :

(i) If $f(x)$ is unbounded at $x = b$ only *i.e.*, if $f(x) \to \infty$ as $x \to b$ only, then we define

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x)\,dx,$$

provided that the limit exists finitely. Here ε is a small positive number.

(ii) If $f(x) \to \infty$ as $x \to a$ only, then we define

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b} f(x)\,dx,$$

provided that the limit exists finitely.

(iii) If $f(x) \to \infty$ as $x \to c$ only, where $a < c < b$, then we define

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)\,dx + \lim_{\varepsilon' \to 0} \int_{c+\varepsilon'}^{b} f(x)\,dx,$$

provided that both these limits exist finitely.

(iv) If $f(x)$ is unbounded at both the points a and b of the interval (a, b) and is bounded at each other point of this interval, we write

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

where $a < c < b$ and the value of the integral exists only if each of the integrals on the right hand side exists.

2 Convergence of Improper Integrals

When the limit of an improper integral as defined above, is a definite finite number, we say that the given integral is **convergent** and the value of the integral is equal to the value of that limit. When the limit is ∞ or $-\infty$, the integral is said to be **divergent** *i.e.*, the value of the integral does not exist.

In case the limit is neither a definite number nor ∞ or $-\infty$, the integral is said to be **oscillatory** and in this case also the value of the integral does not exist *i.e.*, the integral is not convergent. We can define the convergence of the infinite integral $\int_a^\infty f(x)\,dx$ as follows :

Definition: *The integral* $\int_a^\infty f(x)\,dx$ *is said to converge to the value I, if for any arbitrarily chosen positive number* ε*, however small but not zero, there exists a corresponding positive number N such that*

$$\left| \int_a^b f(x)\,dx - I \right| < \varepsilon \text{ for all values of } b \geq N.$$

Similarly we can define the convergence of an integral, when the lower limit is infinite, or when the integrand becomes infinite at the upper or lower limit.

Illustrative Examples

Example 1: *Discuss the convergence of the following integrals by evaluating them*

(*i*) $\int_1^\infty \frac{dx}{\sqrt{x}}$, (*ii*) $\int_1^\infty \frac{dx}{x^{3/2}}$.

Solution: (i) We have

$$\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{x\to\infty} \int_1^x \frac{dx}{\sqrt{x}}, \text{ (By def.)}$$

$$= \lim_{x\to\infty} \int_1^x x^{-1/2}\,dx = \lim_{x\to\infty} \left[\frac{x^{1/2}}{1/2}\right]_1^x$$

$$= \lim_{x\to\infty} [2\sqrt{x} - 2] = \infty.$$

Thus the limit does not exist finitely and therefore the given integral is divergent (*i.e.*, the integral does not exist).

(ii) We have

$$\int_1^\infty \frac{dx}{x^{3/2}} = \lim_{x\to\infty} \int_1^x \frac{dx}{x^{3/2}}, \quad \text{(By def.)}$$

$$= \lim_{x\to\infty} \int_1^x x^{-3/2}\,dx = \lim_{x\to\infty} \left[\frac{x^{-1/2}}{-1/2}\right]_1^x = \lim_{x\to\infty} \left[-\frac{2}{\sqrt{x}}\right]_1^x$$

$$= \lim_{x\to\infty} \left[-\frac{2}{\sqrt{x}} + 2\right] = 2.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent and its value is 2.

Example 2: *Test the convergence of* $\int_0^\infty e^{-mx}\,dx$, $(m > 0)$.

Solution: We have $\int_0^\infty e^{-mx}\,dx = \lim_{x\to\infty} \int_0^\infty e^{-mx}\,dx$, (by def.)

$$= \lim_{x\to\infty} \left[\frac{e^{-mx}}{-m}\right]_0^x = \lim_{x\to\infty} \left\{-\frac{1}{m}(e^{-mx} - 1)\right\}$$

$$= -\frac{1}{m}[0 - 1] = \frac{1}{m}.$$

Thus the limit exists and is unique and finite, therefore the given integral is convergent.

Example 3: *Test the convergence of* $\int_0^\infty \dfrac{4a\,dx}{x^2+4a^2}$.

Solution: We have $\int_0^\infty \dfrac{4a\,dx}{x^2+4a^2} = \lim_{x\to\infty} \int_0^x \dfrac{4a\,dx}{x^2+(2a)^2}$, (By def.)

$$= \lim_{x\to\infty} \left[4a\cdot\frac{1}{2a}\tan^{-1}\frac{x}{2a}\right]_0^x = 2\lim_{x\to\infty}\left[\tan^{-1}\frac{x}{2a}\right]_0^x$$

$$= 2\,.\lim_{x\to\infty}\left[\tan^{-1}\frac{x}{2a} - 0\right] = 2\,.\,[\tan^{-1}\infty] = 2\,.\,\frac{\pi}{2} = \pi.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent.

Example 4: *Test the convergence of (i)* $\int_{-\infty}^0 e^x\,dx$; *(ii)* $\int_{-\infty}^0 e^{-x}\,dx$.

Solution: (i) We have $\int_{-\infty}^0 e^x\,dx = \lim_{x\to\infty}\int_{-x}^0 e^x\,dx$, (By def.)

$$= \lim_{x\to\infty}[e^x]_{-x}^0 = \lim_{x\to\infty}[1-e^{-x}] = [1-0] = 1.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent.

(ii) We have $\int_{-\infty}^0 e^{-x}\,dx = \lim_{x\to\infty}\int_{-x}^0 e^{-x}\,dx$, (By def.)

$$= \lim_{x\to\infty}\left[\frac{e^{-x}}{-1}\right]_{-x}^0 = -\lim_{x\to\infty}[e^0 - e^x] = \infty.$$

Thus the limit does not exist finitely and therefore the given integral is divergent(*i.e.*, the integral does not exist).

Example 5: *Test the convergence of* $\int_{-\infty}^\infty \dfrac{dx}{1+x^2}$.

(Kanpur 2008; Gorakhpur 11)

Solution: We have

$$\int_{-\infty}^\infty \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^\infty \frac{dx}{1+x^2}$$

$$= \lim_{x\to\infty}\int_{-x}^0 \frac{dx}{1+x^2} + \lim_{x\to\infty}\int_0^x \frac{dx}{1+x^2}$$

$$= \lim_{x\to\infty}[\tan^{-1}x]_{-x}^0 + \lim_{x\to\infty}[\tan^{-1}x]_0^x$$

$$= \lim_{x\to\infty}[0-\tan^{-1}(-x)] + \lim_{x\to\infty}[\tan^{-1}x - 0]$$

$$= -(-\pi/2) + \pi/2 = \pi.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent.

Example 6: *Evaluate* $\int_0^1 \dfrac{dx}{\sqrt{x}}$.

(Gorakhpur 2010)

Solution: In the given integral, the integrand $1/\sqrt{x}$ becomes infinite at the lower limit $x = 0$. Therefore we have

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0} \int_{0+\varepsilon}^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0} [2\sqrt{x}]_\varepsilon^1$$

$$= \lim_{\varepsilon \to 0} [2 - 2\sqrt{\varepsilon}] = 2.$$

Hence the given integral is convergent and its value is 2.

Example 7: *Evaluate* $\int_0^1 \frac{dx}{\sqrt{(1-x)}}$.

Solution: Here the integrand *i.e.*, $1/\sqrt{(1-x)}$ becomes unbounded *i.e.*, infinite at the upper limit (*i.e.*, $x = 1$).

$$\therefore \quad \int_0^1 \frac{dx}{\sqrt{(1-x)}} = \lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} \frac{dx}{\sqrt{(1-x)}}$$

$$= \lim_{\varepsilon \to 0} [-2\sqrt{(1-x)}]_0^{1-\varepsilon} = \lim_{\varepsilon \to 0} [-2\sqrt{\varepsilon} + 2] = 2,$$

which is a definite real number. Hence the given integral is convergent and its value is 2.

Example 8: *Evaluate* $\int_{-1}^1 \frac{dx}{x^2}$.

Solution: Here the integrand becomes infinite at $x = 0$ and $-1 < 0 < 1$.

$$\therefore \quad \int_{-1}^1 \frac{dx}{x^2} = \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} \frac{dx}{x^2} + \lim_{\varepsilon' \to 0} \int_{\varepsilon'}^1 \frac{dx}{x^2}$$

$$= \lim_{\varepsilon \to 0} \left[-\frac{1}{x}\right]_{-1}^{-\varepsilon} + \lim_{\varepsilon' \to 0} \left[-\frac{1}{x}\right]_{\varepsilon'}^1$$

$$= \lim_{\varepsilon \to 0} \left[\frac{1}{\varepsilon} - 1\right] + \lim_{\varepsilon' \to 0} \left[-1 + \frac{1}{\varepsilon'}\right].$$

Since both the limits do not exist finitely, therefore the integral does not exist and is divergent.

Comprehensive Exercise 1

Evaluate the following integrals and discuss their convergence :

1. $\int_1^\infty \frac{dx}{x}$.
2. $\int_3^\infty \frac{dx}{(x-2)^2}$.
3. $\int_0^\infty e^{2x}\, dx$.
4. $\int_0^\infty \frac{dx}{(1+x)^{2/3}}$
5. $\int_{-\infty}^0 \sinh x\, dx$;
6. $\int_{-\infty}^0 \cosh x\, dx$.
7. $\int_0^\infty \cos x\, dx$
8. $\int_{-\infty}^\infty e^{-x}\, dx$.
9. $\int_{-\infty}^\infty \frac{dx}{x^2 + 2x + 2}$.

10. $\int_0^1 \frac{dx}{x^3}$. **11.** $\int_0^1 \frac{dx}{1-x}$. **12.** $\int_{-1}^1 \frac{dx}{x^{2/3}}$.

(Gorakhpur 2011)

Answers 1

1.	∞, divergent	2.	1, convergent	3.	∞, divergent
4.	∞, divergent	5.	$-\infty$, divergent	6.	∞, divergent
7.	Oscillates and so not convergent				
8.	∞, divergent	9.	π, convergent	10.	∞, divergent
11.	∞, divergent	12.	6, convergent		

3 Tests for Convergence of Improper Integrals of the First Kind

To test the convergence of improper integrals in which the range of integration is infinite and the integrand is bounded.

If an integral of the form $\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^b f(x)\,dx$ cannot be actually integrated, its convergence is determined with the help of the following tests :

4 Comparison Test

(Meerut 2012)

Let $f(x)$ and $g(x)$ be two functions which are bounded and integrable in the interval (a, ∞). Also let $g(x)$ be positive and $|f(x)| \le g(x)$ when $x \ge a$. Then, if $\int_a^\infty g(x)\,dx$ is **convergent**, *$\int_a^\infty f(x)\,dx$ is also* **convergent**.

Similarly if $|f(x)| \ge g(x)$ for all values of x greater than some number $x_0 > a$ and $\int_a^\infty g(x)\,dx$ is **divergent**, *then $\int_a^\infty f(x)\,dx$ is also* **divergent**.

Alternative form of the above comparison test:

If $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ is a definite number, other than zero, the integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both converge or both diverge.

Note: While applying comparison test, we generally take $g(x) = \frac{1}{x^n}$ *i.e.*, $\int_a^\infty \frac{dx}{x^n}$ is generally taken as the comparison integral.

Theorem: *The comparison integral* $\int_a^\infty \frac{dx}{x^n}$, *where* $a > 0$, *is convergent when* $n > 1$ *and divergent when* $n \leq 1$.

Proof: By the definition of an improper integral, we have

$$\int_a^\infty \frac{dx}{x^n} = \lim_{x \to \infty} \int_a^x \frac{dx}{x^n} = \lim_{x \to \infty} \int_a^x x^{-n}\, dx$$

$$= \lim_{x \to \infty} \left[\frac{x^{1-n}}{1-n}\right]_a^x, \text{ if } n \neq 1$$

$$= \lim_{x \to \infty} \left[\frac{x^{1-n}}{1-n} - \frac{a^{1-n}}{1-n}\right]. \qquad \ldots(1)$$

If $n > 1$, then $1 - n$ is negative and so $n - 1$ is positive.

Therefore in this case

$$\lim_{x \to \infty} x^{1-n} = \lim_{x \to \infty} \frac{1}{x^{n-1}} = \frac{1}{\infty} = 0.$$

Hence from (1), we have

$$\int_a^\infty \frac{dx}{x^n} = \frac{a^{1-n}}{n-1}, \text{ if } n > 1.$$

Hence the given integral is convergent when $n > 1$.

If $n < 1$, then $1 - n$ is positive and so $\lim_{x \to \infty} x^{1-n} = \infty$.

$\therefore$ from (1), we have $\int_a^\infty \frac{dx}{x^n} = \infty$.

Hence the given integral is divergent when $n < 1$.

When $n = 1$, we have

$$\int_a^\infty \frac{dx}{x^n} = \int_a^\infty \frac{dx}{x} = \lim_{x \to \infty} \int_a^x \frac{dx}{x} = \lim_{x \to \infty} [\log x]_a^x$$

$$= \lim_{x \to \infty} [\log x - \log a] = \infty - \log a = \infty.$$

Hence the given integral is divergent when $n = 1$.

$\therefore$ $\int_a^\infty \frac{dx}{x^n}$ converges when $n > 1$ and diverges when $n \leq 1$.

In other words $\int_a^\infty \frac{dx}{x^n}$ converges if and only if $n > 1$.

Illustrative Examples

Example 9: *Test the convergence of the integral*

$$\int_0^\infty \frac{\cos mx}{x^2 + a^2}\, dx.$$

(Rohilkhand 2011)

Solution: Here $f(x) = \dfrac{\cos mx}{x^2 + a^2}\cdot$ Let $g(x) = \dfrac{1}{x^2 + a^2}\cdot$

Obviously $g(x)$ is positive in the interval $(0, \infty)$.

We have $|f(x)| = \left|\dfrac{\cos mx}{x^2 + a^2}\right| = \dfrac{|\cos mx|}{x^2 + a^2} \le \dfrac{1}{x^2 + a^2}$, since $|\cos mx| \le 1$.

Thus $|f(x)| \le g(x)$ when $x \ge 0$.

$\therefore$ by comparison test, $\int_0^\infty \dfrac{\cos mx}{x^2 + a^2}\,dx$ is convergent if $\int_0^\infty \dfrac{dx}{x^2 + a^2}$ is convergent.

But
$$\int_0^\infty \frac{dx}{x^2 + a^2} = \lim_{x \to \infty} \int_0^\infty \frac{dx}{x^2 + a^2} = \lim_{x \to \infty} \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^x$$
$$= \lim_{x \to \infty} \left[\frac{1}{a}\tan^{-1}\frac{x}{a} - 0\right] = \frac{1}{a}\cdot\frac{\pi}{2}$$
$=$ a definite real number.

$\therefore$ $\displaystyle\int_0^\infty \frac{dx}{x^2 + a^2}$ is convergent.

Hence $\displaystyle\int_0^\infty \frac{\cos mx}{x^2 + a^2}\,dx$ is also convergent.

Example 10: *Test the convergence of the integral*
$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx.$$

Solution: Let $a > 0$. Then we can write
$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \int_0^a \frac{\sin^2 x}{x^2}\,dx + \int_a^\infty \frac{\sin^2 x}{x^2}\,dx.$$

Since $\displaystyle\lim_{x \to 0} \frac{\sin^2 x}{x^2} = 1$, therefore the integrand $\dfrac{\sin^2 x}{x^2}$ is bounded throughout the finite interval $(0, a)$.

So $\displaystyle\int_0^a \frac{\sin^2 x}{x^2}\,dx$ is a proper integral and we need to check the convergence of the integral $\displaystyle\int_a^\infty \frac{\sin^2 x}{x^2}\,dx$ only.

Here $f(x) = \dfrac{\sin^2 x}{x^2}\cdot$ Take $g(x) = \dfrac{1}{x^2}\cdot$

Obviously $g(x)$ is positive in the interval (a, ∞).

We have $|f(x)| = \left|\dfrac{\sin^2 x}{x^2}\right| = \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$, since $\sin^2 x \le 1$.

$\therefore$ by comparison test, $\displaystyle\int_a^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent if $\displaystyle\int_a^\infty \frac{dx}{x^2}$ is convergent.

But the comparison integral $\int_a^{\infty} \frac{dx}{x^2}$ is convergent because here $n = 2$ which is > 1.

$\therefore$ $$\int_a^{\infty} \frac{\sin^2 x}{x^2} \, dx \text{ is convergent.}$$

Hence $\int_0^{\infty} \frac{\sin^2 x}{x^2} \, dx$ is convergent.

Example 11: *Show that the integral* $\int_a^{\infty} \frac{dx}{x \sqrt{(1 + x^2)}}$ *converges, where* $a > 0$.

Solution: Let $f(x) = \frac{1}{x \sqrt{(1 + x^2)}}$.

Then $f(x)$ is bounded in the interval (a, ∞). Take $g(x) = 1 / x^2$. Then $g(x)$ is positive in the interval (a, ∞). We have

$$|f(x)| = \left| \frac{1}{x \sqrt{(1 + x^2)}} \right| = \frac{1}{x^2 \sqrt{\{1 + (1 / x^2)\}}}$$

$$< \frac{1}{x^2}, \text{ since } \frac{1}{\sqrt{\{1 + (1 / x^2)\}}} < 1.$$

$\therefore$ by comparison test, $\int_a^{\infty} \frac{dx}{x \sqrt{(1 + x^2)}}$ is convergent if $\int_a^{\infty} \frac{dx}{x^2}$ is convergent.

But the comparison integral $\int_a^{\infty} \frac{dx}{x^2}$ is convergent because here $n = 2$ which is > 1.

Hence $\int_a^{\infty} \frac{dx}{x \sqrt{(1 + x^2)}}$ is also convergent.

Alternative Method: Here $f(x) = \frac{1}{x \sqrt{(1 + x^2)}} = \frac{1}{x^2 \sqrt{\{1 + (1 / x^2)\}}}$.

Take $g(x) = \frac{1}{x^2}$.

We have $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{1}{\sqrt{\{1 + (1 / x^2)\}}} = 1$, which is finite and non-zero.

Therefore $\int_a^{\infty} f(x) \, dx$ and $\int_a^{\infty} g(x) \, dx$ either both converge or both diverge. But $\int_a^{\infty} g(x) \, dx = \int_a^{\infty} \frac{dx}{x^2}$ is convergent because here $n = 2$ which is > 1.

Hence $\int_a^{\infty} f(x) \, dx$ *i.e.*, $\int_a^{\infty} \frac{1}{x \sqrt{(1 + x^2)}} \, dx$ is also convergent.

Example 12: *Test the convergence of* $\int_0^{\infty} e^{-x} \frac{\sin x}{x} \, dx$.

(Kanpur 2011)

Solution: We can write

$$\int_0^{\infty} e^{-x} \frac{\sin x}{x} \, dx = \int_0^{1} e^{-x} \frac{\sin x}{x} \, dx + \int_1^{\infty} e^{-x} \frac{\sin x}{x} \, dx .$$

Since $\lim_{x \to 0} e^{-x} \dfrac{\sin x}{x} = 1$, therefore the integrand $e^{-x} \dfrac{\sin x}{x}$ is bounded throughout the finite interval $(0, 1)$. So $\int_0^1 e^{-x} \dfrac{\sin x}{x}\, dx$ is a proper integral and therefore it is convergent. Thus we need to check the convergence of $\int_1^\infty e^{-x} \dfrac{\sin x}{x}\, dx$ only.

Let $f(x) = e^{-x} \dfrac{\sin x}{x}$. Then $f(x)$ is bounded in the interval $(1, \infty)$.

Take $g(x) = e^{-x}$. Then $g(x)$ is positive in the interval $(1, \infty)$.

We have

$$|f(x)| = \left| e^{-x} \frac{\sin x}{x} \right| = e^{-x} . |\sin x| \cdot \frac{1}{x}$$

$$\leq e^{-x}, \text{since } |\sin x| \leq 1 \text{ and } \frac{1}{x} \leq 1.$$

Thus $|f(x)| \leq g(x)$ throughout the interval $(1, \infty)$.

$\therefore$ by comparison test $\int_1^\infty f(x)\, dx$ is convergent if $\int_1^\infty g(x)\, dx$ is convergent.

Now $$\int_1^\infty g(x)\, dx = \int_1^\infty e^{-x}\, dx = \lim_{x \to \infty} \int_1^\infty e^{-x}\, dx = \lim_{x \to \infty} [-e^{-x}]_1^x$$

$$= \lim_{x \to \infty} [-e^{-x} + e^{-1}] = 0 + e^{-1} = 1/e,$$

which is a definite finite number. Hence $\int_1^\infty g(x)\, dx$ is convergent.

$\therefore$ $\int_1^\infty f(x)\, dx$ is also convergent.

Hence $\int_0^\infty e^{-x} \dfrac{\sin x}{x}\, dx$ is convergent because the sum of two convergent integrals is also convergent.

Example 13: *Show that the integral* $\int_0^\infty e^{-x^2}\, dx$ *is convergent.*

(Rohilkhand 2010; Meerut 12)

Solution: We have

$$\int_0^\infty e^{-x^2} dx = \int_0^1 e^{-x^2}\, dx + \int_1^\infty e^{-x^2}\, dx.$$

Obviously $\int_0^1 e^{-x^2}\, dx$ is a proper integral because here the interval of integration $(0, 1)$ is finite and the integrand e^{-x^2} is bounded throughout this interval. Therefore this integral is convergent. So we need to check the convergence of $\int_1^\infty e^{-x^2}\, dx$ only.

Let $f(x) = e^{-x^2}$. Take $g(x) = xe^{-x^2}$ so that $g(x)$ is positive throughout the interval $(1, \infty)$. We have

$$|f(x)| = e^{-x^2} \le xe^{-x^2}, \text{ since } x \ge 1.$$

Thus $|f(x)| \le g(x)$ throughout the interval $(1, \infty)$.

$\therefore$ by comparison test $\int_1^\infty e^{-x^2}\,dx$ is convergent if $\int_1^\infty xe^{-x^2}\,dx$ is convergent.

Now
$$\int_1^\infty xe^{-x^2}\,dx = \lim_{x \to \infty} \int_1^x xe^{-x^2}\,dx$$
$$= \lim_{x \to \infty} \left(-\frac{1}{2}e^{-x^2}\right)_1^x$$
$$= \lim_{x \to \infty} \left(-\frac{1}{2}e^{-x^2} + \frac{1}{2}e^{-1}\right)$$
$$= \frac{1}{2}e^{-1}, \text{ which is a definite number.}$$

$\therefore$ $\int_1^\infty xe^{-x^2}\,dx$ is convergent and so $\int_1^\infty e^{-x^2}\,dx$ is also convergent.

Hence the given integral $\int_0^\infty e^{-x^2}\,dx$ is also convergent as it is the sum of two convergent integrals.

5 The μ-Test

(Gorakhpur 2012)

Let $f(x)$ be bounded and integrable in the interval (a, ∞) where $a > 0$.

If there is a number $\mu > 1$, such that $\lim_{x \to \infty} x^\mu f(x)$ exists, then $\int_a^\infty f(x)\,dx$ is convergent.

If there is a number $\mu \le 1$, such that $\lim_{x \to \infty} x^\mu f(x)$ exists and is non-zero, then $\int_a^\infty f(x)\,dx$ is divergent and the same is true if $\lim_{x \to \infty} x^\mu f(x)$ is $+\infty$ or $-\infty$.

While applying the μ-test, the value of μ is usually taken to be equal to the highest power of x in the denominator of the integrand minus the highest power of x in the numerator of the integrand.

Illustrative Examples

Example 14: *Examine the convergence of* $\int_1^\infty \frac{dx}{x^{1/3}(1 + x^{1/2})}$. **(Gorakhpur 2012)**

Solution: Let
$$f(x) = \frac{1}{x^{1/3}(1 + x^{1/2})} = \frac{1}{x^{1/3}\,x^{1/2}\,\{1 + (1/x^{1/2})\}}$$
$$= \frac{1}{x^{5/6}\,\{1 + (1/x^{1/2})\}}.$$

Obviously $f(x)$ is bounded in the interval $(1, \infty)$.

Take $\mu = \frac{5}{6} - 0 = \frac{5}{6} \cdot$ We have

$$\lim_{x \to \infty} x^{\mu} f(x) = \lim_{x \to \infty} x^{5/6} \cdot \frac{1}{x^{5/6} \{1 + (1 / x^{1/2})\}}$$

$$= \lim_{x \to \infty} \frac{1}{1 + (1 / x^{1/2})} = 1,$$

which is finite and non-zero. Since $\mu = \frac{5}{6}$ *i.e.*, < 1, it follows from the μ-test that the given integral is divergent.

Example 15: *Examine the convergence of* $\int_0^{\infty} \frac{x\, dx}{(1 + x)^3}$.

(Rohilkhand 2011; Purvanchal 11)

Solution: Let $a > 0$. Then we have

$$\int_0^{\infty} \frac{x\, dx}{(1 + x)^3} = \int_0^{a} \frac{x\, dx}{(1 + x)^3} + \int_a^{\infty} \frac{x\, dx}{(1 + x)^3}.$$

The first integral on the right hand side is convergent because it is a proper integral. We observe that in this integral the range of integration $(0, a)$ is finite and the integrand $x / (1 + x)^3$ is bounded throughout the interval $(0, a)$. So we need to check the convergence of $\int_a^{\infty} \frac{x\, dx}{(1 + x)^3}$ only.

Let $f(x) = \frac{x}{(1 + x)^3} \cdot$ Then $f(x)$ is bounded in the interval (a, ∞).

Take $\mu = 3 - 1 = 2$. Then

$$\lim_{x \to \infty} x^{\mu} f(x) = \lim_{x \to \infty} x^2 \cdot \frac{x}{(1 + x)^3} = \lim_{x \to \infty} \frac{1}{\{1 + (1 / x)\}^3} = 1,$$

which exists *i.e.*, is equal to a definite real number.

Since $\mu = 2$ *i.e.*, > 1, therefore by μ-test the integral $\int_a^{\infty} \frac{x\, dx}{(1 + x)^3}$ is convergent.

Hence $\int_0^{\infty} \frac{x\, dx}{(1 + x)^3}$ is also convergent because it is the sum of two convergent integrals.

Example 16: *Examine the convergence of* $\int_a^{\infty} \frac{dx}{x (\log x)^{n + 1}}$, *where* $a > 1$.

Solution: Let $\log x = t$ so that $(1 / x)\, dx = dt$.

$$\therefore \qquad \int_a^{\infty} \frac{dx}{x (\log x)^{n + 1}} = \int_{\log a}^{\infty} \frac{dt}{t^{n + 1}}.$$

Let $\qquad f(t) = 1 / t^{n + 1}$.

Then $f(t)$ is bounded in the interval $(\log a, \infty)$.

Take $\mu = (n + 1) - 0 = n + 1$. Then

$$\lim_{t \to \infty} t^{\mu} f(t) = \lim_{t \to \infty} \frac{t^{n+1}}{t^{n+1}} = \lim_{t \to \infty} 1 = 1,$$

which is finite and non-zero.

Therefore by μ-test, the given integral is convergent if

$$\mu > 1 \text{ } i.e., n + 1 > 1 \text{ } i.e., n > 0$$

and divergent if $\mu \leq 1$ *i.e.*, $n + 1 \leq 1$ *i.e.*, $n \leq 0$.

Example 17: *Show that the integral* $\int_1^{\infty} x^{n-1} e^{-x} \, dx$ *is convergent.*

Solution: Let $f(x) = x^{n-1} e^{-x}$. Then $f(x)$ is bounded in the interval $(1, \infty)$. We have

$$\lim_{x \to \infty} x^{\mu} f(x) = \lim_{x \to \infty} \frac{x^{\mu} . x^{n-1}}{e^{x}} = \lim_{x \to \infty} \frac{x^{\mu+n-1}}{1 + x + \frac{x^2}{2!} + \dots}$$

$$= 0 \qquad \text{for all values of } \mu \text{ and } n.$$

Taking $\mu > 1$, we see by μ-test that the integral $\int_1^{\infty} x^{n-1} e^{-x} \, dx$

is convergent for all values of n.

Example 18: *Test the convergence of the integral* $\int_0^{\infty} \frac{x^{2m}}{1 + x^{2n}} \, dx$,

where m and n are positive integers.

(Purvanchal 2007; Rohilkhand 12; Gorakhpur 13, 15)

Solution: Let $a > 0$. We have

$$\int_0^{\infty} \frac{x^{2m}}{1 + x^{2n}} \, dx = \int_0^{a} \frac{x^{2m}}{1 + x^{2n}} \, dx + \int_a^{\infty} \frac{x^{2m}}{1 + x^{2n}} \, dx.$$

The first integral on the right hand side is a proper integral and so it is convergent. Therefore the given integral is convergent or divergent according as $\int_a^{\infty} \frac{x^{2m}}{1 + x^{2n}} \, dx$ is convergent or divergent.

To test the convergence of $\int_a^{\infty} \frac{x^{2m}}{1 + x^{2n}} \, dx$, let us take $\mu = 2n - 2m$.

We have

$$\lim_{x \to \infty} x^{\mu} \cdot \frac{x^{2m}}{1 + x^{2n}} = \lim_{x \to \infty} x^{2n - 2m} \cdot \frac{x^{2m}}{x^{2n} \{1 + (1/x^{2n})\}}$$

$$= \lim_{x \to \infty} \frac{1}{1 + (1/x^{2n})} = 1,$$

which is finite and non-zero.

$\therefore$ by μ-test, the given integral is convergent if $\mu > 1$ *i.e.*, if $2n - 2m > 1$ which is possible if $n > m$ since m and n are positive integers. Also by μ-test, the given integral is divergent if $\mu \leq 1$ *i.e.*, if $2n - 2m \leq 1$ *i.e.*, if $n \leq m$ since n and m are positive integers.

6 Abel's Test for the Convergence of Integral of a Product

If $\int_a^{\infty} f(x)\,dx$ converges and $\phi(x)$ is bounded and monotonic for $x > a$, then $\int_a^{\infty} f(x)\,\phi(x)\,dx$ is convergent. **(Rohilkhand 2008, 11; Purvanchal 07, 10, 11; Kanpur 12; Gorakhpur 14, 15)**

Illustrative Examples

Example 19: *Test the convergence of $\int_a^{\infty} (1 - e^{-x}) \dfrac{\cos x}{x^2}\,dx$, when $a > 0$.*

(Rohilkhand 2009; Gorakhpur 15)

Solution: Let $f(x) = \dfrac{\cos x}{x^2}$ and $\phi(x) = 1 - e^{-x}$.

We have $\left|\dfrac{\cos x}{x^2}\right| \le \dfrac{1}{x^2}$ as $|\cos x| \le 1$.

Since $\int_a^{\infty} \dfrac{1}{x^2}\,dx$ is convergent, therefore by comparison test $\int_a^{\infty} \dfrac{\cos x}{x^2}\,dx$ is also convergent.

Again $\phi(x) = 1 - e^{-x}$ is monotonic increasing and bounded function for $x > a$.

Hence by Abel's test $\int_a^{\infty} (1 - e^{-x}) \dfrac{\cos x}{x^2}\,dx$ is convergent.

Example 20: *Test the convergence of $\int_a^{\infty} e^{-x} \dfrac{\sin x}{x^2}\,dx$, where $a > 0$.*

Solution: Let $f(x) = \dfrac{\sin x}{x^2}$ and $\phi(x) = e^{-x}$.

Since $\left|\dfrac{\sin x}{x^2}\right| \le \dfrac{1}{x^2}$ and $\int_a^{\infty} \dfrac{1}{x^2}\,dx$ is convergent, therefore by comparison test $\int_a^{\infty} \dfrac{\sin x}{x^2}\,dx$ is also convergent.

Again e^{-x} is monotonic decreasing and bounded function for $x > a$.

Hence by Abel's test $\int_a^{\infty} e^{-x} \dfrac{\sin x}{x^2}\,dx$ is convergent.

7 Dirichlet's test for the Convergence of Integral of a Product

If $f(x)$ be bounded and monotonic in the interval $a \le x < \infty$ and if $\lim_{x \to \infty} f(x) = 0$, then the integral $\int_a^{\infty} f(x)\,\phi(x)\,dx$ converges provided $\left|\int_a^{x} \phi(x)\,dx\right|$ is bounded as x takes all finite values. **(Purvanchal 2012)**

Illustrative Examples

Example 21: *Test the convergence of the integral*

$$\int_a^\infty \frac{\sin x}{\sqrt{x}}\, dx, \text{ where } a > 0.$$

(Garhwal 2006, 09)

Solution: Let $f(x) = \dfrac{1}{\sqrt{x}}$ and $\phi(x) = \sin x$.

Now $\dfrac{1}{\sqrt{x}}$ is bounded and monotonic decreasing for all $x \geq a$ and $\lim_{x \to \infty} \dfrac{1}{\sqrt{x}} = 0$.

Also $\left| \int_a^x \phi(x)\, dx \right| = \left| \int_a^x \sin x\, dx \right| = |\cos a - \cos x| \leq 2$, for all finite values of x.

[Note that the value of $\cos x$ lies between -1 and 1].

$\therefore$ $\left| \int_a^x \phi(x)\, dx \right|$ is bounded for all finite values of x.

Hence by Dirichlet's test the integral $\int_a^\infty \dfrac{\sin x}{\sqrt{x}}\, dx$ is convergent.

Example 22: *Show that* $\int_0^\infty \sin x^2\, dx$ *is convergent.*

(Agra 2012)

Solution: We have $\int_0^\infty \sin x^2\, dx = \int_0^1 \sin x^2\, dx + \int_1^\infty \sin x^2\, dx$.

But $\int_0^1 \sin x^2\, dx$ is a proper integral and hence convergent.

Now it remains to test the convergence of $\int_1^\infty \sin x^2\, dx$. We can write

$$\int_1^\infty \sin x^2\, dx = \int_1^\infty 2x\,.(\sin x^2)\,.\frac{1}{2x}\, dx.$$

Let $f(x) = \dfrac{1}{2x}$ and $\phi(x) = 2x \sin x^2$.

The function $f(x) = \dfrac{1}{2x}$ is bounded and monotonic decreasing for all $x \geq 1$ and

$\lim_{x \to \infty} \dfrac{1}{2x} = 0$.

Also $\left| \int_1^x \phi(x)\, dx \right| = \int_1^x 2x \sin x^2\, dx$

$= |\cos 1^2 - \cos x^2| \leq 2$, for all finite values of x.

$\therefore$ $\left| \int_1^x \phi(x)\, dx \right|$ is bounded for all finite values of x.

Hence by Dirichlet's test

$$\int_1^\infty \frac{1}{2x} \cdot (\sin x^2)\, 2x\, dx \ \ i.e., \int_1^\infty \sin x^2\, dx \text{ is convergent.}$$

Since the sum of two convergent integrals is convergent, therefore the integral $\int_0^\infty \sin x^2\, dx$ is convergent.

Example 23: *Show that the integral* $\int_0^\infty \frac{\sin x}{x}\, dx$ *is convergent.*

(Garhwal 2007; Purvanchal 10, 12)

Solution: We have $\int_0^\infty \frac{\sin x}{x}\, dx = \int_0^a \frac{\sin x}{x}\, dx + \int_a^\infty \frac{\sin x}{x}\, dx$, where $a > 0$.

Since $\lim_{x \to 0} \frac{\sin x}{x} = 1$, the integral $\int_0^a \frac{\sin x}{x}\, dx$ is a proper integral and hence convergent.

Now to test the convergence of $\int_a^\infty \frac{\sin x}{x}\, dx$.

Let $f(x) = 1/x$ and $\phi(x) = \sin x$.

The function $f(x) = 1/x$ is bounded and monotonic decreasing for all $x \geq a$ and $\lim_{x \to \infty} \frac{1}{x} = 0$.

Also $\left| \int_a^x \phi(x)\, dx \right| = \left| \int_a^x \sin x\, dx \right| = |\cos a - \cos x| \leq 2$, for all finite values of x.

$\therefore$ $\left| \int_a^x \phi(x)\, dx \right|$ is bounded for all finite values of x.

Hence by Dirichlet's test the integral $\int_a^\infty \frac{\sin x}{x}\, dx$ is convergent.

Since the sum of two convergent integrals is convergent, therefore $\int_0^\infty \frac{\sin x}{x}\, dx$ is convergent.

Example 24: *Prove that* $\int_a^\infty \frac{\cos \alpha x - \cos \beta x}{x}\, dx$ *is convergent where* $a > 0$.

(Rohilkhand 2011)

Solution: We have

$$\int_a^\infty \frac{\cos \alpha x - \cos \beta x}{x}\, dx = \int_a^\infty \frac{\cos \alpha x}{x}\, dx - \int_a^\infty \frac{\cos \beta x}{x}\, dx.$$

The function $f(x) = 1/x$ is bounded and monotonic decreasing for all $x \geq a$ and $\lim_{x \to \infty} \frac{1}{x} = 0$.

Also $$\left| \int_a^x \cos \alpha x\, dx \right| = \left| \frac{1}{\alpha} (\sin \alpha x - \sin \alpha a) \right| \leq \frac{2}{|\alpha|} \cdot$$

$\therefore$ $\left| \int_a^x \cos \alpha x\, dx \right|$ is bounded for all finite values of x.

Similarly $\left| \int_a^x \cos \beta x \, dx \right|$ is bounded for all finite values of x.

$\therefore$ by Dirichlet's test both the integrals

$$\int_a^\infty \frac{\cos \alpha x}{x} \, dx \text{ and } \int_a^\infty \frac{\cos \beta x}{x} \, dx \text{ are convergent.}$$

Hence the given integral is convergent.

Example 25: *Show that the integral*

$$\int_0^\infty e^{-ax} \frac{\sin x}{x} \, dx, a \geq 0 \textit{ is convergent.}$$

(Kanpur 2009; Garhwal 10)

Solution: We have

$$\int_0^\infty e^{-ax} \frac{\sin x}{x} \, dx = \int_0^\alpha e^{-ax} \frac{\sin x}{x} \, dx + \int_\alpha^\infty e^{-ax} \frac{\sin x}{x} \, dx, \text{ where } \alpha > 0.$$

Since $\lim_{x \to 0} e^{-ax} \frac{\sin x}{x} = 1$, the integral $\int_0^\alpha e^{-ax} \frac{\sin x}{x} \, dx$ is a proper integral and hence convergent.

Now it remains to test the convergence of

$$\int_\alpha^\infty e^{-ax} \frac{\sin x}{x} \, dx. \text{ Let } f(x) = \frac{e^{-ax}}{x} \text{ and } \phi(x) = \sin x.$$

Obviously the function $f(x) = \dfrac{1}{x e^{ax}}$ is bounded and monotonic decreasing for all $x \geq \alpha$ and $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \dfrac{1}{x e^{ax}} = 0.$

Moreover $\left| \int_\alpha^x \phi(x) \, dx \right| = \left| \int_\alpha^x \sin x \, dx \right| = |\cos \alpha - \cos x| \leq 2$, for all finite values of x.

$\therefore$ $\left| \int_\alpha^x \phi(x) \, dx \right|$ is bounded for all finite values of x.

$\therefore$ by Dirichlet's test $\int_\alpha^\infty e^{-ax} \frac{\sin x}{x} \, dx$ is convergent.

Since the sum of two convergent integrals is convergent, therefore $\int_0^\infty e^{-ax} \frac{\sin x}{x} \, dx$ is convergent.

8 Absolute Convergence

The infinite integral $\int_a^\infty f(x) \, dx$ *is said to be absolutely convergent if the integral* $\int_a^\infty |f(x)| \, dx$ *is convergent.*

If the integral $\int_a^\infty f(x) \, dx$ is absolutely convergent, it is necessarily convergent. But if the integral $\int_a^\infty f(x) \, dx$ is convergent, it is not necessarily absolutely convergent. Thus absolute convergence gives a sufficient but not a necessary condition for the convergence of an infinite integral.

Illustrative Examples

Example 26: *Show that* $\int_1^\infty \frac{\sin x}{x^4}\, dx$ *is absolutely convergent.*

Solution: The integral $\int_1^\infty \frac{\sin x}{x^4}\, dx$ will be absolutely convergent if $\int_1^\infty \left|\frac{\sin x}{x^4}\right| dx$ is convergent.

Let $f(x) = \left|\frac{\sin x}{x^4}\right|$. Then $f(x)$ is bounded in the interval $(1, \infty)$. We have

$$f(x) = \left|\frac{\sin x}{x^4}\right| = \frac{|\sin x|}{x^4} \le \frac{1}{x^4}, \text{ since } |\sin x| \le 1.$$

$\therefore$ by comparison test, $\int_0^\infty f(x)\, dx$ is convergent if $\int_1^\infty \frac{1}{x^4}\, dx$ is convergent. But the comparison integral $\int_1^\infty \frac{1}{x^4}\, dx$ is convergent because here $n = 4$ which is > 1.

Hence $\int_1^\infty f(x)\, dx$ is convergent and so the given integral is absolutely convergent.

Example 27: *Show that* $\int_0^\infty \frac{\sin mx}{a^2 + x^2}\, dx$ *converges absolutely.*

(Garhwal 2008)

Solution: The integral $\int_0^\infty \frac{\sin mx}{a^2 + x^2}\, dx$ will be absolutely convergent if $\int_0^\infty \left|\frac{\sin mx}{a^2 + x^2}\right| dx$ is convergent.

Let $f(x) = \left|\frac{\sin mx}{a^2 + x^2}\right|$. Then $f(x)$ is bounded in the interval $(0, \infty)$. We have

$$f(x) = \left|\frac{\sin mx}{a^2 + x^2}\right| = \frac{|\sin mx|}{a^2 + x^2} \le \frac{1}{a^2 + x^2}, \text{ since } |\sin mx| \le 1.$$

$\therefore$ by comparison test, $\int_0^\infty f(x)\, dx$ is convergent if $\int_0^\infty \frac{1}{a^2 + x^2}\, dx$ is convergent.

But $$\int_0^\infty \frac{dx}{a^2 + x^2} = \lim_{x \to \infty} \int_0^x \frac{dx}{a^2 + x^2} = \lim_{x \to \infty} \left[\frac{1}{a} \tan^{-1} \frac{x}{a}\right]_0^x$$

$$= \lim_{x \to \infty} \left[\frac{1}{a} \tan^{-1} \frac{x}{a} - 0\right] = \frac{1}{a} \cdot \frac{\pi}{2},$$

which is a definite real number.

$\therefore \quad \int_0^\infty \frac{dx}{a^2 + x^2}$ is convergent. Hence $\int_0^\infty f(x)\, dx$ is also convergent and so the given integral is absolutely convergent.

Example 28: *Show that the integral $\int_0^\infty e^{-x} \cos mx \, dx$ converges absolutely.*

Solution: The integral $\int_0^\infty e^{-x} \cos mx \, dx$ will be absolutely convergent if $\int_0^\infty |e^{-x} \cos mx| \, dx$ is convergent.

Let $f(x) = |e^{-x} \cos mx|$. Then $f(x)$ is bounded in the interval $(0, \infty)$. We have

$$f(x) = |e^{-x} \cos mx| = e^{-x} |\cos mx|$$
$$\leq e^{-x}, \quad \text{since} |\cos mx| \leq 1.$$

$\therefore$ by comparison test, $\int_0^\infty f(x) \, dx$ is convergent if $\int_0^\infty e^{-x} \, dx$ is convergent.

But $$\int_0^\infty e^{-x} \, dx = \lim_{x \to \infty} \int_0^x e^{-x} \, dx = \lim_{x \to \infty} [-e^{-x}]_0^\infty$$

$$= \lim_{x \to \infty} [-e^{-x} + 1] = 1, \text{ which is a definite real number.}$$

$\therefore$ $\int_0^\infty e^{-x} \, dx$ is convergent.

Hence $\int_0^\infty f(x) \, dx$ is convergent and so the given integral is absolutely convergent.

9 Tests for Convergence of Improper Integrals of The Second Kind

Now we shall make a study of the tests for the convergence of a definite integral of the type $\int_a^b f(x) \, dx$ in which the range of integration is finite and the integrand $f(x)$ is unbounded at one or more points of the given interval $[a, b]$. It is sufficient to consider the case when $f(x)$ becomes unbounded at $x = a$ and bounded for all other values of x in the interval $[a, b]$. In this case we have

$$\int_a^b f(x) \, dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b f(x) \, dx.$$

In the articles to follow we give a few important tests for the convergence of the above integral.

10 Comparison Test

Consider the improper integral $\int_a^b f(x) \, dx$, where the range of integration (a, b) is finite and $f(x)$ is unbounded only at $x = a$. Let $g(x)$ be positive in the interval $(a + \varepsilon, b)$ and $|f(x)| \leq g(x)$ in the interval $(a + \varepsilon, b)$. Then $\int_a^b f(x) \, dx$ is convergent if $\int_a^b g(x) \, dx$ is convergent.

Similarly if $|f(x)| \geq g(x)$ *for all values of x in the interval* $(a+\varepsilon, b)$, *then* $\int_a^b f(x)\,dx$ *is divergent provided* $\int_a^b g(x)\,dx$ *is divergent.*

Alternative form of the above comparison test:

If $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ is a definite number, other than zero, the integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ either both converge or both diverge.

Note: While applying the above comparison test, we generally take $g(x) = \dfrac{1}{(x-a)^n}$ *i.e.*, $\int_a^b \dfrac{dx}{(x-a)^n}$ is generally taken as the comparison integral.

Theorem: *The comparison integral* $\int_a^b \dfrac{dx}{(x-a)^n}$ *is convergent when* $n < 1$ *and divergent when* $n \geq 1$. **(Purvanchal 2011; Gorakhpur 13)**

Proof: We have

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b (x-a)^{-n}\,dx$$

$$= \lim_{\varepsilon \to 0} \left[\frac{(x-a)^{-n+1}}{1-n}\right]_{a+\varepsilon}^b, \text{ if } n \neq 1$$

$$= \lim_{\varepsilon \to 0} \left[\frac{(b-a)^{1-n}}{1-n} - \frac{\varepsilon^{1-n}}{1-n}\right]. \qquad \ldots(1)$$

If $n < 1$, then $1-n$ is positive and so $\lim_{\varepsilon \to 0} \varepsilon^{1-n} = 0$. Therefore from (1), we have

$$\int_a^b \frac{dx}{(x-a)^n} = \frac{(b-a)^{1-n}}{1-n}, \text{ if } n < 1.$$

Hence the given integral converges when $n < 1$.

If $n > 1$, then $1-n$ is negative and so $n-1$ is positive. Therefore in this case, from (1), we have

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0} \left[\frac{(b-a)^{1-n}}{1-n} + \frac{1}{(n-1)\,\varepsilon^{n-1}}\right] = \infty.$$

Hence the given integral diverges when $n > 1$.

When $n = 1$, we have

$$\int_a^b \frac{dx}{(x-a)^n} = \int_a^b \frac{dx}{(x-a)} = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b \frac{dx}{x-a}$$

$$= \lim_{\varepsilon \to 0} \left[\log(x-a)\right]_{a+\varepsilon}^b = \lim_{\varepsilon \to 0} [\log(b-a) - \log \varepsilon]$$

$$= \infty \qquad [\because \log 0 = -\infty]$$

Hence the given integral diverges when $n = 1$.

Illustrative Examples

Example 29: *Show that the integral* $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ *is convergent.*

Solution: In the given integral, the integrand $f(x) = \frac{1}{x^{1/3}(1+x^2)}$ is unbounded at the lower limit of integration $x = 0$.

Take $g(x) = 1/x^{1/3}$.

Then $\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{1}{1+x^2} = 1$, which is finite and non-zero.

$\therefore$ by comparison test

$$\int_0^1 f(x)\,dx \text{ and } \int_0^1 g(x)\,dx$$

either both converge or both diverge. But the comparison integral $\int_0^1 \frac{dx}{x^{1/3}}$ is convergent because here $n = 1/3$ which is less than 1. Hence the integral $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ is also convergent.

Example 30: *Test the convergence of the integral* $\int_1^2 \frac{dx}{\sqrt{(x^4-1)}}$.

Solution: In the given integral the integrand $f(x) = 1/\sqrt{(x^4 - 1)}$ is unbounded at the lower limit of integration $x = 1$.

Take $g(x) = 1/\sqrt{(x^2-1)}$.

Then $$\lim_{x \to 1} \frac{f(x)}{g(x)} = \lim_{x \to 1} \left\{ \frac{1}{\sqrt{(x^4-1)}} \cdot \sqrt{(x^2-1)} \right\} = \lim_{x \to 1} \frac{1}{\sqrt{(x^2+1)}}$$

$$= 1/\sqrt{2}, \text{ which is finite and non-zero.}$$

Therefore by comparison test,

$$\int_1^2 f(x)\,dx \text{ and } \int_1^2 g(x)\,dx$$

are either both convergent or both divergent.

But $$\int_1^2 g(x)\,dx = \int_1^2 \frac{dx}{\sqrt{(x^2-1)}} = \lim_{\varepsilon \to 0} \int_{1+\varepsilon}^2 \frac{dx}{\sqrt{(x^2-1)}}$$

$$= \lim_{\varepsilon \to 0} [\log\{x + \sqrt{(x^2-1)}\}]_{1+\varepsilon}^2$$

$$= \lim_{\varepsilon \to 0} [\log(2+\sqrt{3}) - \log\{1+\varepsilon+\sqrt{(\varepsilon^2+\varepsilon)}\}]$$

$$= \log(2+\sqrt{3}), \quad \text{which is a definite real number.}$$

$\therefore \qquad \int_1^2 g(x)\, dx$ is convergent.

Hence $\int_1^2 \dfrac{1}{\sqrt{(x^4 - 1)}}\, dx$ is also convergent.

Example 31: *Show that the integral* $\int_0^1 \dfrac{\sec x}{x}\, dx$ *is divergent.*

Solution: In the given integral the integrand $f(x) = \dfrac{\sec x}{x}$ is unbounded at the lower limit of integration $x = 0$. Take $g(x) = 1/x$.

Then $$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \left\{ \frac{\sec x}{x} \cdot x \right\} = \lim_{x \to 0} \sec x = 1,$$

which is finite and non-zero.

Therefore, by comparison test, $\int_0^1 f(x)\, dx$ and $\int_0^1 g(x)\, dx$ either both converge or both diverge. But the comparison integral $\int_0^1 \dfrac{1}{x}\, dx$ is divergent because here $n = 1$.

Hence the given integral $\int_0^1 \dfrac{\sec x}{x}\, dx$ is also divergent.

Example 32: *Show that* $\int_0^1 x^{n-1} e^{-x}\, dx$ *is convergent if* $n > 0$.

Solution: If $n \geq 1$, then $\int_0^1 x^{n-1} e^{-x}\, dx$ is a proper integral because the integrand $f(x) = x^{n-1} e^{-x}$ is bounded in the interval $(0, 1)$. So the given integral is convergent when $n \geq 1$.

If $0 < n < 1$, the integrand $f(x) = x^{n-1} e^{-x}$ is unbounded at $x = 0$. Take $g(x) = x^{n-1}$.

Then $\lim_{x \to 0} \dfrac{f(x)}{g(x)} = \lim_{x \to 0} e^{-x} = 1$, which is finite and non-zero.

$\therefore$ by comparison test, $\int_0^1 f(x)\, dx$ and $\int_0^1 g(x)\, dx$ either both converge or both diverge.

But $$\int_0^1 g(x)\, dx = \int_0^1 x^{n-1}\, dx = \lim_{\varepsilon \to 0} \int_\varepsilon^1 x^{n-1}\, dx = \lim_{\varepsilon \to 0} \left[\frac{x^n}{n} \right]_\varepsilon^1$$

$$= \lim_{\varepsilon \to 0} \left[\frac{1}{n} - \frac{\varepsilon^n}{n} \right] = \frac{1}{n},$$ which is a definite real number.

$\therefore \qquad \int_0^1 g(x)\, dx$ is convergent.

Hence $\int_0^1 x^{n-1} e^{-x}\, dx$ is also convergent.

Example 33: *Show that the integral* $\int_0^\infty x^{n-1} e^{-x}\, dx$ *is convergent if* $n > 0$.

(Garhwal 2006)

Solution: We have

$$\int_0^{\infty} x^{n-1} e^{-x}\, dx = \int_0^1 x^{n-1} e^{-x}\, dx + \int_1^{\infty} x^{n-1} e^{-x}\, dx.$$

Let $$I_1 = \int_0^1 x^{n-1} e^{-x}\, dx \text{ and } I_2 = \int_1^{\infty} x^{n-1} e^{-x}\, dx.$$

The integral I_2 is convergent for all values of n.

[For proof see Ex. 17 after article 5]

Also the integral I_1 is convergent if $n > 0$. [For proof see Ex. 32 above]

Hence the given integral is convergent if $n > 0$ because then it is the sum of two convergent integrals.

11 The μ-Test

Let $f(x)$ be unbounded at $x = a$ and be bounded and integrable in the arbitrary interval $(a + \varepsilon, b)$, where $0 < \varepsilon < b - a$. If there is a number μ between 0 and 1 such that

$$\lim_{x \to a+0} (x - a)^{\mu} f(x) \text{ exists, then } \int_a^b f(x)\, dx$$

is convergent.

If there is a number $\mu \geq 1$ such that $\lim_{x \to a+0} (x - a)^{\mu} f(x)$ exists and is non-zero, then $\int_a^b f(x)\, dx$ is divergent and the same is true if

$$\lim_{x \to a+0} (x - a)^{\mu} f(x) = +\infty \quad or \quad -\infty.$$

In case $f(x)$ is unbounded at $x = b$, we should find

$$\lim_{x \to b-0} (b - x)^{\mu} . f(x),$$

the other conditions of the test remaining the same.

Illustrative Examples

Example 34: *Prove that the integral* $\int_0^1 \frac{dx}{\sqrt{\{x(1-x)\}}}$ *converges.*

(Garhwal 2009, 12)

Solution: In the given integral the integrand $f(x) = 1/\sqrt{\{x(1-x)\}}$ is unbounded both at $x = 0$ and at $x = 1$. If $0 < a < 1$, we can write

$$\int_0^1 \frac{dx}{\sqrt{\{x(1-x)\}}} = \int_0^a \frac{dx}{\sqrt{\{x(1-x)\}}} + \int_a^1 \frac{dx}{\sqrt{\{x(1-x)\}}} = I_1 + I_2, \text{ say.}$$

In the integral I_1 the integrand $f(x)$ is unbounded at the lower limit of integration $x = 0$ and in the integral I_2 the integrand $f(x)$ is unbounded at the upper limit of integration $x = 1$.

To test the convergence of I_1. Take $\mu = \frac{1}{2}\cdot$ We have

$$\lim_{x \to 0} x^{\mu} f(x) = \lim_{x \to 0} x^{1/2} \cdot \frac{1}{\sqrt{\{x(1-x)\}}} = \lim_{x \to 0} \frac{1}{\sqrt{(1-x)}}$$

$$= 1 \text{ i.e., the limit exists.}$$

Since $0 < \mu < \frac{1}{2}$, therefore by μ-test I_1 is convergent.

To test the convergence of I_2 Take $\mu = \frac{1}{2} \cdot$ We have

$$\lim_{x \to 1-0} (1-x)^{\mu} . f(x) = \lim_{x \to 1-0} (1-x)^{1/2} \cdot \frac{1}{\sqrt{\{x(1-x)\}}}$$

$$= \lim_{x \to 1-0} \frac{1}{\sqrt{x}} = \lim_{\varepsilon \to 0} \frac{1}{\sqrt{(1-\varepsilon)}} = 1.$$

Hence by μ-test I_2 is convergent since $0 < \mu < 1$.

Thus the given integral is the sum of two convergent integrals. Hence the given integral itself is convergent.

Example 35: *Test the convergence of* $\int_0^1 \frac{\log x}{\sqrt{(2-x)}} dx$.

(Agra 2012)

Solution: Let $f(x) = \frac{\log x}{\sqrt{(2-x)}} \cdot$ Then $f(x)$ is unbounded both at $x = 0$ and $x = 2$. If $0 < a < 2$, we can write

$$\int_0^2 \frac{\log x}{\sqrt{(2-x)}} dx = \int_0^a \frac{\log x}{\sqrt{(2-x)}} dx + \int_a^2 \frac{\log x}{\sqrt{(2-x)}} dx = I_1 + I_2 \text{, say.}$$

To test the convergence of I_1. We have

$$\lim_{x \to 0} x^{\mu} f(x) = \lim_{x \to 0} \left\{ x^{\mu} \cdot \frac{\log x}{\sqrt{(x-2)}} \right\} = 0 \text{ if } \mu > 0.$$

Therefore taking μ between 0 and 1, it follows by μ-test that I_1 is convergent.

To test the convergence of I_2.

Take $\mu = \frac{1}{2} \cdot$ We have

$$\lim_{x \to 2-0} (2-x)^{\mu} . f(x) = \lim_{x \to 2-0} (2-x)^{1/2} \cdot \frac{\log x}{\sqrt{(2-x)}}$$

$$= \lim_{x \to 2-0} \log x = \lim_{\varepsilon \to 0} \log(2-\varepsilon)$$

$$= \log 2.$$

$\therefore$ by μ-test I_2 is convergent because $0 < \mu < 1$.

Hence the given integral is also convergent, it being the sum of two convergent integrals.

Example 36: *Test the convergence of* $\int_0^1 x^{p-1} e^{-x} dx$.

Solution: Let $f(x) = x^{p-1} e^{-x}$

and $I = \int_0^1 x^{p-1} e^{-x} dx.$

If $p \ge 1$, $f(x)$ is bounded throughout the interval $(0, 1)$ and so I is a proper integral and hence it is convergent if $p \ge 1$.

If $p < 1$, $f(x)$ is unbounded at $x = 0$. In this case, we have

$$\lim_{x \to 0} x^{\mu} f(x) = \lim_{x \to 0} x^{\mu} . x^{p-1} e^{-x} = \lim_{x \to 0} x^{\mu + p - 1} e^{-x}$$

$$= 1 \text{ if } \mu + p - 1 = 0 \text{ i.e., } \mu = 1 - p.$$

So by μ-test when $0 < \mu < 1$ *i.e.*, $0 < p < 1$, the given integral is convergent and when $\mu \ge 1$ *i.e.*, $p \le 0$, the given integral is divergent.

Hence I is convergent if $p > 0$ and is divergent if $p \le 0$.

12 Abel's Test

If $\int_a^b f(x)\, dx$ converges and $\phi(x)$ is bounded and monotonic for $a \le x \le b$, then $\int_a^b f(x)\, \phi(x)\, dx$ converges.

13 Dirichlet's Test

If $\int_{a+\varepsilon}^{b} f(x)\, dx$ be bounded and $\phi(x)$ be bounded and monotonic on the interval $a \le x \le b$, converging to zero as x tends to a, then $\int_a^b f(x)\, \phi(x)\, dx$ converges.

Illustrative Examples

Example 37: *Test the convergence of $\int_0^{\pi/2} \frac{\cos x}{x^n}\, dx$.*

Solution: When $n \le 0$, the given integral is a proper integral and hence convergent.

When $n > 0$, the integrand becomes unbounded at $x = 0$.

Let $$f(x) = \frac{\cos x}{x^n}.$$

Then $$\lim_{x \to 0} x^{\mu} f(x) = \lim_{x \to 0} x^{\mu - n} \cos x = 1, \text{ if } \mu = n.$$

Hence by μ-test it follows that the given integral is convergent when $0 < n < 1$, and divergent when $n \ge 1$.

From the above discussion we conclude that the given integral is convergent when $n < 1$, and divergent when $n \ge 1$.

Example 38: *Show that the integral $\int_0^{\pi/2} \log \sin x\, dx$ converges.*

(Meerut 2012; Rohilkhand 12)

Solution: The only point of infinite discontinuity of the integrand is $x = 0$.

Now $\lim_{x \to 0} x^{\mu} \log \sin x$, when $\mu > 0$

$$= \lim_{x \to 0} \frac{\log \sin x}{x^{-\mu}}, \qquad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x \to 0} \frac{\cot x}{-\mu\, x^{-\mu - 1}}$$

$$= \lim_{x \to 0} -\frac{1}{\mu} \cdot \frac{x^{\mu + 1}}{\tan x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} -\frac{1}{\mu} \cdot \frac{(\mu + 1)\, x^{\mu}}{\sec^2 x}, \qquad \text{[by L'Hospital's rule]}$$

$$= 0, \text{ if } \mu > 0.$$

Taking μ between 0 and 1, it follows from μ-test that the given integral is convergent.

Example 39: *Discuss the convergence of the integral*

$$\int_0^1 x^{n-1} \log x \, dx. \qquad \textbf{(Purvanchal 2009; Garhwal 12)}$$

Solution: (i) Since $\lim_{x \to 0} x^r \log x = 0$ where $r > 0$, the integral is a proper integral, when $n > 1$.

(ii) When $n = 1$, we have

$$\int_0^1 \log x \, dx = \lim_{\varepsilon \to 0} \int_{\varepsilon}^1 \log x \, dx = \lim_{\varepsilon \to 0} [x \log x - x]_{\varepsilon}^{1}$$

$$= \lim_{\varepsilon \to 0} [-1 - \varepsilon \log \varepsilon + \varepsilon] = -1.$$

$\therefore$ the integral is convergent if $n = 1$.

(iii) Let $n < 1$ and $f(x) = x^{n-1} \log x$.

Then $\lim_{x \to 0} x^{\mu} f(x) = \lim_{x \to 0} x^{\mu + n - 1} \log x$

$$= 0 \quad \text{if } \mu > 1 - n \qquad \text{...(1)}$$

and

$$= -\infty \quad \text{if } \mu \le 1 - n. \qquad \text{...(2)}$$

Hence when $0 < n < 1$, we can choose μ between 0 and 1 and satisfying (1). The integral is therefore convergent by μ-test when $0 < n < 1$.

Again when $n \le 0$, we can take $\mu = 1$ and satisfying (2). Hence by μ-test the integral is divergent when $n \le 0$.

Therefore from (i), (ii) and (iii), we conclude that the given integral is convergent when $n > 0$ and divergent when $n \le 0$.

Example 40: *Discuss the convergence or divergence of the integral*

$$\int_0^{\infty} \frac{x^{a-1}}{1 + x} \, dx.$$

(Garhwal 2006, 11)

Solution: Let $f(x)=\dfrac{x^{a-1}}{1+x}\cdot$ If $b>0$, we can write

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx=\int_0^b \frac{x^{a-1}}{1+x}\,dx+\int_b^\infty \frac{x^{a-1}}{1+x}\,dx$$

$$=I_1+I_2,\ \text{say}.$$

Let $a\geq 1$. Then $f(x)$ is bounded throughout the interval $(0,b)$ and so the integral I_1 is a proper integral and hence it is convergent. To test the convergence of the infinite integral I_2 in this case, we have

$$\lim_{x\to\infty} x^\mu f(x)=\lim_{x\to\infty} x^\mu\cdot\frac{x^{a-1}}{1+x}=\lim_{x\to\infty}\frac{x^{\mu+a-1}}{x+1}$$

$$=1,\ \text{if } \mu+a-1=1\quad i.e.,\quad \text{if } \mu=2-a$$

which is ≤ 1 since $a\geq 1$.

Hence by μ-test I_2 is divergent.

$\therefore$ the given integral is divergent if $a\geq 1$.

Let $a<1$. Then in the interval $(0,b)$, $f(x)$ is unbounded only at $x=0$. Also $f(x)$ is bounded throughout the interval (b,∞). Therefore in this case I_1 is an improper integral of the second kind and I_2 is an improper integral of the first kind. To test the convergence of I_1, we have

$$\lim_{x\to 0} x^\mu\cdot\frac{x^{a-1}}{x+1}=\lim_{x\to 0}\frac{x^{\mu+a-1}}{x+1}=1,$$

if $\mu+a-1=0$ *i.e.*,

if $\mu=1-a$.

If we take $0<a<1$, then we have $0<\mu<1$ and so by μ-test I_1 is convergent. If we take $a\leq 0$, then $\mu\geq 1$ and so by μ-test I_1 is divergent.

To test the convergence of I_2 when $a<1$, we have

$$\lim_{x\to\infty} x^\mu\cdot\frac{x^{a-1}}{x+1}=\lim_{x\to\infty}\frac{x^{\mu+a+1}}{x+1}=1,\ \text{if } \mu+a-1=1\ i.e.,$$

if $\mu=2-a$ which is >1 since $a<1$.

Hence by μ-test I_2 is convergent if $a<1$.

Thus I_2 is convergent if $a<1$. But I_1 is convergent if $0<a<1$ and is divergent if $a\leq 0$.

$\therefore$ the given integral is convergent if $0<a<1$ and is divergent if $a\leq 0$.

Hence the given integral is convergent if $0<a<1$ and is divergent if $a\geq 1$ or if $a\leq 0$.

Example 41: *Discuss the convergence of the Beta function*

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

(Purvanchal 2007, 12; Kanpur 12)

Solution: Let $f(x)=x^{m-1}(1-x)^{n-1}$.

The following different cases arise :

(i) When m and n are both ≥ 1, the integrand $f(x)$ is bounded throughout the interval $(0,1)$ and so the given integral is a proper integral and is convergent.

(ii) When m and n are both < 1, the integrand $f(x)$ becomes infinite both at $x = 0$ and at $x = 1$. In this case we take $0 < a < 1$ and we write

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^a x^{m-1}(1-x)^{n-1}\,dx + \int_a^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= I_1 + I_2, \text{ say.}$$

In the case of the integral I_1, the interval of integration is $(0, a)$ and so the integrand is unbounded at $x = 0$ only. To test the convergence of I_1, we have

$$\lim_{x \to 0} x^{\mu} \cdot f(x) = \lim_{x \to 0} x^{\mu} \cdot x^{m-1}(1-x)^{n-1}$$

$$= \lim_{x \to 0} x^{\mu + m - 1}(1-x)^{n-1}$$

$$= 1, \text{ if } \mu + m - 1 = 0 \text{ } i.e., \text{ if } \mu = 1 - m.$$

If we take $0 < m < 1$, we have $0 < \mu < 1$ and so by μ-test I_1 is convergent. If we take $m \le 0$, we have $\mu \ge 1$ and so by μ-test I_1 is divergent.

Again in the case of the integral I_2, the interval of integration is $(a, 1)$ and so the integrand is unbounded at $x = 1$ only. To test the convergence of I_2, we have

$$\lim_{x \to 1-0} (1-x)^{\mu} \cdot f(x) = \lim_{x \to 1-0} (1-x)^{\mu} x^{m-1}(1-x)^{n-1}$$

$$= \lim_{x \to 1-0} (1-x)^{\mu + n - 1} x^{m-1}$$

$$= \lim_{\varepsilon \to 0} \{1-(1-\varepsilon)\}^{\mu + n - 1}(1-\varepsilon)^{m-1}$$

$$= \lim_{\varepsilon \to 0} \varepsilon^{\mu + n - 1}(1-\varepsilon)^{m-1}$$

$$= 1, \text{ if } \mu + n - 1 = 0 \text{ } i.e., \text{ if } \mu = 1 - n.$$

If we take $0 < n < 1$, we have $0 < \mu < 1$ and so by μ-test I_2 is convergent. If we take $n \le 0$, we have $\mu \ge 1$ and so by μ-test I_2 is divergent.

Thus if m and n are both < 1, the given integral is convergent only if $0 < m < 1$ and $0 < n < 1$.

(iii) When $m < 1$ and $n \ge 1$, the integrand $f(x)$ is unbounded only at $x = 0$. In this case by μ-test, the given integral is convergent if $0 < m < 1$ and is divergent if $m \le 0$.

Again if $m \ge 1$ and $n < 1$, the integrand $f(x)$ is unbounded only at $x = 1$. In this case by μ-test, the given integral is convergent if $0 < n < 1$ and is divergent if $n \le 0$.

Hence from (i), (ii) and (iii) it follows that the given integral is convergent if both m and n are > 0 and divergent otherwise.

Example 42: *Discuss the convergence of the Gamma function*

$$\int_0^\infty x^{n-1} e^{-x}\,dx.$$

(Kanpur 2008; Garhwal 10; Purvanchal 08, 11, 12; Rohilkhand 10, 11)

Solution: We can write

$$\int_0^{\infty} x^{n-1} e^{-x}\, dx = \int_0^{1} x^{n-1} e^{-x}\, dx + \int_1^{\infty} x^{n-1} e^{-x}\, dx$$

$$= I_1 + I_2 \text{, say.}$$

Let us first discuss the convergence of I_1.

Let $\quad f(x) = x^{n-1} e^{-x}$.

If $n \geq 1$, $f(x)$ is bounded throughout the interval $[0, 1]$ and so I_1 is a proper integral and hence it is convergent if $n \geq 1$.

If $n < 1$, $f(x)$ is unbounded at $x = 0$. In this case we have

$$\lim_{x \to 0} x^{\mu} f(x) = \lim_{x \to 0} x^{\mu} \cdot x^{n-1} e^{-x} = \lim_{x \to 0} x^{\mu + n - 1} e^{-x}$$

$$= 1 \text{, if } \mu + n - 1 = 0 \text{ } i.e., \mu = 1 - n.$$

So by μ-test when $0 < \mu < 1$ *i.e.*, $0 < n < 1$, the integral I_1 is convergent and when $\mu \geq 1$ *i.e.* $n \leq 0$, the integral I_1 is divergent.

$\therefore$ I_1 is convergent if $n > 0$ and is divergent if $n \leq 0$.

Now let us discuss the convergence of the integral I_2. The function $f(x) = x^{n-1} e^{-x}$ is bounded for all values of x in the interval $(1, \infty)$. We have

$$\lim_{x \to \infty} x^{\mu} f(x) = \lim_{x \to \infty} \frac{x^{\mu} \cdot x^{n-1}}{e^x} = \lim_{x \to \infty} \frac{x^{\mu + n - 1}}{1 + x + \dfrac{x^2}{2!} + \ldots}$$

$$= 0 \text{ for all values of } \mu \text{ and } n.$$

Taking $\mu > 1$, we see by μ-test that the integral

$$I_2 = \int_1^{\infty} x^{n-1} e^{-x}\, dx \text{ is convergent for all values of } n.$$

Hence the given integral is convergent if $n > 0$ and is divergent if $n \leq 0$.

Comprehensive Exercise 2

1. Show that the integral $\displaystyle\int_{\pi}^{\infty} \frac{\sin^2 x}{x^2}\, dx$ is convergent.

2. Test the convergence of the following integrals:

(i) $\displaystyle\int_0^{\infty} \frac{\cos x}{1 + x^2}\, dx$

(Purvanchal 2010; Bundelkhand 11; Rohilkhand 12; Gorakhpur 12)

(ii) $\displaystyle\int_{\pi}^{\infty} \frac{\sin x}{x^2}\, dx$

.(iii) $\displaystyle\int_0^{\infty} \frac{\sin mx}{x^2 + a^2}\, dx$

(Garhwal 2008)

(iv) $\int_0^\infty \frac{x^3}{(x^2+a^2)^2}\,dx$

(v) $\int_1^\infty \frac{dx}{\sqrt{(x^3+1)}}$ **(Gorakhpur 2012, 15)**

(vi) $\int_0^\infty \frac{1-\cos x}{x^2}\,dx$

(vii) $\int_2^\infty \frac{dx}{\sqrt{(x^2-x-1)}}$

(viii) $\int_2^\infty \frac{dx}{\sqrt{(x^2-1)}}$

(ix) $\int_0^\infty \frac{x^2\,dx}{(1+x)^3}$

(x) $\int_0^\infty \frac{x^{3/2}}{(b^2x^2+c)}\,dx.$

3. Show that the following integrals are convergent:

(i) $\int_0^\infty \frac{x^2}{(a^2+x^2)^2}\,dx$ (ii) $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}.$

4. Test the convergence of $\int_b^\infty \frac{x^{3/2}\,dx}{\sqrt{(x^4-a^4)}}$, where $b > a$.

5. Show that the integral $\int_a^\infty x^{n-1}e^{-x}\,dx$ is convergent, where $a > 0$.

6. Test the convergence of the following integrals:

(i) $\int_0^1 \frac{dx}{x^3(1+x^2)}$ (ii) $\int_0^1 \frac{dx}{(x+1)\sqrt{(1-x^2)}}$

(iii) $\int_0^{\pi/2} \frac{\cos x}{x^2}\,dx$ (iv) $\int_0^{\pi/4} \frac{1}{\sqrt{(\tan x)}}\,dx$

(Agra 2012)

(v) $\int_0^{\pi/2} \frac{\sin x}{x^{1+n}}\,dx.$

7. Test the convergence of the following integrals :

(i) $\int_0^\infty \frac{dx}{x^{1/3}(1+x^{1/2})}$ (ii) $\int_0^\infty \frac{dx}{x\sqrt{(1+x^2)}}$

(iii) $\int_0^\infty \frac{x^{1/2}}{x^2+4}\,dx$ (iv) $\int_0^\infty \frac{x}{1+x^2}\sin x\,dx.$

8. Examine the convergence of the integral $\int_0^\infty \frac{\sin x}{x^{3/2}}\,dx.$

9. Show that the integral $\int_0^\infty e^{-a^2x^2}\cos bx\,dx$ is absolutely convergent.

Answers 2

2. (i) Convergent (ii) Convergent
(iii) Convergent (iv) Divergent
(v) Convergent (vi) Convergent
(vii) Divergent (viii) Divergent
(ix) Divergent (x) Divergent
4. Divergent
6. (i) Divergent (ii) Convergent
(iii) Divergent (iv) Convergent
(v) Convergent if $n<1$ and divergent if $n \geq 1$
(vi) Convergent
7. (i) Divergent (ii) Divergent
(iii) Divergent (iv) Convergent
8. Convergent

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The integral $\int_a^\infty \frac{dx}{x^n}$, where $a>0$, is convergent when

(a) $n=1$ (b) $n<1$
(c) $n \leq 1$ (d) $n>1$
(Garhwal 2009; Rohilkhand 11)

2. The integral $\int_a^b \frac{dx}{(x-a)^n}$ is convergent when

(a) $n<1$ (b) $n>1$
(c) $n=1$ (d) $n \geq 1$ **(Garhwal 2006, 10, 11)**

3. The integral $\int_0^1 x^{m-1}(1-x)^{n-1}\, dx$ is convergent when

(a) $m>0$ (b) $n>0$
(c) $m>0, n>0$ (d) $m=0, n>1$

4. The integral $\int_0^\infty x^{n-1} e^{-x}\, dx$ is divergent when

(a) $n>0$ (b) $n>1$
(c) $n \leq 0$ (d) $n=\frac{1}{2}$

5. The integral $\int_a^\infty \frac{\sin^2 x}{x^2} dx, a > 0$

(a) convergent (b) divergent

(c) uniformly convergent (d) none of these

(Rohilkhand 2012)

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. The definite integral $\int_a^b f(x)\, dx$ is said to be a ...... if the range of integration (a, b) is finite and the integrand $f(x)$ is bounded over (a, b).
2. The definite integral $\int_a^b f(x)\, dx$ is said to be an improper integral if the interval (a, b) is finite and $f(x)$ is not ...... over this interval.
3. The definite integral $\int_a^b f(x)\, dx$ is said to be an ...... if the interval (a, b) is not finite and $f(x)$ is bounded over (a, b).
4. A definite integral $\int_a^b f(x)\, dx$ in which the range of integration (a, b) is finite but the integrand $f(x)$ is unbounded at one or more points of the interval $a \leq x \leq b$, is called an improper integral of the ...... kind.
5. The integral $\int_0^\infty \frac{dx}{1 + x^2}$ is an improper integral of the ...... kind.
6. The integral $\int_0^4 \frac{dx}{(x-2)(x-3)}$ is an improper integral of the ...... kind.
7. The integral $\int_a^\infty \frac{dx}{x^n}$, where $a > 0$, is convergent when ....
8. The integral $\int_a^b \frac{dx}{(x-a)^n}$ is divergent when ....
9. The integral $\int_0^\infty e^{-x} x^{n-1}\, dx$ is convergent if ....
10. The integral $\int_0^1 x^{m-1} (1-x)^{n-1}\, dx$ is convergent if both m and n are > ......

True or False

Write 'T' for true and 'F' for false statement.

1. The integral $\int_3^\infty \frac{dx}{(x-2)^2}$ is divergent.
2. The integral $\int_0^\infty \frac{\cos x}{1 + x^2} dx$ is convergent.

(Garhwal 2012)

3. The integral $\int_a^\infty \frac{dx}{x^n}$, where $a > 0$, is convergent when $n \leq 1$.
4. The integral $\int_a^b \frac{dx}{(x-a)^n}$ is divergent when $n < 1$.

5. The integral $\int_0^1 \frac{dx}{x^3 (1+x^2)}$ is convergent.

6. The integral $\int_0^1 \frac{dx}{x^{1/3} (1+x^2)}$ is convergent.

7. The integral $\int_1^\infty \frac{dx}{\sqrt{(x^3+1)}}$ is convergent.

8. The integral $\int_2^\infty \frac{dx}{\sqrt{(x^2-1)}}$ is convergent.

Answers

Multiple Choice Questions

1. (d) 2. (a) 3. (c) 4. (c) 5. (a)

Fill in the Blank(s)

1. proper integral 2. bounded
3. improper integral of the first kind 4. second 5. first
6. second 7. $n > 1$ 8. $n \geq 1$ 9. $n > 0$ 10. 0

True or False

1. *F* 2. *T* 3. *F* 4. *F* 5. *F*
6. *T* 7. *T* 8. *F*

□

Chapter 10

Indeterminate Forms

1 Indeterminate Forms

The form 0/0 has got no definite value. For if we write $0/0 = y$, then the equation $0\,y = 0$ reduces to an identity in y, *i.e.*, it is true for all values of y. We cannot cancel 0 from both sides. Therefore the form 0/0 is meaningless.

Now suppose $\lim\limits_{x \to a} \phi(x) = 0$ and $\lim\limits_{x \to a} \psi(x) = 0$.

Then we cannot write $\lim\limits_{x \to a} \dfrac{\phi(x)}{\psi(x)} = \dfrac{\lim\limits_{x \to a} \phi(x)}{\lim\limits_{x \to a} \psi(x)}$

because in that case $\lim\limits_{x \to a} \dfrac{\phi(x)}{\psi(x)}$ takes the form 0/0 which is meaningless. It, however, does not mean that if $\lim\limits_{x \to a} \dfrac{\phi(x)}{\psi(x)}$ takes the form 0/0, then the limit itself does not exist.

For example, $\lim\limits_{x \to a} \dfrac{x^2 - a^2}{x - a}$ takes the form 0/0 if we write it as $\dfrac{\lim\limits_{x \to a} (x^2 - a^2)}{\lim\limits_{x \to a} (x - a)}$.

But, we have $\lim\limits_{x \to a} \dfrac{x^2 - a^2}{x - a} = \lim\limits_{x \to a} \dfrac{(x-a)(x+a)}{(x-a)} = \lim\limits_{x \to a} (x+a) = 2a$ and thus the limit exists.

Here it should not be confused that we have made an attempt to find the value of 0/0. We have simply evaluated the limit of a function which is the quotient of two functions such that if we take their limits separately, then the combination takes the form 0/0.

The form 0/0 is an **indeterminate form**. It has no definite value. The other indeterminate forms are ∞/∞, $\infty - \infty$, $0 \times \infty$, $1^{\infty}, 0^0, \infty^0$. In this chapter we shall discuss methods which enable us to evaluate the limits of indeterminate forms.

2 The Form 0/0 (L' Hospital's Rule)

Suppose $\phi(x)$ and $\psi(x)$ are functions which can be expanded by Taylor's theorem in the neighbourhood of $x = a$. Also let $\phi(a) = 0$, and $\psi(a) = 0$. Then

$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}.$$

We have, by Taylor's theorem, $\lim\limits_{x \to a} \dfrac{\phi(x)}{\psi(x)}$

$$= \lim_{x \to a} \frac{\phi(a) + (x-a)\phi'(a) + \dfrac{(x-a)^2}{2!}\phi''(a) + \ldots + R_1}{\psi(a) + (x-a)\psi'(a) + \dfrac{(x-a)^2}{2!}\psi''(a) + \ldots + R_2}$$

where $\quad R_1 = \dfrac{(x-a)^n}{n!}\phi^{(n)}\{a + \theta_1(x-a)\}, 0 < \theta_1 < 1,$

and $\quad R_2 = \dfrac{(x-a)^n}{n!}\psi^{(n)}\{a + \theta_2(x-a)\}, 0 < \theta_2 < 1.$

But, by hypothesis, $\phi(a) = 0$ and $\psi(a) = 0$.

Therefore, $\quad \lim\limits_{x \to a} \dfrac{\phi(x)}{\psi(x)} = \lim\limits_{x \to a} \dfrac{\phi(a) + (x-a)\phi'(a) + \dfrac{(x-a)^2}{2!}\phi''(a) + \ldots + R_1}{\psi(a) + (x-a)\psi'(a) + \dfrac{(x-a)^2}{2!}\psi''(a) + \ldots + R_2}$

Dividing the numerator and denominator by $x - a$, we have

$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{\phi'(a) + (x-a)\left\{\dfrac{1}{2!}\phi''(a) + \dfrac{1}{3!}(x-a)\phi''(a) + \ldots\right\}}{\psi'(a) + (x-a)\left\{\dfrac{1}{2!}\psi''(a) + \dfrac{1}{3!}(x-a)\psi'''(a) + \ldots\right\}}$$

$$= \frac{\phi'(a)}{\psi'(a)}, \quad \text{if } \phi'(a) \text{ and } \psi'(a) \text{ are not both zero}$$

$$= \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}.$$

This proves the theorem which is generally known as *L'Hospital's Rule*.

It can be easily seen that if $\phi'(a)$, $\phi''(a), \ldots\ldots, \phi^{(n-1)}(a)$ and $\psi'(a)$, $\psi''(a), \ldots, \psi^{(n-1)}(a)$ are all zero, but $\phi^{(n)}(a)$ and $\psi^{(n)}(a)$ are not both zero, then

$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{\phi^{(n)}(x)}{\psi^{(n)}(x)}.$$

The theorem of this article is true even if x tends to ∞ or $-\infty$ instead of a, *i.e.*, if

$$\lim_{x \to \infty} \phi(x) = 0, \ \lim_{x \to \infty} \psi(x) = 0.$$

then $$\lim_{x \to \infty} \frac{\phi(x)}{\psi(x)} = \lim_{x \to \infty} \frac{\phi'(x)}{\psi'(x)}.$$

Writing $x = 1/y$, we have as $x \to \infty$, $y \to 0$.

$$\therefore \quad \lim_{x \to \infty} \frac{\phi(x)}{\psi(x)} = \lim_{y \to 0} \frac{\phi(1/y)}{\psi(1/y)}$$

$$= \lim_{y \to 0} \frac{\phi'(1/y)\, y^{-2}}{\psi'(1/y)\, y^{-2}}, \quad \text{by L'Hospital's rule}$$

$$= \lim_{y \to 0} \frac{\phi'(1/y)}{\psi'(1/y)} = \lim_{x \to \infty} \frac{\phi'(x)}{\psi'(x)}$$

Note 1: L'Hospital's rule implies

$$\lim_{x \to a^+} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a^-} \frac{\phi(x)}{\psi(x)}.$$

Note 2: While applying L'Hospital's rule we are not to differentiate $\dfrac{\phi(x)}{\psi(x)}$ by the rule for finding the differential coefficient of the quotient of two functions. But we are to differentiate the numerator and denominator separately.

Note 3: **Important:** Before applying L'Hospital's rule we must satisfy ourselves that the form is 0/0. Sometimes it happens that at some stage the resulting function is not indeterminate of the type 0/0 and we still apply L'Hospital's rule which is not justified in that case. This is a fairly common error.

3 Method of Expansion (Algebraic Methods)

In many cases the limit of an indeterminate form can be easily obtained by using some well known algebraic and trigonometrical expansions. We can also make use of some well-known limits in order to solve the problems or to shorten the work. The following expansions should be remembered :

1. $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots$

2. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots, |x| < 1$

3. $a^x = 1 + x \log a + \dfrac{x^2}{2!}(\log a)^2 + \dfrac{x^3}{3!}(\log a)^3 + \ldots$

4. $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$

5. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots$

6. $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots$

7. $\tan x = x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + \ldots$

8. $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots, |x| < 1$

9. $\log(1-x) = -\left(x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \ldots\right), |x| < 1$

10. $\sin^{-1} x = x + \dfrac{x^3}{6} + \dfrac{3x^5}{40} + \ldots$

11. $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \ldots$

12. $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots$

13. $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots$

Also **remember** that

$$\log 1 = 0, \log e = 1, \log \infty = \infty; \log 0 = -\infty.$$

Sometimes the use of the following limits shortens the work :

(i) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1,$ **(ii)** $\lim\limits_{x \to 0} \cos x = 1,$

(iii) $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1,$ **(iv)** $\lim\limits_{x \to 0} (1+x)^{1/x} = e,$

(v) $\lim\limits_{x \to 0} (1+nx)^{1/x} = e^n,$ **(vi)** $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^x = e,$

(vii) $\lim\limits_{x \to \infty} \left(1 + \dfrac{a}{x}\right)^x = e^a.$

Illustrative Examples

Example 1: *Evaluate* $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x} - 2 \log (1+x)}{x \sin x}$.

Solution: We have $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x} - 2 \log (1+x)}{x \sin x}$ $\left[\text{form } \dfrac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{e^x + e^{-x} - \dfrac{2}{(1+x)}}{\sin x + x \cos x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} + \dfrac{2}{(1+x)^2}}{\cos x + \cos x - x \sin x} = \frac{1-1+2}{1+1-0} = \frac{2}{2} = 1.$$

Example 2: *Evaluate* $\lim\limits_{x \to 0} \dfrac{\sin x - x + \dfrac{1}{6} x^3}{x^5}$.

Solution: We have $\lim\limits_{x \to 0} \dfrac{\sin x - x + \dfrac{1}{6} x^3}{x^5}$, $\left[\text{form } \dfrac{0}{0}\right]$

$$= \lim_{x \to 0} \frac{\cos x - 1 + \dfrac{1}{2} x^2}{5 x^4} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{-\sin x + x}{20 x^3} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{-\cos x + 1}{60 x^2} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{\sin x}{120 x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{\cos x}{120} = \frac{1}{120}.$$

Example 3: *Evaluate* $\lim\limits_{x \to 0} \dfrac{x \cos x - \log (1+x)}{x^2}$.

Solution: We have $\lim\limits_{x \to 0} \dfrac{x \cos x - \log (1+x)}{x^2}$

$$= \lim_{x \to 0} \frac{x\left(1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots\right) - \left(x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots\right)}{x^2}$$

$$= \lim_{x \to 0} \frac{\dfrac{x^2}{2} - \dfrac{5}{6} x^3 + \ldots}{x^2}$$

$$= \lim_{x \to 0} \left(\frac{1}{2} - \frac{5}{6} x + \text{ terms containing higher powers of } x \right) = \frac{1}{2}.$$

Example 4: *Evaluate* $\lim_{x \to 0} \dfrac{x^{1/2} \tan x}{(e^x - 1)^{3/2}}$. **(Rohilkhand 2005; Kanpur 07)**

Solution: Here it should be noted that we cannot apply Hospital's rule since $x^{1/2}$ cannot be expanded by Taylor's theorem in the neighbourhood of $x = 0$. However, we can get the result by the use of algebraic methods. We thus have

$$\lim_{x \to 0} \frac{x^{1/2} \tan x}{(e^x - 1)^{3/2}}$$

$$= \lim_{x \to 0} \frac{x^{1/2} \tan x}{[\{1 + x + (x^2 / 2\,!) + \ldots\ldots\} - 1]^{3/2}}$$

$$= \lim_{x \to 0} \frac{x^{1/2} \tan x}{[x + (x^2 / 2) + \ldots\ldots]^{3/2}}$$

$$= \lim_{x \to 0} \frac{x^{1/2} \tan x}{x^{3/2} [1 + (x / 2) + \ldots\ldots]^{3/2}}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} \cdot \frac{1}{\left(1 + \frac{x}{2\,!} + \ldots\right)^{3/2}}$$

$$= 1. \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1 \text{ and } \lim_{x \to 0} \cos x = 1.\right]$$

Example 5: *Evaluate* $\lim_{x \to 0} \dfrac{(1 + x)^{1/x} - e}{x}$. **(Kashi 2013; Rohilkhand 13)**

Solution: Here $\lim_{x \to 0} \dfrac{(1 + x)^{1/x} - e}{x}$ is of the form $\dfrac{0}{0}$ because $\lim_{x \to 0} (1 + x)^{1/x} = e$. First we shall obtain an expansion for $(1 + x)^{1/x}$ in ascending powers of x.

Let $y = (1 + x)^{1/x}$. Then

$$\log y = \frac{1}{x} \log (1 + x) = \frac{1}{x} \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots \right) = 1 - \frac{x}{2} + \frac{x^2}{3} - \ldots$$

$$= 1 + z, \text{ where } z = -(x / 2) + (x^2 / 3) - \ldots$$

$$\therefore \qquad y = e^{1 + z} = e \,.\, e^z = e \cdot \left(1 + \frac{z}{1\,!} + \frac{z^2}{2\,!} + \ldots \right)$$

$$= e \left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \ldots \right) + \frac{1}{2} \left(-\frac{x}{2} + \frac{x^2}{3} - \ldots \right)^2 + \ldots \right]$$

$$= e \left[1 - \frac{x}{2} + \frac{x^2}{3} + \frac{1}{8} x^2 + \text{ terms containing powers of higher than } 3 \right]$$

$$= e\left[1 - \frac{1}{2}x + \frac{11}{24}x^2 + \ldots\ldots\right].$$

Now $$\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x} = \lim_{x \to 0} \frac{e\left[1 - \frac{1}{2}x + \frac{11}{24}x^2 + \ldots\ldots\right] - e}{x}$$

$$= \lim_{x \to 0} \frac{e\left[-\frac{1}{2}x + \frac{11}{24}x^2 + \ldots\ldots\right]}{x}$$

$$= \lim_{x \to 0} e\left[-\frac{1}{2} + \frac{11}{24}x + \ldots\ldots\right] = -\frac{1}{2}e.$$

Comprehensive Exercise 1

1. State L' Hospital's rule.

Evaluate the following limits :

2. (i) $\lim_{x \to 0} \frac{\sin x}{x}$ (ii) $\lim_{x \to 0} \frac{x - \sin x}{x^3}$

(iii) $\lim_{x \to 0} \frac{e^x - 1}{x}$ (iv) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$.

(Meerut 2012B)

3. (i) $\lim_{x \to 1} \frac{x^5 - 2x^3 - 4x^2 + 9x - 4}{x^4 - 2x^3 + 2x - 1}$ (ii) $\lim_{x \to 0} \frac{a^x - b^x}{x}$ **(Agra 2003)**

(iii) $\lim_{x \to 0} \frac{\log(1 - x^2)}{\log \cos x}$ (iv) $\lim_{x \to 0} \frac{xe^x - \log(1+x)}{x^2}$.

4. (i) $\lim_{x \to 0} \frac{x - \sin x}{\tan^3 x}$

(ii) $\lim_{x \to 0} \frac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}$

(iii) $\lim_{x \to 0} \frac{(1+x)^n - 1}{x}$ (iv) $\lim_{x \to 0} \frac{\log x}{x - 1}$. **(Garhwal 2001)**

5. (i) $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$ (ii) $\lim_{x \to 0} \frac{\{1 - \sqrt{(1 - x^2)}\}}{x^2}$

(iii) $\lim_{x \to 0} \frac{\tan x - x}{x^2 \tan x}$ **(Avadh 2014)**

(iv) $\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$.

6. (i) $\lim_{x \to 0} \frac{\{\cosh x + \log(1-x) - 1 + x\}}{x^2}$

(ii) $\lim_{x \to 0} \frac{5 \sin x - 7 \sin 2x + 3 \sin 3x}{\tan x - x}$ **(Meerut 2001)**

(iii) $\lim_{x \to 0} \frac{e^x + \log\left(\frac{1-x}{e}\right)}{\tan x - x}$ (iv) $\lim_{x \to 1} \frac{x \sqrt{(3x - 2x^4)} - x^{6/5}}{1 - x^{2/3}}$.

7. (i) $\lim_{x \to \pi/2} \frac{\cos x}{x - \frac{1}{2}\pi}$ (ii) $\lim_{x \to a} \frac{a^x - x^a}{x^x - a^a}$

(iii) $\lim_{x \to 0} \frac{(1+x)^{1/x} - e + \frac{1}{2} ex}{x^2}$ **(Avadh 2006; Purvanchal 14)**

(iv) $\lim_{x \to 0} \frac{\sin x \,.\, \sin^{-1} x - x^2}{x^6}$.

8. (i) $\lim_{x \to 0} \frac{x^2 + 2 \cos x - 2}{x \sin^3 x}$ (ii) $\lim_{x \to 0} \frac{e^x - e^{x \cos x}}{x - \sin x}$

(iii) $\lim_{x \to 0} \frac{\cosh x - \cos x}{x \sin x}$ **(Meerut 2012)**

(iv) $\lim_{x \to 0} \frac{\sin 2x + a \sin x}{x^2}$.

9. (i) $\lim_{x \to 0} \frac{\sin^2 x - x^2}{x^4}$ (ii) $\lim_{x \to 0} \frac{\sin x \sin^{-1} x}{x^2}$.

10. Find the values of a and b in order that $\lim_{x \to 0} \frac{x(1 + a \cos x) - b \sin x}{x^3}$, may be equal to 1. **(Meerut 2013B)**

11. Find the values of a, b, c so that $\lim_{x \to 0} \frac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2$.

12. Find the values of a, b and c so that $\lim_{x \to 0} \frac{x(a + b \cos x) - c \sin x}{x^5} = 1$.

Answers 1

2.	(i)	1	(ii)	1/6	(iii)	1	(iv)	1/2
3.	(i)	4	(ii)	$\log(a/b)$	(iii)	2	(iv)	3/2
4.	(i)	1/6	(ii)	4	(iii)	n	(iv)	1
5.	(i)	1	(ii)	$\frac{1}{2}$	(iii)	1/3	(iv)	2
6.	(i)	0	(ii)	-15	(iii)	$-\frac{1}{2}$	(iv)	81/20

7. (i) -1 (ii) $\dfrac{\log a - 1}{\log a + 1}$ (iii) $11e/24$ (iv) $1/18$

8. (i) $1/12$ (ii) 3 (iii) 1
(iv) Infinite if $a \surd -2$ and 0 if $a = -2$

9. (i) $-\dfrac{1}{3}$ (ii) 1

10. $a = -5/2,\ b = -3/2$

11. $a = 1,\ b = 2,\ c = 1$

12. $a = 120,\ b = 60,\ c = 180$

4 The Form $\dfrac{\infty}{\infty}$

Suppose $\lim\limits_{x \to a} \phi(x) = \infty$ *and* $\lim\limits_{x \to a} \psi(x) = \infty$.

Then
$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}.$$

We have
$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{1/\psi(x)}{1/\phi(x)} \qquad \text{[Form 0/0]}$$
$$= \lim_{x \to a} \frac{\dfrac{-\psi'(x)}{[\psi(x)]^2}}{\dfrac{-\phi'(x)}{[\phi(x)]^2}} \qquad \text{[by L'Hospital's rule]}$$
$$= \lim_{x \to a} \left[\frac{\psi'(x)}{\phi'(x)} \cdot \left\{ \frac{\phi(x)}{\psi(x)} \right\}^2 \right].$$

Thus,
$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{\psi'(x)}{\phi'(x)} \cdot \left\{ \lim_{x \to a} \frac{\phi(x)}{\psi(x)} \right\}^2. \qquad \ldots(1)$$

Now suppose
$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lambda. \qquad \ldots(2)$$

Then three cases arise.

Case I : λ is neither zero nor infinite. In this case dividing both sides of (1) by λ^2, we get
$$\lambda^{-1} = \lim_{x \to a} \frac{\psi'(x)}{\phi'(x)} \quad \text{or} \quad \lambda = \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}.$$

Case II : $\lambda = 0$. In this case adding 1 to each side of equation (2), we get
$$\lambda + 1 = \lim_{x \to a} \frac{\phi(x)}{\psi(x)} + 1 = \lim_{x \to a} \left\{ \frac{\phi(x)}{\psi(x)} + 1 \right\}$$
$$= \lim_{x \to a} \frac{\phi(x) + \psi(x)}{\psi(x)} = \lim_{x \to a} \frac{\phi'(x) + \psi'(x)}{\psi'(x)}$$
$$\left\{ \text{by case I, since form is } \frac{\infty}{\infty} \text{ and } \lambda + 1 \neq 0 \right\}$$

$$= \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)} + 1$$

Therefore $\quad \lambda = \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}.$

Case III : $\lambda = \infty$. In this case, we have

$$\frac{1}{\lim_{x \to a} \frac{\phi(x)}{\psi(x)}} = \lim_{x \to a} \frac{\psi(x)}{\phi(x)} = \lim_{x \to a} \frac{\psi'(x)}{\phi'(x)}. \qquad \text{[by case II]}$$

Therefore $\quad \lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}.$

Hence in every case in which $\lim_{x \to a} \phi(x) = \infty$ and $\lim_{x \to a} \psi(x) = \infty$, we get

$$\lim_{x \to a} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a} \frac{\phi'(x)}{\psi'(x)}.$$

Note 1: By writing $x = 1/y$, we can show as in article 2 that the proposition of this article is also true when $x \to \infty$ or $-\infty$ in place of a.

Note 2: Obviously the proposition of this article is true when one or both the limits are $-\infty$.

Important : We have seen that in both cases when the form is ∞/∞ or 0/0 the rule of evaluating the limit by differentiating the numerator and denominator separately holds good. Also we can easily convert the form ∞/∞ to the form 0/0 and vice-versa. Therefore at every stage we should note carefully that which form will be more suitable to evaluate the limit most quickly. Moreover in some cases it will be necessary to convert the form ∞/∞ to the form 0/0, otherwise the process of differentiating the numerator and the denominator would never terminate.

Illustrative Examples

Example 6: *Evaluate* $\lim_{x \to 0} \frac{\log x}{\cot x}$.

(Agra 2014; Purvanchal 14)

Solution: We have, $\lim_{x \to 0} \frac{\log x}{\cot x}$, $\qquad$ [form ∞/∞]

$$= \lim_{x \to 0} \frac{1/x}{-\operatorname{cosec}^2 x}, \qquad \text{[form } \infty/\infty]$$

$$= \lim_{x \to 0} \frac{-\sin^2 x}{x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{-2 \sin x \cos x}{1} = \frac{-2 \times 0 \times 1}{1} = 0.$$

Example 7: *Evaluate* $\lim\limits_{x \to 0} \dfrac{\log \sin 2x}{\log \sin x}$. **(Agra 2001)**

Solution: We have $\lim\limits_{x \to 0} \dfrac{\log \sin 2x}{\log \sin x}$, [form ∞/∞]

$$= \lim_{x \to 0} \frac{(1/\sin 2x)(2\cos 2x)}{(1/\sin x)\cos x} = \lim_{x \to 0} \frac{2 \cot 2x}{\cot x}, \qquad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x \to 0} \frac{2 \tan x}{\tan 2x}, \qquad \text{[form 0/0]}$$

$$= \lim_{x \to 0} \frac{2 \sec^2 x}{2 \sec^2 2x}, \qquad \textbf{[By L' Hospital's rule]}$$

$$= 1.$$

Example 8: *Evaluate* $\lim\limits_{x \to 0} \dfrac{\log \log (1 - x^2)}{\log \log \cos x}$. **(Kumaun 2000; Avadh 13)**

Solution: We have, $\lim\limits_{x \to 0} \dfrac{\log \log (1 - x^2)}{\log \log \cos x}$, [form ∞/∞]

$$= \lim_{x \to 0} \frac{\dfrac{1}{\log (1 - x^2)} \cdot \dfrac{1}{1 - x^2} \cdot (-2x)}{\dfrac{1}{\log \cos x} \cdot \dfrac{1}{\cos x} \cdot (-\sin x)}$$

$$= 2 \lim_{x \to 0} \frac{x \cos x \log \cos x}{\sin x \,.\, (1 - x^2) \log (1 - x^2)}$$

$$= 2 \lim_{x \to 0} \frac{x}{\sin x} \cdot \lim_{x \to 0} \frac{\cos x}{1 - x^2} \cdot \lim_{x \to 0} \frac{\log \cos x}{\log (1 - x^2)}$$

$$= 2 \times 1 \times 1 \times \lim_{x \to 0} \frac{\log \cos x}{\log (1 - x^2)}, \qquad \text{[form 0/0]}$$

$$= 2 \lim_{x \to 0} \frac{\dfrac{1}{\cos x} \cdot (-\sin x)}{\dfrac{1}{1 - x^2} \cdot (-2x)} = 2 \times \frac{1}{2} \cdot \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1 - x^2}{\cos x} \right) = 1.$$

Comprehensive Exercise 2

Evaluate the following limits :

1. (i) $\lim\limits_{x \to \infty} \dfrac{x}{e^x}$ (ii) $\lim\limits_{x \to \infty} \dfrac{e^x}{x^3}$.

2. (i) $\lim\limits_{x \to 1} \dfrac{\log (1 - x)}{\cot \pi x}$ (ii) $\lim\limits_{x \to \infty} \dfrac{\log x}{a^x}$, $a > 1$.

3. (i) $\lim_{x \to a} \frac{\log (x - a)}{\log (e^x - e^a)}$ (ii) $\lim_{x \to \frac{1}{2}} \frac{\sec \pi x}{\tan 3 \pi x}$.

4. (i) $\lim_{x \to \infty} \left\{ \frac{(\log x)^3}{x} \right\}$ (ii) $\lim_{x \to \infty} \frac{x (\log x)^3}{1 + x + x^2}$.

5. (i) $\lim_{x \to \infty} x \tan \frac{1}{x}$ **(Bundelkhand 2001)**

(ii) $\lim_{x \to 0} \frac{\log \tan 2x}{\log \tan 3x}$.

6. (i) $\lim_{x \to \pi/2} \frac{\log \left(x - \frac{1}{2} \pi \right)}{\tan x}$ (ii) $\lim_{x \to a} \frac{c \{e^{1/(x-a)} - 1\}}{\{e^{1/(x-a)} + 1\}}$.

7. (i) $\lim_{x \to 1} \left\{ \frac{2}{x^2 - 1} - \frac{1}{x - 1} \right\}$ **(Garhwal 2002)**

(ii) $\lim_{x \to \pi/2} (\sec x - \tan x)$.

Answers 2

1.	(i) 0	(ii) ∞	**2.**	(i) 0	(ii) 0
3.	(i) 1	(ii) 3	**4.**	(i) 0	(ii) 0
5.	(i) 1	(ii) 1	**6.**	(i) 0.	(ii) c
7.	(i) $-\frac{1}{2}$	(ii) 0			

5 The Form $\infty - \infty$

This form can be easily reduced to the form $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$.

Suppose $\lim_{x \to a} \phi(x) = \infty$ *and* $\lim_{x \to a} \psi(x) = \infty$.

Then $\lim_{x \to a} \{\phi(x) - \psi(x)\}$ [form $\infty - \infty$]

$$= \lim_{x \to a} \left\{ \frac{1}{1/\phi(x)} - \frac{1}{1/\psi(x)} \right\}$$

$$= \lim_{x \to a} \frac{\frac{1}{\psi(x)} - \frac{1}{\phi(x)}}{\frac{1}{\phi(x) \cdot \phi(x)}}$$, which is of the form $\frac{0}{0}$.

Illustrative Examples

Example 9: *Evaluate* $\lim\limits_{x \to \pi/2} \left(\sec x - \dfrac{1}{1 - \sin x} \right)$.

Solution: We have $\lim\limits_{x \to \pi/2} \left(\sec x - \dfrac{1}{1 - \sin x} \right)$, [form $\infty - \infty$]

$$= \lim_{x \to \pi/2} \left(\frac{1}{\cos x} - \frac{1}{1 - \sin x} \right) = \lim_{x \to \pi/2} \left[\frac{1 - \sin x - \cos x}{\cos x\,(1 - \sin x)} \right], \quad \text{[form } 0/0\text{]}$$

$$= \lim_{x \to \pi/2} \left[\frac{-\cos x + \sin x}{-\sin x\,(1 - \sin x) + \cos x\,(-\cos x)} \right]$$

$$= \lim_{x \to \pi/2} \left[\frac{\sin x - \cos x}{-\sin x + \sin^2 x - \cos^2 x} \right] = \frac{1}{-1 + 1} = \infty.$$

Example 10: *Evaluate* $\lim\limits_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x} \right)$.

(Garhwal 2000, 02)

Solution: We have $\lim\limits_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x} \right)$ [form $\infty - \infty$]

$$= \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^2 \sin^2 x}, \quad \text{[form } 0/0\text{]}$$

$$= \lim_{x \to 0} \frac{\left(x - \dfrac{x^3}{3!} + \ldots \right)^2 - x^2}{x^2 \left(x - \dfrac{x^3}{3!} + \ldots \right)^2}$$

$$= \lim_{x \to 0} \frac{-\dfrac{2x^4}{3!} + \text{terms containing higher powers of } x}{x^4 + \text{terms containing higher powers of } x}$$

$$= \lim_{x \to 0} \frac{-\dfrac{2}{3!} + \text{terms containing powers of } x \text{ only in the numerator}}{1 + \text{terms containing powers of } x \text{ only in the numerator}}$$

$$= \frac{-2}{3!} = -\frac{1}{3}.$$

6 The Form $0 \times \infty$

This form can be easily reduced to the form $\dfrac{0}{0}$ *or to the form* $\dfrac{\infty}{\infty}$.

Suppose $\lim\limits_{x \to a} \phi(x) = 0$ *and* $\lim\limits_{x \to a} \psi(x) = \infty$.

Then
$$\lim_{x \to a} \phi(x) \cdot \psi(x) \qquad [\text{form } 0 \times \infty]$$
$$= \lim_{x \to a} \frac{\phi(x)}{\dfrac{1}{\psi(x)}} \qquad [\text{form } 0/0]$$
$$= \lim_{x \to a} \frac{\psi(x)}{\dfrac{1}{\phi\{x\}}} \qquad [\text{form } \infty/\infty]$$

We shall reduce the form $0 \times \infty$ to the form $0/0$ or ∞/∞ according to our convenience.

Illustrative Examples

Example 11: *Evaluate* $\lim_{x \to 0} x \log \sin x$.

Solution: We have, $\lim_{x \to 0} x \log \sin x$ $\qquad [\text{form } 0 \times \infty]$

$$= \lim_{x \to 0} \frac{\log \sin x}{1/x} \qquad [\text{form } \infty/\infty]$$
$$= \lim_{x \to 0} \frac{(1/\sin x) \cdot \cos x}{-1/x^2} \qquad [\text{form } \infty/\infty]$$
$$= \lim_{x \to 0} \frac{-x^2 \cos x}{\sin x}$$
$$= \lim_{x \to 0} \frac{x^2 \sin x - 2x \cos x}{\cos x} = 0.$$

Comprehensive Exercise 3

Evaluate the following limits:

1. (i) $\lim_{x \to 1} \left(\frac{1}{\log x} - \frac{x}{\log x} \right)$ (ii) $\lim_{x \to 0} \left(\frac{\cot x - \dfrac{1}{x}}{x} \right)$.

2. (i) $\lim_{x \to 0} \left(\frac{\operatorname{cosec} x - \cot x}{x} \right)$ (ii) $\lim_{x \to 0} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right)$.

3. (i) $\lim_{x \to \pi/2} \left(x \tan x - \frac{\pi}{2} \sec x \right)$ (ii) $\lim_{x \to 0} x \log x$.

4. (i) $\lim_{x \to 0} \sin x . \log x$ (ii) $\lim_{x \to \infty} x (a^{1/x} - 1)$.

5. (i) $\lim\limits_{x \to \infty} 2^x \sin \dfrac{a}{2^x}$ **(Agra 2003)**

(ii) $\lim\limits_{x \to \pi/2} (1 - \sin x) \tan x$.

6. $\lim\limits_{x \to 0} x^m (\log x)^n$, where m and n are positive integers.

Answers 3

1.	(i) -1	(ii) $-\dfrac{1}{3}$		2.	(i) $\dfrac{1}{2}$	(ii) $-\dfrac{1}{2}$	
3.	(i) -1	(ii) 0		4.	(i) 0	(ii) $\log a$	
5.	(i) a	(ii) 0		6.	0		

7 The Forms $1^\infty, 0^0, \infty^0$

Suppose $\lim\limits_{x \to a} [\phi(x)]^{\psi(x)}$ *takes any one of these three forms.*

Then let $$y = \lim_{x \to a} [\phi(x)]^{\psi(x)}.$$

Taking logarithm of both sides, we get

$$\log y = \lim_{x \to a} \psi(x) \,.\, \log \phi(x).$$

Now in any of the above three cases, $\log y$ takes the form $0 \times \infty$ which cna be evaluated by the process of article 6.

Illustrative Examples

Example 12: *Evaluate* $\lim\limits_{x \to 0} (\cos x)^{\cot^2 x}$.

(Kumaun 2001; Bundelkhand 14)

Solution: Let $y = \lim\limits_{x \to 0} (\cos x)^{\cot^2 x}$. [form 1^∞]

$\therefore$ $$\log y = \lim_{x \to 0} (\cot^2 x) \,.\, (\log \cos x), \quad \text{[form } \infty \times 0]$$

$$= \lim_{x \to 0} \frac{\log \cos x}{\tan^2 x}, \quad \text{[form } 0/0]$$

$$= \lim_{x \to 0} \frac{\{(1/\cos x) \,.\, (-\sin x)\}}{2 \tan x \,.\, \sec^2 x}, \quad \text{[by L' Hospital's rule]}$$

$$= \lim_{x \to 0} \frac{-\tan x}{2 \tan x \,.\, \sec^2 x} = \lim_{x \to 0} \frac{-1}{2 \sec^2 x} = -\frac{1}{2}.$$

$\therefore \qquad y = e^{-1/2}.$

Example 13: *Evaluate* $\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x$.

Solution: Let $y = \lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^x$. [form 1^∞]

$$\therefore \qquad \log y = \lim_{x \to \infty} \left\{x \log \left(1 + \frac{a}{x}\right)\right\}, \qquad \text{[form } \infty \times 0]$$

$$= \lim_{x \to \infty} \frac{\log \{1 + (a/x)\}}{1/x}, \qquad \text{[form } 0/0]$$

$$= \lim_{x \to \infty} \frac{[1/\{1 + (a/x)\}] . (-a/x^2)}{-(1/x^2)} = \lim_{x \to \infty} \frac{a}{1 + (a/x)} = a.$$

$\therefore \qquad y = e^a.$

8 Compound Forms

Suppose a function is the product of two or more factors the limit of each of which can be easily found. Then the limit of the entire function will be equal to the product of the limits of the factors provided that the product is not in itself an indeterminate form. A similar rule is applicable in the case of a sum, difference, quotient or power.

Illustrative Examples

Example 14: *Evaluate* $\lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{x^2} \log (1 + x)\right]$.

Solution: We have $\lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{x^2} \log (1 + x)\right]$

$$= \lim_{x \to 0} \frac{x - \log (1 + x)}{x^2}, \qquad \text{[form } 0/0]$$

$$= \lim_{x \to 0} \frac{1 - \{1/(1 + x)\}}{2x} = \lim_{x \to 0} \frac{1 + x - 1}{2x (1 + x)}$$

$$= \lim_{x \to 0} \frac{1}{2 (1 + x)} = \frac{1}{2}.$$

Example 15: *Evaluate* $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$.

(Garhwal 2001; Agra 03; Kumaun 03; Kashi 14; Purvanchal 14)

Solution: Let $y = \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$

$$\therefore \quad \log y = \lim_{x \to 0} \frac{1}{x^2} \log \left(\frac{\tan x}{x} \right) = \lim_{x \to 0} \frac{1}{x^2} \log \left[\frac{1}{x} \left(x + \frac{x^3}{3} + \frac{2}{15} x^5 + \dots \right) \right]$$

$$= \lim_{x \to 0} \frac{1}{x^2} \log \left[1 + \left(\frac{x^2}{3} + \frac{2}{15} x^4 + \dots \right) \right]$$

$$= \lim_{x \to 0} \frac{\left(\frac{x^2}{3} + \frac{2}{15} x^4 + \dots \right) - \frac{1}{2} \left(\frac{x^2}{3} + \frac{2}{15} x^4 + \dots \right)^2 + \dots}{x^2}$$

$$= \lim_{x \to 0} \frac{\frac{x^2}{3} + \text{terms containing higher powrs of } x}{x^2}$$

$$= \lim_{x \to 0} \left[\frac{1}{3} + \text{terms containing powers of } x \text{ only in the numerator} \right]$$

$$= \frac{1}{3}.$$

$$\therefore \quad y = e^{1/3}.$$

Comprehensive Exercise 4

Evaluate the following limits:

1. (i) $\lim_{x \to 0} x^x$ **(Agra 2002; Kanpur 04)**

(ii) $\lim_{x \to 0} (\cos x)^{1/x}$.

2. (i) $\lim_{x \to 0} (\cos x)^{1/x^2}$ **(Garhwal 2003)**

(ii) $\lim_{x \to \pi/2} (\sin x)^{\tan x}$.

3. (i) $\lim_{x \to \pi/2} (\sec x)^{\cot x}$

(ii) $\lim_{x \to \pi/4} (\tan x)^{\tan 2x}$.

4. (i) $\lim_{x \to a} \left(2 - \frac{x}{a} \right)^{\tan (\pi x / 2a)}$ **(Rohilkhand 2012)**

(ii) $\lim_{x \to 1} (1 - x^2)^{1/\log (1 - x)}$.

5. (i) $\lim_{x \to 1} x^{1/(1 - x)}$

(ii) $\lim_{x \to \infty} (a_0 x^m + a_1 x^{m-1} + \dots + a_m)^{1/x}$.

6. (i) $\lim_{x \to 0} \left(\frac{\tan x}{x} \right)^{1/x}$ **(Garhwal 2001, 03)**

(ii) $\lim_{x \to 0} \left(\frac{\tan x}{x} \right)^{1/x^3}$.

7. (i) $\lim_{x \to 0} \left(\frac{\sinh x}{x} \right)^{1/x^2}$ **(Kumaun 2008)**

(ii) $\lim_{x \to 0} \left(\frac{\sin x}{x} \right)^{1/x^2}$. **(Kumaun 2003)**

8. (i) $\lim_{x \to 0} \left\{ \frac{2(\cosh x - 1)}{x^2} \right\}^{1/x^2}$

(ii) $\lim_{x \to \infty} \left\{ \frac{\log x}{x} \right\}^{1/x}$.

9. (i) $\lim_{x \to \infty} \left(\frac{\pi}{2} - \tan^{-1} x \right)^{1/x}$

(ii) $\lim_{x \to 0} (\operatorname{cosec} x)^{1/\log x}$.

10. $\lim_{x \to 1} (1 - x) \tan \frac{\pi x}{2}$.

Answers 4

1. (i) 1	(ii) 1	**2.** (i) $e^{-1/2}$	(ii) 1
3. (i) 1	(ii) $1/e$	**4.** (i) $e^{2/\pi}$	(ii) e
5. (i) $1/e$	(ii) 1	**6.** (i) 1	(ii) ∞
7. (i) $e^{1/6}$	(ii) $e^{-1/6}$	**8.** (i) $e^{1/12}$	(ii) 1
9. (i) 1	(ii) $1/e$	**10.** $2/\pi$	

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The value of $\lim_{x \to 0} \frac{1 - \cos x}{3x^2}$ is

(a) 0 (b) $\frac{2}{3}$

(c) $\frac{1}{6}$ (d) 1 **(Garhwal 2002)**

2. Which of the following is not an indeterminate form ?

(a) $\frac{\infty}{\infty}$ (b) $0 \times \infty$

(c) 1^0 (d) 0^0

3. The value of the $\lim\limits_{x \to 0} \frac{\log \tan x}{\log x}$ is

(a) 0 (b) 1

(c) –1 (d) None of these

4. Which of the following is an indeterminate form ?

(a) $\infty + \infty$ (b) $\infty \times \infty$

(c) 1^∞ (d) 0^∞

5. The value of $\lim\limits_{x \to 0} \frac{a^x - 1 - x \log_e a}{x^2}$ is

(a) $(\log_e a)^2$ (b) $\frac{(\log_e a)}{2}$

(c) $a - \log_e a$ (d) $\frac{(\log_e a)^2}{2}$ **(Garhwal 2001)**

6. The value of $\lim\limits_{x \to 1} \frac{\log x}{x-1}$ is

(a) –1 (b) ∞

(c) 1 (d) 0

7. The value of $\lim\limits_{x \to \infty} \frac{\log_e x}{a^x}, a > 1$ is

(a) $\frac{1}{\log_e a}$ (b) a

(c) 1 (d) 0 **(Garhwal 2003)**

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. $\lim\limits_{x \to 0} \frac{\sin ax}{\sin bx} = \ldots\ldots$

2. $\lim\limits_{x \to 0} \frac{\log(1 + kx^2)}{1 - \cos x} = \ldots\ldots$

3. $\lim\limits_{x \to \infty} \frac{x^2 + 2x}{5 - 3x^2} = \ldots\ldots$

4. $\lim\limits_{x \to 1} \left(\sec \frac{\pi}{2x}\right) \cdot \log x = \ldots\ldots$

5. The value of $\lim\limits_{x \to 0} \frac{x - \tan x}{x^3} = \ldots\ldots$ **(Agra 2002)**

6. The value of $\lim_{x \to 0} \frac{a^x - b^x}{x} = \ldots\ldots$ **(Agra 2003)**

True or False

Write 'T' for true and 'F' for false statement.

1. While applying L' Hospital's rule to evaluate $\lim_{x \to a} \frac{f(x)}{\phi(x)}$, if the form is $\frac{0}{0}$, we are to differentiate $f(x) / \phi(x)$ as a fraction.
2. The indeterminate form $\frac{\infty}{\infty}$ can be easily converted to the form $\frac{0}{0}$ and vice-versa.
3. If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} \phi(x) = \infty$, then $\lim_{x \to a} \frac{f(x)}{\phi(x)} = \lim_{x \to a} \frac{f'(x)}{\phi'(x)}$.
4. $\lim_{x \to a} \frac{(1+x)^{1/x} - e + \frac{ex}{2}}{x^2}$ is of the form $\frac{\infty}{0}$. **(Meerut 2001)**

Answers

Multiple Choice Questions

1.	(c)	2.	(c)	3.	(b)	4.	(c)	5.	(d)
6.	(c)	7.	(d)						

Fill in the Blank(s)

1.	$\frac{a}{b}$	2.	$2k$	3.	$-\frac{1}{3}$	4.	$\frac{2}{\pi}$	5.	$-\frac{1}{3}$
6.	$\log \frac{a}{b}$								

True or False

1.	*F*	2.	*T*	3.	*T*	4.	*F*

□