Krishna's

TEXT BOOK on

REAL ANALYSIS

(For B.A. and B.Sc. IVth Semester students of Kumaun University)

Kumaun University Semester Syllabus *w.e.f.* 2017-18

By

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Dedicated

to

Lord

Krishna

Authors & Publishers

Preface

This book on **Real Analysis** has been specially written according to the latest **Syllabus** to meet the requirements of **B.A. and B.Sc. Semester-IV Students** of all colleges affiliated to Kumaun University.

The subject matter has been discussed in such a simple way that the students will find no difficulty to understand it. The proofs of various theorems and examples have been given with minute details. Each chapter of this book contains complete theory and a fairly large number of solved examples. Sufficient problems have also been selected from various university examination papers. At the end of each chapter an exercise containing objective questions has been given.

We have tried our best to keep the book free from misprints. The authors shall be grateful to the readers who point out errors and omissions which, inspite of all care, might have been there.

The authors, in general, hope that the present book will be warmly received by the students and teachers. We shall **indeed** be very thankful to our colleagues for their recommending this book to their students.

The authors wish to express their thanks to Mr. S.K. Rastogi, *M.D.,* Mr. Sugam Rastogi, *Executive Director,* Mrs. Kanupriya Rastogi, *Director* and entire team of **KRISHNA Prakashan Media (P) Ltd., Meerut** for bringing out this book in the present nice form.

The authors will feel amply rewarded if the book serves the purpose for which it is meant. Suggestions for the improvement of the book are always welcome.

— Authors

Syllabus

Real Analysis

B.A./B.Sc. IV Semester

Kumaun University

Fourth Semester – Second Paper

B.A./B.Sc. Paper-II M.M.-60

Continuity and Differentiability of functions: Continuity of functions, Uniform continuity, Differentiability, Taylor's theorem with various forms of remainders.

Integration: Riemann integral-definition and properties, integrability of continuous and monotonic functions, Fundamental theorem of integral calculus, Mean value theorems of integral calculus.

Improper Integrals: Improper integrals and their convergence, Comparison test, Dritchlet's test, Absolute and uniform convergence, Weierstrass M-Test, Infinite integral depending on a parameter.

Sequence and Series: Sequences, theorems on limit of sequences, Cauchy's convergence criterion, infinite series, series of non-negative terms, Absolute convergence, tests for convergence, comparison test, Cauchy's root Test, ratio Test, Rabbe's, Logarithmic test, De Morgan's Test, Alternating series, Leibnitz's theorem.

Uniform Convergence: Point wise convergence, Uniform convergence, Test of uniform convergence, Weierstrass M-Test, Abel's and Dritchlet's test, Convergence and uniform convergence of sequences and series of functions.

Brief Contents

Krishna's

REAL ANALYSIS

Chapters

Chapter

1

Continuity

1 Continuity

(Purvanchal 2010, 11; Gorakhpur 11; Avadh 14)

The intuitive concept of continuity is derived from geometrical considerations. If the graph of the function $y = f(x)$ is a continuous curve, it is natural to call the function continuous. This requires that there should be no sudden changes in the value of the function. A small change in x should produce only a small change in y. Moreover for the graph to be a continuous running curve, it should possess a definite direction at each point.

But the continuity as defined in pure analysis is quite distinct from the intuitive or the geometrical concept of the term. Sometimes drawing a graph is difficult. We now give the arithmetical definition of continuity given by Cauchy.

Cauchy's definition of continuity.

A real valued function f defined on an open interval I is said to be continuous at $a \in I$ iff for any arbitrarily chosen positive number ε, however small, we can find a corresponding number $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \text{ whenever } |x - a| < \delta. \quad \ldots(1)$$

(Bundelkhand 2010; Kanpur 11)

We say that f is a **continuous function** if it is continuous at every $x \in I$.

In other words, f is continuous at a if for any given $\varepsilon > 0$, we can find a $\delta > 0$ such that

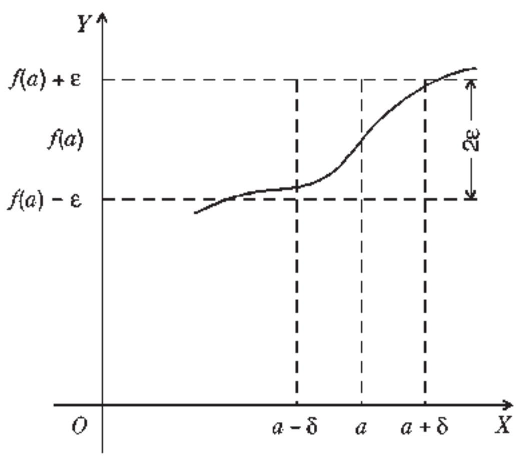
$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

This means that the function f will be continuous at $x = a$ if the difference between $f(a)$ and the value of $f(x)$ at any point in the interval $]a - \delta, a + \delta[$ can be made less than a pre-assigned positive number ε. Note that we choose δ after we have chosen ε.

A geometrical interpretation of the above definition is immediate. Corresponding to any pre-assigned positive number ε, we can determine an interval of width 2δ about the point $x = a$ (see the figure) such that for any point x lying in the interval $]a - \delta, a + \delta[$, $f(x)$ is confined to lie between

$$f(a) - \varepsilon \text{ and } f(a) + \varepsilon.$$

The inequality (1) may be written in the form of an equality as $f(x) = f(a) + \eta$, where $|\eta| < \varepsilon$.



Note 1: For a function $f(x)$ to be continuous at $x = a$, it is necessary that $\lim_{x \to a} f(x)$ must exist.

Note 2: $|f(x) - f(x)| \Rightarrow f(a) - \in < f(x) < f(a) + \in$

and $|x - a| < \delta \Rightarrow a - \delta < x < a + \delta$

Note 3: The function must be defined at the point of continuity.

Note 4: The value of δ depends upon the values of ε and a.

Note 5: The interval I may be of any one of the forms :

$$]a, b[\,,]-\infty, b\,[\,,]a, \infty[\,,]-\infty, \infty[.$$

An alternative definition of continuity of a Function At a Point: *A function f is said to be continuous at $a \in I$ iff* $\lim_{x \to a} f(x)$ *exists, is finite and is equal to* $f(a)$ *otherwise the function is discontinuous at* $x = a$.

This definition of continuity follows immediately from the definition of limit and the definition of continuity. Thus a function f is said to be continuous at a, if $f(a+0) = f(a-0) = f(a)$. This is a working formula for testing the continuity of a function at a given point. **(Bundelkhand 2008, 10; Kashi 12)**

Polynomial Function:

Theorem 1: *A polynomial function is always a continuous function.*

Proof: If $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$ is a polynomial in x of degree n, then by the above definition it can be easily seen that $f(x)$ is continuous for all $x \in \mathbf{R}$.

If c be any real number, then

$$\lim_{x \to c} f(x) = \lim_{x \to c} \{a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n\}$$

$$= a_0 \lim_{x \to c} x^n + a_1 \lim_{x \to c} x^{n-1} + \ldots + a_{n-1} \lim_{x \to c} x + \lim_{x \to c} a_n$$

$$= a_0 c^n + a_1 c^{n-1} + \ldots + a_{n-1} c + a_n \qquad \left[\because \lim_{x \to c} x = c\right]$$

$$= f(c).$$

Since $\lim_{x \to c} f(x) = f(c)$, therefore $f(x)$ is continuous at $x = c$.

Thus $f(x)$ is continuous at every real number c and so $f(x)$ is continuous for all $x \in \mathbf{R}$.

Thus remember that a polynomial function $f(x)$ is always continuous at each point of its domain.

Continuity from left and continuity from right:

Let f be a function defined on an open interval I and let $a \in I$. We say that f is continuous from the left at a if $\lim_{x \to a-0} f(x)$ *exists and is equal to* $f(a)$*. Similarly f is said to be continuous from the right at a if* $\lim_{x \to a+0} f(x)$ *exists and is equal to* $f(a)$.

From the above definitions it is clear that for a function f to be continuous at a, it is necessary as well as sufficient that f be continuous from the left as well as from the right at a.

Continuous function: *A function f is said to be a continuous function if it is continuous at each point of its domain.*

Continuity in an open interval: A function f is said to be continuous in the open interval $]a, b[$ if it is continuous at each point of the interval. **(Bundelkhand 2009)**

Continuity in a closed interval: Let f be a function defined on the closed interval $[a, b]$. We say that f is continuous at a if it is continuous from the right at a and also that f is continuous at b if it is continuous from the left at b. Further, f is said to be continuous on the closed interval $[a, b]$, if (i) it is continuous from the right at a, (ii) continuous from the left at b and (iii) continuous on the open interval $]a, b[$.

Thus if a function f is defined on the closed interval $[a, b]$, then

(i) it is continuous at the left end point a if $f(a) = f(a+0)$

i.e., $$f(a) = \lim_{x \to a+0} f(x)$$

(ii) it is continuous at the right end point b if $f(b) = f(b-0)$

i.e., $$f(b) = \lim_{x \to b-0} f(x)$$

and (iii) it is continuous at an interior point c of $[a, b]$ *i.e.*, at $c \in]a, b[$

if $$f(c-0) = f(c) = f(c+0)$$

i.e., if $$\lim_{x \to c-0} f(x) = f(c) = \lim_{x \to c+0} f(x).$$

2 Discontinuity

Definition: *If a function is not continuous at a point, then it is said to be discontinuous at that point and the point is called a point of discontinuity of this function.*

Types of Discontinuity:

(i) Removable discontinuity: **(Meerut 2011; Avadh 14)**

A function f is said to have a *removable discontinuity* at a point a if $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$ *i.e.*, if

$$f(a+0) = f(a-0) \neq f(a).$$

The function can be made continuous by defining it in such a way that $\lim_{x \to a} f(x) = f(a)$.

(ii) Discontinuity of the first kind or ordinary discontinuity: **(Meerut 2010B)**

A function f is said to have a *discontinuity of the first kind* or *ordinary discontinuity* at a if $f(a+0)$ and $f(a-0)$ both exist but are not equal. The point a is said to be a point of discontinuity from the left or right according as $f(a-0) \neq f(a) = f(a+0)$ or $f(a-0) = f(a) \neq f(a+0)$.

(iii) Discontinuity of the second kind: **(Meerut 2003, 10B)**

A function f is said to have a *discontinuity of the second kind*, at a if none of the limits $f(a+0)$ and $f(a-0)$ exist. The point a is said to be a point of discontinuity of the second kind from the left or right according as $f(a-0)$ or $f(a+0)$ does not exist.

(iv) Mixed discontinuity: **(Meerut 2012B)**

A function f is said to have a *mixed discontinuity* at a, if f has a discontinuity of second kind on one side of a and on the other side a discontinuity of first kind or may be continuous.

(v) Infinite discontinuity:

A function f is said to have an *infinite discontinuity* at a if $f\ (a+0)$ or $f\ (a-0)$ is $+\infty$ or $-\infty$. Obviously, if f has a discontinuity at a and is unbounded in every neighbourhood of a, then f is said to have an infinite discontinuity at a.

3 Jump of a Function at a Point

If both $f\ (a+0)$ and $f\ (a-0)$ exist, then the **jump** in the function at a is defined as the non-negative difference $f\ (a+0) \sim f\ (a-0)$. A function having a finite number of jumps in a given interval is **called piecewise continuous.**

4 Algebra of Continuous Functions

Theorem 1: *Let f and g be defined on an interval I. If f and g are continuous at $a \in I$, then $f + g$ is also continuous at a.*

Theorem 2: *Let f and g be defined on an interval I. If f and g are continuous at $a \in I$, then fg is continuous at a.*

Theorem 3: *If f is continuous at a point a and $c \in \mathbf{R}$, then cf is continuous at a.*

Theorem 4: *Let f and g be defined on an interval I, and let $g\ (a) \neq 0$. If f and g are continuous at $a \in I$, then $f\ /\ g$ is continuous at a.*

Theorem 5: *If f is continuous at a then $|\,f\,|$ is also continuous at a.*

Note: The converse is not true. For example, if

$$f\ (x) = -1, \text{ for } x < a \text{ and } f\ (x) = 1 \text{ for } x \geq a \text{ then}$$

$$\lim_{x \to a} |\,f\ (x)\,| = 1 = |\,f\ (a)\,|, \text{ but } \lim_{x \to a} f\ (x) \text{ does not exist.}$$

Thus $|\,f\,|$ is continuous at a while f is not continuous at a.

Illustrative Examples

Example 1: *Test the following functions for continuity :*

(i) $f\ (x) = x \sin (1\ /\ x),\ x \neq 0,\ f\ (0) = 0$ *at* $x = 0$. **(Kanpur 2005; Avadh 08; Meerut 09B; Purvanchal 09; Kashi 12; Rohilkhand 14; Gorakhpur 12, 14)**

Also draw the graph of the function. **(Lucknow 2007)**

(ii) $f\ (x) = 2^{1/x}$ *when* $x \neq 0,\ f\ (0) = 0$ *at* $x = 0$.

(iii) $f\ (x) = 1\ /\ (1 - e^{-1/x}),\ x \neq 0,\ f\ (0) = 0$ *at* $x = 0$.

Solution: (i) Here $f(0+0)=\lim_{h\to 0} f(0+h), h>0$

$$=\lim_{h\to 0} f(h)=\lim_{h\to 0} h\sin\frac{1}{h}=0.$$

$$\left[\because \lim_{h\to 0} h=0 \text{ and } \left|\sin\frac{1}{h}\right|\le 1 \text{ for all } h\ne 0 \text{ i.e., } \sin(1/h) \text{ is bounded in some deleted neighbourhood of zero}\right]$$

Similarly $f(0-0)=\lim_{h\to 0} f(0-h), h>0$

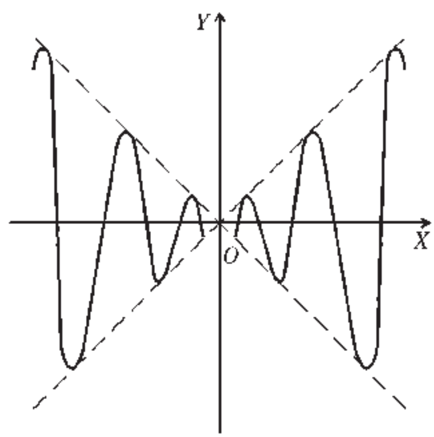
$$=\lim_{h\to 0} f(-h)=\lim_{h\to 0}(-h)\sin\left(\frac{1}{-h}\right)$$

$$=\lim_{h\to 0} h\sin\frac{1}{h}=0, \text{ as before.}$$

Also $f(0)=0$. Thus $f(0-0)=f(0)=f(0+0)$.

$\therefore$ the function $f(x)$ is continuous at $x=0$. To draw the graph of the function we put $y=f(x)$.

So the graph of the function is the curve $y=x\sin(1/x), x\ne 0$ and $y=0$ when $x=0$.



If we put $-x$ in place of x, the equation of this curve does not change and so this curve is symmetrical about the y-axis and it is sufficient to draw the graph when $x>0$.

Also

$|f(x)|=|x\sin(1/x)|=|x|\cdot|\sin(1/x)|$

$\le|x|$. $\qquad [\because |\sin(1/x)|\le 1]$

$\therefore$ for all x the curve $y=x\sin(1/x)$ lies between the lines $y=x$ and $y=-x$.

Excluding origin the curve meets the y-axis at the points where

$$\sin\frac{1}{x}=0 \text{ i.e., where } \frac{1}{x}=\pi,2\pi,3\pi,\ldots \text{ i.e., where } x=\frac{1}{\pi},\frac{1}{2\pi},\frac{1}{3\pi},\ldots.$$

Also $\qquad y=x$ at the points where $\sin\frac{1}{x}=1$ *i.e.*, $\frac{1}{x}=\frac{\pi}{2},\frac{5\pi}{2},\frac{9\pi}{2},\ldots$

i.e., $x=\frac{2}{\pi},\frac{2}{5\pi},\frac{2}{9\pi},\ldots$

and $\qquad y=-x$ at the points where $\sin\frac{1}{x}=-1$ *i.e.*, $\frac{1}{x}=\frac{3\pi}{2},\frac{7\pi}{2},\ldots$

i.e., $x=\frac{2}{3\pi},\frac{2}{7\pi},\ldots.$

We have $$\frac{dy}{dx} = \sin\frac{1}{x} + x\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}.$$

So at the points where $\sin(1/x) = 1$, we have $\cos(1/x) = 0$ and $dy/dx = 1$ *i.e.*, at these points the curve touches the straight line $y = x$. Similarly at the points where $\sin(1/x) = -1$, the curve touches the straight line $y = -x$.

Also $$\lim_{x\to\infty} x\sin\frac{1}{x} \qquad [\text{Form } \infty\times 0]$$

$$= \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{\theta\to 0} \frac{\sin\theta}{\theta}, \text{ putting } \frac{1}{x} = \theta \text{ so that } \theta\to 0 \text{ as } x\to\infty$$

$$= 1.$$

Thus $y \to 1$ as $x \to \infty$ and so the straight line $y = 1$ is an asymptote of the curve.

Although the function is continuous at the origin, yet the graph of the function in the vicinity of the origin cannot be drawn, since the function oscillates infinitely often in any interval containing the origin.

From the graph it is clear that the function makes an infinite number of oscillations in the neighbourhood of $x = 0$. The oscillations, however, go on diminishing in length as $x \to 0$.

Note 1: If we are to check the continuity of $f(x)$ at any point $x = c$, where $c \neq 0$, then we see that

$$\lim_{x\to c} f(x) = \lim_{x\to c} x\sin\frac{1}{x} = c\sin\frac{1}{c} = f(c)$$

and so $f(x)$ is continuous at $x = c$.

Thus $f(x)$ is continuous for all $x \in \mathbf{R}$ *i.e.*, $f(x)$ is continuous on the whole real line.

Note 2: If we take $f(0) = 2$, the function becomes discontinuous at $x = 0$ and has a **removable discontinuity** at $x = 0$.

(ii) Here $f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} 2^{1/h} = 2^{\infty} = \infty$,

$$f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} 2^{-1/h} = 2^{-\infty} = 0 \quad \text{and} \quad f(0) = 0.$$

Since $f(0+0) \neq f(0-0)$, therefore the function is discontinuous at the origin.

It has an **infinite discontinuity** there.

(iii) Here $f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \dfrac{1}{1-e^{-1/h}} = 1$,

$$f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{1}{1-e^{1/h}} = 0.$$

Since $f(0+0) \neq f(0-0)$, hence $f(x)$ is discontinuous at $x = 0$ and has discontinuity of the first kind. This function has a jump of one unit at 0 since $f(0+0) - f(0-0) = 1$.

Example 2: *Consider the function f defined by $f(x) = x - [x]$, where x is a positive variable and $[x]$ denotes the integral part of x and show that it is discontinuous for integral values of x and continuous for all others. Draw its graph.*

Solution: From the definition of the function $f(x)$, we have

$$f(x) = x - (n-1) \qquad \text{for } n-1 < x < n,$$

$$f(x) = 0 \qquad \text{for } x = n,$$

$$f(x) = x - n \qquad \text{for } n < x < n+1, \text{ where } n \text{ is an integer.}$$

We shall test the function $f(x)$ for continuity at $x = n$.

We have $f(n) = 0$;

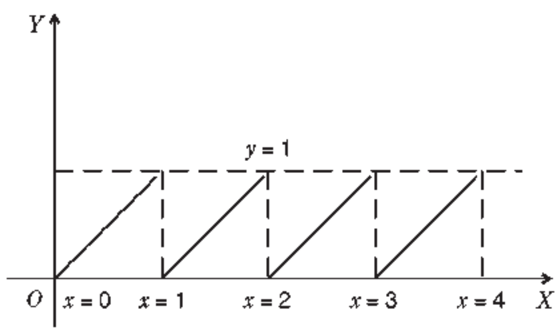
$$f(n+0) = \lim_{h \to 0} f(n+h) = \lim_{h \to 0} \{(n+h) - n\} \qquad [\because n < n+h < n+1]$$

$$= \lim_{h \to 0} h = 0;$$

and $$f(n-0) = \lim_{h \to 0} f(n-h) = \lim_{h \to 0} \{(n-h) - (n-1)\} \qquad [\because n-1 < n-h < n]$$

$$= \lim_{h \to 0} (1-h) = 1.$$

Since $f(n+0) \neq f(n-0)$, the function $f(x)$ is discontinuous at $x = n$. Thus $f(x)$ is discontinuous for all integral values of x. It is obviously continuous for all other values of x.



Since x is a positive variable, putting $n = 1, 2, 3, 4, 5, \ldots$ we see that the graph of $f(x)$ consists of the following straight lines :

$y = x$	when $0 < x < 1$,	$y = 0$ when $x = 1$
$y = x - 1$	when $1 < x < 2$,	$y = 0$ when $x = 2$
$y = x - 2$	when $2 < x < 3$,	$y = 0$ when $x = 3$
$y = x - 3$	when $3 < x < 4$,	$y = 0$ when $x = 4$ and so on.

The graph of the function thus obtained is shown by thick lines from $x = 0$ to $x = 4$. From the graph it is evident that :

(i) The function is discontinuous for all integral values of x but continuous for other values of x.

(ii) The function is bounded between 0 and 1 in every domain which includes an integer.

(iii) The lower bound 0 is attained but the upper bound 1 is not attained since $f(x) \neq 1$ for any value of x.

Example 3: *Show that the function $f(x) = [x] + [-x]$ has removable discontinuity for integral values of x.* **(Kanpur 2009)**

Solution: We observe that $f(x) = 0$, when x is an integer and $f(x) = -1$, when x is not an integer. Hence if n is any integer, we have $f(n-0) = f(n+0) = -1$ and $f(n) = 0$. So the function $f(x)$ has a removable discontinuity at $x = n$, where n is an integer.

Example 4: *Let $y = E(x)$, where $E(x)$ denotes the integral part of x. Prove that the function is discontinuous where x has an integral value. Also draw the graph.*

Solution: From the definition of $E(x)$, we have

$$E(x) = n-1 \quad \text{for} \quad n-1 \leq x < n,$$
$$E(x) = n \quad \text{for} \quad n \leq x < n+1$$
$$E(x) = n+1 \quad \text{for} \quad n+1 \leq x < n+2,$$

and so on where n is an integer.

We consider $x = n$.

Then $E(n) = n$, $E(n-0) = n-1$ and $E(n+0) = n$.

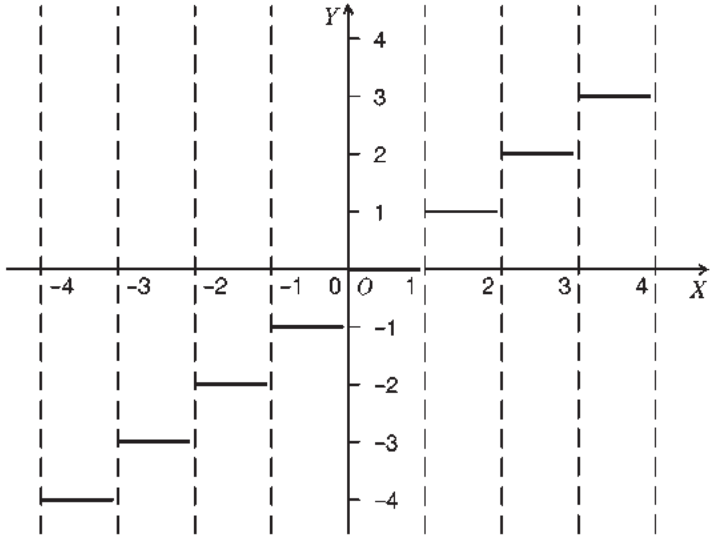
Since $E(n+0) \neq E(n-0)$, the function $E(x)$ is discontinuous at $x = n$ *i.e.,* when x has an integral value.

Evidently it is continuous for all other values of x.

To draw the graph, we put $n = \ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$, so that

$$y = -4, \quad \text{when} \quad -4 \leq x < -3,$$
$$y = -3, \quad \text{when} \quad -3 \leq x < -2,$$
$$y = -2, \quad \text{when} \quad -2 \leq x < -1,$$
$$y = -1, \quad \text{when} \quad -1 \leq x < 0,$$
$$y = 0, \quad \text{when} \quad 0 \leq x < 1$$
$$y = 1, \quad \text{when} \quad 1 \leq x < 2$$
$$y = 2, \quad \text{when} \quad 2 \leq x < 3$$
$$y = 3, \quad \text{when} \quad 3 \leq x < 4$$
$$y = 4, \quad \text{when} \quad 4 \leq x < 5 \text{ and so on.}$$

The graph is shown by thick lines.



Example 5: *Show that the function* ϕ *defined as*

$$\phi(x) = \begin{cases} 0 & \text{for} \quad x = 0 \\ \frac{1}{2} - x & \text{for} \quad 0 < x < \frac{1}{2} \\ \frac{1}{2} & \text{for} \quad x = \frac{1}{2} \\ \frac{3}{2} & \text{for} \quad \frac{1}{2} < x < 1 \\ 1 & \text{for} \quad x = 1 \end{cases}$$

has three points of discontinuity which you are required to find. Also draw the graph of the function.

(Rohilkhand 2009; Avadh 10, 13)

Solution: Here the domain of the function $\phi(x)$ is the closed interval $[0, 1]$.

When $0 < x < \frac{1}{2}$, $\phi(x) = \frac{1}{2} - x$ which is a polynomial in x of degree 1. We know that a polynomial function is continuous at each point of its domain and so $\phi(x)$ is continuous at each point of the open interval $0 < x < \frac{1}{2}$.

Again when $\frac{1}{2} < x < 1$, $\phi(x) = \frac{3}{2} - x$ which is also a polynomial in x and so $\phi(x)$ is also continuous at each point of the open interval $\frac{1}{2} < x < 1$.

Now it remains to test the function $\phi(x)$ for continuity at $x = 0, \frac{1}{2}$ and 1.

(i) For $x = 0$, we have $\phi(0) = 0$,

$$\phi(0+0) = \lim_{h \to 0} \phi(0+h) = \lim_{h \to 0} \phi(h) = \lim_{h \to 0} \left(\frac{1}{2} - h\right) = \frac{1}{2}.$$

Since $\phi(0) \neq \phi(0+0)$, the function $\phi(x)$ is discontinuous at $x = 0$ and the discontinuity is ordinary.

(ii) For $x = \frac{1}{2}$, we have $\phi\left(\frac{1}{2}\right) = \frac{1}{2}$,

$$\phi\left(\frac{1}{2} - 0\right) = \lim_{h \to 0} \phi\left(\frac{1}{2} - h\right) = \lim_{h \to 0} \left[\frac{1}{2} - \left(\frac{1}{2} - h\right)\right],$$

$$\left[\text{Note that } 0 < \frac{1}{2} - h < \frac{1}{2}\right]$$

$$= \lim_{h \to 0} h = 0.$$

Since $\phi\left(\frac{1}{2} - 0\right) \neq \phi\left(\frac{1}{2}\right)$, the function $\phi(x)$ is discontinuous from the left at $x = 1/2$.

Again $\quad \phi\left(\frac{1}{2} + 0\right) = \lim_{h \to 0} \phi\left(\frac{1}{2} + h\right), h > 0$

$$= \lim_{h \to 0} \left[\frac{3}{2} - \left(\frac{1}{2} + h\right)\right] \qquad \left[\because \frac{1}{2} < \frac{1}{2} + h < 1\right]$$

$$= \lim_{h \to 0} (1 - h) = 1 \neq \phi\left(\frac{1}{2}\right) = \frac{1}{2}.$$

Thus the function $\phi(x)$ is discontinuous from the right also at $x = \frac{1}{2}$.

In this way $\phi(x)$ has discontinuity of the first kind *i.e.*, ordinary discontinuity at $x = \frac{1}{2}$ and the jump of the function at $x = 1/2$ is $\phi\left(\frac{1}{2} + 0\right) - \phi\left(\frac{1}{2} - 0\right)$ *i.e.*, $1 - 0$ *i.e.*, 1.

(iii) For $x = 1$, we have $\phi(1) = 1$,

$$\phi(1 - 0) = \lim_{h \to 0} \phi(1 - h)$$

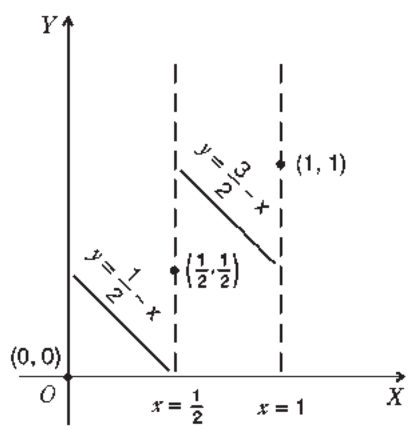
$$= \lim_{h \to 0} [(3/2) - (1 - h)], \qquad \left[\text{Note that } \frac{1}{2} < 1 - h < 1\right]$$

$$= \lim_{h \to 0} \left(\frac{1}{2} + h\right) = \frac{1}{2}.$$

Since $\phi(1) \neq \phi(1 - 0)$, $\phi(x)$ is discontinuous at $x = 1$ and the discontinuity is ordinary.

Hence the function $\phi(x)$ has three points of discontinuity at $x = 0, \frac{1}{2}$ and 1.

The graph of the function consists of the point $(0, 0)$; the segment of the line $y = \frac{1}{2} - x$, $0 < x < \frac{1}{2}$; the point $\left(\frac{1}{2}, \frac{1}{2}\right)$; the segment of the line $y = \frac{3}{2} - x$, $\frac{1}{2} < x < 1$; and the point $(1, 1)$.



Thus the graph is as shown in the figure. From the graph we observe that the function is discontinuous at $x = 0, \frac{1}{2}$ and 1.

Example 6: *Determine the values of a, b, c for which the function*

$$f(x)=\begin{cases}\dfrac{\sin(a+1)x+\sin x}{x} & \text{for } x<0\\ c & \text{for } x=0\\ \dfrac{(x+bx^2)^{1/2}-x^{1/2}}{bx^{3/2}} & \text{for } x>0\end{cases}$$

is continuous at $x=0$.

Solution: Here $f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0}\dfrac{(h+bh^2)^{1/2}-h^{1/2}}{bh^{3/2}}$

$$=\lim_{h\to 0}\frac{(1+bh)^{1/2}-1}{bh}=\lim_{h\to 0}\frac{\{1+\frac{1}{2}bh+\ldots\}-1}{bh}=\frac{1}{2},$$

which is independent of b and so b may have any real value except 0.

Again $f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0}\dfrac{\sin(a+1)(-h)+\sin(-h)}{(-h)}$

$$=\lim_{h\to 0}\frac{\sin(a+1)h+\sin h}{h}$$

$$=\lim_{h\to 0}\frac{2\sin\left(\frac{1}{2}a+1\right)h\cos(ah/2)}{h}$$

$$=\lim_{h\to 0}\frac{\sin\{(a+2)/2\}h}{\{(a+2)/2\}h}(a+2)\cos(ah/2)=a+2.$$

For continuity at $x=0$, we have $f(0+0)=f(0-0)=f(0)$

i.e., $\dfrac{1}{2}=a+2=c.\ \therefore\ c=\dfrac{1}{2}$ and $a=-\dfrac{3}{2}\cdot$

Example 7: *A function* $f(x)$ *is defined as follows :*

$$f(x)=\begin{cases}(x^2/a)-a, & \text{when } x<a\\ 0, & \text{when } x=a\\ a-(a^2/x), & \text{when } x>a.\end{cases}$$

Prove that the function $f(x)$ *is continuous at* $x=a$.

(Bundelkhand 2007; Avadh 09; Rohilkhand 13)

Solution: We have

$$f(a+0)=\lim_{h\to 0} f(a+h)=\lim_{h\to 0}\left[a-\frac{a^2}{(a+h)}\right],$$

$[\because f(x)=a-(a^2/x)$ for $x>a]$

$$=[a-(a^2/a)]=a-a=0;$$

$$f(a-0)=\lim_{h\to 0} f(a-h)=\lim_{h\to 0}\left[\frac{(a-h)^2}{a}-a\right],$$

$[\because f(x)=(x^2/a)-a \text{ for } x<a]$

$$=[(a^2/a)-a]=a-a=0.$$

Also, we have $f(a)=0$.

Since $f(a+0)=f(a-0)=f(a)$, therefore $f(x)$ is continuous at $x=a$.

Example 8: *Examine the function defined below for continuity at* $x=a$ *:*

$$f(x)=\frac{1}{x-a}\,cosec\left(\frac{1}{x-a}\right), x\neq a$$

and $f(x)=0,\ x=a.$ **(Avadh 2004; Lucknow 08)**

Solution: We have $f(a+0)=\lim_{h\to 0} f(a+h)$

$$=\lim_{h\to 0}\frac{1}{a+h-a}\operatorname{cosec}\frac{1}{a+h-a}$$

$$=\lim_{h\to 0}\frac{1}{h\sin(1/h)}$$

$=+\infty$, since $h\sin(1/h)\to 0$ as $h\to 0$.

$$f(a-0)=\lim_{h\to 0} f(a-h)=\lim_{h\to 0}\frac{1}{a-h-a}\operatorname{cosec}\left(\frac{1}{a-h-a}\right)$$

$$=\lim_{h\to 0}-\left[\frac{1}{h}\cdot\frac{1}{\sin\{-(1/h)\}}\right]=\lim_{h\to 0}\frac{1}{h\sin(1/h)}$$

$=+\infty$, since $h\sin(1/h)\to 0$ as $h\to 0$.

Also, we have $f(a)=0$.

Since $f(a+0)=f(a-0)\neq f(a)$, the function $f(x)$ is discontinuous at $x=a$, having an infinite discontinuity of the second kind.

Example 9: *Examine the function defined below for continuity at* $x=0$:

$$f(x)=\frac{sin^2\,ax}{x^2}\ for\ x\neq 0,\ f(x)=1\,for\ x=0.$$

(Lucknow 2006, 07; Meerut 10)

Solution: We have $f(0)=1$;

$$f(0+0)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)$$

$$=\lim_{h\to 0}\frac{\sin^2 ah}{h^2}$$

$$= \lim_{h \to 0} \left(\frac{\sin ah}{ah}\right)^2 . a^2 = 1 . a^2 = a^2 ;$$

and $$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} \frac{\sin^2(-ah)}{(-h)^2}$$

$$= \lim_{h \to 0} \frac{\sin^2 ah}{h^2} = a^2.$$

Since $f(0+0) = f(0-0) \neq f(0)$.

Hence $f(x)$ is discontinuous at $x = 0$.

Example 10: *A function $f(x)$ is defined as follows :*

$$f(x) = 1 + x \text{ if } x \leq 2 \text{ and } f(x) = 5 - x \text{ if } x \geq 2.$$

Is the function continuous at $x = 2$? **(Meerut 2002, 06; Lucknow 09)**

Solution: Here $f(2) = 1 + 2$ or $5 - 2 = 3$;

$$f(2+0) = \lim_{h \to 0} f(2+h), \text{ where } h \text{ is + ive and sufficiently small}$$

$$= \lim_{h \to 0} [5 - (2+h)], \qquad [\because 2 + h > 2 \text{ and } f(x) = 5 - x \text{ if } x > 2]$$

$$= \lim_{h \to 0} (3 - h) = 3 ;$$

and $$f(2-0) = \lim_{h \to 0} f(2-h), \text{ where } h \text{ is + ive and sufficiently small}$$

$$= \lim_{h \to 0} [1 + (2-h)], \qquad [\because 2 - h < 2 \text{ and } f(x) = 1 + x \text{ if } x < 2]$$

$$= \lim_{h \to 0} (3 - h) = 3.$$

Thus $f(2+0) = f(2-0) = f(2)$. Hence the function $f(x)$ is continuous at $x = 2$.

Example 11: *Discuss the continuity of the function $f(x)$ defined as follows:*

$$f(x) = x^2 \text{ for } x < -2, \quad f(x) = 4 \text{ for } -2 \leq x \leq 2, \quad f(x) = x^2 \text{ for } x > 2.$$

Solution: We shall test the continuity of $f(x)$ only at the points $x = -2$ and 2. Obviously it is continuous at all other points.

At $x = -2$. We have $f(-2) = 4$;

$$f(-2+0) = \lim_{h \to 0} f(-2+h) = \lim_{h \to 0} 4 = 4 ;$$

$$f(-2-0) = \lim_{h \to 0} f(-2-h) = \lim_{h \to 0} (-2-h)^2, \qquad [\because -2 - h < -2]$$

$$= 4.$$

Since $f(-2+0) = f(-2-0) = f(-2)$, the function is continuous at $x = -2$.

At $x = 2$. We have $f(2) = 4$;

$$f(2+0) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} (2+h)^2 = 4;$$

$$f(2-0) = \lim_{h \to 0} f(2-h) = \lim_{h \to 0} 4 = 4.$$

Since $f(2+0) = f(2-0) = f(2)$, the function is continuous at $x = 2$.

Comprehensive Exercise 1

1. Discuss the continuity and discontinuity of the following functions :

(i) $f(x) = x^3 - 3x$

(ii) $f(x) = x + x^{-1}$

(iii) $f(x) = e^{-1/x}$

(iv) $f(x) = \sin x$.

(v) $f(x) = \cos(1/x)$ when $x \neq 0$, $f(0) = 0$. **(Lucknow 2005)**

(vi) $f(x) = \sin(1/x)$ when $x \neq 0$, $f(0) = 0$. **(Lucknow 2011)**

(vii) $f(x) = \dfrac{\sin x}{x}$ when $x \neq 0$ and $f(0) = 1$. **(Kanpur 2007; Avadh 08)**

(viii) $f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ when $x \neq 0$ and $f(0) = 1$. **(Meerut 2004B; Kumaun 10)**

(ix) $f(x) = \dfrac{e^{1/x}}{1 + e^{1/x}}$ when $x \neq 0$, $f(0) = 0$. **(Lucknow 2011; Bundelkhand 11)**

(x) $f(x) = \dfrac{x\, e^{1/x}}{1 + e^{1/x}} + \sin(1/x)$ when $x \neq 0$, $f(0) = 0$.

(xi) $f(x) = \sin x \cos(1/x)$ when $x \neq 0$, $f(0) = 0$.

2. (i) Examine at $x = 0$, the continuity of $f(x) = \begin{cases} \dfrac{e^{1/x^2}}{1 - e^{1/x^2}}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0. \end{cases}$

(Meerut 2008)

(ii) If $f(x) = \dfrac{1}{x - a} \sin \dfrac{1}{x - a}$, find $f(a+0)$ and $f(a-0)$.

Is the function continuous at $x = a$?

3. Find out the points of discontinuity of the following functions :

(i) $f(x) = (2 + e^{1/x})^{-1} + \cos e^{1/x}$ for $x \neq 0, f(0) = 0$.

(ii) $f(x) = 1/2^n$ for $1/2^{n+1} < x \leq 1/2^n, n = 0, 1, 2, \ldots$ and $f(0) = 0$.

4. If $f(x) = \dfrac{1}{x} \sin \dfrac{1}{x}$ for $x \neq 0$ and $f(0) = 0$, show that $f(x)$ is finite for every value of x in the interval $[-1, 1]$ but is not bounded. Determine the points of discontinuity of the function if any.

5. A function f defined on $[0, 1]$ is given by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational.} \end{cases}$$

Show that f takes every value between 0 and 1 (both inclusive), but it is continuous only at the point $x = \dfrac{1}{2}$.

(Rohilkhand 2012B)

6. Prove that the function f defined by $f(x) = \begin{cases} \dfrac{1}{2}, & \text{if } x \text{ is rational} \\ \dfrac{1}{3}, & \text{if } x \text{ is irrational} \end{cases}$

is discontinuous everywhere.

7. (i) Show that the function f defined by $f(x) = \dfrac{x\, e^{1/x}}{1 + e^{1/x}}, x \neq 0, f(0) = 1$ is not continuous at $x = 0$ and also show how the discontinuity can be removed. **(Rohilkhand 2006; Lucknow 08; Meerut 11)**

(ii) Show that the function $f(x) = 3x^2 + 2x - 1$ is continuous for $x = 2$.

(iii) Show that the function $f(x) = (1 + 2x)^{1/x}, x \neq 0$,

and $f(x) = e^2, x = 0$ is continuous at $x = 0$.

8. Examine the continuity of the function $f(x) = \begin{cases} -x^2 & \text{if } x \leq 0 \\ 5x - 4 & \text{if } 0 < x \leq 1 \\ 4x^2 - 3x & \text{if } 1 < x < 2 \\ 3x + 4 & \text{if } x \geq 2 \end{cases}$

at $x = 0, 1$ and 2. **(Meerut 2004, 06B, 07B; Lucknow 06; Avadh 06; Purvanchal 06, 10; Gorakhpur 15)**

9. (i) Show that the function

$$f(x) = \frac{e^{1/x} - 1}{e^{1/x} + 1}, x \neq 0 \text{ and } f(0) = 0 \text{ is discontinuous at } x = 0.$$

(ii) Show that the following function is continuous at $x = 0$:

$$f(x) = \frac{\sin^{-1} x}{x}, x \neq 0, f(0) = 1.$$

(Agra 2003)

10. Discuss the continuity of the function $f(x) = \dfrac{1}{1 - e^{1/x}}$ when $x \neq 0$ and $f(0) = 0$ for all values of x. **(Meerut 2004; Rohilkhand 10B; Lucknow 10)**

11. Prove that the function $f(x) = \dfrac{|x|}{x}$ for $x \neq 0, f(0) = 0$ is continuous at all points except $x = 0$. **(Kanpur 2008; Meerut 09; Gorakhpur 11)**

12. Test the continuity of the function $f(x)$ at $x = 0$ if

$$f(x) = \frac{e^{1/x} \sin(1/x)}{1 + e^{1/x}}, x \neq 0 \text{ and } f(x) = 0, x = 0.$$

(Meerut 2005)

13. Examine the following function for continuity at $x = 0$ and at $x = 1$:

$$f(x) = \begin{cases} x^2 & \text{for } x \leq 0 \\ 1 & \text{for } 0 < x \leq 1 \\ 1/x & \text{for } x > 1. \end{cases}$$

(Meerut 2001,03, 04B, 05)

14. Discuss the continuity of the following function at $x = 0$:

$$f(x) = \begin{cases} \cos x, & x \geq 0 \\ -\cos x, & x < 0. \end{cases}$$

15. Test the continuity of the following functions at $x = 0$:

(i) $f(x) = x \cos(1/x)$, when $x \neq 0, f(0) = 0$. **(Meerut 2007)**

(ii) $f(x) = x \log x$, for $x > 0, f(0) = 0$.

16. Discuss the nature of discontinuity at $x = 0$ of the function $f(x) = [x] - [-x]$ where $[x]$ denotes the integral part of x.

17. Discuss the continuity of $f(x) = (1/x) \cos(1/x)$.

18. Give an example of each of the following types of functions :

(i) The function which possesses a limit at $x = 1$ but is not defined at $x = 1$.

(ii) The function which is neither defined at $x = 1$ nor has a limit at $x = 1$.

(iii) The function which is defined at two points but is nevertheless discontinuous at both the points.

19. In the closed interval $[-1, 1]$ let f be defined by

$$f(x) = x^2 \sin(1/x^2) \text{ for } x \neq 0 \text{ and } f(0) = 0.$$

In the given interval (i) Is the function bounded ? (ii) Is it continuous ?

Answers 1

1. (i) Continuous for all x (ii) Discontinuous at $x=0$
(iii) Discontinuous at $x=0$ (iv) Continuous for all x
(v) Discontinuous at $x=0$ (vi) Discontinuous at 0
(vii) Continuous for all x (viii) Discontinuous at 0
(ix) Discontinuous at 0 (x) Discontinuous at 0
(xi) Continuous for all x
2. (i) Discontinuous at $x=0$
(ii) No, it has a discontinuity of second kind. Here both $f(a+0)$ and $f(a-0)$ do not exit
3. (i) Discontinuous at $x=0$ (ii) Discontinuous at $x=1/2^n, = 1,2,3,...$
4. Discontinuous at 0
8. Continuous at $x=1,2$ and discontinuous at $x=0$
10. Discontinuous only at $x=0$ and the discontinuity is ordinary
12. Discontinuity of the second kind at $x=0$
13. Discontinuous at $x=0$ and continuous at $x=1$
14. Discontinuous at $x=0$
15. (i) Continuous (ii) Continuous
16. Discontinuity of the first kind
17. Continuous for all x, except at $x=0$ where it has discontinuity of the second kind
18. (i) $f(x)=x^2$ for $x>1$, $f(x)=x^3$ for $x<1$
(ii) $f(x)=-x^2$ for $x<1$, $f(x)=x^2$ for $x>1$
(iii) $f(x)=0$ for $x\le 0$, $f(x)=\frac{3}{2}-x$ for $0<x\le\frac{1}{2}$, $f(x)=\frac{3}{2}+x$ for $x>\frac{1}{2}$
19. (i) Yes (ii) Yes

5 Criteria for Continuity or Equivalent Definitions of Continuity

Theorem 1: (Heine's definition of continuity). Sequential Continuity: *The necessary and sufficient condition for a function f defined on I $\subset \mathbf{R}$ to be continuous at $a \in I$ is that for each sequence $<a_n>$ in I which converges to a, we have*

$$\lim_{n\to\infty} f(a_n) = f(a).$$

Proof: The condition is necessary: Let f be continuous at a and let $<a_n>$ be a sequence in I such that $\lim_{n\to\infty} a_n = a$.

Let ε be any positive number. Since f is continuous at a, therefore, for a given $\varepsilon>0$, we can find a number $\delta>0$ such that

$$|f(x)-f(a)|<\varepsilon \text{ whenever } |x-a|<\delta. \qquad \ldots(1)$$

Also, since $\lim_{n \to \infty} a_n = a$, therefore, there exists a positive integer m such that

$$|a_n - a| < \delta \text{ whenever } n > m. \quad \ldots(2)$$

Setting $x = a_n$ in (1), we get

$$|f(a_n) - f(a)| < \varepsilon \text{ whenever } |a_n - a| < \delta. \quad \ldots(3)$$

From (2) and (3), we get

$$|f(a_n) - f(a)| < \varepsilon \text{ whenever } n > m.$$

$$\therefore \quad \lim_{n \to \infty} f(a_n) = f(a).$$

Hence the condition is necessary.

The condition is sufficient:

Suppose for every sequence $< a_n >$ in I converging to a, we have

$$\lim_{n \to \infty} f(a_n) = f(a).$$

Then we have to show that f is continuous at a.

Let us suppose that f is not continuous at a. Then there exists an $\varepsilon > 0$ such that for every $\delta > 0$ there is an x such that $|x - a| < \delta$ but $|f(x) - f(a)| \geq \varepsilon$.

If we take $\delta = 1/n$, we find that for each positive integer n, there is an a_n such that

$$|a_n - a| < 1/n \text{ but } |f(a_n) - f(a)| \geq \varepsilon.$$

Then $\lim_{n \to \infty} a_n = a$, but $f(a_n)$ does not converge to $f(a)$ *i.e.,*

$$\lim_{n \to \infty} f(a_n) \neq f(a).$$

But this is a contradiction. Hence f must be continuous at $x = a$.

Theorem 2: *A function $f : \mathbf{R} \to \mathbf{R}$ is continuous iff for every open set G in $\mathbf{R}$, the inverse image $f^{-1}(G)$ is an open set in $\mathbf{R}$.*

Proof: The 'only if' part: Let f be continuous and let G be any open set in $\mathbf{R}$. If $f^{-1}(G)$ is empty, then it is open. If $f^{-1}(G)$ is not empty, let $a \in f^{-1}(G)$. Then $f(a) \in G$. Since G is an open set containing $f(a)$, therefore, there exists an $\varepsilon > 0$ such that

$$]f(a) - \varepsilon, f(a) + \varepsilon[\subset G.$$

Now f is continuous at a, so we can find a number $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

i.e., $\quad x \in]a - \delta, a + \delta[\Rightarrow f(x) \in]f(a) - \varepsilon, f(a) + \varepsilon[\subset G$

or $\quad x \in]a - \delta, a + \delta[\Rightarrow f(x) \in G$

$$\Rightarrow x \in f^{-1}(G)$$

Hence $]a - \delta, a + \delta[\subset f^{-1}(G)$. Thus $f^{-1}(G)$ is a neighbourhood of a. Since a is any point of $f^{-1}(G)$, therefore, it follows that $f^{-1}(G)$ is open.

The 'if' part: Let the inverse image $f^{-1}(G)$ of every open set G be open. To show that f is continuous. Consider any point $a \in \mathbf{R}$. Let $\varepsilon > 0$ be given. Then $]f(a)-\varepsilon, f(a)+\varepsilon[$ is an open set containing $f(a)$ and hence by hypothesis $f^{-1}(]f(a)-\varepsilon, f(a)+\varepsilon[)$ is an open set containing a. Consequently, we can find a number $\delta > 0$ such that

$$]a-\delta, a+\delta[\subset f^{-1}(]f(a)-\varepsilon, f(a)+\varepsilon[), \text{ so that}$$

$$f(]a-\delta, a+\delta[) \subset]f(a)-\varepsilon, f(a)+\varepsilon[.$$

Thus, for any given $\varepsilon > 0$ we have found a number $\delta > 0$ such that

$$x \in]a-\delta, a+\delta[\Rightarrow f(x) \in]f(a)-\varepsilon, f(a)+\varepsilon[\text{ i.e.,}$$

$$|x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon.$$

This shows that f is continuous at a. Since a is any point of $\mathbf{R}$, hence, f is continuous on $\mathbf{R}$.

Theorem 3: *A function $f : \mathbf{R} \to \mathbf{R}$ is continuous on $\mathbf{R}$ iff for every closed set H in $\mathbf{R}$, $f^{-1}(H)$ is closed in $\mathbf{R}$.*

Proof: First, let f be continuous and let H be any closed set in $\mathbf{R}$. Then $\mathbf{R} \sim H$ is an open set in $\mathbf{R}$, so that by the preceding theorem, $f^{-1}(\mathbf{R} \sim H)$ is an open set in $\mathbf{R}$. Since

$$f^{-1}(\mathbf{R} \sim H) = \mathbf{R} \sim f^{-1}(H),$$

hence it follows that $\mathbf{R} \sim f^{-1}(H)$ is an open set in $\mathbf{R}$ and consequently $f^{-1}(H)$ is a closed set in $\mathbf{R}$.

Conversely, let $f^{-1}(H)$ be closed for every closed set H. To show that f is continuous. Let G be any open set in $\mathbf{R}$. Then $\mathbf{R} \sim G$ is a closed set in $\mathbf{R}$ and hence by hypothesis $f^{-1}(\mathbf{R} \sim G)$ is a closed set in $\mathbf{R}$.

Since $f^{-1}(\mathbf{R} \sim G) = \mathbf{R} \sim f^{-1}(G)$, therefore, this means that $\mathbf{R} \sim f^{-1}(G)$ is a closed set in $\mathbf{R}$ and hence $f^{-1}(G)$ is an open set in $\mathbf{R}$. Thus $f^{-1}(G)$ is open in $\mathbf{R}$ whenever G is open in $\mathbf{R}$. Therefore f is continuous by the preceding theorem.

Illustrative Examples

Example 12: *A function f defined on $[0,1]$ is given by*

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational}. \end{cases}$$

Show that f takes every value between 0 and 1 (both inclusive), but it is continuous only at the point $x = \frac{1}{2}$.

Solution: Let $c \in [0,1]$.

If c is rational, then $f(c) = c$.

If c is irrational, then $1-c$ is also irrational

and $\quad 0 < 1-c < 1$ i.e., $1-c \in [0,1]$.

We have $f(1-c)=1-(1-c)=c$.
Thus f takes every value c in $[0,1]$.
Now to show that f is continuous only at the point $x=\frac{1}{2}\cdot$

Let x_0 be any point of $[0,1]$. For each positive integer n we select a rational number a_n and an irrational number b_n, both in $[0,1]$, such that

$$|a_n - x_0| < 1/n, |b_n - x_0| < 1/n.$$

$$\therefore \qquad \lim_{n\to\infty} a_n = x_0 = \lim_{n\to\infty} b_n.$$

If f is to be continuous at x_0, then we must have

$$\lim_{n\to\infty} f(a_n) = f(x_0) = \lim_{n\to\infty} f(b_n).$$

Now $f(a_n) = a_n$ for all n and $f(b_n) = 1 - b_n$ for all n.

$$\therefore \qquad \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} a_n = x_0$$

and

$$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} (1-b_n) = 1 - x_0.$$

So for f to be continuous at x_0, we must have

$$x_0 = f(x_0) = 1 - x_0 \quad i.e., \quad x_0 = \frac{1}{2}\cdot$$

Thus $x=\frac{1}{2}$ is the only possible point of [0, 1] where f can be continuous.
Now we shall show that f is actually continuous at the point $x = 1/2$.
We have $f(1/2) = \frac{1}{2}\cdot$
Let $\varepsilon > 0$ be given.
Take a positive real number $\delta = \frac{1}{2}\varepsilon$. Then if x is rational, we have

$$\left|x-\frac{1}{2}\right| < \delta \Rightarrow \left|f(x) - f\left(\frac{1}{2}\right)\right| = \left|x - \frac{1}{2}\right| < \delta = \frac{1}{2}\varepsilon < \varepsilon$$

and if x is irrational, we have

$$\left|x-\frac{1}{2}\right| < \delta \Rightarrow \left|f(x) - f\left(\frac{1}{2}\right)\right| = \left|(1-x) - \frac{1}{2}\right| = \left|x - \frac{1}{2}\right| < \delta = \frac{1}{2}\varepsilon < \varepsilon.$$

Thus, we have

$$\left|x-\frac{1}{2}\right| < \delta \Rightarrow \left|f(x) - f\left(\frac{1}{2}\right)\right| < \varepsilon,$$

so that f is continuous at $x = \frac{1}{2}\cdot$
Hence f is continuous only at the point $x = 1/2$.

Example 13: *Let f be a function defined on* $]0,1[$ *by setting* $f(x) = 0$ *when x is irrational and* $f(x) = 1/q$ *when x is a rational number of the form* p/q *where p and q are positive integers having no factor in common. Show that f is continuous at each irrational point and discontinuous at each rational point.*

Solution: First, let $a = p/q$ be any rational number in $]0,1[$, where p and q are positive integers having no factor in common. For each positive integer n, we choose a positive irrational number x_n such that $|x_n - a| < 1/n$. Then the sequence $< x_n >$ converges to a. Also it is given that $f(x_n) = 0$ for each n, so that $\lim_{n \to \infty} f(x_n) = 0 \neq f(a)$, since $f(a) = f(p/q) = 1/q \neq 0$.

It follows from the preceding theorem that f is not continuous at a rational point.

Next, let b be an irrational number in $]0,1[$ so that $f(b) = 0$. Let $\varepsilon > 0$ be given. Choose a positive integer n such that $1/n < \varepsilon$. Evidently there can be only a finite number of rational numbers p/q in $]0,1[$ such that $q < n$. Hence, we can find a number $\delta > 0$ such that no rational number in $]b-\delta, b+\delta[\subset]0,1[)$ has its denominator less than n.

Thus we have shown that

$$|x - b| < \delta \Rightarrow |f(x) - f(b)| = |f(x)| = 0 < \varepsilon, \quad \ldots(1)$$

if x is irrational

and

$$|x - b| < \delta \Rightarrow |f(x) - f(b)| = |f(x)| \leq 1/n < \varepsilon, \quad \ldots(2)$$

if x is rational.

From (1) and (2), it follows that $|f(x) - f(b)| < \varepsilon$ whenever $|x - b| < \delta$.

Hence f is continuous at b.

6 Boundedness and Intermediate Value Properties of Continuous Functions

Theorem 1: (Borel's Theorem) *If f is a continuous function on the closed interval $[a,b]$, then the interval can always be divided up into a finite number of subintervals such that, given $\varepsilon > 0$, $|f(x_1) - f(x_2)| < \varepsilon$, where x_1 and x_2 are any two points in the same sub-interval.*

Proof: Let us assume that the theorem is false. Then for any mode of sub-division, there must be at least one of the subintervals for which the theorem is false. Let us divide $[a,b]$ into two equal parts. Let c be the point of division. Then the theorem must be false in at least one of the two parts. Suppose it is false in $[c,b]$. Renaming the interval $[c,b]$ as $[a_1, b_1]$, we divide $[a_1, b_1]$ into two equal parts. Again, the theorem must be false in at least one of these two parts. Continuing this process of repeated bisection indefinitely, we get a sequence of closed intervals

$[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n], \ldots$ such that

$$a \leq a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots \leq b_n \leq \ldots \leq b_2 \leq b_1 \leq b, \quad \ldots(1)$$

$$b_1 - a_1 = \frac{1}{2}(b-a), b_2 - a_2 = \frac{1}{2}(b_1 - a_1) = \frac{1}{2^2}(b-a), \ldots\ldots,$$

$$b_n - a_n = \frac{1}{2^n}(b-a), \ldots \quad \ldots(2)$$

and the theorem is false for each interval $[a_n, b_n]$. ...(3)

From (1), we find that the sequence $< a_n >$ is increasing and bounded above by b and $< b_n >$ is decreasing and bounded below by a. Hence both $< a_n >$ and $< b_n >$ are convergent. So there exist $l_1, l_2 \in \mathbf{R}$ such that $\lim a_n = l_1$ and $\lim b_n = l_2$.

From (2), $\lim (b_n - a_n) = \lim \frac{1}{2^n}(b-a) = (b-a) \lim \frac{1}{2^n} = (b-a) . 0 = 0$

i.e., $\lim b_n - \lim a_n = 0$ *i.e.,* $l_2 - l_1 = 0$ *i.e.,* $l_2 = l_1$.

Now $l_1 = \sup < a_n >,$

$< a_n >$ being an increasing and bounded above sequence

$= \inf < b_n >,$

$< b_n >$ being a decreasing and bounded below sequence.

$\therefore$ $a_n \le l_1$ and $l_1 \le b_n$ *i.e*, $a_n \le l_1 \le b_n \;\forall n \in \mathbf{N}$

$\Rightarrow$ $a \le l_1 \le b$ *i.e.,* either $a < l_1 < b$ or $a = l_1$ or $b = l_1$.

Case I: $a < l_1 < b$. Since f is continuous in $[a, b]$, it is also continuous at $x = l_1$. Then for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(l_1)| < \frac{1}{2}\varepsilon, \text{ whenever } |x - l_1| < \delta.$$

Now we can choose n so large that the interval $[a_n, b_n]$ lies completely inside the interval $]l_1 - \delta, l_1 + \delta[$.

Hence if x_1 and x_2 are any two points in $[a_n, b_n]$, then

$$|f(x_1) - f(l_1)| < \varepsilon/2, |f(x_2) - f(l_1)| < \varepsilon/2.$$

Now $|f(x_1) - f(x_2)| = |f(x_1) - f(l_1) + f(l_1) - f(x_2)|$

$\le |f(x_1) - f(l_1)| + |f(l_1) - f(x_2)|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon \Rightarrow$ the result is true in $[a_n, b_n]$.

This is a contradiction to the fact (3) and so our initial assumption that the theorem is not true is wrong. Hence the theorem must be true.

Similarly we can prove the cases for $l_1 = a$ or $l_1 = b$.

Note: The possibility $l_1 = a$ or $l_1 = b$ shows that the interval considered in the theorem must be closed so that f is continuous at l_1.

Theorem 2: (Boundedness Theorem) *If a function $f(x)$ is continuous in a closed interval $[a, b]$, then it is bounded in that interval.*

Proof: By the above theorem, for a given $\varepsilon > 0$, we can sub-divide the interval $[a, b]$ into a finite number of sub-intervals say $[a = a_0, a_1], [a_1, a_2], \ldots, [a_{n-1}, a_n = b]$ such that

$$|f(x_1) - f(x_2)| < \varepsilon \quad \ldots(1)$$

for any two points x_1, x_2 in the same sub-interval. Let x be any point in the first sub-interval $[a, a_1]$. Then by (1), we have, $\forall\, x \in [a, a_1]$

$$|f(x) - f(a)| < \varepsilon \text{ i.e., } f(a) - \varepsilon < f(x) < f(a) + \varepsilon. \quad \ldots(2)$$

In particular, for $x = a_1, |f(a_1) - f(a)| < \varepsilon.$ ...(3)

Again $\forall\, x \in [a_1, a_2], |f(x) - f(a_1)| < \varepsilon.$ ...(4)

$\therefore$ $\forall\ x \in [a_1, a_2]$, we have

$$|f(x) - f(a)| = |f(x) - f(a_1) + f(a_1) - f(a)|$$
$$\leq |f(x) - f(a_1)| + |f(a_1) - f(a)|$$
$$< \varepsilon + \varepsilon, \text{ from (3) and (4)}$$
$$= 2\varepsilon.$$

Thus $\forall\ x \in [a_1, a_2]$, we have $|f(x) - f(a)| < 2\varepsilon$

i.e., $$f(a) - 2\varepsilon < f(x) < f(a) + 2\varepsilon. \qquad \ldots(5)$$

From (2) and (5), we see that all the values of $f(x)$ in the first two sub-intervals lie between

$$f(a) - 2\varepsilon \text{ and } f(a) + 2\varepsilon.$$

Proceeding in the same way, we can show that $\forall\, x \in [a_{n-1}, a_n = b]$, we have

$$f(a) - n\varepsilon < f(x) < f(a) + n\varepsilon.$$

Hence all the values of $f(x)$ in the interval $[a, b]$ will lie between $f(a) - n\varepsilon$ and $f(a) + n\varepsilon$.

Thus $f(x)$ is bounded in $[a, b]$.

Note: A bounded function in $[a, b]$ need not be continuous in $[a, b]$. For example, the function

$$f(x) = \sin(1/x) \text{ for } x \neq 0,\ f(0) = 0$$

is bounded in $[0, 1]$ but not continuous in $[0, 1]$ since it is discontinuous at $x = 0$.

Theorem 3: (The Mostest Theorem) *If a function $f(x)$ is continuous in $[a, b]$, then it attains its supremum and infimum at least once in $[a, b]$.*

Proof: Since $f(x)$ is continuous in $[a, b]$, $f(x)$ is bounded in $[a, b]$. Let M and m be the supremum and infimum of f in $[a, b]$ respectively. We shall show that f attains its supremum M at least once in this interval. Let, if possible, $f(x)$ does not attain M, then

$$f(x) \neq M\ \ \forall\, x \in [a, b] \quad \text{i.e.,} \quad M - f(x) \neq 0\ \forall\, x \in [a, b].$$

Now, M, being a constant, is always a continuous function and f is given to be continuous in $[a, b]$.

$\therefore$ $$M - f(x) \text{ is also continuous in } [a, b]$$

$\Rightarrow$ $$\frac{1}{M - f(x)} \text{ is also continuous in } [a, b] \quad [\because M - f(x) \neq 0\ \forall\, x \in [a, b]]$$

Consequently $1/\{M - f(x)\}$ is bounded in $[a, b]$. Let $G \in \mathbf{R}_+$ be its upper bound in $[a, b]$ so that $\forall\ x \in [a, b]$

$$\frac{1}{M - f(x)} \leq G \text{ i.e., } M - f(x) \geq \frac{1}{G} \quad \text{i.e.,} \quad f(x) \leq M - (1/G) < M$$

i.e., $$M - (1/G) < M \text{ is also an upper bound of } f.$$

But this contradicts the fact that M is the supremum of f in $[a, b]$. Hence $f(x) = M$ for at least one value of x in $[a, b]$. Similarly we can show that f attains its infimum at least once in $[a, b]$.

Theorem 4: *If f is continuous at $x = x_0$ where $f(x_0) \neq 0$, then a positive number δ can be found such that $f(x)$ has the same sign as $f(x_0)$ for every value of x in $]x_0 - \delta, x_0 + \delta[$.*

Proof: Since f is continuous at $x = x_0$, hence for a given $\varepsilon > 0$, we can find a number $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$, whenever $|x - x_0| < \delta$

i.e., $\quad f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$, whenever $x_0 - \delta < x < x_0 + \delta$.

Now $\quad f(x_0) \neq 0 \Rightarrow |f(x_0)| > 0$.

If we choose ε such that $0 < \varepsilon < |f(x_0)|$, we see that $f(x_0) - \varepsilon$ and $f(x_0) + \varepsilon$ have the same sign as $f(x_0)$. It implies that $f(x)$ has the same sign as $f(x_0)$ for all x in the interval $]x_0 - \delta, x_0 + \delta[$.

Theorem 5: (**(Bolzano's Theorem)**) *If f is continuous in $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then there is at least one value of x for which $f(x)$ vanishes.*

Proof: For definiteness, let $f(a) < 0$ and $f(b) > 0$. Consider a subset S of $[a, b]$ defined as follows :

$$S = \{x : a \leq x \leq b \text{ and } f(x) < 0\}.$$

Then $S \neq \varnothing$, since $a \in S$, $f(a)$ being < 0. By definition, b is an upper bound of S. It follows that S has a supremum, say u, by the completeness property of real numbers. Obviously $a \leq u \leq b$. Now, we shall show that $f(u) = 0$.

Step I: First we shall show that $u \neq a$. Since $f(a) < 0$, we can find a positive number δ such that $f(x) < 0$ whenever $a \leq x < a + \delta$.

It follows that $[a, a + \delta[$ S and hence the supremum of S must be greater than or equal to $a + \delta$.

$\therefore \quad u \geq a + \delta$ *i.e.,* $u \neq a$.

Step II: We shall show that $u \neq b$. In fact, since $f(b) > 0$, therefore, there exists a positive number δ_1 such that $f(x) > 0$ whenever $b - \delta_1 < x \leq b$.

It gives that $b - \delta_1$ is an upper bound of S, and hence

$$u = \sup S \leq b - \delta_1 < b \quad \textit{i.e., } u < b.$$

Step III: We shall show that $f(u) \not> 0$. Since $a < u < b$, therefore, if $f(u) > 0$, then we can find a positive number δ_2 such that $f(x) > 0$ whenever

$$u - \delta_2 < x < u + \delta_2 .$$

Also, since $u = \sup S$, there exists $x_0 \in S$ such that $u - \delta_2 < x_0 \leq u$. This means that $f(x_0) > 0$.

Again, $x_0 \in S \Rightarrow f(x_0) < 0$, by definition of S. This contradiction implies that $f(u) \not> 0$.

Step IV: We shall show that $f(u) \not< 0$. Since, if $f(u) < 0$, then we can find a positive number δ_3 such that $u + \delta_3 < b$ and $f(x) < 0$ whenever $u - \delta_3 < x < u + \delta_3$.

If x_1 is any point such that $u < x_1 < u + \delta_3$, then $f(x_1) < 0$. But this is a contradiction to the fact that u is the supremum of all those points of $[a, b]$ for which f is negative. Consequently $f(u) \not< 0$.

It follows from steps III and IV that $f(u) = 0$.

Theorem 6: (The Intermediate Value Theorem) *If a function f is continuous in the closed interval $[a, b]$, then $f(x)$ must take at least once all values between $f(a)$ and $f(b)$.*

Proof: Let $f(a) < d < f(b)$. Let us define a function F such that

$$F(x) = f(x) - d.$$

Since *f* is continuous in the closed interval $[a, b]$, F must also be continuous in $[a, b]$. Also $F(a) < 0$ and $F(b) > 0$ *i.e.,* $F(a)$ and $F(b)$ are of opposite signs. It follows that there exists x_0 in $]a, b[$ such that $F(x_0) = 0$ *i.e.,* $f(x_0) - d = 0$ or $f(x_0) = d$.

Since d is any value between $f(a)$ and $f(b)$ it follows that *f* takes all values between $f(a)$ and $f(b)$ at least once.

The converse of the above theorem is not true. For example, let *f* be the function defined as $f(x) = \sin(1/x), x \neq 0$ and $f(0) = 0$.

In the interval $[-2/\pi, 2/\pi]$ this function takes all values between $f(-2/\pi)$ and $f(2/\pi)$ *i.e.,* between -1 and 1 an infinite number of times as x varies from $-2/\pi$ to $2/\pi$, but this function is not continuous in $[-2/\pi, 2/\pi]$ as it is discontinuous at $x = 0$.

Corollary 1: *Let f be continuous on $[a, b]$ and let $k \in [m, M]$ where $m = \inf f$ and $M = \sup f$ on $[a, b]$. Then there exists $c \in [a, b]$ such that $f(c) = k$.*

Proof: Since *f* is continuous on $[a, b]$ and every function defined and continuous on a closed interval attains its supremum and infimum, therefore, there exist $x_1, x_2 \in [a, b]$ such that

$$m = f(x_1) \text{ and } M = f(x_2).$$

If $x_1 = x_2$, then *f* is a constant function on $[a, b]$ and the result follows.

Let $x_1 < x_2$. Then $[x_1, x_2] \subset [a, b]$ and *f* is continuous on $[x_1, x_2]$.

Hence by the above theorem there exists $c \in [x_1, x_2] \subset [a, b]$ such that $f(c) = k$.

Similarly we can prove the result if $x_1 > x_2$.

Corollary 2: *Let f be continuous on $[a, b]$. Then $f([a, b]) = [m, M]$, where $m = \inf f$ and $M = \sup f$ on $[a, b]$ and thus $f([a, b])$ is a closed set.*

Proof: By Corollary 1, f takes all values between m and M and hence $[m, M] \subset f([a, b])$. Since every value of *f* on $[a, b]$ lies between m and M, hence, $f([a, b]) \subset [m, M]$. Thus $f([a, b]) = [m, M]$ which is a closed set because every closed interval is a closed set.

7 Uniform Continuity

Let a function *f* be continuous for every value of x in $[a, b]$. It means that if $x_0 \in [a, b]$, then, given $\varepsilon > 0$, there exists $\delta > 0$, such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$.

The number δ will depend upon x_0 as well as ε and so we may write it symbolically as $\delta(\varepsilon, x_0)$. For some functions it may happen that given $\varepsilon > 0$ the same δ serves for all $x_0 \in [a, b]$ in the condition of continuity. Such functions are called uniformly continuous on $[a, b]$.

Definition: *A function f defined on an interval I is said to be uniformly continuous on I if given* $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that* $|f(x) - f(y)| < \varepsilon$ *whenever* $|x - y| < \delta$*, where x, y are in I.*

(Meerut 2001)

It should be noted carefully that uniform continuity is a property associated with an interval and not with a single point. The concept of continuity is local in character whereas the concept of uniform continuity is global in character.

Note: A function f is not uniformly continuous on I, if there exists some $\varepsilon > 0$ for which no $\delta > 0$ serves *i.e.,* for any $\delta > 0$, there exist $x, y \in I$ such that $|f(x) - f(y)| \geq \varepsilon$ and $|x - y| < \delta$.

The following two theorems express the relation between continuity and uniform continuity. The first one gives that uniform continuity always implies continuity. The second one gives a sufficient condition under which continuity implies uniform continuity.

Theorem 1: *If f is uniformly continuous on an interval I, then it is continuous on I.*

Proof: Let x_0 be any point of I and let $\varepsilon > 0$ be given. Since f is uniformly continuous on I, therefore, given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta, \ \forall x, y \text{ in } I.$$

Taking $y = x_0$, we have in particular,

$$|f(x) - f(x_0)| < \varepsilon \text{ whenever } |x - x_0| < \delta \ \ \forall x \text{ in } I.$$

This means that f is continuous at x_0.

Since x_0 is an arbitrary point of I, therefore f is continuous at every point of I.

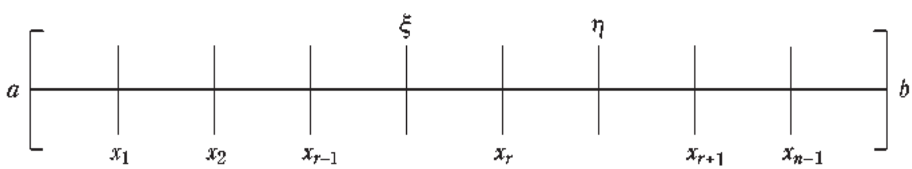
Hence f is continuous on I.

Theorem 2: *A function which is continuous in a closed and bounded interval* $I = [a, b]$ *is uniformly continuous in* $[a, b]$*.*

Proof: Take any given $\varepsilon > 0$. By theorem 1 of article 3, the interval $[a, b]$ can be divided up into sub-intervals $[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]$ such that for any two points α, β in the same sub-interval, we have $|f(\alpha) - f(\beta)| < \frac{1}{2}\varepsilon$. ...(1)

Let δ be a positive number which does not exceed the least of the numbers

$$x_1 - a, x_2 - x_1, \ldots, b - x_{n-1}.$$



Let ξ, η be any two points in $[a, b]$ such that $|\xi - \eta| \not> \delta$.

If these two points are in the same sub-interval, then by (1), we have

$$|f(\xi) - f(\eta)| < \frac{1}{2}\varepsilon.$$

If ξ, η do not lie in the same sub-interval, then surely they lie one in each of the two consecutive intervals. Let x_r be the point of division such that $x_{r-1} < \xi < x_r < \eta < x_{r+1}$. Then we have

$$\begin{aligned} |f(\xi) - f(\eta)| &= |f(\xi) - f(x_r) + f(x_r) - f(\eta)| \\ &\leq |f(\xi) - f(x_r)| + |f(x_r) - f(\eta)| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \text{ by (1).} \end{aligned}$$

Thus we have shown that, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(\xi) - f(\eta)| < \varepsilon \text{ whenever } |\xi - \eta| < \delta, \ \forall \xi, \eta \text{ in } [a, b].$$

Hence f is uniformly continuous in $[a, b]$.

Illustrative Examples

Example 14: *Give an example to show that a function continuous in an open interval may fail to be uniformly continuous in the interval.*

Solution: Consider the function f defined on the open interval $]0,1[$ as follows :

$$f(x) = 1/x, \forall x \in]0,1[.$$

First we shall show that f is continuous in $]0,1[$ *i.e.*, f is continuous at each point c in $]0,1[$.

We have $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} \frac{1}{x} = \frac{1}{c} = f(c)$.

So $f(x)$ is continuous at each point c in $]0,1[$ and hence $f(x)$ is continuous in $]0,1[$.

Now we shall show that $f(x)$ is not uniformly continuous in $]0,1[$.

For any $\delta > 0$, we can find a positive integer m such that $1/m < \delta$.

Let $x_1 = 1/m$ and $x_2 = 1/2m$. Then $0 < x_1 < 1$ and $0 < x_2 < 1$ so that $x_1, x_2 \in]0,1[$.

We have $|x_1 - x_2| = \left|\frac{1}{m} - \frac{1}{2m}\right| = \left|\frac{1}{2m}\right| = \frac{1}{2m} < \frac{1}{m} < \delta$

and $|f(x_1) - f(x_2)| = \left|\frac{1}{x_1} - \frac{1}{x_2}\right| = |m - 2m| = |-m| = m > \frac{1}{2}$.

[$\because$ m is a + ive integer]

Thus if we take $\varepsilon = \frac{1}{2} > 0$, then what ever $\delta > 0$ we try there exist $x_1, x_2 \in]0,1[$ such that

$$|x_1 - x_2| < \delta \text{ but } |f(x_1) - f(x_2)| > \varepsilon = \frac{1}{2}.$$

In this way for $\varepsilon = \frac{1}{2} > 0$, there exists no $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \varepsilon \text{ whenever } |x_1 - x_2| < \delta, x_1, x_2 \in]0,1[.$$

Hence $f(x) = 1/x$ is not uniformly continuous in $]0,1[$.

Example 15: *Prove that the function f defined on* $\mathbf{R}^+$ *as* $f(x) = \sin\frac{1}{x}$, $\forall\ x > 0$ *is continuous but not uniformly continuous on* $\mathbf{R}^+$.

Solution: Let a be any positive real number.

We have $$f(a-0) = \lim_{h\to 0} f(a-h) = \lim_{h\to 0} \sin\frac{1}{a-h} = \sin\frac{1}{a},$$

$$f(a+0) = \lim_{h\to 0} f(a+h) = \lim_{h\to 0} \sin\frac{1}{a+h} = \sin\frac{1}{a}\cdot$$

Also $$f(a) = \sin\frac{1}{a}\cdot$$

Since $f(a+0) = f(a-0) = f(a)$, f is continuous at a.

But a is an arbitrary point of $\mathbf{R}^+$, so f is continuous on $\mathbf{R}^+$.

It remains to show that f is not uniformly continuous on $\mathbf{R}^+$.

We shall show that no δ works for $\varepsilon = \frac{1}{2}\cdot$

Let δ be any positive number. Take $x_1 = \frac{1}{n\pi}$, $x_2 = \frac{1}{n\pi + (\pi/2)} = \frac{2}{(2n+1)\pi}$,

where n is a positive integer such that $x_1 - x_2 = \frac{1}{n\pi} - \frac{2}{(2n+1)\pi} < \delta\cdot$

(We can always choose such a positive integer n for each positive δ.)

Now $|x_1 - x_2| < \delta$ but $|f(x_1) - f(x_2)| = |\sin n\pi - \sin\frac{1}{2}(2n+1)\pi| = 1 > \varepsilon$.

This shows that for this choice of ε, we are unable to find $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \varepsilon \text{ whenever } |x_1 - x_2| < \delta, \quad \forall\, x_1, x_2 \in \mathbf{R}^+.$$

Hence f is not uniformly continuous on $\mathbf{R}^+$.

Example 16: *Show that the function f defined by* $f(x) = x^3$, *is uniformly continuous in* $[-2, 2]$.

Solution: Let $x_1, x_2 \in [-2, 2]$. We have

$$|f(x_2) - f(x_1)| = |x_2^3 - x_1^3| = |(x_2 - x_1)(x_2^2 + x_1^2 + x_1x_2)|$$

$$\le |x_2 - x_1| \{|x_2|^2 + |x_1|^2 + |x_1||x_2|\}$$

$$\le 12|x_2 - x_1|.$$

$$[\because x_1, x_2 \in [-2, 2] \Rightarrow |x_1| \le 2 \text{ and } |x_2| \le 2]$$

$\therefore$ $$|f(x_2) - f(x_1)| < \varepsilon \text{ whenever } |x_2 - x_1| < \varepsilon/12.$$

Thus given $\varepsilon > 0$, there exists $\delta = \varepsilon/12$ such that

$$|f(x_2) - f(x_1)| < \varepsilon \text{ whenever } |x_2 - x_1| < \delta\ \forall\ x_1, x_2 \in [-2, 2].$$

Hence $f(x)$ is uniformly continuous in $[-2, 2]$.

Example 17: *Let $f:\mathbf{R}\to\mathbf{R}$ given by $f(x)=x^2$. Show that f is not uniformly continuous on* **R**.

Solution: Let $\varepsilon>0$ be given. The function $f(x)$ will be uniformly continuous on **R** if we are able to find $\delta>0$ such that

$$x_1, x_2 \in \mathbf{R}, |x_1-x_2|<\delta \Rightarrow |f(x_1)-f(x_2)|<\varepsilon. \qquad \ldots(1)$$

The function $f(x)$ will not be uniformly continuous on **R** if we produce some $\varepsilon>0$ for which no δ works *i.e.*, for which for every $\delta>0$, there exist $x_1, x_2\in\mathbf{R}$ such that $|x_1-x_2|<\delta$ but $|f(x_1)-f(x_2)|\ge\varepsilon$.

So here we shall show that for some given $\varepsilon>0$, there exists no $\delta>0$ which satisfies the condition (1).

By the axiom of Archimedes, for any $\delta>0$, there exists a positive integer n such that

$$n\delta^2>\varepsilon. \qquad \ldots(2)$$

If we take $x_1=n\delta$ and $x_2=n\delta+\frac{1}{2}\delta$, then $|x_1-x_2|=\frac{1}{2}\delta<\delta$,

but
$$\begin{aligned}|f(x_1)-f(x_2)| &= |x_1^2-x_2^2| = |x_1-x_2||x_1+x_2|\\ &= \frac{1}{2}\delta\left(2n\delta+\frac{1}{2}\delta\right)\\ &= n\delta^2+\frac{1}{4}\delta^2>\varepsilon, \text{ by (2).}\end{aligned}$$

Hence for these two points x_1, x_2, we would always have $|f(x_1)-f(x_2)|>\varepsilon$, whatever $\delta>0$ we take. This contradicts (1). Hence f is not uniformly continuous on **R**.

Example 18: *In the closed interval $[-1,1]$ let f be defined by*

$$f(x)=x^2 \sin(1/x^2) \text{ for } x\ne 0 \text{ and } f(0)=0.$$

In the given interval (i) Is the function bounded ? (ii) Is it continuous ? (iii) Is it uniformly continuous ?

Solution: (i) If $x\in[-1,1]$ and $x\ne0$, we have

$$\begin{aligned}|f(x)| &= |x^2\sin(1/x^2)| = |x^2|\cdot|\sin(1/x^2)|\\ &= |x|^2\cdot|\sin(1/x^2)|\le 1\cdot1=1.\end{aligned}$$

$$[\because |\sin(1/x^2)|\le 1 \text{ and } -1\le x\le 1\Rightarrow |x|\le 1]$$

Also $f(0)=0\Rightarrow |f(0)|=0<1$.

Thus $|f(x)|\le 1, \forall\ x\in[-1,1]$ and so f is bounded in $[-1,1]$.

(ii) Let $c\in[-1,1]$ and $c\ne0$.

We have $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c} x^2\sin\frac{1}{x^2}=c^2\sin\frac{1}{c^2}=f(c)$.

$\therefore$ $f(x)$ is continuous at every point c of $[-1,1]$ if $c\ne0$.

Now to check the continuity of $f(x)$ at $x=0$.

We have $f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h), h > 0$

$$= \lim_{h \to 0} (-h)^2 \sin\left\{\frac{1}{(-h)^2}\right\}$$

$$= \lim_{h \to 0} h^2 \sin\frac{1}{h^2} = 0.$$

$$\left[\because \lim_{h \to 0} h^2 = 0 \text{ and } \left|\sin\frac{1}{h^2}\right| \le 1 \text{ if } h \ne 0\right]$$

Again $f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$

$$= \lim_{h \to 0} h^2 \sin\frac{1}{h^2} = 0.$$

Also $f(0) = 0.$

Since $f(0-0) = f(0) = f(0+0)$, therefore $f(x)$ is continuous at $x = 0$.

Thus $f(x)$ is continuous at each point of $[-1, 1]$ and so it is continuous in $[-1, 1]$.

(iii) Since f is continuous in the closed interval $[-1, 1]$, therefore it is also uniformly continuous in $[-1, 1]$.

Example 19: *Prove that if f and g are bounded and uniformly continuous on an interval I, then the product function fg is also uniformly continuous on I. Is boundedness of each function necessary for the uniform continuity of the product ? If not so, give a counter example.*

Solution: It is given that the functions f and g are bounded and uniformly continuous on I.

To prove that fg is also uniformly continuous on I.

Since f is bounded on I, therefore there exists $k_1 > 0$ such that

$$|f(x)| \le k_1, \forall x \in I. \qquad ...(1)$$

Again g is also bounded on I and so there exists $k_2 > 0$ such that

$$|g(x)| \le k_2, \forall x \in I. \qquad ...(2)$$

Now take any given $\varepsilon > 0$.

Since f is uniformly continuous on I, therefore there exists $\delta_1 > 0$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{2k_2},$$

whenever $|x - y| < \delta_1$, where $x, y \in I$. ...(3)

Again g is also uniformly continuous on I and so there exists $\delta_2 > 0$ such that

$$|g(x) - g(y)| < \frac{\varepsilon}{2k_1},$$

whenever $|x - y| < \delta_2$, where $x, y \in I$. ...(4)

Take $\delta = \text{mini}(\delta_1, \delta_2)$. Then from (3) and (4), we have

$$|f(x) - f(y)| < \frac{\varepsilon}{2k_2}$$

and $$|g(x) - g(y)| < \frac{\varepsilon}{2k_1}, \qquad \text{...(5)}$$

whenever $|x - y| < \delta$, where $x, y \in I$.

Now $\delta > 0$ is such that if $x, y \in I$ and $|x - y| < \delta$, then

$$|(fg)(x) - (fg)(y)| = |f(x)\, g(x) - f(y)\, g(y)|$$

$$= |f(x)\, g(x) - f(y)\, g(x) + f(y)\, g(x) - f(y)\, g(y)|$$

$$= |\{f(x) - f(y)\}\, g(x) + \{g(x) - g(y)\}\, f(y)|$$

$$\leq |f(x) - f(y)| . |g(x)| + |g(x) - g(y)| . |f(y)|$$

$$< \frac{\varepsilon}{2k_2} \cdot k_2 + \frac{\varepsilon}{2k_1} \cdot k_1, \text{ using (1), (2) and (5)}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$|(fg)(x) - (fg)(y)| < \varepsilon$, whenever $|x - y| < \delta$ where $x, y \in I$.

Hence fg is uniformly continuous on I.

Second part: Boundedness of each function is not necessary for the uniform continuity of the product as is obvious from the following examples.

Example Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = x, \forall\, x \in \mathbf{R} \text{ and } g(x) = 1, \forall\, x \in \mathbf{R}.$$

Here both the functions f and g are uniformly continuous on $\mathbf{R}$. The function f is not bounded. But the product function $fg = f$ is uniformly continuous on $\mathbf{R}$.

Example Consider the functions f and g defined on $[0, \infty[$ by

$$f(x) = g(x) = \sqrt{x}, \forall\, x \in [0, \infty[.$$

Both the functions f and g are not bounded. The product function fg is given by

$$(fg)(x) = x, \forall\, x \in [0, \infty[$$

which is obviously uniformly continuous on $[0, \infty[$.

Example 20: *Find the points of discontinuity of the function $f : \mathbf{R} \to \mathbf{R}$ defined by*

$$f(x) = \lim_{m \to \infty} \left\{ \lim_{n \to \infty} (\cos m!\, \pi\, x)^{2n} \right\}.$$

Solution: Let x be a rational number, say p/q, where p, q are integers prime to each other. Choosing m sufficiently large, $m!\, \pi\, x$ can be made an integral multiple of π so that $\cos(m!\, \pi\, x) = \pm 1$.

$$\therefore \qquad \lim_{n \to \infty} (\cos m!\, \pi x)^{2n} = \lim_{n \to \infty} (\pm 1)^{2n} = 1.$$

Hence $f(x) = 1$, when x is rational.

If x is irrational, then for any integral value of m, $\cos m\,!\,\pi\, x$ will always lie between -1 and $+1$.

$\therefore$ $(\cos m\,!\,\pi x)^{2n} = (r_m)^{2n}$ where $|r_m| < 1$, for a fixed value of m.

Hence $f(x) = \lim\limits_{m \to \infty} \lim\limits_{n \to \infty} (r_m)^{2n} = 0$, when x is irrational.

Since $f(x)$ is 1 for rational values of x and 0 for irrational values of x, f is totally discontinuous *i.e.,* discontinuous for every value of x. This is so because at any point a (rational or irrational) the limits $f(a+0)$ as well as $f(a-0)$ do not exist. Note that there are infinite number of rational and infinite number of irrational points in any neighbourhood $]\,a-h, a+h\,[$ of a, however small h may be and at these points the functional values differ widely.

Here $\overline{f(a+0)} = 1$, $\underline{f(a+0)} = 0$, $\overline{f(a-0)} = 1$, $\underline{f(a-0)} = 0$.

Hence there is a discontinuity of second kind at every point.

Example 21: *Show that the function* $f : \mathbf{R} \to \mathbf{R}$ *defined by*

$$f(x) = \lim_{n \to \infty} \left[\lim_{t \to 0} \frac{\sin^2 (n\,!\,\pi\, x)}{\sin^2 (n\,!\,\pi\, x) + t^2} \right]$$

is equal to 0 *when x is rational and to* 1 *when x is irrational. Hence show that the function is totally discontinuous.*

Solution: Let x be a rational number say p/q, where p, q are integers prime to each other. By taking n sufficiently large, $n\,!\,\pi x$ can be made an integral multiple of π so that $\sin (n\,!\,\pi x) = 0$.

Hence $f(x) = \lim\limits_{t \to 0} \dfrac{0}{0 + t^2} = 0$, when x is rational.

If x is irrational, then $0 < \sin^2 (n\,!\,\pi\, x) < 1$.

Hence
$$f(x) = \lim_{n \to \infty} \lim_{t \to 0} \frac{1}{1 + t^2 / \sin^2 (n\,!\,\pi x)}$$
$$= \frac{1}{1+0} = 1\text{, when } x \text{ is irrational.}$$

Thus $f(x) = 0$ when x is rational and 1 when x is irrational. Hence f is totally discontinuous.

Example 22: *Show that the function* $f : \mathbf{R} \to \mathbf{R}$ *defined by*

$$f(x) = \lim_{t \to \infty} \frac{(1 + \sin \pi\, x)^t - 1}{(1 + \sin \pi\, x)^t + 1}$$

is discontinuous at the points $x = 0, 1, 2, \ldots, n, \ldots$

Solution: For $x = 0, 1, 2, 3, \ldots, n, \ldots$ we have $\sin \pi\, x = 0$, so that at these values of x

$$f(x) = \lim_{n \to \infty} \frac{(1+0)^t - 1}{(1+0)^t + 1} = 0.$$

Now if $2m < x < 2m+1$ (m being an integer), then $\sin \pi x$ is positive. Hence for such values of x, we have

$$f(x) = \lim_{t \to \infty} \frac{1 - \dfrac{1}{(1+\sin \pi x)^t}}{1 + \dfrac{1}{(1+\sin \pi x)^t}} = \frac{1 - \dfrac{1}{\infty}}{1 + \dfrac{1}{\infty}} = 1. \qquad \left[\because \frac{1}{\infty} = 0\right]$$

Again if $2m+1 < x < 2m+2$, $\sin \pi x$ is negative and so

$$\lim_{t \to \infty} (1 + \sin \pi x)^t = 0.$$

Hence for such values of x, $f(x) = \dfrac{0-1}{0+1} = -1$.

From the values of $f(x)$ mentioned above, we observe that

(i) if x is an even integer, then

$$f(x) = 0,\ f(x+0) = 1 \text{ and } f(x-0) = -1$$

and (ii) if x is an odd integer, then

$$f(x) = 0,\ f(x+0) = -1 \text{ and } f(x-0) = 1.$$

Hence f has discontinuities of the first kind at

$$x = 0, 1, 2, \ldots, n, \ldots$$

Example 23: *Let a function $f : \mathbf{R} \to \mathbf{R}$ satisfy the equation*

$$f(x+y) = f(x) + f(y),\ \forall\ x, y \in \mathbf{R}. \textit{ Show that}$$

(i) If f is continuous at the point $x = a$, then it is continuous for all $x \in \mathbf{R}$.

(ii) If f is continuous then $f(x) = kx$, for some constant k.

Solution: (i) Let f be continuous at the point $x = a$.

We have $f(a+0) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} [f(a) + f(h)]$, by definition of f

$$= \lim_{h \to 0} f(a) + \lim_{h \to 0} f(h) = f(a) + \lim_{h \to 0} f(h).$$

Since f is continuous at a, we have $f(a) = f(a+0)$

$$\Rightarrow \quad f(a) = f(a) + \lim_{h \to 0} f(h) \Rightarrow \lim_{h \to 0} f(h) = 0. \quad \ldots(1)$$

Similarly $f(a) = f(a-0) \Rightarrow \lim_{h \to 0} f(-h) = 0. \quad \ldots(2)$

Now let α be any real number. We shall show that f is continuous at α.

We have $f(\alpha+0) = \lim_{h \to 0} f(\alpha+h) = \lim_{h \to 0} [f(\alpha) + f(h)]$

$$= \lim_{h \to 0} f(\alpha) + \lim_{h \to 0} f(h) = f(\alpha), \text{ using (1).}$$

Similarly, using (2) we have $f(\alpha - 0) = f(\alpha)$.

Thus $\qquad f(\alpha + 0) = f(\alpha - 0) = f(\alpha)$.

Hence f is continuous at $x = \alpha$. Since α is arbitrary so f is continuous for all $x \in \mathbf{R}$.

(ii) Here we consider the following cases :

Case I: Let $x = 0$. Since $f(x + x) = f(x) + f(x)$, we have

$$f(0) = f(0 + 0) = f(0) + f(0) \text{ and so } f(0) = 0.$$

Hence $f(x) = kx$ for every constant k in this case.

Case II: Let x be any positive integer. Then

$$f(x) = f(1 + 1 + \ldots x \text{ times}) = f(1) + f(1) + \ldots x \text{ times}$$
$$= x f(1) = kx, \text{ where } k = f(1) \text{ } i.e., \text{ a constant.}$$

Case III: Let x be any negative integer. Put $x = -y$ so that y is a positive integer.

Now $\qquad 0 = f(0)$, by Case I

$$= f(y - y) = f(y) + f(-y).$$

$\therefore \qquad f(-y) = -f(y)$.

Hence $\qquad f(x) = -f(y) = -y\, f(1)$, by Case II

$$= kx, \text{ where } k = f(1) \text{ } i.e., \text{ a constant.}$$

Case IV: Let x be any rational number. Put $x = p/q$ where q is a positive integer and p is any integer, positive, negative or zero.

Now $$f\left(q \cdot \frac{p}{q}\right) = f\left(\frac{p}{q} + \frac{p}{q} + \ldots q \text{ times}\right)$$
$$= f\left(\frac{p}{q}\right) + f\left(\frac{p}{q}\right) + \ldots q \text{ times.}$$

$\therefore$ $$f(p) = q f\left(\frac{p}{q}\right).$$

But $f(p) = kp$, by previous cases.

Hence $$q f\left(\frac{p}{q}\right) = kp \quad \text{or} \quad f\left(\frac{p}{q}\right) = k\frac{p}{q}$$

or $\qquad f(x) = kx$, in this case also.

Case V: Finally let x be any real number. Let $< x_n >$ be a sequence of rational numbers converging to x. Since f is continuous at x, the sequence $< f(x_n) >$ converges to $f(x)$. Thus, we have

$$\lim_{n \to \infty} x_n = x \quad \text{and} \quad \lim_{n \to \infty} f(x_n) = f(x)$$

Since x_n is a rational number, we have by case IV,

$$f(x_n) = kx_n.$$

$\therefore$ $$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} kx_n = k \lim_{n \to \infty} x_n = kx$$

or $\qquad f(x) = kx$.

Hence $\qquad f(x) = kx, \ \forall \ x \in \mathbf{R}$.

8 Meaning of the Sign of Derivative

Let $f'(c) > 0$ where c is an interior point of the domain of the function f; then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) > 0.$$

If $\varepsilon > 0$ be any number $< f'(c)$, there exists $\delta > 0$ such that

$$|x - c| \le \delta \Rightarrow \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon$$

i.e., $\quad x \in [c - \delta, c + \delta], x \ne c$

$$\Rightarrow \quad \frac{f(x) - f(c)}{x - c} \in \,]\, f'(c) - \varepsilon, f'(c) + \varepsilon\,[\qquad \ldots(1)$$

Since ε was chosen smaller than $f'(c)$ we conclude from (1) that

$$\frac{f(x) - f(c)}{x - c} > 0 \text{ when } x \in [c - \delta, c + \delta], x \ne c.$$

We then have, $f(x) - f(c) > 0$ when $c < x \le c + \delta$

and $\quad f(x) - f(c) < 0$ when $c - \delta \le x < c$.

Thus we have shown that if $f'(c) > 0$, there exists a neighbourhood $[c - \delta, c + \delta]$ of c such that

$$f(x) > f(c) \;\forall\, x \in \,]\, c, c + \delta\,[\text{ and } f(x) < f(c) \;\forall\, x \in [c - \delta, c\,].$$

If $f'(c) < 0$, it can be similarly shown that there exists a neighbourhood $]\, c - \delta, c + \delta\,[$ of c such that

$$f(x) > f(c) \;\forall\, x \in [c - \delta, c\,[\text{ and } f(x) < f(c) \;\forall\, x \in \,]\, c, c + \delta\,[.$$

Similarly, it can be shown for the end points a and b that there exist intervals, $]\, a, a + \delta]$ and $[\, b - \delta, b\,[$ such that

$$f'(a) > 0 \Rightarrow f(x) > f(a) \;\forall\, x \in \,]\, a, a + \delta\,]$$

$$f'(a) < 0 \Rightarrow f(x) < f(a) \;\forall\, x \in \,]\, a, a + \delta\,]$$

$$f'(b) > 0 \Rightarrow f(x) < f(b) \;\forall\, x \in [\, b - \delta, b\,[$$

and $\quad f'(b) < 0 \Rightarrow f(x) > f(b) \;\forall\, x \in [\, b - \delta, b\,[$

Comprehensive Exercise 2

1. Let f be the function defined on $[-1, 1]$ by

   $f(x) = x$, if x is irrational, $\;f(x) = 0$, if x is rational.

   Show that f is continuous only at $x = 0$.

2. Let $f : \mathbf{R} \to \mathbf{R}$ be such that

   $f(x) = x \quad$ when x is irrational

   $\quad\;\;\; = -x$ when x is rational.

   Show that $f(x)$ is continuous only at $x = 0$.

3. Show that the function f defined on **R** by

$$f(x) = 1 \text{ when } x \text{ is rational}, \; f(x) = -1 \text{ when } x \text{ is irrational}$$

is discontinuous at every point of **R**.

4. Show that $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$ is continuous but not uniformly continuous on **R**.

5. Explain fully uniform continuity and discuss the uniform continuity of the function $f(x) = x^2, \forall\, x \in \mathbf{R}$ in $]0, 1[$.

6. Let $f :]-1, 1[\to [0, 1]$ be a function defined by $f(x) = x^2$. Using definition show that f is uniformly continuous on its domain.

7. Show that the function $f(x) = 1/x, x > 0$ is continuous in $(0, 1)$ but not uniformly continuous.

8. Define uniform continuity and show that the function $f(x) = x^2 + 3x, x \in [-1, 1]$ is uniformly continuous in $[-1, 1]$.

9. Show that a function which is continuous in a closed interval is bounded in that interval. Verify the theorem for

$$f(x) = \cos x \;\text{ in } [-\frac{1}{2}\pi, \frac{1}{2}\pi] .$$

10. Give an example to show that a function continuous on an open interval need not be bounded on that interval.

11. If the function f is continuous in the closed interval $[a, b]$, prove that it attains its least upper bound and greatest lower bound in $[a, b]$. Verify the theorem for the function $f(x) = \sin x$ in $[0, 2\pi]$.

12. If f is continuous in $[a, b]$ and $f(a) \cdot f(b) < 0$, show that $f(c) = 0$ for at least one $c \in [a, b]$.

13. Let f be continuous on $[a, b]$ and suppose that $f(x) = 0$ for every rational x in $[a, b]$. Prove that $f(x) = 0$ for all x in $[a, b]$.

14. If a function is continuous on a closed interval $[a, b]$, then it attains its bounds at least once in $[a, b]$. Give an example of a function which is continuous and bounded, and attains its supremum but does not attain its infimum.

15. Prove that the identity mapping of any interval I is uniformly continuous on I.

16. Show that the function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \frac{1}{1 + e^{1/\sin(n!\,\pi x)}}$$

can be made discontinuous at any rational point in the interval $[0, 1]$ by a proper choice of n.

17. If $f : \mathbf{R} \to \mathbf{R}$ is a continuous function and satisfies the relation $f(x + y) = f(x)\, f(y), \forall\; x, y \in \mathbf{R}$, then either $f(x) = 0 \;\; \forall\; x \in \mathbf{R}$

or there exists an $a > 0$ such that

$$f(x) = a^x, \forall\; x \in \mathbf{R}.$$

18. Show that a function f is continuous at a iff for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x_1, x_2 \in]a-\delta, a+\delta[\Rightarrow |f(x_1) - f(x_2)| < \varepsilon.$$

19. Discuss the nature of discontinuity of the function f, defined by

$$f(x) = \lim_{n \to \infty} \frac{\log(2+x) - x^{2n} \sin x}{1 + x^{2n}}$$

at $x = 1$. Show that $f(0)$ and $f(\pi/2)$ differ in sign.

20. Let f and g be continuous on $[a, b]$ and let $f(a) < g(a)$ but $f(b) > g(b)$. Prove that $f(c) = g(c)$ for some $c \in]a, b[$.

21. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and let f be zero on a dense set (*i.e.*, a set whose intersection with every interval is non-empty). Then f is identically zero.

22. Determine the discontinuities of the function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \lim_{n \to \infty} \frac{\{1 + \sin(\pi/x)\}^n - 1}{\{1 + \sin(\pi/x)\}^n + 1}, 0 < x \le 1.$$

23. Show that the function

$$\phi(x) = \lim_{n \to \infty} \frac{x^{2n+2} - \cos x}{x^{2n} + 1}$$

does not vanish anywhere in the interval $[0, 2]$ though $\phi(0)$ and $\phi(2)$ differ in sign. Discuss the continuity of the function at $x = 1$.

24. Examine for continuity the function f defined by

$$f(x) = \lim_{n \to \infty} \frac{e^x - x^n \sin x}{1 + x^n} \left(0 \le x \le \frac{\pi}{2}\right)$$

at $x = 1$. Explain why the function f does not vanish anywhere in $[0, \pi/2]$ although $f(0) \,.\, f(\pi/2) < 0$.

Answers 2

5. Uniformly continuous in $]0, 1[$
19. Discontinuity of the first kind at $x = 1$
22. f has a discontinuity of second kind at $x = 0$ and ordinary discontinuous at $x = 1, \frac{1}{2}, \frac{1}{3}, \ldots$
23. Discontinuity of the first kind at $x = 1$
24. Discontinuous at $x = 1$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The value of K for which $f(x) = \begin{cases} \dfrac{\sin 5x}{3x}, & \text{if } x \neq 0 \\ K, & \text{if } x = 0 \end{cases}$

is continuous at $x = 0$, shall be

(a) 1/3 (b) 3/5
(c) 0 (d) 5/3 **(Kumaun 2008)**

2. The necessary and sufficient condition for a function f defined on $I \subset R$ to be continuous at $a \in I$ is that for each sequence $< a_n >$ in I which converges to a, we have $\lim\limits_{n \to \infty} f(a_n) =$

(a) $f'(a)$ (b) $f^2(a)$
(c) $f(a)$ (d) none of these

3. The function f defined on $[-1, 1]$ by

$$f(x) = x, \text{ if } x \text{ is irrational}$$
$$f(x) = 0, \text{ if } x \text{ is rational.}$$

is continuous at $x =$

(a) 0 (b) 1
(c) $-1, 1$ (d) none of these

4. At $x = 1$, the function $f(x) = \lim\limits_{n \to \infty} \dfrac{\{1 + \sin(\pi / x)\}^n - 1}{\{1 + \sin(\pi / x)\}^n + 1}, 0 < x \leq 1.$

(a) is continuous
(b) has discontinuity of first kind
(c) has discontinuity of second kind
(d) is uniformly continuous.

5. For all real values of x, the function $f(x) = x^2$ is

(a) continuous (b) discontinuous
(c) uniformly continuous (d) not uniformly continuous.

6. In the interval $[-1, 1]$ the function f defined by $f(x) = x^2 \sin(1 / x^2)$ for $x \neq 0$ and $f(0) = 0$ is

(a) bounded (b) uniformly continuous
(c) unbounded (d) none of these

Fill in the Blank(s)

Fill in the blanks "......", so that the following statements are complete and correct.

1. A function $f(x)$ is continuous at a point $x = a$ if $\lim_{x \to a} f(x) = \ldots\ldots$

(Bundelkhand 2008; Kumaun 14)

2. If $f(x) = x - [x]$, where $[x]$ denotes the greatest integer less than or equal to x, then $f(x) = \ldots\ldots$, for $3 < x < 4$.
3. A function $f(x)$ has a removable discontinuity at $x = a$ if $\lim_{x \to a} f(x)$ exists but is not equal to ....
4. The domain of the function $f(x) = \dfrac{\sin x}{x}$ is ....
5. The domain of the function $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

   is ....
6. If a function $f(x)$ is continuous in a closed interval $[a, b]$, it is .... in that interval.
7. If f is continuous in a closed and bounded interval I, it is .... on I.
8. The function $f : R \to R$ defined by

$$f(x) = \lim_{t \to \infty} \frac{(1 + \sin \pi x)^t - 1}{(1 + \sin \pi x)^t + 1}$$

   is .... at the points $x = 0, 1, 2, \ldots$
9. The function $f(x) = x^3$ is .... continuous in $[-2, 2]$.
10. The function f defined on R^+ as $f(x) = \sin \dfrac{1}{x}$, $\forall x > 0$ is .... but not .... on R^+.
11. The function $f(x) = \lim_{n \to \infty} \dfrac{e^x - x^n \sin x}{1 + x^n} \left(0 \leq x \leq \dfrac{\pi}{2}\right)$ is .... at $x = 1$.

True or False

Write 'T' for true and 'F' for false statement.

1. The function $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$

   is continuous at $x = 0$.

2. The function $f(x) = \begin{cases} \sin x, & x \geq 0 \\ -\sin x, & x < 0 \end{cases}$

is continuous at $x = 0$.

3. The function $f(x) = \begin{cases} x\cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

is discontinuous at $x = 0$.

4. If a function f is continuous at a, then $|f|$ is also continuous at a.

5. The function $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

is continuous at $x = 0$.

6. The function $f(x) = \begin{cases} 1, & x < 1 \\ 2 - x, & 1 \leq x < 2 \\ 2, & x \geq 2 \end{cases}$

is discontinuous at $x = 1$.

7. If a function $f(x)$ is continuous in a closed interval $[a, b]$, then it is bounded in $[a, b]$.

8. If a function $f(x)$ is uniformly continuous in an interval I, then it is also continuous in I.

9. If a function $f(x)$ is continuous in an open interval I, then it is also uniformly continuous in I.

10. If a function $f(x)$ is continuous in a closed interval $[a, b]$, then it is also uniformly continuous in $[a, b]$.

11. A function continuous on an open interval is bounded on that interval.

12. The function $f(x) = x^2 + 3x, x \in [-1, 1]$ is uniformly continuous in this interval.

13. The function $f(x) = x^2$ is uniformly continuous on $]0, 1[$.

14. A function defined on $[0, 1]$ and given by $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational} \end{cases}$,

is discontinuous at $x = \dfrac{1}{2}$.

Answers

Multiple Choice Questions

1. (d) 2. (c) 3. (a) 4. (b) 5. (a), (d)
6. (a), (b)

Fill in the Blank(s)

1. $f(a)$
2. $x - 3$
3. $f(a)$
4. $\mathbf{R} - \{0\}$
5. $\mathbf{R}$
6. bounded
7. uniformly continuous
8. discontinuous
9. uniformly
10. continuous, uniformly continuous
11. discontinuous

True or False

1. F	2. T	3. F	4. T	5. T
6. F	7. T	8. T	9. F	10. T
11. F	12. T	13. T	14. F	

□

Chapter

2

Differentiability

1 Definitions

Derivative at a Point. **(Bundelkhand 2010)**

Let I denote the open interval $]a, b[$ *in* $\mathbf{R}$ *and let* $x_0 \in I$. *Then a function* $f : I \to \mathbf{R}$ *is said to be* **differentiable** (*or* **derivable**) *at* x_0 *iff*

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \text{ or equivalently } \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists finitely and this limit, if it exists finitely, is called the **differential coefficient** *or* **derivative** *of f with respect to x at* $x = x_0$.

It is denoted by $f'(x_0)$ *or by* $D f(x_0)$.

Progressive and regressive derivatives.

The *progressive derivative* of f at $x = x_0$ is given by

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}, h > 0.$$

It is also called the **right hand differential coefficient** of f at $x = x_0$ and is denoted by $R\,f\,'(x_0)$ or by $f'(x_0 + 0)$.

The *regressive derivative* of f at $x = x_0$ is given by

$$\lim_{h \to 0} \frac{f\,(x_0 - h) - f\,(x_0)}{-h}, h > 0.$$

It is also called the **left hand differential coefficient** of f at $x = x_0$ and is denoted by $L\,f'(x_0)$ or by $f'(x_0 - 0)$.

It is obvious that *f is derivable at x_0 iff $L\,f'(x_0)$ and $R\,f'(x_0)$ both exist and are equal.*

Remark: If $f\,(x) = a_0\,x^{\,n} + a_1\,x^{\,n-1} + \ldots + a_{n-1}\,x + a_n$ is a polynomial in x of degree n, then $f\,(x)$ is differentiable at every point a of **R**.

Differentiability in an interval: **(Meerut 2003)**

Open interval] *a*, *b* [. *A function $f :]a, b[\to$ **R** is said to be differentiable in $]a, b[$ iff it is differentiable at every point of $]a, b[$.*

Closed interval] *a*, *b* [. *A function $f : [a, b] \to$ **R** is said to be differentiable in $[a, b]$ iff $R\,f\,'(a)$ exists, $L\,f\,'(b)$ exists and f is differentiable at every point of $]a, b[$.*

Derivative of a function: **(Gorakhpur 2010)**

Let f be a function whose domain is an interval I. If I_1 be the set of all those points x of I at which f is differentiable *i.e.,* $f\,'(x)$ exists and if $I_1 \neq \varnothing$, we get another function $f\,'$ with domain I_1. It is called the *first derivative* of f (or simply the derivative of f). Similarly 2nd, 3rd, ..., nth derivatives of f are defined and are denoted by $f'', f''', \ldots, f^{(n)}$ respectively.

Note: The derivative of a function at a point and the derivative of a function are two different but related concepts. The derivative of f at a point a is a number while the derivative of f is a function. However, very often the term derivative of f is used to denote both number and function and it is left to the context to distinguish what is intended.

An alternate definition of differentiability:

Let f be a function defined on an interval I and let a be an interior point of I. Then, by the definition of $f\,'(a)$, assuming it to exist, we have

$$f'(a) = \lim_{x \to a} \frac{f\,(x) - f\,(a)}{x - a}$$

i.e., $f'(a)$ exists if for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\left| \frac{f\,(x) - f\,(a)}{x - a} - f\,'(a) \right| < \varepsilon \text{ whenever } 0 < |x - a| < \delta$$

or equivalently

$$x \in \,]\,a - \delta, a + \delta\,[\Rightarrow f\,'(a) - \varepsilon < \frac{f\,(x) - f\,(a)}{x - a} < f\,'(a) + \varepsilon.$$

2 Geometrical Meaning of a Derivative

We take two neighbouring points $P[a, f(a)]$ and $Q[a+h, f(a+h)]$ on the curve $y = f(x)$.

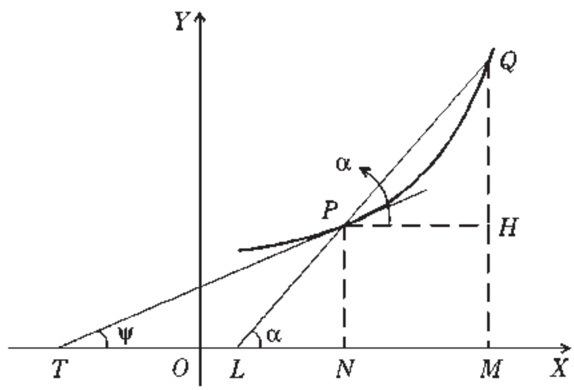
Let the chord PQ and the tangent at P meet the x-axis in L and T respectively. Let $\angle QLX = \alpha$ and $\angle PTX = \psi$. Draw PN and $QM \perp$ to OX and $PH \perp$ to QM.

Then $\quad PH = NM = OM - ON = a + h - a = h,$

and $\quad QH = QM - MH = QM - PN = f(a+h) - f(a).$

$\therefore \quad \tan\alpha = \dfrac{QH}{PH} = \dfrac{f(a+h) - f(a)}{h}.$ ...(1)

As $h \to 0$, the point Q moving along the curve approaches the point P, the chord PQ approaches the tangent line TP as its limiting position and the angle α approaches the angle ψ.



Hence taking limits as $h \to 0$, the equation (1) gives

$$\tan\psi = f'(a).$$

Hence $f'(a)$ is the tangent of the angle which the tangent line to the curve $y = f(x)$ at the point $P[a, f(a)]$ makes with x-axis.

3 Meaning of the Sign of Derivative

Let $f'(c) > 0$ where c is an interior point of the domain of the function f; then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) > 0.$$

If $\varepsilon > 0$ be any number $< f'(c)$, there exists $\delta > 0$ such that

$$|x - c| \leq \delta \Rightarrow \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon$$

i.e., $\quad x \in [c - \delta, c + \delta], x \neq c$

$\Rightarrow \quad \dfrac{f(x) - f(c)}{x - c} \in \,]f'(c) - \varepsilon, f'(c) + \varepsilon[\cdot$ ...(1)

Since ε was chosen smaller than $f'(c)$, we conclude from (1) that

$$\frac{f(x) - f(c)}{x - c} > 0 \text{ when } x \in [c - \delta, c + \delta], x \neq c.$$

We then have, $f(x) - f(c) > 0$ when $c < x \le c + \delta$

and $\quad f(x) - f(c) < 0$ when $c - \delta \le x < c$.

Thus we have shown that if $f'(c) > 0$, there exists a neighbourhood $[c - \delta, c + \delta]$ of c such that

$$f(x) > f(c) \quad \forall\, x \in]c, c + \delta] \text{ and } f(x) < f(c) \quad \forall\, x \in [c - \delta, c[.$$

If $f'(c) < 0$, it can be similarly shown that there exists a neighbourhood $[c - \delta, c + \delta]$ of c such that

$$f(x) > f(c) \quad \forall\, x \in [c - \delta, c[\text{ and } f(x) < f(c) \quad \forall\, x \in]c, c + \delta].$$

Similarly, it can be shown for the end points a and b that there exist intervals $]a, a + \delta]$ and $[b - \delta, b[$ such that

$$f'(a) > 0 \Rightarrow f(x) > f(a) \;\; \forall \;\; x \in]a, a + \delta]$$

$$f'(a) < 0 \Rightarrow f(x) < f(a) \;\; \forall \;\; x \in]a, a + \delta]$$

$$f'(b) > 0 \Rightarrow f(x) < f(b) \;\; \forall \;\; x \in [b - \delta, b[$$

and $\quad f'(b) < 0 \Rightarrow f(x) > f(b) \;\; \forall \;\; x \in [b - \delta, b[$.

4 A Necessary Condition for the Existence of a Finite Derivative

Theorem 1. *Continuity is a necessary but not a sufficient condition for the existence of a finite derivative.* **(Kanpur 2007, 12; Meerut 10, 10B, 11; Avadh 10; Kashi 14; Gorakhpur 13, 14)**

Proof: Let f be differentiable at x_0. Then $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ exists and equals $f'(x_0)$. Now, we can write $f(x) - f(x_0) = \dfrac{f(x) - f(x_0)}{x - x_0}(x - x_0)$, if $x \ne x_0$.

Taking limits as $x \to x_0$, we get

$$\lim_{x \to x_0} [f(x) - f(x_0)] = \lim_{x \to x_0} \left\{ \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \right\}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0) = f'(x_0) \,.\, 0 = 0,$$

so that $\quad \lim\limits_{x \to x_0} f(x) = f(x_0)$.

Hence f is continuous at x_0. Thus continuity is a necessary condition for differentiability but it is not a sufficient condition for the existence of a finite derivative. The following example illustrates this fact :

Let $f(x) = x \sin(1/x), x \ne 0$ and $f(0) = 0$. **(Gorakhpur 2014)**

This function is continuous at $x = 0$ but not differentiable at $x = 0$.

Since $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} x \sin \frac{1}{x} = 0 = f(0)$, therefore the function $f(x)$ is continuous at $x = 0$.

Now $Rf'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{f(h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{h \sin(1/h) - 0}{h} = \lim_{h \to 0} \sin \frac{1}{h},$$

which does not exist. Similarly $Lf'(0)$ does not exist.

Thus $f(x)$ is not differentiable at $x = 0$, though it is continuous there.

5 Algebra of Derivatives

Now we shall establish some fundamental theorems regarding the differentiability of the sum, product and quotient of differentiable functions.

Theorem 1: *If a function f is differentiable at a point x_0 and c is any real number, then the function cf is also differentiable at x_0 and $(cf)'(x_0) = cf'(x_0)$.*

Proof: By the definition of $f'(x_0)$, we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0).$$

Now $$\lim_{x \to x_0} \frac{(cf)(x) - (cf)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{cf(x) - cf(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \left\{ c \cdot \frac{f(x) - f(x_0)}{x - x_0} \right\} = c \cdot \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = cf'(x_0).$$

Hence cf is differentiable at x_0 and $(cf)'(x_0) = cf'(x_0)$.

Theorem 2: *Let f and g be defined on an interval I. If f and g are differentiable at $x_0 \in I$, then so also is $f + g$ and*

$$(f+g)'(x_0) = f'(x_0) + g'(x_0).$$

Proof: Since f and g are differentiable at x_0, therefore

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0), \qquad \ldots(1)$$

and $$\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = g'(x_0). \qquad \ldots(2)$$

Now $$\lim_{x \to x_0} \frac{(f+g)(x) - (f+g)(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{[f(x) + g(x)] - [f(x_0) + g(x_0)]}{x - x_0}$$

$$= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} \right]$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0},$$

as the limit of a sum is equal to the sum of the limits

$= f'(x_0) + g'(x_0)$, using (1) and (2).

Hence $f + g$ is differentiable at x_0 and

$$(f + g)'(x_0) = f'(x_0) + g'(x_0).$$

Theorem 3: *Let f and g be defined on an interval I. If f and g are differentiable at $x_0 \in I$, then so also is fg and* $(fg)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$.

Proof: Since f and g are differentiable at x_0, we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \qquad \ldots(1)$$

and

$$\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = g'(x_0) \qquad \ldots(2)$$

Now

$$\lim_{x \to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x)\, g(x) - f(x_0)\, g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x)\, g(x) - f(x_0)\, g(x) + f(x_0)\, g(x) - f(x_0)\, g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} \cdot g(x) + f(x_0) \frac{g(x) - g(x_0)}{x - x_0} \right]$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} . \lim_{x \to x_0} g(x) + f(x_0) . \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$

$= f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$,

using (1), (2) and the fact that $\lim\limits_{x \to x_0} g(x) = g(x_0)$.

Note that $g(x)$ is differentiable at $x = x_0$ implies that $g(x)$ is continuous at x_0 and so

$$\lim_{x \to x_0} g(x) = g(x_0).$$

Hence fg is differentiable at x_0 and $(fg)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$.

Theorem 4: *If f is differentiable at x_0 and $f(x_0) \neq 0$, then the function $1/f$ is differentiable at x_0 and* $(1/f)'(x_0) = -f'(x_0)/\{f(x_0)\}^2$.

Proof: Since f is differentiable at x_0, therefore

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0). \qquad \ldots(1)$$

Since f is differentiable at x_0, it is continuous at x_0, therefore

$$\lim_{x \to x_0} f(x) = f(x_0) \neq 0. \qquad \ldots(2)$$

Also, since $f(x_0) \neq 0$, hence, $f(x_0) \neq 0$ in some neighbourhood N of x_0. Now, we have for $x \in N$

$$\lim_{x \to x_0} \frac{\frac{1}{f(x)} - \frac{1}{f(x_0)}}{x - x_0} = \lim_{x \to x_0} \left\{ -\frac{f(x) - f(x_0)}{x - x_0} \cdot \frac{1}{f(x)} \cdot \frac{1}{f(x_0)} \right\}$$

$$= -\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} \frac{1}{f(x)} \cdot \frac{1}{f(x_0)}$$

$$= -f'(x_0) \cdot \frac{1}{f(x_0)} \cdot \frac{1}{f(x_0)}, \text{ using (1) and (2)}$$

$$= -f'(x_0) / \{f(x_0)\}^2.$$

Hence $1/f$ is differentiable at x_0 and $(1/f)'(x_0) = -f'(x_0)/\{f(x_0)\}^2$.

Theorem 5: *Let f and g be defined on an interval I. If f and g are differentiable at $x_0 \in I$, and $g(x_0) \neq 0$, then the function f/g is differentiable at x_0 and*

$$\left(\frac{f}{g}\right)'(x_0) = \frac{[g(x_0) f'(x_0) - f(x_0) g'(x_0)]}{[g(x_0)]^2}.$$

Proof: Use theorems 3 and 4 of article 5.

6 The Chain Rule of Differentiability

Theorem 1. *Let f and g be functions such that the range of f is contained in the domain of g. If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then $g \circ f$ is differentiable at x_0, and*

$$(g \circ f)'(x_0) = g'(f(x_0)) . f'(x_0).$$

Proof: Let $y = f(x)$ and $y_0 = f(x_0)$.

Since f is differentiable at x_0, we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

or $$f(x) - f(x_0) = (x - x_0)[f'(x_0) + \lambda(x)] \qquad \ldots(1)$$

where $\lambda(x) \to 0$ as $x \to x_0$.

Further since g is differentiable at y_0, we have

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = g'(y_0)$$

or $\quad g(y) - g(y_0) = (y - y_0)[g'(y_0) + \mu(y)]$ ...(2)

where $\quad \mu(y) \to 0$ as $y \to y_0$.

Now $\quad (g \circ f)(x) - (g \circ f)(x_0) = g(f(x)) - g(f(x_0)) = g(y) - g(y_0)$

$= (y - y_0)[g'(y_0) + \mu(y)]$, by (2)

$= [f(x) - f(x_0)][g'(y_0) + \mu(y)]$

$= (x - x_0)[f'(x_0) + \lambda(x)][g'(y_0) + \mu(y)]$, by (1).

Thus if $x \neq x_0$, then

$$\frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = [g'(y_0) + \mu(y)].[f'(x_0) + \lambda(x)]. \qquad ...(3)$$

Also f being differentiable at x_0, is continuous at x_0 and hence as $x \to x_0$, $f(x) \to f(x_0)$ *i.e.*, $y \to y_0$.

Consequently $\mu(y) \to 0$ as $x \to x_0$ and $\lambda(x) \to 0$ as $x \to x_0$.

Taking the limits as $x \to x_0$, we get from (3)

$$\lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = g'(y_0) . f'(x_0).$$

Hence the function gof is differentiable at x_0 and $(gof)'(x_0) = g'(f(x_0)) f'(x_0)$.

7 Derivative of the Inverse Function

Theorem: *If f be a continuous one-to-one function defined on an interval and let f be differentiable at x_0, with $f'(x_0) \neq 0$, then the inverse of the function f is differentiable at $f(x_0)$ and its derivative at $f(x_0)$ is $1/f'(x_0)$.*

Proof: Before proving the theorem we remind that if the domain of f be X and its range be Y, then the inverse function g of f usually denoted by f^{-1} is the function with domain Y and range X such that $f(x) = y \Leftrightarrow g(y) = x$. Also g exists if f is one-one.

Let $\quad y = f(x)$ and $y_0 = f(x_0)$.

Since f is differentiable at x_0, we have

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

or $\quad f(x) - f(x_0) = (x - x_0)[f'(x_0) + \lambda(x)]$ ...(1)

where $\lambda(x) \to 0$ as $x \to x_0$. Further, we have

$g(y) - g(y_0) = x - x_0$, by definition of g.

$$\therefore \quad \frac{g(y) - g(y_0)}{y - y_0} = \frac{x - x_0}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0) + \lambda(x)}, \text{ by (1).}$$

It can be easily seen that if $y \to y_0$, then $x \to x_0$.

In fact, f is continuous at x_0 implies that $g = f^{-1}$ is continuous at $f(x_0) = y_0$ and consequently

$$g(y) \to g(y_0) \text{ as } y \to y_0 \text{ i.e., } x \to x_0 \text{ as } y \to y_0,$$

so that $\lambda(x) \to 0$ as $y \to y_0$.

$$\therefore \quad \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{1}{f'(x_0) + \lambda(x)} = \frac{1}{f'(x_0)}$$

or $\quad g'(y_0) = \dfrac{1}{f'(x_0)} \quad$ or $\quad g'(f(x_0)) = \dfrac{1}{f'(x_0)}$.

Illustrative Examples

Example 1: *Prove that the function $f(x) = |x|$ is continuous at $x = 0$, but not differentiable at $x = 0$ where $|x|$ means the numerical value or the absolute value of x.*

(Rohilkhand 2007; Bundelkhand 08; Meerut 13B; Avadh 11)

Also draw the graph of the function.

Solution: We have $f(0) = |0| = 0$,

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} |h| = \lim_{h \to 0} h = 0$$

and

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} |-h| = \lim_{h \to 0} h = 0.$$

$$\therefore \quad f(0) = f(0+0) = f(0-0).$$

Hence $f(x)$ is continuous at $x = 0$.

Also, we have $Rf'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \dfrac{f(h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{|h| - 0}{h} = \lim_{h \to 0} \frac{h}{h}, \ (h \text{ being positive})$$

$$= \lim_{h \to 0} 1 = 1,$$

and

$$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{|-h| - 0}{-h} = \lim_{h \to 0} \frac{h}{-h}, \ (h \text{ being positive})$$

$$= \lim_{h \to 0} -1 = -1.$$

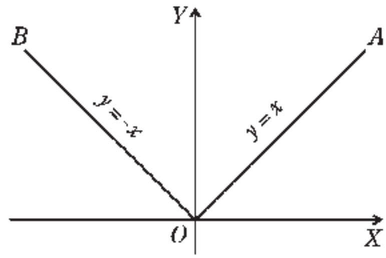
Since $Rf'(0) \neq Lf'(0)$, the function $f(x)$ is not differentiable at $x = 0$.

To draw the graph of the function $f(x) = |x|$.

We have $f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0. \end{cases}$

Let $y = f(x)$. Then the graph of the function consists of the following straight lines :

$$y = x, \quad x \geq 0$$
$$y = -x, \quad x \leq 0.$$



The graph is as shown in the figure. From the graph we observe that the function is continuous at the point O *i.e.*, at the point $x = 0$ but it is not differentiable at this point. The tangent to the curve at the point O from the right is the straight line OA and from the left is the straight line OB. Thus the tangent to the curve at O does not exist and so the function is not differentiable at O.

Example 2: *Show that the function $f(x) = |x| + |x - 1|$ is not differentiable at $x = 0$ and $x = 1$.* **(Meerut 2005B, 08; Kashi 14)**

Solution: We first observe that if $x < 0$, then

$$|x| = -x \text{ and } |x - 1| = |1 - x| = 1 - x \;;$$

if $0 \leq x \leq 1$, then $|x| = x$ and $|x - 1| = |1 - x| = 1 - x$;

and if $x > 1$, then $|x| = x$ and $|x - 1| = x - 1$.

$\therefore$ the function $f(x)$ is given by

$$f(x) = \begin{cases} 1 - 2x, & \text{if } x < 0 \\ 1, & \text{if } 0 \leq x \leq 1 \\ 2x - 1, & \text{if } x > 1. \end{cases}$$

At $x = 0$. We have $Rf'(0) = \lim_{h \to 0} \dfrac{f(0 + h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{1 - 1}{h}, \text{ as } f(x) = 1 \text{ if } 0 \leq x \leq 1$$

$$= \lim_{h \to 0} 0 = 0,$$

and $\quad Lf'(0) = \lim_{h \to 0} \dfrac{f(0 - h) - f(0)}{-h}$

$$= \lim_{h \to 0} \frac{f(-h) - f(0)}{-h} = \lim_{h \to 0} \frac{[1 - 2(-h)] - 1}{-h}$$

$[\because f(x) = 1 - 2x, \text{ if } x < 0]$

$$= \lim_{h \to 0} \frac{2h}{-h} = \lim_{h \to 0} -2 = -2 \cdot$$

$\because$ $Rf'(0) \neq Lf'(0)$, so the given function is not differentiable at $x = 0$.

At $x = 1$. We have

$$Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{[2(1+h) - 1] - 1}{h}$$

$$= \lim_{h \to 0} \frac{2 + 2h - 1 - 1}{h} = \lim_{h \to 0} \frac{2h}{h} = \lim_{h \to 0} 2 = 2,$$

and $$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{1-1}{-h} = \lim_{h \to 0} 0 = 0.$$

$\because$ $Rf'(1) \neq Lf'(1)$, so the given function $f(x)$ is not differentiable at $x = 1$.

Example 3: *Let $f(x)$ be an even function. If $f'(0)$ exists, find its value.*

(Kanpur 2010)

Solution: $f(x)$ is an even function, so $f(-x) = f(x)$ $\forall x$.

$f'(0)$ exists $\Rightarrow Rf'(0) = Lf'(0) = f'(0)$.

Now $$f'(0) = Rf'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}, h > 0$$

$$= \lim_{h \to 0} \frac{f(-h) - f(0)}{h} \qquad [\because f(-x) = f(x)]$$

$$= -\lim_{h \to 0} \frac{f(-h) - f(0)}{-h} = -Lf'(0) = -f'(0).$$

$\therefore$ $2f'(0) = 0 \Rightarrow f'(0) = 0.$

Example 4: *Let* $f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ x - 1, & 0 < x \le 2, \end{cases}$ *and* $g(x) = f(|x|) + |f(x)|$. *Test the differentiability of $g(x)$ in $]-2, 2[$.*

Solution: When $-2 \le x \le 0, |x| = -x$ and when $0 < x \le 2, |x| = x$.

Now $-2 \le x \le 0 \Rightarrow |x| = -x$

$\Rightarrow$ $f(|x|) = f(-x) = -x - 1.$ $\qquad [\because 0 < -x \le 2]$

So we have $$f(|x|) = \begin{cases} x - 1, & 0 < x \le 2 \\ -x - 1, & -2 \le x \le 0 \end{cases}$$

and $$|f(x)| = \begin{cases} 1, & -2 \le x \le 0 \\ -x + 1, & 0 < x \le 1 \\ x - 1, & 1 < x \le 2 \cdot \end{cases}$$

$\therefore$ $$g(x) = f(|x|) + |f(x)| = \begin{cases} -x, & -2 \le x \le 0 \\ 0, & 0 < x \le 1 \\ 2x - 2, & 1 < x \le 2. \end{cases}$$

We see that $g(x)$ is differentiable $\forall\, x \in\,]-2, 2[$, except possibly at $x = 0$ and 1.

$$L\, g'(0) = \lim_{h \to 0} \frac{g(0-h) - g(0)}{-h} = \lim_{h \to 0} \frac{g(-h) - g(0)}{-h}$$

$$= \lim_{h \to 0} \frac{h - 0}{-h} = -1,$$

$$R\, g'(0) = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0.$$

Since $L\, g'(0) \neq R\, g'(0)$, $g(x)$ is not differentiable at $x = 0$.

Again
$$R\, g'(1) = \lim_{h \to 0} \frac{g(1+h) - g(1)}{h} = \lim_{h \to 0} \frac{2(1+h) - 2 - 0}{h} = 2,$$

$$L\, g'(1) = \lim_{h \to 0} \frac{g(1-h) - g(1)}{-h} = \lim_{h \to 0} \frac{0 - 0}{-h} = 0 \neq R\, g'(1).$$

$\therefore$ g is not differentiable at $x = 1$.

Example 5: *Suppose the function f satisfies the conditions :*

(*i*) $f(x + y) = f(x)\, f(y)\ \forall\, x, y$

(*ii*) $f(x) = 1 + x\, g(x)$ *where* $\lim_{x \to 0} g(x) = 1$.

Show that the derivative $f'(x)$ *exists and* $f'(x) = f(x)$ *for all x.*

Solution: Putting δx for y in the first condition, we have

$$f(x + \delta x) = f(x)\, f(\delta x).$$

Then
$$f(x + \delta x) - f(x) = f(x)\, f(\delta x) - f(x)$$

or
$$\frac{f(x + \delta x) - f(x)}{\delta x} = \frac{f(x)\,[f(\delta x) - 1]}{\delta x}$$

$$= \frac{f(x)\, \delta x\, g(\delta x)}{\delta x}, \text{ by given condition (ii)}$$

$$= f(x)\, g(\delta x).$$

$$\therefore \quad \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} f(x)\, g(\delta x) = f(x) \,.\, 1.$$

$$\left[\because \lim_{\delta x \to 0} g(\delta x) = 1\right]$$

$\therefore$ $f'(x) = f(x)$. Since $f(x)$ exists, $f'(x)$ also exists.

Example 6: *Show that the function f given by* $f(x) = x \tan^{-1}(1/x)$ *for* $x \neq 0$ *and* $f(0) = 0$ *is continuous but not differentiable at* $x = 0$. **(Purvanchal 2008; Lucknow 11; Bundelkhand 12; Meerut 13)**

Solution: Since $\lim_{x \to 0} f(x) = \lim_{x \to 0} x \tan^{-1} \frac{1}{x} = 0 = f(0)$, therefore the function f is continuous at $x = 0$.

Now $$R f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h \tan^{-1}(1/h) - 0}{h} = \lim_{h \to 0} \tan^{-1}\left(\frac{1}{h}\right)$$

$$= \tan^{-1} \infty = \frac{\pi}{2}$$

and $$L f'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-h \tan^{-1}(-1/h) - 0}{-h}$$

$$= \lim_{h \to 0} \tan^{-1}\left(-\frac{1}{h}\right) = -\tan^{-1} \infty = -\frac{\pi}{2}.$$

Since $R f'(0) \neq L f'(0)$, f is not differentiable at $x = 0$.

Example 7: *Investigate the following function from the point of view of its differentiability. Does the differential coefficient of the function exist at $x = 0$ and $x = 1$?*

$$f(x) = \begin{cases} -x & \text{if} \quad x < 0 \\ x^2 & \text{if} \quad 0 \leq x \leq 1 \\ x^3 - x + 1 & \text{if} \quad x > 1. \end{cases}$$

(Meerut 2006)

Solution: We check the function $f(x)$ for differentiability at $x = 0$ and $x = 1$ only. For other values of x, obviously $f(x)$ is differentiable because it is a polynomial function. It can be seen that $f(x)$ is continuous at $x = 0$ and $x = 1$.

Now $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{-(0-h) - 0}{-h} = -1$$

and $$R f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{(0+h)^2 - 0}{h} = 0.$$

$\because L f'(0) \neq R f'(0)$, the function is not differentiable at $x = 0$.

Again $$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{(1-h)^2 - 1}{-h}$$

$$= \lim_{h \to 0} \frac{-2h + h^2}{-h} = \lim_{h \to 0} (2 - h) = 2$$

and $$R f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)^3 - (1+h) + 1 - 1}{h}$$

$$= \lim_{h \to 0} \frac{2h + 3h^2 + h^3}{h} = \lim_{h \to 0} (2 + 3h + h^2) = 2 = L f'(1).$$

Hence $f'(1)$ exists *i.e.,* the function is differentiable at $x = 1$.

Example 8: *Find* $f'(1)$ *if* $f(x)=\begin{cases}\dfrac{x-1}{2x^2-7x+5}, & when\;\; x\neq 1\\ -1/3, & when\;\; x=1.\end{cases}$

Solution: We have $f'(1)=\lim\limits_{h\to 0}\dfrac{f(1+h)-f(1)}{h}$

$$=\lim_{h\to 0}\left[\frac{1+h-1}{2(1+h)^2-7(1+h)+5}-\left(\frac{-1}{3}\right)\right]\Big/ h$$

$$=\lim_{h\to 0}\frac{3h+2(1+h)^2-7(1+h)+5}{3h\,[2(1+h)^2-7(1+h)+5]}$$

$$=\lim_{h\to 0}\frac{2h^2}{3h(-3h+2h^2)}=\lim_{h\to 0}\frac{2}{-9+6h}=-\frac{2}{9}.$$

Example 9: *Test the continuity and differentiability in* $-\infty<x<\infty$, *of the following function :*

$$\begin{aligned} f(x)&=1 && in\;\; -\infty<x<0\\ &=1+\sin x && in\;\; 0\le x<\tfrac{1}{2}\pi\\ &=2+\left(x-\tfrac{1}{2}\pi\right)^2 && in\;\; \tfrac{1}{2}\pi\le x<\infty.\end{aligned}$$

(Avadh 2009)

Solution: We shall test $f(x)$ for continuity and differentiability at $x=0$ and $\pi/2$. It is obviously continuous as well as differentiable at all other points.

(i) Continuity and differentiability of $f(x)$ at $x=0$

We have $f(0)=1+\sin 0=1;$

$$f(0+0)=\lim_{h\to 0}f(0+h)=\lim_{h\to 0}f(h)=\lim_{h\to 0}(1+\sin h)=1;$$

and $$f(0-0)=\lim_{h\to 0}f(0-h)=\lim_{h\to 0}f(-h)=\lim_{h\to 0}1=1.$$

Since $f(0)=f(0+0)=f(0-0)$, $f(x)$ is continuous at $x=0$.

Now $$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{(1+\sin h)-(1+\sin 0)}{h}=\lim_{h\to 0}\frac{\sin h}{h}=1;$$

and $$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\frac{1-(1+\sin 0)}{-h}=\lim_{h\to 0}\frac{0}{-h}=\lim_{h\to 0}0=0.$$

Since $Rf'(0)\neq Lf'(0)$, $f(x)$ is not differentiable at $x=0$.

(ii) Continuity and differentiability of $f(x)$ at $x = \frac{1}{2}\pi$.

We have
$$f\left(\frac{1}{2}\pi\right) = 2 + \left(\frac{1}{2}\pi - \frac{1}{2}\pi\right)^2 = 2;$$

$$f\left(\frac{1}{2}\pi + 0\right) = \lim_{h \to 0} f\left(\frac{1}{2}\pi + h\right)$$
$$= \lim_{h \to 0}\left[2 + \left\{\left(\frac{1}{2}\pi + h\right) - \frac{1}{2}\pi\right\}^2\right]$$
$$= \lim_{h \to 0}(2 + h^2) = 2;$$

and
$$f\left(\frac{1}{2}\pi - 0\right) = \lim_{h \to 0} f\left(\frac{1}{2}\pi - h\right)$$
$$= \lim_{h \to 0}\left[1 + \sin\left(\frac{1}{2}\pi - h\right)\right]$$
$$= \lim_{h \to 0}(1 + \cos h) = 1 + 1 = 2.$$

Since $f\left(\frac{1}{2}\pi\right) = f\left(\frac{1}{2}\pi + 0\right) = f\left(\frac{1}{2}\pi - 0\right)$, f is continuous at $x = \frac{1}{2}\pi$.

Now
$$R f'\left(\frac{1}{2}\pi\right) = \lim_{h \to 0} \frac{f\left(\frac{1}{2}\pi + h\right) - f\left(\frac{1}{2}\pi\right)}{h}$$
$$= \lim_{h \to 0} \frac{\left[2 + \left\{\frac{1}{2}\pi + h - \frac{1}{2}\pi\right\}^2\right] - \left[2 + \left(\frac{1}{2}\pi - \frac{1}{2}\pi\right)^2\right]}{h}$$
$$= \lim_{h \to 0} \frac{2 + h^2 - 2}{h} = \lim_{h \to 0} h = 0;$$

and
$$L f'\left(\frac{1}{2}\pi\right) = \lim_{h \to 0} \frac{f\left(\frac{1}{2}\pi - h\right) - f\left(\frac{1}{2}\pi\right)}{-h}$$
$$= \lim_{h \to 0} \frac{1 + \sin\left(\frac{1}{2}\pi - h\right) - 2}{-h} = \lim_{h \to 0} \frac{-1 + \cos h}{-h}$$
$$= \lim_{h \to 0} \frac{1 - \cos h}{h} = \lim_{h \to 0} \frac{2\sin^2(h/2)}{h}$$
$$= \lim_{h \to 0}\left[\frac{\sin(h/2)}{h/2} . \sin(h/2)\right] = 1 \times 0 = 0.$$

Since $R f'(0) = L f'(0)$, $f(x)$ is differentiable at $x = \frac{1}{2}\pi$.

Example 10: *If $f(x) = x^2 \sin(1/x)$, for $x \neq 0$ and $f(0) = 0$, then show that $f(x)$ is continuous and differentiable everywhere and that $f'(0) = 0$. Also show that the function $f'(x)$ has a discontinuity of second kind at the origin.* **(Meerut 2006B; Avadh 06; Kumaun 12; Kanpur 14)**

Solution: We have $f(0+0) = \lim_{h \to 0} (0+h)^2 \sin \frac{1}{0+h} = \lim_{h \to 0} h^2 \sin \frac{1}{h} = 0;$

$$f(0-0) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} (-h)^2 \sin(-1/h) = -\lim_{h \to 0} h^2 \sin \frac{1}{h} = 0.$$

$\because$ $f(0+0) = f(0-0) = f(0)$, so the function is continuous at $x = 0$.

Now $$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin(1/h) - 0}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0;$$

and $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{(-h)^2 \sin(-1/h) - 0}{-h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0.$$

Thus $Rf'(0) = Lf'(0)$ implies that $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$.

For all other values of x, $f(x)$ is easily seen to be continuous and differentiable.

Now $$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x} \text{ at } x \neq 0 \text{ and } f'(0) = 0.$$

$\therefore$ $$f'(0+0) = \lim_{h \to 0} f'(0+h) = \lim_{h \to 0} f'(h)$$

$$= \lim_{h \to 0} \left(2h \sin \frac{1}{h} - \cos \frac{1}{h}\right), \text{ which does not exist.}$$

Similarly it can be shown that $f'(0-0)$ does not exist.

Hence f' is discontinuous at the origin. Since both the limits $f'(0-0)$ and $f'(0+0)$ do not exist, therefore the discontinuity is of the second kind.

Example 11: *A function f is defined by* $f(x) = x^p \cos(1/x), x \neq 0; f(0) = 0.$

What conditions should be imposed on p so that f may be

(i) continuous at $x = 0$ *(ii) differentiable at $x = 0$?*

Solution: We have

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} [(0+h)^p \cos\{1/(0+h)\}]$$

$$= \lim_{h \to 0} h^p \cos(1/h) \qquad ...(1)$$

and $$f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0}[(0-h)^p\cos\{1/(0-h)\}]$$

$$=\lim_{h\to 0}(-h)^p\cos(1/h). \quad \ldots(2)$$

Now if the function $f(x)$ is to be continuous at $x=0$, then

$$f(0+0)=f(0)=0=f(0-0)$$

i.e., the limits given in (1) and (2) must both tend to zero.

This is possible only if $p>0$, which is the required condition.

Now $$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{h^p\cos(1/h)-0}{h}=\lim_{h\to 0}h^{p-1}\cos\frac{1}{h} \quad \ldots(3)$$

and $$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{(-h)^p\cos(-1/h)-0}{-h}$$

$$=\lim_{h\to 0}-(-1)^p h^{p-1}\cos(1/h). \quad \ldots(4)$$

Now if $f'(x)$ exists at $x=0$ then we must have $Rf'(0)=Lf'(0)$ and this is possible only if $p-1>0$ *i.e.,* $p>1$ which gives $Rf'(0)=0=Lf'(0)$. Hence in order that f is differentiable at $x=0$, p must be greater than 1.

Example 12: *For a real number y, let* $[y]$ *denote the greatest integer less than or equal to y. Then if* $f(x)=\dfrac{\tan(\pi[x-\pi])}{1+[x]^2}$, *show that* $f'(x)$ *exists for all x.*

Solution: From the definition of $[y]$, we see that $[x-\pi]$ is an integer for all values of x. Then $\pi(x-\pi)$ is an integral multiple of π and so $\tan(\pi[x-\pi])=0$ $\forall x$. Since $[x]$ is an integer so $1+[x]^2\neq 0$ for any x. Thus $f(x)=0$ for all x *i.e.,* $f(x)$ is a constant function and so it is continuous and differentiable *i.e.,* $f'(x)$ exists $\forall x$ and is equal to zero.

Example 13: *Determine the set of all points where the function* $f(x)=x/(1+|x|)$ *is differentiable.*

Solution: Since $|x|=x$, $x>0$, $|x|=-x$, $x<0$, $|x|=0$, $x=0$,

$$\therefore \quad f(x)=\frac{x}{1-x}, x<0;\ f(x)=0,\ x=0;\ f(x)=\frac{x}{1+x}, x>0.$$

We have $$f(0+0)=\lim_{h\to 0}f(0+h)=\lim_{h\to 0}f(h)=\lim_{h\to 0}\frac{h}{1+h}=0;$$

$$f(0-0)=\lim_{h\to 0}f(0-h)=\lim_{h\to 0}f(-h)=\lim_{h\to 0}\frac{-h}{1+h}=0.$$

Since $f(0+0)=f(0)=f(0-0)=0$ so the function is continuous at $x=0$.

Further $$L f'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{[-h/(1+h)] - 0}{-h} = \lim_{h \to 0} \frac{1}{1+h} = 1;$$

$$R f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{[h/(1+h)] - 0}{h} = 1.$$

Since $L f'(0) = R f'(0)$, so the function is differentiable at $x = 0$. It is obviously differentiable for all other real values of x. Hence it is differentiable in the interval $]-\infty, \infty[$.

Example 14: *Let $f(x) = \sqrt{(x)}\,\{1 + x \sin(1/x)\}$ for $x > 0$, $f(0) = 0$,*

$$f(x) = -\sqrt{(-x)}\,\{1 + x \sin(1/x)\} \text{ for } x < 0.$$

Show that $f'(x)$ exists everywhere and is finite except at $x = 0$ where its value is $+\infty$.

Solution: We have

$$R f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{h})\{1 + h \sin(1/h)\} - 0}{h}$$

$$= \lim_{h \to 0} \left[\frac{1}{\sqrt{h}} + (\sqrt{h}) \sin(1/h)\right] = \infty + 0 = \infty$$

and $$L f'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-\sqrt{[-(-h)]}\left[1 + (-h) \sin \frac{1}{-h}\right] - 0}{-h}$$

$$= \lim_{h \to 0} \left[\frac{1}{\sqrt{h}} + (\sqrt{h}) \sin \frac{1}{h}\right] = \infty + 0 = \infty.$$

Since $R f'(0) = L f'(0) = \infty$, $\therefore f'(0) = \infty$.

We have $$f'(x) = \frac{1}{2\sqrt{x}} + \frac{3}{2}\sqrt{x} \sin \frac{1}{x} - \frac{1}{\sqrt{x}} \cos \frac{1}{x} \text{ for } x > 0$$

and $$f'(x) = \frac{1}{2\sqrt{(-x)}} + \frac{3}{2}\sqrt{(-x)} \sin \frac{1}{x} - \frac{1}{\sqrt{(-x)}} \cos \frac{1}{x} \text{ for } x < 0.$$

Hence $f'(a)$ is finite for all $a \neq 0$.

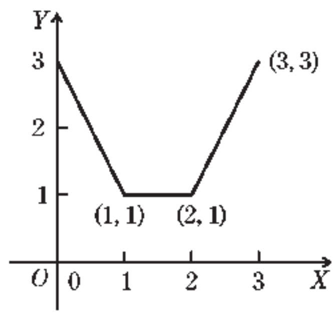
Example 15: *Draw the graph of the function* $y = |x-1| + |x-2|$ *in the interval* $[0,3]$ *and discuss the continuity and differentiability of the function in this interval.*

(Garhwal 2008; Meerut 07B, 09; Gorakhpur 12)

Solution: From the given definition of the function, we have

$$y = 1 - x + 2 - x = 3 - 2x \quad \text{when} \quad x \le 1$$
$$y = x - 1 + 2 - x = 1 \quad \text{when} \quad 1 \le x \le 2$$
$$y = x - 1 + x - 2 = 2x - 3 \quad \text{when} \quad x \ge 2.$$

Thus the graph consists of the segments of the three straight lines $y = 3 - 2x$, $y = 1$ and $y = 2x - 3$ corresponding to the intervals $[0,1], [1,2], [2,3]$ respectively. The graph of the function for the interval $[0,3]$ is as given in the figure.



The graph shows that the function is continuous throughout the interval but is not differentiable at $x = 1, 2$ because the slopes at these points are different on the left and right hand sides.

To test it analytically, we write $y = f(x)$. Then

$$f(x) = 3 - 2x \quad \text{when} \quad x \le 1$$
$$= 1 \quad \text{when} \quad 1 \le x \le 2$$
$$= 2x - 3 \quad \text{when} \quad x \ge 2.$$

This function is obviously continuous and differentiable at all points of the interval $[0,3]$ except possibly at $x = 1$ and at $x = 2$.

At $x = 1$, we have $f(1) = 1$;

$$f(1-0) = \lim_{h \to 0} [3 - 2(1-h)] = 1; \; f(1+0) = \lim_{h \to 0} (1) = 1.$$

Since $f(1-0) = f(1+0) = f(1)$, f is continuous at $x = 1$.

Again $Rf'(1) = \lim_{h \to 0} \dfrac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \dfrac{1-1}{h} = 0$

and $Lf'(1) = \lim_{h \to 0} \dfrac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \dfrac{3 - 2(1-h) - 1}{-h} = -2.$

Since $Rf'(1) \ne Lf'(1)$, f is not differentiable at $x = 1$.

At $x = 2$, we have $f(2) = 1$;

$$f(2-0) = \lim_{h \to 0} (1) = 1; \; f(2+0) = \lim_{h \to 0} [2(2+h) - 3] = 1.$$

Since $f(2-0) = f(2+0) = f(2)$, f is continuous at $x = 2$.

Again $Rf'(2) = \lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \dfrac{2(2+h) - 3 - 1}{h} = 2$

and $Lf'(2) = \lim_{h \to 0} \dfrac{f(2-h) - f(2)}{-h} = \lim_{h \to 0} \dfrac{1-1}{-h} = 0.$

Since $Rf'(2) \ne Lf'(2)$, f is not differentiable at $x = 2$.

Example 16: *Show that the function* $f : \mathbf{R} \to \mathbf{R}$ *defined by*

$$f(x) = x\left[1 + \frac{1}{3}\sin\log x^2\right],\ x \neq 0 \text{ and } f(0) = 0$$

is everywhere continuous but has no differential coefficient at the origin. **(Garhwal 2009)**

Solution: Obviously the function $f(x)$ is continuous at every point of **R** except possibly at $x = 0$. We test at $x = 0$. Given $f(0) = 0$.

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0}\left[(0+h)\left\{1 + \frac{1}{3}\sin\log(0+h)^2\right\}\right]$$

$$= \lim_{h \to 0}[h + (h/3)\sin\log h^2] = 0 + 0 \times \text{ a finite quantity} = 0.$$

[$\because$ $\sin\log h^2$ oscillates between -1 and $+1$ as $h \to 0$]

Similarly we can show that $f(0-0) = 0$.

Hence f is continuous at $x = 0$.

Now $$Rf'(0) = \lim_{h \to 0}\frac{(0+h)\left\{1 + \frac{1}{3}\sin\log(0+h)^2\right\} - 0}{h}$$

$$= \lim_{h \to 0}\left\{1 + \frac{1}{3}\sin\log h^2\right\},$$ which does not exist since $\sin\log h^2$ oscillates between -1 and 1 as $h \to 0$.

$$Lf'(0) = \lim_{h \to 0}\frac{(0-h)\left\{1 + \frac{1}{3}\sin\log(0-h)^2\right\} - 0}{-h}$$

$$= \lim_{h \to 0}\left[1 + \frac{1}{3}\sin\log h^2\right],$$ which does not exist as above.

Hence f has no differential coefficient at $x = 0$.

Example 17: *Let* $f(x) = x\dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}},\ x \neq 0;\ f(0) = 0.$

Show that $f(x)$ *is continuous but not derivable at* $x = 0$.

(Meerut 2005; Purvanchal 07;Kanpur 08; Lucknow 09; Gorakhpur 10; Bundelkhand 14)

Solution: We have $f(0) = 0$;

$$f(0+0) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} h\frac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}}$$

$$= \lim_{h \to 0} h\frac{1 - e^{-2/h}}{1 + e^{-2/h}},$$ dividing the Nr. and Dr. by $e^{1/h}$

$$= 0 \times \frac{1-0}{1+0} = 0 \times 1 = 0;$$

and $\quad f(0-0)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)$

$$=\lim_{h\to 0} -h\,\frac{e^{1/-h}-e^{-1/-h}}{e^{1/-h}+e^{-1/-h}}=\lim_{h\to 0} -h\,\frac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}}$$

$$=\lim_{h\to 0} -h\,\frac{e^{-2/h}-1}{e^{-2/h}+1}=0\times\frac{0-1}{0+1}=0.$$

Since $f(0+0)=f(0-0)=f(0)$, the function is continuous at $x=0$.

Now $\quad Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}$

$$=\lim_{h\to 0}\left[h\,\frac{e^{1/h}-e^{-1/h}}{e^{1/h}+e^{-1/h}}-0\right]\Big/ h=\lim_{h\to 0}\frac{1-e^{-2/h}}{1+e^{-2/h}}=\frac{1-0}{1+0}=1.$$

and $\quad Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$

$$=\lim_{h\to 0}\left[(-h)\,\frac{e^{-1/h}-e^{1/h}}{e^{-1/h}+e^{1/h}}-0\right]\Big/(-h)$$

$$=\lim_{h\to 0}\frac{e^{-2/h}-1}{e^{-2/h}+1}=\frac{0-1}{0+1}=-1.$$

Since $Rf'(0)\neq Lf'(0)$, the function is not derivable at $x=0$.

Example 18: *Let $f(x)=e^{-1/x^2}\sin(1/x)$ when $x\neq 0$ and $f(0)=0$. Show that at every point f has a differential coefficient and this is continuous at $x=0$.* **(Avadh 2006)**

Solution: We test differentiability at $x=0$.

$$Rf'(0)=\lim_{h\to 0}\frac{e^{-1/h^2}\sin(1/h)-0}{h}=\lim_{h\to 0}\frac{\sin(1/h)}{he^{1/h^2}}$$

$$=\lim_{h\to 0}\frac{\sin(1/h)}{h\left\{1+\dfrac{1}{h^2}+\dfrac{1}{2!\,h^4}+\dots\right\}}=\lim_{h\to 0}\frac{\sin(1/h)}{h+\dfrac{1}{h}+\dfrac{1}{2!}\cdot\dfrac{1}{h^3}+\dots}$$

$$=\frac{\text{a finite quantity lying between }-1\text{ and }+1}{\infty}=0.$$

Similarly $Lf'(0)=0$.

Since $Rf'(0)=Lf'(0)=0$, hence the function $f(x)$ is differentiable at $x=0$ and $f'(0)=0$.

If x is any point other than zero, then

$$f'(x)=(2/x^3)\,e^{-1/x^2}\sin(1/x)-(1/x^2)\,e^{-1/x^2}\cos(1/x)$$

$$=\{(2/x)\sin(1/x)-\cos(1/x)\}(1/x^2)(1/e^{1/x^2}) \qquad \dots(1)$$

Now $\quad f'(0+0) = \lim_{h \to 0} f'(0+h) = \lim_{h \to 0} \left(\frac{2}{h} \sin \frac{1}{h} - \cos \frac{1}{h} \right) \cdot \frac{1}{h^2 e^{1/h^2}}$

$$= \lim_{h \to 0} \left(\frac{2 \sin (1/h)}{h^3 e^{1/h^2}} - \frac{\cos (1/h)}{h^2 e^{1/h^2}} \right)$$

$$= \lim_{h \to 0} \left[\frac{2 \sin (1/h)}{h^3 \left(1 + \frac{1}{h^2} + \frac{1}{2!.h^4} + \dots \right)} - \frac{\cos (1/h)}{h^2 \left(1 + \frac{1}{h^2} + \frac{1}{2!.h^4} + \dots \right)} \right]$$

$$= \frac{\text{some finite quantity}}{\infty} - \frac{\text{some finite quantity}}{\infty} = 0 \cdot$$

Similarly $f'(0-0) = 0$. Hence f' is continuous at $x = 0$.

Comprehensive Exercise 1

1. Show that $f(x) = |x - 1|, 0 \le x \le 2$ is not derivable at $x = 1$. Is it continuous in $[0, 2]$?

2. (a) If $f(x) = \frac{x}{1 + e^{1/x}}, x \ne 0, f(0) = 0,$ show that f is continuous at $x = 0,$ but $f'(0)$ does not exist.
(Lucknow 2005, 10; Gorakhpur 13; Purvanchal 14)

   (b) If $f(x) = \frac{x e^{1/x}}{1 + e^{1/x}}$ for $x \ne 0$ and $f(0) = 0,$ show that $f(x)$ is continuous at $x = 0,$ but $f'(0)$ does not exist. **(Lucknow 2006)**

3. A function ϕ is defined as follows :

   $$\phi(x) = -x \text{ for } x \le 0, \phi(x) = x \text{ for } x \ge 0.$$

   Test the character of the function at $x = 0$ as regards continuity and differentiability.

4. Show that the function f defined on **R** by

   $$f(x) = |x - 1| + 2|x - 2| + 3|x - 3|$$

   is continuous but not differentiable at the points $1, 2,$ and 3.
(Bundelkhand 2009)

5. Show that the function $f(x) = x, 0 < x \le 1$
$\qquad = x - 1, 1 < x \le 2$

   has no derivative at $x = 1$.

6. Show that the function $f(x) = x^2 - 1, x \ge 1$
$\qquad = 1 - x, x < 1$

   has no derivative at $x = 1$.

7. The following limits are derivatives of certain functions at a certain point. Determine these functions and the points.

(i) $\lim_{x \to 2} \dfrac{\log x - \log 2}{x - 2}$ (ii) $\lim_{h \to 0} \dfrac{\sqrt{(a+h)} - \sqrt{a}}{h}$.

8. Let $f(x) = x^2 \sin(x^{-4/3})$ except when $x = 0$ and $f(0) = 0$. Prove that $f(x)$ has zero as a derivative at $x = 0$.

9. A function ϕ is defined as : $\phi(x) = 1 + x$ if $x \le 2$, $\phi(x) = 5 - x$ if $x > 2$.
Test the character of the function at $x = 2$ as regards its continuity and differentiability. **(Avadh 2007)**

10. Examine the following curve for continuity and differentiability :

$$y = x^2 \quad \text{for} \quad x \le 0$$
$$y = 1 \quad \text{for} \quad 0 < x \le 1$$
$$y = 1/x \quad \text{for} \quad x > 1.$$

Also draw the graph of the function. **(Meerut 2003, 04B, 09B)**

11. A function $f(x)$ is defined as follows :

$$f(x) = 1 + x \quad \text{for} \quad x \le 0,$$
$$f(x) = x \quad \text{for} \quad 0 < x < 1,$$
$$f(x) = 2 - x \quad \text{for} \quad 1 \le x \le 2,$$
$$f(x) = 3x - x^2 \quad \text{for} \quad x > 2.$$

Discuss the continuity of $f(x)$ and the existence of $f'(x)$ at $x = 0, 1$ and 2.

12. Discuss the continuity and differentiability of the following function :

$$f(x) = x^2 \quad \text{for} \quad x < -2$$
$$f(x) = 4 \quad \text{for} \quad -2 \le x \le 2$$
$$f(x) = x^2 \quad \text{for} \quad x > 2.$$

Also draw the graph. **(Meerut 2007, 10B)**

13. A function $f(x)$ is defined as follows :

$$f(x) = x \text{ for } 0 \le x \le 1, \; f(x) = 2 - x \text{ for } x \ge 1.$$

Test the character of the function at $x = 1$ as regards the continuity and differentiability. **(Meerut 2003)**

14. Examine the function defined by $f(x) = x^2 \cos(e^{1/x}), x \ne 0, f(0) = 0$ with regard to (i) continuity (ii) differentiability in the interval $]-1, 1[$.

15. (a) Define continuity and differentiability of a function at a given point. If a function possesses a finite differential coefficient at a point, show that it is continuous at this point. Is the converse true ? Give example in support of your answer.

(b) What do you understand by the derivative of a real valued function at the point $b \in \mathbf{R}$? Apply your definition to discuss the derivative of $f(x) = |x|, x \in \mathbf{R}$ *at* $x = 0$.

(c) Prove that if a function $f(x)$ possesses a finite derivative in a closed interval $[a, b]$, then $f(x)$ is continuous in $[a, b]$.

Answers 1

1. Yes

3. Continuous at $x = 0$ but not differentiable at $x = 0$

7. (i) The function is $\log x$ and the point is $x = 2$

(ii) The function is $\sqrt{x}$ and the point is $x = a$

9. Continuous but not differentiable at $x = 2$

10. Discontinuous and non-differentiable at $x = 0$, continuous and non-differentiable at $x = 1$

11. Discontinuous and non-differentiable at $x = 0, 2$ and continuous but not differentiable at $x = 1$

12. Continuous but not differentiable at $x = -2, 2$

13. Continuous but non-differentiable at $x = 1$

14. Continuous and differentiable throughout $\mathbf{R}$

8 Rolle's Theorem

If a function $f(x)$ is such that

(*i*) *$f(x)$ is continuous in the closed interval $[a, b]$,*

(*ii*) *$f'(x)$ exists for every point in the open interval $]a, b[$,*

(*iii*) *$f(a) = f(b)$, then there exists at least one value of x, say c, where $a < c < b$, such that $f'(c) = 0$.* **(Lucknow 2007; Purvanchal 07; Kanpur 08, 12; Meerut 12B; Kashi 13, 14; Gorakhpur 12, 13, 14; Avadh 08, 11, 14)**

Proof: Since f is continuous on $[a, b]$, it is bounded on $[a, b]$. Let M and m be the supremum and infimum of f respectively in the closed interval $[a, b]$.

Now either $M = m$ or $M \neq m$.

If $M = m$, then f is a constant function over $[a, b]$ and consequently $f'(x) = 0$ for all x in $[a, b]$. Hence the theorem is proved in this case.

If $M \neq m$, then at least one of the numbers M and m must be different from the equal values $f(a)$ and $f(b)$. For the sake of definiteness, let $M \neq f(a)$.

Since every continuous function on a closed interval attains its supremum, therefore, there exists a real number c in $[a, b]$ such that $f(c) = M$. Also, since $f(a) \neq M \neq f(b)$, therefore, c is different from both a and b. This implies that $c \in]a, b[$.

Now $f(c)$ is the supremum of f on $[a, b]$, therefore,

$$f(x) \leq f(c) \quad \forall \quad x \text{ in } [a, b]. \qquad \ldots(1)$$

In particular, for all positive real numbers h such that $c - h$ lies in $[a, b]$,

$$f(c-h) \leq f(c)$$

$$\Rightarrow \qquad \frac{f(c-h) - f(c)}{-h} \geq 0. \qquad \ldots(2)$$

Since $f'(x)$ exists at each point of $]a, b[$, and hence, in particular $f'(c)$ exists, so taking limit as $h \to 0$, (2) gives $\quad Lf'(c) \geq 0$. ...(3)

Similarly, from (1), for all positive real numbers h such that $c + h$ lies in $[a, b]$, we have

$$f(c+h) \leq f(c).$$

By the same argument as above, we get

$$Rf'(c) \leq 0. \qquad \ldots(4)$$

Since $f'(c)$ exists, hence, $Lf'(c) = f'(c) = Rf'(c)$. ...(5)

From (3), (4) and (5) we conclude that $f'(c) = 0$.

In the same manner we can consider the case $M = f(a) \neq m$.

Note 1: There may be more than one point like c at which $f'(x)$ vanishes.

Note 2: Rolle's theorem will not hold good

(i) if $f(x)$ is discontinuous at some point in the interval $a \leq x \leq b$,

or (ii) if $f'(x)$ does not exist at some point in the interval $a < x < b$,

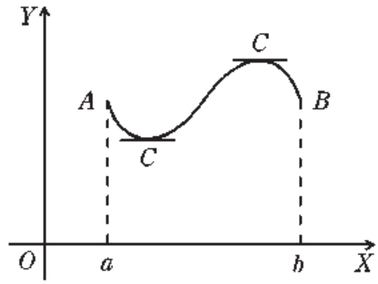
or (iii) if $f(a) \neq f(b)$.

Note 3: It can be seen that the conditions of Rolle's theorem are not necessary for $f'(x)$ to vanish at some point in $]a, b[$. For example, $f(x) = \cos(1/x)$ is discontinuous at $x = 0$ in the interval $[-1, 2]$ but $f'(x)$ vanishes at an infinite number of points in the interval.

Geometrical interpretation of Rolle's Theorem: **(Gorakhpur 2015)**

Suppose the function $f(x)$ is not constant and satisfies the conditions of Rolle's theorem in the interval $[a, b]$. Then its geometrical interpretation is that the curve representing the graph of the function f must have a tangent parallel to x-axis, at least at one point between a and b.

Algebraical interpretation of Rolle's Theorem:



Rolle's theorem leads to a very important result in the theory of equations, when $f(a)=f(b)=0$ and $f:[a,b]\to\mathbf{R}$ is a polynomial function $f(x)$. Here a and b are the roots of the equation $f(x)=0$. Since a polynomial function $f(x)$ is continuous and differentiable at every point of its domain and we have taken $f(a)=f(b)$, therefore, all the three conditions of Rolle's theorem are satisfied and consequently there exists a point $c\in\,]a,b[$ such that $f'(c)=0$ *i.e., if a and b are any two roots of the polynomial equation* $f(x)=0$, *then there exists at least one root of the equation* $f'(x)=0$ *which lies between a and b.*

Illustrative Examples

Example 19: *Discuss the applicability of Rolle's theorem for* $f(x)=2+(x-1)^{2/3}$ *in the interval* $[0,2]$. **(Meerut 2012)**

Solution: We have $f(x)=2+(x-1)^{2/3}$. Here $f(0)=3=f(2)$, which shows that the third condition of Rolle's theorem is satisfied.

Since $f(x)$ is an algebraic function of x, it is continuous in the closed interval $[0,2]$. Thus the first condition of Rolle's theorem is satisfied.

Now $f'(x)=\dfrac{2}{3}\cdot[1/(x-1)^{1/3}]$. We see that for $x=1$, $f'(x)$ is not finite while $x=1$ is a point of the open interval $0<x<2$. Thus the second condition of Rolle's theorem is not satisfied.

Hence the Rolle's theorem is not applicable for the function $f(x)=2+(x-1)^{2/3}$ in the interval $[0,2]$.

Example 20: *Discuss the applicability of Rolle's theorem in the interval* $[-1,1]$ *to the function* $f(x)=|x|$.

Solution: Given $f(x)=|x|$. Here $f(-1)=|-1|=1$, $f(1)=|1|=1$, so that $f(-1)=f(1)$.

Further the function $f(x)$ is continuous throughout the closed interval $[-1,1]$ but it is not differentiable at $x=0$ which is a point of the open interval $]-1,1[$. Thus the second condition of Rolle's theorem is not satisfied. Hence the Rolle's theorem is not applicable here.

Example 21: *Are the conditions of Rolle's theorem satisfied in the case of the following functions ?*

(i) $f(x)=x^2$ *in* $2\le x\le 3$, *(ii)* $f(x)=\cos(1/x)$ *in* $-1\le x\le 1$,

(iii) $f(x)=\tan x$ *in* $0\le x\le\pi$.

Solution: (i) The function $f(x) = x^2$ is continuous and differentiable in the interval $[2, 3]$. Also $f(2) = 4$ and $f(3) = 9$, so that $f(2) \neq f(3)$.
Thus the first two conditions of Rolle's theorem are satisfied and the third condition is not satisfied.

(ii) The function $f(x) = \cos(1/x)$ is discontinuous at $x = 0$ and consequently is not differentiable there. Thus the first two conditions of Rolle's theorem are not satisfied.

Here $f(-1) = \cos(-1) = \cos 1$ and $f(1) = \cos 1$. Thus $f(-1) = f(1)$ *i.e.*, the third condition is satisfied.

(iii) The function $f(x) = \tan x$ is not continuous at $x = \pi/2$ and consequently is not differentiable there. Thus the first two conditions of Rolle's theorem are not satisfied here.

Further $f(0) = \tan 0 = 0$ and $f(\pi) = \tan \pi = 0$. Thus $f(0) = f(\pi)$ *i.e.*, the third condition is satisfied.

Example 22: *Discuss the applicability of Rolle's theorem to* $f(x) = \log\left[\dfrac{x^2 + ab}{(a+b)\,x}\right]$, *in the interval* $[a, b]$, $0 < a < b$. **(Lucknow 2008, 11; Rohilkhand 14)**

Solution: Here $f(a) = \log\left[\dfrac{a^2 + ab}{(a+b)\,a}\right] = \log 1 = 0,$

and $$f(b) = \log\left[\frac{b^2 + ab}{(a+b)\,b}\right] = \log 1 = 0.$$

Thus $$f(a) = f(b) = 0.$$

Also
$$Rf'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log\left\{\frac{(x+h)^2 + ab}{(a+b)(x+h)}\right\} - \log\left\{\frac{x^2 + ab}{(a+b)\,x}\right\}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log \frac{(x^2 + 2xh + h^2 + ab)(a+b)\,x}{(a+b)(x+h)(x^2 + ab)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log\left\{\frac{(x^2 + 2xh + h^2 + ab)}{x^2 + ab} \times \frac{x}{x+h}\right\}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log\left\{1 + \frac{2xh + h^2}{x^2 + ab}\right\} - \log\left\{1 + \frac{h}{x}\right\}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{2xh + h^2}{x^2 + ab} - \frac{h}{x} + \ldots\right], \qquad \ldots(1)$$

$$\left[\because \log(1 + y) = y - \frac{1}{2}y^2 + \ldots\right]$$

$$= \frac{2x}{x^2 + ab} - \frac{1}{x}.$$

Again $\quad L\, f\,'(x) = \lim_{h \to 0} \left[\frac{f\,(x - h) - f\,(x)}{-h} \right]$

$$= \lim_{h \to 0} \frac{1}{(-h)} \left[\frac{-2hx + h^2}{x^2 + ab} - \frac{(-h)}{x} + \dots \right],$$

replacing h by $-h$ in (1)

$$= \frac{2x}{x^2 + ab} - \frac{1}{x}.$$

Since $R\, f\,'(x) = L\, f\,'(x)$, $f\,(x)$ is differentiable for all values of x in $[a, b]$. This implies that $f\,(x)$ is also continuous for all values of x in $[a, b]$. Thus all the three conditions of Rolle's theorem are satisfied. Hence $f\,'(x) = 0$ for at least one value of x in the open interval $]a, b[$.

Now $\quad f'(x) = 0 \Rightarrow \frac{2x}{x^2 + ab} - \frac{1}{x} = 0$

or $\quad 2\,x^2 - (x^2 + ab) = 0$

or $\quad x^2 = ab \quad$ or $\quad x = \sqrt{(ab)},$

which being the geometric mean of a and b lies in the open interval $]\,a, b[$. Hence the Rolle's theorem is verified.

Remark: In this question to find $f\,'(x)$, we can also proceed as follows :
We have

$$f\,(x) = \log\,(x^2 + ab) - \log\,(a + b) - \log\, x.$$

$\therefore \quad f'(x) = \frac{2\,x}{x^2 + ab} - \frac{1}{x}.$

Obviously $f\,'(x)$ exists for all values of x in $[a, b]$.

Example 23: *Verify Rolle's theorem in the case of the functions*

(i) $\quad f\,(x) = 2x^3 + x^2 - 4x - 2,$ **(Lucknow 2009)**

(ii) $\quad f\,(x) = \sin x$ *in* $[0, \pi],$

(iii) $\quad f\,(x) = (x - a)^m\,(x - b)^n$, *where m and n are +ive integers, and x lies in the interval* $[a, b]$.

Solution: (i) Since $f\,(x)$ is a rational integral function of x, therefore, it is continuous and differentiable for all real values of x. Thus the first two conditions of Rolle's theorem are satisfied in any interval.

Here $\quad f\,(x) = 0$ gives $2x^3 + x^2 - 4x - 2 = 0$

or $\quad (x^2 - 2)\,(2x + 1) = 0$ *i.e.,* $x = \pm \sqrt{2}, -\frac{1}{2}.$

Thus $\quad f\,(\sqrt{2}) = f\,(-\sqrt{2}) = f\left(-\frac{1}{2}\right) = 0.$

If we take the interval $[-\sqrt{2}, \sqrt{2}]$, then all the three conditions of Rolle's theorem are satisfied in this interval. Consequently there is at least one value of x in the open interval $]-\sqrt{2}, \sqrt{2}[$ for which $f'(x)=0$.

Now $\qquad f'(x)=0 \Rightarrow 6x^2+2x-4=0 \Rightarrow 3x^2+x-2=0$

or $\qquad (3x-2)(x+1)=0 \quad$ or $\quad x=-1, 2/3 \quad$ *i.e.,* $\quad f'(-1)=f'(2/3)=0$.

Since both the points $x=-1$ and $x=2/3$ lie in the open interval $]-\sqrt{2}, \sqrt{2}[$, Rolle's theorem is verified.

(ii) The function $f(x)=\sin x$ is continuous and differentiable in $[0, \pi]$.

Also $f(0)=0=f(\pi)$. Thus all the three conditions of Rolle's theorem are satisfied. Hence $f'(x)=0$ for at least one value of x in the open interval $]0, \pi[$.

Now $\qquad f'(x)=0 \Rightarrow \cos x=0 \Rightarrow x=\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$.

Since $x=\pi/2$ lies in the open interval $]0, \pi[$, the Rolle's theorem is verified.

(iii) We have $f(x)=(x-a)^m(x-b)^n$.

As m and n are positive integers, $(x-a)^m$ and $(x-b)^n$ are polynomials in x on being expanded by binomial theorem. Hence $f(x)$ is also a polynomial in x. Consequently $f(x)$ is continuous and differentiable in the closed interval $[a, b]$. Also $f(a)=f(b)=0$.

Thus all the three conditions of Rolle's theorem are satisfied so that there is at least one value of x in the open interval $]a, b[$ where $f'(x)=0$.

Now $\quad f'(x)=(x-a)^m \cdot n(x-b)^{n-1}+m(x-a)^{m-1}(x-b)^n$.

Solving the equation $f'(x)=0$, we get $x=a, b, (na+mb)/(m+n)$

Out of these values the value $(na+mb)/(m+n)$ is a point which lies in the open interval $]a, b[$, since it divides the interval $]a, b[$ internally in the ratio $m:n$. Hence the Rolle's theorem is verified.

Example 24: *Verify Rolle's theorem for*

$$f(x)=x(x+3)e^{-x/2} \text{ in } [-3, 0].$$

(Gorakhpur 2015)

Solution: We have $f(x)=x(x+3)e^{-x/2}$.

$$\therefore \qquad f'(x)=(2x+3)e^{-x/2}+(x^2+3x)e^{-x/2} \cdot \left(-\frac{1}{2}\right)$$

$$=e^{-x/2}\left[2x+3-\frac{1}{2}(x^2+3x)\right]=-\frac{1}{2}(x^2-x-6)e^{-x/2},$$

which exists for every value of x in the interval $[-3, 0]$. Hence $f(x)$ is differentiable and so also continuous in the interval $[-3, 0]$. Also $f(-3)=f(0)=0$.

Thus all the three conditions of Rolle's theorem are satisfied. So $f'(x)=0$ for at least one value of x lying in the open interval $]-3, 0[$.

Now $\qquad f'(x)=0 \Rightarrow -\frac{1}{2}(x^2-x-6)e^{-x/2}=0 \quad$ or $\quad x^2-x-6=0$

or $\qquad (x-3)(x+2)=0 \quad$ or $\quad x=3, -2$.

Since the value $x=-2$ lies in the open interval $]-3, 0[$, the Rolle's theorem is verified.

Comprehensive Exercise 2

1. (i) State Rolle's Theorem. **(Kanpur 2005, 08; Lucknow 07)**

(ii) Verify Rolle's theorem when $f(x) = e^x \sin x, a = 0, b = \pi$. **(Gorakhpur 2012)**

2. Verify Rolle's theorem for the following functions :

(i) $f(x) = (x-4)^5 (x-3)^4$ in the interval $[3, 4]$.

(ii) $f(x) = x^3 - 6x^2 + 11x - 6$.

(iii) $f(x) = x^3 - 4x$ in $[-2, 2]$. **(Kanpur 2007)**

(iv) $f(x) = e^x (\sin x - \cos x)$ in $[\pi/4, 5\pi/4]$. **(Meerut 2013B)**

(v) $f(x) = 10x - x^2$ in $[0, 10]$. **(Kanpur 2006)**

3. Discuss the applicability of Rolle's theorem to the function

$$f(x) = x^2 + 1, \text{ when } 0 \le x \le 1$$
$$= 3 - x, \text{ when } 1 < x \le 2.$$

4. Show that between any two roots of $e^x \cos x = 1$ there exists at least one root of $e^x \sin x - 1 = 0$.

5. State and prove Rolle's theorem. Interpret it geometrically. Verify Rolle's theorem for the function $f(x) = x^2$ in $[-1, 1]$. **(Lucknow 2010)**

6. Verify the truth of Rolle's theorem for the function $f(x) = x^2 - 3x + 2$ on the interval $[1, 2]$.

7. Does the function $f(x) = |x - 2|$ satisfy the conditions of Rolle's theorem in the interval $[1, 3]$? Justify your answer with correct reasoning.

8. The function f is defined in $[0, 1]$ as follows :

$$f(x) = 1 \quad \text{for } 0 \le x < \frac{1}{2}$$
$$= 2 \quad \text{for } \frac{1}{2} \le x \le 1.$$

Show that $f(x)$ satisfies none of the conditions of Rolle's theorem, yet $f'(x) = 0$ for many points in $[0, 1]$.

9. If $a + b + c = 0$, then show that the quadratic equation $3ax^2 + 2bx + c = 0$ has at least one root in $]0, 1[$.

10. Let $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots + \frac{a_{n-1}}{2} + a_n = 0$. Show that there exists at least one real x between 0 and 1 such that $a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0$.

(Lucknow 2009)

11. If $f(x) = \begin{vmatrix} \sin x & \sin \alpha & \sin \beta \\ \cos x & \cos \alpha & \cos \beta \\ \tan x & \tan \alpha & \tan \beta \end{vmatrix}$ where $0 < \alpha < \beta < \pi / 2$,

show that $f'(\xi) = 0$, where $\alpha < \xi < \beta$.

12. Show that there is no real number k for which the equation $x^3 - 3x + k = 0$, has two distinct roots in $]0, 1[$.

Answers 2

3. The given function is not differentiable at $x = 1$ and so Rolle's theorem is not applicable to the given function in the interval [0, 2].
7. The function does not satisfy the third condition that $f(x)$ must be differentiable in the open interval $]1, 3[$.

9 Lagrange's Mean Value Theorem

Theorem: *If a function $f(x)$ is* **(Lucknow 2006, 09; Avadh 07, 12, 14; Meerut 12; Kanpur 11; Rohilkhand 12, 12B; Gorakhpur 10, 12, 14; Kashi 14)**

(i) continuous in a closed interval $[a, b]$,

and (ii) differentiable in the open interval $]a, b[$ i.e., $a < x < b$, then there exists at least one value 'c' of x lying in the open interval $]a, b[$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof: Consider the function $\phi(x)$ defined by $\phi(x) = f(x) + Ax$, ...(1)

where A is a constant to be chosen such that $\phi(a) = \phi(b)$

i.e., $f(a) + Aa = f(b) + Ab$

or $A = -\dfrac{f(b) - f(a)}{b - a}$. ...(2)

(i) Now the function f is given to be continuous on $[a, b]$ and the mapping $x \to Ax$ is continuous on $[a, b]$, therefore ϕ is continuous on $[a, b]$.

(ii) Also, since f is given to be differentiable on $]a, b[$ and the mapping $x \to Ax$ is differentiable on $]a, b[$, therefore, ϕ is differentiable on $]a, b[$.

(iii) By our choice of A, we have $\phi(a) = \phi(b)$.

From (i), (ii) and (iii), we find that ϕ satisfies all the conditions of Rolle's theorem on $[a, b]$. Hence there exists at least one point, say $x = c$, of the open interval $]a, b[$, such that $\phi'(c) = 0$.

But $\qquad \phi'(x) = f'(x) + A$, from (1).

$\therefore \qquad \phi'(c) = 0 \Rightarrow f'(c) + A = 0$

or $\qquad f'(c) = -A = \dfrac{f(b) - f(a)}{b - a}$, from (2).

This proves the theorem. It is usually known as the 'First Mean Value Theorem of Differential Calculus'.

Another form of Lagrange's mean value theorem:

If in the above theorem, we take $b = a + h$, then a number c, lying between a and b can be written as $c = a + \theta h$, where θ is some real number such that $0 < \theta < 1$.

Now Lagrange's theorem can be stated as follows :

If f be defined and continuous on $[a, a + h]$ and differentiable on $]a, a + h[$, then there exists a point $c = a + \theta h$ $(0 < \theta < 1)$ in the open interval $]a, a + h[$ such that

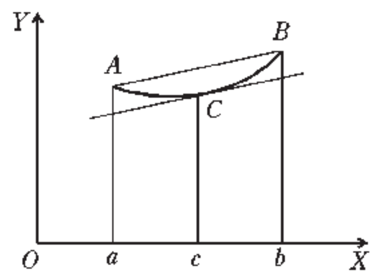
$$\frac{f(a + h) - f(a)}{h} = f'(a + \theta h)$$

or $$\boldsymbol{f(a + h) - f(a) = hf'(a + \theta h).}$$

Geometrical interpretation of the mean value theorem:

(Gorakhpur 2012, 14)

Let $y = f(x)$ and let ACB be the graph of $y = f(x)$ in $[a, b]$. The coordinates of the point A are $(a, f(a))$ and those of B are $(b, f(b))$. If the chord AB makes an angle α with the x-axis, then



$$\tan\alpha = \frac{f(b) - f(a)}{b - a}$$

$$= f'(c),$$

by Lagrange's mean value theorem where $a < c < b$.

Thus Lagrange's mean value theorem says that there is some point c in $]a, b[$ such that the tangent to the curve at this point is parallel to the chord joining the points on the graph with abscissae a and b.

10 Some Important Deductions from the Mean Value Theorem

Theorem 1: *If a function f is continuous on $[a, b]$, differentiable on $]a, b[$ and if $f'(x) = 0$ for all x in $]a, b[$, then $f(x)$ has a constant value throughout $[a, b]$.*

Proof: Let c be any point of $]a, b]$. Then the function f is continuous on $[a, c]$ and differentiable on $]a, c[$. Thus f satisfies all the conditions of Lagrange's mean value theorem on $[a, c]$. Consequently there exists a real number d between a and c *i.e.,* $a < d < c$ such that

$$f(c) - f(a) = (c - a) f'(d).$$

But by hypothesis $f'(x)=0$ throughout the interval $]a,b[$, therefore, in particular $f'(d)=0$ and hence $f(c)-f(a)=0$ or $f(c)=f(a)$. Since c is any point of $]a,b]$, therefore, it gives that $f(x)=f(a)\ \forall\, x$ in $]a,b]$. Thus $f(x)$ has a constant value throughout $[a,b]$.

Theorem 2: *If $f(x)$ and $\phi(x)$ are functions continuous on $[a,b]$ and differentiable on $]a,b[$ and if $f'(x)=\phi'(x)$ throughout the interval $]a,b[$, then $f(x)$ and $\phi(x)$ differ only by a constant.*

Proof: Consider the function $F(x)=f(x)-\phi(x)$. Throughout the interval $]a,b[$, we have

$$F'(x)=f'(x)-\phi'(x)=0, \text{ because } f'(x)=\phi'(x).$$

Consequently, from theorem 1, we get

$$F(x)=\text{constant or } f(x)-\phi(x)=\text{constant}.$$

Theorem 3: *If $f'(x)=k$ for each point x of $[a,b]$, k being a constant, then*

$$f(x)=kx+C\ \forall\, x\in[a,b], \text{ where } C \text{ is a constant.}$$

Proof: Consider the interval $[a,x]$ such that $[a,x]$ lies in the interval $[a,b]$ *i.e.,* $[a,x]\subset[a,b]$. Since $f'(x)$ exists $\forall\, x\in[a,b]$, f is differentiable on $[a,b]$ and hence on $[a,x]$ and consequently continuous on $[a,x]$. Thus f satisfies all the conditions of Lagrange's mean value theorem on $[a,x]$ and hence there is a point $c\in\,]a,x[$ such that

$$f(x)-f(a)=(x-a)\,f'(c).$$

But by hypothesis $f'(x)=k\ \forall\, x\in[a,b]$, therefore, in particular $f'(c)=k$ as $a<c<x<b$ *i.e.,* $a<c<b$.

Hence $f(x)-f(a)=(x-a)k$ or $f(x)=kx+f(a)-ak$

or $\quad f(x)=kx+C$ where $C=f(a)-ak$ is a constant.

Theorem 4: *If f is continuous on $[a,b]$ and $f'(x)\ge 0$ in $]a,b[$, then f is increasing in $[a,b]$.*

Proof: Let x_1 and x_2 be any two distinct points of $[a,b]$ such that $x_1<x_2$. Then f satisfies the conditions of the Lagrange's mean value theorem in $[x_1,x_2]$. Consequently there exists a number c such that $x_1<c<x_2$, and

$$f(x_2)-f(x_1)=(x_2-x_1)\,f'(c).$$

Now $x_2-x_1>0$ and $f'(c)\ge 0$ (as $f'(x)\ge 0\ \forall\ x\in\,]a,b[$ and c is a point of $]a,b[$), therefore

$$f(x_2)-f(x_1)\ge 0 \text{ i.e., } f(x_1)\le f(x_2).$$

Thus $\quad x_1<x_2\Rightarrow f(x_1)\le f(x_2)\ \forall\ x_1,x_2\in[a,b].$

Hence f is an increasing function in the interval $[a,b]$.

Similarly, we can prove that if $f'(x)\le 0$ in $]a,b[$, then f is decreasing in $[a,b]$.

Corollary: *If f is continuous on $[a,b]$, then f is strictly increasing or strictly decreasing on $[a,b]$ according as*

$$f'(x)>0 \text{ or } <0 \text{ in }]a,b[\cdot$$

11 Cauchy's Mean Value Theorem

(Kanpur 2007; Lucknow 10; Avadh 12; Rohilkhand 14)

Theorem: *If two functions* $f(x)$ *and* $g(x)$ *are*

(*i*) *continuous in a closed interval* $[a, b]$,

(*ii*) *differentiable in the open interval* $]a, b[$,

(*iii*) $g'(x) \neq 0$ *for any point of the open interval* $]a, b[$, *then there exists at least one value* c *of* x *in the open interval* $]a, b[$, *such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}, \ a < c < b.$$

Proof: First we observe that as a consequence of condition (iii), $g(b) - g(a) \neq 0$. For if $g(b) - g(a) = 0$ *i.e.*, $g(b) = g(a)$, then the function $g(x)$ satisfies all the conditions of Rolle's theorem in $[a, b]$ and consequently there is some x in $]a, b[$ for which $g'(x) = 0$, thus contradicting the hypothesis that $g'(x) \neq 0$ for any point of $]a, b[$.

Now consider the function $F(x)$ defined on $[a, b]$, by setting

$$F(x) = f(x) + Ag(x), \qquad \ldots(1)$$

where A is a constant to be chosen such that $F(a) = F(b)$

i.e., $f(a) + Ag(a) = f(b) + Ag(b)$

or $$-A = \frac{f(b) - f(a)}{g(b) - g(a)}. \qquad \ldots(2)$$

Since $g(b) - g(a) \neq 0$, therefore A is a definite real number.

(i) Now f and g are continuous on $[a, b]$, therefore, F is also continuous on $[a, b]$.

(ii) Again, since f and g are differentiable on $]a, b[$, therefore F is also differentiable on $]a, b[$.

(iii) By our choice of A, $F(a) = F(b)$.

Thus the function $F(x)$ satisfies the conditions of Rolle's theorem in the interval $[a, b]$. Consequently there exists, at least one value, say c, of x in the open interval $]a, b[$ such that $F'(c) = 0$.

But $F'(x) = f'(x) + Ag'(x)$, from (1).

$\therefore$ $F'(c) = 0 \Rightarrow f'(c) + Ag'(c) = 0$

or $$-A = \frac{f'(c)}{g'(c)}. \qquad \ldots(3)$$

From (2) and (3), we get

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Another form: If $b = a + h$, then $a + \theta h = a$ when $\theta = 0$ and $a + \theta\, h = b$ when $\theta = 1$. Therefore, if $0 < \theta < 1$, then $a + \theta h$ means some value between a and b. So putting $b = a + h$ and $c = a + \theta\, h$, the result of the above theorem can be written as

$$\frac{f\,(a+h) - f\,(a)}{g\,(a+h) - g\,(a)} = \frac{f'\,(a+\theta h)}{g'\,(a+\theta h)},\ 0 < \theta < 1.$$

Note 1: If we take $g\,(x) = x$ for all x in $[a, b]$, then Cauchy's mean value theorem gives Lagrange's mean value theorem as a particular case. For $g\,(x) = x$ means $g\,(b) = b$, $g\,(a) = a$, $g'\,(x) = 1$ and so $g'\,(c) = 1$. Putting these values in Cauchy's mean value theorem, we get Lagrange's mean value theorem. Thus Cauchy's mean value theorem is more general than Lagrange's mean value theorem.

Note 2: Cauchy's mean value theorem cannot be obtained by applying Lagrange's mean value theorem to the functions f and g.

For applying Lagrange's mean value theorem to $f\,(x)$ and $g\,(x)$ separately, we get

$$f\,(b) - f\,(a) = (b - a)\, f\,'\,(c_1), \text{ where } a < c_1 < b$$

and $$g\,(b) - g\,(a) = (b - a)\, g\,'\,(c_2), \text{ where } a < c_2 < b.$$

Dividing, we have $\dfrac{f\,(b) - f\,(a)}{g\,(b) - g\,(a)} = \dfrac{f\,'\,(c_1)}{g'\,(c_2)}$.

Note that here c_1 is not necessarily equal to c_2.

Illustrative Examples

Example 25: *If $f\,(x) = (x-1)(x-2)(x-3)$ and $a = 0$, $b = 4$, find 'c' using Lagrange's mean value theorem.* **(Lucknow 2006, 07, 11; Rohilkhand 14)**

Solution: We have

$$f\,(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6.$$

$\therefore$ $$f\,(a) = f\,(0) = -6 \text{ and } f\,(b) = f\,(4) = 6.$$

$\therefore$ $$\frac{f\,(b) - f\,(a)}{b - a} = \frac{6 - (-6)}{4 - 0} = \frac{12}{4} = 3.$$

Also $f\,'\,(x) = 3x^2 - 12x + 11$ gives $f\,'\,(c) = 3c^2 - 12c + 11$.

Putting these values in Lagrange's mean value theorem

$$\frac{f\,(b) - f\,(a)}{b - a} = f\,'\,(c),\ (a < c < b), \text{ we get}$$

$$3 = 3c^2 - 12c + 11 \quad \text{or} \quad 3c^2 - 12c + 8 = 0$$

or $$c = \frac{12 \pm \sqrt{(144 - 96)}}{6} = 2 \pm \frac{2\sqrt{3}}{3}.$$

As both of these values of c lie in the open interval $]0, 4[$, hence both of these are the required values of c.

Example 26: *Let* $f : [0, 1] \to \mathbf{R}$ *be defined by*

$$f(x) = (x-1)^2 + 2 \;\; \forall \; x \in [0, 1].$$

Find the equation of the tangent to the graph of this curve which is parallel to the chord joining the points $(0, 3)$ *and* $(1, 2)$ *of the curve.*

Solution: Since $f(x)$ is a polynomial function, therefore it is continuous on $[0, 1]$ and differentiable in $]0, 1[$. Hence, by Lagrange's mean value theorem, there is some $c \in \,]0, 1[$ such that

$$\frac{f(1) - f(0)}{1 - 0} = f'(c)$$

or $$\frac{2-3}{1} = f'(c) \quad \text{or} \quad -1 = f'(c).$$

Now $f'(x) = 2(x-1)$ gives $f'(c) = 2(c-1)$.

Thus $2(c-1) = -1$ *i.e.*, $c = \frac{1}{2}\cdot$

$\therefore\; f(c) = \frac{9}{4}$, so that the point of contact of the tangent is $\left(\frac{1}{2}, \frac{9}{4}\right)$ and its slope is $f'(c) = -1$. Hence the equation of the required tangent is

$$y - \frac{9}{4} = -1\left(x - \frac{1}{2}\right)$$

or $$4x + 4y = 11.$$

Example 27: *Compute the value of* θ *in the first mean value theorem*

$$f(x+h) = f(x) + hf'(x + \theta h), \text{ if } f(x) = ax^2 + bx + c.$$

Solution: Here $f(x) = ax^2 + bx + c$.

$\therefore$ $$f(x+h) = a(x+h)^2 + b(x+h) + c,$$

$$f'(x) = 2ax + b,\; f'(x + \theta h) = 2a(x + \theta h) + b.$$

Substituting all these values in the Lagrange's mean value theorem, we get

$$a(x+h)^2 + b(x+h) + c = ax^2 + bx + c + h[2a(x + \theta h) + b] \qquad \dots(1)$$

The relation (1) being identically true for all values of x, hence when $x \to 0$, we have

$$ah^2 + bh + c = c + h[2a\theta h + b]$$

or $$ah^2 = 2a\theta h^2 \quad \text{or} \quad \theta = 1/2.$$

Example 28: *A function* $f(x)$ *is continuous in the closed interval* $[0, 1]$ *and differentiable in the open interval* $]0, 1[$, *prove that*

$$f'(x_1) = f(1) - f(0), \text{ where } 0 < x_1 < 1.$$

Solution: Here $a = 0, b = 1$ so that

$$\frac{f(b) - f(a)}{b - a} = \frac{f(1) - f(0)}{1 - 0} = f(1) - f(0).$$

If we take $c = x_1$, and substitute these values in the result of Lagrange's mean value theorem, we get

$$f(1) - f(0) = f'(x_1) \text{ where } 0 < x_1 < 1.$$

This is a particular case of Lagrange's mean value theorem. Students can give an independent proof of this.

Example 29: *Separate the intervals in which the polynomial*

$$2x^3 - 15x^2 + 36x + 1 \textit{ is increasing or decreasing.}$$

Solution: We have $f(x) = 2x^3 - 15x^2 + 36x + 1.$

$\therefore$ $\quad f'(x) = 6x^2 - 30x + 36 = 6(x-2)(x-3).$

Now $\quad f'(x) > 0$ for $x < 2$ or for $x > 3$,

$\quad f'(x) < 0$ for $2 < x < 3$, and $f'(x) = 0$ for $x = 2, 3$.

Thus $f'(x)$ is +ive in the intervals $]-\infty, 2[$ and $]3, \infty[$ and negative in the interval $]2, 3[$.

Hence f is monotonically increasing in the intervals $]-\infty, 2], [3, \infty[$ and monotonically decreasing in the interval $[2, 3]$.

Example 30: *Show that*

$$\frac{x}{1+x} < \log(1+x) < x \textit{ for } x > 0.$$

(Bundelkhand 2011)

Solution: Let $f(x) = \log(1+x) - \dfrac{x}{1+x}$.

$\therefore$ $\quad f(0) = 0.$

Then $\quad f'(x) = \dfrac{1}{1+x} - \dfrac{1.(1+x) - x.1}{(1+x)^2} = \dfrac{1}{1+x} - \dfrac{1}{(1+x)^2} = \dfrac{x}{(1+x)^2}.$

We observe that $f'(x) > 0$ for $x > 0$. Hence $f(x)$ is monotonically increasing in the interval $[0, \infty[$. Therefore

$$f(x) > f(0) \text{ for } x > 0 \textit{ i.e., } \left[\log(1+x) - \frac{x}{1+x}\right] > 0 \text{ for } x > 0$$

i.e., $\quad \log(1+x) > \dfrac{x}{1+x}$ for $x > 0$. ...(1)

Again, let $\quad \phi(x) = x - \log(1+x).$

$\therefore$ $\quad \phi(0) = 0.$

Then $\quad \phi'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x}.$

We observe that $\phi'(x) > 0$ for $x > 0$. Hence $\phi(x)$ is monotonically increasing in the interval $[0, \infty[$. Therefore

$$\phi(x) > \phi(0) \text{ for } x > 0 \textit{ i.e., } [x - \log(1+x)] > 0 \text{ for } x > 0$$

i.e., $\quad x > \log(1+x)$ for $x > 0$. ...(2)

From (1) and (2), we get

$$\frac{x}{1+x} < \log(1+x) < x \text{ for } x > 0.$$

Example 31: *Verify Cauchy's mean value theorem for the functions x^2 and x^3 in the interval* $[1, 2]$. **(Avadh 2013)**

Solution: Let $f(x) = x^2$ and $g(x) = x^3$. Then $f(x)$ and $g(x)$ are continuous in the closed interval $[1, 2]$ and differentiable in the open interval $]1, 2[$. Also $g'(x) = 3x^2 \neq 0$ for any point in the open interval $]1, 2[$. Hence by Cauchy's mean value theorem there exists at least one real number c in the open interval $]1, 2[$, such that

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)}. \qquad \ldots(1)$$

Now $$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{4 - 1}{8 - 1} = \frac{3}{7}.$$

Also $f'(x) = 2x$, $g'(x) = 3x^2$.

$\therefore \quad \dfrac{f'(c)}{g'(c)} = \dfrac{2c}{3c^2} = \dfrac{2}{3c}$. Putting these values in (1), we get $\dfrac{3}{7} = \dfrac{2}{3c}$ or $c = \dfrac{14}{9}$ which lies in the open interval $]1, 2[$. Hence Cauchy's mean value theorem is verified.

Example 32: *If in the Cauchy's mean value theorem, we write $f(x) = e^x$ and $g(x) = e^{-x}$, show that 'c' is the arithmetic mean between a and b.*

Solution: Here $$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{e^b - e^a}{e^{-b} - e^{-a}} = -e^a e^b = -e^{a+b},$$

and $$\frac{f'(x)}{g'(x)} = \frac{e^x}{-e^{-x}} \text{ so that } \frac{f'(c)}{g'(c)} = \frac{e^c}{-e^{-c}} = -e^{2c}.$$

Putting these values in Cauchy's mean value theorem, we get

$$-e^{a+b} = -e^{2c} \quad \text{or} \quad 2c = a + b \quad \text{or} \quad c = \frac{1}{2}(a + b).$$

Thus c is the arithmetic mean between a and b.

Example 33: *If in the Cauchy's mean value theorem, we write*

(i) $f(x) = \sqrt{x}$ *and* $g(x) = 1/\sqrt{x}$, *then c is the geometric mean between a and b,* **(Rohilkhand 2014)**

and if

(ii) $f(x) = 1/x^2$ *and* $g(x) = 1/x$, *then c is the harmonic mean between a and b.* **(Rohilkhand 2005)**

Solution: (i) Here $$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{\sqrt{b} - \sqrt{a}}{(1/\sqrt{b}) - (1/\sqrt{a})} = -\sqrt{(ab)},$$

and $$\frac{f'(x)}{g'(x)} = \frac{\frac{1}{2}x^{-1/2}}{-\frac{1}{2}x^{-3/2}} \text{ so that } \frac{f'(c)}{g'(c)} = -\frac{c^{-1/2}}{c^{-3/2}} = -c.$$

Putting these values in Cauchy's mean value theorem, we get

$$-\sqrt{(ab)} = -c \quad \text{or} \quad c = \sqrt{(ab)}$$

i.e., c is the geometric mean between a and b.

(ii) Here $$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{(1/b^2)-(1/a^2)}{(1/b)-(1/a)}=\frac{a+b}{ab}$$

and $$\frac{f'(x)}{g'(x)}=\frac{-2x^{-3}}{-x^{-2}} \text{ so that } \frac{f'(c)}{g'(c)}=\frac{-2c^{-3}}{-c^{-2}}=\frac{2}{c}\cdot$$

Putting these values in Cauchy's mean value theorem, we get

$$\frac{a+b}{ab}=\frac{2}{c} \qquad \text{or} \qquad c=\frac{2ab}{a+b}$$

i.e., c is the harmonic mean between a and b.

Comprehensive Exercise 3

1. State Lagrange's mean value theorem. Test if Lagrange's mean value theorem holds for the function $f(x)=|x|$ in the interval $[-1,1]$.
(Kanpur 2010; Rohilkhand 13B)

2. If $f(x)=1/x$ in $[-1,1]$, will the Lagrange's mean value theorem be applicable to $f(x)$? **(Meerut 2012B)**

3. Verify Lagrange's mean value theorem for the function
$$f:[-1,1]\to \mathbf{R} \text{ given by } f(x)=x^3.$$

4. Find 'c' of the mean value theorem, if
$$f(x)=x(x-1)(x-2)\,;\, a=0, b=\frac{1}{2}\cdot$$
(Kumaun 2012)

5. Find 'c' of Mean value theorem when
(i) $f(x)=x^3-3x-2$ in $[-2,3]$
(ii) $f(x)=2x^2+3x+4$ in $[1,2]$
(iii) $f(x)=x(x-1)$ in $[1,2]$ **(Meerut 2013B)**
(iv) $f(x)=x^2-3x-1$ in $\left[-\frac{11}{7},\frac{13}{7}\right]\cdot$

6. Show that any chord of the parabola $y=Ax^2+Bx+C$ is parallel to the tangent at the point whose abscissa is same as that of the middle point of the chord.

7. If $f''(x)$ exists for all points in $[a,b]$ and $\dfrac{f(c)-f(a)}{c-a}=\dfrac{f(b)-f(c)}{b-c}$ where $a<c<b$, then there is a number ξ such that $a<\xi<b$ and $f''(\xi)=0$.

8. State the conditions for the validity of the formula
$$f(x+h)=f(x)+h\,f'(x+\theta h)$$
and investigate how far these conditions are satisfied and whether the result is true, when $f(x)=x\sin(1/x)$ (being defined to be zero at $x=0$) and $x<0<x+h$.

9. (a) Show that $x^3 - 3x^2 + 3x + 2$ is monotonically increasing in every interval.

(b) Show that $\log(1+x) - \dfrac{2x}{2+x}$ is increasing when $x > 0$.

10. Determine the intervals in which the function

$$(x^4 + 6x^3 + 17x^2 + 32x + 32)\, e^{-x}$$

is increasing or decreasing.

11. Use the function $f(x) = x^{1/x}$, $x > 0$ to determine the bigger of the two numbers e^{π} and π^{e}.

12. If $a = -1, b \geq 1$ and $f(x) = 1/|x|$, show that the conditions of Lagrange's mean value theorem are not satisfied in the interval $[a, b]$, but the conclusion of the theorem is true if and only if $b > 1 + \sqrt{2}$.

13. (a) State Cauchy's mean value theorem. **(Kanpur 2007)**

(b) Verify Cauchy's mean value theorem for $f(x) = \sin x$, $g(x) = \cos x$ in $[-\pi/2, 0]$. **(Lucknow 2007)**

14. If $f(x) = x^2$, $g(x) = \cos x$, then find the point $c \in \,]0, \pi/2[$ which gives the result of Cauchy's mean value theorem in the interval $[0, \pi/2]$ for the functions $f(x)$ and $g(x)$.

15. Show that $\dfrac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta$, where $0 < \alpha < \theta < \beta < \dfrac{\pi}{2}\cdot$

16. Use Cauchy's mean value theorem to evaluate $\lim\limits_{x \to 1} \left[\dfrac{\cos \frac{1}{2}\pi x}{\log(1/x)}\right]$.

Answers 3

1. The mean value theorem does not hold since the given function is not differentiable at $x = 0$
2. not applicable
4. $1 - \dfrac{\sqrt{21}}{6}$
5. (i) $\pm\sqrt{(7/3)}$ (ii) 3/2 (iii) 3/2 (iv) 1/7
8. Condition of differentiability is not satisfied in $x < 0 < x + h$ since $f(x)$ is non-differentiable at $x = 0$.
10. Increasing in the intervals $[-2, -1]$ and $[0, 1]$ and decreasing in the intervals $]-\infty, -2[$, $[-1, 0]$ and $[1, \infty[$.
11. e^{π} is bigger than π^{e}.
14. Root of the equation $\sin c - (8c/\pi^2) = 0$ in the open interval $]\pi/6, \pi/2[$.
16. $\pi/2$.

12 Taylor's Theorem with Lagrange's form of Remainder After n Terms

Theorem: *If $f(x)$ is a single-valued function of x such that*

(*i*) *all the derivatives of $f(x)$ upto $(n-1)$ th are continuous in $a \le x \le a+h$,*

(*ii*) *$f^{(n)}(x)$ exists in $a < x < a+h$, then*

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a+\theta h), \text{ where } 0<\theta<1.$$

Proof: Consider the function ϕ defined by

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) + \frac{A}{n!}(a+h-x)^n,$$

where A is a constant to be suitably chosen.

We choose A such that $\phi(a) = \phi(a+h)$.

Now $\quad \phi(a) = f(a) + h f'(a) + \dfrac{h^2}{2!} f''(a) + \ldots + \dfrac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \dfrac{A}{n!} h^n,$

and $\quad \phi(a+h) = f(a+h).$

Hence A is given by

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} A. \quad \ldots(1)$$

Now, by hypothesis, all the functions

$$f(x), f'(x), f''(x), \ldots\ldots, f^{(n-1)}(x)$$

are continuous in the closed interval $[a, a+h]$ and differentiable in the open interval $]a, a+h[$.

Further $(a+h-x), (a+h-x)^2/2!, \ldots, (a+h-x)^n/n!$, all being polynomials, are continuous in the closed interval $[a, a+h]$ and differentiable in the open interval $]a, a+h[$. Also A is a constant.

$\therefore \quad \phi(x)$ is continuous in the closed interval $[a, a+h]$ and differentiable in the open interval $]a, a+h[$.

By our choice of A, $\phi(a) = \phi(a+h)$. Hence $\phi(x)$ satisfies all the conditions of Rolle's theorem.

Consequently $\phi'(a+\theta h) = 0$, where $0<\theta<1$.

Now $\phi'(x) = f'(x) - f'(x) + (a+h-x) f''(x) - (a+h-x) f''(x)$

$$+ \ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(x) - \frac{A}{(n-1)!}(a+h-x)^{n-1}$$

$$= \frac{(a+h-x)^{n-1}}{(n-1)!}[f^{(n)}(x) - A],$$

since other terms cancel in pairs.

$\therefore$ $\phi'(a+\theta h) = 0$ gives

$$\frac{[a+h-(a+\theta h)]^{n-1}}{(n-1)!}[f^{(n)}(a+\theta h) - A] = 0$$

or $$\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}[f^{(n)}(a+\theta h) - A] = 0$$

or $f^{(n)}(a+\theta h) - A = 0$ or $A = f^{(n)}(a+\theta h)$.

$[\because h \neq 0, (1-\theta) \neq 0 \text{ as } 0 < \theta < 1]$

Putting this value of A in (1), we get

$$\boldsymbol{f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h).}$$

This is **Taylor's development** of $f(a+h)$ in ascending integral powers of h. The $(n+1)$th term $\boldsymbol{\frac{h^n}{n!}f^{(n)}(a+\theta h)}$ is called **Lagrange's form of remainder** after n terms in Taylor's expansion of $f(a+h)$.

Note: If we take $n = 1$, we see that Lagrange's mean value theorem is a particular case of the above theorem.

Corollary. (Maclaurin's development):

If we take the interval $[0, x]$ instead of $[a, a+h]$, so that changing a to 0 and h to x in Taylor's theorem, we get

$$\boldsymbol{f(x) = f(0) + xf'(0 + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{n!}f^{(n)}(\theta x)}$$

which is known as **Maclaurin's theorem** or **Maclaurin's development** of $f(x)$ in the interval $[0, x]$ with **Lagrange's form of remainder** $\boldsymbol{\frac{x^n}{n!}f^{(n)}(\theta x)}$ after n terms.

13 Taylor's Theorem with Cauchy's Form of Remainder

Theorem: *If $f(x)$ is a single-valued function of x such that*

(*i*) *all the derivatives of $f(x)$ upto $(n-1)$th are continuous in $a \le x \le a+h$,*

(*ii*) *$f^{(n)}(x)$ exists in $a < x < a+h$, then*

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a)$$
$$+ \frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(a+\theta h), \text{ where } 0 < \theta < 1.$$

Proof: Consider the function ϕ defined by

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + \ldots$$
$$+ \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) + (a+h-x) A,$$

where A is a constant to be suitably chosen. We choose A such that $\phi(a) = \phi(a+h)$.

Now $\quad \phi(a) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + h A,$

and $\quad \phi(a+h) = f(a+h).$

Hence A is given by

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + hA. \quad \ldots(1)$$

As explained earlier in article 12, it can be easily seen that $\phi(x)$ satisfies all the conditions of Rolle's theorem. Consequently

$$\phi'(a+\theta h) = 0, \text{ where } 0 < \theta < 1.$$

Now $\phi'(x) = \dfrac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(x) - A$, since other terms cancel in pairs.

$\therefore \quad \phi'(a+\theta h) = 0$ gives $\dfrac{[a+h-(a+\theta h)]^{n-1}}{(n-1)!} f^{(n)}(a+\theta h) - A = 0$

or $\quad A = \dfrac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(a+\theta h).$

Putting this value of A in (1), we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a)$$
$$+ \frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(a+\theta h).$$

The $(n+1)$th term $\frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(a+\theta h)$ is called **Cauchy's form of remainder** after n terms in the **Taylor's expansion** of $f(a+h)$ in ascending integral powers of h.

Corollary. (Maclaurin's development with Cauchy's form of remainder):
If we change a to 0 and h to x in the above result, we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0)$$
$$+ \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(\theta x),$$

which is **Maclaurin's theorem with Cauchy's form of remainder**. The $(n+1)$th term $\frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(\theta x)$ is known as Cauchy's form of remainder after n terms in Maclaurin's development of $f(x)$ in the interval $[0, x]$.

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The function $f(x) = |x-1|$ is not differentiable at
   (a) $x = 0$ (b) $x = -1$
   (c) $x = 1$ (d) $x = 2$
2. The function $f(x) = |x+3|$ is not differentiable at
   (a) $x = 3$ (b) $x = -3$
   (c) $x = 0$ (d) $x = 1$
3. A function $f(x)$ is differentiable at $x = a$ if
   (a) $Rf'(a) = Lf'(a)$ (b) $Rf'(a) = 0$
   (c) $Lf'(a) = 0$ (d) $Rf'(a) \neq Lf'(a)$

4. A function $\phi(x)$ is defined as follows :

$\phi(x) = 1 + x$ if $x \le 2$

$\phi(x) = 5 - x$ if $x > 2$.

Then

(a) $\phi(x)$ is continuous but not differentiable at $x = 2$

(b) $\phi(x)$ is differentiable at every point of **R**

(c) $\phi(x)$ is neither continuous nor differentiable at $x = 2$

(d) $\phi(x)$ is differentiable at $x = 2$ but is not continuous at $x = 2$.

5. Out of the following four functions tell the function for which the conditions of Rolle's theorem are satisfied.

(a) $f(x) = |x|$ in $[-1, 1]$

(b) $f(x) = x^2$ in $2 \le x \le 3$

(c) $f(x) = \sin x$ in $[0, \pi]$

(d) $f(x) = \tan x$ in $0 \le x \le \pi$

6. The function $f(x) = \sin x$ is increasing in the interval

(a) $[0, \pi]$

(b) $\left[0, \frac{\pi}{2}\right]$

(c) $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

(d) $\left[\frac{\pi}{2}, \pi\right]$

(Kumaun 2014)

7. The value of 'c' of Lagrange's mean value theorem for $f(x) = x(x-1)$ in $[1, 2]$ is given by

(a) $c = \frac{5}{4}$

(b) $c = \frac{3}{2}$

(c) $c = \frac{7}{4}$

(d) $c = \frac{11}{6}$

8. The value of 'c' of Rolle's theorem for the function $f(x) = e^x \sin x$ in $[0, \pi]$ is given by

(a) $c = \frac{3\pi}{4}$

(b) $c = \frac{\pi}{4}$

(c) $c = \frac{\pi}{2}$

(d) $c = \frac{5\pi}{6}$

9. The function $f(x) = |x|$ at $x = 0$ shall be

(a) differentiable

(b) continuous but not differentiable

(c) discontinuous

(d) none of these

(Kumaun 2009)

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. A function $f(x)$ is said to be differentiable at $x = a$ if

$$\lim_{x \to a} \frac{f(x) - \ldots}{x - a} \text{ exists.}$$

2. The right hand derivative of $f(x)$ at $x = a$ is given by

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{\dots}, h > 0,$$

provided the limit exists.

3. The left hand derivative of $f(x)$ at $x = a$ is given by

$$\lim_{h \to 0} \frac{f(a-h) - f(a)}{\dots}, h > 0,$$

provided the limit exists.

4. A function $f :]a, b[\to \mathbf{R}$ is said to be differentiable in $]a, b[$ if and only if it is differentiable at every point in … .

5. If a function $f(x)$ is differentiable at $x = a$, then $f'(a)$ is the tangent of the angle which the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ makes with … .

6. Continuity is a necessary but not a … condition for the existence of a finite derivative.

7. The function $f(x) = |x|$ is differentiable at every point of $\mathbf{R}$ except at $x = \dots$.

8. If a function $f(x)$ is such that

   (i) $f(x)$ is continuous in the closed interval $[a, b]$,

   (ii) $f'(x)$ exists for every point in the open interval $]a, b[$,

   (iii) $f(a) = f(b)$, then there exists at least one value of x, say c, where $a < c < b$, such that $f'(c) = 0$.

   The above theorem is known as … .

9. If a function $f(x)$ is

   (i) continuous in the closed interval $[a, b]$, and

   (ii) differentiable in the open interval $]a, b[$ *i.e.*, $a < x < b$, then there exists at least one value 'c' of x lying in the open interval $]a, b[$ such that

$$\frac{f(b) - f(a)}{b - a} = \dots.$$

10. If two functions $f(x)$ and $g(x)$ are

   (i) continuous in a closed interval $[a, b]$

   (ii) differentiable in the open interval $]a, b[$, and

   (iii) $g'(x) \neq 0$ for any point of the open interval $]a, b[$, then there exists at least one value c of x in the open interval $]a, b[$, such that

$$\frac{f(b) - f(a)}{\dots} = \frac{f'(c)}{g'(c)}.$$

11. If f is continuous in $[a, b]$ and $f'(x) \geq 0$ in $]a, b[$, then f is … in $[a, b]$.

12. If $f(x) = \sin x$, then

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \dots.$$

True or False

Write 'T' for true and 'F' for false statement.

1. If a function $f(x)$ is continuous at $x = a$, it must also be differentiable at $x = a$.
2. If a function $f(x)$ is differentiable at $x = a$, it must be continuous at $x = a$.
3. If a function $f(x)$ is differentiable at $x = a$, it may or may not be continuous at $x = a$.
4. The function $f(x) = |x|$ is differentiable at every point of **R**.
5. Rolle's theorem is applicable for $f(x) = \sin x$ in $[0, 2\pi]$.
6. Rolle's theorem is applicable for $f(x) = |x|$ in $[-1, 1]$.
7. Lagrange's mean value theorem is applicable for $f(x) = |x|$ in $[-1, 1]$.
8. The function $f(x) = \sin x$ is increasing in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
9. If $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has no root in $]0, 1[$.
10. If f is continuous on $[a, b]$ and $f'(x) \leq 0$ in $]a, b[$, then f is increasing in $[a, b]$.
11. The function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is decreasing in the interval $[2, 3]$.
12. Let $f(x) = |x| + |x - 1|$. Then $Rf'(0) = 0$.
13. Rolle's theorem is not applicable for the function $f(x) = x(x+2)e^{-x/2}$ in $[-2, 0]$.
14. The value of 'c' of Lagrange's mean value theorem for the function $f(x) = 2x^2 + 3x + 4$ in $[1, 2]$ is given by $c = \frac{5}{4}$.
15. If $f(x) = x^n$, then $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} = n\, x^{n-1}$.
16. If $f(x) = \cos x$, then $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = -\sin a$.
17. If $f(x) = e^x$, then $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = e^x$.

Answers

Multiple Choice Questions

1.	(c)	2.	(b)	3.	(a)	4.	(a)	5.	(c)
6.	(b)	7.	(b)	8.	(a)	9.	(b)		

Fill in the Blank(s)

1.	$f(a)$	**2.**	h	**3.**	$-h$	**4.**	$]a, b[$	**5.**	the x-axis
6.	sufficient	**7.**	0	**8.**	Rolle's theorem			**9.**	$f'(c)$
10.	$g(b) - g(a)$			**11.**	increasing	**12.**	$\cos x$		

True or False

1.	*F*	**2.**	*T*	**3.**	*F*	**4.**	*F*	**5.**	*T*
6.	*F*	**7.**	*F*	**8.**	*T*	**9.**	*F*	**10.**	*F*
11.	*T*	**12.**	*T*	**13.**	*F*	**14.**	*F*	**15.**	*T*
16.	*T*	**17.**	*F*						

□

Chapter 3

Sequences

1 Introduction

In the present chapter we shall study a special class of functions, namely sequences. The study of sequences plays an important role in Analysis.

2 Sequence

Definition: *Let S be any non-empty set. A function whose domain is the set* **N** *of natural numbers and whose range is a subset of S, is called a sequence in the set S.*

In other words a sequence in a set S is a rule which assigns to each natural number a unique element of S.

Real Sequence: A sequence whose range is a subset of **R** is called a *real sequence* or a sequence of real numbers.

In this chapter we shall study only real sequences. Therefore the term sequence will be used to denote a real sequence.

If s is a sequence, then the image $s(n)$ of $n \in \mathbf{N}$ is usually denoted by s_n. It is customary to denote the sequence s by the symbol $< s_n >$ or by $\{s_n\}$. The image s_n of n is called the nth term of the sequence.

A sequence can be described in several different ways.

1. Listing in order, the first few elements of a sequence, till the rule for writing down different elements becomes clear. For example, $< 1, 8, 27, 64, \dots >$ is the sequence whose nth term is n^3.
2. Defining a sequence by a formula for its nth term. For example, the sequence $< 1, 8, 27, 64, \dots >$ can also be written as $< 1, 8, \dots, n^3, \dots >$ or as $< n^3 : n \in \mathbf{N} >$ or simply as $< n^3 >$.
3. Defining a sequence by a Recursion formula *i.e.* by a rule which expresses the nth term in terms of the $(n-1)$th term. For example, let
$$a_1 = 1, a_{n+1} = 3a_n, \text{ for all } n \geq 1.$$

These relations define a sequence whose nth term is 3^{n-1}.

Illustrations:

1. $< \frac{1}{n} >$ is the sequence $< 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots >$.
2. $< \frac{n}{n+1} >$ is the sequence $< \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots >$.
3. $< (-1)^n / n >$ is the sequence $< -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots >$.
4. Let $s_1 = 1, s_2 = 1$ and $s_{n+2} = s_{n+1} + s_n$ for all $n \geq 1$.

   From the above formula, $s_3 = s_2 + s_1 = 2$,
$$s_4 = s_3 + s_2 = 3, s_5 = s_4 + s_3 = 5 \text{ and so on.}$$

$\therefore$ $$< s_n > = < 1, 1, 2, 3, 5, \dots >.$$

Range of a sequence: *The set of all distinct terms of a sequence is called its range.*

$\therefore$ The range of a sequence $< s_n >$ = the set $\{s_1, s_2, \dots\}$.

Illustration 1: The range of the sequence $< (-1)^n > = \{-1, 1\}$, a finite set.

Illustration 2: The range of the sequence
$$\frac{1}{n} = \left\{ \frac{1}{n} : n \in \mathbf{N} \right\},$$
is an infinite set.

Constant sequence: *A sequence $< s_n >$ defined by $s_n = a$ for all $n \in \mathbf{N}$ is called a constant sequence.*

The sequence $< s_n > = < a, a, a, \dots >$ is a constant sequence.

Its range = the singleton $\{a\}$ is a finite set.

Equality of two sequences: *Two sequences $< s_n >$ and $< t_n >$ are said to be equal if $s_n = t_n \ \forall$ $n \in \mathbf{N}$.*

3 Operations on Sequences

Since sequences of real numbers are real valued functions, we define the sum, difference, product etc. of two sequences as follows :

Let $<s_n>, <t_n>$ be two sequences. Then the sequences having nth terms $s_n + t_n, s_n - t_n$ and $s_n t_n$ are respectively called the *sum, difference* and *product* of the sequences $<s_n>$ and $<t_n>$. These sequences are denoted by $<s_n + t_n>$, $<s_n - t_n>$ and $<s_n t_n>$ respectively.

If $t_n \neq 0$ for all $n \in \mathbf{N}$, then the sequence whose nth term is $1/t_n$ is known as the *reciprocal of the sequence* $<t_n>$ and is denoted by $<1/t_n>$. Also the sequence whose nth term is s_n / t_n is called the *quotient* of the sequence $<s_n>$ by the sequence $<t_n>$ and it is denoted by $<s_n / t_n>$.

If $c \in \mathbf{R}$, then the sequence having nth term cs_n is called the *scalar multiple* of $<s_n>$ by c. This sequence is denoted by $<cs_n>$.

4 Subsequences and Order Preservation

Subsequence: *Let $<s_n>$ be any sequence. If $<n_1, n_2, \ldots, n_k, \ldots>$ be a strictly increasing sequence of positive integers i.e., $i > j \Rightarrow n_i > n_j$, then the sequence*

$$<s_{n_1}, s_{n_2}, \ldots\ldots, s_{n_k}, \ldots>$$

is called a subsequence of $<s_n>$.

From the condition $i > j \Rightarrow n_i > n_j$, we conclude that the order of the various terms in the subsequence is the same as it is in the sequence.

Illustrations:

1. Let $<s_n> = <1, 0, 1, 0, 1, 0, \ldots>$ *i.e.*, $s_1 = 1, s_2 = 0, s_3 = 1, s_4 = 0, \ldots\ldots$.

Take $n_1 = 1, n_2 = 3, n_3 = 5, \ldots..$. Then $<n_r>$ is a sequence of positive integers such that $n_1 < n_2 < n_3 < \ldots..$

Hence $<1, 1, 1, \ldots>$ is a subsequence of $<s_n>$.

Similarly if we take $n_1 = 2, n_2 = 4, n_3 = 6, \ldots$, then the sequence $<0, 0, 0, \ldots>$ is also a subsequence of $<s_n>$.

2. The sequence of primes $<2, 3, 5, 7, 11, \ldots>$ is a subsequence of the sequence of natural numbers $<1, 2, 3, 4, \ldots..>$.

3. The sequence $<7^2, 3^2, 15^2, 11^2, 19^2, \ldots>$ is not a subsequence of the sequence $<1^2, 2^2, 3^2, 4^2, \ldots>$.

Here $n_1 = 7, n_2 = 3, n_3 = 15, n_4 = 11, n_5 = 19$. But $<7, 3, 15, 11, 19, \ldots>$ is not a strictly increasing sequence of positive integers.

Order Preservation: Consider the sequence

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots \quad \ldots (1)$$

There can be found many sequences contained in this sequence, for example the sequences

$$\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots\ldots$$

and

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\ldots \quad \ldots (2)$$

Now to see if the order of the original sequence is preserved we need to examine the subscripts of the terms of the sequences under consideration.

Let the sequence given in (1) be denoted by $\{a_n\}_{n=1}^{\infty}$. Then the sequence given by (2) can be denoted by $\{a_{2^n}\}_{n=1}^{\infty}$.

Here we say that the order of the terms of sequence (1) is preserved in the sequence (2) as the sequence of subscripts $\{2^n\}_{n=1}^{\infty}$ is strictly increasing.

A sequence is not only a countably infinite set, instead it is defined as a countably infinite set expressed in a specific order.

A sequence is said to be contained in another if the order of the terms is preserved. The sequence $\frac{1}{4}, \frac{1}{2}, \frac{1}{8}, \frac{1}{6}, \ldots\ldots$ is not considered to be contained as sequence, in the sequence (1) even though it is contained as a subset.

5 Bounded Sequences

(Kanpur 2008)

Definition 1: *A sequence $< s_n >$ is said to be bounded above if the range set of $< s_n >$ is bounded above i.e., if there exists a real number k_1 such that*

$$s_n \le k_1 \text{ for all } n \in \mathbf{N}.$$

The number k_1 is called an **upper bound** of the sequence $< s_n >$.

Definition 2: *A sequence $< s_n >$ is said to be bounded below if the range set of $< s_n >$ is bounded below i.e., if there exists a real number k_2 such that*

$$s_n \ge k_2 \text{ for all } n \in \mathbf{N}.$$

The number k_2 is called a **lower bound** of the sequence $< s_n >$.

Definition 3: *A sequence $< s_n >$ is said to be bounded if the range set of $< s_n >$ is both bounded above and bounded below i.e., if there exist two real numbers k_1 and k_2 such that*

$$k_2 \le s_n \le k_1 \text{ for all } n \in \mathbf{N}.$$

Equivalently, a sequence $< s_n >$ is bounded if and only if there exists a real number $K > 0$ such that

$$|s_n| \le K \text{ for all } n \in \mathbf{N}.$$

It is not necessary that a sequence be bounded above or bounded below.

A sequence $< s_n >$ is said to be **unbounded** if it is either unbounded below or unbounded above.

Definition 4: *The least number say, M, if it exists, of the set of the upper bounds of $< s_n >$ is called the* **least upper bound** (**l.u.b.**) *or the* **supremum** (**sup**) *of the sequence $< s_n >$.*

The greatest number say, m, if it exists, of the set of the lower bounds of $< s_n >$ is called the **greatest lower bound** (**g.l.b.**) *or the* **infimum** (**inf.**) *of the sequence $< s_n >$.*

Note 1: If the range of a sequence is a finite set, then the sequence is bounded because a finite set is always bounded.

Note 2: Every subsequence of a bounded sequence is bounded.

Illustrations:

1. The sequence $< \frac{1}{n} >$ is bounded since $\left|\frac{1}{n}\right| \le 1$ for all $n \in \mathbf{N}$.
2. The sequence $< \frac{n}{n+1} >$ is bounded since $\frac{1}{2} \le \frac{n}{n+1} < 1$ for all $n \in \mathbf{N}$.
3. The sequence $< (-1)^n >$ is bounded since $|(-1)^n| \le 1$ for all $n \in \mathbf{N}$. In fact $|(-1)^n| = 1$ for all $n \in \mathbf{N}$.
4. The sequence $< (-1)^n / n >$ is bounded since $|(-1)^n / n| \le 1$ for all $n \in \mathbf{N}$.
5. The sequence $< s_n >$ defined by $s_n = 1 + (-1)^n$ for all $n \in \mathbf{N}$, is bounded since the range set of the sequence is $\{0, 2\}$, which is a finite set.
6. The sequence $< n^2 >$ is bounded below by 1 but not bounded above.
7. The sequence $< -n^2 >$ is bounded above by -1 but not bounded below.
8. The sequence $< s_n > = < (-1)^n n >$ is neither bounded below nor bounded above.

For any positive real number K, there exists a positive integer $2m$ such that $2m > K$. It gives that $s_{2m} > K$. Hence $< s_n >$ is not bounded above.

Similarly it can be shown that $< s_n >$ is not bounded below.

Theorem: *A sequence $< s_n >$ is bounded iff there exist $m \in \mathbf{N}$, $l \in \mathbf{R}$ and $a > 0$ such that*

$$| s_n - l | < a \text{ for all } n \ge m.$$

Proof: Let $< s_n >$ be a bounded sequence. Then there exist two real numbers k_1, k_2 such that $k_1 < s_n < k_2$ for all $n \in \mathbf{N}$

$$\Rightarrow \quad k_1 - \frac{k_1 + k_2}{2} < s_n - \frac{k_1 + k_2}{2} < k_2 - \frac{k_1 + k_2}{2} \;\forall\; n \in \mathbf{N}$$

$$\Rightarrow \quad \frac{k_1 - k_2}{2} < s_n - \frac{k_1 + k_2}{2} < \frac{k_2 - k_1}{2} \;\forall\; n \in \mathbf{N}$$

$$\Rightarrow \quad -a < s_n - l < a \;\forall\; n \in \mathbf{N} \quad \text{where} \quad l = \frac{k_1 + k_2}{2} \quad \text{and} \quad a = \frac{k_2 - k_1}{2}$$

$$\Rightarrow \quad | s_n - l | < a \;\forall\; n \in \mathbf{N}$$

$$\Rightarrow \quad |s_n - l| < a \;\forall\; n \ge m, \text{ where } m = 1 \in \mathbf{N}, l \in \mathbf{R} \text{ and } a > 0.$$

Conversely, let there exist $l \in \mathbf{R}, a > 0$ and $m \in \mathbf{N}$ such that $|s_n - l| < a$ for all $n \ge m$.

This gives $l - a < s_n < l + a$ for all $n \ge m$.

Choose $k_1 = \min\{s_1, s_2, \ldots, s_{m-1}, l - a\}$.

Then $k_1 \le s_n$, for all $n \in \mathbf{N}$.

Again choose $k_2 = \max\{s_1, s_2, \ldots, s_{m-1}, l + a\}$.

Then $s_n \le k_2$ for all $n \in \mathbf{N}$.

$\therefore \quad k_1 \le s_n \le k_2$ for all $n \in \mathbf{N}$.

Hence $< s_n >$ is a bounded sequence.

Note: In view of the above theorem we conclude that a sequence is bounded even if $k_1 \le s_n \le k_2$ for $n \ge m$. However in such a case it is not necessary that k_1 is a lower bound and k_2 is an upper bound of the sequence $< s_n >$.

6 Convergent Sequences

Definition: *A sequence* $< s_n >$ *is said to converge to a number* l, *if for any given* $\varepsilon > 0$ *there exists a positive integer* m *such that*

$$|s_n - l| < \varepsilon \text{ for all } n \geq m.$$

The number l *is called the* **limit** *of the sequence* $< s_n >$ *and we write* $s_n \to l$ *as* $n \to \infty$ *or* $\lim_{n \to \infty} s_n = l$ *or simply* $\lim s_n = l$.

(Gorakhpur 2011)

The positive integer m depends on the value of ε.

The phrase '$|s_n - l| < \varepsilon$ for all $n \geq m$' expresses the fact that the absolute value of the difference between s_n and l can be made less than ε from some stage onwards *i.e.*, $l - \varepsilon < s_n < l + \varepsilon$ from some stage onwards.

The truth of the statement that the sequence $< s_n >$ converges to l depends upon showing that all except a finite number of terms of the sequence must lie in the open interval $]\, l - \varepsilon, l + \varepsilon\, [$, whatever $\varepsilon > 0$ we take. If we can find even one ε for which infinitely many terms of the sequence lie outside $]\, l - \varepsilon, l + \varepsilon\, [$, the sequence will not converge to l. The number of terms lying outside $]\, l - \varepsilon, l + \varepsilon\, [$ depends upon ε. The smaller the ε, the larger the number of terms which lie outside $]\, l - \varepsilon, l + \varepsilon\, [$.

From the above discussion, we conclude that '*a sequence converges to l iff it lies ultimately in each open interval around l*'.

Theorem 1: *If* $< s_n >$ *is a sequence of non-negative numbers such that* $\lim s_n = l$, *then* $l \geq 0$.

Proof: Suppose, if possible, $l < 0$. Then $-l > 0$.

Since $\lim s_n = l$, for $\varepsilon = -(l/2) > 0$, there exists $m \in \mathbf{N}$ such that $|s_n - l| < -l/2$ for all $n \geq m$.

In particular, $|s_m - l| < -l/2$

i.e., $$l + \frac{l}{2} < s_m < l - \frac{l}{2}$$

i.e., $s_m < l/2$ or $s_m < 0$, because by assumption $l < 0$.

But by hypothesis we have $s_m \geq 0$ because it is given that $s_n \geq 0$ for all n.

Hence our assumption is wrong. So we must have $l \geq 0$.

Remark: In the above proof ε can be taken any positive real number such that $0 < \varepsilon \leq -l$, say $\varepsilon = -l$ or $\varepsilon = -l/2$ or $\varepsilon = -l/3$ etc.

Note 1: A negative number cannot be the limit of a sequence of non-negative numbers.

2. If $\lim s_n = l$ and $l < 0$ then there exists a positive integer m such that $s_n < 0$ for all $n \geq m$.

3. A sequence $< s_n >$ is called a null sequence if $\lim s_n = 0$.

Theorem 2: *A sequence cannot converge to more than one limit i.e., the limit of a sequence is unique.* **(Gorakhpur 2011)**

Proof: If possible, suppose a sequence $< s_n >$ converges to two distinct numbers l and l'. Since $l \neq l'$, therefore $|l - l'| > 0$.

Let $\varepsilon = \frac{1}{2}|l - l'|$, then $\varepsilon > 0$.

Now $< s_n >$ converges to $l \Rightarrow$ there exists $m_1 \in \mathbf{N}$ such that

$$|s_n - l| < \varepsilon \text{ for all } n \geq m_1. \quad \text{...(1)}$$

Similarly $< s_n >$ converges to $l' \rightarrow$ there exists $m_2 \in \mathbf{N}$ such that

$$|s_n - l'| < \varepsilon \text{ for all } n \geq m_2. \quad \text{...(2)}$$

Let $m = \max \{m_1, m_2\}$.

Then (1) and (2) hold for all $n \geq m$.

We have for all $n \geq m$

$$|l - l'| = |(s_n - l) - (s_n - l')| \leq |s_n - l| + |s_n - l'|$$
$$< \varepsilon + \varepsilon = 2\varepsilon = |l - l'|.$$

Thus $|l - l'| < |l - l'|$, which is absurd. Hence our initial assumption that $l \neq l'$ is wrong and we must have $l = l'$ *i.e.*, the limit of a sequence is unique.

Note: After taking $\varepsilon = \frac{1}{2}|l - l'|$, we can also give the following argument :

Since $< s_n >$ converges to both l and l', therefore it lies ultimately in both the intervals $]l - \varepsilon, l + \varepsilon[$ and $]l' - \varepsilon, l' + \varepsilon[$. This is impossible since these two intervals have no real number in common. Hence our assumption is wrong and thus the sequence cannot converge to more than one limit.

Theorem 3: *If $< s_n >$ converges to l, then any subsequence of $< s_n >$ also converges to l.*

Proof: Let $< s_{n_k} >$ be any subsequence of $< s_n >$. Then by definition of subsequence, $n_1, n_2, \ldots, n_k, \ldots$. are positive integers such that

$$n_1 < n_2 < \ldots < n_k < \ldots.$$

Now $n_1 \geq 1 \Rightarrow n_k \geq k$ (by induction).

Since $< s_n >$ converges to l, so given $\varepsilon > 0$, there exists a positive integer m such that

$$|s_k - l| < \varepsilon \text{ for all } k \geq m.$$

For $k \geq m$ we have $n_k \geq k \geq m$.

$\therefore$ $|s_{n_k} - l| < \varepsilon$ for all $n_k \geq m$.

$\therefore$ $< s_{n_k} >$ converges to l.

Corollary: *All subsequences of a convergent sequence converge to the same limit.*

Proof: By theorem 3, any subsequence of a sequence converges to the same limit as the limit of the sequence and by theorem 2, the limit of a sequence is unique. This shows that all subsequences of a convergent sequence have the same limit.

Note: To show that a given sequence is not convergent it is enough to show that two of its subsequences converge to different limits.

Illustration: The sequence $< (-1)^n >$ is not convergent.

The two subsequences $< 1, 1, 1, \ldots >$ and $< -1, -1, -1, \ldots >$ of the given sequence converge respectively to 1 and -1 which are different.

Theorem 4: *If the subsequences* $< s_{2n-1} >$ *and* $< s_{2n} >$ *of the sequence* $< s_n >$ *converge to the same limit* l*, then the sequence* $< s_n >$ *converges to* l*.*

Proof: Let $\varepsilon > 0$ be given. Then, since $\lim s_{2n-1} = l$, there exists $m_1 \in \mathbf{N}$ such that $| s_{2n-1} - l | < \varepsilon \;\; \forall \; n \geq m_1$.

Similarly $\lim s_{2n} = l \Rightarrow$ for $\varepsilon > 0$, there exists $m_2 \in \mathbf{N}$ such that

$$| s_{2n} - l | < \varepsilon \;\; \forall \; n \geq m_2 \;.$$

Let $\quad m = \max. \{ m_1, m_2 \}$.

Then $\quad | s_{2n-1} - l | < \varepsilon$ and $| s_{2n} - l | < \varepsilon \; \forall \; n \geq m$.

$\therefore \quad | s_n - l | < \varepsilon \;\; \forall \; n \geq 2m - 1$.

Hence $< s_n >$ converges to l.

Theorem 5: *Every convergent sequence is bounded.*

(Kanpur 2008; Gorakhpur 10, 13, 14)

Proof: Let $< s_n >$ be a sequence which converges to l. Take $\varepsilon = 1$. Then there exists a positive integer m such that

$$| s_n - l | < 1, \text{ for all } n \geq m,$$

i.e., $\quad l - 1 < s_n < l + 1$, for all $n \geq m$.

Let $\quad k = \min. \{ s_1, s_2, \ldots\ldots, s_{m-1}, l - 1 \}$,

and $\quad K = \max. \{ s_1, s_2, \ldots\ldots, s_{m-1}, l + 1 \}$.

$\therefore \quad k \leq s_n \leq K$ for all $n \in \mathbf{N}$.

Hence the sequence $< s_n >$ is bounded.

Note: The converse of the above theorem need not be true. That is, a bounded sequence need not be convergent. For example, the sequence $< (-1)^n >$ is bounded but is not convergent. **(Gorakhpur 2013, 15)**

Illustrative Examples

Example 1: *Show that the sequence* $< 1/n >$ *has the limit* 0*.*

Solution: For any given $\varepsilon > 0$, we have $\left| \frac{1}{n} - 0 \right| < \varepsilon$ when $\frac{1}{n} < \varepsilon$ *i.e.,* when $n > \frac{1}{\varepsilon} \cdot$

Let us choose a positive integer $m > 1/\varepsilon$. Then for all $n \geq m$, we have

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{m} < \varepsilon.$$

$\therefore \quad < \frac{1}{n} >$ converges to 0 *i.e.,* $< \frac{1}{n} >$ has the limit 0.

Example 2: *If* $s_n = k \; (\in \mathbf{R})$ *is a constant sequence, then* $\lim s_n = k$ *.*

Solution: We have $| s_n - k | = | k - k | = 0$ for all $n \in \mathbf{N}$.

Given any $\varepsilon > 0$, $| s_n - k | = 0 < \varepsilon$ for all n

i.e., $\quad | s_n - k | < \varepsilon$ for all $n \geq m = 1$.

Hence $\quad \lim s_n = k$.

Example 3: *The sequence $< s_n >$ where $s_n = 1/2^n$ converges to '0'.*

Solution: We have $|s_n - 0| = \dfrac{1}{2^n}\cdot$

$\therefore$ For any $\varepsilon > 0, |s_n - 0| < \varepsilon$ if $\dfrac{1}{2^n} < \varepsilon$

i.e., if $2^n > \dfrac{1}{\varepsilon}$ *i.e.*, if $n > \dfrac{\log(1/\varepsilon)}{\log 2}\cdot$

Let us choose a positive integer $m > \left(\log \dfrac{1}{\varepsilon}\right) \Big/ \log 2.$

Then for all $n \geq m, \left| s_n - 0 \right| < \varepsilon.$

$\therefore$ $< s_n >$ converges to 0.

Example 4: *Show that the sequence $< s_n >$ defined by $s_n = r^n$ converges to zero if $|r| < 1$.*

Solution: If $|r| < 1$, then we can write $|r| = \dfrac{1}{1+h}$, where $h > 0$. Since $h > 0$, therefore

$$(1+h)^n = 1 + nh + \frac{n(n-1)}{2} h^2 + \ldots + h^n \geq 1 + nh \text{ for all } n.$$

Now $|s_n - 0| = |r^n| = |r|^n = \dfrac{1}{(1+h)^n} \leq \dfrac{1}{1+nh}, \;\forall\; n.$

Let $\varepsilon > 0$ be given. Then $|s_n - 0| < \varepsilon$ if $\dfrac{1}{1+nh} < \varepsilon$ *i.e.*, if $n > \left(\dfrac{1}{\varepsilon} - 1\right) \Big/ h.$

If we take a positive integer $m > \left(\dfrac{1}{\varepsilon} - 1\right) \Big/ h$, then for all $n \geq m$, $|s_n - 0| < \varepsilon.$

Hence $< s_n >$ converges to zero.

Example 5: *Let $< s_n >$ be a sequence such that $s_n \neq 0$ for any n, and $\dfrac{s_{n+1}}{s_n} \to l$. Prove that if $|l| < 1$, then $s_n \to 0$.*

Solution: Since $|l| < 1$, hence there exists $\varepsilon_0 > 0$ such that

$$|l| + \varepsilon_0 = h < 1.$$

Now $\dfrac{s_{n+1}}{s_n} \to l \Rightarrow$ there exists a positive integer m such that

$$\left| \frac{s_{n+1}}{s_n} - l \right| < \varepsilon_0 \text{ for all } n \geq m.$$

We have $\left| \dfrac{s_{n+1}}{s_n} \right| = \left| \left(\dfrac{s_{n+1}}{s_n} - l \right) + l \right| \leq \left| \dfrac{s_{n+1}}{s_n} - l \right| + |l|,$

$$< \varepsilon_0 + |l|, \text{ for all } n \geq m$$

i.e., $\left| \dfrac{s_{n+1}}{s_n} \right| < h$, for all $n \geq m$. ...(1)

Replacing n by $m, m+1, \ldots, n-1$ successively in (1) and multiplying the corresponding sides of the resulting $n - m$ inequalities, we get

$$\left|\frac{s_{m+1}}{s_m}\right| \cdot \left|\frac{s_{m+2}}{s_{m+1}}\right| \cdots\cdots \left|\frac{s_n}{s_{n-1}}\right| < h^{n-m},$$

or
$$\left|\frac{s_{m+1}}{s_m} \cdot \frac{s_{m+2}}{s_{m+1}} \cdots\cdots\cdots \frac{s_n}{s_{n-1}}\right| < h^{n-m},$$

or
$$|s_n| < h^n \left(\frac{|s_m|}{h^m}\right), \text{ for all } n > m. \quad \ldots(2)$$

Again, since $0 < h < 1$, therefore, $h^n \to 0$ and hence, given $\varepsilon > 0$, there exists a positive integer m_1 such that

$$|h^n| < h^m \varepsilon / |s_m|, \text{ for all } n \geq m_1. \quad \ldots(3)$$

Let us choose a positive integer p such that $p > \max.\{m, m_1\}$.

From (2) and (3), we get

$$|s_n| < \varepsilon, \text{ for all } n \geq p.$$

Hence $s_n \to 0$.

Example 6: *Find an $m \in \mathbf{N}$ such that $\left|\dfrac{2n}{n+3} - 2\right| < \dfrac{1}{5}$ for all $n \geq m$.*

Solution: We have $\left|\dfrac{2n}{n+3} - 2\right| < \dfrac{1}{5} \Rightarrow \left|\dfrac{2n - 2n - 6}{n+3}\right| < \dfrac{1}{5}$

$$\Rightarrow \quad \frac{6}{n+3} < \frac{1}{5} \Rightarrow \frac{n+3}{6} > 5 \Rightarrow n > 27.$$

If we take a positive integer $m > 27$, we have

$$\left|\frac{2n}{n+3} - 2\right| < \frac{1}{5}, \text{ for all } n \geq m.$$

Hence for $\varepsilon = 1/5$, the required least value of $m = 28$.

In fact, we can find $m \in \mathbf{N}$ for each $\varepsilon > 0$ such that

$$\left|\frac{2n}{n+3} - 2\right| < \varepsilon \text{ for all } n \geq m.$$

$$\therefore \quad \lim \frac{2n}{n+3} = 2.$$

Example 7: *Show that the sequence $< s_n >$ where $s_n = \dfrac{2n^2+1}{2n^2-1}$, $\forall\ n \in \mathbf{N}$ converges to 1.*

Solution: Let $\varepsilon > 0$ be given.

We have
$$|s_n - 1| = \left|\frac{2n^2+1}{2n^2-1} - 1\right| = \left|\frac{2}{2n^2-1}\right| = \frac{2}{2n^2-1} < \varepsilon \text{ if } n > \sqrt{\left(\frac{2+\varepsilon}{2\varepsilon}\right)}.$$

If we choose a positive integer $m > \sqrt{\left(\dfrac{2+\varepsilon}{2\varepsilon}\right)}$, then for all $n \geq m$, $|s_n - 1| < \varepsilon$.

$$\therefore \quad \lim s_n = 1.$$

Example 8: *Show that the sequence* $< s_n >$ *where* $s_n = \dfrac{3n}{n+5n^{1/2}}$ *has the limit* 3.

Solution: Let $\varepsilon > 0$ be given.

We have $$\left|\frac{3n}{n+5n^{1/2}} - 3\right| = \left|\frac{3n-3n-15n^{1/2}}{n+5n^{1/2}}\right| = \frac{15n^{1/2}}{n+5n^{1/2}}$$

which is less than $\dfrac{15n^{1/2}}{n} = \dfrac{15}{n^{1/2}}$.

Thus $\dfrac{15n^{1/2}}{n+5n^{1/2}}$ will be $< \varepsilon$ if $\dfrac{15}{n^{1/2}} < \varepsilon$ *i.e.* if $n > \dfrac{225}{\varepsilon^2}$.

If we choose a positive integer $m > \dfrac{225}{\varepsilon^2}$, then $|s_n - 3| < \varepsilon$ for all $n \geq m$.

Hence $$\lim_{n \to \infty} s_n = 3.$$

Example 9: *Prove that lim* $(1/n^p) = 0, p > 0$.

Solution: Let $\varepsilon > 0$ be given.

We have $$\left|\frac{1}{n^p} - 0\right| < \varepsilon \Rightarrow \frac{1}{n^p} < \varepsilon \Rightarrow n^p > \frac{1}{\varepsilon}$$
$$\Rightarrow n > \left(\frac{1}{\varepsilon}\right)^{1/p}.$$

By Archimedean property, for $(1/\varepsilon)^{1/p} \in \mathbf{R}$ there exists a positive integer $m > (1/\varepsilon)^{1/p}$. If we choose $m > (1/\varepsilon)^{1/p}$, then we have

$$\left|\frac{1}{n^p} - 0\right| < \varepsilon \text{ for all } n \geq m.$$

Hence $$\lim \frac{1}{n^p} = 0, \text{ when } p > 0.$$

Example 10: *Show that the sequence* $< s_n >$, *where* $s_n = (-1)^{n-1}/n$, *converges to* 0.

Solution: We have $s_n = \dfrac{1}{n}$ if n is odd, $s_n = -\dfrac{1}{n}$ if n is even. The sequences $< s_{2n-1} >$ and $< s_{2n} >$ are subsequences of $< s_n >$.

$$< s_{2n-1} > = < 1, \frac{1}{3}, \frac{1}{5}, \ldots. >, < s_{2n} > = < -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, \ldots >.$$

But $< 1, \dfrac{1}{3}, \dfrac{1}{5}, \ldots >$ and $< \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{6}, \ldots >$ are subsequences of the sequence $< 1/n >$ which converges to 0.

$\therefore$ $< s_{2n-1} >$ and $< s_{2n} >$ both converge to the same limit 0.

Hence $< s_n >$ converges to 0.

[By theorem 4, article 6]

Alternative Solution: Take any given $\varepsilon > 0$.

We have $| s_n - 0 | = | s_n | = \left| \dfrac{(-1)^{n-1}}{n} \right| = \dfrac{1}{n} < \varepsilon$ if $n > \dfrac{1}{\varepsilon}$.

Now by Archimedean property of real numbers, for a given real number $1/\varepsilon$ there exists a positive integer m such that $m > 1/\varepsilon$ or $1/m < \varepsilon$.

Then for all $n \geq m$, we have

$$| s_n - 0 | = \frac{1}{n} \leq \frac{1}{m} < \varepsilon.$$

Hence $< s_n >$ converges to zero.

Example 11: *Show that* $\lim \sqrt[n]{n} = 1$.

Solution: Let $\sqrt[n]{n} = 1 + h_n$, where $h_n \geq 0$.

$$\therefore \quad n = (1 + h_n)^n = 1 + nh_n + \frac{n(n-1)}{1.2} h_n^2 + \ldots + h_n^n$$

$$> \frac{n(n-1)}{2} h_n^2 \text{ for all } n, \text{ since } h_n \geq 0.$$

$$\therefore \quad h_n^2 < \frac{2}{n-1} \text{ for } n \geq 2 \quad i.e. \quad |h_n| < \sqrt{\left(\frac{2}{n-1}\right)} \text{ for } n \geq 2.$$

Let $\varepsilon > 0$ be given. Then $|h_n| < \sqrt{\left(\dfrac{2}{n-1}\right)} < \varepsilon$, provided $\dfrac{2}{n-1} < \varepsilon^2$ *i.e.* $n > \dfrac{2}{\varepsilon^2} + 1$.

If we choose $m \in \mathbf{N}$ such that $m > \dfrac{2}{\varepsilon^2} + 1$, then we have

$$|h_n| < \varepsilon \ \forall \ n \geq m$$

i.e.
$$|\sqrt[n]{n} - 1| < \varepsilon \ \forall \ n \geq m.$$

$$\therefore \quad \lim \sqrt[n]{n} = 1.$$

Example 12: *Show that the sequence* $< s_n >$ *defined by*

$$s_n = \{\sqrt{(n+1)} - \sqrt{n}\}, \forall \ n \in \mathbf{N} \textit{ is convergent.}$$

Solution: We have $s_n = \sqrt{(n+1)} - \sqrt{n}$

$$= \{\sqrt{(n+1)} - \sqrt{n}\} \frac{\{\sqrt{(n+1)} + \sqrt{n}\}}{\{\sqrt{(n+1)} + \sqrt{n}\}}$$

$$= \frac{1}{\sqrt{(n+1)} + \sqrt{n}} < \frac{1}{\sqrt{n} + \sqrt{n}} = \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{n}} \quad i.e., \quad s_n < \frac{1}{\sqrt{n}}.$$

Let $\varepsilon > 0$ be given. Then $|s_n - 0| < \dfrac{1}{\sqrt{n}} < \varepsilon$, provided $\sqrt{n} > \dfrac{1}{\varepsilon}$ *i.e.,* $n > \dfrac{1}{\varepsilon^2}$.

If m is a positive integer greater than $1/\varepsilon^2$, then

$$|s_n - 0| < \varepsilon \text{ for all } n \geq m.$$

Hence
$$\lim s_n = 0.$$

Example 13: *Show that the sequence* $< s_n >$ *where* $s_n = \sin n\pi\theta$ *and* θ *is a rational number such that* $0 < \theta < 1$, *is not convergent.*

Solution: Let $\theta = \frac{p}{q}$, where p and q are integers. Since $0 < \theta < 1$, we must have $q \geq 2$. For $n = q, 2q, 3q, \ldots$, the terms of $< s_n >$ are $\sin \pi p, \sin 2\pi p, \sin 3\pi p, \ldots$, *i.e.*, 0, 0, 0, ... Thus $< s_n >$ contains a subsequence < 0, 0, 0, ...> which converges to 0. Now for $n = q + 1$, $2q + 1, 3q + 1$, ... the terms of $< s_n >$ are

$$\sin\left(\pi p + \frac{\pi p}{q}\right), \sin\left(2\pi p + \frac{\pi p}{q}\right), \sin\left(3\pi p + \frac{\pi p}{q}\right), \ldots.$$

i.e., $$(-1)^p \sin(\pi p / q), (-1)^{2p} \sin(\pi p / q), (-1)^{3p} \sin(\pi p / q), \ldots.$$

All these terms have absolute value $\sin(\pi p / q)$ and do not tend to zero since $0 < \pi p / q < \pi$. $[\because\ 0 < p / q < 1]$.

Thus the sequence $< s_n >$ contains a subsequence whose limit is 0 and a subsequence which (may or may not converge but certainly) does not have the limit zero. Hence we conclude that the given sequence is not convergent. [See corollary to theorem 3]

Remark: The sequence $< \sin n\pi\theta >$ obviously converges to 0 for $\theta = 0$ or $\theta = 1$.

7 Divergent Sequences

Definition 1: *A sequence $< s_n >$ is said to diverge to $+\infty$ if for any given $k > 0$ (however large), there exists $m \in \mathbf{N}$ such that $s_n > k$ for all $n \geq m$.*

If $< s_n >$ diverges to infinity, we write $s_n \to \infty$ as $n \to \infty$ or $\lim s_n = +\infty$.

Definition 2: *A sequence $< s_n >$ is said to diverge to $-\infty$ if for any given $k < 0$ (however small), there exists $m \in \mathbf{N}$ such that $s_n < k$ for all $n \geq m$.*

If $< s_n >$ diverges to minus infinity, we write

$$s_n \to -\infty \text{ as } n \to \infty \quad \text{or} \quad \lim s_n = -\infty.$$

A sequence is said to be a divergent sequence if it diverges to either $+\infty$ or $-\infty$.

Illustrations:

1. $< 2, 4, 6, \ldots, 2n, \ldots >$ diverges to $+\infty$.
2. $< 3, 3^2, 3^3, \ldots, 3^n, \ldots >$ diverges to $+\infty$.
3. $< x, x^2, x^3, \ldots, x^n, \ldots >, x > 1$, diverges to $+\infty$.
4. $< -2, -4, -6, \ldots, -2n, \ldots >$ diverges to $-\infty$.
5. $< -3, -3^2, -3^3, \ldots, -3^n, \ldots >$ diverges to $-\infty$.
6. $< -x, -x^2, -x^3, \ldots, -x^n, \ldots >, x > 1$, diverges to $-\infty$.

8 Oscillatory Sequences

Definition: *A sequence $< s_n >$ is said to be an oscillatory sequence if it is neither convergent nor divergent.*

An oscillatory sequence is said to oscillate finitely or infinitely according as it is bounded or unbounded.

Illustrations:

1. The sequence $<(-1)^n>$ oscillates finitely.
2. The sequence $<(-1)^n n>$ oscillates infinitely.

Theorem 1: *If a sequence $<s_n>$ diverges to infinity then any subsequence of $<s_n>$ also diverges to infinity.*

Proof: Let $<s_{n_k}>$ be any subsequence of the sequence $<s_n>$. Then by the definition of a subsequence $<n_1, n_2, \ldots\ldots, n_k, \ldots\ldots>$ is a strictly increasing sequence of positive integers. This implies $n_1 \geq 1 \Rightarrow n_k \geq k$ (by induction).

Take any given positive real number k_1.

Now $<s_n>$ diverges to $\infty \Rightarrow$ for $k_1 > 0$ there exists $m \in \mathbf{N}$ such that $s_n > k_1$ for all $n \geq m$ *i.e.*, $s_k > k_1 \ \forall\, k \geq m$.

For $k \geq m$, we have $n_k \geq k \geq m$ *i.e.*, $n_k \geq m$.

$\therefore$ $\quad s_{n_k} > k_1$ for all $n_k \geq m$.

$\therefore$ $\quad <s_{n_k}>$ diverges to infinity.

Note: If $s_{2n-1} \to \infty$ as $n \to \infty$ and $s_{2n} \to \infty$ as $n \to \infty$, then $s_n \to \infty$ as $n \to \infty$.

Theorem 2: *If $s_n > 0$ for all $n \in \mathbf{N}$, then*

$$s_n \to \infty \text{ as } n \to \infty \Leftrightarrow \frac{1}{s_n} \to 0 \text{ as } n \to \infty.$$

Proof: Let $s_n \to \infty$ as $n \to \infty$.

Let $\varepsilon > 0$ be given. Since $s_n \to \infty$ as $n \to \infty$, hence for $1/\varepsilon > 0$ there exists $m \in \mathbf{N}$ such that

$$s_n > 1/\varepsilon \text{ for all } n \geq m$$

$$\Rightarrow \quad \frac{1}{s_n} < \varepsilon \ \forall\, n \geq m \Rightarrow \left|\frac{1}{s_n}\right| < \varepsilon \ \forall\, n \geq m \qquad [\because s_n > 0]$$

$$\Rightarrow \quad \left|\frac{1}{s_n} - 0\right| < \varepsilon \ \forall\, n \geq m \ \Rightarrow \ \frac{1}{s_n} \to 0 \text{ as } n \to \infty.$$

Conversely, let $\dfrac{1}{s_n} \to 0$ as $n \to \infty$.

Take any given $k > 0$.

Now $\dfrac{1}{s_n} \to 0$ as $n \to \infty \Rightarrow$ for $1/k > 0$ there exists $m \in \mathbf{N}$ such that

$$\left|\frac{1}{s_n} - 0\right| < \frac{1}{k} \text{ for all } n \geq m$$

$$\Rightarrow \quad \frac{1}{s_n} < \frac{1}{k} \ \forall\, n \geq m \Rightarrow s_n > k \ \forall\, n \geq m$$

$$\Rightarrow \quad s_n \to \infty \text{ as } n \to \infty.$$

Theorem 3: *If the sequences $<s_n>, <t_n>$ diverge to infinity then $<s_n + t_n>$ and $<s_n t_n>$ diverge to infinity.*

Proof: Take any given $k > 0$. The sequence $< s_n >$ diverges to infinity $\Rightarrow$ for $k > 0$, there exists $m_1 \in \mathbf{N}$ such that $s_n > k \ \forall \ n \geq m_1$. Again, the sequence $< t_n >$ diverges to infinity $\Rightarrow$ for $1 > 0$, there exists $m_2 \in \mathbf{N}$ such that $t_n > 1 \ \forall \ n \geq m_2$.

Take $m = \max. \{m_1, m_2\}$.

$\therefore \quad s_n + t_n > k + 1 > k \quad \forall \ n \geq m \quad$ and $\quad s_n \, t_n > k \, .1 = k \quad \forall \ n \geq m.$

$\therefore$ Both $< s_n + t_n >$ and $< s_n \, t_n >$ diverge to infinity.

Theorem 4: *If $< s_n >$ diverges to infinity and $< t_n >$ is bounded then $< s_n + t_n >$ diverges to infinity.*

Proof: The sequence $< t_n >$ is bounded $\Rightarrow$ there exists $k_1 > 0$ such that

$$|t_n| < k_1 \ \forall \ n \in \mathbf{N}.$$

The sequence $< s_n >$ diverges to infinity $\Rightarrow$ for $k > 0$ there exists $m \in \mathbf{N}$ such that $s_n > k + k_1 \ \forall n \geq m.$

$\therefore$ For all $n \geq m$, we have

$$s_n + t_n \geq s_n - |t_n| \qquad [\because \ x \geq -|x|]$$

$$> k + k_1 - k_1 = k.$$

Thus for $k > 0$, there exists $m \in \mathbf{N}$ such that $s_n + t_n > k$ for all $n \geq m$.

Hence $< s_n + t_n >$ diverges to infinity.

Corollary: *If $< s_n >$ diverges to infinity and $< t_n >$ converges then $< s_n + t_n >$ diverges to infinity.*

Illustrative Examples

Example 14: *Prove that the sequence $< n^p >$ where $p > 0$ diverges to infinity.*

Solution: Let $s_n = n^p$. Then $s_n > 0$ for all n as $n \in \mathbf{N}$ and $p > 0$.

$\therefore$ The sequence $< \dfrac{1}{s_n} > = < \dfrac{1}{n^p} >$ exists.

Since we know that $\dfrac{1}{n^p} \to 0$ as $n \to \infty$,

$\therefore \quad n^p \to \infty$ as $n \to \infty$.

Hence $< n^p >$ diverges to ∞.

Example 15: *Show that the sequence $< \log \dfrac{1}{n} >$ diverges to $- \infty$.*

Solution: Let $\quad s_n = \log \dfrac{1}{n} \cdot$ Take any given $k < 0$.

Then $s_n < k$ if $\log \dfrac{1}{n} < k$ *i.e.,* if $- \log n < k$

i.e., if $\log n > -k$ *i.e.,* if $n > e^{-k}$.

If we take $m \in \mathbf{N}$ such that $m > e^{-k}$, then $s_n < k$ for all $n \geq m$.

Hence $\quad s_n \to -\infty$ as $n \to \infty$.

Example 16: *If $< t_n >$ diverges to ∞ and $s_n > t_n \ \forall \ n$, then $< s_n >$ diverges to ∞.*

Solution: Take any given $k > 0$.

Since $< t_n >$ diverges to ∞, therefore, for $k > 0$ there exists $m \in \mathbf{N}$ such that

$$t_n > k \text{ for all } n \geq m$$

$\Rightarrow$ $s_n > k$ for all $n \geq m$. $\quad [\because s_n > t_n \ \forall \ n \in \mathbf{N}]$

Hence $< s_n >$ diverges to ∞.

9 Algebra of Convergent Sequences

Theorem 1: *If $\lim s_n = l$ and $\lim t_n = l'$ then $\lim (s_n + t_n) = l + l'$. In other words the limit of the sum of two convergent sequences is the sum of their limits.*

Proof: Take any given $\varepsilon > 0$.

Since $\lim s_n = l$, therefore for a given positive real number $\varepsilon / 2$ there exists $m_1 \in \mathbf{N}$ such that

$$| s_n - l | < \varepsilon / 2 \text{ for all } n \geq m_1.$$

Similarly, since $\lim t_n = l'$, there exists $m_2 \in \mathbf{N}$ such that $| t_n - l' | < \varepsilon / 2$ for all $n \geq m_2$.

Let $m = \max. \{m_1, m_2\}$. Then

$$| s_n - l | < \varepsilon / 2 \text{ and } | t_n - l' | < \varepsilon / 2 \text{ for all } n \geq m.$$

$\therefore$ for all $n \geq m$, we have

$$| (s_n + t_n) - (l + l') | = | (s_n - l) + (t_n - l') |$$

$$\leq | s_n - l | + | t_n - l' | < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus for any given $\varepsilon > 0$, there exists a positive integer m such that

$$| (s_n + t_n) - (l + l') | < \varepsilon \text{ for all } n \geq m.$$

$\therefore$ The sequence $< s_n + t_n >$ is convergent and

$$\lim (s_n + t_n) = l + l' = \lim s_n + \lim t_n.$$

Note: The converse of the above theorem need not be true.

Let $s_n = (-1)^n$ and $t_n = (-1)^{n+1}$.

Then $s_n + t_n = (-1)^n + (-1)^{n+1} = (-1)^n [1 + (-1)] = 0$.

Hence the sequence $< s_n + t_n >$ converges to 0, while $< s_n >$ and $< t_n >$ oscillate finitely.

Theorem 2: *If $\lim s_n = l$ and $c \in \mathbf{R}$, then $\lim (cs_n) = cl$.*

Proof: If $c = 0$, the theorem is obvious because then $\lim (cs_n) = 0 = 0 \, . \, l$.

Let $c \neq 0$. Take any given $\varepsilon > 0$.

Since $\lim s_n = l$, hence for a given positive real number $\varepsilon / | c |$, there exists $m \in \mathbf{N}$ such that

$$| s_n - l | < \frac{\varepsilon}{| c |} \text{ for all } n \geq m.$$

Now for all $n \geq m$, we have

$$|cs_n - cl| = |c\,(s_n - l)| = |c||s_n - l| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$

$\therefore$ $\lim (cs_n) = cl.$

Theorem 3: *If $\lim s_n = l$ and $\lim t_n = l'$, then $\lim (s_n - t_n) = l - l'$.*

Proof: By theorem 2,

$$\lim (-t_n) = \lim [(-1)\, t_n] = (-1)\, l' = -l'.$$

We have $\lim (s_n - t_n) = \lim [s_n + (-t_n)] = \lim s_n + \lim (-t_n)$, [by theorem 1]

$$= l - l'.$$

Corollary: *If $< s_n >$ and $< t_n >$ are convergent sequences such that $s_n \leq t_n$ for all $n \in \mathbf{N}$ and $\lim s_n = l$, $\lim t_n = l'$, then $l \leq l'$.*

Proof: By theorem 3, we have $\lim (t_n - s_n) = l' - l$. By hypothesis $t_n - s_n \geq 0$ for all $n \in \mathbf{N}$. Hence $l' - l \geq 0$, by theorem 1 of article 6. Thus $l' \geq l$ *i.e.*, $l \leq l'$.

Theorem 4: *If $\lim s_n = 0$ and the sequence $< t_n >$ is bounded then $\lim (s_n t_n) = 0$.*

Proof: The sequence $< t_n >$ is bounded $\Rightarrow$ there exists $k \in \mathbf{R}^+$ such that

$$|t_n| < k \;\; \forall \;\; n \in \mathbf{N}.$$

Take any given $\varepsilon > 0$.

Since $\lim s_n = 0$, therefore for a given positive real number ε / k there exists $m \in \mathbf{N}$ such that $|s_n - 0| = |s_n| < \dfrac{\varepsilon}{k} \;\forall\; n \geq m$.

Now for all $n \geq m$, we have

$$|s_n t_n - 0| = |s_n t_n| = |s_n||t_n| < \frac{\varepsilon}{k} \cdot k = \varepsilon.$$

Thus for any given $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that $|s_n t_n - 0| < \varepsilon$ for all $n \geq m$.

Hence $\lim (s_n t_n) = 0.$

Theorem 5: *If $\lim s_n = l$ and $\lim t_n = l'$, then $\lim (s_n t_n) = ll'$.*

Proof: Let $\varepsilon > 0$ be given. We have

$$|s_n t_n - ll'| = |s_n t_n - l t_n + l t_n - ll'| = |t_n (s_n - l) + l\,(t_n - l')|$$
$$\leq |t_n (s_n - l)| + |l\,(t_n - l')|$$
$$= |t_n||s_n - l| + |l||t_n - l'|. \qquad ...(1)$$

Since the sequence $< t_n >$ is convergent, therefore it is bounded *i.e.* there exists a positive real number k such that

$$|t_n| \leq k \text{ for all } n \in \mathbf{N}.$$

Since the sequences $< s_n >$ and $< t_n >$ are convergent, therefore there exist positive integers m_1 and m_2, such that

$$|s_n - l| < \frac{\varepsilon}{2k} \text{ for all } n \geq m_1 \qquad ...(2)$$

and $$|t_n - l'| < \frac{\varepsilon}{2(|l| + 1)} \text{ for all } n \geq m_2. \qquad ...(3)$$

Let $m = \max.\{m_1, m_2\}$. From (1), (2) and (3), we have for all $n \geq m$

$$|s_n t_n - ll'| < k \cdot \frac{\varepsilon}{2k} + |l| \frac{\varepsilon}{2(|l|+1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\lim (s_n t_n) = l\, l'$.

Note 1: In the inequality (3) we have taken $(|l|+1)$. Had we not done so, this inequality would have failed in case $l = 0$. Hence to include this case also we used this device.

Note 2: The converse of the above theorem need not be true.

Let $s_n = (-1)^n$ and $t_n = (-1)^{n+1}$.

Then $s_n t_n = (-1)^n (-1)^{n+1} = (-1)^{2n+1} = -1.$

Thus $< s_n t_n >$ is a constant sequence and converges to -1 while $< s_n >$ and $< t_n >$ oscillate finitely.

Theorem 6: *If $\lim s_n = l$ and $l \neq 0$, then there exists a positive number k and a positive integer m, such that $|s_n| > k$ for all $n \geq m$.*

Proof: Let us choose $\varepsilon = \frac{1}{2}|l|$. Then $\varepsilon > 0$, since $l \neq 0$. Since $\lim s_n = l$, therefore there must exist a positive integer m, such that $|s_n - l| < \varepsilon$ for all $n \geq m$.

We can write $l = l - s_n + s_n$.

$\therefore$ $|l| = |(l - s_n) + s_n|$

$\leq |l - s_n| + |s_n| < \varepsilon + |s_n|$ for all $n \geq m$.

$\therefore$ $|s_n| > |l| - \varepsilon$ for all $n \geq m$

or $|s_n| > |l| - \frac{1}{2}|l| = \frac{1}{2}|l|$ for all $n \geq m$.

Thus we have found a positive number $k = \frac{1}{2}|l|$ and a positive integer m, such that $|s_n| > k$ for all $n \geq m$.

Theorem 7: *If $\lim t_n = l', l' \neq 0$ and $t_n \neq 0\ \forall\ n$, then $\lim (1/t_n) = 1/l'$.*

Proof: We have $\left|\frac{1}{t_n} - \frac{1}{l'}\right| = \frac{|l' - t_n|}{|t_n| \cdot |l'|}$. ...(1)

Since $l' \neq 0$, therefore by theorem 6, there exists a positive number k and a positive integer m_1, such that

$$|t_n| > k \text{ or } \frac{1}{|t_n|} < \frac{1}{k} \text{ for all } n \geq m_1. \qquad ...(2)$$

Take any given $\varepsilon > 0$.

Since $\lim t_n = l'$, therefore for a given positive real number $k|l'|\varepsilon$, there exists $m_2 \in \mathbf{N}$ such that

$$|t_n - l'| < k|l'|\varepsilon \text{ for all } n \geq m_2. \qquad ...(3)$$

Let $m = \max.\{m_1, m_2\}$. From (1), (2) and (3), we have

$$\left|\frac{1}{t_n} - \frac{1}{l'}\right| < \frac{1}{|l'|} \cdot \frac{1}{k} \cdot k|l'|\varepsilon \text{ for all } n \geq m$$

i.e., $\left|\frac{1}{t_n} - \frac{1}{l'}\right| < \varepsilon$ for all $n \ge m$.

Hence $\lim (1/t_n) = 1/l'$.

Theorem 8: *If $\lim s_n = l$ and $\lim t_n = l'$ $(\neq 0)$, $t_n \neq 0$ for all n, then*

$$\lim (s_n / t_n) = l / l'.$$

Proof: Since $t_n \neq 0\ \forall\ n$ and $l' \neq 0$, therefore, by theorem 7, $\lim (1/t_n) = 1/l'$.

Now $\lim\left(\frac{s_n}{t_n}\right) = \lim\left(s_n \cdot \frac{1}{t_n}\right) = (\lim s_n) \lim\left(\frac{1}{t_n}\right)$ [by theorem 5, article 9]

$$= l \cdot \frac{1}{l'} = \frac{l}{l'}.$$

Hence $\lim \frac{s_n}{t_n} = \frac{l}{l'} = \frac{\lim s_n}{\lim t_n}$.

Theorem 9: Squeeze Theorem (Sandwich Theorem): *If $< s_n >, < t_n >$ and $< u_n >$ are three sequences such that*

(i) for some positive integer k, $s_n \le u_n \le t_n$ for $n \ge k$,

(ii) $\lim s_n = \lim t_n = l$, then $\lim u_n = l$.

Proof: Let $\varepsilon > 0$ be given.

Since $\lim s_n = l$, therefore, there exists $m_1 \in \mathbf{N}$ such that

$|s_n - l| < \varepsilon$ for all $n \ge m_1$

i.e., $l - \varepsilon < s_n < l + \varepsilon$ for all $n \ge m_1$.

Similarly $\lim t_n = l$, therefore, there exists $m_2 \in \mathbf{N}$ such that

$|t_n - l| < \varepsilon$ for all $n \ge m_2$

i.e., $l - \varepsilon < t_n < l + \varepsilon$ for all $n \ge m_2$.

Let $m = \max.\{m_1, m_2, k\}$. Then, for $n \ge m$, we have

$l - \varepsilon < s_n \le u_n \le t_n < l + \varepsilon$

or $l - \varepsilon < u_n < l + \varepsilon$.

Thus $|u_n - l| < \varepsilon$ for all $n \ge m$.

Hence $\lim u_n = l$.

Corollary: *If $< s_n >$ and $< t_n >$ are two sequences such that $|s_n| \le |t_n|\ \forall n \ge k$ where $k \in \mathbf{N}$ and $\lim t_n = 0$, then $\lim s_n = 0$.*

Proof: Lim $t_n = 0 \Rightarrow \lim |t_n| = 0$ and $\lim (-|t_n|) = 0$.

We have $|s_n| \le |t_n|\ \forall\ n \ge k$

$\Rightarrow$ $-|t_n| \le s_n \le |t_n|\ \forall\ n \ge k$.

Hence by Sandwich theorem, $\lim s_n = 0$.

Note: If $|s_n| \le \alpha |t_n|\ \forall n \ge k$ where $k \in \mathbf{N}$ and α is a positive real number, then

$$\lim t_n = 0 \Rightarrow \lim s_n = 0.$$

For example, let $s_n = \dfrac{\cos n\pi}{n} \cdot$

Then $\quad |s_n| = \left|\dfrac{\cos n\pi}{n}\right| \le \dfrac{1}{n} \cdot$ $\quad$ [$\because -1 \le \cos n\pi \le 1$]

$\therefore \quad |s_n| \le \left|\dfrac{1}{n}\right|$ and $\lim \dfrac{1}{n} = 0.$

Hence $\quad \lim s_n = \lim \dfrac{\cos n\pi}{n} = 0.$

Theorem 10: (Cauchy's first theorem on limits)

If $\lim\limits_{n \to \infty} s_n = l,$ *then* $\lim\limits_{n \to \infty} \dfrac{s_1 + s_2 + \ldots. + s_n}{n} = l.$

(Gorakhpur 2015)

Proof: Define a sequence $< t_n >$ such that

$$s_n = l + t_n \;\forall\; n \in \mathbf{N}.$$

$\therefore \quad \lim t_n = 0$

and $$\frac{s_1 + s_2 + \ldots + s_n}{n} = l + \frac{t_1 + t_2 + \ldots + t_n}{n} \cdot \qquad \ldots(1)$$

In order to prove the theorem we wish to show that

$$\lim_{n \to \infty} \frac{t_1 + t_2 + \ldots + t_n}{n} = 0.$$

Let $\varepsilon > 0$ be given. Since $\lim t_n = 0$, therefore, there exists a positive integer m, such that

$$|t_n - 0| = |t_n| < \varepsilon / 2 \;\;\forall\;\; n \ge m. \qquad \ldots(2)$$

Also, since every convergent sequence is bounded, hence there exists a real number $k > 0$ such that

$$|t_n| \le k \;\forall\; n \in \mathbf{N}. \qquad \ldots(3)$$

Now for all $n \ge m$, we have

$$\left|\frac{t_1 + t_2 + \ldots + t_n}{n}\right| = \left|\frac{t_1 + t_2 + \ldots + t_m}{n} + \frac{t_{m+1} + t_{m+2} + \ldots + t_n}{n}\right|$$

$$\le \frac{|t_1| + |t_2| + \ldots + |t_m|}{n} + \frac{|t_{m+1}| + |t_{m+2}| + \ldots + |t_n|}{n}$$

$$\le \frac{mk}{n} + \frac{n-m}{n} \cdot \frac{\varepsilon}{2} \qquad \text{[From (2) and (3)]}$$

$$< \frac{mk}{n} + \frac{\varepsilon}{2} \cdot \qquad \ldots(4)$$

$$\left[\because \; 0 \le \frac{n-m}{n} < 1\right]$$

If m is fixed, then $\dfrac{mk}{n} < \dfrac{1}{2}\varepsilon$ if $n > \dfrac{2mk}{\varepsilon} \cdot$

Let us choose a positive integer $p > \dfrac{2mk}{\varepsilon} \cdot$ Then

$$\frac{mk}{n} < \frac{1}{2}\varepsilon \text{ for } n \ge p. \qquad \ldots(5)$$

Let $M = \max.\{m, p\}$. From (4) and (5), we have

$$\left|\frac{t_1 + t_2 + \ldots + t_n}{n}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \;\forall\; n \geq M.$$

Thus $\lim \dfrac{t_1 + t_2 + \ldots + t_n}{n} = 0$ and consequently (1) gives

$$\lim \frac{s_1 + s_2 + \ldots + s_n}{n} = l.$$

Note: The converse of the above theorem need not be true.

Consider the sequence $< s_n >$ where $s_n = (-1)^n$.

For this sequence,

$$\frac{s_1 + s_2 + \ldots + s_n}{n} = 0, \text{ if } n \text{ is even, and } \quad = -\frac{1}{n}, \text{ if } n \text{ is odd.}$$

$\therefore \quad \lim \dfrac{s_1 + s_2 + \ldots + s_n}{n} = 0$, but $< s_n >$ is not convergent.

Theorem 11: (Cauchy's second theorem on limits). *If $< s_n >$ is a sequence such that $s_n > 0$ for all n and $\lim s_n = l$, then $\lim (s_1\, s_2 \ldots s_n)^{1/n} = l$.*

Proof: Let us define a sequence $< t_n >$ such that

$$t_n = \log s_n \text{ for all } n.$$

Since $\lim s_n = l$, therefore $\lim t_n = \log l$.

By Cauchy's first theorem on limits, we have

$$\lim \frac{t_1 + t_2 + \ldots + t_n}{n} = \log l$$

i.e., $$\lim \frac{\log s_1 + \log s_2 + \ldots + \log s_n}{n} = \log l$$

i.e., $$\lim \log (s_1\, s_2 \ldots s_n)^{1/n} = \log l$$

and hence, $\lim (s_1\, s_2 \ldots s_n)^{1/n} = l$.

Note: While proving this theorem, we have used the following fact :

$$\lim s_n = l \Leftrightarrow \lim \log s_n = \log l,$$

provided $s_n > 0$ for all n and $l > 0$.

Theorem 12: *If $< s_n >$ is a sequence such that $s_n > 0$ for all $n \in \mathbf{N}$ and $\lim \dfrac{s_{n+1}}{s_n} = l$, then $\lim \sqrt[n]{s_n} = l$.*

Proof: Let us define a sequence $< t_n >$ such that

$$t_1 = s_1,\; t_2 = \frac{s_2}{s_1},\; t_3 = \frac{s_3}{s_2}, \ldots,\; t_n = \frac{s_n}{s_{n-1}}, \ldots.$$

$$\therefore \quad t_1\, t_2 \ldots t_n = s_n.$$

Also $\lim \dfrac{s_{n+1}}{s_n} = l \Rightarrow \lim \dfrac{s_n}{s_{n-1}} = l \Rightarrow \lim t_n = l.$

Since $s_n > 0$ for all n, hence $t_n > 0$ for all n.

Thus we have a sequence $< t_n >$ such that $t_n > 0$ for all n and $\lim t_n = l$.
Hence by theorem 11 of article 9, we have $\lim (t_1 t_2 \ldots t_n)^{1/n} = l$

i.e., $\lim (s_n)^{1/n} = l.$

Theorem 13: (**Cesaro's theorem**)
If $\lim s_n = l$ *and* $\lim t_n = l'$, *then*

$$\lim \frac{s_1 t_n + s_2 t_{n-1} + \ldots + s_n t_1}{n} = ll'.$$

Proof: Let $s_n = l + x_n$ and $|x_n| = X_n$. Then $\lim x_n = 0$ and hence $\lim X_n = 0$. Therefore by theorem 10 of article 9, we have

$$\lim \frac{1}{n}(X_1 + X_2 + \ldots + X_n) = 0.$$

Now $\quad \frac{1}{n}(s_1\, t_n + s_2\, t_{n-1} + \ldots + s_n\, t_1)$

$$= \frac{l}{n}(t_1 + t_2 + \ldots + t_n) + \frac{1}{n}(x_1\, t_n + x_2\, t_{n-1} + \ldots + x_n\, t_1), \qquad \ldots(1)$$

on substituting for $s_1, s_2, \ldots, s_n$.
Now the sequence $< t_n >$ is convergent and every convergent sequence is bounded. Therefore there exists a positive real number k, such that

$$|t_n| < k \ \forall\ n.$$

$$\therefore \quad 0 \le \left|\frac{1}{n}(x_1\, t_n + x_2\, t_{n-1} + \ldots + x_n\, t_1)\right|$$

$$\le \frac{1}{n}[|x_1|.|t_n| + |x_2|.|t_{n-1}| + \ldots + |x_n|.|t_1|]$$

$$< \frac{k}{n}(|x_1| + |x_2| + \ldots + |x_n|)$$

$$= \frac{k}{n}(X_1 + X_2 + \ldots + X_n)$$

$\to 0$, since k is fixed for all n.

$$\therefore \quad \lim \frac{1}{n}(x_1 t_n + x_2 t_{n-1} + \ldots + x_n\, t_1) = 0, \text{ by Sandwich theorem.}$$

Now since $\lim t_n = l'$, we have by theorem 10 of article 9,

$$\lim \frac{1}{n}(t_1 + t_2 + \ldots + t_n) = l'.$$

Hence finally, we get from (1)

$$\lim \frac{1}{n}\left(s_1 t_n + s_2\, t_{n-1} + \ldots + s_n\, t_1\right) = ll'.$$

Note: Theorems of article 9 provide an easier method for evaluating the limits of sequences than the method for evaluating these limits directly by definition. Later on we shall illustrate the use of the theorems of this section to evaluate the limits of sequences.

10 Monotonic Sequences

Definition 1: *A sequence $< s_n >$ is said to be monotonically increasing (or non-decreasing), if $s_n \le s_{n+1}$ for all n i.e., $s_n \le s_m$ for $n < m$.*

Definition 2: *A sequence $< s_n >$ is said to be strictly increasing if $s_n < s_{n+1}$ for all $n \in \mathbf{N}$.*

Definition 3: *A sequence $< s_n >$ is said to be monotonically decreasing (or non-increasing) if $s_n \ge s_{n+1}$ for all n i.e., $s_n \ge s_m$ for $n < m$.*

Definition 4: *A sequence $< s_n >$ is said to be strictly decreasing if $s_n > s_{n+1}$ for all $n \in \mathbf{N}$.*

Definition 5: *A sequence $< s_n >$ is said to be monotonic if it is either monotonically increasing or monotonically decreasing.*

Illustrations:

1. The sequence $< 1, 2, 3, \ldots, n, \ldots >$ is strictly increasing.
2. $< 2, 2, 4, 4, 6, 6, \ldots >$ is monotonically increasing.
3. $< -\frac{1}{n} >$ is strictly increasing.
4. $< 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots >$ is strictly decreasing.
5. $< 1, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \ldots >$ is monotonically decreasing.
6. $< -2, -4, -6, -8, \ldots >$ is strictly decreasing.
7. $< 0, 1, 0, 1, 0, 1, \ldots >$ is not monotonic.
8. $< -2, 2, -4, 4, -6, 6, \ldots >$ is not monotonic.
9. $< 1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \ldots >$ is not monotonic.

Note: If $< s_n >$ is a sequence of positive terms, then $< s_n >$ is increasing $\Leftrightarrow < 1/s_n >$ is decreasing.

Theorem 1: (Monotone convergence theorem). *Every bounded monotonically increasing sequence converges.* **(Gorakhpur 2011, 13)**

Proof: Let $< s_n >$ be a bounded monotonically increasing sequence.

Let $S = \{s_n : n \in \mathbf{N}\}$ be the range of the sequence $< s_n >$. Then S is a non-empty set which is bounded above. Hence by the completeness axiom for **R**, there exists a number $l = \sup S$. We shall show that $< s_n >$ converges to l.

Let $\varepsilon > 0$ be given. Then $l - \varepsilon < l$, so that $l - \varepsilon$ is not an upper bound of S. Hence there exists a positive integer m such that $s_m > l - \varepsilon$. Since $< s_n >$ is monotonically increasing, therefore,

$$s_n \ge s_m > l - \varepsilon \text{ for all } n \ge m. \qquad \ldots(1)$$

Also, since l is the supremum of S, therefore,

$$s_n \le l < l + \varepsilon \text{ for all } n. \qquad \ldots(2)$$

From (1) and (2), we get $l - \varepsilon < s_n < l + \varepsilon$ for all $n \ge m$

i.e., $\quad |s_n - l| < \varepsilon$ for all $n \ge m$.

$\therefore \quad \lim s_n = l$.

Corollary 1: *Every bounded monotonically decreasing sequence converges.*

Proof: Let $< s_n >$ be a bounded monotonically decreasing sequence. Define a sequence $< t_n >$ such that $t_n = - s_n$, for all $n \in \mathbf{N}$. Then $< t_n >$ is a bounded monotonically increasing sequence and hence by the above theorem, it converges. If $\lim t_n = l$, then $\lim s_n = \lim (- t_n) = - \lim t_n = - l$.

Note: We can prove this result independently by taking infimum of the set S.

Corollary 2: *Every bounded monotonic sequence converges.* **(Kanpur 2012)**

Proof: In order to prove this theorem we are to prove the following two results.

(i) Every bounded monotonically increasing sequence is convergent. (Give complete proof of theorem 1)

(ii) Every bounded monotonically decreasing sequence is convergent. (Deduce it from the result of theorem 1 as deduced in corollary 1)

Theorem 2: *A non-decreasing (i.e., monotonically increasing) sequence which is not bounded above diverges to infinity.*

Proof: Let $< s_n >$ be a non-decreasing sequence which is not bounded above.

Take any real number $k > 0$.

Now $< s_n >$ is not bounded above $\Rightarrow$ there exists $m \in \mathbf{N}$ such that $s_m > k$.

Also $< s_n >$ is non-decreasing $\Rightarrow s_n \geq s_m$ for $n > m$.

$\therefore \quad s_n \geq s_m > k$ for $n > m$ or $s_n > k$ for $n > m$.

Hence $< s_n >$ diverges to infinity.

Theorem 3: *A non-increasing (i.e., monotonically decreasing) sequence which is not bounded below diverges to minus infinity.*

Proof: Let $< s_n >$ be a non-increasing sequence which is not bounded below.

Take any real number $k < 0$.

Now $< s_n >$ is not bounded below $\Rightarrow$ there exists $m \in \mathbf{N}$ such that $s_m < k$.

Also $< s_n >$ is non-increasing $\Rightarrow s_n \leq s_m$ for $n > m$.

$\therefore \quad s_n \leq s_m < k$ for $n > m$ or $s_n < k$ for $n > m$.

Hence $< s_n >$ diverges to $- \infty$.

Theorem 4: *Every sequence has a monotonic subsequence.*

Proof: Consider the sequence $a_0 = < s_n >$. Let $a_1, a_2, a_3, \ldots$ denote the subsequences

$$< s_2, s_3, s_4, \ldots >, < s_3, s_4, s_5, \ldots >, < s_4, s_5, s_6, \ldots >, \ldots \text{ respectively.}$$

There arise two different cases.

(i) **Each of the sequences** $a_0, a_1, a_2, \ldots$ **has a greatest term.** Let $s_{n_1}, s_{n_2}, s_{n_3}, \ldots$ denote the greatest terms of $a_0, a_1, a_2, \ldots$ respectively.

Then $n_1 \leq n_2 \leq n_3 \leq \ldots$ and $s_{n_1} \geq s_{n_2} \geq s_{n_3} \geq \ldots$.

Consequently $< s_{n_1}, s_{n_2}, s_{n_3}, \ldots >$ is a monotonically decreasing subsequence of $< s_n >$.

(ii) **At least one of the sequences $a_0, a_1, a_2, \ldots$ has no greatest term.** Suppose a_m has no greatest term. Then each term of a_m is ultimately followed by some term of a_m that exceeds it. For, if there is a term of a_m which exceeds all the terms following it, then it can be exceeded by finitely many terms at the most and hence, a_m must have a greatest term. Now s_{m+1} is the first term of a_m. Let s_{n_2} be the first term of a_m exceeding s_{m+1}, s_{n_3} the first term of a_m that follows s_{n_2} and exceeds it, s_{n_4} the first term of a_m that follows s_{n_3} and exceeds it, and so on.

Thus $< s_{m+1}, s_{n_2}, s_{n_3}, s_{n_4}, \ldots >$ is a monotonically increasing subsequence of $< s_n >$.

Note: In the above proof, we have used the concept of a greatest term of a sequence. *A term s_k of a sequence $< s_n >$ is said to be a greatest term of $< s_n >$ if $s_n \le s_k$ for all n.*

It is not necessary for a sequence to have a greatest term; *e.g.*, the sequence $< 1, 3, 5, \ldots >$ has no greatest term. Also, it is not necessary that a greatest term of a sequence be unique; *e.g.*, for the sequence $< s_n >$ defined by $s_n = (-1)^{n-1}$, each of the terms $s_1, s_3, s_5, \ldots$ is a greatest term.

If $< s_{n_k} >$ be a subsequence of $< s_n >$, s_{n_p} be a greatest term of $< s_{n_k} >$ and s_m be a greatest term of $< s_n >$, then $s_{n_p} \le s_m$, because s_{n_p} is also a term of $< s_n >$, and s_m is a greatest term of $< s_n >$.

11 Limit Points of a Sequence

Definition: *A real number p is said to be a limit point (or a cluster point) of a sequence $< s_n >$ if every neighbourhood of p contains infinite number of terms of the sequence.*

(Gorakhpur 2012)

Since every open interval $]\, p - \varepsilon, p + \varepsilon\, [$, $\varepsilon > 0$, is a neighbourhood of p and also every neighbourhood of p contains an open interval $]\, p - \varepsilon, p + \varepsilon\, [$ for some $\varepsilon > 0$, therefore we can say that a real number p is a limit point of a sequence $< s_n >$ iff given any $\varepsilon > 0$, $s_n \in \,]\, p - \varepsilon, p + \varepsilon\, [$ for infinitely many values of n *i.e.*, $|s_n - p| < \varepsilon$ for infinitely many values of n.

It can be easily seen that a real number p is a limit point of a sequence $< s_n >$ iff given any neighbourhood N of p and $m \in \mathbf{N}$ we can find $k \in \mathbf{N}$ such that $k > m$ and $s_k \in N$.

Remarks: 1. Limit point of a sequence is different from the limit of a sequence. The limit of a sequence is a limit point of the sequence, while a limit point of a sequence need not be the limit of the sequence.

2. Limit point of a sequence need not be a term of the sequence.

3. If $s_n = l$ for infinitely many values of n then l is a limit point of $< s_n >$.

4. A real number p is not a limit point of $< s_n >$ if there exists even one neighbourhood of p containing finite number of terms of the sequence.

Illustrations:

1. The sequence $< (-1)^n >$ has 1 and -1 as limit points. Here $s_n = -1$, if n is odd and $s_n = 1$, if n is even. Any neighbourhood of -1 will contain all the odd terms of the sequence hence -1 is a limit point.

Similarly any neighbourhood of 1 will contain all the even terms of the sequence, so 1 is a limit point.

2. The sequence $< \frac{1}{n} >$ has only one limit point, namely 0.

Given $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that $\frac{1}{m} < \varepsilon$.

For $n \geq m, 0 < \frac{1}{n} \leq \frac{1}{m} < \varepsilon$ *i.e.,* $-\varepsilon < 0 < \frac{1}{n} < \varepsilon$ for all $n \geq m$.

Hence $\frac{1}{n} \in \,]-\varepsilon, \varepsilon[$ for all $n \geq m$. Thus every ε-nhd of 0 contains infinitely many points of the sequence. Hence 0 is a limit point of this sequence.

3. The sequence $< 1, 2, 3, \ldots, n, \ldots >$ has no limit point.

Let $p \in \mathbf{R}$. Whatever ε we take, the neighbourhood $]\, p - \varepsilon, p + \varepsilon \,[$ of p contains at the most a finite number of terms of this sequence. Hence p is not a limit point of this sequence.

Theorem 1: *If l is a limit point of the range of a sequence $< s_n >$, then l is a limit point of the sequence $< s_n >$.*

Proof: Let S be the range set of the sequence $< s_n >$. Since l is a limit point of S, therefore, every nhd. of l contains infinite number of distinct elements of the set S. But each element of the set S is a term of the sequence $< s_n >$. Hence every nhd of l contains infinite number of terms of the sequence $< s_n >$. Thus l is a limit point of the sequence $< s_n >$.

Note 1: The converse of the above theorem need not be true.

Consider the sequence $< s_n >$ where $s_n = 1 + (-1)^n$.

We have $s_n = 0$, if n is odd and $s_n = 2$, if n is even.

$\therefore$ 0, 2 are limit points of the sequence $< s_n >$. But the range of this sequence is the set $\{0, 2\}$, which is a finite set.

Now a finite set has no limit points, hence the range of $< s_n >$ has no limit points.

Note 2: If a sequence has all its terms distinct, then the limit points of the sequence and the limit points of the range set are same.

Theorem 2: *If $s_n \to l$, then l is the only limit point of $< s_n >$.*

Proof: First we shall show that l is a limit point of $< s_n >$. Let $\varepsilon > 0$ be given. Since $s_n \to l$, therefore, there exists a positive integer m such that

$$|s_n - l| < \varepsilon \text{ for all } n \geq m$$

i.e., $$|s_n - l| < \varepsilon \text{ for infinitely many values of } n.$$

This shows that l is a limit point of $< s_n >$.

Now we shall show that if l' be any limit point of $< s_n >$, then we must have $l' = l$.

Let $\varepsilon > 0$ be arbitrary. Since l is the limit of $< s_n >$, therefore, there exists a positive integer p such that

$$|s_n - l| < \varepsilon / 2 \text{ for all } n \geq p. \quad \ldots(1)$$

Since l' is a limit point of $< s_n >$, therefore, there must exist a positive integer $q > p$ such that $$|s_q - l'| < \varepsilon / 2. \quad \ldots(2)$$

Putting $n = q$ in (1), $|s_q - l| < \varepsilon / 2. \quad \ldots(3)$

Now $\quad |l - l'| = |(s_q - l') - (s_q - l)| \leq |s_q - l'| + |s_q - l|$
$< \varepsilon / 2 + \varepsilon / 2$, from (2) and (3)

i.e., $\quad |l - l'| < \varepsilon$.

Since ε is arbitrary, hence we must have $|l - l'| = 0$ *i.e.,* $l = l'$.

Note: The converse of the above theorem need not be true; *i.e.,* a sequence having only one limit point may not converge.

Consider the sequence $< s_n >$, where s_n is given by

$$s_n = \begin{cases} \dfrac{1}{n}, & \text{if } n \text{ is even ;} \\ n, & \text{if } n \text{ is odd.} \end{cases}$$

There is only one limit point of $< s_n >$, namely 0, and yet the sequence does not converge.

Theorem 3: (Bolzano-Weierstrass Theorem for sequences): *Every bounded sequence has at least one limit point.*

Proof: Let $< s_n >$ be a bounded sequence and S be its range set. Then $S = \{s_n : n \in \mathbf{N}\}$. Since the sequence $< s_n >$ is bounded, therefore, S is a bounded set.

Case I: Let S be a finite set. Then for infinitely many indices n, $s_n = p$, where p is some real number. Obviously p is a limit point of $< s_n >$.

Case II: Let S be an infinite set. Since S is bounded, by Bolzano-Weierstrass theorem for sets of real numbers, S has a limit point, say p. Hence every nbd of p contains infinitely many distinct points of S or every nbd of p contains infinitely many terms of the sequence $< s_n >$ and consequently p is a limit point of the sequence $< s_n >$.

Corollary 1: *If F is a closed and bounded set of real numbers, then every sequence in F has a limit point in F.*

Proof: Let $< s_n >$ be a sequence in F. Then $< s_n >$ is bounded and hence, it has a limit point, say p. Also $p \in F$, since p cannot belong to $\mathbf{R} - F$. If $p \in \mathbf{R} - F$, then $\mathbf{R} - F$ is an open set containing p. Thus it is a neighbourhood of p that contains no term of the sequence $< s_n >$ and hence we get a contradiction to the fact that p is a limit point of the sequence $< s_n >$.

Corollary 2: *If I is a closed interval, then every sequence in I has a limit point in I.*

Proof: Since every closed interval is a closed and bounded set, hence the result follows from corollary 1.

We have proved earlier that every convergent sequence is bounded and also it has only one limit point. Now we shall prove the converse.

Theorem 4: *If a sequence $< s_n >$ is bounded and has only one limit point, say l, then $s_n \to l$.*

Proof: Since $< s_n >$ is bounded, so it has at least one limit point. But l is the only limit point of $< s_n >$. Hence, for any $\varepsilon > 0$, $]\, l - \varepsilon, l + \varepsilon\, [$ contains s_n for all except a finite number of values of n.

Let $s_{m_1}, s_{m_2}, s_{m_3}, \ldots, s_{m_p}$ be the finite number of terms of the sequence $< s_n >$ that lie outside $]\, l - \varepsilon, l + \varepsilon [$.

If $\quad m-1=\max.\{m_1, m_2, \ldots, m_p\}$, then $s_n \in]\, l-\varepsilon, l+\varepsilon\,[\ \ \forall n \geq m$.
Hence for any $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$|s_n - l| < \varepsilon \quad \forall n \geq m.$$

$\therefore$ the sequence $< s_n >$ converges to l.

Theorem 5: *A real number p is a limit point of a sequence* $< s_n >$ *iff there exists a subsequence of* $< s_n >$ *converging to p.*

Proof: First, let p be a limit point of a sequence $< s_n >$.

We shall use the result that a real number p is a limit point of a sequence $< s_n >$ if given any $\varepsilon > 0$ and any positive integer m, there exists a positive integer $k > m$ such that $s_k \in]\, p-\varepsilon, p+\varepsilon\,[$.

By choosing $\varepsilon = 1$ and $m = 1$, there must exist a positive integer $n_1 > 1$, such that

$$|s_{n_1} - p| < 1. \qquad \ldots(1)$$

Choosing $\varepsilon = \dfrac{1}{2}$, $m = n_1$, there must exist a positive integer $n_2 > n_1$, such that

$$|s_{n_2} - p| < \frac{1}{2}. \qquad \ldots(2)$$

Continuing in this way, we can inductively define a subsequence $< s_{n_1}, s_{n_2}, \ldots, s_{n_k}, \ldots >$ such that $\quad |s_{n_k} - p| < \dfrac{1}{k}$.

In fact, if we assume that $s_{n_1}, s_{n_2}, \ldots, s_{n_k}$ have been obtained, by choosing

$$\varepsilon = \frac{1}{k+1}, m = n_k,$$

we can get a positive integer $n_{k+1} > n_k$ such that

$$|s_{n_{k+1}} - p| < \frac{1}{k+1}.$$

But s_{n_1} has already been obtained. Thus the construction of $< s_{n_k} >$ is complete by induction.

We now claim that the sequence $< s_{n_k} > \to p$. In fact, for any $\varepsilon > 0$, we can choose a positive integer j, such that $1/j < \varepsilon$. For this choice of j, we get $|s_{n_k} - p| < 1/j < \varepsilon \,\forall k \geq j$.
This shows that the sequence $< s_{n_k} > \to p$.

Conversely, let $< s_{n_k} >$ be a subsequence of $< s_n >$ converging to p. We have to show that p is a limit point of $< s_n >$.

Since $s_{n_k} \to p$, therefore, given any $\varepsilon > 0$ there exists positive integer j such that

$$|s_{n_k} - p| < \varepsilon \text{ for all } k \geq j.$$

Thus every neighbourhood of p contains infinitely many terms of $< s_{n_k} >$ *i.e.,* infinitely many terms of $< s_n >$ and consequently p is a limit point of $< s_n >$.

Theorem 6: *The set of limit points of a bounded sequence is bounded.*

Proof: Let $< s_n >$ be a bounded sequence. Then there exist $k_1, k_2 \in \mathbf{R}$ such that

$$k_1 \leq s_n \leq k_2 \text{ for all } n \in \mathbf{N}$$

i.e., $s_n \notin]-\infty, k_1[$ and $s_n \notin]\, k_2, \infty\,[$ for any n. Hence if $l \in \mathbf{R}$ and $l \notin]-\infty, k_1[\cup]\, k_2, \infty\,[$, then l is not a limit point of the sequence. Thus if $l \in \mathbf{R}$ is a limit point of the sequence, then $l \in [k_1, k_2]$. Consequently the set of limit points of $< s_n >$ is bounded.

Theorem 7: *Every bounded sequence has the greatest and the least limit points.*

Proof: Let $< s_n >$ be a bounded sequence. Then the set L of limit points of $< s_n >$ is bounded.

Now $L \neq \varnothing$ and L is bounded, hence by completeness axiom the set L has infimum and supremum.

If $\inf L = u$ and $\sup L = v$, then we have to show that $u, v \in L$.

For $\varepsilon > 0$, $]\, v - \varepsilon, v + \varepsilon\,[$ is a neighbourhood of v.

Since $v = \sup L$, therefore, there exists some $x \in L$ such that

$$v - \varepsilon < x \le v < v + \varepsilon \quad i.e., \quad x \in \,]\, v - \varepsilon, v + \varepsilon\,[$$

or $\quad]\, v - \varepsilon, v + \varepsilon\,[$ is a neighbourhood of x.

Since x is a limit point of $< s_n >$, hence $]\, v - \varepsilon, v + \varepsilon\,[$ contains infinite number of terms of the sequence. It holds for every $\varepsilon > 0$. Thus every neighbourhood of v contains infinite number of terms of the sequence $< s_n >$.

$\therefore \quad v$ is a limit point of the sequence $< s_n >$.

$\therefore \quad v \in L$.

Similarly, we can show that $u \in L$.

12 Cauchy Sequences

Cauchy Convergence Criterion for Sequences:

In this section we shall establish an important criterion, known as *Cauchy Convergence Criterion,* which will help us to decide whether a sequence is convergent or divergent without knowing its limit or limit point. It involves only the elements of the sequence to which we wish to apply it.

Definition: *A sequence $< s_n >$ is said to be a Cauchy sequence if given $\varepsilon > 0$ there exists $m \in$ **N** such that*

$$|s_n - s_m| < \varepsilon \text{ for all } n \ge m$$

or $\quad |s_{n+p} - s_n| < \varepsilon$ *for all* $n \ge m$ *and every* $p \ge 0$

or $\quad |s_{m+p} - s_m| < \varepsilon$ *for all* $p \ge 0$

or $\quad |s_p - s_q| < \varepsilon$ *for all* $p, q \ge m$. **(Gorakhpur 2012)**

Remark: $|s_p - s_q| < \varepsilon$ for all $p, q \ge m$ means that s_p and s_q are arbitrarily close together for large values of p and q.

Illustrations:

1. $< 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots >$ is a Cauchy sequence.

Let the given sequence be $< s_n >$, where $s_n = 1/n$.

Take any given $\varepsilon > 0$.

If $n \ge m$, then

$$|s_n - s_m| = \left|\frac{1}{n} - \frac{1}{m}\right| = \left|\frac{m-n}{nm}\right| = \frac{n-m}{nm} = \frac{n-m}{n} \cdot \frac{1}{m} < \frac{1}{m} \cdot$$

$$\left[\because\ 0 \le \frac{n-m}{n} < 1\right]$$

If we take $m \in \mathbf{N}$ such that $m > \frac{1}{\varepsilon}$ *i.e,* $\frac{1}{m} < \varepsilon$, then

$$|s_n - s_m| = \left| \frac{1}{n} - \frac{1}{m} \right| < \varepsilon, \forall \ n \geq m.$$

Hence the given sequence is a Cauchy sequence.

2. The sequence $< n^2 >$ is not a Cauchy sequence.

If $n > m$, then $n^2 - m^2 = (n - m)(n + m) > 2m > 1$, for any value of m. Taking $\varepsilon = 1$, we cannot find a positive integer m such that $|n^2 - m^2| < \varepsilon$ for all $n \geq m$.

Theorem 1: *If $< s_n >$ is a Cauchy sequence, then $< s_n >$ is bounded.*
(Kanpur 2008; Gorakhpur 14)

Proof: Let $< s_n >$ be a Cauchy sequence. For $\varepsilon = 1$, there exists $m \in \mathbf{N}$ such that $|s_n - s_m| < 1$ for all $n \geq m$

i.e., $s_m - 1 < s_n < s_m + 1$ for all $n \geq m$.

Let $k_1 = \min. \{s_1, s_2, \ldots, s_{m-1}, s_m - 1\}$

and $k_2 = \max. \{s_1, s_2, \ldots, s_{m-1}, s_m + 1\}$.

$\therefore$ $k_1 \leq s_n \leq k_2$ for all $n \in \mathbf{N}$.

Hence $< s_n >$ is bounded.

Note: The converse of the above theorem need not be true. The sequence $< (-1)^n >$ is bounded but is not a Cauchy sequence. **(Gorakhpur 2014)**

Theorem 2: (Cauchy Convergence Criterion): *A sequence converges if and only if it is a Cauchy sequence.*

Proof: First, let $< s_n >$ be a convergent sequence which converges to, say, l.

Since $s_n \to l$, therefore, for given $\varepsilon > 0$ there must exist $m \in \mathbf{N}$ such that

$$|s_n - l| < \varepsilon / 2 \ \forall \ n \geq m.$$

In particular, $|s_m - l| < \varepsilon / 2$.

Now
$$|s_n - s_m| = |(s_n - l) - (s_m - l)| \leq |s_n - l| + |s_m - l|$$
$$< \varepsilon / 2 + \varepsilon / 2 \text{ for all } n \geq m.$$

Thus $|s_n - s_m| < \varepsilon \ \forall \ n \geq m$, showing that $< s_n >$ is a Cauchy sequence.

Conversely, let $< s_n >$ be a Cauchy sequence. Then $< s_n >$ is bounded. By Bolzano-Weierstrass theorem, $< s_n >$ has a limit point, say l. We shall show that $s_n \to l$.

Let $\varepsilon > 0$ be given. Since $< s_n >$ is a Cauchy sequence, there exists $m \in \mathbf{N}$ such that

$$|s_n - s_m| < \varepsilon / 3 \ \forall \ n \geq m.$$

Since l is a limit point of $< s_n >$, therefore every nbd of l contains infinite terms of the sequence $< s_n >$. In particular the open interval $]\, l - \frac{1}{3}\varepsilon, l + \frac{1}{3}\varepsilon \,[$ contains infinite terms of $< s_n >$. Hence there exists a positive integer $k > m$ such that

$$l - \frac{1}{3}\varepsilon < s_k < l + \frac{1}{3}\varepsilon \quad i.e., \ |s_k - l| < \varepsilon / 3.$$

Now $\quad |s_n - l| = |(s_n - s_m) + (s_m - s_k) + (s_k - l)|$

$\leq |s_n - s_m| + |s_m - s_k| + |s_k - l|$

$< \varepsilon / 3 + \varepsilon / 3 + \varepsilon / 3$ for all $n \geq m$.

Thus $|s_n - l| < \varepsilon$ for all $n \geq m$.

$\therefore \quad < s_n >$ converges to l.

13 Limit Superior and Limit Inferior of a Sequence

Let $< s_n >$ be a sequence which is bounded above. Then, for each fixed $n \in \mathbf{N}$, the set $\{s_n, s_{n+1}, \ldots\}$ is bounded above and hence it must have a supremum. Let

$$\bar{s}_n = \sup \{s_n, s_{n+1}, \ldots\}.$$

Since $\{s_{n+1}, s_{n+2}, \ldots\}$ is a subset of $\{s_n, s_{n+1}, \ldots\}$, therefore, it is obvious that $\bar{s}_n \geq \bar{s}_{n+1}$. Thus the sequence $< \bar{s}_n >$ is a monotonically decreasing sequence and consequently, it either converges or else it diverges to $-\infty$.

Similarly, if the sequence $< s_n >$ is bounded below, then the set $\{s_n, s_{n+1}, \ldots\}$ has an infimum. Let $\underline{s}_n = \inf \{s_n, s_{n+1}, \ldots\}$, then the sequence $< \underline{s}_n >$ is monotonically increasing and hence it either converges or diverges to ∞.

Keeping these notations in mind we now define limit superior and limit inferior.

Definition 1: *Let $< s_n >$ be a sequence of real numbers which is bounded above and let*

$$\bar{s}_n = \sup \{s_n, s_{n+1}, \ldots\}.$$

If $< \bar{s}_n >$ converges we define the **limit superior** *of $< s_n >$ by*

$$\lim_{n \to \infty} \sup s_n = \lim_{n \to \infty} \bar{s}_n$$

If $< \bar{s}_n >$ diverges to $-\infty$, we write $\lim\limits_{n \to \infty} \sup s_n = -\infty$

If a sequence $< s_n >$ is not bounded above, we write

$$\lim_{n \to \infty} \sup s_n = \infty$$

Definition 2: *Let $< s_n >$ be a sequence of real numbers which is bounded below and let*

$$\underline{s}_n = \inf \{s_n, s_{n+1}, \ldots\}.$$

If $< \underline{s}_n >$ converges we define the **limit inferior** *of $< s_n >$ by*

$$\lim_{n \to \infty} \inf s_n = \lim_{n \to \infty} \underline{s}_n .$$

If $< \underline{s}_n >$ diverges to ∞, we write $\lim\limits_{n \to \infty} \inf s_n = \infty$.

If a sequence $< s_n >$ is not bounded below, we write

$$\lim_{n \to \infty} \inf s_n = -\infty.$$

Note 1: The notations $\overline{\lim}\, s_n$ and $\underline{\lim}\, s_n$ are also used for lim sup s_n and lim inf s_n respectively. In future, we shall use these notations.

Note 2: The limit superior and the limit inferior are also called the **upper limit** and the **lower limit** of $< s_n >$ respectively.

Note 3: We have $\overline{\lim}\, s_n = \inf\{\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n, \ldots\}$

and $\qquad \underline{\lim}\, s_n = \sup\{\underline{s_1}, \underline{s_2}, \ldots, \underline{s_n}, \ldots\}$.

Illustrations:

1. Let $< s_n >$ be the sequence defined by $s_n = (-1)^n \;\forall\, n \in \mathbf{N}$.

It is bounded above by 1 and bounded below by -1. For this sequence, $\bar{s}_n = 1$ and $\underline{s_n} = -1$ for all $n \in \mathbf{N}$.

Hence $\overline{\lim}\, s_n = 1$ and $\underline{\lim}\, s_n = -1$.

2. Let $< s_n >$ be the sequence defined by $s_n = -n \;\forall\, n \in \mathbf{N}$. It is bounded above by -1 but it is not bounded below.

$$\bar{s}_n = \sup\{-n, -n-1, \ldots\} = -n$$

Since $\bar{s}_n \to -\infty$ as $n \to \infty$, hence $\overline{\lim}\, s_n = -\infty$. Also, since $< s_n >$ is not bounded below, by definition $\underline{\lim}\, s_n = -\infty$. Thus in this sequence both the limit superior and the limit inferior are $-\infty$.

3. Let $< s_n >$ be the sequence defined by $s_n = n \;\forall\, n \in \mathbf{N}$. It is bounded below but not bounded above.

$$\underline{s_n} = \inf\{n, n+1, \ldots\} = n.$$

Since $\underline{s_n} \to \infty$ as $n \to \infty$, hence $\underline{\lim}\, s_n = \infty$.

Also, since $< s_n >$ is not bounded above, by definition $\overline{\lim}\, s_n = \infty$. Thus in this sequence both the limit superior and the limit inferior are ∞.

4. Let $< s_n >$ be the sequence defined by $s_n = (-1)^n \left(1 + \dfrac{1}{n}\right)$.

Then $\qquad < s_n > = < -2, \dfrac{3}{2}, -\dfrac{4}{3}, \dfrac{5}{4}, -\dfrac{6}{5}, \dfrac{7}{6}, \ldots >$.

In this case $\qquad \bar{s}_1 = \dfrac{3}{2}, \bar{s}_2 = \dfrac{3}{2}, \bar{s}_3 = \dfrac{5}{4}, \bar{s}_4 = \dfrac{5}{4}, \bar{s}_5 = \dfrac{7}{6}$ etc.

and $\qquad \underline{s_1} = -2, \underline{s_2} = -\dfrac{4}{3}, \underline{s_3} = -\dfrac{4}{3}, \underline{s_4} = -\dfrac{6}{5}$ etc.

Hence $\qquad \overline{\lim}\, s_n = \inf\left\{\dfrac{3}{2}, \dfrac{5}{4}, \dfrac{7}{6}, \ldots\right\} = 1$

and $\qquad \underline{\lim}\, s_n = \sup\left\{-2, -\dfrac{4}{3}, -\dfrac{6}{5}, \ldots\right\} = -1.$

Theorem 1: *If $< s_n >$ is a convergent sequence of real numbers and if $\lim s_n = l$, then $\overline{\lim}\, s_n = \underline{\lim}\, s_n = l$. Conversely, if*

$$\overline{\lim}\, s_n = \underline{\lim}\, s_n = l \in \mathbf{R},$$

then $< s_n >$ is convergent and $\lim\limits_{n \to \infty} s_n = l$.

Proof: First suppose that the sequence $< s_n >$ converges with $\lim s_n = l$. Let $\varepsilon > 0$ be given. Since $s_n \to l$, therefore, we can find a positive integer m, such that

$$|s_n - l| < \varepsilon \text{ for all } n \geq m$$

i.e., $$l - \varepsilon < s_n < l + \varepsilon \text{ for all } n \geq m.$$

This inequality shows that for all $n \geq m$, $l + \varepsilon$ is an upper bound of $\{s_n, s_{n+1}, \ldots\}$ and $l - \varepsilon$ is not an upper bound of $\{s_n, s_{n+1}, \ldots\}$.

Since $\bar{s}_n = \{s_n, s_{n+1}, \ldots\}$, it follows that

$$l - \varepsilon < \bar{s}_n \leq l + \varepsilon,\ n \geq m.$$

Taking limits as $n \to \infty$, we get

$$l - \varepsilon \leq \lim_{n \to \infty} \bar{s}_n \leq l + \varepsilon.$$

Since ε is arbitrary, it follows that $\overline{\lim}\, s_n = l$.

Similarly, we can show that $\underline{\lim}\, s_n = l$.

Thus $$\overline{\lim}\, s_n = \underline{\lim}\, s_n = l.$$

Conversely, let $\overline{\lim}\, s_n = \underline{\lim}\, s_n = l$.

Since $l = \lim\limits_{n \to \infty} \bar{s}_n$, given any $\varepsilon > 0$, there exists $m_1 \in \mathbf{N}$ such that

$$|\bar{s}_n - l| < \varepsilon \quad \text{for } n \geq m_1 \qquad \textit{i.e.,} \qquad l - \varepsilon < \bar{s}_n < l + \varepsilon \quad \text{for } n \geq m_1.$$

The definition of $\bar{s}_n$ then gives that

$$s_n < l + \varepsilon \quad \text{for } n \geq m_1. \qquad \ldots(1)$$

Similarly, since $l = \lim\limits_{n \to \infty} \underline{s_n}$, there exists $m_2 \in \mathbf{N}$ such that

$$|\underline{s_n} - l| < \varepsilon \text{ for } n \geq m_2$$

which implies as above that $s_n > l - \varepsilon$ for $n \geq m_2$. ...(2)

Let $m = \max.\{m_1, m_2\}$. Then from (1) and (2) we find that

$$|s_n - l| < \varepsilon \text{ for } n \geq m.$$

This proves that the sequence $< s_n >$ converges and that

$$\lim_{n \to \infty} s_n = l.$$

Similar results hold good for divergent sequences. Below we state them without proof.

Theorem 2: *A sequence $< s_n >$ diverges to $+\infty$ iff* $\overline{\lim}\, s_n = \underline{\lim}\, s_n = \infty$.

Theorem 3: *A sequence $< s_n >$ diverges to $-\infty$ iff* $\overline{\lim}\, s_n = \underline{\lim}\, s_n = -\infty$.

Theorem 4: *If $< s_n >$ and $< t_n >$ are bounded sequences of real numbers such that $s_n \leq t_n$ for all $n \in \mathbf{N}$, then $\overline{\lim}\, s_n \leq \overline{\lim}\, t_n$ and $\underline{\lim}\, s_n \leq \underline{\lim}\, t_n$.*

Proof: Since $s_n \leq t_n$, therefore it is easy to see that

$$\bar{s}_n \leq \bar{t}_n \quad \text{and } \underline{s_n} \leq \underline{t_n},$$

where $\bar{s}_n, \bar{t}_n, \underline{s_n}, \underline{t_n}$ have their usual meanings as defined earlier.

Then we have from the corollary to theorem 3 of article 9,

$$\lim \bar{s}_n \leq \lim \bar{t}_n \quad \text{and} \quad \lim \underline{s_n} \leq \lim \underline{t_n}$$

or $$\overline{\lim}\, s_n \leq \overline{\lim}\, t_n \quad \text{and} \quad \underline{\lim}\, s_n \leq \underline{\lim}\, t_n.$$

Theorem 5: *If $< s_n >$ and $< t_n >$ are bounded sequences of real numbers, then*

(*i*) $\overline{lim}\ (s_n + t_n) \le \overline{lim}\ s_n + \overline{lim}\ t_n$; (*ii*) $\underline{lim}\ (s_n + t_n) \ge \underline{lim}\ s_n + \underline{lim}\ t_n$.

Proof: Let $\bar{s}_n = \{s_n, s_{n+1}, \ldots\}$, and $\bar{t}_n = \sup\{t_n, t_{n+1}, \ldots\}$.

Then $s_k \le \bar{s}_n, (k \ge n),\ t_k \le \bar{t}_n, (k \ge n)$.

$\therefore$ $s_k + t_k \le \bar{s}_n + \bar{t}_n$ for $k \ge n$.

Thus $\bar{s}_n + \bar{t}_n$ is an upper bound for $\{s_n + t_n, s_{n+1} + t_{n+1}, \ldots\}$.

Hence $\overline{(s_n + t_n)} = \sup\{s_n + t_n, s_{n+1} + t_{n+1}, \ldots\} \le \bar{s}_n + \bar{t}_n$.

$\therefore$ $\lim \overline{(s_n + t_n)} \le \lim (\bar{s}_n + \bar{t}_n) = \lim \bar{s}_n + \lim \bar{t}_n$

i.e., $\overline{\lim}\ (s_n + t_n) \le \overline{\lim}\ s_n + \overline{\lim}\ t_n$.

Thus the result (i) has been proved. Similarly (ii) can be proved.

Note: It can be shown that there exist sequences for which the inequalities in the above theorem are strict inequalities.

Let $s_n = (-1)^n, n \in \mathbf{N}$ and $t_n = (-1)^{n+1}, n \in \mathbf{N}$.

Then $s_n + t_n = 0, n \in \mathbf{N}$. Here $\overline{\lim}\ s_n = 1 = \overline{\lim}\ t_n$ and $\overline{\lim}\ (s_n + t_n) = 0$.

Hence, in this case $\overline{\lim}\ (s_n + t_n) < \overline{\lim}\ s_n + \overline{\lim}\ t_n$.

14 Nested Interval Theorem or Cantor's Intersection Theorem

Definition: *A sequence of sets $< A_n >$ is called a nested sequence of sets if* $A_1 \supset A_2 \supset \ldots .\ A_n \supset A_{n+1} \supset \ldots .$

Theorem: *For each $n \in \mathbf{N}$, let $I_n = [a_n, b_n]$ be a (non-empty) closed and bounded interval on $\mathbf{R}$, such that $< I_n >$ is a nested sequence with $\lim\limits_{n \to \infty}$ (length of I_n) i.e., $\lim\limits_{n \to \infty} (b_n - a_n) = 0$. Then*

$\bigcap\limits_{n=1}^{\infty} I_n$ contains precisely one point.

(Gorakhpur 2014)

Proof: Since $< I_n >$ is nested, we have

$$I_n \supset I_{n+1} \text{ for all } n \in \mathbf{N}$$

i.e., $[a_n, b_n] \supset [a_{n+1}, b_{n+1}]$ for all $n \in \mathbf{N}$. ...(1)

It follows from (1) that $a_n \le a_{n+1} \le b_{n+1} \le b_n\ \forall\ n \in \mathbf{N}$.

This shows that the sequence $< a_n >$ is a monotonically increasing sequence bounded above by b_1, and $< b_n >$ is a monotonically decreasing sequence bounded below by a_1. Hence, $< a_n >$ and $< b_n >$, both converge.

Now we have $\lim\limits_{n \to \infty}$ (length of I_n) $= \lim\limits_{n \to \infty} (b_n - a_n) = 0$

$\Rightarrow$ $\lim\limits_{n \to \infty} b_n = \lim\limits_{n \to \infty} a_n = l$, (say).

Since a monotonically increasing bounded sequence converges to its supremum, it follows that l is the supremum of the range set of the sequence $< a_n >$. Also, a monotonically decreasing bounded sequence converges to its infimum, hence l is the infimum of the range set of the sequence $< b_n >$.

Thus $a_n \le l \le b_n \; \forall \; n$. Hence, $l \in I_n$ for all n. It follows that $l \in \bigcap_{n=1}^{\infty} I_n$.

Clearly no $l' \ne l$ can belong to $\bigcap_{n=1}^{\infty} I_n$.

For let $l' \in \bigcap_{n=1}^{\infty} I_n$, then $0 \le |l' - l| \le |b_n - a_n| \; \forall \; n$.

Since $|b_n - a_n| \to 0$, hence we get $|l' - l| = 0$ *i.e.*, $l' = l$.

Hence $\bigcap_{n=1}^{\infty}$ consists of exactly one point.

Illustrative Examples

Example 17: *Show that the sequence* $< s_n >$ *defined by*

$$s_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n},$$

converges.

Solution: We have

$$s_{n+1} - s_n = \left(\frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n+2}\right) - \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right)$$

$$= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2n+2} > 0 \text{ for all } n.$$

Hence, the sequence $< s_n >$ is monotonically increasing.

Now $$|s_n| = s_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n} < \frac{1}{n} + \ldots + \frac{1}{n} \text{ (upto } n \text{ terms)}$$

$$= n \cdot \frac{1}{n} = 1$$

i.e., $|s_n| < 1$ for all n.

Hence, the sequence $< s_n >$ is bounded.

Since $< s_n >$ is a bounded, monotonically increasing sequence, hence it converges.

Example 18: *Show that the sequence* $< s_n >$ *defined by the relation*

$$s_1 = 2, s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{(n-1)!} \quad (n \ge 2), \textit{ converges.}$$

Solution: We have $s_{n+1} - s_n = \dfrac{1}{n!} > 0$ for all n.

Hence, the sequence $< s_n >$ is monotonically increasing.

Now, we shall show that $< s_n >$ is bounded.

For $n \geq 2$, $n! = 1.2.3 \ldots n$ contains $(n-1)$ factors each of which is greater than or equal to 2. Hence $n! \geq 2^{n-1}$ for all $n \geq 2$.

$\therefore$ $\dfrac{1}{n!} \leq \dfrac{1}{2^{n-1}}$, for all $n \geq 2$.

Thus
$$s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{(n-1)!}$$
$$\leq 1 + \frac{1}{1!} + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-2}}$$
$$= 1 + \frac{1 - \left(\frac{1}{2}\right)^{n-1}}{1 - \frac{1}{2}} < 3, \text{ for all } n \geq 2.$$

Also $s_1 = 2 < 3$.

$\therefore$ $2 \leq s_n < 3$ for all $n \in \mathbf{N}$

i.e., $< s_n >$ is bounded.

Since $< s_n >$ is a bounded, monotonically increasing sequence, consequently, it converges.

Example 19: *Show that* $\lim_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n$ *exists and lies between 2 and 3.*

Solution: Here $s_n = \left(1 + \dfrac{1}{n}\right)^n$. Obviously $s_1 = 2$.

By the binomial theorem, we get

$$s_n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \ldots + \frac{1}{n^n}$$
$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots$$
$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \ldots \left(1 - \frac{n-1}{n}\right). \quad \ldots(1)$$

Similarly,
$$s_{n+1} = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \frac{1}{3!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) + \ldots$$
$$+ \frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) \ldots \left(1 - \frac{n}{n+1}\right)$$
$$> 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots$$
$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \ldots \left(1 - \frac{n-1}{n}\right).$$

[$\because$ each term on the R.H.S. of s_{n+1} is $\geq$ the corresponding term on the R.H.S. of s_n and moreover the number of terms in the expansion of s_{n+1} is $n+2$ *i.e.*, one more than the number of terms $n+1$ in the expansion of s_n]

$\therefore \qquad s_{n+1} > s_n$ for all $n \in \mathbf{N}$.

Hence, the sequence $< s_n >$ is monotonically increasing.

$\therefore \qquad s_n \geq s_1 = 2, \forall \; n \in \mathbf{N}$.

From (1), we see that

$$s_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} \qquad \text{[See Ex. 18]}$$

$$= 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} < 1 + \frac{1}{1 - \frac{1}{2}} = 3, \text{ for all } n.$$

Thus $2 \leq s_n < 3$, for all n.

Hence the sequence $< s_n >$ is bounded.

Since $< s_n >$ is a bounded, monotonically increasing sequence, consequently, it converges *i.e.*, $\lim\limits_{n \to \infty} s_n$ exists and $\lim s_n = \sup < s_n >$.

Now $2 \leq s_n < 3$ for all $n \Rightarrow 2 \leq \lim\limits_{n \to \infty} s_n \leq 3$, which shows that the limit lies between 2 and 3.

Note: The actual value of the $\lim \left(1 + \frac{1}{n}\right)^n$ is defined to be equal to e. Hence e lies between 2 and 3. Taking limit of both sides of (1) as $n \to \infty$, we see that

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots \infty.$$

Example 20: *Prove that* $< s_n >$ *is convergent where* $s_n = 2 - \dfrac{1}{2^{n-1}}$.

Solution: We have

$$s_{n+1} - s_n = \left(2 - \frac{1}{2^n}\right) - \left(2 - \frac{1}{2^{n-1}}\right) = \frac{1}{2^{n-1}} - \frac{1}{2^n} > 0 \text{ for all } n.$$

Hence, the sequence $< s_n >$ is monotonically increasing.

Also, $s_n = 2 - \dfrac{1}{2^{n-1}} < 2$ for all n. $\qquad \left[\because \dfrac{1}{2^{n-1}} > 0\right]$

$\therefore \qquad < s_n >$ is bounded above.

Since $< s_n >$ is a bounded above, monotonically increasing sequence, hence it converges.

We have $\qquad \lim\limits_{n \to \infty} s_n = \lim\limits_{n \to \infty} \left(2 - \dfrac{1}{2^{n-1}}\right) = 2.$

Example 21: *Show that the sequence* $< s_n >$ *defined by*

$$s_1 = \frac{1}{2},\ s_{n+1} = \frac{2\,s_n + 1}{3}\ \forall\ n \in \mathbf{N}$$ *is convergent. Also find its limit.*

Solution: We have $s_1 = \frac{1}{2}$ and $s_{n+1} = \frac{2\,s_n+1}{3}\ \forall\ n \in \mathbf{N}$.

First applying mathematical induction we shall show that

$$s_{n+1} > s_n\ \forall\ n \in \mathbf{N}.$$

We have $$s_2 = \frac{2\,.\,s_1 + 1}{3} = \frac{2.\left(\frac{1}{2}\right) + 1}{3} = \frac{2}{3}.$$

$\therefore$ $s_2 > s_1$.

Now assume as our induction hypothesis that $s_{n+1} > s_n$ for some positive integer n.

Then $s_{n+1} > s_n \Rightarrow 2\,s_{n+1} > 2\,s_n \Rightarrow 2\,s_{n+1} + 1 > 2\,s_n + 1$

$$\Rightarrow \frac{2\,s_{n+1}+1}{3} > \frac{2\,s_n + 1}{3} \Rightarrow s_{n+2} > s_{n+1}.$$

Thus $s_2 > s_1$ and if $s_{n+1} > s_n$, then $s_{n+2} > s_{n+1}$.

$\therefore$ by induction $s_{n+1} > s_n\ \forall\ n \in \mathbf{N}$.

Thus the sequence $< s_n >$ is monotonic increasing.

Now we shall show that the sequence $< s_n >$ is also bounded above.

We have $s_{n+1} > s_n\ \forall\ n \in \mathbf{N}$

$$\Rightarrow \frac{2\,s_n+1}{3} > s_n\ \forall\ n \in \mathbf{N} \quad \Rightarrow \quad 2\,s_n + 1 > 3\,s_n\ \forall\ n \in \mathbf{N}$$

$$\Rightarrow s_n < 1\ \forall\ n \in \mathbf{N}.$$

$\therefore$ the sequence $< s_n >$ is bounded above by 1.

Since the sequence $< s_n >$ is monotonic increasing and bounded above, therefore by monotone convergence theorem $< s_n >$ converges to its supremum.

Let $\lim\limits_{n\to\infty} s_n = l$. Then $\lim\limits_{n\to\infty} s_{n+1} = l$.

Now $$s_{n+1} = \frac{2\,s_n+1}{3} \Rightarrow \lim s_{n+1} = \lim \frac{2\,s_n+1}{3}$$

$$\Rightarrow l = \frac{2\lim s_n + 1}{3} \Rightarrow l = \frac{2l+1}{3} \Rightarrow l = 1.$$

Hence $s_n \to 1$. We have $\inf < s_n > = s_1 = \frac{1}{2}$ and $\sup < s_n > = \lim s_n = 1$.

Example 22: *Show that the sequence* $< s_n >$ *defined by*

$$s_1 = \sqrt{2},\ s_{n+1} = \sqrt{(2\,s_n)}$$

converges to 2.

Solution: We have $s_2 = \sqrt{(2\sqrt{2})}$.

Since $1 < \sqrt{2} \Rightarrow 2 < 2\sqrt{2} \Rightarrow \sqrt{2} < \sqrt{(2\sqrt{2})}$,

$\therefore$ $s_1 < s_2$.

Let us suppose that $s_m < s_{m+1}$.

Then $\sqrt{(2s_m)} < \sqrt{(2s_{m+1})} \quad \Rightarrow \quad s_{m+1} < s_{m+2}$.

Hence, by mathematical induction, we have

$$s_n < s_{n+1} \text{ for all } n \in \mathbf{N}$$

i.e., $< s_n >$ is monotonically increasing.

Also, we have $s_1 = \sqrt{2} < 2$.

Let us suppose that $s_m < 2$. Then $\sqrt{(2s_m)} < \sqrt{(2.2)} = 2 \Rightarrow s_{m+1} < 2$.

Hence, by mathematical induction, we have $s_n < 2$ for all $n \in \mathbf{N}$

i.e., $< s_n >$ is bounded above by 2.

Thus $< s_n >$ is a monotonically increasing sequence bounded above by 2, hence, it converges.

Let $\lim\limits_{n \to \infty} s_n = l$. Then $\lim\limits_{n \to \infty} s_{n+1} = l$.

Now $\quad s_{n+1} = \sqrt{(2s_n)} \Rightarrow \lim s_{n+1} = \lim \sqrt{(2s_n)}$

$\Rightarrow \quad l = \sqrt{(2l)} \Rightarrow l(l-2) = 0 \Rightarrow l = 0, 2.$

But $\quad s_n \geq s_1 = \sqrt{2} \ \forall \ n \in \mathbf{N} \Rightarrow s_n - \sqrt{2} \geq 0 \ \forall \ n \in \mathbf{N}$.

$\therefore \quad \lim (s_n - \sqrt{2}) \geq 0 \quad$ *i.e.*, $\lim s_n \geq \sqrt{2}$.

Hence l cannot be zero. Therefore $l = 2$.

Example 23: *Show that the sequence* $< s_n >$ *defined by* $s_1 = 1$ *and* $s_{n+1} = \sqrt{(2 + s_n)}, \forall$ $n \in \mathbf{N}$ *is monotonically increasing and bounded. Also find its limit.*

Solution: We have $s_1 = 1$ and $(s_{n+1})^2 = 2 + s_n, \forall n \in \mathbf{N}$.

$\therefore \quad s_2 = \sqrt{3}, s_3 = \sqrt{(2 + \sqrt{3})}, \ldots\ldots$

Now $\quad 1 < \sqrt{3} \quad \Rightarrow \quad s_1 < s_2$.

Let us suppose that $s_m < s_{m+1}$. Then $\sqrt{(2 + s_m)} < \sqrt{(2 + s_{m+1})}$

$\Rightarrow \quad s_{m+1} < s_{m+2}$.

Hence, by mathematical induction, we have

$$s_n < s_{n+1} \text{ for all } n \in \mathbf{N}$$

i.e., $< s_n >$ is monotonically increasing.

Again, $\quad s_{n+1} > s_n \quad \Rightarrow \quad \sqrt{(2 + s_n)} > s_n$

$\Rightarrow \quad 2 + s_n - s_n^2 > 0 \quad \Rightarrow \quad (2 - s_n)(1 + s_n) > 0$

$\Rightarrow \quad (2 - s_n) > 0 \qquad [\because \ 1 + s_n > 0]$

or $\quad s_n < 2 \ \forall \ n \in \mathbf{N}$.

Hence $< s_n >$ is bounded.

Thus $< s_n >$ is a monotonically increasing sequence bounded above by 2; consequently it converges.

Let $\lim s_n = l$. Then $\lim s_{n+1} = l$.

Now $\qquad s_{n+1} = \sqrt{(2+s_n)} \;\Rightarrow\; \lim s_{n+1} = \lim \sqrt{(2+s_n)}$

$\Rightarrow \qquad l = \sqrt{(2+l)} \;\Rightarrow\; l^2 - l - 2 = 0 \;\Rightarrow\; (l+1)(l-2) = 0$

$\Rightarrow \qquad l = -1, 2.$

But l cannot be -1 since all terms of the sequence are positive. Hence $l = 2$.

Note: If a sequence $< s_n >$ is monotonically increasing then there is no need to show that $< s_n >$ has a lower bound because s_1 is always its lower bound.

Similarly, for a monotonically decreasing sequence there is no need to find an upper bound, because s_1 will always be its upper bound.

Example 24: *Show that the sequence* $< s_n >$ *defined by*

$$s_1 = 1, s_{n+1} = \frac{4+3s_n}{3+2s_n}, n \in \mathbf{N}$$

is convergent and find its limit.

Solution: We observe that all the terms of the given sequence are positive.

First by mathematical induction we shall show that

$$s_{n+1} > s_n \;\forall\; n \in \mathbf{N}.$$

We have $\qquad s_1 = 1, s_2 = \dfrac{4+3s_1}{3+2s_1} = \dfrac{4+3\cdot 1}{3+2\cdot 1} = \dfrac{7}{5}\cdot$

$\therefore \qquad s_2 > s_1.$

Now assume as our induction hypothesis that for some positive integer n,

$$s_{n+1} > s_n. \qquad ...(1)$$

Then $\qquad s_{n+2} - s_{n+1} = \dfrac{4+3\,s_{n+1}}{3+2\,s_{n+1}} - \dfrac{4+3s_n}{3+2s_n}$

$$= \frac{(4+3s_{n+1})(3+2s_n) - (4+3s_n)(3+2\,s_{n+1})}{(3+2\,s_{n+1})(3+2\,s_n)}$$

$$= \frac{s_{n+1} - s_n}{(3+2\,s_{n+1})(3+2s_n)} > 0, \text{ by (1).}$$

$\therefore \qquad s_{n+2} > s_{n+1}.$

Thus $s_2 > s_1$ and if $s_{n+1} > s_n$, then we have also $s_{n+2} > s_{n+1}$.

$\therefore$ by mathematical induction $s_{n+1} > s_n, \forall\; n \in \mathbf{N}$.

Thus the sequence $< s_n >$ is monotonic increasing.

Now we shall show that the sequence $< s_n >$ is also bounded above.

We have $\qquad s_{n+1} = \dfrac{3s_n + 4}{2\,s_n + 3} = \dfrac{\frac{3}{2}(2\,s_n + 3) - \frac{1}{2}}{2\,s_n + 3} = \dfrac{3}{2} - \dfrac{1}{2\,(2\,s_n + 3)},$

showing that $\quad s_{n+1} < \dfrac{3}{2}, \forall\; n \in \mathbf{N}.$

Also $\qquad s_1 = 1 < \dfrac{3}{2}\cdot$

Thus $s_n < \frac{3}{2}$, $\forall\, n \in \mathbf{N}$. Therefore the sequence $< s_n >$ is bounded above by $\frac{3}{2}$.

Since the sequence $< s_n >$ is monotonic increasing and bounded above, therefore by monotone convergence theorem it converges to its supremum.

Let $\lim s_n = l$. Then $\lim s_{n+1} = l$.

Now $$s_{n+1} = \frac{4 + 3s_n}{3 + 2s_n} \Rightarrow \lim s_{n+1} = \frac{4 + 3 \lim s_n}{3 + 2 \lim s_n}$$

$$\Rightarrow \quad l = \frac{4 + 3l}{3 + 2l} \Rightarrow l^2 = 2 \Rightarrow l = \pm\sqrt{2}.$$

Since all terms of the sequence are positive so l cannot be negative. Hence $l = \sqrt{2}$. We have $\inf < s_n > = s_1 = 1$ and $\sup < s_n > = \lim s_n = \sqrt{2}$.

Example 25: *A sequence $< s_n >$ of positive terms is defined by*

$$s_1 = k > 0 \; ; s_{n+1} = \frac{3 + 2s_n}{2 + s_n}, \forall\, n \in \mathbf{N}.$$

Show that the sequence converges to a limit independent of k and find the limit.

Solution: We have $s_1 = k > 0$, and $s_{n+1} = \frac{3 + 2\,s_n}{2 + s_n}$, $\forall\; n \in \mathbf{N}$.

Then $s_2 > 0, s_3 > 0$ and so on.

Therefore the terms of the sequence are all positive.

Now first by mathematical induction we shall show that

$$s_{n+1} > s_n \;\forall\; n \in \mathbf{N}.$$

We have $$s_2 - s_1 = \frac{3 + 2s_1}{2 + s_1} - s_1 = \frac{3 + 2k}{2 + k} - k = \frac{3 - k^2}{2 + k} > 0 \text{ if } 0 < k < \sqrt{3}.$$

Thus $\quad s_2 > s_1$ if $0 < k < \sqrt{3}$.

Now assume as our induction hypothesis that for some positive integer n,

$$s_{n+1} > s_n. \qquad ...(1)$$

Then $$s_{n+2} - s_{n+1} = \frac{3 + 2\,s_{n+1}}{2 + s_{n+1}} - \frac{3 + 2\,s_n}{2 + s_n}$$

$$= \frac{s_{n+1} - s_n}{(2 + s_n)(2 + s_{n+1})} > 0, \text{ by (1).}$$

$\therefore \quad s_{n+2} > s_{n+1}$.

Thus $s_2 > s_1$ and if $s_{n+1} > s_n$, then we have also $s_{n+2} > s_{n+1}$.

$\therefore$ by induction $s_{n+1} > s_n$, $\forall\; n \in \mathbf{N}$.

Thus the sequence $< s_n >$ is monotonic increasing.

Now we shall show that the sequence $< s_n >$ is also bounded above.

We have $s_{n+1} > s_n \ \forall \ n \in \mathbf{N}$

$$\Rightarrow \quad \frac{3+2s_n}{2+s_n} > s_n \Rightarrow \frac{3+2s_n}{2+s_n} - s_n > 0$$

$$\Rightarrow \quad \frac{3-s_n^2}{2+s_n} > 0 \Rightarrow 3 - s_n^2 > 0 \Rightarrow s_n^2 < 3$$

$$\Rightarrow \quad s_n < \sqrt{3}, \forall \ n \in \mathbf{N}.$$

$\therefore$ $< s_n >$ is bounded above by $\sqrt{3}$.

Thus $< s_n >$ is a bounded monotonically increasing sequence. Hence it converges.

Let $\lim s_n = l$.

Then $\lim s_{n+1} = l$.

Now $$s_{n+1} = \frac{3+2s_n}{2+s_n} \Rightarrow \lim s_{n+1} = \frac{3+2\lim s_n}{2+\lim s_n}$$

$$\Rightarrow \quad l = \frac{3+2l}{2+l} \Rightarrow l^2 = 3 \Rightarrow l = \pm\sqrt{3}.$$

But l cannot be negative because the terms of the sequence $< s_n >$ are all positive.

Hence $l = \sqrt{3}$ which is independent of k.

Example 26: *If u_1, v_1 are given unequal numbers and*

$$u_n = \frac{1}{2}(u_{n-1} + v_{n-1}),\ v_n = \sqrt{(u_{n-1}\, v_{n-1})}, \text{ where } n \geq 2,$$

prove that (i) u_n decreases, and v_n increases as n increases,

(ii) $< u_n >, < v_n >$ are both convergent and have the same limit, where $u > v > 0$, and $u_1 = \frac{1}{2}(u+v)$ and $v_1 = \sqrt{(uv)}$.

Solution: Since $u > v > 0, u_1 = \frac{1}{2}(u+v)$ and $v_1 = \sqrt{(uv)}$, therefore, u_1 and v_1 are positive and $u_1 > v_1$ since A.M. > G.M.

$\therefore$ $v_1 < u_2 < u_1$ $\quad$ [$\because u_2 = \frac{1}{2}(u_1 + v_1)$ so that v_1, u_2, u_1 are in A.P.]

and $v_1 < v_2 < u_1$. $\quad$ [$\because v_2 = \sqrt{(u_1 v_1)}$ so that v_1, v_2, u_1 are in G.P.]

Since u_2 is the A.M. and v_2 is the G.M. of u_1 and v_1, therefore, we have $u_2 > v_2$.

Hence as above, we get

$$v_2 < u_3 < u_2 \text{ and } v_2 < v_3 < u_2, \text{ and so on.}$$

Thus $v_1 < v_2 < v_3 < \ldots < \ldots < u_3 < u_2 < u_1$.

Therefore the sequences $< u_n >$ and $< v_n >$ are monotonically decreasing and monotonically increasing respectively. Obviously both are bounded so that they are convergent.

Now we have to show that $< u_n >$ and $< v_n >$ have the same limits.

Let $\lim u_n = A$ and $\lim v_n = B$.

Now $$u_n = \frac{1}{2}(u_{n-1} + v_{n-1}) \Rightarrow \lim u_n = \lim \frac{1}{2}(u_{n-1} + v_{n-1})$$

or $$A = \frac{1}{2}(A+B) \quad i.e., \ A = B.$$

Example 27: *If x_1, x_2 are + ive and $x_{n+2} = \sqrt{(x_{n+1} \cdot x_n)}$, prove that the sequences $x_1, x_3, x_5, \ldots$ and $x_2, x_4, x_6, \ldots$ are one an increasing and the other a decreasing sequence and show that their common limit is $(x_1 x_2^2)^{1/3}$.*

Solution: Let $x_1 > x_2$. Then since $x_3 = \sqrt{(x_2 x_1)}$, we have

$$x_1 > x_3 > x_2,$$

and since $x_3 > x_2$, we have $x_3 > x_4 > x_2$. $[\because x_4 = \sqrt{(x_3 x_2)}]$

Similarly, we have

$$x_3 > x_5 > x_4 ; x_5 > x_6 > x_4 ;$$

$$x_5 > x_7 > x_6 ; x_7 > x_8 > x_6 ;$$

and so on.

Thus $\quad x_2 < x_4 < x_6 < \ldots < \ldots < x_5 < x_3 < x_1$.

Hence $< x_1, x_3, x_5, \ldots >$ is monotonic decreasing and

$$< x_2, x_4, x_6, \ldots >$$

is monotonic increasing and both being bounded, are convergent.

Let $\lim x_n = A$, if n is even and $\lim x_n = B$, if n is odd.

Now $\quad x_{n+2} = \sqrt{(x_{n+1} \cdot x_n)} \Rightarrow \lim x_{n+2} = \lim \sqrt{(x_{n+1} \cdot x_n)}$.

$\therefore \quad B = \sqrt{(A \cdot B)}$ or $A = B$, if n is odd

and $\quad A = \sqrt{(B \cdot A)}$ or $A = B$, if n is even.

Hence in either case $A = B$.

Now $$\frac{x_3}{x_2} = \frac{1}{x_2}\sqrt{(x_1 x_2)} = \sqrt{\left(\frac{x_1}{x_2}\right)},$$

$$\frac{x_4}{x_2} = \frac{x_4}{x_3} \cdot \frac{x_3}{x_2} = \sqrt{\left(\frac{x_2}{x_3}\right)}\sqrt{\left(\frac{x_1}{x_2}\right)} = \sqrt{\left(\frac{x_1}{x_3}\right)},$$

$$\frac{x_5}{x_2} = \frac{x_5}{x_4} \cdot \frac{x_4}{x_3} \cdot \frac{x_3}{x_2} = \sqrt{\left(\frac{x_3}{x_4}\right)}\sqrt{\left(\frac{x_2}{x_3}\right)}\sqrt{\left(\frac{x_1}{x_2}\right)} = \sqrt{\left(\frac{x_1}{x_4}\right)}.$$

In a like manner, $\dfrac{x_n}{x_2} = \sqrt{\left(\dfrac{x_1}{x_{n-1}}\right)}$

or $\quad \lim (x_n \sqrt{x_{n-1}}) = \sqrt{(x_1 x_2^2)}$

i.e., $\quad A \cdot A^{1/2} = \sqrt{(x_1 x_2^2)}$ or $A = (x_1 x_2^2)^{1/3}$.

Example 28: *If k is positive and $\alpha, -\beta$ are the positive and negative roots of $x^2 - x - k = 0$, prove that if $u_n = \sqrt{(k + u_{n-1})}$ and $u_1 > 0$, then $u_n \to \alpha$.*

Solution: Since u_1 is positive, hence by virtue of the relation

$$u_n = \sqrt{(k + u_{n-1})},\ u_2, u_3, \ldots, u_n, \ldots \text{ are all positive.}$$

Thus $\quad u_n > 0, \forall\ n \in \mathbf{N}$.

We have $u_n^2 - u_{n-1}^2 = (k + u_{n-1}) - (k + u_{n-2}) = u_{n-1} - u_{n-2}$ so that $u_n >$ or $< u_{n-1}$ according as $u_{n-1} >$ or $< u_{n-2}$ and hence $< u_n >$ is a monotonic sequence, it is an increasing or a decreasing sequence according as $u_2 >$ or $< u_1$.

Now $\qquad x^2 - x - k \equiv (x - \alpha)(x + \beta)$ ...(1)

$\Rightarrow \qquad u_1^2 - u_1 - k = (u_1 - \alpha)(u_1 + \beta).$ ...(2)

Let $u_1 > \alpha$. Then from (2), we have

$$u_1^2 - u_1 - k > 0$$

$$\Rightarrow \qquad u_1 + k < u_1^2 \Rightarrow \sqrt{(u_1 + k)} < u_1 \Rightarrow u_2 < u_1.$$

Hence in this case $< u_n >$ is a decreasing sequence.

Since $u_n > 0, \forall\ n \in \mathbf{N}$, therefore $< u_n >$ is bounded below by 0.

Thus $< u_n >$ is a monotonically decreasing sequence and is bounded below and hence $< u_n >$ is convergent.

Again let $u_1 < \alpha$. Then from (2), we have

$$u_1^2 - u_1 - k < 0 \quad \Rightarrow \quad u_1 + k > u_1^2 \Rightarrow \sqrt{(u_1 + k)} > u_1 \Rightarrow u_2 > u_1.$$

Hence is this case $< u_n >$ is an increasing sequence.

Now $\qquad u_n^2 = u_{n-1} + k < u_n + k, \qquad [\because u_{n-1} < u_n]$

i.e., $\qquad u_n^2 - u_n - k < 0 \quad$ or $\quad (u_n - \alpha)(u_n + \beta) < 0$, using (1).

$\therefore \qquad u_n < \alpha. \qquad [\because\ u_n > 0 \text{ and } \beta > 0 \Rightarrow u_n + \beta > 0]$

Thus in this case $u_n < \alpha, \forall\ n \in \mathbf{N}$.

Hence in this case $< u_n >$ is a monotonically increasing sequence and is bounded above by α and so $< u_n >$ is convergent.

Thus $< u_n >$ is convergent whether $u_1 >$ or $< \alpha$.

Let $\qquad \lim u_n = l.$

Now $\qquad (u_n - \alpha)(u_n + \beta) = u_n^2 - u_n - k = (u_{n-1} + k) - u_n - k = u_{n-1} - u_n.$

Taking limits, we get $(l - \alpha)(l + \beta) = l - l = 0$.

This gives $l = \alpha$ or $l = -\beta$.

Since the terms of the sequence $< u_n >$ are all positive, so its limit cannot be negative. Therefore we cannot have $l = -\beta$. Therefore we must have $l = \alpha$.

In case $u_1 = \alpha$, then from (2), we have

$$u_1^2 - u_1 - k = 0 \Rightarrow u_2 = \sqrt{(u_1 + k)} = u_1.$$

Now repeatedly using the relation

$$u_n^2 - u_{n-1}^2 = u_{n-1} - u_{n-2},$$

we observe that $u_3 = u_2$, $u_4 = u_3$, $u_5 = u_4$, and so on.

Thus in this case $u_n = u_1 = \alpha, \forall\ n \in \mathbf{N}$.

So in this case also the sequence $< u_n >$ converges to α.

Hence in all cases $< u_n >$ converges to α which is the positive root of the equation $x^2 - x - k = 0$.

Example 29: *Show that the sequence $< s_n >$ defined by*

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \textit{ does not converge.}$$

Solution: We shall show that the given sequence is not a Cauchy sequence. For this we shall show that if we take $\varepsilon = \frac{1}{2} > 0$, then there exists no positive integer m such that

$$|s_n - s_m| < \varepsilon \ \forall \ n \geq m.$$

Whatever positive integer m may be, if we take $n = 2m$, then $n > m$ and we have

$$|s_n - s_m| = |s_{2m} - s_m|$$

$$= \left|\left(1 + \frac{1}{2} + \ldots + \frac{1}{m} + \frac{1}{m+1} + \ldots + \frac{1}{2m}\right) - \left(1 + \frac{1}{2} + \ldots + \frac{1}{m}\right)\right|$$

$$= \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{2m}$$

$$> \frac{1}{2m} + \frac{1}{2m} + \frac{1}{2m} + \ldots \text{ upto } m \text{ terms}$$

$$= m \cdot \frac{1}{2m} = \frac{1}{2}.$$

Thus if we take $\varepsilon = \frac{1}{2}$, then whatever positive integer m we take, we have $n = 2m > m$ and

$$|s_n - s_m| = |s_{2m} - s_m| > \frac{1}{2} \quad i.e., \ |s_n - s_m| > \varepsilon.$$

In this way for $\varepsilon = \frac{1}{2} > 0$, there exists no positive integer m such that

$$|s_n - s_m| < \varepsilon \ \forall \ n \geq m.$$

$\therefore$ the given sequence is not a Cauchy sequence.

Hence by Cauchy convergence criterion $< s_n >$ is not convergent.

Example 30: *If $< s_n >$ be a sequence of positive numbers such that*

$$s_n = \frac{1}{2}(s_{n-1} + s_{n-2}), \textit{ for all } n > 2,$$

then show that $< s_n >$ converges and find $\lim s_n$.

Solution: In case $s_1 = s_2$, it can be easily seen that $s_n = s_1$ for all n, therefore $< s_n >$ converges to s_1. Now we consider the case $s_1 \neq s_2$.

We first find that

$$|s_n - s_{n-1}| = \left|\frac{1}{2}(s_{n-1} + s_{n-2}) - s_{n-1}\right| = \frac{1}{2}|s_{n-1} - s_{n-2}|$$

$$= \frac{1}{2} \cdot \frac{1}{2}|s_{n-2} - s_{n-3}| = \frac{1}{2^2}|s_{n-2} - s_{n-3}|$$

$$= \frac{1}{2^{n-2}}|s_2 - s_1|, \text{ for } n \geq 2. \qquad \ldots(1)$$

Now for $n \geq m$, we have

$$|s_n - s_m| = |(s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \ldots + (s_{m+1} - s_m)|$$

$$\leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \ldots + |s_{m+1} - s_m|$$

$$= \left(\frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \ldots + \frac{1}{2^{m-1}} \right) | s_2 - s_1 | \text{ by (1)}$$

$$= \frac{1}{2^{m-1}} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-m-1}} \right) | s_2 - s_1 |$$

$$< \frac{1}{2^{m-2}} | s_2 - s_1 |. \qquad \ldots(2)$$

$$\left[\because \ 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots + \left(\frac{1}{2}\right)^{n-m-1} < 2 \right]$$

Let $\varepsilon > 0$ be given. We can choose a positive integer m such that $\frac{1}{2^{m-2}} | s_2 - s_1 | < \varepsilon$. For this value of m, we have from (2)

$$| s_n - s_m | < \varepsilon \text{ for } n \geq m.$$

Hence $< s_n >$ is a Cauchy sequence and therefore by Cauchy's convergence criterion it converges.

Let $\lim s_n = l$. Putting $n = 3, 4, \ldots, k$ in the relation $s_n = \frac{1}{2}(s_{n-1} + s_{n-2})$, we get

$$\left.\begin{aligned} s_3 &= \frac{1}{2}(s_2 + s_1) \\ s_4 &= \frac{1}{2}(s_3 + s_2) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ s_{k-1} &= \frac{1}{2}(s_{k-2} + s_{k-3}) \\ s_k &= \frac{1}{2}(s_{k-1} + s_{k-2}). \end{aligned}\right\} \qquad \ldots(3)$$

Adding the corresponding sides of the relations in (3), we get

$$s_k + \frac{1}{2} s_{k-1} = \frac{1}{2}(s_1 + 2s_2).$$

Proceeding to the limit as $k \to \infty$, we get

$$\frac{3}{2} l = \frac{1}{2}(s_1 + 2s_2) \quad \textit{i.e.,} \quad l = \frac{1}{3}(s_1 + 2s_2).$$

Example 31: *Let $< u_n >$ be a sequence and $s_n = u_1 + u_2 + \ldots + u_n$.*

If $t_n = | u_1 | + | u_2 | + \ldots + | u_n |$ for each $n \in \mathbf{N}$ and $< t_n >$ is a Cauchy sequence, then $< s_n >$ is also a Cauchy sequence.

Solution: Let $\varepsilon > 0$ be given. Since $< t_n >$ is a Cauchy sequence, therefore, for given $\varepsilon > 0$ there exists $m \in \mathbf{N}$ such that

$$| t_n - t_m | < \varepsilon \ \forall \ n \geq m$$

$\Rightarrow$ $\quad | u_{m+1} | + | u_{m+2} | + \ldots + | u_n | < \varepsilon \ \forall \ n \geq m.$

But $\quad | u_{m+1} | + | u_{m+2} | + \ldots + | u_n | \geq | u_{m+1} + u_{m+2} + \ldots + u_n |.$

$\therefore$ $\quad | u_{m+1} + u_{m+2} + \ldots + u_n | < \varepsilon \ \forall \ n \geq m$

or $\quad | s_n - s_m | < \varepsilon \ \forall \ n \geq m.$

Hence $< s_n >$ is a Cauchy sequence.

Example 32: *Find* (i) $\lim \sqrt{\left(\frac{n+1}{n}\right)}$ (ii) $\lim \frac{\sin(n\pi/3)}{\sqrt{n}}$.

Solution: (i) We have $\sqrt{\left(\frac{n+1}{n}\right)} = \sqrt{\left(1+\frac{1}{n}\right)} < 1 + \frac{1}{2n}$.

$$\left[\because \left(1+\frac{1}{2n}\right)^2 = 1 + \frac{1}{n} + \frac{1}{4n^2} > 1 + \frac{1}{n}\right].$$

Since $1+\frac{1}{n} > 1$, hence $\sqrt{\left(1+\frac{1}{n}\right)} > 1$.

$\therefore \quad 1 < \sqrt{\left(\frac{n+1}{n}\right)} < 1 + \frac{1}{2n}$ for all $n \in \mathbf{N}$.

But $\lim 1 = 1$ and $\lim \left(1+\frac{1}{2n}\right) = 1$.

Hence, by Sandwich theorem, $\lim \sqrt{\left(\frac{n+1}{n}\right)} = 1$.

(ii) Let $s_n = \sin(n\pi/3)$, $t_n = 1/\sqrt{n}$.

We have $-1 \le \sin\frac{n\pi}{3} \le 1$ for all $n \in \mathbf{N}$.

$\therefore$ $< s_n >$ is a bounded sequence.

Also $\qquad \lim t_n = \lim \frac{1}{\sqrt{n}} = 0.$

Hence by theorem 4 of article 9, $\lim (s_n t_n) = 0$ *i.e.,* $\lim \frac{\sin(n\pi/3)}{\sqrt{n}} = 0$.

Example 33: *Show that* $\lim \left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(n+n)^2}\right] = 0$.

Solution: Let $\quad s_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(n+n)^2}$.

For $1 \le m \le n$, we have $(n+1)^2 \le (n+m)^2 \le (n+n)^2$

or $$\frac{1}{(n+1)^2} \ge \frac{1}{(n+m)^2} \ge \frac{1}{(n+n)^2}.$$

Putting $m = 1, 2, \ldots, n$ and adding the corresponding sides of the n inequalities thus obtained, we get

$$\frac{n}{(n+1)^2} \ge s_n \ge \frac{n}{(n+n)^2}$$

i.e., $$\frac{n}{4n^2} \le s_n \le \frac{n}{(n+1)^2} < \frac{n}{n^2} \text{ for all } n \in \mathbf{N}$$

i.e., $$\frac{1}{4n} \le s_n < \frac{1}{n} \text{ for all } n \in \mathbf{N}.$$

But $\lim \frac{1}{4n} = 0$ and $\lim \frac{1}{n} = 0$.

Hence, by Sandwich theorem, $\lim s_n = 0$.

Example 34: *If $r > 0$, show that* $\lim r^{1/n} = 1$.

Solution: **Case 1:** When $r > 1$.

Let $s_n = r^{1/n} - 1$. Then $s_n > 0$, for all n.

Now $\qquad s_n = r^{1/n} - 1 \in r^{1/n} = 1 + s_n \Rightarrow r = (1 + s_n)^n$

$\Rightarrow \qquad r = 1 + n\, s_n + \ldots + s_n^{\,n} \geq 1 + n\, s_n$ for all $n \in \mathbf{N}$

$\Rightarrow \qquad \dfrac{r-1}{n} \leq s_n \;\forall\; n \in \mathbf{N}.$

$\therefore \qquad 0 < s_n \leq \dfrac{r-1}{n} \;\forall\; n \in \mathbf{N}.$

Hence, by Sandwich theorem, $\lim s_n = 0$ *i.e.,* $\lim (r^{1/n} - 1) = 0$ or $\lim r^{1/n} = 1$.

Case 2: When $r = 1$.

In this case $r^{1/n} = 1 \;\forall\; n$ and hence $< r^{1/n} >$ converges to 1.

Case 3: When $0 < r < 1$.

Since $\dfrac{1}{r} > 1$, therefore, by Case 1, $\lim \left(\dfrac{1}{r}\right)^{1/n} = 1$ *i.e.,* $\lim \dfrac{1}{r^{1/n}} = 1$.

$\therefore$ By theorem 7 of article 9, $\lim r^{1/n} = 1$.

Example 35: *Show that* $\lim \dfrac{1}{n}\left(1 + \dfrac{1}{3} + \ldots + \dfrac{1}{2n-1}\right) = 0.$

Solution: Let $s_n = \dfrac{1}{2n-1}\cdot$ Then $\lim s_n = \lim \dfrac{1}{2n-1} = 0.$

$\therefore$ By Cauchy's first theorem on limits, $\lim \dfrac{s_1 + s_2 + \ldots + s_n}{n} = 0\cdot$

Since $\qquad s_1 = 1,\; s_2 = \dfrac{1}{3}, \ldots, s_n = \dfrac{1}{2n-1},$

$$\therefore \qquad \lim \frac{1}{n}\left\{1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}\right\} = 0\cdot$$

Example 36: *Prove that* $\lim \dfrac{1}{n}(1 + 2^{1/2} + 3^{1/3} + \ldots + n^{1/n}) = 1\cdot$

Solution: Let $s_n = n^{1/n}$. Then we know that $\lim n^{1/n} = 1$.

Hence, by Cauchy's first theorem on limits,

$$\lim \frac{1}{n}(s_1 + s_2 + \ldots + s_n) = 1$$

or

$$\lim \frac{1}{n}(1 + 2^{1/2} + 3^{1/3} + \ldots + n^{1/n}) = 1.$$

Example 37: *Prove that* $\lim \left[\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3 \ldots \left(\frac{n+1}{n}\right)^n\right]^{1/n} = e.$

Solution: Let $s_n = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$. Then $\lim s_n = e$.

Clearly $s_n > 0$ for all n.

Hence, by theorem 11 of article 9, $\lim (s_1 s_2 \ldots s_n)^{1/n} = e$.

Since $s_1 = \frac{2}{1}, s_2 = \left(\frac{3}{2}\right)^2, s_3 = \left(\frac{4}{3}\right)^3, \ldots s_n = \left(\frac{n+1}{n}\right)^n,$

$\therefore \quad \lim \left[\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3 \ldots \left(\frac{n+1}{n}\right)^n\right]^{1/n} = e.$

Example 38: *Show that*

(*i*) $\lim \frac{n}{(n!)^{1/n}} = e,$

(*ii*) $\lim [\{(n+1)(n+2)\ldots(n+n)\}^{1/n} / n] = 4/e.$ **(Gorakhpur 2012)**

Solution: (i) Let $s_n = \frac{n^n}{n!}$, then $s_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$.

$$\therefore \quad \frac{s_{n+1}}{s_n} = \frac{(n+1)^{n+1}}{n+1} \cdot \frac{1}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n.$$

Also $s_n > 0$ for all $n \in \mathbf{N}$.

Hence by theorem 12 of article 9, we have

$$\lim s_n^{1/n} = \lim \frac{s_{n+1}}{s_n} = \lim \left(1+\frac{1}{n}\right)^n = e$$

i.e., $$\lim \frac{n}{(n!)^{1/n}} = e.$$

(ii) Let $s_n = (n+1)(n+2)\ldots(n+n)/n^n$.

Then $$\frac{s_{n+1}}{s_n} = \frac{2(2n+1)}{(n+1)}\left(\frac{n}{n+1}\right)^n,$$

so that $$\lim \frac{s_{n+1}}{s_n} = \lim \left[\frac{2(2n+1)}{n+1} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n}\right] = 4 \cdot \frac{1}{e} = \frac{4}{e}.$$

Also $s_n > 0$ for all $n \in \mathbf{N}$.

Hence by theorem 12 of article 9, we have

$$\lim s_n^{1/n} = \lim [\{(n+1)(n+2)\ldots(n+n)\}^{1/n} / n] = 4/e.$$

Example 39: *If $p>0$ and c is real, then find the $\lim \dfrac{n^c}{(1+p)^n}$.*

Solution: Let k be an integer such that $k>c, k>0$.

We have, for $n>2k$

$$(1+p)^n > {}^nC_k\, p^k = \frac{n(n-1)\ldots(n-k+1)}{k!}\, p^k > \frac{n^k}{2^k}\cdot\frac{p^k}{k!}$$

$$\left[\because\ n>2k \Rightarrow (n-r+1)>\frac{n}{2} \text{ for } r=1,2,\ldots,k\right]$$

or $$0<\frac{1}{(1+p)^n}<\frac{2^k\,k!}{n^k\,p^k}, \text{ for } n>2k.$$

$\therefore$ $$0<\frac{n^c}{(1+p)^n}<\frac{2^k\,k!}{p^k}\cdot\frac{1}{n^{k-c}}, \text{ for } n>2k. \qquad \ldots(1)$$

Since $k-c>0$, therefore, $\dfrac{1}{n^{k-c}}\to 0$ as $n\to\infty$.

Hence from (1), we get $\displaystyle\lim_{n\to\infty}\frac{n^c}{(1+p)^n}=0.$

Example 40: *The usual definition of e is given by $e=\sum\limits_{n=0}^{\infty}\dfrac{1}{n!}$. Show that e is irrational.*

Solution: Let $s_n = 1+\dfrac{1}{1!}+\dfrac{1}{2!}+\dfrac{1}{3!}+\ldots+\dfrac{1}{n!}$.

Then $$e-s_n=\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\frac{1}{(n+3)!}+\ldots\ldots$$

$$<\frac{1}{(n+1)!}\left[1+\frac{1}{n+1}+\frac{1}{(n+1)^2}+\ldots\right]$$

$$=\frac{1}{(n+1)!}\cdot\frac{1}{1-\dfrac{1}{n+1}}=\frac{1}{n(n!)}.$$

Thus $$0<e-s_n<\frac{1}{n(n!)}. \qquad \ldots(1)$$

Let, if possible, e be rational. Then e can be put in the form p/q where p and q are positive integers.

From (1), we have $0<e-s_q<\dfrac{1}{q(q!)}$

or $$0<(q!)(e-s_q)<\frac{1}{q} \qquad \ldots(2)$$

Now $$(q!)\,s_q=(q!)\left\{1+1+\frac{1}{2!}+\ldots+\frac{1}{q!}\right\}$$

is an integer. Also, by our assumption $e(q!)$ is an integer. It follows that $e(q!)-s_q(q!)=q!(e-s_q)$ is an integer. Since $q\ge 1$, therefore, (2) shows the existence of an integer between 0 and 1 which is absurd. Hence our initial assumption is wrong. So e is irrational.

Comprehensive Exercise 1

1. Write a formula or formulae for the nth term s_n for each of the following sequences :

(a) $1, -4, 9, -16, 25, -36, \ldots$ (b) $1, 0, 1, 0, 1, 0, \ldots$

(c) $1, 3, 6, 10, 15, \ldots$

2. Which of the sequences (a), (b), (c) in the above problem are subsequences of the sequence $< s_n >$ defined by $s_n = n$?

3. Find whether the following sequences are bounded above or below :

(i) $< \dfrac{(-1)^n}{n} >$ (ii) $< 2^n >$ (iii) $< n! >$.

4. Are the sequences $< s_n >$ defined as follows, bounded ?

(i) $s_n = 1 + \dfrac{(-1)^n}{n}$ (ii) $s_n = \left(1 + \dfrac{1}{n}\right)^n$

(iii) $s_n = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \ldots + \dfrac{1}{n!}$.

(iv) $s_n = 1$, if n is divisible by 3 and $s_n = 0$, otherwise.

5. Use the definition of the limit of a sequence to show that the limit of the sequence $< s_n >$ where $s_n = 2n / (n + 3)$, is 2.

6. Show that the sequence $< s_n >$ where $s_n = n / (n + 1)$ converges to 1.

7. If the sequence $< s_n >$ converges to l, then prove that the sequence $| s_n |$ converges to $| l |$.

8. Show by considering the sequence $< s_n = (-1)^n >$ that $< | s_n | >$ may converge but $< s_n >$ may not.

9. If $\lim s_n = l$ and $s_n \le m$ for all $n \in \mathbf{N}$, prove that $l \le m$.

10. Let $< s_n >$ be a sequence such that $< s_n^2 >$ converges to zero. Is it necessary that $< s_n >$ should converge to zero ?

11. If $s_n = \dfrac{2n}{n + 4n^{1/2}}$, prove that $< s_n >$ is convergent.

12. If $s_n = \dfrac{n}{2^n}$, prove that $< s_n > \to 0$.

13. If $< s_n >$ converges to $l \, (\ne 0)$, prove that $< (-1)^n s_n >$ oscillates.

14. If $< s_n >$ diverges and $c \, (\ne 0) \in \mathbf{R}$, prove that $< c s_n >$ diverges.

15. Show that the sequence $< n + (-1)^n n >$ oscillates infinitely.

16. Show that $< s_n >$ converges to e, where s_n is

(i) $\left(1 + \dfrac{1}{n}\right)^{n+1}$ (ii) $\left(1 + \dfrac{1}{n+1}\right)^n$ (iii) $\left(1 - \dfrac{1}{n}\right)^{-n}$.

17. Show that the sequence $<s_n>$, where $s_n = \left(1+\frac{2}{n}\right)^n$, converges to e^2.

18. If $s_n = \frac{n^3 - 2n + 1}{n^3 + 2n^2 - 1}$, prove that $\lim s_n = 1$.

19. If $s_n = \frac{(3n-1)(n^4 - n)}{(n^2+2)(n^3+1)}$, prove that $\lim s_n = 3$.

20. Prove that the sequence $< \frac{n^2 + 3n + 5}{2n^2 + 5n + 7} >$ converges to $\frac{1}{2}$.

21. Prove that the sequence $<s_n>$ where $s_n = \frac{n}{n^2+1}$ is convergent.

22. Show that the sequence $<s_n>$ defined by

$$s_n = \frac{1}{\sqrt{(n^2+1)}} + \frac{1}{\sqrt{(n^2+2)}} + \ldots + \frac{1}{\sqrt{(n^2+n)}} \text{ converges to } 1.$$

23. Prove that $\lim \left[\frac{1}{\sqrt{(2n^2+1)}} + \frac{1}{\sqrt{(2n^2+2)}} + \ldots + \frac{1}{\sqrt{(2n^2+n)}}\right] = \frac{1}{\sqrt{2}}$.

24. Show that $\lim \left[\frac{1}{(n+1)^\lambda} + \frac{1}{(n+2)^\lambda} + \ldots + \frac{1}{(2n)^\lambda}\right] = 0, \lambda > 1$.

25. A sequence $<s_n>$ is defined as follows :

$$s_1 = a > 0, s_{n+1} = \sqrt{\{(ab^2 + s_n^2)/(a+1)\}}, b > a, n \geq 1.$$

Show that $<s_n>$ is a bounded monotonically increasing sequence and $\lim s_n = b$.

26. Prove that $\lim_{n \to \infty} \left\{\frac{(3n)!}{(n!)^3}\right\}^{1/n} = 27$.

27. Prove that $\lim \left[\frac{(n!)^{1/n}}{n}\right] = \frac{1}{e}$.

28. (i) Let $<s_n>$ be a sequence defined as follows :

$$s_1 = \frac{3}{2}; s_{n+1} = 2 - \frac{1}{s_n}, n \geq 1.$$

Show that $<s_n>$ is monotonic and bounded. Find the limit of the sequence.

(ii) Show that the sequence $<s_n>$ defined by the formula $s_1 = 1$, $s_{n+1} = \sqrt{(3s_n)}$ converges to 3.

29. If $<s_n>$ is a sequence such that $s_n > 0$ and $s_{n+1} \leq k\, s_n$ for all $n \geq m$ and $0 < k < 1$, m being a fixed positive integer, then $\lim s_n = 0$.

30. Prove that if x be any real number, then $\lim_{n \to \infty} \frac{x^n}{n!} = 0$.

31. If $s_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \ldots + \frac{1}{3^n}$, prove that $<s_n>$ converges. Also find $\sup <s_n>$ and $\inf <s_n>$.

32. What do you understand by a monotonic sequence ? Prove that the sequence $< s_n >$, where $s_n = \dfrac{2n-7}{3n+1}$, is :

(i) monotonic increasing (ii) bounded above.

(iii) bounded below.

Also show that the sequence $< s_n >$ is convergent and find sup $< s_n >$ and inf $< s_n >$.

33. If $s_n = \dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \ldots + \dfrac{1}{n\,(n+1)}$, prove that $< s_n >$ is increasing and convergent. Find also sup $< s_n >$ and inf $< s_n >$.

34. If $s_n = 1 + \dfrac{1}{2} + \ldots + \dfrac{1}{n} - \log n$, prove that $< s_n >$ is decreasing and bounded.

35. Test the sequence $< s_n >$, defined by $s_n = (-1)^n\, n$, for limit points.

36. Find the limit points of the sequence $< s_n >$ defined by

$$s_n = (-1)^n \left(1 + \frac{1}{n}\right).$$

37. State and prove Cauchy's general principle of convergence for real sequences. Hence prove that the sequence $< s_n >$, where $s_n = \dfrac{n+1}{n}$ converges.

38. Show, by applying Cauchy's convergence criterion, that the sequence $< s_n >$ defined by

$$s_n = 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1} \text{ does not converge.}$$

39. If the sequence $< s_n >$ converges and if $< t_n >$ is a sequence such that

$$| t_n - t_m | \le | s_n - s_m |,$$

for all positive integers m and n, then prove that $< t_n >$ converges.

40. If $< s_n >$ is a Cauchy sequence of real numbers which has a subsequence converging to l, prove that $< s_n >$ itself converges to l.

41. Show that $\lim\, [(n\,!)\,(a\,/\,n)^n] = 0$ or $+\infty$ according as $a < e$ or $a > e$, where a is any non-negative real number.

42. If $0 < u_1 < u_2$ and $u_n = \dfrac{2u_{n-1}\, u_{n-2}}{u_{n-1} + u_{n-2}}$

(*i.e.*, u_n is the harmonic mean of u_{n-1} and u_{n-2}), show that

$$\lim u_n = 3u_1\, u_2\,/\,(2\,u_1 + u_2).$$

43. If the sequences $< s_n >$ and $< t_n >$ converge to zero and if $< t_n >$ is a strictly decreasing sequence so that $t_{n+1} < t_n \;\forall\; n \in \mathbf{N}$, then

$$lim\, \frac{s_n}{t_n} = lim\, \frac{s_n - s_{n+1}}{t_n - t_{n+1}}$$

provided that the limit on the right exists, whether finite or infinite.

44. If $s_n = \dfrac{a}{1 + s_{n-1}}$, where a, s_n are positive, show that the sequence $< s_n >$ tends to a definite limit l, the positive root of the equation $x^2 + x = a$.

45. Prove that the set of limit points of every sequence is a closed set.

Answers 1

1. (a) $s_n = (-1)^{n-1} n^2$ (b) $s_n = 1$ if n is odd, $s_n = 0$ if n is even,
   (c) $s_n = \dfrac{n(n+1)}{2}$
2. (c).
3. (i) Bounded above as well as bounded below
   (ii) Bounded below but not above
   (iii) Bounded below but not above
4. (i) Yes (ii) Yes (iii) Yes (iv) Yes
10. Yes
28. (i) 1
35. No limit point
36. 1 and – 1

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Every bounded monotonically increasing sequence converges to
   (a) its supremum (b) its infimum
   (c) 0 (d) 1

2. If $< s_n >$ is a sequence of non-negative numbers such that $\lim s_n = l$, then
   (a) $l < 0$ (b) $l > 0$
   (c) $l \geq 0$ (d) $l = 1$

3. Every subsequence of a convergent sequence is
   (a) divergent (b) convergent
   (c) may be convergent or divergent (d) oscillatory

4. The sequence $< s_n >$ where $s_n = \dfrac{5n}{n + 3n^{1/2}}$ has the limit

(a) 3 (b) 1

(c) $\dfrac{1}{3}$ (d) 5

5. The sequence $< 1, -1, 1, -1, 1, -1, \ldots >$ has

(a) no limit point

(b) only one limit point

(c) two limit points

(d) an infinite number of limit points

6. The sequence $< s_n >$, where $s_n = \left(1 + \dfrac{2}{n}\right)^{n+3}$, converges to

(a) e (b) e^2

(c) $e + 3$ (d) $e^2 + 3$

7. If $s_n = 1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \ldots + \dfrac{1}{2^n}$, then the sequence $< s_n >$ is

(a) unbounded (b) convergent

(c) divergent (d) oscillatory

8. The sequence $< s_n >$, where $s_n = \dfrac{3n^2 + 1}{3n^2 - 1}$, converges to

(a) 1 (b) 3

(c) -1 (d) 0

9. Every Cauchy sequence is

(a) oscillatory (b) divergent

(c) unbounded (d) convergent

10. The value of the limit $\lim_{n \to \infty} \dfrac{1}{n}(1 + 2^{1/2} + 3^{1/3} + \ldots + n^{1/n})$ is equal to

(a) 1 (b) 0

(c) 2 (d) 3

11. Every convergent sequence is :

(a) oscillatory (b) unbounded

(c) bounded (d) oscillates finitely

(Rohilkhand 2011)

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. The range of the sequence $< (-1)^n >$ is the set ... .

2. The sequence $< 1/n >$ converges to ... .

3. If $s_n = \dfrac{1}{2^n}$ for all $n \in \mathbf{N}$, then

   (i) $\sup < s_n > = \ldots..$ and (ii) $\inf < s_n > = \ldots.$

4. The supremum of the sequence $< \dfrac{n}{n+1} >$ is ... .

   (Rohilkhand 2012)

5. The nth term of the sequence $< 1, -1, 1, -1, \ldots >$ is ... .

6. If the subsequences $< s_{2n-1} >$ and $< s_{2n} >$ of the sequence $< s_n >$ converge to the same limit l, then $\lim\limits_{n \to \infty} s_n = \ldots.$

7. If $s_n = \sqrt[n]{n}$, then $\lim\limits_{n \to \infty} s_n = \ldots.$

8. The sequence $< -2, -4, -6, \ldots, -2n, \ldots >$ diverges to ... .

9. If $\lim s_n = l$ and $\lim t_n = l'$, then $\lim (s_n t_n) = \ldots.$

10. If $\lim\limits_{n \to \infty} s_n = l$, then $\lim\limits_{n \to \infty} \dfrac{s_1 + s_2 + \ldots + s_n}{n} = \ldots.$

11. Every bounded monotonic sequence is ... .

12. The sequence $< \dfrac{n^2 + 4n + 7}{3n^2 + 5n - 9} >$ converges to ... .

13. If $s_n = \left(1 + \dfrac{1}{n}\right)^{n+2}$, then $\lim\limits_{n \to \infty} s_n = \ldots.$

14. If $s_n = \sqrt{(n+1)} - \sqrt{n}$, then $\lim\limits_{n \to \infty} s_n = \ldots.$

15. If $s_n = \left(1 + \dfrac{1}{n}\right)^n$, then $\lim\limits_{n \to \infty} (s_1 s_2 \ldots s_n)^{1/n} = \ldots.$

True or False

Write 'T' for true and 'F' for false statement.

1. The sequence $< 2, -2, 2, -2, 2, -2, \ldots >$ is bounded.

2. The sequence $< 1, -1, 1, -1, 1, -1, \ldots >$ is convergent.

3. The sequence $< 3, -3, 3, -3, 3, -3, \ldots >$ is a Cauchy sequence.

4. Every Cauchy sequence is always convergent.

5. Every subsequence of a divergent sequence is always divergent.

6. A sequence $< s_n >$ is said to converge to a number l, if for any given $\varepsilon > 0$ there exists a positive integer m such that $|s_n - l| > \varepsilon$ for all $n \geq m$.

7. If a sequence $< s_n >$ is convergent, then $\lim\limits_{n \to \infty} s_n$ is unique.

8. Every convergent sequence is always bounded.

9. Every bounded sequence is always convergent.

10. The sequence $< 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots >$ is a Cauchy sequence.

11. If $\lim s_n = l$ and $\lim t_n = l'$, then $\lim (s_n + t_n) = l + l'$.

12. Every bounded monotonically decreasing sequence converges to its infimum.

13. A monotonically decreasing sequence which is not bounded below diverges to minus infinity.

14. A monotonically increasing sequence which is not bounded above is a convergent sequence.

15. Every bounded sequence has at least one limit point.

16. The sequence $< 1, -1, 1, -1, \ldots >$ has no limit points.

17. If a sequence $< s_n >$ is convergent, then it may or may not be a Cauchy sequence.

18. The sequence $< s_n >$, where $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$, is a Cauchy sequence.

19. The sequence $< s_n >$, where $s_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{(n-1)!}$, is a convergent sequence.

20. If a sequence $< s_n >$ is convergent, then the sequence $< |s_n| >$ is also convergent.

21. If a sequence $< |s_n| >$ is convergent, then the sequence $< s_n >$ is also convergent.

22. The sequence $< s_n >$, where $s_n = \frac{n}{n^3 + 1}$, converges to 0.

23. The sequence $< s_n >$, where $s_n = \left(1 + \frac{1}{n}\right)^n$, is a bounded sequence.

24. The sequence $< 1, -1, 1, -1, 1, -1, \ldots >$ is a monotonic sequence.

25. The sequence $< s_n >$, where $s_n = 3 - \frac{1}{3^{n-1}}$, converges to 2 .

Answers

Multiple Choice Questions

1.	(a)	**2.**	(c)	**3.**	(b)	**4.**	(d)	**5.**	(c)
6.	(b)	**7.**	(b)	**8.**	(a)	**9.**	(d)	**10.**	(a)
11.	(c)								

Fill in the Blank(s)

1.	$\{1, -1\}$	**2.**	0	**3.**	(i) $\frac{1}{2}$ (ii) 0	**4.**	1	**5.**	$(-1)^{n-1}$
6.	l	**7.**	1	**8.**	$-\infty$	**9.**	ll'	**10.**	l
11.	convergent	**12.**	$\frac{1}{3}$	**13.**	e	**14.**	0	**15.**	e

True or False

1.	*T*	**2.**	*F*	**3.**	*F*	**4.**	*T*	**5.**	*T*
6.	*F*	**7.**	*T*	**8.**	*T*	**9.**	*F*	**10.**	*T*
11.	*T*	**12.**	*T*	**13.**	*T*	**14.**	*F*	**15.**	*T*
16.	*F*	**17.**	*F*	**18.**	*F*	**19.**	*T*	**20.**	*T*
21.	*F*	**22.**	*T*	**23.**	*T*	**24.**	*F*	**25.**	*F*

□

Chapter

4

Uniform Convergence of Sequences and Series of Functions

1 Pointwise Convergence

In the present chapter we shall study the uniform convergence of sequences and series of real-valued functions defined on an interval.

Sequence of real-valued functions: *Let f_n be a real-valued function defined on the interval I for all positive integral values n. Then $< f_1, f_2, \ldots, f_n, \ldots >$ or simply $< f_n >$ is called a sequence of real valued functions on I.*

Uniformly bounded sequence: *A sequence $< f_n >$ is said to be uniformly bounded on an interval I if*

$$| f_n (x) | < M \text{ for every } x \in I$$

and for every positive integer n.

Illustrations : (i) The sequence $< f_n >$ where $f_n (x) = 1 / nx$ is bounded but not uniformly bounded.

(ii) The sequence defined by $< \sin nx >$ is uniformly bounded on $\mathbf{R}$ since $| \sin nx | \leq 1$ for all $x \in \mathbf{R}$ and for all $n \in \mathbf{N}$.

Limit function of a sequence of functions: Let f_n be a real valued function defined on the interval I for all positive integral values n. Suppose that the function f_n tends to a definite limit for all values of x in I as $n \to \infty$.

This means that, for $c \in I$, the sequence $< f_n(c) >$ *i.e.*, $< f_1(c), f_2(c), \ldots, f_n(c), \ldots >$ of real numbers is convergent. The limiting values of each of the sequences of real numbers for various points in I will define a function of x, say f, such that the limiting value of the convergent sequence $< f_n(c) >$ is equal to the value $f(c)$ of the function f at c. This function f is called the **limit function** or **the limit of the convergent sequence** $< f_n >$ of functions on I.

Pointwise convergence: The sequence $< f_n >$ converges pointwise to f, written as

$$\lim_{n \to \infty} f_n = f \quad \text{iff} \quad \lim_{n \to \infty} f_n(x) = f(x)$$

for every x in the domain. Here f is said to be the pointwise limit function of $< f_n(x) >$.

Sum function of a series: Let $\quad u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots \quad$ ...(1)

be the series of real valued functions defined on the interval I. Corresponding to this series, we find a sequence x of functions where

$$f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x).$$

We say that the series (1) is convergent if the sequence $< f_n >$ is convergent and the limit function f of the sequence is called the **sum function** of the series.

2 Uniform Convergence of Sequences

Let the sequence $< f_n(x) >$ converge for every point x in I *i.e.*, the function f_n tends to a definite limit as $n \to \infty$ for every x in I. This limit will be a function of x, say f. It follows from the definition of a limit that for every $\varepsilon > 0$, there exists a positive integer m such that

$$|f_n(x) - f(x)| < \varepsilon \text{ whenever } n \geq m.$$

This integer m will depend upon x as well as ε and so we can write it as $m(x, \varepsilon)$. Now if we keep ε fixed and vary x, then for a given point x in I there corresponds a value of $m(x, \varepsilon)$. Thus, we shall find a set of values of $m(x, \varepsilon)$. This set may or may not have an upper bound. If this set has an upper bound, say M, then for every point x in I, we get

$$|f_n(x) - f(x)| < \varepsilon \text{ whenever } n \geq M.$$

In this case, the sequence $< f_n >$ is said to converge uniformly to f on I.

Definition: *Let $< f_n >$ be a sequence of functions defined on an interval I. The sequence $< f_n >$ is said to converge uniformly to the function f on I if for every $\varepsilon > 0$, there can be found a positive integer m such that*

$$|f_n(x) - f(x)| < \varepsilon \textit{ for all } n \geq m \textit{ and for all } x \in I.$$

(Garhwal 2008; Avadh 09)

The function f is called uniform limit of the sequence $< f_n >$ on I.

Note 1: It should be noted carefully that in the above definition the emphasis is on the phrase "one m for all x" whereas for ordinary convergence it is "one m for each x".

Note 2: The sequence $< f_n >$ does not converge uniformly to f on an interval I iff there exists some $\varepsilon > 0$ such that there is no positive integer m for which

$$| f_n(x) - f(x) | < \varepsilon \ \forall \ n \geq m \text{ and } x \in I.$$

Note 3: It can be easily observed that uniform convergence of a sequence $< f_n >$ on I implies pointwise convergence of the sequence $< f_n >$ at every point of I but point-wise convergence does not necessarily ensure its uniform convergence on I. A sequence of functions may be convergent at every point on I and yet may not be uniformly convergent on I. For example, consider the sequence $< f_n >$ defined by

$$f_n(x) = x^n \ (0 \leq x \leq 1).$$

We have $\lim\limits_{n \to \infty} x^n = 0$ if $0 \leq x < 1$

and $\lim\limits_{n \to \infty} x^n = 1$ if $x = 1$.

Thus the limit function f is defined as follows :

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Since the function f_n has a definite limit for every value of x in $[0, 1]$ as $n \to \infty$ so the sequence $< f_n(x) >$ converges for every $x \in [0, 1]$.

Now we shall check whether the convergence is uniform or not. For this we take the interval $[0, 1[$. Let $\varepsilon > 0$ be given. Then

$$| f_n(x) - f(x) | < \varepsilon \Rightarrow | x^n - 0 | < \varepsilon \Rightarrow x^n < \varepsilon \Rightarrow \left(\frac{1}{x}\right)^n > \frac{1}{\varepsilon}$$

$$\Rightarrow \qquad n \log \frac{1}{x} > \log \frac{1}{\varepsilon} \Rightarrow n > \frac{\log(1/\varepsilon)}{\log(1/x)} \qquad \ldots(1)$$

Thus $m(x, \varepsilon)$ is an integer just greater than $\log(1/\varepsilon)/\log(1/x)$ for $x \neq 1$. In particular, when $x = 0, m(x, \varepsilon) = 1$.

Now from (1), we see that $n \to \infty$ as x, starting from 0, increases and approaches 1 and hence it is not possible to find a positive integer m such that

$$| f_n(x) - f(x) | < \varepsilon \text{ for all } n \geq m \text{ and } \forall \ x \in [0, 1[.$$

Consequently $< f_n >$ is not uniformly convergent in $[0, 1[$.

However, if we take the interval $[0, k]$, where $0 < k < 1$, we find that the greatest value of $\log(1/\varepsilon)/\log(1/x)$ is $\log(1/\varepsilon)/\log(1/k)$ so that taking m equal to any positive integer greater than this greatest value, we get

$$| f_n(x) - f(x) | < \varepsilon \text{ for all } n \geq m \text{ and } \forall \ x \in [0, k].$$

Hence $< f_n(x) >$ converges uniformly on $[0, k]$.

Point of non-uniform convergence of the sequence

It is a point, such that the sequence does not converge uniformly in any neighbourhood of it, however small.

In the above example, the point $x = 1$ is such a point. **(Avadh 2009)**

Uniform convergence of a series of functions

Definition: *Let* $\sum_{n=1}^{\infty} u_n(x)$ *be a series of functions defined on the interval* I *and let*

$$f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) \text{ for } n \in \mathbf{N}.$$

The series $\Sigma\, u_n$ *is said to converge uniformly on* I *if the sequence* $< f_n >$ *converges uniformly on* I.

3 Cauchy's General Principle of Uniform Convergence or A Necessary and Sufficient Condition for Uniform Convergence

It helps us in proving the uniform convergence of $< f_n >$ without having any idea about the limit function f.

Theorem: *Let* $< f_n >$ *be a sequence of real-valued functions defined on an interval* I. *Then* $< f_n >$ *converges uniformly on* I *if and only if for every* $\varepsilon > 0$, *there exists a positive integer* m *such that*

$$|f_n(x) - f_p(x)| < \varepsilon \text{ for all } n, p \geq m \text{ and } \forall\, x \in I. \qquad \ldots(1)$$

(Garhwal 2006, 11, 12; Rohilkhand 07)

Proof: The only if part (Necessary part): Let the sequence $< f_n >$ converge uniformly to f on I. By definition, given $\varepsilon > 0$, there exists a positive integer m such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2} \;\forall\, n \geq m, \text{and } \forall\, x \in I.$$

Hence, if $n, p \in \mathbf{N}$ are such that $n, p \geq m$ then $\forall\, x \in I$, we have

$$\begin{aligned} |f_n(x) - f_p(x)| &= |f_n(x) - f(x) + f(x) - f_p(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_p(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus (1) holds for this m.

The if part (Sufficient part): Let $< f_n >$ be any sequence of functions on I such that, given $\varepsilon > 0$, there exists a positive integer m such that the result (1) holds. Now we shall show the existence of a function f on I such that the sequence $< f_n >$ converges uniformly to f on I.

We see, from (1) that for each fixed $x \in I$, the sequence $< f_n(x) >$ of real numbers is a Cauchy sequence. Since every Cauchy sequence of real numbers is convergent so $\lim_{n \to \infty} f_n(x)$ exists for every $x \in I$.

Let us define a function f by $f(x) = \lim_{n \to \infty} f_n(x) \;\forall\, x \in I$

Keeping p fixed in (1) and letting $n \to \infty$, we have

$$|f(x) - f_p(x)| < \varepsilon \;\forall\; p \geq m \text{ and } \forall\; x \in I.$$

$\therefore \quad < f_n(x) >$ converges uniformly to f on I.

Note: The above theorem can be also stated as below :

A sequence $< f_n >$ *is uniformly convergent on* I *if given* $\varepsilon > 0$ *there exists a positive integer* m *such that*

$$| f_{n+p}(x) - f_n(x) | < \varepsilon \ \forall \ n \geq m, \ \forall \ x \in I \text{ and } \forall \ p \in \mathbf{N}.$$

Corollary: *A series* $\Sigma \, u_n(x)$ *will converge uniformly on* I *iff for every* $\varepsilon > 0$ *, there exists a positive integer* m *such that*

$$| u_{n+1}(x) + u_{n+2}(x) + \ldots + u_{n+p}(x) | < \varepsilon$$

for all $n \geq m$, *for all* $x \in I$, *and for all* $p = 1, 2, \ldots$.

This result follows by the above theorem since the uniform convergence of a series $\Sigma \, u_n(x)$ depends upon the uniform convergence of the sequence $< f_n >$ where

$$f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x).$$

4 Tests for Uniform Convergence

Theorem 1: ($\mathbf{M}_n$-test): *Let* $< f_n >$ *be a sequence of functions defined on an interval* I.

Let $\lim\limits_{n \to \infty} f_n(x) = f(x)$ *for all* $x \in I$.

Set $M_n = \sup \{ | f_n(x) - f(x) | : x \in I \}$.

Then $< f_n >$ *converges uniformly to* f *if and only if* $M_n \to 0$ *as* $n \to \infty$.

Proof: The only if part: Let $< f_n >$ converge uniformly to f on I. Then for every $\varepsilon > 0$, there exists a positive integer m, independent of x, such that

$$| f_n(x) - f(x) | < \varepsilon \ \forall \ n \geq m, \forall \ x \in I.$$

Since $M_n = \sup \{ | f_n(x) - f(x) | : x \in I \}$, we have

$M_n \leq \varepsilon$ for all $n \geq m$.

$\therefore$ $M_n \to 0$ as $n \to \infty$.

The if part: Let $M_n \to 0$ as $n \to \infty$. Then for given $\varepsilon > 0$ there exists $m \in \mathbf{N}$ such that $M_n < \varepsilon \ \forall \ n \geq m$.

Since $M_n = \sup \{ | f_n(x) - f(x) | : x \in I \}$,

$\therefore$ $| f_n(x) - f(x) | \leq M_n < \varepsilon \ \forall \ n \geq m$ and $\forall \ x \in I$

i.e., $| f_n(x) - f(x) | < \varepsilon \ \forall \ n \geq m$ and $\forall \ x \in I$.

$\therefore$ $< f_n >$ converges uniformly to f on I.

Note: The M_n-test is a very powerful test. Therefore reader is advised to use it whenever it is applicable.

Theorem 2: (Weierstrass's M-test). *A series* $\sum\limits_{n=1}^{\infty} u_n(x)$ *of functions will converge uniformly on* I *if there exists a convergent series* $\sum\limits_{n=1}^{\infty} M_n$ *of positive constants such that*

$$| u_n(x) | \leq M_n$$

for all n *and for all* $x \in I$. **(Rohilkhand 2007; Kanpur 09)**

Proof: Since $\Sigma\, M_n$ is convergent, by Cauchy's principle, for given $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$M_{n+1} + M_{n+2} + \ldots + M_{n+p} < \varepsilon \qquad \ldots(1)$$

for all $n \geq m$, $p = 1, 2, \ldots$.

By hypothesis $|u_n(x)| \leq M_n \ \forall\, n \in \mathbf{N}$ and $\forall\, x \in I$. ...(2)

From (1) and (2) we conclude that

$$\begin{aligned} &|u_{n+1}(x) + u_{n+2}(x) + \ldots + u_{n+p}(x)| \\ &\leq |u_{n+1}(x)| + |u_{n+2}(x)| + \ldots + |u_{n+p}(x)| \\ &\leq M_{n+1} + M_{n+2} + \ldots + M_{n+p} < \varepsilon \end{aligned}$$

for all $n \geq m$ and for all $x \in I$.

$\therefore$ $\Sigma\, u_n(x)$ converges uniformly and absolutely on I.

Note: It can be easily seen that the same proof would hold if M_n were a function of x and if the series $\Sigma\, M_n(x)$ were uniformly convergent on I.

In the next two results we shall use Abel's Lemma which is stated below :

Abel's Lemma: *If the sequence $< v_n >$ of positive terms is monotonic decreasing and numbers $u_1, u_2, \ldots, u_n$ and k_1, k_2 are such that*

$$k_1 < u_1 + u_2 + \ldots + u_r < k_2 \quad \textit{for } 1 \leq r \leq n$$

then

$$k_1 v_1 \leq \sum_{n=1}^{\infty} u_r v_r \leq k_2 v_1.$$

Theorem 3: (Abel's Test). *The series $\Sigma\, u_n(x)\, v_n(x)$ will converge uniformly on $[a, b]$ if*

(*i*) *$\Sigma\, u_n(x)$ is uniformly convergent on $[a, b]$*

(*ii*) *the sequence $< v_n(x) >$ is monotonic for every x in $[a, b]$*

(*iii*) *$< v_n(x) >$ is uniformly bounded in $[a, b]$ i.e., there is a positive number k, independent of x and n, such that $|v_n(x)| < k$ for every value of x in $[a, b]$ and every positive integer n.*

Proof: Let $f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x), n \in \mathbf{N}$.

$\Sigma\, u_n(x)$ is uniformly convergent on $[a, b]$

$\Rightarrow$ by Cauchy's principle for given $\varepsilon > 0$ there exists a positive integer m such that

$$|f_q(x) - f_p(x)| < \frac{\varepsilon}{k} \ \forall\, q, p \in \mathbf{N}, q > p > m \text{ and } \forall\, x \in [a, b]$$

$$\left| \sum_{n=p+1}^{q} u_n(x) \right| < \frac{\varepsilon}{k} \ \forall\, q > p > m \text{ and } \forall\, x \in [a, b].$$

$\therefore$ ε / k is an upper bound of $\displaystyle\sum_{n=p+1}^{q} u_n(x)$.

By hypothesis, the sequence $< v_n(x) >$ is monotonic in $[a, b]$. Hence by Abel's lemma, we have

$$\left| \sum_{n=p+1}^{q} u_n(x)\, v_n(x) \right| \leq \frac{\varepsilon}{k} v_{p+1}(x) < \frac{\varepsilon}{k} \cdot k = \varepsilon$$

$\forall\, p, q \in \mathbf{N}, q > p > m$ and $\forall\, x \in [a, b]$.

Consequently, by Cauchy's principle, $\Sigma\, u_n(x)\, v_n(x)$ is uniformly convergent on $[a, b]$.

Theorem 4: (Dirichlet's Test): *The series $\Sigma\, u_n(x)\, v_n(x)$ will be uniformly convergent on $[a, b]$ if*

(i) $< v_n(x) >$ *is a positive monotonic decreasing sequence converging uniformly to zero on* $[a, b]$

(ii) $|f_n(x)| = \left| \sum_{r=1}^{n} u_r(x) \right| < k$, *for every value of x in* $[a, b]$ *and for all integral values of n, where k is a fixed number independent of x.* **(Avadh 2008)**

Proof: We have $|f_n(x)| < k \;\forall\, x \in [a, b]$ and $\forall\, n \in \mathbf{N}$.

$\therefore$ for all $x \in [a, b]$ and for all $p, q \in \mathbf{N}, q > p > m_1$, we get

$$|f_q(x) - f_p(x)| \le |f_q(x)| + |f_p(x)|$$
$$< k + k = 2k$$

i.e.
$$\left| \sum_{n=p+1}^{q} u_n(x) \right| < 2k \;\; \forall\, x \in [a, b] \text{ and } \forall\, q > p > m_1.$$

$\therefore$ $2k$ is an upper bound of $\sum_{n=p+1}^{q} u_n(x)$.

Also sequence $< v_n(x) >$ is positive monotonic decreasing, so by Abel's lemma

$$\left| \sum_{n=p+1}^{q} u_n(x)\, v_n(x) \right| < 2k\, v_{p+1}(x). \quad \text{...(1)}$$

Again $< v_n(x) >$ converges uniformly to zero on $[a, b]$

$\Rightarrow$ given $\varepsilon > 0$ there exists $m_2 \in \mathbf{N}$ such that

$$|v_n(x)| < \frac{\varepsilon}{2k} \;\forall\, n \ge m_2 \text{ and } \forall\, x \in [a, b]. \quad \text{...(2)}$$

Let $m = \max.\{m_1, m_2\}$. Then (1) and (2) hold for all $n > m$.

$\therefore$
$$\left| \sum_{n=p+1}^{q} u_n(x)\, v_n(x) \right| < 2k\, \frac{\varepsilon}{2k} = \varepsilon$$

for all $x \in [a, b]$, and for all $q > p > m$.

Hence $\Sigma\, u_n(x)\, v_n(x)$ is uniformly convergent on $[a, b]$.

Illustrative Examples

Example 1: *Let* $f_n(x) = \dfrac{nx}{1 + n^2 x^2}$ *when* $0 \le x \le 1$ *and* $n = 1, 2, 3, \ldots$. *Examine as to whether the sequence* $< f_n >$ *is uniformly convergent on* $\mathbf{R}$.

(Garhwal 2007, 11; Avadh 08; Rohilkhand 09, 12)

Solution: We have

$$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{nx}{1 + n^2 x^2} = 0 \;\forall\, x \in \mathbf{R}.$$

Let, if possible, the sequence converge uniformly on **R**. Then for a given $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$|f_n(x) - f(x)| = \frac{n|x|}{1 + n^2 x^2} < \varepsilon \;\forall\; n \geq m \text{ and } \forall\; x \in \mathbf{R}. \qquad \ldots(1)$$

If we take $x = \dfrac{1}{n}$ $(n = 1, 2, 3, \ldots)$, then

$$|f_n(x) - f(x)| = \frac{n \cdot \dfrac{1}{n}}{1 + n^2 \cdot \dfrac{1}{n^2}} = \frac{1}{2}.$$

Hence, we note that for $\varepsilon = \dfrac{1}{2}$, there is no single m such that (1) holds simultaneously $\forall\; x \in \mathbf{R}$.

For if such an m existed, we would have

$$|f_m(x) - f(x)| < \frac{1}{2} \;\forall\; x \in \mathbf{R}.$$

But if $x = \dfrac{1}{m}$, this is a contradiction because we would have $\dfrac{1}{2} < \dfrac{1}{2}$ in that case. Thus convergence is non-uniform. In fact 0 is a point of non-uniform convergence.

Note: It can be easily seen that this sequence will converge uniformly on any interval which does not include 0.

Example 2: *The sum to n terms of a series is*

$$f_n(x) = \frac{n^2 x}{1 + n^4 x^2}.$$

Show that it converges non-uniformly in the interval $[0, 1]$.

Solution: We have $f(x) = \lim\limits_{n \to \infty} \dfrac{n^2 x}{1 + n^4 x^2} = 0 \;\forall\; x \in [0, 1]$

$$\therefore \qquad |f_n(x) - f(x)| = \frac{n^2 |x|}{1 + n^4 x^2}.$$

Take $x = 1/n^2$ and proceed as in above example.

Another Method: M_n-test readily proves the non-uniform convergence. We have

$$M_n = \sup \{|f_n(x) - f(x)| : x \in \mathbf{R}\} = \sup \left\{ \frac{n^2 |x|}{1 + n^4 x^2} : x \in \mathbf{R} \right\}$$

$$\geq \frac{n^2 \cdot \dfrac{1}{n^2}}{1 + n^4 \cdot \dfrac{1}{n^4}} = \frac{1}{2} \left(\text{Taking } x = \frac{1}{n^2} \in \mathbf{R} \right).$$

Since M_n cannot tend to zero as $n \to \infty$ hence by M_n-test the sequence is non-uniformly convergent.

Example 3: *Show that 0 is a point of non-uniform convergence of the sequence* $< f_n >$ *where* $f_n(x) = 1 - (1 - x^2)^n$.

Solution: We have

$$f(x) = \lim_{n \to \infty} f_n(x) = \begin{cases} 0 & \text{when } x = 0 \\ 1 & \text{when } 0 < |x| < \sqrt{2} \end{cases}$$

$$\begin{aligned} \therefore \quad M_n &= \sup\{|f_n(x) - f(x)| : x \in]0, \sqrt{2}[\} \\ &= \sup\{(1 - x^2)^n : x \in]0, \sqrt{2}[\} \\ &\geq \left(1 - \frac{1}{n}\right)^n \qquad \left\{\text{taking } x = \frac{1}{\sqrt{n}} \in]0, \sqrt{2}[\right\} \\ &\to \frac{1}{e} \text{ as } n \to \infty. \end{aligned}$$

Since M_n does not tend to zero as $n \to \infty$, the sequence is non-uniformly convergent. Also as $n \to \infty$, $x \to 0$ and hence 0 is a point of non-uniform convergence.

Example 4: *Prove that the exponential series*

$$\Sigma\, u_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

converges uniformly on every bounded subset of **R**.

Solution: Let $[-k, k]$ be a bounded subset of **R**, for $k > 0$. Let $\varepsilon > 0$ be given. Then there exists $m \in \mathbf{N}$ such that for any $p \in \mathbf{N}$ and $x \in [-k, k]$, we get

$$\left|\frac{x^n}{n!}\right| \leq \left|\frac{k^n}{n!}\right| < \frac{\varepsilon}{p} \quad \forall \ n \geq m.$$

$\therefore$ for all $n \geq m$ and for all $p \in \mathbf{N}$, we have

$$\begin{aligned} &\left|\frac{x^{n+1}}{(n+1)!} + \frac{x^{n+2}}{(n+2)!} + \ldots + \frac{x^{n+p}}{(n+p)!}\right| \\ &\leq \left|\frac{x^{n+1}}{(n+1)!}\right| + \ldots + \left|\frac{x^{n+p}}{(n+p)!}\right| < \frac{\varepsilon}{p} \cdot p = \varepsilon. \end{aligned}$$

Thus for any $\varepsilon > 0$ there exists $m \in \mathbf{N}$ such that for any $p \in \mathbf{N}$ and for all $x \in [-k, k]$, we get

$$|f_{n+p} - f_n| < \varepsilon \text{ where } f_n = 1 + \frac{x}{1!} + \ldots + \frac{x^n}{n!}.$$

Hence, by Cauchy's principle $\Sigma\, u_n(x)$ converges uniformly on $[-k, k]$.
Since $k > 0$ is arbitrary, $\Sigma\, u_n(x)$ converges uniformly on any bounded subset of **R**.

Note: This series is not uniformly convergent on **R**. Since $\forall\ m \in \mathbf{N}$ there exists $x \in \mathbf{R}$ such that $\frac{x^m}{m!} > 1$ and hence for any $\varepsilon > 0$,

$1 < \frac{x^m}{m!} < \varepsilon$ is not true.

Example 5: *Show that the series* $\cos x + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots$

converges uniformly on **R**.

Solution: Here, we have

$$\left|\frac{\cos nx}{n^2}\right| \leq \frac{1}{n^2} \;\forall\, x \in \mathbf{R}.$$

But we know that the series $\Sigma \dfrac{1}{n^2}$ is convergent.

Hence by Weierstrass's M-test the given series is uniformly convergent on $\mathbf{R}$.

Example 6: *Prove that* $\Sigma\, a_n\, n^{-x}$ *is uniformly convergent on* $[0,1]$ *if* $\Sigma\, a_n$ *converges uniformly on* $[0,1]$.

Solution: Take $v_n(x) = n^{-x} = \dfrac{1}{n^x}$ and $u_n(x) = a_n$.

The sequence $< n^{-x} >$ is monotonic decreasing on $[0,1]$.

Since $< n^{-x} >$ decreases on $[0,1]$,

$\therefore \quad \dfrac{1}{n^x} \leq \dfrac{1}{n^0} = 1 \;\forall\, n \in \mathbf{N}$ and $\forall\, x \in [0,1]$.

$\therefore \quad |v_n(x)| = |n^{-x}| \leq 1 \;\forall\, n \in \mathbf{N}$ and $\forall\, x \in [0,1]$.

Thus $< v_n(x) >$ is uniformly bounded and monotonic decreasing sequence on $[0,1]$.
Also $\Sigma\, u_n(x) = \Sigma\, a_n$ is uniformly convergent on $[0,1]$.
Hence by Abel's test, $\Sigma\, u_n(x)\, v_n(x) = \Sigma\, a_n\, n^{-x}$ is uniformly convergent on $[0,1]$.

Example 7: *Test for uniform convergence the series*

$$\sum_{n=0}^{\infty} x\, e^{-nx} \text{ in the closed interval } [0,1].$$

(Garhwal 2007; Rohilkhand 12)

Solution: Here $f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$

$$= \sum_{n=0}^{n-1} xe^{-nx} = \frac{x\,(1 - 1/e^{nx})}{1 - 1/e^x} = \frac{xe^x}{e^x - 1}\left(1 - \frac{1}{e^{nx}}\right).$$

$$\therefore \quad f(x) = \lim_{n \to \infty} f_n(x) = \begin{cases} 0 & \text{when } x = 0 \\ \dfrac{xe^x}{e^x - 1} & \text{when } 0 < x \leq 1. \end{cases}$$

Let us consider $0 < x \leq 1$. We have

$$M_n = \sup\,\{\,|f_n(x) - f(x)| : x \in]0,1]\}$$

$$= \sup\left\{\frac{xe^x}{(e^x - 1)\,e^{nx}} : x \in]0,1]\right\}$$

$$\geq \frac{(1/n)\,.\,e^{1/n}}{(e^{1/n} - 1)\,e}. \qquad \left(\text{Taking } x = \frac{1}{n} \in]0,1]\right)$$

Now $\displaystyle \lim_{n \to \infty} \frac{(1/n)\,.\,e^{1/n}}{e\,(e^{1/n} - 1)} \qquad \left(\text{Form } \frac{0}{0}\right)$

$$= \lim_{n \to \infty} \frac{(1/n) \cdot e^{1/n}(-1/n^2) + (-1/n^2)\, e^{1/n}}{e \cdot e^{1/n}(-1/n^2)}$$

$$= \lim_{n \to \infty} \frac{\{(1/n) + 1\}}{e} = \frac{1}{e}.$$

Hence M_n cannot tend to zero as $n \to \infty$ and consequently the sequence is non-uniformly convergent by M_n-test. Here 0 is a point of non-uniform convergence.

Example 8: *Test for uniform convergence the series*

(*i*) $\Sigma \dfrac{x}{(n + x^2)^2}$ **(Purvanchal 2008; Garhwal 09)**

(*ii*) $\Sigma \dfrac{x}{n(1 + nx^2)}$.

Solution: (i) Here $u_n(x) = \dfrac{x}{(n + x^2)^2}$.

For maximum or minimum of $u_n(x)$, $\dfrac{du_n(x)}{dx} = 0$

or $(n + x^2)^2 - 4x^2(n + x^2) = 0$ or $3x^4 + 2nx^2 - n^2 = 0$

or $x^2 = \dfrac{n}{3}$ *i.e.*, $x = \sqrt{\left(\dfrac{n}{3}\right)}$.

It can be easily seen that $\dfrac{d^2u_n(x)}{dx^2}$ is – ive when $x = \sqrt{\left(\dfrac{n}{3}\right)}$.

$$\therefore \qquad M_n = \max. u_n(x) = \frac{\sqrt{(n/3)}}{\left(n + \frac{n}{3}\right)^2} = \frac{3\sqrt{3}}{16n^{3/2}}.$$

Thus $|u_n(x)| \leq M_n$. But ΣM_n is convergent.

Hence, by Weierstrass's M-test, the given series is uniformly convergent for all values of x.

(ii) Here $u_n(x) = \dfrac{x}{n(1 + nx^2)}$ is max. or min. when

$$n(1 + nx^2) - 2n^2x^2 = 0 \quad \text{or} \quad x = \pm 1/\sqrt{n}.$$

It can be easily seen that $u_n(x)$ is maximum for $x = 1/\sqrt{n}$.

$$\therefore \qquad M_n = \max. u_n(x) = \frac{1/\sqrt{n}}{n(1+1)} = \frac{1}{2n^{3/2}}.$$

But ΣM_n is convergent. Hence, by Weierstrass's M-test, the given series is uniformly convergent for all values of x.

Comprehensive Exercise 1

1. Prove that the sequence $< f_n >$ defined by $f_n(x) = \dfrac{x}{1+nx}$ $(0 \le x < \infty)$ is uniformly convergent to 0 on $[0, \infty[$.

2. Show that the sequence $< f_n >$ where

$$f_n(x) = nx(1-x)^n$$

does not converge uniformly on $[0, 1]$. **(Garhwal 2007)**

3. Show that 0 is a point of non-uniform convergence of the sequence $< f_n(x) >$, where $f_n(x) = nxe^{-nx^2}, x \in \mathbf{R}$. **(Rohilkhand 2007)**

4. Show that the sequence $< f_n >$ where $f_n(x) = x^{n-1}(1-x)$ converges uniformly in the interval $[0, 1]$.

5. If $\sum_{n=0}^{\infty} a_n$ converges absolutely prove that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[0, 1]$.

6. Show that $\sum_{n=1}^{\infty} \dfrac{1}{n^2 + x^2}$ $(0 \le x < \infty)$ is uniformly convergent on $[0, \infty[$.

7. Prove that $\sum_{n=1}^{\infty} \dfrac{x}{n(n+1)}$ is uniformly convergent on $]0, k[, k > 0$ but not on $]0, \infty[$.

8. Prove that if δ is any fixed positive number less than unity, the series $\Sigma \dfrac{x^n}{n+1}$ is uniformly convergent in $[-\delta, \delta]$.

9. Show that the series $\sin x + \dfrac{1}{2}\sin 2x + \dfrac{1}{3}\sin 3x + \dots$

converges uniformly in $0 < a \le x \le b < 2\pi$.

10. Show that the series

$$\frac{x^2}{1+x} + \left(\frac{2x^2}{1+2x} - \frac{x^2}{1+x}\right) + \dots + \left(\frac{nx^2}{1+nx} - \frac{(n-1)x^2}{1+(n-1)x}\right) + \dots$$

converges uniformly on $[0, 1]$.

11. Consider the series $\Sigma \dfrac{(-1)^{n-1}}{n} x^n$ for uniform convergence in $[0, 1]$.

12. Consider the series $\Sigma \dfrac{(-1)^{n-1}}{(n+x^2)}$ for uniform convergence for all values of x.

13. Show that the series $\sum_{n=1}^{\infty} (-1)^{n-1} x^n$ converges uniformly in $0 \le x \le k < 1$.

14. Show that the series $\sum_{n=1}^{\infty} \dfrac{1}{1+n^2 x}$ converges uniformly in $[1, \infty[$.

5 Uniform Convergence and Continuity

Let us consider the sequence $< f_n >$ defined on $[0, 1]$ where $f_n(x) = x^n \ (0 \le x \le 1)$.

Since $f_n(x) = x^n \ \forall \ n \in \mathbf{N}$ is a polynomial function, f_n is continuous on $[0, 1] \ \ \forall \ n \in \mathbf{N}$.

The limit function f is defined by

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x = 1. \end{cases}$$

The sequence $< f_n >$ converges pointwise to the function f which is discontinuous at $x = 1$ *i.e.* discontinuous on $[0, 1]$.

Hence we see that a sequence $< f_n >$ of continuous functions may converge pointwise to discontinuous function.

This is also true for a series of continuous functions $\Sigma \ u_n(x)$ converging pointwise to $f(x)$.

However, in the following theorem we prove that a sequence $< f_n >$ of continuous functions on $[a, b]$ converging uniformly to f on $[a, b]$ is such that the uniform limit function f is also continuous on $[a, b]$.

Theorem 1: *Let $< f_n >$ be a sequence of real-valued functions on $[a, b]$ which converges uniformly to the function f on $[a, b]$. If each $f_n \ (n = 1, 2, 3, \ldots)$ is continuous on $[a, b]$, then f is also continuous on $[a, b]$.*

Proof: Let c be an arbitrary element of $[a, b]$. It is sufficient to prove that f is continuous at c. Since each f_n is continuous on $[a, b]$, it implies it is continuous at c. Now let $\varepsilon > 0$ be given. Since $< f_n >$ converges uniformly to f on $[a, b]$, there exists $m \in \mathbf{N}$ such that

$$| f_n(x) - f(x) | < \frac{\varepsilon}{3} \ \forall \ n \ge m \text{ and } \ \forall \ x \in [a, b].$$

In particular, we have

$$| f_m(x) - f(x) | < \frac{\varepsilon}{3} \qquad \ldots(1)$$

and
$$| f_m(c) - f(c) | < \frac{\varepsilon}{3}. \qquad \ldots(2)$$

Also since f_m is continuous at c, there exists $\delta > 0$ such that

$$| f_m(x) - f_m(c) | < \frac{\varepsilon}{3} \text{ whenever } | x - c | < \delta. \qquad \ldots(3)$$

Hence if $| x - c | < \delta$, we get

$$\begin{aligned} | f(x) - f(c) | &= | f(x) - f_m(x) + f_m(x) - f_m(c) + f_m(c) - f(c) | \\ &\le | f(x) - f_m(x) | + | f_m(x) - f_m(c) | + | f_m(c) - f(c) | \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ by (1), (2) and (3).} \end{aligned}$$

Thus for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$| f(x) - f(c) | < \varepsilon \text{ whenever } | x - c | < \delta.$$

It follows that f is continuous at c, which proves the theorem.

Corollary: *Let* $\sum_{n=1}^{\infty} u_n(x)$ *be a series of real valued continuous functions defined on* $[a,b]$. *If the series converges uniformly to the function* f *on* $[a,b]$, *then* f *is continuous on* $[a,b]$. *In other words, the sum function of a uniformly convergent series of continuous functions is itself continuous.*

Proof: We know that the uniform convergence of the series $\sum_{n=1}^{\infty} u_n(x)$ is the same thing as the uniform convergence of the sequence $< f_n(x) >$ where

$$f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x).$$

Also the sum $f_m(x)$ of the finite number of continuous functions $u_1(x), u_2(x), \ldots, u_m(x)$ is continuous. For the rest of the proof we can proceed as in the above theorem.

Note 1: The reader should note that uniform convergence is only a sufficient but not a necessary condition for the continuity of the sum function *i.e.*, if the sum function is continuous on $[a,b]$, it is not necessary that the series is uniformly convergent on $[a,b]$. For example, let us consider the series

$$\sum_{n=1}^{\infty} \left[\frac{n^2 x}{1+n^3 x^2} - \frac{(n-1)^2 x}{1+(n-1)^3 x^2}\right] \text{ defined on } [0,1].$$

Here, we have $f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$

$$= \left(\frac{x}{1+x^2} - 0\right) + \left(\frac{2^2 x}{1+2^3 x^2} - \frac{x}{1+x^2}\right) + \ldots$$

$$+ \left(\frac{n^2 x}{1+n^3 x^2} - \frac{(n-1)^2 x}{1+(n-1)^3 x^2}\right) = \frac{n^2 x}{1+n^3 x^2}.$$

$$\therefore \qquad f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{n^2 x}{1+n^3 x^2} = 0 \ \forall \ x.$$

Hence $f(x)$ is continuous on $[0,1]$.

Now $\qquad M_n = \sup \{ | f_n(x) - f(x) : x \in [0,1] \}$

$$= \sup \left\{ \frac{n^2 x}{1+n^3 x^2} : x \in [0,1] \right\}$$

$$\geq \frac{n^2 \cdot \frac{1}{n^{3/2}}}{1 + n^3 \cdot \frac{1}{n^3}} = \frac{\sqrt{n}}{2} \qquad \left(\text{taking } x = \frac{1}{n^{3/2}}\right)$$

$$\to \infty \text{ as } n \to \infty.$$

Thus M_n cannot tend to zero as $n \to \infty$. Hence the series is non-uniformly convergent on $[0,1]$ by M_n-test. Here 0 is a point of non-uniform convergence.

Note 2: From the above theorem we also conclude that if the series of continuous functions defined on $[a,b]$ has discontinuous sum, it cannot be uniformly convergent on a subset A of $[a,b]$ which contains a point of discontinuity. For, if the series were uniformly convergent, we have seen above that its sum must be continuous on the domain of uniform convergence.

Consequently the theorem provides a very good negative test for uniform convergence.
For example, consider the series

$$\sum_{n=0}^{\infty} x^n (1-x) \text{ in } [0,1].$$

Here $\quad f_n(x) = 1 - x^n.$

$$\therefore \quad f(x) = \begin{cases} 0, & \text{if } x = 1 \\ 1, & \text{if } 0 \le x < 1. \end{cases}$$

For each value of n, $x^n (1-x)$ is continuous on $[0,1]$ although $f(x)$ is discontinuous at 1. It gives that the series cannot be uniformly convergent on $[0,1]$.

Illustrative Examples

Example 9: *Test for uniform convergence and continuity of the sum function of the series for which*

(*i*) $f_n(x) = \dfrac{1}{1+nx}$ $(0 \le x \le 1)$, **(Garhwal 2008)**

(*ii*) $f_n(x) = nx(1-x)^n$ $(0 \le x \le 1)$.

Solution: (i) We have $f_n(x) = \dfrac{1}{1+nx}$.

$$\therefore \quad f(x) = \lim_{n \to \infty} f_n(x) = \begin{cases} 0, & \text{if } 0 < x \le 1 \\ 1, & \text{if } x = 0. \end{cases}$$

The sum function f is discontinuous at 0. Hence the series will be non-uniformly convergent in $[0,1]$.
By the previous methods it can be verified that zero is a point of non-uniform convergence.
(ii) Here $f_n(x) = nx(1-x)^n, (0 \le x \le 1)$.

If $\quad 0 < x < 1, \lim_{n \to \infty} nx(1-x)^n$ $\quad$ [Form: $\infty \times 0$]

$$= \lim_{n \to \infty} \frac{nx}{(1-x)^{-n}} \qquad \left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \lim_{n \to \infty} \frac{x}{-(1-x)^{-n} \log(1-x)}$$

$$= \lim_{n \to \infty} -\frac{x(1-x)^n}{\log(1-x)} = 0.$$

Also $f_n(x) = 0$ if $x = 0$ or 1.
Hence $f(x) = \lim_{n \to \infty} f_n(x) = 0 \;\; \forall\, x \in [0,1]$.

The sum function $f(x)$ is continuous for all $x \in [0,1]$.
However the sequence $< f_n(x) >$ is not uniformly convergent on $[0,1]$.

Example 10: *Test the series* $\Sigma\ xe^{-nx}$ *for uniform convergence and continuity of its sum function near* $x = 0$.

Solution: We have $\quad f_n(x) = x\,\dfrac{1-e^{-nx}}{1-e^{-x}}\cdot$

$$\therefore \qquad f(x) = \begin{cases} \lim\limits_{n \to \infty} f_n(x) = \dfrac{x}{1-e^{-x}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Now $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{x}{1-e^{-x}}$

$$= \lim_{x \to 0} \frac{x}{1-\left(1-x+\dfrac{x^2}{2!}-\dfrac{x^3}{3!}+\ldots\right)}$$

$$= \lim_{x \to 0} \frac{1}{1-\dfrac{x}{2!}+\ldots} = 1.$$

Also $\qquad f(0) = 0.$

Since $\lim\limits_{x \to 0} f(x) \neq f(0)$, the function $f(x)$ is not continuous at $x = 0$. It implies that the series does not converge uniformly in any interval containing 0.

6 Uniform Convergence and Integration

Theorem 1: *Let* $< f_n >$ *be a sequence of real valued functions defined on the closed and bounded interval* $[a, b]$ *and let* $f_n \in \mathbf{R}[a, b]$, *for* $n = 1, 2, 3, \ldots$. *If* $< f_n >$ *converges uniformly to the function* f *on* $[a, b]$, *then* $f \in \mathbf{R}[a, b]$ *and*

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \int_a^b f_n(x)\,dx.$$

(Garhwal 2010)

Proof: Let $\varepsilon > 0$ be given.

The sequence $< f_n >$ converges uniformly to f on $[a, b]$

$\Rightarrow$ there exists $m \in \mathbf{N}$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3(b-a)} \ \forall\ n \geq m \text{ and } \forall\ x \in [a, b]. \qquad \ldots(1)$$

In particular, for $n = m$, $|f_m(x) - f(x)| < \dfrac{\varepsilon}{3(b-a)}$

i.e., $$f_m(x) - \frac{\varepsilon}{3(b-a)} < f(x) < f_m(x) + \frac{\varepsilon}{3(b-a)}\cdot \qquad \ldots(2)$$

Since $f_m \in \mathbf{R}[a, b]$, there exists a partition

$P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ of $[a, b]$ such that

$$U(P, f_m) - L(P, f_m) < \frac{1}{3}\varepsilon. \qquad \ldots(3)$$

Let $m_r^{(m)}$, $M_r^{(m)}$ and m_r, M_r be the infima and suprema of f_m and f respectively on $[x_{r-1}, x_r]$.

Now from (2), we get

$$f_m(x) < f(x) + \frac{\varepsilon}{3(b-a)} \quad \forall\, x \in [a, b].$$

In particular, $f_m(x) < f(x) + \dfrac{\varepsilon}{3(b-a)}\ \forall\, x \in [x_{r-1}, x_r]$.

It follows that $m_r^{(m)} \le m_r + \dfrac{\varepsilon}{3(b-a)}$

$$\Rightarrow \quad \sum_{r=1}^{n} m_r^{(m)} \Delta x_r \le \sum_{r=1}^{n} m_r \Delta x_r + \frac{\varepsilon}{3(b-a)} \sum_{r=1}^{n} \Delta x_r$$

$$\Rightarrow \quad L(P, f_m) \le L(P, f) + \frac{1}{3}\varepsilon. \qquad \text{...(4)}$$

Again from (2), we also have

$$f(x) < f_m(x) + \frac{\varepsilon}{3(b-a)} \quad \forall\, x \in [x_{r-1}, x_r].$$

It follows that $M_r \le M_r^{(m)} + \dfrac{\varepsilon}{3(b-a)}$

$$\Rightarrow \quad \sum_{r=1}^{n} M_r \Delta x_r \le \sum_{r=1}^{n} M_r^{(m)} \Delta x_r + \frac{\varepsilon}{3(b-a)} \sum_{r=1}^{n} \Delta x_r$$

$$\Rightarrow \quad U(P, f) \le U(P, f_m) + \frac{1}{3}\varepsilon. \qquad \text{...(5)}$$

Adding (4) and (5), we have

$$U(P, f) + L(P, f_m) \le L(P, f) + U(P, f_m) + \frac{2\varepsilon}{3}$$

or

$$U(P, f) - L(P, f) \le U(P, f_m) - L(P, f_m) + \frac{2\varepsilon}{3}$$

$$< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon, \text{ using (3).}$$

Thus for a given $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$.

Hence $f \in \mathbf{R}[a, b]$. This proves the first result.

To prove the second result, we have for all $n \ge m$,

$$\left| \int_a^b f_n - \int_a^b f \right| = \left| \int_a^b (f_n - f) \right| \le \int_a^b |f_n - f|$$

$$< \int_a^b \frac{\varepsilon}{3(b-a)}\, dx, \text{ using (1)}$$

$$= \frac{\varepsilon}{3(b-a)} \int_a^b dx = \frac{\varepsilon}{3(b-a)}(b-a) = \frac{\varepsilon}{3}.$$

It follows that $\lim\limits_{n \to \infty} \left[\int_a^b f_n(x)\, dx \right] = \int_a^b f(x)\, dx.$

This completes the proof of the theorem.

Corollary 1: *Let* $< f_n >$ *be a sequence of real valued continuous functions defined on* $[a, b]$ *such that* $f_n \to f$ *uniformly on* $[a, b]$*. Then* $f \in \mathbf{R}\,[a, b]$ *and*

$$\lim_{n \to \infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proof: Since every continuous function is Riemann integrable so $f_n \in \mathbf{R}\,[a, b]$ for $n = 1, 2, 3, \ldots$ and hence the symbol $\int_a^b f_n(x)\,dx$ is meaningful.

Again if the sequence $< f_n >$ converges uniformly to f on $[a, b]$ and f_n is continuous on $[a, b]$ for each $n \in \mathbf{N}$ then (by theorem 1 of article 5) f is continuous on $[a, b]$ and hence $f \in \mathbf{R}\,[a, b]$. Thus the first result is proved. For the second result the proof is the same as that given in the above theorem.

Term by Term Integration

Corollary 2: *Let* $\sum_{n=1}^{\infty} u_n(x)$ *be a series of real valued functions defined on* $[a, b]$ *such that* $u_n(x) \in \mathbf{R}\,[a, b]$, *for* $n = 1, 2, 3, \ldots$ *If the series converges uniformly to* f *on* $[a, b]$, *then*

$$f \in \mathbf{R}\,[a, b]$$

and

$$\int_a^b \left[\sum_{n=1}^{\infty} u_n(x) \right] dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx.$$

Proof: Let $u_1(x) + u_2(x) + \ldots + u_n(x) = f_n(x)$. Then $f_n \in \mathbf{R}\,[a, b]$ for each fixed n because the sum of a finite number of R-integrable functions is R-integrable. Also we know that the uniform convergence of the series $\Sigma\, u_n(x)$ is the same thing as the uniform convergence of the sequence $< f_n >$ so that $< f_n >$ converges uniformly to f on $[a, b]$. Hence $f \in \mathbf{R}\,[a, b]$, by the above theorem.

Now

$$\int_a^b \left[\sum_{n=1}^{\infty} u_n(x) \right] dx = \int_a^b f(x)\,dx = \lim_{n \to \infty} \int_a^b f_n(x)\,dx$$

$$= \lim_{n \to \infty} \int_a^b \left[\sum_{m=1}^{n} u_m(x) \right] dx$$

$$= \lim_{n \to \infty} \sum_{m=1}^{n} \int_a^b u_m(x)\,dx$$

$$= \sum_{n=1}^{\infty} \left[\int_a^b u_n(x)\,dx \right].$$

Note: The condition of uniform convergence of the series $\Sigma\, u_n(x)$ is only sufficient but not necessary for the validity of term by term integration. For example, consider the series

$$\sum_{n=1}^{\infty} u_n(x), \quad \text{where} \quad u_n(x) = \frac{nx}{1 + n^2 x^2} - \frac{(n-1)\,x}{1 + (n-1)^2\,x^2}.$$

We have

$$f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x) = \frac{nx}{1 + n^2 x^2}.$$

$\therefore$

$$f(x) = \lim_{n \to \infty} f_n(x) = 0 \text{ for all } x.$$

Now $$\int_0^1 \left[\sum_{n=1}^{\infty} u_n(x) \right] dx = \int_0^1 f(x)\, dx = \int_0^1 0\, dx = 0,$$

and $$\sum_{n=1}^{\infty} \int_0^1 u_n(x)\, dx = \lim_{n \to \infty} \sum_{m=1}^{n} \int_0^1 u_m(x)\, dx$$

$$= \lim_{n \to \infty} \int_0^1 \left[\sum_{m=1}^{n} u_m(x) \right] dx$$

$$= \lim_{n \to \infty} \int_0^1 f_n(x)\, dx$$

$$= \lim_{n \to \infty} \int_0^1 \frac{nx}{1+n^2x^2}\, dx$$

$$= \lim_{n \to \infty} \frac{1}{2n} \log(1+n^2) = 0.$$

Hence $\sum_1^{\infty} \int_0^1 u_n(x)\, dx = \int_0^1 \sum_1^{\infty} u_n(x)\, dx$, although 0 is a point of non-uniform convergence of the series as we have seen earlier.

Illustrative Examples

Example 11: *Show that*

$$\int_0^1 \left(\sum_1^{\infty} \frac{x^n}{n^2} \right) dx = \sum_1^{\infty} \frac{1}{n^2(n+1)}.$$

Solution: By Weierstrass's M-test, the series $\Sigma \frac{x^n}{n^2}$ is uniformly convergent for $0 \le x \le 1$. Therefore it can be integrated term by term.

Hence $$\int_0^1 \left(\sum_1^{\infty} \frac{x^n}{n^2} \right) dx = \sum_1^{\infty} \int_0^1 \frac{x^n}{n^2}\, dx = \sum_1^{\infty} \left[\frac{x^{n+1}}{(n+1)\, n^2} \right]_0^1 = \sum_1^{\infty} \frac{1}{n^2(n+1)}.$$

Example 12: *Examine for term by term integration the series the sum of whose first n terms is* $n^2 x(1-x)^n, (0 \le x \le 1)$.

Solution: Here $f_n(x) = n^2 x(1-x)^n$.

Obviously $f_n(x) = 0$ when $x = 0$ or 1.

When $0 < x < 1$, we have

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{n^2 x}{(1-x)^{-n}} \qquad \left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \lim_{n \to \infty} \frac{2nx}{-(1-x)^{-n} \log(1-x)} \qquad \left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \lim_{n \to \infty} \frac{2x}{(1-x)^{-n} [\log(1-x)]^2} = 0.$$

Hence $f(x) = \lim_{n \to \infty} f_n(x) = 0$ for $0 \le x \le 1$.

$\therefore$ $$\int_0^1 f(x)\, dx = 0.$$

But $$\int_0^1 f_n(x)\, dx = \int_0^1 n^2 x (1-x)^n\, dx$$

$$= n^2 \left[-\frac{x(1-x)^{n+1}}{n+1} - \frac{(1-x)^{n+2}}{(n+1)(n+2)} \right]_0^1$$

$$= \frac{n^2}{(n+1)(n+2)} \to 1 \text{ as } n \to \infty.$$

Consequently term by term integration over the interval $[0, 1]$ is not justified. In fact, 0 is a point of non-uniform convergence of the series *i.e.* the series is non-uniformly convergent in $0 \le x \le 1$.

For, let if possible, the series be uniformly convergent.

Then for a given $\varepsilon > 0$, we have

$$|f_n(x) - f(x)| = n^2 x (1-x)^n < \varepsilon \qquad \text{...(1)}$$

for all values of $n \ge m$ and $\forall\ x \in [0, 1]$.

Taking $x = 1/n$, we get

$$|f_n(x) - f(x)| = n^2 \cdot \frac{1}{n}\left(1 - \frac{1}{n}\right)^n = n\left(1 - \frac{1}{n}\right)^n$$

$$\to \infty \text{ as } n \to \infty, \textit{ i.e.}, \text{as } x \to 0.$$

This contradicts (1). Hence the series does not converge uniformly in $[0, 1]$. Here 0 is a point of non-uniform convergence of the series.

Example 13: *Show that the series*

$$2xe^{-x^2} = \sum_1^{\infty} 2x\left[\frac{1}{n^2} e^{-x^2/n^2} - \frac{1}{(n+1)^2} e^{-x^2/(n+1)^2}\right]$$

can be integrated term by term between any two finite limits. Can the function defined by the series be integrated between the limits 0 and ∞? If so, what is the value of this integral given by integrating the series term by term between these limits ?

Solution: Here $$u_n(x) = 2x\left[\frac{e^{-x^2/n^2}}{n^2} - \frac{e^{-x^2/(n+1)^2}}{(n+1)^2}\right].$$

$\therefore$ $$f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$$

$$= 2x\left[e^{-x^2} - \frac{e^{-x^2/2^2}}{2^2}\right] + 2x\left[\frac{e^{-x^2/2^2}}{2^2} - \frac{e^{-x^2/3^2}}{3^2}\right] + \ldots$$

$$+ 2x\left[\frac{e^{-x^2/n^2}}{n^2} - \frac{e^{-x^2/(n+1)^2}}{(n+1)^2}\right]$$

$$= 2x\left[e^{-x^2} - \frac{e^{-x^2/(n+1)^2}}{(n+1)^2}\right].$$

Then $\qquad f(x) = \lim_{n\to\infty} f_n(x) = 2xe^{-x^2}$ for all values of x.

Now $\qquad \int_a^b f(x)\,dx = \int_a^b 2xe^{-x^2}\,dx = e^{-a^2} - e^{-b^2}$

and $\qquad \int_a^b f_n(x)\,dx = e^{-a^2} - e^{-b^2} + \left[e^{-b^2/(n+1)^2} - e^{-a^2/(n+1)^2}\right]$

$$\to e^{-a^2} - e^{-b^2} \text{ as } n\to\infty.$$

Since $\qquad \int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx,$

the series can be integrated term by term between any two finite limits.

Again $\qquad \int_0^\infty f(x)\,dx = \int_0^\infty 2xe^{-x^2}\,dx = 1$

and $\qquad \int_0^\infty f_n(x)\,dx = \left[-e^{-x^2} + e^{-x^2/(n+1)^2}\right]_0^\infty = 0.$

Since $\qquad \int_0^\infty f(x)\,dx \neq \lim_{n\to\infty} \int_0^\infty f_n(x)\,dx,$

the series cannot be integrated term by term between the limits 0 and ∞.

7 Uniform Convergence and Differentiation

Theorem 1: *Let $< f_n >$ be a sequence of real valued functions defined on $[a, b]$ which converges uniformly to the function f on $[a, b]$. Let c be any point of $[a, b]$ and suppose that*

$$\lim_{x\to c} f_n(x) = L_n \quad (n = 1, 2, 3, \ldots).$$

Then the sequence $< L_n >$ of real constants converges, and

$$\lim_{x\to c} f(x) = \lim_{n\to\infty} L_n.$$

In other words, the conclusion of the theorem is that

$$\lim_{x\to c}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to c} f_n(x).$$

Proof: Let $\varepsilon > 0$ be given. Since $< f_n >$ converges uniformly to f on $[a, b]$, there exists $m \in \mathbf{N}$ such that

$$|f_n(x) - f_p(x)| < \varepsilon \text{ for all } n, p \geq m \text{ and } \forall\, x \in [a, b] \qquad \ldots(1)$$

In (1), letting $x \to c$, we have $|L_n - L_p| < \varepsilon$
for all $n \geq m, p \geq m$. Hence by Cauchy's general principle of convergence for real sequences, the sequence $< L_n >$ converges. This proves the first assertion.

Let $\lim_{n \to \infty} L_n = L.$

Now since $< L_n >$ converges to L and $< f_n >$ converges uniformly to f on $[a, b]$, there exists a positive integer k such that

$$|L_k - L| < \frac{1}{3}\varepsilon \qquad \text{...(2)}$$

and $$|f_k(x) - f(x)| < \frac{1}{3}\varepsilon, \qquad \text{...(3)}$$

for all $x \in [a, b]$.

Also since $\lim_{n \to c} f_k(x) = L_k$, there exists $\delta > 0$ such that

$$|f_k(x) - L_k| < \frac{1}{3}\varepsilon \text{ whenever } |x - c| < \delta. \qquad \text{...(4)}$$

Thus for all x satisfying $|x - c| < \delta$, we have

$$\begin{aligned} |f(x) - L| &= |f(x) - f_k(x) + f_k(x) - L_k + L_k - L| \\ &\le |f(x) - f_k(x)| + |f_k(x) - L_k| + |L_k - L| \\ &< \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon, \text{ by (2), (3) and (4).} \end{aligned}$$

Hence for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } |x - c| < \delta.$$

By the definition of limit, it follows that

$$\lim_{x \to c} f(x) = L = \lim_{n \to \infty} L_n$$

or $$\lim_{n \to c} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{n \to c} f_n(x).$$

The theorem is now completely established.

Corollary: *Let* $\sum_{n=1}^{\infty} u_n(x)$ *be a series of real valued functions defined on* $[a, b]$ *and let* $\lim_{x \to c} u_n(x)$ *exist* $(n = 1, 2, 3, \ldots)$ *where* c *is any point of* $[a, b]$. *If the series* $\Sigma\, u_n(x)$ *converges uniformly on* $[a, b]$, *then*

$$\lim_{x \to c} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \left[\lim_{x \to c} u_n(x) \right].$$

Proof: Let $f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$. Then

$$\lim_{x \to c} \sum_{n=1}^{\infty} u_n(x) = \lim_{x \to c} \lim_{n \to \infty} \sum_{m=1}^{n} u_m(x) = \lim_{x \to c} \lim_{n \to \infty} f_n(x) \qquad \text{...(5)}$$

and $$\sum_{n=1}^{\infty} \left[\lim_{x \to c} u_n(x) \right]$$

$$= \lim_{n \to \infty} \sum_{m=1}^{n} \left[\lim_{x \to c} u_m(x) \right]$$

$$= \lim_{n \to \infty} \lim_{x \to c} \sum_{m=1}^{n} u_m(x).$$

[$\because$ the limit of the sum of a finite number of functions is the sum of their limits]

$$= \lim_{n \to \infty} \lim_{x \to c} f_n(x). \qquad ...(6)$$

Now the result follows from (5), (6) and the above theorem.

Theorem 2: *Let $< f_n >$ be a sequence of real valued functions defined on $[a, b]$ such that*

(i) f_n is differentiable on $[a, b]$ for $n = 1, 2, 3, \ldots$,

(ii) the sequence $< f_n(c) >$ converges for some point c of $[a, b]$

(iii) the sequence $< f_n' >$ converges uniformly on $[a, b]$. Then the sequence $< f_n >$ converges uniformly to a differentiable limit f and

$$\lim_{n \to \infty} f_n'(x) = f'(x) \quad (a \le x \le b).$$

Proof: Let $\varepsilon > 0$ be given. Since $< f_n(c) >$ converges and $< f_n' >$ converges uniformly on $[a, b]$, there exists a positive integer m such that for all $n, p \ge m$,

$$|f_n(c) - f_p(c)| < \frac{1}{2}\varepsilon \qquad ...(1)$$

and
$$|f_n'(x) - f_p'(x)| < \frac{\varepsilon}{2(b-a)} \quad (a \le x \le b). \qquad ...(2)$$

Applying the mean value theorem of differential calculus to the function $f_n - f_p$ where $n, p \ge m$, for any x, y in $[a, b]$ there exists some ξ between x and y such that

$$\frac{(f_n - f_p)(x) - (f_n - f_p)(y)}{x - y} = (f_n - f_p)'(\xi).$$

$\therefore$
$$|f_n(x) - f_p(x) - f_n(y) + f_p(y)| = |x - y||f_n'(\xi) - f_p'(\xi)|$$

$$< \frac{|x-y|\varepsilon}{2(b-a)}, \text{ using (2)} \qquad ...(3)$$

$$< \frac{1}{2}\varepsilon \qquad [\because |x - y| \le (b - a)] \qquad ...(4)$$

for all $n, p \ge m$ and $\forall\ x, y \in [a, b]$.

Hence, for all $n, p \ge m$ and $\forall\ x \in [a, b]$,

$$|f_n(x) - f_p(x)| = |f_n(x) - f_p(x) - f_n(c) + f_p(c) + f_n(c) - f_p(c)|$$

$$\le |f_n(x) - f_p(x) - f_n(c) + f_p(c)| + |f_n(c) - f_p(c)|$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \text{ using (1) and (4).}$$

Thus for a given $\varepsilon > 0$, there exists a positive integer m such that $|f_n(x) - f_p(x)| < \varepsilon$ for all $n, p \ge m$ and $\forall\ x \in [a, b]$.

It follows that the sequence $< f_n >$ converges uniformly on $[a, b]$, say to a function f,

i.e., $$f(x) = \lim_{n \to \infty} f_n(x), (a \le x \le b).$$

Thus the first result is proved.

Now for an arbitrary but for the moment a fixed $x \in [a, b]$, define

$$F_n(y) = \frac{f_n(y) - f_n(x)}{y - x}, F(y) = \frac{f(y) - f(x)}{y - x} \quad ...(5)$$

for $a \leq y \leq b, y \neq x$. Then

$$\lim_{y \to x} F_n(y) = \lim_{y \to x} \frac{f_n(y) - f_n(x)}{y - x} = f_n'(x) \quad ...(6)$$

for $n = 1, 2, 3, \ldots$.

Also for $n, p \geq m$, we have

$$|F_n(y) - F_p(y)| = \left| \frac{f_n(y) - f_n(x) + f_p(y) - f_p(x)}{y - x} \right|$$

$$< \frac{\varepsilon}{2(b-a)}, \text{ using (3).}$$

It implies that $< F_n >$ converges uniformly for $a \leq y \leq b, y \neq x$. Since $< f_n >$ converges to f, it follows from (5) that

$$\lim_{n \to \infty} F_n(y) = \lim_{n \to \infty} \frac{f_n(y) - f_n(x)}{y - x}$$

$$= \frac{f(y) - f(x)}{y - x} = F(y). \quad ...(7)$$

Now applying theorem 1 of article 7 to $< F_n >$ with $L_n = f_n'(x)$, and using (6), (7), we get

$$\lim_{y \to x} F(y) = \lim_{n \to \infty} f_n'(x)$$

or $$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = \lim_{n \to \infty} f_n'(x), \text{ using (5)}$$

or $$f'(x) = \lim_{n \to \infty} f_n'(x), \forall x \in [a, b].$$

Thus the theorem is completely established.

Term by Term Differentiation

Corollary: *Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of real valued differentiable functions on $[a, b]$ such that $\sum_{n=1}^{\infty} u_n(c)$ converges for some point c of $[a, b]$ and $\sum_{n=1}^{\infty} u_n'(x)$ converges uniformly on $[a, b]$. Then the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on $[a, b]$ to a differentiable sum function f and*

$$f'(x) = \lim_{n \to \infty} \sum_{m=1}^{n} u_m'(x), \quad (a \leq x \leq b)$$

i.e., $$\frac{d}{dx}\left[\sum_{n=1}^{\infty} u_n(x)\right] = \sum_{n=1}^{\infty} \left[\frac{d}{dx} u_n(x)\right], \quad (a \leq x \leq b).$$

(Purvanchal 2009)

Proof: Let $f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$.

Since the differential coefficient of the sum of a finite number of differentiable functions is equal to the sum of the their differential coefficients, hence we have

$$f_n'(x) = u_1'(x) + u_2'(x) + \ldots + u_n'(x).$$

Thus the series $\sum_{n=1}^{\infty} u_n(x)$ and $\sum_{n=1}^{\infty} u_n'(x)$ are equivalent to the sequences $< f_n >$ and $< f_n' >$ respectively.

Now the proof is the same as in the above theorem.

Note: If in addition to the hypothesis of the above theorem, the continuity of f_n' is assumed then a much shorter proof based on theorem 1 of article 5, theorem 1 of article 6 and the fundamental theorem of integral calculus is given in the following theorem.

Theorem 3: *Let $< f_n >$ be a sequence of real valued functions defined on $[a, b]$ such that*

(i) f_n is differentiable on $[a, b]$ for $n = 1, 2, 3, \ldots$

(ii) the sequence $< f_n >$ converges to f on $[a, b]$,

(iii) the sequence $< f_n' >$ converges uniformly to g on $[a, b]$,

(iv) each f_n' is continuous on $[a, b]$.

Then $\quad g(x) = f'(x) \quad (a \le x \le b)$

i.e., $\quad \lim_{n \to \infty} f_n'(x) = f'(x), \quad (a \le x \le b).$

Proof: Since the sequence $< f_n' >$ of continuous functions converges uniformly to g on $[a, b]$, it follows by theorem 1 of article 5 that g is continuous on $[a, b]$. Also $< f_n' >$ converges uniformly to g on $[a, x]$, where x is any point of $[a, b]$. It follows, by theorem 1 of article 6, that

$$\lim_{n \to \infty} \int_a^x f_n'(t)\, dt = \int_a^x g(t)\, dt. \qquad \ldots(1)$$

By the fundamental theorem of integral calculus, we have

$$\int_a^x f_n'(t)\, dt = f_n(x) - f_n(a). \qquad \ldots(2)$$

But by hypothesis,

$$\lim_{n \to \infty} f_n(x) = f(x) \text{ and } \lim_{n \to \infty} f_n(a) = f(a). \qquad \ldots(3)$$

Combining (1), (2) and (3), we get

$$f(x) - f(a) = \int_a^x g(t)\, dt, \qquad (a \le x \le b).$$

It implies that

$$f'(x) = g(x), \qquad (a \le x \le b)$$

or

$$f'(x) = \lim_{n \to \infty} f_n'(x), \qquad (a \le x \le b).$$

Corollary: *Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of functions defined on $[a, b]$ such that*

(i) $u_n(x)$ is differentiable on $[a, b]$ for $n = 1, 2, 3, \ldots$

(*ii*) *the series* $\sum_{n=1}^{\infty} u_n(x)$ *converges to* f *on* $[a, b]$

(*iii*) *the series* $\sum_{n=1}^{\infty} u_n'(x)$ *converges uniformly to* g *on* $[a, b]$

(*iv*) *each* u_n' *is continuous on* $[a, b]$.

Then $\quad f'(x) = g(x), \qquad (a \le x \le b)$

i.e., $\quad f'(x) = \sum_{n=1}^{\infty} u_n'(x), \qquad (a \le x \le b)$

Note: For term by term differentiation, the derived series must be uniformly convergent and the original series need only be simply convergent.

Illustrative Examples

Example 14: *Show that*

$$\lim_{x \to 1} \sum_{n=1}^{\infty} \frac{nx^2}{n^3 + x^3} = \sum_{n=1}^{\infty} \frac{n}{n^3 + 1}.$$

Solution: First we shall show that the series $\sum_{n=1}^{\infty} \frac{nx^2}{n^3 + x^3}$ is uniformly convergent on $[0, k]$ for any $k > 0$.

Let $\quad u_n(x) = \frac{1}{n^3 + x^3}$ and $v_n(x) = nx^2$.

Then $\quad |u_n(x)| \le \frac{1}{n^3} \ \forall \ x \in [0, k]$.

But $\Sigma \frac{1}{n^3}$ is a convergent series. Hence by Weierstrass's M-test, the series $\Sigma\, u_n(x)$ is uniformly convergent on $[0, k]$. Also, for every $x \in [0, k]$, the sequence $< v_n(x) >$ is monotonically increasing.

It follows (by Abel's test) that the series

$$\Sigma\, u_n(x)\, v_n(x) = \Sigma \frac{nx^2}{n^3 + x^3}$$

converges uniformly on $[0, k]$. Hence by the above corollary, we get

$$\lim_{x \to 1} \left(\sum_{n=1}^{\infty} \frac{nx^2}{n^3 + x^3} \right) = \sum_{n=1}^{\infty} \left(\lim_{x \to 1} \frac{nx^2}{n^3 + x^3} \right) = \sum_{n=1}^{\infty} \frac{n}{n^3 + 1}.$$

Example 15: *Show that the series* $\Sigma\, u_n(x)$ *for which*

$$f_n(x) = \frac{1}{2n^2} \log(1 + n^4 x^2)$$

can be differentiated term by term although the series $\Sigma\, u_n'(x)$ *does not converge uniformly.*

Solution: Here

$$f(x) = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{\log(1 + n^4 x^2)}{2n^2} \qquad \left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \lim_{n \to \infty} \frac{\dfrac{4n^3 x^2}{1 + n^4 x^2}}{4n} = \lim_{n \to \infty} \frac{n^2 x^2}{1 + n^4 x^2} = 0 \text{ for } 0 \le x \le 1.$$

Hence $\quad f'(x) = 0.$

Also, $\quad \lim_{n \to \infty} f_n'(x) = \lim_{n \to \infty} \dfrac{n^2 x}{1 + n^4 x^2} = 0 \text{ for } 0 \le x \le 1.$

Since $\quad f'(x) = \lim_{n \to \infty} f_n'(x),$

therefore, the given series can be differentiated term by term. But the series $\Sigma\, u_n'(x)$ does not converge uniformly in $[0, 1]$ since the sequence $< f_n'(x) >$ has 0 as a point of non-uniform convergence as we have seen earlier.

This example shows that term by term differentiation holds even though $\Sigma\, u_n'(x)$ is non-uniformly convergent.

Example 16: *Show that, if* $f(x) = \sum_1^{\infty} \dfrac{1}{n^3 + n^4 x^2}$, *then it has a differential coefficient equal to* $-2x \sum_1^{\infty} \dfrac{1}{n^2 (1 + nx^2)^2}$ *for all values of x.*

(Garhwal 2010)

Solution: Here $u_n(x) = \dfrac{1}{n^3 + n^4 x^2}$, so that $u_n'(x) = -\dfrac{2x}{n^2 (1 + nx^2)^2}$.

Now $u_n'(x)$ is maximum when $\dfrac{d}{dx} u_n'(x) = 0$,

i.e., $\quad (1 + nx^2)^2 - 4nx^2 (1 + nx^2) = 0$

or $\quad 1 - 3nx^2 = 0 \quad$ or $\quad x = \pm 1/\sqrt{(3n)}.$

It can be easily seen that $\dfrac{d^2}{dx^2} u_n'(x)$ is – ive when

$$x = -1/\sqrt{(3n)}.$$

$$\therefore \quad \text{Max.} |u_n'(x)| = \frac{2}{\sqrt{3} \,.\, n^{5/2} \left(1 + \frac{1}{3}\right)^2} = \frac{3\sqrt{3}}{8n^{5/2}}$$

so that $\quad |u_n'(x)| < \dfrac{1}{n^{5/2}}$ for all values of x.

Since $\Sigma \dfrac{1}{n^{5/2}}$ is convergent, it follows by Weierstrass's M-test that the series $\Sigma\, u_n'(x)$ converges uniformly for all values of x. Hence the series $\Sigma\, u_n(x)$ can be differentiated term by term.

$$\therefore \quad f'(x) = \sum_{n=1}^{\infty} u_n'(x) = -2x \sum_1^{\infty} \frac{1}{n^2 (1 + nx^2)^2}.$$

Comprehensive Exercise 2

1. Examine for term by term integration the series for which
$$f_n(x) = nxe^{-nx^2}$$
indicating the interval over which your conclusion holds. **(Purvanchal 2009)**

2. Show that the series for which
(i) $f_n(x) = \dfrac{1}{1+nx}$, (ii) $f_n(x) = nx(1-x)^n$
can be integrated term by term in $0 \le x \le 1$, although they are not uniformly convergent in this interval. **(Purvanchal 2008; Rohilkhand 10)**

3. Examine for the continuity the sum function and for term by term integration the series whose nth term is
$$n^2 xe^{-n^2x^2} - (n-1)^2 \, xe^{-(n-1)^2 x^2}$$
x having all values in the interval $[0, 1]$.

4. Show that near $x = 0$, the series $u_1(x) + u_2(x) + u_3(x) + \ldots$ where $u_1(x) = x, u_n(x) = x^{1/(2n-1)} - x^{1/(2n-3)}$ and real values of x are concerned, is discontinuous and non-uniformly convergent. Can the series be integrated term by term ?

5. Examine for term by term integration the series $\Sigma \, x^{n-1}(1 - 2x^n)$ in the interval $0 \le x \le 1$.

6. Show that the series, for which $f_n(x) = \dfrac{nx}{1+n^2x^2}, 0 \le x \le 1$
cannot be differentiated term by term at $x = 0$.

7. Show that the function represented by $\sum_{n=1}^{\infty} \dfrac{\sin nx}{n^3}$ is differentiable for every x and its derivative is $\sum_{1}^{\infty} \dfrac{\cos nx}{n^2}$.

8. Let $u_n(x) = x^2 \, [x^{1/(2n-1)} - x^{1/(2n-3)}] \sin \dfrac{1}{x}$ for $x \ne 0$, $u_n(0) = 0$
for any positive integer greater than unity and
$$u_1(x) = x^3 \sin(1/x) \text{ for } x \ne 0, u_1(0) = 0.$$
Show that $\sum_{1}^{\infty} u_n(x)$ converges for all values of x to $f(x)$, where $f(x) = x^3 \sin(1/x)$ for $x \ne 0$ and $f(0) = 0$. Also that f' is discontinuous at $x = 0$, that $\sum_{1}^{\infty} u_n'(x)$ is not uniformly convergent in any interval including the origin, and that
$$f'(x) = \sum_{1}^{\infty} u_n'(x) \text{ for all values of } x.$$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. On every bounded subset of **R**, the exponential series :

$$\Sigma u_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

(a) converges uniformly (b) converges non-uniformly
(c) diverges uniformly (d) none of these.

2. For all real values of x, the series $\cos x + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots$

(a) converges uniformly (b) diverges uniformly
(c) converges non uniformly (d) none of these.

3. The series $\Sigma \frac{x}{n(1+nx^2)}$:

(a) converges uniformly (b) diverges uniformly
(c) converges non uniformly (d) none of these.

4. The sum function $f(x)$, for which $f_n(x) = nx(1-x)^n, 0 \le x \le 1$, is

(a) not continuous (b) continuous
(c) uniformly convergent (d) none of these.

5. $\lim\limits_{x \to 1} \sum\limits_{n=1}^{\infty} \frac{nx^2}{n^3 + x^3} =$

(a) $\sum\limits_{n=1}^{\infty} \frac{1}{n^3+1}$ (b) $\sum\limits_{n=1}^{\infty} \frac{n}{n^3+1}$

(c) $\sum\limits_{n=1}^{\infty} \frac{1}{2n^3}$ (d) none of these.

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. $\Sigma\, a_n n^{-x}$ is uniformly convergent on $[0,1]$ if Σa_n converges .... on $[0,1]$.

2. The sequence $< f_n >$ defined by $f_n(x) = \frac{x}{1+nx}, 0 \le x < \infty$ is uniformly convergent to .... on $[0, \infty[$.

3. $\sum\limits_{n=1}^{\infty} \frac{1}{n^2+x^2}, 0 \le x < \infty$ is .... convergent on $[0, \infty[$.

4. The series $\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x +$ ....
converges .... in $0 < a \le x \le b < 2\pi$.

5. The series $\sum_{n=1}^{\infty} \frac{1}{1+n^2 x}$ is .... convergent in $[1, \infty[$.

6. Uniform convergence is only a .... but not a .... condition for the continuity of the sum function.

True or False

Write 'T' for true and 'F' for false statement.

1. The sequence $< f_n >$ is said to converge uniformly to the function f on an interval I if for every $\in > 0$, these exists a positive integer m such that

$$| f_n (x) - f (x) | > \in \text{ for all } n \geq m \text{ and for all } x \in I.$$

2. 0 is a point of uniform convergence of the sequence $< f_n >$ where $f_n (x) = 1 - (1 - x^2)^n$.

3. The series $\sum_{n=0}^{\infty} xe^{-nx}$ is non uniformly convergent in $[0, 1]$.

4. 0 is a point of non uniform convergence of the sequence $< f_n(x) >$ where

$$f_n (x) = nxe^{-nx^2}, \quad x \in R.$$

5. $\sum_{n=1}^{\infty} \frac{x}{n(n+1)}$ is uniformly convergent on $]0, \infty[$.

6. $\int_0^1 \left(\sum_1^{\infty} \frac{x^n}{n^2} \right) dx = \sum_1^{\infty} \frac{1}{n^2}$.

7. The series, for which $f_n (x) = \frac{nx}{1 + n^2 x^2}, 0 \leq x \leq 1$.

can not be differentiated term by term at $x = 0$.

8. If $f (x) = \sum_1^{\infty} \frac{1}{n^3 + n^4 x^2}$, $f'(x) = -2x \sum_1^{\infty} \frac{1}{n^2 (1 + nx^2)^2}$ for all x.

9. The series, the sum of whose first n terms is $n^2 x (1 - x)^n, 0 \leq x \leq 1$, can be integrated term by term.

Answers

Multiple Choice Questions

1. (a)	2. (a)	3. (a)	4. (b)	5. (b)

Fill in the Blank(s)

1. uniformly	2. 0	3. uniformly	4. uniformly
5. uniformly	6. sufficient, necessary		

True or False

1. *F*	2. *F*	3. *T*	4. *T*	5. *F*
6. *F*	7. *T*	8. *T*	9. *F*	

□

Chapter 5

Infinite Series

1 Infinite Series

An expression of the form $u_1 + u_2 + \ldots + u_n + \ldots$ *in which every term is followed by another according to some definite law is called a* **series.**

The series is called a **finite series**, if the number of terms is *finite*. Symbolically, the finite series $u_1 + u_2 + \ldots + u_n$ having n terms is denoted by $\sum_{r=1}^{n} u_r$.

The series is called an infinite series, if the number of terms is infinite. Symbolically, the infinite series $u_1 + u_2 + \ldots + u_n + \ldots$ is denoted by $\sum_{n=1}^{\infty} u_n$ or simply by $\Sigma\, u_n$.

Since we are going to deal with infinite series only, therefore we shall simply use the term '**series**' *to denote an infinite series.*

2 Convergence and Divergence of Series

Convergent Series : **(Gorakhpur 2011; Kashi 14)**

A series Σu_n is said to be convergent if S_n, the sum of its first n terms, tends to a definite finite limit S as n tends to infinity.

We write $S = \lim_{n \to \infty} S_n$.

The finite limit S to which S_n tends is called the sum of the series.

Divergent Series: *A series Σu_n is said to be divergent if S_n, the sum of its first n terms, tends to either $+\infty$ or $-\infty$ as n tends to infinity,*

i.e., if $\lim_{n \to \infty} S_n = \infty$ *or* $-\infty$.

Oscillatory Series: *A series Σu_n is said to be an oscillatory series if S_n, the sum of its first n terms, neither tends to a definite finite limit nor to $+\infty$ or $-\infty$ as n tends to ∞.*

The series is said to *oscillate finitely,* if the value of S_n as $n \to \infty$ fluctuates within a finite range. It is said to *oscillate infinitely,* if S_n tends to infinity and its sign is alternately positive and negative.

Sequence of Partial Sums of a Series :

If S_n denotes the sum of the first n terms of the series Σu_n, so that

$$S_n = u_1 + u_2 + \ldots + u_n,$$

then S_n is called the **partial sum** of the first n terms of the series and the sequence $< S_n > = < S_1, S_2, \ldots, S_n, \ldots >$ is called the **sequence of partial sums** of the given series. We can define the convergent, divergent and oscillatory series in terms of the sequence of partial sums.

Definition: *A series Σu_n is said to be convergent, divergent or oscillatory according as the sequence $< S_n >$ of its partial sums is convergent, divergent or oscillatory.*

If the sequence $< S_n >$ of partial sums of a series Σu_n converges to S then S is said to be the sum of the series Σu_n.

Note: Since the limits for infinite series will be taken as $n \to \infty$, so throughout this chapter we shall write $\lim_{n \to \infty}$ as 'lim' only.

Illustration 1:

The series $1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots + \left(\frac{2}{3}\right)^{n-1} + \ldots$ is convergent.

Here the given series is a geometric series with common ratio $2/3 < 1$.

$$\therefore \quad S_n = \frac{1.\{1-(2/3)^n\}}{1-(2/3)} = 3\{1-(2/3)^n\}.$$

Now, $\lim S_n = \lim 3\{1-(2/3)^n\} = 3(1-0)$ $[\because 2/3 < 1]$

$= 3$, a definite finite number.

Consequently the given series is convergent.

Illustration 2:

The series $1 + 2 + 3 + \ldots + n + \ldots$ is divergent.

Here, $$S_n = 1 + 2 + 3 + \ldots + n = \frac{1}{2} n (n+1).$$

$\therefore$ $$\lim S_n = \lim \frac{1}{2} n (n+1) = \infty.$$

Consequently the given series is divergent.

Illustration 3:

The series $2 - 2 + 2 - 2 + \ldots$ is oscillatory.

Here, $$S_n = 0 \text{ if } n \text{ is even,}$$
$$= 2, \text{ if } n \text{ is odd.}$$

Therefore, the sequence $< S_n >$ of partial sums of the series, and consequently the given series, is oscillatory.

Below we give some results which will be found useful and can be easily proved.

1. The nature of a series remains unaltered if
   (i) the signs of all the terms are changed;
   (ii) a finite number of terms are added or omitted;
   (iii) each term of the series is multiplied or divided by the same fixed number c which is not zero.
2. If $\Sigma\, u_n$ converges to A and $\Sigma\, v_n$ converges to B, then $\Sigma\,(u_n + v_n)$ converges to $A + B$.
3. If $\Sigma\, u_n$ converges to A and $c \in \mathbf{R}$, then $\Sigma\, c\, u_n$ converges to $c\,A$.
4. If $\Sigma\, u_n$ converges to A and $\Sigma\, v_n$ converges to B and $p, q \in \mathbf{R}$, then $\Sigma\,(pu_n + qv_n)$ converges to $pA + qB$.
5. If $\Sigma\, u_n$ diverges and $c \in \mathbf{R}, c \neq 0$, then $\Sigma\, cu_n$ diverges.
6. If $\Sigma\, u_n$ and $\Sigma\, v_n$ are two divergent series having all terms positive, then $\Sigma\,(u_n + v_n)$ also diverges.

3 A Necessary Condition for Convergence

For a series $\Sigma\, u_n$ to be convergent, it is necessary that $\lim u_n = 0$.

Or *For every convergent series $\Sigma\, u_n$, we must have* $\lim u_n = 0$. **(Gorakhpur 2010, 13)**

Let the series $\Sigma\, u_n$ be convergent. Let S_n denote the sum of n terms of the series $\Sigma\, u_n$.

Then $$S_n = u_1 + u_2 + \ldots + u_n \text{ and } S_{n-1} = u_1 + u_2 + \ldots + u_{n-1}.$$

$\therefore$ $$u_n = S_n - S_{n-1}. \qquad \ldots(1)$$

Since the series $\Sigma\, u_n$ is convergent, therefore, S_n and S_{n-1} both will tend to the same finite limit, say S, as $n \to \infty$.

Taking limits of both sides of (1), we get

$$\lim u_n = \lim S_n - \lim S_{n-1} = S - S = 0.$$

Hence for a convergent series, it is necessary that $\lim u_n = 0$.

Note: It is to be noted that the above condition is only necessary but not sufficient for a series to be convergent *i.e.,* if $\lim u_n = 0$, then the series Σu_n may or may not be convergent. **(Gorakhpur 2010, 13)**

For example, consider the series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots + \frac{1}{\sqrt{n}} + \ldots$$

Here $u_n = \dfrac{1}{\sqrt{n}}$, so that $\lim u_n = \lim \dfrac{1}{\sqrt{n}} = 0$. But the series does not converge as shown below.

We have $S_n = 1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \ldots + \dfrac{1}{\sqrt{n}} > \dfrac{1}{\sqrt{n}} + \dfrac{1}{\sqrt{n}} + \dfrac{1}{\sqrt{n}} + \ldots + \dfrac{1}{\sqrt{n}} = \dfrac{n}{\sqrt{n}} = \sqrt{n}$,

i.e., $S_n > \sqrt{n}$, which tends to infinity as n tends to infinity. Hence the series is divergent.

Again consider the geometric series $\dfrac{1}{2} + \left(\dfrac{1}{2}\right)^2 + \left(\dfrac{1}{2}\right)^3 + \ldots$, for which $\lim u_n = \lim \left(\dfrac{1}{2}\right)^n = 0$ and the series is convergent.

Thus if $u_n \to 0$, we cannot say anything about the behaviour of the series but if u_n does not tend to zero, the series definitely does not converge. The more useful form of the above test is as follows:

If a series Σu_n be such that u_n does not tend to zero as n tends to infinity, then the series does not converge.

4 Cauchy's General Principle of Convergence for Series

Sometimes it is either impossible or difficult to find the sequence of partial sums of a given series and yet we want to know whether the series converges or not. Now we shall establish a fundamental principle, for dealing with the convergence of such series, known as *Cauchy's general principle of convergence.*

Theorem: *A necessary and sufficient condition for a series Σu_n to converge is that for each $\varepsilon > 0$, there exists a positive integer m, such that*

$$|u_{m+1} + u_{m+2} + \ldots + u_n| < \varepsilon \text{ for all } n > m$$

Or $\quad |u_{p+1} + u_{p+2} + \ldots + u_q| < \varepsilon$ *for all* $q \geq p \geq m$

Or $\quad |u_{n+1} + u_{n+2} + \ldots + u_{n+p}| < \varepsilon$ *for all* $n \geq m, p > 0$.

Proof: Let $< S_n >$ be the sequence of partial sums of the series Σu_n. The series Σu_n will converge, iff the sequence $< S_n >$ of its partial sums converges. By Cauchy's general principle of convergence for sequences, we know that a necessary and sufficient condition for the convergence of $< S_n >$ is that for each $\varepsilon > 0$, there exists $m \in \mathbf{N}$ such that

$$|S_n - S_m| < \varepsilon \text{ for all } n > m$$

i.e., $\quad |u_{m+1} + u_{m+2} + \ldots + u_n| < \varepsilon$ for all $n > m$.

Hence the result.

Illustrative Examples

Example 1: *Discuss the convergence of a geometric series.* **(Gorakhpur 2015)**

Solution: Consider the geometric series

$$a + ax + ax^2 + ax^3 + \ldots + ax^{n-1} + \ldots. \qquad \ldots(1)$$

Let S_n be the sum of first n terms of the series (1).

$\therefore \quad S_n = \dfrac{a(1-x^n)}{1-x}$ if $x < 1$ and $S_n = \dfrac{a(x^n - 1)}{x-1}$ if $x > 1$.

Case I: When $|x| < 1$ *i.e.,* $-1 < x < 1$.

If $|x| < 1$, then $x^n \to 0$ as $n \to \infty$.

$$\therefore \quad \lim S_n = \lim \frac{a(1-x^n)}{1-x} = \frac{a(1-0)}{1-x} = \frac{a}{1-x},$$

which is a definite finite number and therefore the series is convergent.

Case II: When $x = 1$.

If $x = 1$, then each term of the series (1) is a.

$\therefore \quad S_n = a + a + \ldots$ to n terms $= na$.

$\therefore \quad \lim S_n = \infty$ or $-\infty$ according as a is positive or negative. Hence the series is divergent.

Case III: When $x > 1$.

If $x > 1$, then $x^n \to \infty$ as $n \to \infty$.

$$\therefore \quad \lim S_n = \lim \frac{a(x^n - 1)}{x - 1} = \infty \text{ or } -\infty \text{ according as } a > \text{ or } < 0.$$

Hence the series is divergent.

Case IV: When $x = -1$.

If $x = -1$, then the series (1) becomes $a - a + a - a + \ldots$.

The sum of n terms of the series is a or 0 according as n is odd or even.

Hence the series is an oscillatory series, the oscillation being finite.

Case V: When $x < -1$.

If $x < -1$, then $-x > 1$.

Let $r = -x$, then $r > 1$ and so $r^n \to \infty$ as $n \to \infty$.

Now $\quad S_n = \dfrac{a(1-x^n)}{1-x} = \dfrac{a\{1-(-r)^n\}}{1-(-r)} \qquad [\because x = -r]$

$$= \frac{a(1+r^n)}{1+r} \quad \text{or} \quad \frac{a(1-r^n)}{1+r}, \text{ according as } n \text{ is odd or even.}$$

$\therefore$ in this case $\lim S_n$ is ∞ or $-\infty$ according as n is odd or even, provided $a > 0$ and if $a < 0$ the results are reversed.

Therefore in this case the series is an oscillatory series, the oscillation being infinite.

Hence a geometric series whose common ratio is x is convergent if $|x| < 1$, divergent if $x \geq 1$ and oscillatory if $x \leq -1$.

Example 2: *Prove that the series* $\Sigma \dfrac{1}{4^n}$ *converges to* $\dfrac{1}{3}\cdot$

Solution: Here $S_n = \dfrac{1}{4} + \dfrac{1}{4^2} + \dfrac{1}{4^3} + \ldots + \dfrac{1}{4^n} = \dfrac{\dfrac{1}{4}\left\{1 - \left(\dfrac{1}{4}\right)^n\right\}}{1 - \dfrac{1}{4}} = \dfrac{1}{3}\left(1 - \dfrac{1}{4^n}\right)\cdot$

$$\therefore \qquad \lim S_n = \lim \frac{1}{3}\left(1 - \frac{1}{4^n}\right) = \frac{1}{3}\cdot \qquad \left[\because \ \lim \frac{1}{4^n} = 0\right]$$

$\therefore$ the sequence $< S_n >$ converges to $\dfrac{1}{3}$ and hence $\Sigma\, u_n$ converges to $\dfrac{1}{3}\cdot$

Example 3: *Test the convergence of the series*

$$log_e\, 2 + log_e \frac{3}{2} + log_e \frac{4}{3} + log_e \frac{5}{4} + \ldots.$$

Solution: Here, $S_n = \log_e 2 + \log_e \dfrac{3}{2} + \log_e \dfrac{4}{3} + \ldots + \log_e \left(\dfrac{n+1}{n}\right)$

$$= \log_e \left\{2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n+1}{n}\right\} = \log_e (n+1).$$

$$\therefore \qquad \lim S_n = \lim \log (n+1) = \log \infty = \infty.$$

Hence the given series is divergent.

Example 4: *Show that the series*

$$\sqrt{\left(\frac{1}{4}\right)} + \sqrt{\left(\frac{2}{6}\right)} + \ldots + \sqrt{\left[\frac{n}{2\,(n+1)}\right]} + \ldots$$

does not converge.

Solution: Here,

$$u_n = \sqrt{\left[\frac{n}{2\,(n+1)}\right]} = \frac{1}{\sqrt{2}} \sqrt{\left(\frac{n}{n+1}\right)} = \frac{1}{\sqrt{2}} \cdot \left[\frac{1}{1 + (1/n)}\right]^{1/2}.$$

$$\therefore \qquad \lim_{n \to \infty} u_n = \frac{1}{\sqrt{2}} \neq 0.$$

Hence the given series does not converge.

Example 5: *Show that the series* $\Sigma \dfrac{1}{n}$ *does not converge.*

Solution: Let the given series converge. Then for $\varepsilon = \dfrac{1}{4}$, by Cauchy's general principle of convergence, we can find a positive integer m such that

$$\frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{n} < \frac{1}{4} \text{ for all } n > m.$$

Taking $n = 2m$, we see that

$$\frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{n} = \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{2m}$$

$$> m \cdot \frac{1}{2m} = \frac{1}{2}.$$

Thus we get a contradiction. Hence the given series does not converge.

5 Series of Positive Terms (or Series of Non-negative Terms)

If Σu_n is a series of positive terms then $u_n > 0$ for all $n \in \mathbf{N}$.

The important aspect of this series is that its sequence of partial sums is increasing.

We have $S_n = u_1 + u_2 + \ldots + u_n$, then $S_n - S_{n-1} = u_n$.

Since $u_n > 0$ for all n, therefore we get $S_n - S_{n-1} > 0$ for all n, *i.e.*, $S_n > S_{n-1}$ for all n *i.e.*, the sequence $< S_n >$ is a monotonically increasing sequence.

Now a monotonic sequence can either converge or diverge but cannot oscillate. Hence, we have only two possibilities for a series of positive terms, either the series converges or it diverges.

We give some fundamental results for series of positive terms.

Theorem 1: *A series Σu_n of positive terms converges iff there exists a number K such that $u_1 + u_2 + \ldots + u_n < K$ for all n.*

Proof: First, suppose that there exists a number K such that

$$u_1 + u_2 + \ldots + u_n < K, \ \forall n \quad i.e., \quad S_n < K, \ \forall \ n.$$

This shows that the sequence $< S_n >$ of partial sums of the series Σu_n is bounded above. Also, the sequence $< S_n >$ is an increasing sequence, since the series Σu_n is of positive terms. We know that every bounded monotonic sequence converges. Therefore $< S_n >$ converges and hence Σu_n converges.

Conversely, we assume that Σu_n converges. Then, the sequence $< S_n >$ of partial sums of the series converges. We know that every convergent sequence is bounded. Therefore $< S_n >$ is bounded and hence there exist real numbers k and K such that $k < S_n < K$, for all n.

It gives $S_n < K$ *i.e.*, $u_1 + u_2 + \ldots + u_n < K$, for all n.

Note: In the light of the above theorem, we conclude that to show that a series of positive terms converges, it is sufficient to show that the sequence of its partial sums is bounded. On the other hand, to show that a series of positive terms diverges, we have to show that the sequence of its partial sums is not bounded, *i.e.*, for any real number A, there exists a positive integer m such that $S_m > A$.

Theorem 2: *A series of positive terms is divergent if each term after a fixed stage is greater than some fixed positive number.*

Proof: Let each term of the series be greater than a fixed positive number. We can assume so because the convergence or divergence of the series is not affected by omitting a finite number of terms.

So let Σu_n be the given series of positive terms and let $u_n > k$ (a fixed positive number) for all n.

Now $\quad S_n = u_1 + u_2 + \ldots + u_n > nk.$

But $\quad \lim nk = \infty.$

$\therefore \quad \lim S_n = \infty.$

Hence the series Σu_n is divergent.

Corollary: *A series of positive terms is divergent if* $\lim u_n > 0$.

Proof: Let $\lim u_n = l$, where $l > 0$. Then for a given $\varepsilon > 0$, there exists a positive integer m such that

$$|u_n - l| < \varepsilon, \text{ for all } n \geq m$$

i.e.,
$$l - \varepsilon < u_n < l + \varepsilon, \text{ for all } n \geq m.$$

Let $l - \varepsilon = a$. Then a is a fixed positive number because ε can be taken as small as we please. For example take $\varepsilon = \frac{1}{2} l$.

Thus $u_n > a$ for all $n \geq m$. Hence the given series is divergent.

Theorem 3: *If each term of a series Σu_n of positive terms, does not exceed the corresponding term of a convergent series Σv_n of positive terms, then Σu_n is convergent.*

While, if each term of Σu_n exceeds (or equals) the corresponding term of a divergent series of positive terms, then Σu_n is divergent.

Proof: Let $u_n \leq v_n$ for all n.

Let S_n and S_n' be the sums of first n terms of the two series Σu_n and Σv_n respectively.

Then $S_n = u_1 + u_2 + \ldots + u_n$ and $S_n' = v_1 + v_2 + \ldots + v_n$.

Since $u_n \leq v_n \;\; \forall \; n$, therefore, $S_n \leq S_n'$.

But Σv_n is convergent, therefore $S_n' \to S'$ (a finite quantity) as $n \to \infty$.

$\therefore \quad \lim S_n \leq S'$ (a finite quantity).

$\therefore \quad S_n$ itself tends to a finite limit as $n \to \infty$.

Hence the series Σu_n is convergent.

Now if $u_n \geq v_n$, for all n, then $S_n \geq S_n'$.

But Σv_n is divergent, therefore $S_n' \to \infty$ as $n \to \infty$ and hence $S_n \to \infty$ as $n \to \infty$. Consequently Σu_n is divergent.

6 The Auxiliary Series $\Sigma 1/n^p$

The infinite series

$$\Sigma \frac{1}{n^p} \quad i.e., \quad \frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}+\ldots$$

is convergent if $p>1$ *and divergent if* $p \leq 1$. **(Kumaun 2001; Avadh 05; Kanpur 07; Kashi 13; Rohilkhand 14; Agra 14)**

Proof: Case I: Let $p>1$. Since the terms of the given series are all positive, we can group them as we like. Hence we write the given series

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots=\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)$$
$$+\left(\frac{1}{8^p}+\frac{1}{9^p}+\ldots+\frac{1}{15^p}\right)+\ldots \qquad \ldots(1)$$

Now since $p>1$,

$\therefore \qquad 3>2 \Rightarrow 3^p>2^p \Rightarrow 1/3^p<1/2^p.$

$\therefore \qquad \dfrac{1}{2^p}+\dfrac{1}{3^p}<\dfrac{1}{2^p}+\dfrac{1}{2^p}$

or $\qquad \dfrac{1}{2^p}+\dfrac{1}{3^p}<\dfrac{2}{2^p}.$

Similarly $\dfrac{1}{4^p}+\dfrac{1}{5^p}+\dfrac{1}{6^p}+\dfrac{1}{7^p}<\dfrac{1}{4^p}+\dfrac{1}{4^p}+\dfrac{1}{4^p}+\dfrac{1}{4^p}=\dfrac{4}{4^p},$

$\qquad \dfrac{1}{8^p}+\dfrac{1}{9^p}+\ldots+\dfrac{1}{15^p}<\dfrac{8}{8^p},$

and so on.

Thus we observe that on being grouped as mentioned in (1), the given series is term by term

$$<\frac{1}{1^p}+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\ldots$$

But the series on the R.H.S. of the above inequality is a geometric series and is convergent since its common ratio is $2/2^p=1/2^{p-1}$ which is less than 1 as $p>1$. Thus the given series on being grouped as in (1) is term by term less than a convergent series. Consequently the given series is convergent when $p>1$.

Case II: Let $p=1$. Then we group terms of the given series as

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$$
$$=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\ldots+\frac{1}{16}\right)+\ldots \qquad \ldots(2)$$

Now as $3<4$, so $\dfrac{1}{3}>\dfrac{1}{4}$ or $\dfrac{1}{3}+\dfrac{1}{4}>\dfrac{1}{4}+\dfrac{1}{4}$

or $\qquad \dfrac{1}{3}+\dfrac{1}{4}>\dfrac{2}{4} \quad i.e., \quad \dfrac{1}{3}+\dfrac{1}{4}>\dfrac{1}{2}.$

Similarly, $\quad \dfrac{1}{5}+\dfrac{1}{6}+\dfrac{1}{7}+\dfrac{1}{8}>\dfrac{1}{8}+\dfrac{1}{8}+\dfrac{1}{8}+\dfrac{1}{8}=\dfrac{4}{8}=\dfrac{1}{2}$,

$\dfrac{1}{9}+\dfrac{1}{10}+\ldots+\dfrac{1}{16}>\dfrac{1}{2}$, and so on.

Thus we observe that on being grouped as in (2), the given series is term by term

$$>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots. \qquad \ldots(3)$$

The series on the R.H.S. of (3) is divergent as the sum of the first n terms of the series

$$=1+(n-1)\cdot\frac{1}{2}=\frac{1}{2}(n+1), \text{ which tends to infinity as } n\to\infty.$$

Thus the given series on being grouped as in (2) is term by term greater than a divergent series.

Consequently the given series is divergent when $p=1$.

Case III: Let $p<1$. Then

$$\frac{1}{n^p}>\frac{1}{n} \text{ for } n=2,3,4,\ldots.$$

In this case the given series

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots$$

is term by term greater than the series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots,$$

which is a divergent series, as proved in case II.

Consequently the given series is divergent when $p<1$.

Hence the proof is complete.

Now we shall give some tests to know whether the given series of positive terms is convergent or divergent without actually finding out the sum of its n terms.

7 Comparison Test

Theorem: First form: *Let $\Sigma\, u_n$ and $\Sigma\, v_n$ be two series of positive terms such that $u_n<Kv_n$ for all n, where K is a fixed positive number. Then if $\Sigma\, v_n$ converges, so does $\Sigma\, u_n$, and if $\Sigma\, u_n$ diverges, then $\Sigma\, v_n$ also diverges.* **(Gorakhpur 2010)**

Proof: Since $u_n<Kv_n$ for all n,

$$\therefore \quad u_1+u_2+\ldots+u_n<K(v_1+v_2+\ldots+v_n),\ \forall\ n. \qquad \ldots(1)$$

Now if $\Sigma\, v_n$ converges, then there must exist a positive real number A, such that

$$v_1+v_2+\ldots+v_n<A,\ \forall\ n. \qquad \ldots(2)$$

From (1) and (2), we get

$$u_1+u_2+\ldots+u_n<K\,A,\ \forall\ n.$$

Thus the sequence of partial sums of the series $\Sigma\, u_n$ is bounded above and hence $\Sigma\, u_n$ converges.

To prove the other result, we assume that $\Sigma\, u_n$ diverges. Then for any positive real number B, there must exist a positive integer m such that

$$u_1 + u_2 + \ldots + u_n > BK, \text{ for all } n > m. \qquad \ldots(3)$$

From (1) and (3), we get

$$v_1 + v_2 + \ldots + v_n > B, \text{ for all } n > m.$$

Hence the series $\Sigma\, v_n$ diverges.

Second form: *Let $\Sigma\, u_n$ and $\Sigma\, v_n$ be two series of positive terms and let k and K be positive real numbers such that*

$$kv_n < u_n < Kv_n, \text{ for all } n.$$

Then the series $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Proof: From $kv_n < u_n < Kv_n$, for all n, we get

$$kv_n < u_n \quad \text{or} \quad v_n < \left(\frac{1}{k}\right) u_n, \text{ for all } n.$$

Now applying the result proved in the first form of the comparison test, we conclude that

(i) if $\Sigma\, u_n$ converges, then $\Sigma\, v_n$ also converges.

(ii) if $\Sigma\, v_n$ diverges, then $\Sigma\, u_n$ also diverges.

Again, applying the result of the first form of the comparison test for the inequality $u_n < Kv_n$, we conclude that

(iii) if $\Sigma\, v_n$ converges, then $\Sigma\, u_n$ also converges.

(iv) if $\Sigma\, u_n$ diverges, then $\Sigma\, v_n$ also diverges.

The desired result now follows from (i), (ii), (iii) and (iv).

Third form: *Let $\Sigma\, u_n$ and $\Sigma\, v_n$ be two series of positive terms and let K be a positive number such that $u_n < Kv_n$ for all $n > m$, m being a fixed positive integer. Then if the series $\Sigma\, v_n$ be convergent, then the series $\Sigma\, u_n$ is also convergent and if the series $\Sigma\, u_n$ is divergent, then the series, $\Sigma\, v_n$ is also divergent.*

Proof: The above result follows from the result of the first form of the comparison test because the convergence or the divergence of a series remains unaffected by omitting a finite number of terms of the series.

Fourth form: *Let $\Sigma\, u_n$ and $\Sigma\, v_n$ be two series of positive terms and let k and K be positive real numbers such that $kv_n < u_n < Kv_n$ for all $n > m$, m being a fixed positive integer. Then the series $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.*

Proof: Since the omission of a finite number of terms of a series has no effect on its convergence or divergence, therefore,

(i) the series $u_1 + u_2 + \ldots$ and the series $u_{m+1} + u_{m+2} + \ldots$ converge or diverge together ; and

(ii) the series $v_1 + v_2 + \ldots$ and the series $v_{m+1} + v_{m+2} + \ldots$ converge or diverge together.

Again, $kv_n < u_n < Kv_n$ for all $n > m \;\Rightarrow\; kv_{m+p} < u_{m+p} < Kv_{m+p}$ for all $p \in \mathbf{N}$, therefore, by the result of the second form of the comparison test, we have

(iii) the series $u_{m+1} + u_{m+2} + \ldots$ and the series $v_{m+1} + v_{m+2} + \ldots$ converge or diverge together.

Hence from (i), (ii) and (iii), we conclude that the series $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Fifth form: (Important from the point of view of application to the solution of problems): *Let $\Sigma\, u_n$ and $\Sigma\, v_n$ be two series of positive terms such that*

$$\lim_{n \to \infty} \frac{u_n}{v_n} = l \ \textbf{(finite and non-zero)};$$

then both the series converge or diverge together i.e., the two series $\Sigma\, u_n$ and $\Sigma\, v_n$ are either both convergent or both divergent.

Proof: We have $\dfrac{u_n}{v_n} > 0$ for all n, therefore

$$\lim_{n \to \infty} \frac{u_n}{v_n} \geq 0 \quad i.e., \quad l \geq 0.$$

Since $l \neq 0$ (given), therefore, $l > 0$.

Choose $\varepsilon > 0$ in such a way that $l - \varepsilon > 0$.

Now $\displaystyle\lim_{n \to \infty} \frac{u_n}{v_n} = l \Rightarrow$ there exists $m \in \mathbf{N}$ such that

$$l - \varepsilon < \frac{u_n}{v_n} < l + \varepsilon, \text{ for all } n > m. \qquad \ldots(1)$$

Since $v_n > 0 \ \forall\ n$, hence multiplying (1) throughout by v_n, we get

$$(l - \varepsilon)\, v_n < u_n < (l + \varepsilon)\, v_n, \text{ for all } n > m. \qquad \ldots(2)$$

Now if $\Sigma\, v_n$ is convergent then $\Sigma\, (l + \varepsilon)\, v_n$ is also convergent. In this case from (2), we see that $\Sigma\, u_n$ is term by term less than a convergent series $\Sigma\, (l + \varepsilon)\, v_n$ except possibly for a finite number of terms. Therefore the series $\Sigma\, u_n$ is also convergent.

Again if $\Sigma\, v_n$ is divergent then $\Sigma\, (l - \varepsilon)\, v_n$ is also divergent. In this case from (2), we see that $\Sigma\, u_n$ is term by term greater than a divergent series $\Sigma\, (l - \varepsilon)\, v_n$ except possibly for a finite number of terms. Therefore the series $\Sigma\, u_n$ is also divergent.

Hence the series $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together.

Sixth form: *Let $\Sigma\, u_n$ and $\Sigma\, v_n$ be two series of positive terms such that*

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}, \textit{for all } n \geq m.$$

Then $\Sigma\, v_n$ converges $\Rightarrow \Sigma\, u_n$ converges and $\Sigma\, u_n$ diverges $\Rightarrow \Sigma\, v_n$ diverges.

Proof: We have $\dfrac{u_n}{u_{n+1}} > \dfrac{v_n}{v_{n+1}}$, for all $n \geq m$.

Putting $n = m + 1, m + 2, \ldots, n - 1$ in the above inequality, we get

$$\frac{u_{m+1}}{u_{m+2}} > \frac{v_{m+1}}{v_{m+2}}, \quad \frac{u_{m+2}}{u_{m+3}} > \frac{v_{m+2}}{v_{m+3}}, \ \ldots, \ \frac{u_{n-1}}{u_n} > \frac{v_{n-1}}{v_n}.$$

Multiplying the corresponding sides of these inequalities, we get

$$\frac{u_{m+1}}{u_n} > \frac{v_{m+1}}{v_n}, \text{ for all } n > m,$$

i.e.,

$$u_n < \left(\frac{u_{m+1}}{v_{m+1}}\right) v_n, \text{ for all } n > m.$$

Now the result follows from the third form.

Note 1: From the point of view of applications, the third and the fifth forms of the comparison test are the most useful.

Note 2: The geometric series $\Sigma \frac{1}{r^n}$ and the auxiliary series $\Sigma \frac{1}{n^p}$ will play a prominent role for comparison.

Working rule for applying comparison test:

The v_n-method: Comparison test is usually applied when the nth term u_n of the given series $\Sigma\, u_n$ contains the powers of n only which may be positive or negative, integral or fractional. The auxiliary series $\Sigma\,(1/n^p)$ is chosen as the series $\Sigma\, v_n$. From article 6, we know that $\Sigma\,(1/n^p)$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Now the question arises that how to choose v_n ? For applying comparison test, it is necessary that $\lim \frac{u_n}{v_n}$ should be finite and non-zero. It will be so if we take $v_n = \frac{1}{n^{p-q}}$, where p and q are respectively the highest indices of n in the denominator and numerator of u_n when it is in the form of a fraction. If u_n, can be expanded in ascending powers of $1/n$, then to get v_n, we should retain only the lowest power of $1/n$. After making a proper choice of v_n, we find $\lim\,(u_n / v_n)$ which should come out to be finite and non-zero. Then the series $\Sigma\, u_n$ and $\Sigma\, v_n$ are either both convergent or both divergent. The whole procedure will be clear from the examples that follow article 8.

Illustrative Examples

Example 6: *Test for convergence the series*

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \ldots + \frac{1}{n^n} + \ldots\ldots$$

Solution: Since $n^n > 2^n$ for all $n > 2$, therefore, $\frac{1}{n^n} < \frac{1}{2^n}$.

Here $\quad u_n = \frac{1}{n^n}$. Let $v_n = \frac{1}{2^n}$.

Since $u_n < v_n$ for all $n > 2$ and $\Sigma\, v_n$ is a convergent series (a geometric series with common ratio $\frac{1}{2}$), therefore, by the comparison test, the given series converges.

Example 7: *Test for convergence the series whose nth terms are*

(i) $\dfrac{\sqrt{n}}{n^2+1}$ **(Kumaun 2002; Kanpur 06; Meerut 13B; Agra 14)**

(ii) $\dfrac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}}$ **(Kanpur 2009; Meerut 13)**

(iii) $\dfrac{n^p}{(1+n)^q}\cdot$

Solution: (i) Here $u_n = \dfrac{\sqrt{n}}{n^2+1}\cdot$

Take $$v_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}},$$

i.e., the auxiliary series is $\Sigma\, v_n = \Sigma \dfrac{1}{n^{3/2}}\cdot$

Now $$\lim \frac{u_n}{v_n} = \lim \left\{\frac{\sqrt{n}}{n^2+1} \,.\, n^{3/2}\right\} = \lim \frac{n^2}{n^2+1}$$

$$= \lim \frac{1}{1+(1/n^2)} = 1, \text{which is finite and non-zero.}$$

Since the auxiliary series $\Sigma\, v_n = \Sigma\,(1/n^{3/2})$ is convergent $\left(p = \dfrac{3}{2} > 1\right)$, therefore, by comparison test the given series $\Sigma\, u_n$ is also convergent.

(ii) Here $$u_n = \frac{(2n^2-1)^{1/3}}{(3n^3+2n+5)^{1/4}} = \frac{n^{2/3}\,(2-1/n^2)^{1/3}}{n^{3/4}\,(3+2/n^2+5/n^3)^{1/4}}$$

$$= \frac{1}{n^{1/12}} \cdot \frac{(2-1/n^2)^{1/3}}{(3+2/n^2+5/n^3)^{1/4}}\cdot$$

Take $$v_n = \frac{1}{n^{1/12}}\cdot$$

Then $$\frac{u_n}{v_n} = \frac{(2-1/n^2)^{1/3}}{(3+2/n^2+5/n^3)^{1/4}}\cdot$$

$\therefore$ $$\lim \frac{u_n}{v_n} = \frac{2^{1/3}}{3^{1/4}}, \text{which is finite and non-zero.}$$

Hence, by comparison test, $\Sigma\, u_n$ and $\Sigma\, v_n$ are either both convergent or both divergent. But the auxiliary series $\Sigma\, v_n$ is divergent because $p = 1/12 < 1$. Hence $\Sigma\, u_n$ is also divergent.

(iii) Here $$u_n = \frac{n^p}{(n+1)^q}\cdot$$

Take $$v_n = \frac{n^p}{n^q} = \frac{1}{n^{q-p}}\cdot$$

Now $$\lim \frac{u_n}{v_n} = \lim \left\{ \frac{n^p}{(n+1)^q} \cdot n^{q-p} \right\} = \lim \frac{1}{(1+1/n)^q} = 1,$$ which is finite and non-zero.

Therefore, by comparison test, Σu_n and Σv_n are either both convergent or both divergent.

But the auxiliary series $\Sigma v_n = \Sigma \dfrac{1}{n^{q-p}}$ is convergent if $q - p > 1$ *i.e.* if $p - q + 1 < 0$ and divergent if $q - p \le 1$ *i.e.* if $p - q + 1 \ge 0$.

Hence by comparison test the given series Σu_n is convergent if $p - q + 1 < 0$ and divergent if $p - q + 1 \ge 0$.

Example 8: *Test for convergence the series whose nth terms are*

(*i*) $\dfrac{1}{1 + 1/n}$ **(Avadh 2012)**

(*ii*) $\sin \dfrac{1}{n}$ **(Kanpur 2012; Gorakhpur 14)**

(*iii*) $\tan^{-1} \dfrac{1}{n}$. **(Kanpur 2008)**

Solution: (i) Here $u_n = \dfrac{1}{1 + 1/n}$.

We have, $$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{1 + (1/n)} = 1,$$ which is > 0.

$\therefore$ the given series is divergent.

(ii) Here, $$u_n = \sin \frac{1}{n} = \frac{1}{n} - \frac{1}{3!} \frac{1}{n^3} + \frac{1}{5!} \frac{1}{n^5} - \ldots\ldots$$

Take $v_n = 1/n$, since the lowest power of $1/n$ in u_n is $1/n$. The auxiliary series $\Sigma v_n = \Sigma (1/n)$ is divergent as here $p = 1$.

Now $$\lim \frac{u_n}{v_n} = \lim \left(1 - \frac{1}{3!} \cdot \frac{1}{n^2} + \frac{1}{5!} \cdot \frac{1}{n^4} - \ldots \right) = 1,$$

which is finite and non-zero.

Hence by comparison test the given series is divergent.

(iii) Here, $$u_n = \tan^{-1} \left(\frac{1}{n} \right) = \frac{1}{n} - \frac{1}{3n^3} + \frac{1}{5n^5} - \ldots \qquad \left[\because \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \right]$$

The lowest power of $1/n$ in u_n is $1/n$. Therefore, to apply the comparison test, the auxiliary series is taken as $\Sigma v_n = \Sigma (1/n)$.

Now, $$\lim \frac{u_n}{v_n} = \lim \left(1 - \frac{1}{3n^2} + \frac{1}{5n^4} - \ldots \right) = 1,$$ which is finite and non-zero.

But the auxiliary series $\Sigma v_n = \Sigma (1/n)$ is divergent as here $p = 1$.

Hence by comparison test the given series is divergent.

Example 9: *Test the convergence of the series*

$$\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\ldots\ldots$$

(Kumaun 2000; Avadh 10)

Solution: Here $u_n=\dfrac{n+1}{n^{\,p}}\cdot$ Take $v_n=\dfrac{n}{n^p}=\dfrac{1}{n^{p-1}}\cdot$

Now $\quad \lim \dfrac{u_n}{v_n}=\lim\left(\dfrac{n+1}{n^p}\,.\,n^{p-1}\right)=\lim\left(1+\dfrac{1}{n}\right)=1,$

which is finite and non-zero.

Hence by comparison test $\Sigma\, u_n$ and $\Sigma\, v_n$ are either both convergent or both divergent.

But the auxiliary series $\Sigma\, v_n=\Sigma\,\dfrac{1}{n^{p-1}}$ is convergent if $p-1>1$ *i.e.*, $p>2$, and divergent if $p-1\le 1$ *i.e.* if $p\le 2$.

Hence the given series $\Sigma\, u_n$ is convergent if $p>2$ and divergent if $p\le 2$.

Example 10: *Test the convergence of the following series*

(i) $\quad 1+\dfrac{1}{2^2}+\dfrac{2^2}{3^3}+\dfrac{3^3}{4^4}+\dfrac{4^4}{5^5}+\ldots$ **(Avadh 2014)**

(ii) $\quad \dfrac{1}{1+\sqrt{2}}+\dfrac{2}{1+2\sqrt{3}}+\dfrac{3}{1+3\sqrt{4}}+\ldots$

Solution: (i) Omitting the first term, if the given series is denoted by $\Sigma\, u_n$, then

$$\Sigma\, u_n=\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\ldots=\Sigma\,\frac{n^n}{(n+1)^{n+1}}\cdot$$

Here, $\quad u_n=\dfrac{n^n}{(n+1)^{n+1}}\cdot$ Take $v_n=\dfrac{n^n}{n^{n+1}}=\dfrac{1}{n}\cdot$

Now $\quad \lim\dfrac{u_n}{v_n}=\lim\left\{\dfrac{n^n}{(n+1)^{n+1}}\,.\,n\right\}$

$$=\lim\left\{\frac{1}{(1+1/n)^n\,.\,(1+1/n)}\right\}=\frac{1}{e},\qquad\left[\because\ \lim\left(1+\frac{1}{n}\right)^n=e\right]$$

which is finite and non-zero.

But the auxiliary series $\Sigma\, v_n=\Sigma\,(1/n)$ is divergent as here $p=1$. Hence by comparison test the given series is divergent.

(ii) Here, $\quad u_n=\dfrac{n}{1+n\sqrt{(n+1)}}\cdot$

Take $\quad v_n=\dfrac{n}{n\sqrt{n}}=\dfrac{1}{n^{1/2}}\cdot$

Now $\quad \lim \dfrac{u_n}{v_n} = \lim \left\{ \dfrac{n}{1 + n\sqrt{(n+1)}} \cdot n^{1/2} \right\}$

$$= \lim \left\{ \frac{1}{1/n^{3/2} + \sqrt{(1 + 1/n)}} \right\} = 1, \text{ which is finite and non-zero.}$$

Since the auxiliary series $\Sigma\, v_n = \Sigma\, (1/n^{1/2})$ is divergent as here $p = 1/2 < 1$, therefore, by comparison test the given series is divergent.

Example 11: *Test the following series for convergence whose nth terms are given by*

(*i*) $\quad (n^3 + 1)^{1/3} - n$ **(Meerut 2013)**

(*ii*) $\sqrt{(n^4 + 1)} - \sqrt{(n^4 - 1)}$. **(Kanpur 2006; Avadh 06, 14; Meerut 13B; Kashi 14)**

Solution: (i) Here, $u_n = (n^3 + 1)^{1/3} - n = (n^3)^{1/3}\,(1 + 1/n^3)^{1/3} - n$

$$= n\left[\left(1 + \frac{1}{n^3}\right)^{1/3} - 1\right] = n\left[1 + \frac{1}{3}\cdot\frac{1}{n^3} + \frac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{2\,!}\cdot\frac{1}{n^6} + \ldots - 1\right]$$

$$= \frac{1}{3n^2} - \frac{1}{9n^5} + \ldots .$$

Taking the lowest power of $1/n$ in u_n, the auxiliary series is given by

$$\Sigma\, v_n = \Sigma\, (1/n^2).$$

Now $\quad \lim \dfrac{u_n}{v_n} = \lim \left\{\left(\dfrac{1}{3n^2} - \dfrac{1}{9n^5} + \ldots\right) . n^2\right\} = \lim \left(\dfrac{1}{3} - \dfrac{1}{9n^3} + \ldots\right) = \dfrac{1}{3},$

which is finite and non-zero.

Since the auxiliary series $\Sigma\, v_n = \Sigma\, (1/n^2)$ is convergent as here $p = 2 > 1$, therefore by comparison test the given series $\Sigma\, u_n$ is also convergent.

(ii) Here $\quad u_n = \sqrt{(n^4 + 1)} - \sqrt{(n^4 - 1)}$

$$= n^2\,[(1 + 1/n^4)^{1/2} - (1 - 1/n^4)^{1/2}]$$

$$= n^2\left[\left\{1 + \frac{1}{2}\cdot\frac{1}{n^4} + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2\,!}\cdot\frac{1}{n^8} + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)}{3\,!}\cdot\frac{1}{n^{12}} + \ldots\right\}\right.$$
$$\left. - \left\{1 - \frac{1}{2}\cdot\frac{1}{n^4} + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2\,!}\cdot\frac{1}{n^8} - \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)}{3\,!}\cdot\frac{1}{n^{12}} + \ldots\right\}\right]$$

$$= n^2\left[2\left\{\frac{1}{2n^4} + \frac{1}{16n^{12}} + \ldots\right\}\right] = \frac{1}{n^2} + \frac{1}{8n^{10}} + \ldots .$$

The lowest power of $1/n$ in u_n is $1/n^2$. Therefore to apply the comparison test we take the auxiliary series as $\Sigma\, v_n = \Sigma\, 1/n^2$, which is convergent as $p = 2 > 1$.

Now $$\lim \frac{u_n}{v_n} = \lim \left[\left\{ \frac{1}{n^2} + \frac{1}{8n^{10}} + \ldots \right\} . n^2 \right]$$

$$= \lim \left\{ 1 + \frac{1}{8n^8} + \ldots \right\} = 1, \text{ which is finite and non-zero.}$$

Therefore, by comparison test, $\Sigma\, u_n$ and $\Sigma\, v_n$ converge or diverge together. Since $\Sigma\, v_n$ is convergent, therefore, $\Sigma\, u_n$ is also convergent.

Alternate solution: We have $u_n = \sqrt{(n^4 + 1)} - \sqrt{(n^4 - 1)}$

$$= \frac{[\sqrt{(n^4+1)} - \sqrt{(n^4-1)}]\,[\sqrt{(n^4+1)} + \sqrt{(n^4-1)}]}{\sqrt{(n^4+1)} + \sqrt{(n^4-1)}}$$

$$= \frac{(n^4+1) - (n^4-1)}{\sqrt{(n^4+1)} + \sqrt{(n^4-1)}} = \frac{2}{\sqrt{(n^4+1)} + \sqrt{(n^4-1)}}$$

$$= \frac{1}{n^2} \cdot \frac{2}{\sqrt{[1 + (1/n^4)]} + \sqrt{[1 - (1/n^4)]}}.$$

Take $$v_n = \frac{1}{n^2}.$$

Then $$\frac{u_n}{v_n} = \frac{2}{\sqrt{[1 + (1/n^4)]} + \sqrt{[1 - (1/n^4)]}}.$$

$\therefore$ $$\lim_{n \to \infty} \frac{u_n}{v_n} = 1 \text{ which is finite and non-zero.}$$

Hence by comparison test $\Sigma\, u_n$ and $\Sigma\, v_n$ are either both convergent or both divergent.

But for $v_n = \dfrac{1}{n^2} = \dfrac{1}{n^p}$, $p = 2 > 1$.

$\therefore$ $\Sigma\, v_n$ is convergent and hence $\Sigma\, u_n$ is also convergent.

Example 12: *Test for convergence of the following series :*

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{(a + b/n)}}$. *(ii)* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + 3^n}$. *(iii)* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} \left(\frac{n+2}{n+3} \right)^n$.

Solution: (i) Here, $u_n = \dfrac{1}{n^{(a + b/n)}} = \dfrac{1}{n^a \,.\, n^{b/n}}$. Let $v_n = \dfrac{1}{n^a}$.

Now $$\lim \frac{u_n}{v_n} = \lim \left[\frac{1}{n^a \,.\, n^{b/n}} . n^a \right] = \lim \left(\frac{1}{n^{b/n}} \right)$$

$$= \lim \frac{1}{(n^{1/n})^b} = \frac{1}{(1)^b} \qquad \left[\because \lim_{n \to \infty} n^{1/n} = 1 \right]$$

$= 1$, which is finite and non-zero.

We know that $\Sigma\, v_n = \Sigma\, (1/n^a)$ is convergent if $a > 1$ and divergent if $a \le 1$.

Hence by comparison test the given series $\Sigma\, u_n$ is convergent if $a > 1$ and divergent if $a \le 1$.

(ii) Here, $u_n = \dfrac{1}{2^n + 3^n} = \dfrac{1}{3^n\left[1+\left(\dfrac{2}{3}\right)^n\right]}\cdot$

Take $v_n = \dfrac{1}{3^n}\cdot$

We know that $\Sigma\, v_n = \Sigma\,(1/3^n)$ is a geometric series with common ratio $1/3 < 1$, hence it is convergent.

Now $\lim \dfrac{u_n}{v_n} = \lim \dfrac{1}{1+\left(\dfrac{2}{3}\right)^n} = 1,$ $[\because \lim r^n = 0, 0 < r < 1]$

which is finite and non-zero.

Hence by comparison test the given series $\Sigma\, u_n$ is convergent.

(iii) Here, $u_n = \dfrac{1}{n^3}\left(\dfrac{n+2}{n+3}\right)^n\cdot$

Take $v_n = \dfrac{1}{n^3}\cdot$ Then $\Sigma\, v_n = \Sigma\,\dfrac{1}{n^3}$ is convergent as $p = 3 > 1$.

Now $$\frac{u_n}{v_n} = \frac{1}{n^3}\left(\frac{n+2}{n+3}\right)^n\cdot n^3 = \left(\frac{n+2}{n+3}\right)^n = \frac{n^n\left(1+\dfrac{2}{n}\right)^n}{n^n\left(1+\dfrac{3}{n}\right)^n} = \frac{\left(1+\dfrac{2}{n}\right)^n}{\left(1+\dfrac{3}{n}\right)^n}\cdot$$

We know that $\lim\left(1+\dfrac{x}{n}\right)^n = e^x$.

$\therefore$ $\lim \dfrac{u_n}{v_n} = \dfrac{e^2}{e^3} = \dfrac{1}{e}$, which is finite and non-zero.

Hence by comparison test, $\Sigma\, u_n$ is convergent.

Example 13: *Test for convergence the series*

$$\frac{1}{1+2^{-1}} + \frac{2}{1+2^{-2}} + \frac{3}{1+2^{-3}} + \frac{4}{1+2^{-4}} + \ldots$$

Solution: Here, $u_n = \dfrac{n}{1+2^{-n}}\cdot$

$\therefore$ $\lim u_n = \lim \dfrac{n}{1+\left(\dfrac{1}{2}\right)^n} = \infty,$

which is > 0. Also $\Sigma\, u_n$ is a series of positive terms.

Hence the given series $\Sigma\, u_n$ is divergent.

Comprehensive Exercise 1

Test for convergence the following series :

1. (i) $\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\dfrac{1}{4\cdot 5}+\dfrac{1}{5\cdot 6}+\ldots..$ **(Kumaun 2002)**

(ii) $\dfrac{1\cdot 2}{3^2\cdot 4^2}+\dfrac{3\cdot 4}{5^2\cdot 6^2}+\dfrac{5\cdot 6}{7^2\cdot 8^2}+\ldots..$ **(Kumaun 2002; Meerut 12B)**

(iii) $\dfrac{1}{1\cdot 2\cdot 3}+\dfrac{3}{2\cdot 3\cdot 4}+\dfrac{5}{3\cdot 4\cdot 5}+\ldots..$ **(Avadh 2011; Meerut 12)**

(iv) $\dfrac{(1+a)(1+b)}{1.2.3}+\dfrac{(2+a)(2+b)}{2.3.4}+\dfrac{(3+a)(3+b)}{3.4.5}+\ldots..$

2. (i) $\dfrac{1}{\sqrt{1}+\sqrt{2}}+\dfrac{1}{\sqrt{2}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{4}}+\ldots..$

(ii) $\dfrac{\sqrt{2}-1}{3^3-1}+\dfrac{\sqrt{3}-1}{4^3-1}+\dfrac{\sqrt{4}-1}{5^3-1}+\dfrac{\sqrt{5}-1}{6^3-1}+\ldots..$

(iii) $1+\dfrac{1}{2\cdot 2^{1/100}}+\dfrac{1}{3\cdot 3^{1/100}}+\dfrac{1}{4\cdot 4^{1/100}}+\ldots..$

3. (i) $\sum\sqrt{\left(\dfrac{n}{n^5+2}\right)}.$ (ii) $\sum\dfrac{1}{(2n-1)^p}.$

(iii) $\sum\left(\dfrac{1}{\sqrt{n}}\sin\dfrac{1}{n}\right).$ (iv) $\sum\cos\dfrac{1}{n}.$ **(Kanpur 2007)**

4. (i) $\Sigma\,[\sqrt{(n+1)}-\sqrt{n}].$ (ii) $\Sigma\,[\sqrt{(n^2+1)}-n].$

(iii) $\Sigma\,[\sqrt{(n^3+1)}-\sqrt{n^3}].$ (iv) $\Sigma\,[\sqrt{(n^4+1)}-n^2].$

5. (i) $\sqrt{\left(\dfrac{1}{2^3}\right)}+\sqrt{\left(\dfrac{2}{3^3}\right)}+\sqrt{\left(\dfrac{3}{4^3}\right)}+\ldots.$

(ii) The series whose nth term is $\dfrac{1}{n}\sin\dfrac{1}{n}.$ **(Kanpur 2005)**

Answers 1

1. (i) Convergent (ii) Convergent

(iii) Convergent (iv) Divergent

2. (i) Divergent (ii) Convergent

(iii) Convergent

3. (i) Convergent
(ii) Convergent if $p > 1$, divergent if $p \leq 1$
(iii) Convergent (iv) Divergent
4. (i) Divergent (ii) Divergent
(iii) Convergent (iv) Convergent
5. (i) Divergent (ii) Convergent

8 Cauchy's Root Test

Theorem 1: *Let Σu_n be a series of positive terms such that $\lim u_n^{1/n} = l$. Then*

(i) Σu_n *converges, if* $l < 1$; *(ii)* Σu_n *diverges, if* $l > 1$;

(iii) the test fails and the series may either converge or diverge, if $l = 1$.

(*Here* $u_n^{1/n}$ *stands for positive nth root of* u_n).

(Kumaun 2001; Kanpur 04, 07; Avadh 06; Meerut 12; Gorakhpur 13)

Proof: Since $u_n > 0$, for all n, and $(u_n)^{1/n}$ stands for positive nth root of u_n, $\lim u_n^{1/n} = l \geq 0$.

Since $\lim u_n^{1/n} = l$, therefore for $\varepsilon > 0$ there exists a positive integer m, such that

$$|u_n^{1/n} - l| < \varepsilon, \text{ for all } n > m,$$

i.e.,
$$l - \varepsilon < u_n^{1/n} < l + \varepsilon, \text{ for all } n > m,$$

i.e.,
$$(l - \varepsilon)^n < u_n < (l + \varepsilon)^n, \text{ for all } n > m. \quad \ldots(1)$$

(i) Let $l < 1$.

Choose $\varepsilon > 0$, such that $r = l + \varepsilon < 1$.

Then $0 \leq l < r < 1$.

From (1), we get $u_n < (l + \varepsilon)^n$ for all $n > m$ *i.e.,* $u_n < r^n$ for all $n > m$.

Since Σr^n is a geometric series with common ratio r less than unity, Σr^n is convergent. Therefore, by comparison test, Σu_n is convergent.

(ii) Let $l > 1$.

Choose $\varepsilon > 0$, such that $r = l - \varepsilon > 1$.

From (1), we get $(l - \varepsilon)^n < u_n$ for all $n > m$ *i.e.,* $u_n > r^n$ for all $n > m$.

Since Σr^n is a geometric series with common ratio greater than unity, Σr^n is divergent. Therefore, by comparison test, Σu_n is divergent.

(iii) Let $l = 1$.

Consider the series $\Sigma\, u_n$, where $u_n = 1/n$.

Then $\qquad u_n^{1/n} = \left(\dfrac{1}{n}\right)^{1/n}$, so that $\lim u_n^{1/n} = 1$. $\qquad$ [Note that $\lim n^{1/n} = 1$].

Since $\Sigma\,(1/n)$ diverges, hence, we observe that if

$$\lim u_n^{1/n} = 1, \text{ the series } \Sigma\, u_n \text{ may diverge.}$$

Now, consider the series $\Sigma\, u_n$, where $u_n = 1/n^2$.

In this case also, $\lim u_n^{1/n} = 1$.

Since $\Sigma\,(1/n^2)$ converges, hence, we observe that if $\lim u_n^{1/n} = 1$, the series $\Sigma\, u_n$ may converge.

Thus the above two examples show that Cauchy's root test fails to decide the nature of the series when $l = 1$.

Note 1: In general the Root test is used when powers are involved.

Another form of Cauchy's Root Test: The root test can also be stated in the form given below :

A series $\Sigma\, u_n$ of positive terms is convergent if for every value of $n \geq m$, m being finite, $(u_n)^{1/n}$ is less than a fixed number which is less than unity.

The series is divergent if $(u_n)^{1/n} \geq 1$ for every value of $n \geq m$.

Proof: **Case 1:** Given $(u_n)^{1/n} < r$, $\forall\, n \geq m$ where r is a fixed positive number such that $r < 1$.

$\therefore \qquad u_n < r^n$, for all $n \geq m$.

Since $\Sigma\, r^n$ is a geometric series with common ratio r less than unity, $\Sigma\, r^n$ is convergent.

Therefore, by comparison test, $\Sigma\, u_n$ is convergent.

Case 2: Given $(u_n)^{1/n} \geq 1, \forall\; n \geq m$.

$\therefore \qquad u_n \geq 1, \forall\, n \geq m$.

Omitting the first $m - 1$ terms of the series because it will not affect the convergence or divergence of the series, we have

$$u_n \geq 1, \forall\, n \in \mathbf{N}$$

$\Rightarrow \qquad S_n = u_1 + \ldots + u_n \geq n$

$\Rightarrow \qquad \lim S_n = \infty$

$\Rightarrow \qquad$ the series is divergent.

Theorem 2: *Let $\Sigma\, u_n$ be a series of positive terms such that $u_n^{1/n} \to \infty$ as $n \to \infty$. Then $\Sigma\, u_n$ diverges.*

Proof: Let $r > 1$. Since $u_n^{1/n} \to \infty$ as $n \to \infty$, therefore, there exists a positive integer m such that $u_n^{1/n} > r$ for all $n \geq m \Rightarrow u_n > r^n$ for all $n \geq m$.

For $r > 1$, the geometric series $\Sigma\, r^n$ is divergent.

Hence, by comparison test, $\Sigma\, u_n$ is divergent.

Some important limits to be remembered:

1. $\lim\limits_{n \to \infty} n^{1/n} = 1.$

2. $\lim\limits_{n \to \infty} \dfrac{\log n}{n} = 0.$

3. $\lim\limits_{n \to \infty} \left(1 + \dfrac{x}{n}\right)^n = e^x.$

4. $\lim\limits_{n \to \infty} \left(1 + \dfrac{x}{n}\right)^p = 1$, if p is finite *i.e.*, if p is a fixed real number.

5. $\lim\limits_{n \to \infty} \left(1 + \dfrac{x}{n}\right)^{n+p} = e^x$, if p is finite.

6. $\lim\limits_{n \to \infty} \dfrac{a_0 n^p + a_1 n^{p-1} + a_2 n^{p-2} + \ldots + a_{p-1} n + a_p}{b_0 n^q + b_1 n^{q-1} + b_2 n^{q-2} + \ldots + b_{q-1} n + b_q}$

$$= \lim_{n \to \infty} \frac{n^p [a_0 + a_1 (1/n) + a_2 (1/n)^2 + \ldots.]}{n^q [b_0 + b_1 (1/n) + b_2 (1/n)^2 + \ldots]}$$

$$= \begin{cases} a_0 / b_0, \text{ if } p = q \\ 0, \text{ if } q > p \\ \infty, \text{ if } p > q \text{ and } a_0 > 0, b_0 > 0. \end{cases}$$

Illustrative Examples

Example 14: *Assuming that $n^{1/n} \to 1$ as $n \to \infty$, show by applying Cauchy's nth root test that the series $\sum\limits_{n=1}^{\infty} (n^{1/n} - 1)^n$ converges.*

Solution: Here, $u_n = (n^{1/n} - 1)^n$.

$\therefore \quad u_n^{1/n} = n^{1/n} - 1.$

$\therefore \quad \lim\limits_{n \to \infty} u_n^{1/n} = \lim\limits_{n \to \infty} (n^{1/n} - 1) = 0 < 1.$

Hence, by Cauchy's root test, the given series converges.

Example 15: *Test the convergence of the following series*

(*i*) $\Sigma\left(1+\dfrac{1}{n}\right)^{-n^2}$ (*ii*) $\Sigma\,\dfrac{x^n}{n\,!}$

(*iii*) $\dfrac{1^3}{3}+\dfrac{2^3}{3^2}+\dfrac{3^3}{3^3}+\dfrac{4^3}{3^4}+\ldots+\dfrac{n^3}{3^n}+\ldots$

Solution: (i) Here $u_n=\left(1+\dfrac{1}{n}\right)^{-n^2}$.

$$\therefore \qquad u_n^{1/n}=\left(1+\frac{1}{n}\right)^{-n}.$$

$$\therefore \qquad \lim u_n^{1/n}=\lim\frac{1}{\left(1+\frac{1}{n}\right)^n}=\frac{1}{e}<1. \qquad [\because\ 2<e<3]$$

Hence by Cauchy's root test the given series is convergent.

(ii) Here $u_n=\dfrac{x^n}{n\,!}$.

$$\therefore \qquad u_n^{1/n}=\frac{x}{(n\,!)^{1/n}}.$$

$$\therefore \qquad \lim u_n^{1/n}=\lim\frac{x}{(n\,!)^{1/n}}=\lim\left[\frac{n}{(n\,!)^{1/n}}\cdot\frac{x}{n}\right]=\lim\left[\frac{(n^n)^{1/n}}{(n\,!)^{1/n}}\cdot\frac{x}{n}\right]$$

$$=\lim\left[\left(\frac{n^n}{n\,!}\right)^{1/n}\cdot\frac{x}{n}\right]=e\,.\lim\frac{x}{n} \qquad \left[\because\ \lim\left(\frac{n^n}{n\,!}\right)^{1/n}=e\right]$$

$$=e\,.\,0=0<1.$$

Hence by Cauchy's root test, the given series is convergent.

(iii) Here $u_n=\dfrac{n^3}{3^n}$.

$$\therefore \qquad u_n^{1/n}=\frac{n^{3/n}}{3}.$$

$$\therefore \qquad \lim u_n^{1/n}=\lim\frac{1}{3}n^{3/n}=\frac{1}{3}\lim\,(n^{1/n})^3=\frac{1}{3}\cdot 1<1.$$

Hence by Cauchy's root test the given series is convergent.

Example 16: *Test the convergence of the series*

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots$$

(Kumaun 2001; Meerut 13B)

Solution: Here $u_n=\left[\dfrac{(n+1)^{n+1}}{n^{n+1}}-\dfrac{n+1}{n}\right]^{-n}$.

$$\therefore \quad u_n^{1/n} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-1} = \left(\frac{n+1}{n}\right)^{-1}\left[\left(\frac{n+1}{n}\right)^{n} - 1\right]^{-1}$$

$$= \left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^{n} - 1\right]^{-1}.$$

$$\therefore \quad \lim u_n^{1/n} = (1+0)^{-1}(e-1)^{-1} = \frac{1}{e-1} < 1. \qquad [\because 2 < e < 3]$$

Hence by Cauchy's root test the given series is convergent.

Example 17: *Test for convergence* $\Sigma\, 3^{-n-(-1)^n}$.

Solution: Here $u_n = 3^{-n-(-1)^n} = \begin{cases} 3^{-n} . 3^{-1}, & \text{if } n \text{ is even} \\ 3^{-n} . 3, & \text{if } n \text{ is odd.} \end{cases}$

$$\therefore \quad u_n^{1/n} = \begin{cases} 3^{-1} . 3^{-1/n} = \dfrac{1}{3} \cdot \dfrac{1}{3^{1/n}}, & \text{if } n \text{ is even} \\ 3^{-1} . 3^{1/n} = \dfrac{1}{3} . 3^{1/n}, & \text{if } n \text{ is odd.} \end{cases}$$

$$\therefore \quad \lim u_n^{1/n} = \frac{1}{3} < 1. \qquad [\because \lim a^{1/n} = 1 \text{ if } a > 0]$$

Hence by Cauchy's root test the given series is convergent.

Example 18: *Test for convergence* $\Sigma \left(\dfrac{n+1}{n+2}\right)^n . x^n, \ (x > 0)$. **(Meerut 2013)**

Solution: Here $u_n = \left(\dfrac{n+1}{n+2}\right)^n x^n.$

$$\therefore \quad u_n^{1/n} = \frac{n+1}{n+2} \cdot x.$$

$$\therefore \quad \lim u_n^{1/n} = \lim \left[\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{2}{n}\right)} \cdot x\right] = x.$$

$\therefore$ By Cauchy's root test, $\Sigma\, u_n$ converges if $x < 1$ and $\Sigma\, u_n$ diverges if $x > 1$.

For $x = 1$, the test fails. When $x = 1$, $u_n = \left(\dfrac{n+1}{n+2}\right)^n$.

$$\therefore \quad \lim u_n = \lim \frac{\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{2}{n}\right)^n} = \frac{e}{e^2} = \frac{1}{e} > 0.$$

$\therefore$ The series $\Sigma\, u_n$ diverges when $x = 1$.

Hence the given series converges if $x < 1$ and diverges if $x \geq 1$.

Example 19: *Test for convergence* $\Sigma \dfrac{1}{(\log n)^n}$.

Solution: Here $u_n = \dfrac{1}{(\log n)^n}$.

$\therefore$ $$u_n^{1/n} = \frac{1}{\log n}.$$

$\therefore$ $$\lim u_n^{1/n} = \lim \frac{1}{\log n} = 0, \text{ which is} < 1.$$

Hence by Cauchy's root test the given series is convergent.

Comprehensive Exercise 2

Test for convergence the following series :

1. (i) $\sum \dfrac{1}{n^{1+(1/n)}}$. **(Kanpur 2008; Avadh 12)**

(ii) $\sum \left(1+\dfrac{1}{n}\right)^{n^2}$.

2. (i) $\sum \left(\dfrac{n}{n+1}\right)^{n^2}$. **(Avadh 2013; Kashi 14)**

(ii) $\sum \left(1+\dfrac{1}{\sqrt{n}}\right)^{-n^{3/2}}$.

3. (i) $2 + \dfrac{3}{2}x + \dfrac{4}{3}x^2 + \dfrac{5}{4}x^3 + \ldots$, where $x > 0$.

(ii) $\sum_{n=1}^{\infty} \dfrac{3n+1}{4n+3}x^n, x > 0$.

4. $\dfrac{2}{1^2}x + \dfrac{3^2}{2^3}x^2 + \dfrac{4^3}{3^4}x^3 + \ldots + \dfrac{(n+1)^n}{n^{n+1}}x^n + \ldots$

5. (i) $1 + \dfrac{x}{2} + \dfrac{x^2}{3^2} + \dfrac{x^3}{4^3} + \ldots \infty,\ x > 0$.

(ii) $x + 2x^2 + 3x^3 + 4x^4 + \ldots$

(iii) $\dfrac{1}{2} + \left(\dfrac{2}{3}\right)x + \left(\dfrac{3}{4}\right)^2 x^2 + \left(\dfrac{4}{5}\right)^3 x^3 + \ldots \infty, x > 0$.

Answers 2

1. (i) Divergent (ii) Divergent
2. (i) Convergent (ii) Convergent
3. (i) Convergent if $x < 1$ and divergent if $x \geq 1$

   (ii) Convergent if $x < 1$ and divergent if $x \geq 1$
4. Convergent if $x < 1$ and divergent if $x \geq 1$
5. (i) Convergent

   (ii) Convergent if $x < 1$ and divergent if $x \geq 1$

   (iii) Convergent if $x < 1$ and divergent if $x \geq 1$

9 D' Alembert's Ratio Test

(Avadh 2003, 05; Kanpur 05; Gorakhpur 11; Meerut 12B; Kashi 14)

Theorem 1: *Let Σu_n be a series of positive terms such that* $\lim \dfrac{u_{n+1}}{u_n} = l$. *Then*

(i) Σu_n *converges if* $l < 1$ *(ii)* Σu_n *diverges if* $l > 1$

and (iii) the test fails to decide the nature of the series if $l = 1$.

Proof: Since $u_n > 0$, for all n, therefore

$$\frac{u_{n+1}}{u_n} > 0 \Rightarrow \lim \frac{u_{n+1}}{u_n} = l \geq 0.$$

Since $\lim \dfrac{u_{n+1}}{u_n} = l$, therefore, for $\varepsilon > 0$, there exists a positive integer m such that

$$\left| \frac{u_{n+1}}{u_n} - l \right| < \varepsilon, \text{ for all } n \geq m$$

i.e.,
$$l - \varepsilon < \frac{u_{n+1}}{u_n} < l + \varepsilon, \text{ for all } n \geq m.$$

Putting $n = m, m+1, \ldots, n-1$ in succession in the above inequality and multiplying the corresponding sides of the $(n-m)$ inequalities thus obtained, we get

$$(l - \varepsilon)^{n-m} < \frac{u_n}{u_m} < (l + \varepsilon)^{n-m} \text{ for all } n > m$$

i.e.,
$$(l - \varepsilon)^n \frac{u_m}{(l - \varepsilon)^m} < u_n < (l + \varepsilon)^n \frac{u_m}{(l + \varepsilon)^m} \text{ for all } n > m. \qquad \ldots(1)$$

(i) Let $l < 1$.

Choose $\varepsilon > 0$ such that $r = l + \varepsilon < 1$.

Then $\quad 0 \le l < r < 1$.

From (1), we get $u_n < \left(\dfrac{u_m}{r^m}\right) r^n$ for all $n > m$

i.e., $\quad u_n < \alpha\, r^n$ for all $n > m$ where $\alpha = \dfrac{u_m}{r^m} \in \mathbf{R}^+$.

Since $\Sigma\, r^n$ is a geometric series with common ratio less than unity, $\Sigma\, r^n$ is convergent.

Hence by comparison test, $\Sigma\, u_n$ is convergent.

(ii) Let $l > 1$.

Choose $\varepsilon > 0$ such that $r = l - \varepsilon > 1$.

From (1), we get $\dfrac{u_m}{r^m} r^n < u_n$, for all $n > m$

i.e., $\quad u_n > \beta r^n$, for all $n > m$ where $\beta = \dfrac{u_m}{r^m} \in \mathbf{R}^+$.

Since $\Sigma\, r^n$ is a geometric series with common ratio greater than unity, $\Sigma\, r^n$ is divergent.

Therefore, by comparison test, $\Sigma\, u_n$ is divergent.

(iii) Let $l = 1$.

Consider the series $\Sigma\, u_n$ where $u_n = 1/n^2$.

Here $$\lim \frac{u_{n+1}}{u_n} = \lim \frac{n^2}{(n+1)^2} = \lim \frac{1}{\left(1 + \frac{1}{n}\right)^2} = 1.$$

Since the series $\Sigma\, (1/n^2)$ converges, we observe that if $l = 1$, the series may be convergent.

Now, consider the series $\Sigma\, u_n$, where $u_n = 1/n$.

Here $$\lim \frac{u_{n+1}}{u_n} = \lim \frac{n}{n+1} = \lim \frac{1}{1 + \frac{1}{n}} = 1 \cdot$$

Since the series $\Sigma\, (1/n)$ diverges, we observe that if $l = 1$, the series may be divergent.

Thus the above two examples show that the test fails to decide the nature when $l = 1$.

Note 1: Taking the reciprocals, the ratio test can also be stated in the form given below.

The series $\Sigma\, u_n$ of positive terms is convergent if lim $\dfrac{u_n}{u_{n+1}} > 1$ and divergent if lim $\dfrac{u_n}{u_{n+1}} < 1$.

If lim $\dfrac{u_n}{u_{n+1}} = 1$, the test fails.

We shall usually apply the ratio test in this form which will in the later part of this chapter be more convenient for further investigation in case the ratio test fails.

The ratio test is generally applied when the nth term of the series involves factorials, products of several factors, or combinations of powers and factorials.

Another form of D' Alembert's Ratio Test: The ratio test can also be stated in the form given below :

An infinite series of positive terms is convergent if from and after some term the ratio of each term to the preceding term is less than a fixed number which is less than unity.

The series is divergent if the above ratio is greater than or equal to unity.

Proof: Case 1: It is given that

$$\frac{u_{n+1}}{u_n} < r \text{ for all } n \geq m, \qquad \ldots(1)$$

where r is a fixed positive number such that $r < 1$.

To prove $\Sigma\, u_n$ is convergent.

Putting $n = m, m+1, \ldots, n-1$ in succession in (1) and multiplying the corresponding sides of the $n - m$ inequalities thus obtained, we get

$$\frac{u_{m+1}}{u_m} \cdot \frac{u_{m+2}}{u_{m+1}} \cdot \frac{u_{m+3}}{u_{m+2}} \cdot \ldots \cdot \frac{u_n}{u_{n-1}} < r^{n-m}$$

$$\Rightarrow \quad \frac{u_n}{u_m} < r^{n-m} \quad \Rightarrow \quad u_n < \frac{u_m}{r^m}\, r^n$$

$$\Rightarrow \quad u_n < \alpha\, r^n, \text{ for all } n > m \text{ where } \alpha = \frac{u_m}{r^m} \in \mathbf{R}^+.$$

Since $\Sigma\, r^n$ is a geometric series with common ratio less than unity, $\Sigma\, r^n$ is convergent. Hence by comparison test, $\Sigma\, u_n$ is also convergent.

Case 2: It is given that

$$\frac{u_{n+1}}{u_n} \geq 1 \text{ for all } n \geq m. \qquad \ldots(2)$$

Putting $n = m, m+1, \ldots, n-1$ in succession in (2) and multiplying the corresponding sides of the $n - m$ inequalities thus obtained, we get

$$\frac{u_n}{u_m} \geq 1 \quad \Rightarrow \quad u_n \geq u_m \text{ for all } n > m.$$

Omitting the first m terms of the series because it will not affect the convergence or divergence of the series, we have

$$u_n \geq u_m \text{ for all } n \in \mathbf{N}$$

$$\Rightarrow \quad S_n = u_1 + \ldots + u_n \geq n\, u_m$$

$$\Rightarrow \quad \lim S_n = \infty$$

$\Rightarrow$ the series is divergent.

Theorem 2: *Let $\Sigma\, u_n$ be a series of positive terms such that $\frac{u_{n+1}}{u_n} \to \infty$ as $n \to \infty$. Then $\Sigma\, u_n$ diverges.*

Proof: Since $\frac{u_{n+1}}{u_n} \to \infty$ as $n \to \infty$, therefore, there exists a positive integer m such that

$$\frac{u_{n+1}}{u_n} > 2 \text{ for all } n \geq m \quad i.e., \quad u_{n+1} > 2\, u_n \text{ for all } n \geq m.$$

Replacing n by $m, m+1, m+2, \ldots, n-1$ and multiplying the $(n-m)$ inequalities, we get

$$u_n > 2^{n-m} \,.\, u_m \text{ for all } n > m$$

i.e.,

$$u_n > \left(\frac{u_m}{2^m}\right) 2^n \text{ for all } n > m.$$

Since the geometric series $\Sigma\, 2^n$ diverges, hence, by comparison test $\Sigma\, u_n$ diverges.

Note: In a similar manner it can be proved that $\Sigma\, u_n$ is convergent if

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \infty.$$

10 Remarks on the Ratio Test

It should be noted that D'Alembert's ratio test does not tell us anything about the convergence of the series $\Sigma\, u_n$ if we only know that $\frac{u_n}{u_{n+1}} > 1\ \forall\, n$.

If $u_n = \frac{1}{n}$, then $\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} > 1$ for all n while the series $\Sigma\, u_n$ is divergent. Also, for the convergence of the series $\Sigma\, u_n$ it is not necessary that $\frac{u_n}{u_{n+1}}$ should have a definite limit.

For a change in the order of the terms of a series of positive terms may affect the value of $\lim \frac{u_n}{u_{n+1}}$ but it does not affect the convergence of the series.

For example, let us consider the series

$$1 + x + x^2 + x^3 + \ldots \text{ where } 0 < x < 1. \qquad \ldots(1)$$

Changing the order of terms, the series becomes

$$x + 1 + x^3 + x^2 + x^5 + x^4 + \ldots \qquad \ldots(2)$$

Since the series (1) is convergent, therefore, the series (2) is also convergent. But in the series (2), the ratio u_n / u_{n+1} is alternately x and $1/x^3$ and consequently $\lim\, (u_n/u_{n+1})$ is not definite.

In comparison with Cauchy's root test, D'Alembert's ratio test is more useful since it is easier to apply than the root test because generally u_n / u_{n+1} is a simpler fraction than u_n. However **the root test is stronger than the ratio test**. To be more precise, whenever the ratio test indicates the nature of the series, the root test does too. But sometimes the ratio test does not apply while the root test succeeds.

Illustrative Examples

Example 20: *Test for convergence the following series :*

(*i*) $1+\dfrac{2^p}{2\,!}+\dfrac{3^p}{3\,!}+\dfrac{4^p}{4\,!}+\ldots$

(Bundelkhand 2006)

(*ii*) $\dfrac{1}{1+2}+\dfrac{2}{1+2^2}+\dfrac{3}{1+2^3}+\ldots$

Solution: (i) Here $u_n=\dfrac{n^p}{n\,!}\cdot$

$$\therefore \qquad u_{n+1}=\frac{(n+1)^p}{(n+1)\,!}\cdot$$

Now
$$\frac{u_n}{u_{n+1}}=\frac{n^p}{n\,!}\cdot\frac{(n+1)\,!}{(n+1)^p}$$

$$=\frac{(n+1)\,n^p}{(n+1)^p}=\frac{n+1}{(1+1/n)^p}\cdot$$

$$\therefore \qquad \lim\frac{u_n}{u_{n+1}}=\lim\frac{n+1}{(1+1/n)^p}=\infty,$$

which is >1 for all values of p.
Hence by ratio test the series $\Sigma\, u_n$ is convergent.

(ii) Here $u_n=\dfrac{n}{1+2^n}\cdot$

$$\therefore \qquad u_{n+1}=\frac{n+1}{1+2^{n+1}}\cdot$$

Now
$$\frac{u_n}{u_{n+1}}=\frac{n}{1+2^n}\cdot\frac{1+2^{n+1}}{n+1}$$

$$=\frac{n\,.\,2^{n+1}\,(1+1/2^{n+1})}{2^n\,(1+1/2^n)\,.\,n\,(1+1/n)}$$

$$=\frac{2\,(1+1/2^{n+1})}{(1+1/2^n\,)\,(1+1/n)}\cdot$$

$$\therefore \qquad \lim\frac{u_n}{u_{n+1}}=2\,\cdot\frac{(1+0)}{(1+0)\,(1+0)}=2,\text{ which is}>1.$$

Therefore, by ratio test, the given series converges.

Example 21: *Test for convergence the series whose nth term is*

(*i*) $\dfrac{n^3+a}{2^n+a}$, (*ii*) $\dfrac{n\,!}{n^n}$, **(Purvanchal 2014)** (*iii*) $\sqrt{\left\{\dfrac{2^n-1}{3^n-1}\right\}}\,\cdot$

Solution: (i) Here $u_n = \dfrac{n^3 + a}{2^n + a}$, $u_{n+1} = \dfrac{(n+1)^3 + a}{2^{n+1} + a}$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{n^3 + a}{2^n + a} \cdot \frac{2^{n+1} + a}{(n+1)^3 + a}$$

$$= \frac{n^3 (1 + a/n^3) \,.\, 2^{n+1} (1 + a/2^{n+1})}{2^n (1 + a/2^n) \,.\, n^3 \{(1 + 1/n)^3 + a/n^3\}}$$

$$= 2 \cdot \frac{(1 + a/n^3)(1 + a/2^{n+1})}{(1 + a/2^n)\{(1 + 1/n)^3 + a/n^3\}}.$$

Now $\quad \lim \dfrac{u_n}{u_{n+1}} = 2 \cdot \dfrac{(1+0)(1+0)}{(1+0)\{(1+0)^3 + 0\}} = 2$, which is > 1.

Therefore, by ratio test the given series converges.

(ii) Here $u_n = \dfrac{n!}{n^n}$ so that $u_{n+1} = \dfrac{(n+1)!}{(n+1)^{n+1}}$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!} = \frac{(n+1)^{n+1}}{n^n \,.\, (n+1)} = \left(1 + \frac{1}{n}\right)^n.$$

$$\therefore \quad \lim \frac{u_n}{u_{n+1}} = \lim \left(1 + \frac{1}{n}\right)^n = e, \text{ which is} > 1.$$

Therefore, by ratio test the given series converges.

(iii) Here $\quad u_n = \sqrt{\left(\dfrac{2^n - 1}{3^n - 1}\right)}$, $u_{n+1} = \sqrt{\left(\dfrac{2^{n+1} - 1}{3^{n+1} - 1}\right)}$.

Now $$\frac{u_n}{u_{n+1}} = \sqrt{\left(\frac{2^n - 1}{3^n - 1} \cdot \frac{3^{n+1} - 1}{2^{n+1} - 1}\right)}$$

$$= \sqrt{\left\{\frac{2^n (1 - 1/2^n) \,.\, 3^n (3 - 1/3^n)}{3^n (1 - 1/3^n) \,.\, 2^n (2 - 1/2^n)}\right\}}$$

$$= \sqrt{\left\{\frac{(1 - 1/2^n)(3 - 1/3^n)}{(1 - 1/3^n)(2 - 1/2^n)}\right\}}.$$

$$\therefore \quad \lim \frac{u_n}{u_{n+1}} = \sqrt{\left(\frac{3}{2}\right)}, \text{ which is } > 1.$$

Therefore, by ratio test the given series converges.

Example 22: *Show that the series*

$$1 + \frac{\alpha + 1}{\beta + 1} + \frac{(\alpha + 1)(2\alpha + 1)}{(\beta + 1)(2\beta + 1)} + \frac{(\alpha + 1)(2\alpha + 1)(3\alpha + 1)}{(\beta + 1)(2\beta + 1)(3\beta + 1)} + \ldots..$$

converges if $\beta > \alpha > 0$ *and diverges if* $\alpha \geq \beta > 0$ $[\alpha > 0, \beta > 0]$.

Solution: Here,

$$u_n = \frac{(\alpha + 1)(2\alpha + 1)\ldots\ldots[(n-1)\alpha + 1]}{(\beta + 1)(2\beta + 1)\ldots\ldots[(n-1)\beta + 1]},$$

so that

$$u_{n+1} = \frac{(\alpha + 1)(2\alpha + 1)\ldots\ldots[(n-1)\alpha + 1](n\alpha + 1)}{(\beta + 1)(2\beta + 1)\ldots\ldots[(n-1)\beta + 1](n\beta + 1)}.$$

Now

$$\frac{u_n}{u_{n+1}} = \frac{n\beta + 1}{n\alpha + 1} = \frac{\beta + 1/n}{\alpha + 1/n}.$$

$\therefore$

$$\lim \frac{u_n}{u_{n+1}} = \lim \frac{\beta + 1/n}{\alpha + 1/n} = \frac{\beta}{\alpha}.$$

Hence by ratio test the series is convergent if $\frac{\beta}{\alpha} > 1$ *i.e.,* if $\beta > \alpha > 0$, divergent if $\frac{\beta}{\alpha} < 1$, *i.e.,* if $\alpha > \beta > 0$, and the test fails if $\frac{\beta}{\alpha} = 1$ *i.e.,* if $\beta = \alpha$.

When $\beta = \alpha$, then the given series becomes

$$1 + 1 + 1 + \ldots\ldots$$

S_n = the sum of n terms of this series = n.

Since $\lim S_n = \infty$, hence the series is divergent.

Thus the given series is convergent if $\beta > \alpha > 0$ and divergent if $\alpha \geq \beta > 0$.

Example 23: *Test for convergence the following series* :

(*i*) $1 + 3x + 5x^2 + 7x^3 + \ldots$ (*ii*) $1 + \frac{x}{2^2} + \frac{x^2}{3^2} + \frac{x^3}{4^2} + \ldots.$

Solution: (i) Here $u_n = (2n-1)x^{n-1}$, $u_{n+1} = (2n+1)x^n$.

$\therefore$

$$\frac{u_n}{u_{n+1}} = \frac{(2n-1)x^{n-1}}{(2n+1)x^n} = \frac{(2-1/n)}{(2+1/n)} \cdot \frac{1}{x}.$$

$\therefore$

$$\lim \frac{u_n}{u_{n+1}} = \frac{2}{2} \cdot \frac{1}{x} = \frac{1}{x}.$$

Hence by ratio test the series is convergent if $1/x > 1$ *i.e.* if

$$1 > x \quad \text{or} \quad x < 1,$$

the series is divergent if $1/x < 1$, *i.e.* if $x > 1$ and the test fails if $1/x = 1$ *i.e.* if $x = 1$.

When $x = 1$, then the given series becomes

$$1 + 3 + 5 + 7 + \ldots$$

S_n = sum of n terms of this series $= \frac{n}{2}(1 + 2n - 1) = n^2$.

Since $\lim S_n = \infty$, hence this series is divergent.

Thus the given series converges if $x < 1$ and diverges if $x \geq 1$.

(ii) Here $u_n = \dfrac{x^{n-1}}{n^2}$, so that $u_{n+1} = \dfrac{x^n}{(n+1)^2}$.

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \frac{x^{n-1}}{n^2} \cdot \frac{(n+1)^2}{x^n} = \left(1 + \frac{1}{n}\right)^2 \cdot \frac{1}{x}.$$

$$\therefore \qquad \lim \frac{u_n}{u_{n+1}} = \lim \left(1 + \frac{1}{n}\right)^2 \cdot \frac{1}{x} = \frac{1}{x}.$$

Hence by ratio test the series converges if $1/x > 1$ *i.e.* if $x < 1$, diverges if $1/x < 1$ *i.e.* if $x > 1$ and the test fails if $1/x = 1$ *i.e.* if $x = 1$.

When $x = 1$, then $u_n = 1/n^2$. We know that $\Sigma\,(1/n^2)$ is convergent because here $p = 2 > 1$.

Thus the given series converges if $x \le 1$ and diverges if $x > 1$.

Example 24: *Test for convergence the series whose nth term is*

(i) $\dfrac{1}{x^n + x^{-n}}$, (ii) $\dfrac{a^n}{x^n + a^n}$.

Solution: Here $u_n = \dfrac{1}{x^n + x^{-n}} = \dfrac{x^n}{x^{2n} + 1}$, $u_{n+1} = \dfrac{x^{n+1}}{x^{2(n+1)} + 1}$.

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \frac{x^n}{x^{2n} + 1} \cdot \frac{x^{2(n+1)} + 1}{x^{n+1}} = \frac{x^{2n+2} + 1}{x^{2n} + 1} \cdot \frac{1}{x}.$$

Now (u_n / u_{n+1}) can be found only if we know that

$x < 1$ or $x > 1$.

Let $x < 1$.

Then
$$\lim \frac{u_n}{u_{n+1}} = \lim \left[\frac{x^{2n+2} + 1}{x^{2n} + 1} \cdot \frac{1}{x}\right]$$
$$= \frac{1}{x}. \qquad [\because \lim x^{2n+2} = 0 = \lim x^{2n} \text{ if } x < 1]$$

But if $x < 1$, then $1/x > 1$.

$\therefore$ if $x < 1$, we have $\lim (u_n / u_{n+1}) > 1$ and hence by ratio test the series converges in this case.

Now let $x > 1$.

Then
$$\lim \frac{u_n}{u_{n+1}} = \lim \left[\frac{x^{2n+2} + 1}{x^{2n} + 1} \cdot \frac{1}{x}\right] = \lim \left[\frac{x^{2n+2}\,(1 + 1/x^{2n+2})}{x^{2n}\,(1 + 1/x^{2n})} \cdot \frac{1}{x}\right]$$
$$= \lim \left[x\,\frac{(1 + 1/x^{2n+2})}{(1 + 1/x^{2n})}\right]$$
$$= x \qquad [\because \lim 1/x^{2n+2} = 0 \text{ if } x > 1]$$

$\therefore$ if $x > 1$, we have $\lim (u_n / u_{n+1}) = x$ *i.e.* > 1 and hence by ratio test the series is convergent in this case also.

Again, if $x = 1$, then $u_n = \dfrac{1}{1+1} = \dfrac{1}{2}$,

i.e., the series becomes $\dfrac{1}{2} + \dfrac{1}{2} + \dfrac{1}{2} + \dots$

S_n = sum of its n terms $= \dfrac{1}{2} \cdot n$.

Since $\lim S_n = \infty$, hence, the series is divergent if $x = 1$.

Thus the given series is convergent if $x > 1$ or $x < 1$ and divergent if $x = 1$.

(ii) Here $u_n = \dfrac{a^n}{x^n + a^n}, u_{n+1} = \dfrac{a^{n+1}}{x^{n+1} + a^{n+1}}$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{a^n}{x^n + a^n} \cdot \frac{x^{n+1} + a^{n+1}}{a^{n+1}} = \frac{x^{n+1} + a^{n+1}}{a\,(x^n + a^n)}.$$

Let $x > a$.

Then
$$\lim \frac{u_n}{u_{n+1}} = \lim \frac{x^{n+1} + a^{n+1}}{a\,(x^n + a^n)} = \lim \frac{x^{n+1}\,[1 + (a/x)^{n+1}]}{ax^n\,[1 + (a/x)^n]}$$
$$= \lim \frac{x}{a} \frac{[1 + (a/x)^{n+1}]}{[1 + (a/x)^n]} = \frac{x}{a}, \text{ which is } > 1 \text{ as } x > a.$$

Hence by ratio test the given series converges if $x > a$.

Let $x < a$.

Then
$$\lim \frac{u_n}{u_{n+1}} = \lim \frac{a^{n+1}\,[1 + (x/a)^{n+1}]}{a \,.\, a^n\,[1 + (x/a)^n]} = \lim \frac{[1 + (x/a)^{n+1}]}{[1 + (x/a)^n]} = 1.$$

$\therefore$ the ratio test fails in this case.

But in this case, $\lim u_n = \lim \dfrac{a^n}{x^n + a^n} = \lim \dfrac{a^n}{a^n\,[1 + (x/a)^n]} = 1$, which is > 0.

$\therefore$ the given series diverges if $x < a$.

Now, if $x = a$, the series is $\dfrac{1}{2} + \dfrac{1}{2} + \dots$, which diverges.

Hence the given series is convergent if $x > a$ and divergent if $x \le a$.

Example 25: *Test for convergence the following series*

(*i*) $\dfrac{1}{2\sqrt{1}} + \dfrac{x^2}{3\sqrt{2}} + \dfrac{x^4}{4\sqrt{3}} + \dfrac{x^6}{5\sqrt{4}} + \dots$ **(Gorakhpur 2013)**

(*ii*) $x + \dfrac{3}{5}x^2 + \dfrac{8}{10}x^3 + \dfrac{15}{17}x^4 + \dots + \dfrac{n^2 - 1}{n^2 + 1}x^n + \dots$. **(Avadh 2012)**

Solution: (i) Here $u_n = \dfrac{x^{2n-2}}{(n+1)\sqrt{n}}, u_{n+1} = \dfrac{x^{2n}}{(n+2)\sqrt{(n+1)}}$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{x^{2n-2}}{(n+1)\sqrt{n}} \cdot \frac{(n+2)\sqrt{(n+1)}}{x^{2n}} = \frac{(1+2/n)}{(1+1/n)} \sqrt{\left(1+\frac{1}{n}\right)} \cdot \frac{1}{x^2}.$$

$$\therefore \quad \lim \frac{u_n}{u_{n+1}} = \frac{1}{1} . \sqrt{1} . \frac{1}{x^2} = \frac{1}{x^2}.$$

$\therefore$ by ratio test the given series is convergent if $1/x^2 > 1$ *i.e.*, if $x^2 < 1$, divergent if $1/x^2 < 1$ *i.e.*, if $x^2 > 1$ and the test fails if $x^2 = 1$.

When $x^2 = 1$, we have $u_n = \dfrac{1}{(n+1)\sqrt{n}}$.

Take $$v_n = \frac{1}{n\sqrt{n}}.$$

$$\therefore \quad \lim \frac{u_n}{v_n} = \lim \frac{n\sqrt{n}}{(n+1)\sqrt{n}} = \lim \frac{1}{(1+1/n)} = 1,$$

which is finite and non-zero. Hence by comparison test $\Sigma\, u_n$ and $\Sigma\, v_n$ are either both convergent or both divergent.

Since $\Sigma\, v_n = \Sigma\,(1/n^{3/2})$ is convergent as $p = 3/2 > 1$, therefore the given series is also convergent if $x^2 = 1$.

Thus the given series is convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$.

(ii) Here $$u_n = \frac{n^2-1}{n^2+1} x^n, \; u_{n+1} = \frac{(n+1)^2-1}{(n+1)^2+1} x^{n+1}.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{n^2-1}{n^2+1} x^n . \frac{(n+1)^2+1}{(n+1)^2-1} \cdot \frac{1}{x^{n+1}}$$

$$= \frac{1-1/n^2}{1+1/n^2} \cdot \frac{1+2/n+2/n^2}{1+2/n} \cdot \frac{1}{x}.$$

$$\therefore \quad \lim \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

$\therefore$ by ratio test the given series is convergent if $1/x > 1$ *i.e.*, if $x < 1$, divergent if $1/x < 1$ *i.e.*, if $x > 1$ and the test fails if $x = 1$.

When $x = 1$, $u_n = \dfrac{n^2-1}{n^2+1} = \dfrac{1-1/n^2}{1+1/n^2}$.

$$\therefore \quad \lim u_n = \lim \frac{1-1/n^2}{1+1/n^2} = 1, \text{ which is } > 0.$$

$\therefore$ the given series is divergent if $x = 1$.

Thus the given series is convergent if $x < 1$ and divergent if $x \geq 1$.

Comprehensive Exercise 3

Test for convergence the following series :

1. (i) $2x + \dfrac{3x^2}{8} + \dfrac{4x^3}{27} + \ldots + \dfrac{(n+1)\,x^n}{n^3} + \ldots$

(Kumaun 2003; Kanpur 11; Meerut 12,12B)

(ii) $\dfrac{x^2}{2\sqrt{1}} + \dfrac{x^3}{3\sqrt{2}} + \dfrac{x^4}{4\sqrt{3}} + \ldots,\ x > 0.$

(iii) $1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots,\ x > 0.$

2. $\dfrac{1^2 \cdot 2^2}{1!} + \dfrac{2^2 \cdot 3^2}{2!} + \dfrac{3^2 \cdot 4^2}{3!} + \ldots$

3. (i) $1 + \dfrac{x}{2} + \dfrac{x^2}{5} + \dfrac{x^3}{10} + \ldots + \dfrac{x^n}{n^2+1} + \ldots$

(ii) $1 + \dfrac{2}{5}x + \dfrac{6}{9}x^2 + \dfrac{14}{17}x^3 + \ldots + \dfrac{2^n - 2}{2^n + 1}x^{n-1} + \ldots$

4. $\dfrac{x}{1\cdot 2} + \dfrac{x^2}{2\cdot 3} + \dfrac{x^3}{3\cdot 4} + \ldots$

(Gorakhpur 2012)

5. $\dfrac{1}{3} + \dfrac{1.2}{3.5} + \dfrac{1.2.3}{3.5.7} + \dfrac{1.2.3.4}{3.5.7.9} + \ldots$

6. (i) $\sum \dfrac{n!\,3^n}{n^n}.$ (ii) $\sum \dfrac{n^3}{(n-1)!}.$

7. (i) $\sum \left(\dfrac{3n-1}{2^n}\right)$ (ii) $\sum \left(\dfrac{x^n}{x+n}\right).$

8. (i) $\sum \dfrac{x^n}{a + \sqrt{n}}$ (ii) $\sum \dfrac{n^n}{n!}.$

9. Test for convergence the series whose n th term is

(i) $\dfrac{n^2(n+1)^2}{n!}$ (ii) $\dfrac{2^n - 1}{3^n + 1}$

(iii) $\dfrac{\sqrt{n}}{\sqrt{(n^2+1)}}x^n, (x > 0)$ (iv) $\dfrac{n^3 - 1}{n^3 + 1}x^n, (x > 0)$

(v) $\sqrt{\left(\frac{n-1}{n^3+1}\right)}\, x^n, (x > 0)$ (vi) $\frac{3^n - 2}{3^n + 1}\, x^{n-1}, (x > 0)$.

(vii) $\frac{x^n}{x^n + a^n}, x > 0, a > 0$.

10. Examine the convergence of the series

$$\frac{1}{1^p} + \frac{x}{3^p} + \frac{x^2}{5^p} + \frac{x^3}{7^p} + \dots$$

Answers 3

1. (i) Convergent if $x \le 1$ and divergent if $x > 1$
   (ii) Convergent if $x \le 1$ and divergent if $x > 1$
   (iii) Convergent for all real values of $x > 0$.
2. Convergent
3. (i) Convergent if $x \le 1$ and divergent if $x > 1$
   (ii) Convergent if $x < 1$ and divergent if $x \ge 1$
4. Convergent if $x \le 1$ and divergent if $x > 1$
5. Convergent
6. (i) Divergent (ii) Convergent
7. (i) Convergent (ii) Convergent if $x < 1$ and divergent if $x \ge 1$
8. (i) Convergent if $x < 1$ and divergent if $x \ge 1$ (ii) Divergent
9. (i) Convergent (ii) Convergent
   (iii) Convergent if $x < 1$ and divergent if $x \ge 1$
   (iv) Convergent if $x < 1$ and divergent if $x \ge 1$
   (v) Convergent if $x < 1$ and divergent if $x \ge 1$
   (vi) Convergent if $x < 1$ and divergent if $x \ge 1$
   (vii) Convergent if $x < a$ and divergent if $x \ge a$
10. Convergent if $x < 1$ and divergent if $x > 1$
    In case $x = 1$, then convergent if $p > 1$ and divergent if $p \le 1$.

11 Cauchy's Condensation Test

(Avadh 2012)

Theorem: *If the function $f(n)$ is positive for all positive integral values of n and continually decreases as n increases, then the two infinite series*

$$f(1) + f(2) + f(3) + \dots + f(n) + \dots$$

and

$$a\, f(a) + a^2 f(a^2) + a^3 f(a^3) + \dots + a^n f(a^n) + \dots$$

are either both convergent or both divergent, a being a positive integer greater than unity.

Proof: The terms in the series $\Sigma\, f(n)$ can be arranged as

$$\{f(1)+f(2)+f(3)+\ldots+f(a)\}$$
$$+\{f(a+1)+f(a+2)+\ldots+f(a^2)\}$$
$$+\{f(a^2+1)+f(a^2+2)+\ldots+f(a^3)\}+\ldots$$
$$\ldots+\{f(a^m+1)+f(a^m+2)+\ldots+f(a^{m+1})\}+\ldots. \qquad \ldots(1)$$

The terms in the $(m+1)$ th group are

$$f(a^m+1)+f(a^m+2)+\ldots+f(a^{m+1}). \qquad \ldots(2)$$

The number of terms in this group is $(a^{m+1}-a^m)$ *i.e.,* $a^m(a-1)$. Also $f(a^{m+1})$ is the smallest term in this group since the terms go on decreasing.

$\therefore \quad f(a^m+1)+f(a^m+2)+\ldots+f(a^{m+1})>a^m(a-1)\,f(a^{m+1})$

or $\quad f(a^m+1)+f(a^m+2)+\ldots+f(a^{m+1})>\dfrac{a-1}{a}\{a^{m+1}f(a^{m+1})\}. \qquad \ldots(3)$

Putting $m=0,1,2,3,\ldots$ successively in (3), we have

$$f(2)+f(3)+\ldots+f(a)>\frac{a-1}{a}\{a\,f(a)\}$$
$$f(a+1)+f(a+2)+\ldots+f(a^2)>\frac{a-1}{a}\{a^2 f(a^2)\}$$
$$f(a^2+1)+f(a^2+2)+\ldots+f(a^3)>\frac{a-1}{a}\{a^3 f(a^3)\}$$
$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$
$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$

Adding all the above inequalities, we get

$$\Sigma\, f(n)-f(1)>\frac{a-1}{a}\,\Sigma\,[a^n f(a^n)].$$

This shows that if the series $\Sigma\, a^n f(a^n)$ is divergent, so also is the series $\Sigma\, f(n)$.

Again, each term of the $(m+1)$ th group given by (2) is less than $f(a^m)$. Hence, we have

$$f(a^m+1)+f(a^m+2)+\ldots+f(a^{m+1})$$
$$<f(a^m)+f(a^m)+\ldots+f(a^m)=a^m(a-1)\,f(a^m)$$

i.e. $\quad f(a^m+1)+f(a^m+2)+\ldots+f(a^{m+1})<(a-1)\{a^m f(a^m)\}. \qquad \ldots(4)$

Putting $m=0,1,2,3,\ldots$ successively in (4), we have

$$f(2)+f(3)+\ldots+f(a)<(a-1)\{f(1)\}$$
$$f(a+1)+f(a+2)+\ldots+f(a^2)<(a-1)\{a\,f(a)\}$$
$$f(a^2+1)+f(a^2+2)+\ldots+f(a^3)<(a-1)\{a^2 f(a^2)\}$$
$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$
$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$

Adding all these inequalities, we get

$$\Sigma\, f(n)-f(1)<(a-1)\,f(1)+(a-1)\,\Sigma\, a^n f(a^n).$$

This shows that if $\Sigma\, a^n f(a^n)$ is convergent, so also is $\Sigma\, f(n)$.

Note: For the validity of the above theorem it is sufficient if $f(n)$ be positive and constantly decreases for values of n greater than a fixed positive integer r.

12 The Auxiliary Series $\Sigma \dfrac{1}{n(\log n)^p}$

Theorem: *The series*

$$1+\frac{1}{2\,(\log 2)^p}+\frac{1}{3\,(\log 3)^p}+\ldots+\frac{1}{n\,(\log n)^p}+\ldots$$

is convergent if $p>1$ *and divergent if* $p\le 1$.

Proof: Case 1: Let $p\le 0$.

Then $\dfrac{1}{n\,(\log n)^p}\ge\dfrac{1}{n}$ for all $n\ge 2$.

Since the series $\Sigma\,(1/n)$ is divergent, therefore by comparison test $\Sigma\,\dfrac{1}{n\,(\log n)^p}$ is also divergent.

Case 2: Let $p>0$. Let $f(n)=\dfrac{1}{n\,(\log n)^p}$.

Obviously $f(n)>0$ for all $n\ge 2$.

Now the given series $\Sigma\,\dfrac{1}{n\,(\log n)^p}=\Sigma\,f(n)$.

Since $<n\,(\log n)^p>$ is an increasing sequence, therefore $<f(n)>$ is a decreasing sequence. Hence by Cauchy's condensation test given in article 11, the series $\Sigma\,f(n)$ is convergent or divergent according as the series $\Sigma\,a^n f(a^n)$ is convergent or divergent.

Now $$a^n f(a^n)=\frac{a^n}{a^n\,(\log a^n)^p}=\frac{1}{(n\log a)^p}=\frac{1}{(\log a)^p}\cdot\frac{1}{n^p}.$$

Since $\dfrac{1}{(\log a)^p}$ is a constant, hence the series $\Sigma\,a^n f(a^n)$ is convergent or divergent according as the series $\Sigma\,(1/n^p)$ is convergent or divergent. But the series $\Sigma\,1/n^p$ is convergent if $p>1$ and divergent if $p\le 1$.

Hence by Cauchy's condensation test the given series is also convergent if $p>1$ and divergent if $p\le 1$.

Illustrative Examples

Example 26: *Test the convergence of the following series :*

(*i*) $\dfrac{1}{\log 2}+\dfrac{1}{\log 3}+\dfrac{1}{\log 4}+\ldots$ (*ii*) $\dfrac{\log 2}{2}+\dfrac{\log 3}{3}+\dfrac{\log 4}{4}+\ldots$

(*iii*) $\dfrac{1}{(log\ 2)^p} + \dfrac{1}{(log\ 3)^p} + \ldots + \dfrac{1}{(log\ n)^p} + \ldots$

Solution: (i) Here $f(n) = \dfrac{1}{\log n} > 0$ for all $n \geq 2$. Also $f(n)$ decreases continually as n increases.

Now $a^n f(a^n) = \dfrac{a^n}{\log(a^n)} = \dfrac{a^n}{n \log a}$, a being taken as some positive integer > 1.

Consider the series $\Sigma\, a^n f(a^n) = \Sigma \{a^n / (n \log a)\} = \Sigma\, v_n$, (say).

Here $v_n = \dfrac{a^n}{n \log a}$, so that $v_{n+1} = \dfrac{a^{n+1}}{(n+1) \log a}$.

$$\therefore \qquad \frac{v_n}{v_{n+1}} = \frac{n+1}{n} \cdot \frac{1}{a} = \left(1 + \frac{1}{n}\right) \cdot \frac{1}{a}.$$

$$\therefore \qquad \lim \frac{v_n}{v_{n+1}} = \frac{1}{a} \text{ which is} < 1 \text{ as by our choice } a > 1.$$

$\therefore$ by ratio test the series $\Sigma\, v_n = \Sigma\, a^n f(a^n)$ is divergent.

Consequently by Cauchy's condensation test the given series

$$\Sigma\, f(n) = \frac{1}{\log 2} + \frac{1}{\log 3} + \ldots, \text{ is also divergent.}$$

(ii) Here $\quad f(n) = \dfrac{\log n}{n} > 0$ for all $n \geq 2$.

Also $f(n)$ decreases continually as n increases.

Now $a^n f(a^n) = a^n \left(\dfrac{\log a^n}{a^n}\right) = n \log a$, a being taken as some + ive integer > 1.

Now the series $\Sigma\, a^n f(a^n) = \Sigma\, (n \log a) = \log a \,.\, \Sigma\, n$

is divergent because the series $\Sigma\, n$ is divergent.

Hence by Cauchy's condensation test the given series $\Sigma\, f(n) = \Sigma\, \dfrac{\log n}{n}$ is also divergent.

(iii) If $p \leq 0$, the given series is obviously divergent. So let us consider the case when $p > 0$. Here $f(n) = \dfrac{1}{(\log n)^p} > 0$ for all $n \geq 2$.

Also $f(n)$ decreases continually as n increases.

Now $a^n f(a^n) = \dfrac{a^n}{(\log a^n)^p} = \dfrac{a^n}{n^p (\log a)^p}$, a being taken > 1.

Consider the series $\Sigma\, a^n f(a^n) = \Sigma\, \dfrac{a^n}{n^p (\log a)^p} = \Sigma\, v_n$, say.

Here $v_n = \dfrac{a^n}{n^p (\log a)^p}$, so that $v_{n+1} = \dfrac{a^{n+1}}{(n+1)^p (\log a)^p}$.

$$\therefore \quad \frac{v_n}{v_{n+1}} = \frac{a^n}{n^p (\log a)^p} \cdot \frac{(n+1)^p (\log a)^p}{a^{n+1}} = \left(1 + \frac{1}{n}\right)^p \cdot \frac{1}{a} \cdot$$

$$\therefore \quad \lim \frac{v_n}{v_{n+1}} = \frac{1}{a} \text{ which is} < 1 \text{ as } a > 1.$$

$\therefore$ by ratio test the series $\Sigma v_n = \Sigma a^n f(a^n)$ is divergent.

Therefore by Cauchy's condensation test the given series $\Sigma f(n)$ is also divergent.

Example 27: *Test for convergence the following series*

(*i*) $\dfrac{(\log 2)^2}{2^2} + \dfrac{(\log 3)^2}{3^2} + \dfrac{(\log 4)^2}{4^2} + \ldots + \dfrac{(\log n)^2}{n^2} + \ldots$

(*ii*) $\dfrac{1}{(2 \log 2)^p} + \dfrac{1}{(3 \log 3)^p} + \ldots + \dfrac{1}{(n \log n)^p} + \ldots$

Solution: (i) Here we can take the first term of the series as

$$\frac{(\log 1)^2}{1^2} \text{ because } \log 1 = 0.$$

$$\therefore \quad u_n = n\text{th term of the series} = \frac{(\log n)^2}{n^2} = f(n), \text{ say.}$$

It is positive for all $n \geq 2$ and decreases continually as n increases.

Now $$a^n f(a^n) = \frac{a^n (\log a^n)^2}{(a^n)^2} = \frac{a^n n^2 (\log a)^2}{(a^n)^2} = \frac{n^2 (\log a)^2}{a^n},$$

a being taken to be a +ive integer > 1.

Consider the series $\Sigma a^n f(a^n) = \Sigma \{ n^2 (\log a)^2 / a^n \} = \Sigma v_n$, (say).

Here $$v_n = \frac{n^2 (\log a)^2}{a^n}, \; v_{n+1} = \frac{(n+1)^2 (\log a)^2}{a^{n+1}}.$$

$$\therefore \quad \frac{v_n}{v_{n+1}} = \frac{n^2 (\log a)^2}{a^n} \cdot \frac{a^{n+1}}{(n+1)^2 (\log a)^2} = \frac{a}{(1 + 1/n)^2}.$$

$$\therefore \quad \lim \frac{v_n}{v_{n+1}} = \lim \frac{a}{(1 + 1/n)^2} = a > 1 \text{ since by our choice } a > 1.$$

$\therefore$ by ratio test the series Σv_n is convergent.

Hence by Cauchy's condensation test the given series $\Sigma f(n)$ is also convergent.

(ii) If $p \leq 0$, obviously the given series is divergent. So it remains to discuss the case when $p > 0$.

When $p > 0$, we have $f(n) = \dfrac{1}{(n \log n)^p} > 0$ for all $n \geq 2$ and it decreases continually as n increases.

Now $$a^n f(a^n) = \frac{a^n}{(a^n \log a^n)^p} = \frac{1}{a^{n(p-1)} \cdot n^p (\log a)^p}, \; a \text{ to be taken} > 1.$$

Case I: Let $p > 1$. Then $a^{n(p-1)} > 1$ as $a > 1$.

$$\therefore \qquad a^n f(a^n) = \frac{1}{a^{n(p-1)} \cdot n^p (\log a)^p} < \frac{1}{(\log a)^p} \cdot \frac{1}{n^p} \qquad \ldots(1)$$

Now $1/(\log a)^p$ is a fixed positive real number and the series $\Sigma (1/n^p)$ is convergent because $p > 1$.

Hence from (1), by comparison test (second form) given in article 7, the series $\Sigma\, a^n f(a^n)$ is convergent.

Now by Cauchy's condensation test it follows that the given series $\Sigma f(n)$ is also convergent.

Case II: Let $p \le 1$. Then $a^{n(p-1)} \le 1$ as $a > 1$.

$$\therefore \qquad a^n f(a^n) \ge \frac{1}{(\log a)^p} \cdot \frac{1}{n^p}. \qquad \ldots(2)$$

Now $1/(\log a)^p$ is a fixed +ive real number and the series $\Sigma (1/n^p)$ is divergent, p being ≤ 1.

Hence from (2), by comparison test the series $\Sigma\, a^n f(a^n)$ is divergent.

Now by Cauchy's condensation test it follows that the given series $\Sigma f(n)$ is also divergent.

Hence the given series is convergent if $p > 1$ and divergent if $p \le 1$.

13 Raabe's Test

Theorem: *The series Σu_n of positive terms is convergent or divergent according as*

$$lim \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right\} > 1 \; or < 1.$$

(Gorakhpur 2014)

Proof: Case I: Let $\lim \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right\} = k$, where $k > 1$.

Choose a number p such that $k > p > 1$.

Compare the series Σu_n with the auxiliary series $\Sigma v_n = \Sigma \frac{1}{n^p}$, which is convergent since $p > 1$.

By article 7, sixth form of comparison test, Σu_n is convergent if after some particular term

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}$$

or

$$\frac{u_n}{u_{n+1}} > \frac{1/n^p}{1/(n+1)^p} = \left(\frac{n+1}{n} \right)^p = \left(1 + \frac{1}{n} \right)^p$$

$$= 1 + p \cdot \frac{1}{n} + \frac{p(p-1)}{2!} \cdot \frac{1}{n^2} + \ldots$$

or $$n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + \frac{p(p-1)}{2!} \cdot \frac{1}{n} + \ldots \qquad \ldots(1)$$

If n be taken sufficiently large the L.H.S and R.H.S. of (1) respectively approach k and p. Also k is greater than p. Therefore (1) is satisfied for sufficiently large values of n. Hence $\Sigma\, u_n$ is convergent if

$$\lim \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) \right\} > 1.$$

Case II: Let $\lim \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) \right\} = l$, where $l < 1$.

Choose a number p such that $l < p < 1$.

Compare the series $\Sigma\, u_n$ with the auxiliary series $\Sigma\, v_n = \Sigma\, (1/n^p)$ which is divergent since $p < 1$.

The series $\Sigma\, u_n$ is divergent if after some particular term

$$\frac{u_n}{u_{n+1}} < \frac{v_n}{v_{n+1}}, \qquad \text{[By article 7, sixth form of comparison test]}$$

or $$n\left(\frac{u_n}{u_{n+1}} - 1\right) < p + \frac{p(p-1)}{2!} \cdot \frac{1}{n} + \ldots \qquad \ldots(2)$$

(Proceeding as in case I)

If n be taken sufficiently large the L.H.S. and R.H.S. of (2) respectively approach l and p. Also $l < p$. Thus (2) is satisfied for sufficiently large values of n. Hence $\Sigma\, u_n$ is divergent if

$$\lim \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) \right\} < 1.$$

However, if $\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = 1$, the Raabe's test fails.

Note: Raabe's test is to be applied when D'Alembert's ratio test fails.

Illustrative Examples

Example 28: *Test the convergence of the series*

(*i*) $\Sigma \dfrac{n!\, x^n}{3 \cdot 5 \cdot 7 \ldots (2n+1)}$ (*ii*) $\sum_{n=1}^{\infty} \dfrac{1}{1 + \log n}$.

Solution: (i) Here $u_n = \dfrac{n!\,x^n}{3.5.7\ldots(2n+1)}$

so that $$u_{n+1} = \frac{(n+1)!\,x^{n+1}}{3.5.7\ldots.(2n+1)(2n+3)}.$$

Now $$\frac{u_n}{u_{n+1}} = \frac{(2n+3)}{(n+1)!}\cdot\frac{n!}{x} = \frac{2n+3}{n+1}\cdot\frac{1}{x} = \left(\frac{2+3/n}{1+1/n}\right)\cdot\frac{1}{x}.$$

$\therefore$ $$\lim\frac{u_n}{u_{n+1}} = \lim\left(\frac{2+3/n}{1+1/n}\right)\cdot\frac{1}{x} = \frac{2}{x}.$$

Hence by D' Alembert's ratio test the series converges if $\dfrac{2}{x} > 1$ *i.e.,* if $x < 2$ and diverges if $2/x < 1$ *i.e.,* if $x > 2$ and the test fails when $2/x = 1$ *i.e.*, when $x = 2$.

In case $x = 2$, we apply Raabe's test.

When $$x = 2,\ \frac{u_n}{u_{n+1}} = \frac{2n+3}{2(n+1)}.$$

$\therefore$ $$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{2n+3}{2n+2} - 1\right) = \frac{n}{2(n+1)} = \frac{1}{2(1+1/n)}.$$

$\therefore$ $$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim\frac{1}{2(1+1/n)} = \frac{1}{2} < 1.$$

Hence by Raabe's test $\Sigma\, u_n$ is divergent if $x = 2$.

Thus the given series $\Sigma\, u_n$ is convergent if $x < 2$ and divergent if $x \geq 2$.

(ii) Here $$u_n = \frac{1}{1+\log n};\ u_{n+1} = \frac{1}{1+\log(n+1)}.$$

Now $$\frac{u_n}{u_{n+1}} = \frac{1+\log(n+1)}{1+\log n}$$
$$= \frac{1+\log\{n(1+1/n)\}}{1+\log n} = \frac{1+\log n+\log(1+1/n)}{1+\log n}$$
$$= \frac{\log(en)+\log(1+1/n)}{\log(en)} = 1+\frac{1}{\log(en)}\log(1+1/n)$$
$$= 1+\frac{1}{\log(en)}\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\right)$$
$$= 1+\frac{1}{n\log(en)} - \frac{1}{2n^2\log(en)} + \ldots.$$

$\therefore$ $\lim\dfrac{u_n}{u_{n+1}} = 1$, and so the ratio test fails.

Now we apply Raabe's test. We have

$$\lim n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim n\left[\frac{1}{n\log(en)}-\frac{1}{2n^2\log(en)}+\ldots\right]$$

$$=\lim\left[\frac{1}{\log(en)}-\frac{1}{2n\log(en)}+\ldots\right]=0<1.$$

Hence by Raabe's test the given series is divergent.

Example 29: *Test the convergence of the series*

(i) $\dfrac{1^2}{4^2}+\dfrac{1^2.5^2}{4^2.8^2}+\dfrac{1^2.5^2.9^2}{4^2.8^2.12^2}+\dfrac{1^2.5^2.9^2.13^2}{4^2.8^2.12^2.16^2}+\ldots$ **(Meerut 2013)**

(*ii*) $1+\dfrac{3}{7}x+\dfrac{3.6}{7.10}x^2+\dfrac{3.6.9}{7.10.13}x^3+\dfrac{3.6.9.12}{7.10.13.16}x^4+\ldots$ **(Meerut 2013B)**

Solution: (i) Here $u_n=\dfrac{1^2.5^2.9^2\ldots\ldots(4n-3)^2}{4^2.8^2.12^2\ldots\ldots(4n)^2}$.

[Note that the nth term of the sequence $1^2,5^2,9^2,\ldots$ is

$$\{1+(n-1)4\}^2 \text{ i.e., } (4n-3)^2$$

and the nth term of the sequence

$$4^2,8^2,12^2,\ldots \text{ is } \{4+(n-1)4\}^2 \text{ i.e., } (4n)^2 \text{].}$$

Then $$u_{n+1}=\frac{1^2.5^2.9^2\ldots\ldots(4n-3)^2.(4n+1)^2}{4^2.8^2.12^2\ldots\ldots(4n)^2.(4n+4)^2}.$$

Now $$\frac{u_n}{u_{n+1}}=\frac{(4n+4)^2}{(4n+1)^2}=\frac{(4+4/n)^2}{(4+1/n)^2}.$$

$\therefore$ $$\lim\frac{u_n}{u_{n+1}}=1, \text{ so that the ratio test fails.}$$

Now we apply Raabe's test. We have

$$\lim n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim n\left[\frac{(4n+4)^2}{(4n+1)^2}-1\right]=\lim n\left[\frac{24n+15}{(4n+1)^2}\right]$$

$$=\lim\left\{\frac{24+15/n}{(4+1/n)^2}\right\}=\frac{24}{4^2}=\frac{3}{2}, \text{ which is } >1.$$

Hence by Raabe's test the series is convergent.

(ii) Omitting the first term of the series, we have
nth term of the sequence 3, 6, 9, … is $3+(n-1)3=3n$

and nth term of the sequence 7, 10, 13, … is $7+(n-1)3=3n+4$.

$\therefore$ $$u_n=\frac{3.6.9\ldots\ldots3n}{7.10.13\ldots\ldots(3n+4)}x^n,$$

and $$u_{n+1}=\frac{3.6.9\ldots\ldots3n.(3n+3)}{7.10.13\ldots\ldots(3n+4)(3n+7)}x^{n+1}.$$

$\therefore \qquad \dfrac{u_n}{u_{n+1}} = \left(\dfrac{3n+7}{3n+3}\right) \cdot \dfrac{1}{x} = \left(\dfrac{3+7/n}{3+3/n}\right) \cdot \dfrac{1}{x}.$

$\therefore \qquad \lim \dfrac{u_n}{u_{n+1}} = \lim \left(\dfrac{3+7/n}{3+3/n}\right) \cdot \dfrac{1}{x} = \dfrac{3}{3} \cdot \dfrac{1}{x} = \dfrac{1}{x}.$

Hence by ratio test, the series is convergent if $1/x > 1$ *i.e.,* if $x < 1$, divergent if $1/x < 1$ *i.e.,* if $x > 1$ and the test fails if $1/x = 1$ *i.e.,* if $x = 1$.

If $x = 1$, then $\dfrac{u_n}{u_{n+1}} = \dfrac{3n+7}{3n+3}$.

Now we apply Raabe's test. We have

$$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim n\left(\frac{3n+7}{3n+3} - 1\right) = \lim \frac{4n}{3n+3}$$

$$= \lim \frac{4}{3+3/n} = \frac{4}{3}, \text{ which is} > 1.$$

Hence the series is convergent when $x = 1$.

Thus the given series is convergent if $x \le 1$ and divergent if $x > 1$.

Example 30: *Test for convergence the following series*

$$1 + a + \frac{a\,(a+1)}{1.2} + \frac{a\,(a+1)\,(a+2)}{1.2.3} + \ldots$$

Solution: Leaving the first term, we have

$$u_n = \frac{a\,(a+1)\,(a+2)\ldots(a+n-1)}{1.2.3\ldots n},$$

and then $\qquad u_{n+1} = \dfrac{a\,(a+1)\,(a+2)\ldots(a+n-1)\,(a+n)}{1.2.3\ldots n\,(n+1)}.$

Now $\qquad \dfrac{u_n}{u_{n+1}} = \dfrac{n+1}{a+n} = \dfrac{1+1/n}{a/n+1}.$

$\therefore \qquad \lim \dfrac{u_n}{u_{n+1}} = 1$, so that the ratio test fails.

Now we apply Raabe's test. We have

$$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim n\left(\frac{n+1}{a+n} - 1\right)$$

$$= \lim \frac{n\,(1-a)}{a+n} = \lim \frac{(1-a)}{1+a/n} = 1-a.$$

Hence by Raabe's test, the given series is convergent if $1 - a > 1$ *i.e.*, if $a < 0$, divergent if $1 - a < 1$ *i.e.*, if $a > 0$ and the test fails if $1 - a = 1$ *i.e.*, if $a = 0$.

In case $a = 0$, the given series becomes $1 + 0 + 0 + 0 + \ldots$

The sum of n terms of this series is always 1. Therefore the series is convergent if $a = 0$.

Thus the given series is convergent if $a \le 0$ and divergent if $a > 0$.

Comprehensive Exercise 4

Test for convergence the following series :

1. $1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\ldots$

2. $1+\frac{1}{2}\cdot\frac{x^2}{4}+\frac{1.3.5}{2.4.6}\cdot\frac{x^4}{8}+\frac{1.3.5.7.9}{2.4.6.8.10}\cdot\frac{x^6}{12}+\ldots$

3. $x^2+\frac{2^2}{3.4}x^4+\frac{2^2.4^2}{3.4.5.6}x^6+\frac{2^2.4^2.6^2}{3.4.5.6.7.8}x^8+\ldots$ **(Kanpur 2014)**

4. (i) $1+\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\frac{x^7}{7}+\ldots$ **(Kashi 2014)**

(ii) $\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\frac{x^7}{7}+\ldots$ **(Gorakhpur 2012, 14)**

(iii) $\frac{x}{1}+\frac{1}{2}\cdot\frac{x^2}{3}+\frac{1.3}{2.4}\cdot\frac{x^3}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^4}{7}+\ldots, x>0.$

5. $\frac{a}{a+3}+\frac{a(a+2)}{(a+3)(a+5)}x+\frac{a(a+2)(a+4)}{(a+3)(a+5)(a+7)}x^2+\ldots$

6. $\sum \frac{4.7\ldots..(3n+1)}{1.2\ldots..n}x^n.$

7. Apply Cauchy's condensation test to discuss the convergence of the series

$$\sum_{n=2}^{\infty}\frac{1}{n\log n(\log\log n)^p}.$$

Answers 4

1. Convergent if $x<1$ and divergent if $x\geq 1$
2. Convergent if $x^2\leq 1$ and divergent if $x^2>1$
3. Convergent if $x^2\leq 1$ and divergent if $x^2>1$
4. (i) Convergent if $x^2\leq 1$ and divergent if $x^2>1$

   (ii) Convergent if $x^2\leq 1$ and divergent if $x^2>1$

   (iii) Convergent if $x\leq 1$ and divergent if $x>1$
5. Convergent if $x\leq 1$ and divergent if $x>1$
6. Convergent if $x<1/3$ and divergent if $x\geq 1/3$
7. Convergent if $p>1$ and divergent if $p\leq 1$

14 Logarithmic Test

Theorem: *The series Σu_n of positive terms is convergent or divergent according as*

$$\lim \left\{ n \log \frac{u_n}{u_{n+1}} \right\} > 1 \text{ or } < 1.$$

Proof: First suppose that

$$\lim \left\{ n \log \frac{u_n}{u_{n+1}} \right\} = k, \text{ where } k > 1.$$

Choose a number p such that $k > p > 1$.

Compare the given series with the auxiliary series Σv_n where $v_n = 1/n^p$, which is convergent as $p > 1$.

The series Σu_n is convergent if after some particular term

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}} \qquad \text{[By article 7, sixth form of comparison test.]}$$

or $$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} = \left(1 + \frac{1}{n}\right)^p$$

or $$\log \frac{u_n}{u_{n+1}} > \log \left(1 + \frac{1}{n}\right)^p = p \log \left(1 + \frac{1}{n}\right) = p \left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots\right]$$

or $$n \log \frac{u_n}{u_{n+1}} > p - \frac{p}{2n} + \frac{p}{3n^2} - \dots \qquad \dots(1)$$

If n is taken sufficiently large the L.H.S. and R.H.S. of (1) respectively approach k and p. Also $k > p$.

Thus (1) is satisfied for sufficiently large values of n. Hence the series Σu_n is convergent if

$$\lim \left\{ n \log \frac{u_n}{u_{n+1}} \right\} > 1 \cdot$$

Similarly, it can be proved that Σu_n is divergent if

$$\lim \left\{ n \log \frac{u_n}{u_{n+1}} \right\} < 1.$$

[The procedure of proof will be the same as given in the proof of Raabe's test]

However, if $\lim \left\{ n \log \frac{u_n}{u_{n+1}} \right\} = 1$, the test fails.

Note: This test is an alternative to Raabe's test and is to be applied when D'Alembert's ratio test fails and when either

(i) n occurs as an exponent in $\frac{u_n}{u_{n+1}}$ so that it is not convenient to find $\frac{u_n}{u_{n+1}} - 1$

(ii) taking logarithm of $\frac{u_n}{u_{n+1}}$ makes the evaluation of limits easier.

Illustrative Examples

Example 31: *Test for convergence the series*

$$1+\frac{2x}{2!}+\frac{3^2 x^2}{3!}+\frac{4^3 x^3}{4!}+\frac{5^4 x^4}{5!}+\ldots\ldots$$

(Kashi 2013; Avadh14)

Solution: Here $u_n=\dfrac{n^{n-1}}{n!}x^{n-1},\ u_{n+1}=\dfrac{(n+1)^n}{(n+1)!}x^n.$

Now $$\frac{u_n}{u_{n+1}}=\frac{n^{n-1}}{n!}\frac{(n+1)!}{(n+1)^n}\cdot\frac{1}{x}=\frac{n^{n-1}(n+1)}{(n+1)^n}\cdot\frac{1}{x}$$

$$=\left(\frac{n}{n+1}\right)^{n-1}\cdot\frac{1}{x}=\frac{1}{(1+1/n)^{n-1}}\frac{1}{x}$$

$$=\frac{1}{(1+1/n)^n}\cdot(1+1/n)\cdot\frac{1}{x}\cdot$$

$\therefore$ $$\lim\frac{u_n}{u_{n+1}}=\lim\left\{\frac{(1+1/n)}{(1+1/n)^n}\cdot\frac{1}{x}\right\}=\frac{1}{ex}\cdot \qquad [\because \lim(1+1/n)^n=e]$$

$\therefore$ by ratio test the series Σu_n converges if $1/ex>1$ *i.e.,* if $x<1/e$, diverges if $1/ex<1$ *i.e.,* if $x>1/e$ and the test fails if $1/ex=1$ *i.e.* if $x=1/e$.

Now if $x=1/e, \dfrac{u_n}{u_{n+1}}=\dfrac{e(1+1/n)}{(1+1/n)^n}\cdot$ Applying log test, we get

$$\lim\left(n\log\frac{u_n}{u_{n+1}}\right)=\lim\left[n\log\left\{\frac{e(1+1/n)}{(1+1/n)^n}\right\}\right]$$

$$=\lim n[\log e+\log(1+1/n)-n\log(1+1/n)]$$

$$=\lim n\left[1+\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\ldots\right)-n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}\ldots\right)\right]$$

$$=\lim n\left[\left(1+\frac{1}{2}\right)\cdot\frac{1}{n}+\left(-\frac{1}{2}-\frac{1}{3}\right)\frac{1}{n^2}+\ldots\right]$$

$$=\lim\left[\frac{3}{2}-\frac{5}{6n}+\ldots\right]=\frac{3}{2}\quad i.e.,\ >1.$$

Therefore the series Σu_n converges when $x=1/e$.

Hence the given series is convergent if $x\le 1/e$ and divergent if $x>1/e$.

Example 32: *Test for convergence the series*

$$\frac{(a+x)}{1!}+\frac{(a+2x)^2}{2!}+\frac{(a+3x)^3}{3!}+\ldots$$

Solution: Here $u_n=\dfrac{(a+nx)^n}{n!},\ u_{n+1}=\dfrac{[a+(n+1)x]^{n+1}}{(n+1)!}\cdot$

Now $$\frac{u_n}{u_{n+1}}=\frac{(a+nx)^n}{n!}\cdot\frac{(n+1)!}{[a+(n+1)x]^{n+1}}=\frac{(a+nx)^n(n+1)}{[a+(n+1)x]^{n+1}}$$

$$= \frac{(n\,x)^n\,(a/nx+1)^n\,n\,(1+1/n)}{(n+1)^{n+1}\,x^{n+1}\,[a/(n+1)\,x+1]^{n+1}}$$

$$= \frac{1}{x}\cdot\frac{n^{n+1}\,(1+a/nx)^n\,(1+1/n)}{n^{n+1}\,(1+1/n)^{n+1}\,[1+a/(n+1)\,x]^{n+1}}$$

$$= \frac{1}{x}\cdot\frac{\left[1+\frac{(a/x)}{n}\right]^n}{\left[1+\frac{1}{n}\right]^n\left[1+\frac{(a/x)}{n+1}\right]^{n+1}}.$$

$\therefore$ $$\lim \frac{u_n}{u_{n+1}} = \frac{1}{x}\,\frac{e^{a/x}}{e.\,e^{a/x}} \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x\right]$$

$$= \frac{1}{ex}.$$

Hence by ratio test the given series is convergent if $1/e\,x > 1$ *i.e.*, if $x < 1/e$, divergent if $1/ex < 1$ *i.e.,* if $x > 1/e$ and the test fails if $1/ex = 1$ *i.e.,* if $x = 1/e$.

If $x = 1/e$, $$\frac{u_n}{u_{n+1}} = \frac{e\left(1+\frac{ea}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n\left[1+\frac{ae}{n+1}\right]^{n+1}}.$$

Applying logarithmic test, we get

$$\lim\left(n\log\frac{u_n}{u_{n+1}}\right) = \lim n\log\left[\frac{e\left(1+\frac{ea}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n\left\{1+\frac{ae}{n+1}\right\}^{n+1}}\right]$$

$$= \lim n\left[\log e + n\log\left(1+\frac{ea}{n}\right) - n\log\left(1+\frac{1}{n}\right) - (n+1)\log\left(1+\frac{ae}{n+1}\right)\right]$$

$$= \lim n\left[1 + n\left(\frac{ea}{n} - \frac{e^2a^2}{2n^2} + \frac{e^3a^3}{3n^3} - \ldots\right) - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\right)\right.$$
$$\left. - (n+1)\left\{\frac{ea}{n+1} - \frac{e^2a^2}{2\,(n+1)^2} + \ldots\right\}\right]$$

$$= \lim n\left[\left(-\frac{e^2a^2}{2} + \frac{1}{2}\right)\frac{1}{n} + \frac{e^2a^2}{2\,(n+1)} + \left(\frac{e^3a^3}{3} - \frac{1}{3}\right)\frac{1}{n^2} + \ldots\right]$$

$$= \lim\left[\left(-\frac{e^2\,a^2}{2} + \frac{1}{2}\right) + \frac{e^2a^2}{2\,(1+1/n)} + \left(\frac{e^3a^3-1}{3n}\right) + \ldots\right]$$

$$= -\frac{e^2a^2}{2} + \frac{1}{2} + \frac{e^2a^2}{2} = \frac{1}{2}, \text{ which is} < 1.$$

$\therefore$ the series is divergent if $x = 1/e$.

Thus the given series is convergent if $x < 1/e$ and divergent if $x \geq 1/e$.

Comprehensive Exercise 5

Test for convergence the following series :

1. $x^2 (\log 2)^q + x^3 (\log 3)^q + x^4 (\log 4)^q + \ldots$
2. $\dfrac{1}{(\log 2)^p} + \dfrac{1}{(\log 3)^p} + \ldots + \dfrac{1}{(\log n)^p} + \ldots$
3. (i) $x + \dfrac{2^2 x^2}{2!} + \dfrac{3^3 x^3}{3!} + \dfrac{4^4 x^4}{4!} + \dfrac{5^5 x^5}{5!} + \ldots$ **(Gorakhpur 2013)**

   (ii) $1 + \dfrac{2^2 x^2}{2!} + \dfrac{3^3 x^3}{3!} + \dfrac{4^4 x^4}{4!} + \dfrac{5^5 x^5}{5!} + \ldots$
4. $1 + \dfrac{1!}{2} x + \dfrac{2!}{3^2} x^2 + \dfrac{3!}{4^3} x^3 + \dfrac{4!}{5^4} x^4 + \ldots$ **(Kanpur 2014)**

Answers 5

1. Convergent if $x < 1$ and divergent if $x \geq 1$
2. Divergent for all values of p
3. (i) Convergent if $x < 1/e$ and divergent if $x \geq 1/e$

   (ii) Convergent if $x < 1/e$ and divergent if $x \geq 1/e$
4. Convergent if $x < e$ and divergent if $x \geq e$

15 De Morgan's and Bertrand's Test

Theorem: *The series Σu_n of positive terms is convergent or divergent according as*

$$lim \left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} log\ n\right] > 1 \ or < 1.$$

Proof: Let $\lim \left[\left\{n\left(\dfrac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n\right] = k$, where $k > 1$.

Take a number p such that $k > p > 1$.

Compare the series Σu_n with the auxiliary series Σv_n, where $v_n = \dfrac{1}{n (\log n)^p}$, which is convergent as $p > 1$.

The series $\Sigma\, u_n$ is convergent if after some particular term

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}, \qquad \text{[By article 7, sixth from of comparison test]}$$

i.e., $$\frac{u_n}{u_{n+1}} > \frac{1}{n\,(\log n)^p} \cdot (n+1)\,\{\log (n+1)\}^p, \qquad \left[\because\ v_n = \frac{1}{n\,(\log n)^p}\right]$$

i.e., $$\frac{u_n}{u_{n+1}} > \left(\frac{n+1}{n}\right)\left[\frac{\log\{n\,(1+1/n)\}}{\log n}\right]^p$$

i.e., $$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left[\frac{\log n + \log (1+1/n)}{\log n}\right]^p$$

i.e., $$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left[\frac{\log n + \frac{1}{n} - \frac{1}{2n^2} + \ldots}{\log n}\right]^p$$

i.e., $$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left[1 + \frac{1}{n \log n} - \frac{1}{2n^2 \log n} + \ldots\right]^p$$

i.e., $$\frac{u_n}{u_{n+1}} > \left(1+\frac{1}{n}\right)\left[1 + \frac{p}{n \log n} + \ldots\right]$$

i.e., $$\frac{u_n}{u_{n+1}} > 1 + \frac{1}{n} + \frac{p}{n \log n} + \ldots$$

i.e., $$n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1 + \frac{p}{\log n} + \ldots$$

i.e., $$n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 > \frac{p}{\log n} + \ldots$$

i.e., $$\left[n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right] \log n > p + \text{terms containing } n \text{ or } \log n$$

in the denominator. ...(1)

Now as n becomes sufficiently large the L.H.S. and R.H.S. of (1) respectively approach k and p. Also $k > p$.

Thus (1) is satisfied for sufficiently large values of n.

Hence the series $\Sigma\, u_n$ is convergent if

$$\lim \left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n\right] > 1.$$

Similarly, it can be proved as in the case of Raabe's test that $\Sigma\, u_n$ is divergent if

$$\lim \left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n\right] < 1.$$

Note: This test is to be applied when both D' Alembert's ratio test and Raabe's test fail.

16 An Alternative to Bertrand's Test

Theorem: *The series $\Sigma\, u_n$ of positive terms is convergent or divergent according as*

$$\lim\left[\left(n \log \frac{u_n}{u_{n+1}} - 1\right) \log n\right] > 1 \text{ or } < 1.$$

Proof: Let $\lim\left[\left(n \log \frac{u_n}{u_{n+1}} - 1\right) \log n\right] = k$, where $k > 1$.

Take a number p such that $k > p > 1$.

Compare the given series $\Sigma\, u_n$ with the auxiliary series $\Sigma\, v_n$ where $v_n = \frac{1}{n\,(\log n)^p}$, which is convergent since $p > 1$. The series $\Sigma\, u_n$ is convergent if after some particular term

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}},$$ by article 7, sixth form of comparison test

i.e. $$\frac{u_n}{u_{n+1}} > 1 + \frac{1}{n} + \frac{p}{n \log n} + \ldots$$ [Proceeding as in article 15]

i.e. $$\log \frac{u_n}{u_{n+1}} > \log \left\{1 + \left(\frac{1}{n} + \frac{p}{n \log n} + \ldots\right)\right\}$$

i.e. $$\log \frac{u_n}{u_{n+1}} > \left(\frac{1}{n} + \frac{p}{n \log n} + \ldots\right) - \frac{1}{2}\left(\frac{1}{n} + \frac{p}{n \log n} + \ldots\right)^2 + \ldots$$

i.e. $$n \log \frac{u_n}{u_{n+1}} > n\left[\frac{1}{n} + \frac{p}{n \log n} - \frac{1}{2n^2} + \ldots\right]$$

i.e. $$n \log \frac{u_n}{u_{n+1}} > 1 + \frac{p}{\log n} - \frac{1}{2n} + \ldots$$

i.e., $$n \log \frac{u_n}{u_{n+1}} - 1 > \frac{p}{\log n} - \frac{1}{2n} + \ldots$$

i.e., $$\left(n \log \frac{u_n}{u_{n+1}} - 1\right) \log n > p - \frac{1}{2}\left\{\frac{\log n}{n}\right\} + \ldots \qquad \ldots(1)$$

Now as n becomes sufficiently large the L.H.S. and R.H.S. of (1) respectively approach k and p. Also $k > p$. Thus (1) is satisfied for sufficiently large values of n. Hence the series $\Sigma\, u_n$ is convergent if

$$\lim\left[\left\{n \log \frac{u_n}{u_{n+1}} - 1\right\} \log n\right] > 1.$$

Similarly, it can be proved that $\Sigma\, u_n$ is divergent if

$$\lim \left[\left\{n \log \frac{u_n}{u_{n+1}} - 1\right\} \log n\right] < 1.$$

Note: This test is to be applied when the log test of article 14 fails *i.e.*, when $\lim \dfrac{u_n}{u_{n+1}} = 1$ and also $\lim n \log \dfrac{u_n}{u_{n+1}} = 1$.

Illustrative Examples

Example 33: *Test for convergence the following series :*

(*i*) $\dfrac{1^2}{2^2} + \dfrac{1^2 . 3^2}{2^2 . 4^2}\, x + \dfrac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2}\, x^2 + \ldots$

(Bundelkhand 2014)

(*ii*) $1 + \dfrac{\alpha . \beta}{1 . \gamma}\, x + \dfrac{\alpha (\alpha + 1) \beta (\beta + 1)}{1 . 2 . \gamma (\gamma + 1)}\, x^2 + \dfrac{\alpha (\alpha + 1)(\alpha + 2) \beta (\beta + 1)(\beta + 2)}{1 . 2 . 3 . \gamma (\gamma + 1)(\gamma + 2)}\, x^3 + \ldots$

Solution: (i) Here $u_n = \dfrac{1^2 . 3^2 \ldots (2n - 1)^2}{2^2 . 4^2 \ldots (2n)^2}\, x^{n-1}$,

and $$u_{n+1} = \frac{1^2 . 3^2 \ldots . (2n - 1)^2 . (2n + 1)^2}{2^2 . 4^2 \ldots . (2n)^2 \, (2n + 2)^2}\, x^n.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(2n + 2)^2}{(2n + 1)^2} \cdot \frac{1}{x} = \left\{\frac{2 + 2/n}{2 + 1/n}\right\}^2 \cdot \frac{1}{x}.$$

$$\therefore \quad \lim \frac{u_n}{u_{n+1}} = \lim \left[\left\{\frac{2 + 2/n}{2 + 1/n}\right\}^2 \cdot \frac{1}{x}\right] = \frac{2^2}{2^2} \cdot \frac{1}{x} = \frac{1}{x}.$$

$\therefore$ by ratio test the given series $\Sigma\, u_n$ is convergent if $1/x > 1$ *i.e.*, $x < 1$, divergent if $1/x < 1$ *i.e.*, $x > 1$ and the test fails if $1/x = 1$ *i.e.*, $x = 1$.

When $x = 1$, we have

$$\frac{u_n}{u_{n+1}} = \frac{(2n + 2)^2}{(2n + 1)^2}.$$

$$\therefore \quad n \left\{\frac{u_n}{u_{n+1}} - 1\right\} = n \left\{\frac{(2n + 2)^2}{(2n + 1)^2} - 1\right\} = \frac{n (4n + 3)}{(2n + 1)^2} = \frac{4 + 3/n}{(2 + 1/n)^2}.$$

$$\therefore \quad \lim n \left\{\frac{u_n}{u_{n+1}} - 1\right\} = \lim \frac{4 + 3/n}{(2 + 1/n)^2} = \frac{4}{2^2} = 1.$$

$\therefore$ Raabe's test also fails when $x = 1$ and so we shall now apply De Morgan's test.

Now $$n \left\{\frac{u_n}{u_{n+1}} - 1\right\} - 1 = \frac{n (4n + 3)}{(2n + 1)^2} - 1 = \frac{-n - 1}{(2n + 1)^2}.$$

$\therefore \quad \lim \left[\left\{n\left(\dfrac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]$

$$= \lim \left[\left\{\frac{-n-1}{(2n+1)^2}\right\}\log n\right] = \lim\left[\frac{-1-1/n}{(2+1/n)^2}\cdot\frac{\log n}{n}\right]$$

$$= \frac{-1}{2^2} \cdot 0 = 0 < 1. \qquad \left[\textbf{Note} \text{ that } \lim \frac{\log n}{n} = 0\right]$$

$\therefore$ by De Morgan's test $\Sigma\, u_n$ is divergent when $x = 1$.

Hence the given series $\Sigma\, u_n$ is convergent if $x < 1$ and divergent if $x \geq 1$.

(ii) Omitting the first term, we have

$$u_n = \frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)\,\beta(\beta+1)(\beta+2)\ldots\ldots(\beta+n-1)}{1.2\ldots n\,.\,\gamma(\gamma+1)(\gamma+2)\ldots(\gamma+n-1)}\,x^n,$$

$$u_{n+1} = \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)(\alpha+n)\,\beta(\beta+1)\ldots(\beta+n-1)(\beta+n)}{1.2\ldots n(n+1)\,.\,\gamma(\gamma+1)\ldots(\gamma+n-1)(\gamma+n)}\,x^{n+1}.$$

Now $\quad \dfrac{u_n}{u_{n+1}} = \dfrac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}\cdot\dfrac{1}{x} = \dfrac{(1+1/n)(\gamma/n+1)}{(\alpha/n+1)(\beta/n+1)}\cdot\dfrac{1}{x}.$

$\therefore \quad \lim \dfrac{u_n}{u_{n+1}} = \dfrac{1.1}{1.1}\cdot\dfrac{1}{x} = \dfrac{1}{x}$ so that by ratio test the series is convergent if $1/x > 1$ *i.e.*, $x < 1$

and divergent if $1/x < 1$ *i.e.*, $x > 1$ and the test fails if $1/x = 1$ *i.e.*, $x = 1$.

When $x = 1$, we have

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} = \frac{n^2+(\gamma+1)n+\gamma}{n^2+(\alpha+\beta)n+\alpha\beta}.$$

$$\therefore \quad n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left[\frac{n^2+(\gamma+1)n+\gamma}{n^2+(\alpha+\beta)n+\alpha\beta}-1\right]$$

$$= \frac{n\{(\gamma+1-\alpha-\beta)n+(\gamma-\alpha\beta)\}}{n^2+(\alpha+\beta)n+\alpha\beta}$$

$$= \frac{(\gamma+1-\alpha-\beta)+(\gamma-\alpha\beta)/n}{1+(\alpha+\beta)/n+\alpha\beta/n^2}.$$

$$\therefore \quad \lim n\left(\frac{u_n}{u_{n+1}}-1\right) = \frac{\gamma+1-\alpha-\beta}{1} = \gamma+1-\alpha-\beta.$$

$\therefore$ if $x = 1$, then by Raabe's test, the series is convergent if $\gamma+1-\alpha-\beta > 1$ *i.e.*, if $\gamma > \alpha+\beta$, divergent if $\gamma+1-\alpha-\beta < 1$ *i.e.*, if $\gamma < \alpha+\beta$, and the test fails if $\gamma+1-\alpha-\beta = 1$ *i.e.*, if $\gamma = \alpha+\beta$.

When $\gamma = \alpha+\beta$, we have

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = \frac{n\{n+\alpha+\beta-\alpha\beta\}}{n^2+(\alpha+\beta)n+\alpha\beta}.$$

Now we shall apply De Morgan's test.

We have

$$\lim\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]=\lim\left[\left\{\frac{n\,(n+\alpha+\beta-\alpha\beta)}{n^2+(\alpha+\beta)\,n+\alpha\beta}-1\right\}\log n\right]$$

$$=\lim\left[\frac{-\alpha\beta n-\alpha\beta}{n^2+(\alpha+\beta)\,n+\alpha\beta}\cdot\log n\right]$$

$$=\lim\left[\frac{-\alpha\beta\,(1+1/n)}{1+(\alpha+\beta)/n+\alpha\beta/n^2}\cdot\frac{\log n}{n}\right]$$

$$=\frac{-\alpha\beta}{1}\cdot 0=0,\text{ which is}<1. \qquad \left[\textbf{Note}\text{ that }\lim\frac{\log n}{n}=0\right]$$

$\therefore$ by De-Morgan's test the series is divergent if $\gamma=\alpha+\beta$.

Thus the given series is convergent if $x<1$, divergent if $x>1$ and for $x=1$, the series is convergent if $\gamma>\alpha+\beta$ and divergent if $\gamma\le\alpha+\beta$.

Example 34: *Test for convergence the series*

$$1^p+\left(\frac{1}{2}\right)^p+\left(\frac{1.3}{2.4}\right)^p+\left(\frac{1.3.5}{2.4.6}\right)^p+\ldots$$

Solution: Omitting the first term 1^p, we have

$$u_n=\left[\frac{1.3.5.\ldots.(2n-1)}{2.4.6\ldots..(2n)}\right]^p,$$

and then $$u_{n+1}=\left[\frac{1.3.5\ldots(2n-1)\,(2n+1)}{2.4.6\ldots(2n)\,(2n+2)}\right]^p.$$

Now $$\frac{u_n}{u_{n+1}}=\left(\frac{2n+2}{2n+1}\right)^p=\left(\frac{1+1/n}{1+1/2n}\right)^p.$$

$\therefore$ $$\lim\frac{u_n}{u_{n+1}}=\left(\frac{1}{1}\right)^p=1\textit{ i.e.}, \text{ the ratio test fails.}$$

Now we apply logarithmic test.

We have $$\log\frac{u_n}{u_{n+1}}=\log\left(\frac{2n+2}{2n+1}\right)^p$$

$$=\log\left(\frac{1+1/n}{1+1/2n}\right)^p$$

$$=p\,[\log(1+1/n)-\log(1+1/2n)]$$

$$=p\left[\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots\right)-\left(\frac{1}{2n}-\frac{1}{2.2^2n^2}+\frac{1}{3.2^3n^3}-\ldots\right)\right]$$

$$=p\left[\left\{1-\frac{1}{2}\right\}\frac{1}{n}-\frac{1}{2}\cdot\left\{1-\frac{1}{4}\right\}\frac{1}{n^2}+\frac{1}{3}\left\{1-\frac{1}{8}\right\}\frac{1}{n^3}-\ldots\right]$$

$$= p\left[\frac{1}{2n} - \frac{3}{8n^2} + \frac{7}{24n^3} - \ldots\right].$$

$\therefore$
$$n \log \frac{u_n}{u_{n+1}} = p\left[\frac{1}{2} - \frac{3}{8n} + \frac{7}{24n^2} - \ldots\right].$$

$\therefore$ $\lim n \log \dfrac{u_n}{u_{n+1}} = \dfrac{p}{2}$, so that the series is convergent if $p/2 > 1$ *i.e.*, if $p > 2$, divergent if $p/2 < 1$ *i.e.*, if $p < 2$ and the test fails if $p/2 = 1$ *i.e.*, if $p = 2$.

If $p = 2$, we have

$$n \log \frac{u_n}{u_{n+1}} = 2\left[\frac{1}{2} - \frac{3}{8n} + \frac{7}{24n^2} - \ldots\right] = 1 - \frac{3}{4n} + \frac{7}{12n^2} - \ldots.$$

$\therefore$
$$\lim \left[\left(n \log \frac{u_n}{u_{n+1}} - 1\right) \log n\right] = \lim \left[\left\{-\frac{3}{4n} + \frac{7}{12n^2} - \ldots\right\} . \log n\right]$$

$$= \lim \left[\left\{-\frac{3}{4} + \frac{7}{12n} - \ldots\right\} \cdot \frac{\log n}{n}\right] = \left\{-\frac{3}{4}\right\} .0 = 0, \text{ which is } < 1.$$

Hence by Alternative to Bertrand's test given in article 16, the series is divergent when $p = 2$.

Thus the given series is convergent if $p > 2$ and divergent if $p \leq 2$.

Comprehensive Exercise 6

Test for convergence the following series :

1. $1 + \dfrac{2^2}{3^2} + \dfrac{2^2 . 4^2}{3^2 . 5^2} + \dfrac{2^2 . 4^2 . 6^2}{3^2 . 5^2 . 7^2} + \ldots$ **(Kashi 2013; Meerut 13)**

2. $\dfrac{1^2}{2^2} + \dfrac{1^2 . 3^2}{2^2 . 4^2} + \dfrac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} + \ldots$ **(Kumaun 2003)**

3. $\dfrac{a}{b} + \dfrac{a(a+1)}{b(b+1)} + \dfrac{a(a+1)(a+2)}{b(b+1)(b+2)} + \ldots$

4. $1 + \dfrac{a(1-a)}{1^2} + \dfrac{(1+a)a(1-a)(2-a)}{1^2 . 2^2} + \dfrac{(2+a)(1+a)a(1-a)(2-a)(3-a)}{1^2 . 2^2 . 3^2} + \ldots$

5. $1 + \dfrac{\alpha}{1 . \beta} x + \dfrac{\alpha(\alpha+1)^2}{1 . 2 \beta(\beta+1)} x^2 + \dfrac{\alpha(\alpha+1)^2 (\alpha+2)^2}{1 . 2 . 3 \beta(\beta+1)(\beta+2)} x^3 + \ldots$

6. $\left\{\dfrac{1}{2 . 4}\right\}^{2/3} + \left\{\dfrac{1 . 3}{2 . 4 . 6}\right\}^{2/3} + \left\{\dfrac{1 . 3 . 5}{2 . 4 . 6 . 8}\right\}^{2/3} + \ldots$

7. $x + x^{1+1/2} + x^{1+1/2+1/3} + x^{1+1/2+1/3+1/4} + \ldots$

Answers 6

1. Divergent
2. Divergent
3. Convergent if $b - a > 1$ and divergent if $b - a \le 1$
4. Divergent
5. Convergent if $x < 1$, divergent if $x > 1$ and when $x = 1$ then convergent if $\beta > 2\alpha$ and divergent if $\beta \le 2\alpha$
6. Divergent
7. Convergent if $x < 1/e$ and divergent if $x \ge 1/e$

17 Summary of Tests

Let the given series of positive terms be $\Sigma\, u_n$. Then to test the series for convergence we proceed as follows :

1. **Find lim u_n:** (a) If $\lim u_n > 0$, the series is divergent.

   (b) If $\lim u_n = 0$, then the series may or may not be convergent. In this case we apply further tests to decide the nature of the series.
2. **If lim $u_n = 0$** and u_n can be arranged as an algebraic fraction in n, then usually **comparison test** should be applied.
3. **If n occurs as an exponent in** u_n and $\lim (u_n)^{1/n}$ can be easily evaluated, then **Cauchy's root test** should be applied.
4. **Cauchy's condensation test** is generally applied when u_n involves $\log n$.

   In case all the above tests are not applicable then we adopt the following scheme of testing in the given order.
5. **D' Alembert's ratio test:** For this we find $\lim \dfrac{u_n}{u_{n+1}}$. The series is convergent or divergent according as this limit is > 1 or < 1. In case this limit is equal to 1 (unity), this test fails. Then we proceed to apply either test 6 (a) or 6 (b) or 6(c) given below depending upon the nature of u_n and u_n / u_{n+1}.
6. **(a) Comparison test:** In some cases when D' Alembert's ratio test fails, the convergence of the series may be decided by comparison test.

   (b) Raabe's test: For this we find $\lim n\left(\dfrac{u_n}{u_{n+1}} - 1\right)$. The series is convergent or divergent according as this limit is > 1 or < 1. In case the limit is equal to 1, this test fails and we apply test 7 (a).

(c) Logarithmic test: If $\frac{u_n}{u_{n+1}} - 1$ cannot be evaluated easily while $\log \frac{u_n}{u_{n+1}}$ can be easily evaluated then we apply logarithmic test. Here we find $\lim \left(n \log \frac{u_n}{u_{n+1}} \right)$. If this limit > 1, the series is convergent and if this limit < 1, the series is divergent. In case the limit = 1, this test fails and we apply test 7 (b).

7. **(a) De Morgan's and Bertrand's test:**

Find $\lim \left[\left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) - 1 \right\} \log n \right]$.

The series is convergent or divergent according as this limit is > 1 or < 1.

Note: When this test is applied, we shall generally find that the limit comes out to be equal to zero and since 0 < 1, the series is divergent.

(b) Alternative to Bertrand's test:

To apply this test we find $\lim \left[\left(n \log \frac{u_n}{u_{n+1}} - 1 \right) \log n \right]$.

The series is convergent or divergent according as this limit is > 1 or < 1.

18 Alternating Series

So far we have mainly dealt with series of positive terms. We have seen that a series of positive terms either converges or diverges and cannot oscillate. But a series which contains an infinite number of positive and an infinite number of negative terms may either converge or diverge or oscillate.

Alternating Series: Definition: *A series whose terms are alternately positive and negative is called an* **alternating series.** Thus an alternating series is of the form

$$u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1} u_n + \ldots$$

where $u_n > 0$ for all n. It is denoted as

$$\sum_{n=1}^{\infty} (-1)^{n-1} u_n .$$

The following are some examples of an alternating series.

(i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots$

(ii) $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + \ldots + \frac{(-1)^{n-1}(n+1)}{2n} + \ldots$

(iii) $1 - \frac{2}{\log 2} + \frac{3}{\log 3} - \frac{4}{\log 4} + \ldots$

Theorem: Alternating Series Test (Leibnitz's Theorem): *An infinite series* $\Sigma\,(-1)^{n-1}\,u_n$ *in which the terms are alternately positive and negative is convergent if each term is numerically less than the preceding term and* $\lim u_n = 0$.

Symbolically, the alternating series

$$u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1}\,u_n + \ldots,\ (u_n > 0 \text{ for all } n)$$

converges if

(*i*) $u_{n+1} \le u_n$ *for all n i.e.,* $u_1 \ge u_2 \ge u_3 \ge u_4 \ge \ldots$

and (*ii*) $\lim u_n = 0$ *i.e.,* $u_n \to 0$ *as* $n \to \infty$.

Proof: Let $S_n = u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1}\,u_n$ so that $< S_n >$ is the sequence of partial sums of the given series.

We shall prove the theorem in **two steps**.

(i) First we shall prove that the subsequences $< S_{2n} >$ and $< S_{2n+1} >$ of the sequence $< S_n >$ converge to the same limit, say S.

We have $\quad S_{2n} = u_1 - u_2 + \ldots + u_{2n-1} - u_{2n}$

and $\quad S_{2n+2} = u_1 - u_2 + \ldots + u_{2n-1} - u_{2n} + u_{2n+1} - u_{2n+2}$.

$\therefore\ S_{2n+2} - S_{2n} = u_{2n+1} - u_{2n+2} \ge 0$ for all n because it is given that $u_{n+1} \le u_n$ for all n.

$\therefore\ S_{2n+2} \ge S_{2n}$ for all n and so the sequence $< S_{2n} >$ is monotonically increasing.

Again for all n,

$$S_{2n} = u_1 - [(u_2 - u_3) + (u_4 - u_5) + \ldots + (u_{2n-2} - u_{2n-1}) + u_{2n}]$$

$= u_1 -$ some positive number because $u_2 - u_3, \ldots, u_{2n-2} - u_{2n-1}, u_{2n}$ are all positive

$\le u_1$.

Thus $S_{2n} \le u_1$ for all n and so the sequence $< S_{2n} >$ is bounded above.

Since the sequence $< S_{2n} >$ is monotonically increasing and bounded above, therefore it converges. Let $\lim S_{2n} = S$.

Now $\quad S_{2n+1} = S_{2n} + u_{2n+1}$.

$$\therefore\quad \lim S_{2n+1} = \lim S_{2n} + \lim u_{2n+1}$$
$$= S + 0 \qquad [\because\ \lim u_n = 0]$$
$$= S.$$

$\therefore$ the sequence $< S_{2n+1} >$ also converges to S.

Thus the subsequences $< S_{2n} >$ and $< S_{2n+1} >$ of the sequence $< S_n >$ converge to the same limit S.

(ii) Now we shall show that the sequence $< S_n >$ also converges to S.

Let $\varepsilon > 0$ be given. Since the sequences $< S_{2n} >$ and $< S_{2n+1} >$ both converge to S, therefore there exist +ive integers m_1 and m_2 such that

$$| S_{2n} - S | < \varepsilon \text{ for all } n \ge m_1$$

and $\quad | S_{2n+1} - S | < \varepsilon$ for all $n \ge m_2$.

Let $m = \max(m_1, m_2)$.

Then $\quad | S_n - S | < \varepsilon$ for all $n \ge 2m$.

$\therefore$ the sequence $< S_n >$ converges to S.

Hence the given series $\Sigma\,(-1)^{n-1}\, u_n$ converges.

Note 1: The above test is equally applicable to the series $\Sigma\,(-1)^n\, u_n, u_n > 0$ for all n, provided both the conditions (i) and (ii) are satisfied.

Note 2: If in the case of an alternating series

$$u_1 - u_2 + u_3 - u_4 + \ldots\ldots\ (u_n > 0 \text{ for all } n),$$

the terms continually decrease, we cannot say that the series is convergent unless $\lim u_n = 0$. Because if $\lim u_n \neq 0$, then $\lim S_{2n}$ and $\lim S_{2n+1}$ will differ and so the series will not be convergent. Such a series is an oscillatory series.

For example, consider the series

$$2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \ldots.$$

Here the terms are alternately positive and negative and each term is numerically less than the preceding term because

$$2 > \frac{3}{2} > \frac{4}{3} > \frac{5}{4} > \ldots\ldots$$

But here $\lim u_n = \lim \dfrac{n+1}{n} = \lim \left(1 + \dfrac{1}{n}\right) = 1 \neq 0$. Hence the given series is not convergent. As a matter of fact it is an oscillatory series.

Illustrative Examples

Example 35: *Show that the series* $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ *converges.* **(Avadh 2012)**

Solution: The given series is an alternating series

$$u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1} u_n + \ldots, (u_n > 0 \text{ for all } n).$$

Here $u_n = 1/n > 0$ for all n.

We have $\quad u_{n+1} - u_n = \dfrac{1}{n+1} - \dfrac{1}{n} = \dfrac{n - n - 1}{n\,(n+1)} = \dfrac{-1}{n\,(n+1)} < 0$ for all n.

Thus $u_{n+1} < u_n$ for all n *i.e.*, each term is numerically less than the preceding term.

Also $\quad \lim u_n = \lim \dfrac{1}{n} = 0.$

Hence by Leibnitz's test for alternating series, the given series is convergent.

Example 36: *Show that the following series are convergent.*

(i) $1^{-p} - 2^{-p} + 3^{-p} - \ldots$ *when* $p > 0$.

(ii) $\dfrac{1}{x+1} - \dfrac{1}{x+2} + \dfrac{1}{x+3} - \ldots$ *except when x is a negative integer.*

Solution: (i) The given series is an alternating series
$$u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1} u_n + \ldots, (u_n > 0 \text{ for all } n).$$

Here $u_n = 1/n^p > 0$ for all n.

Also since $p > 0$, we have $\dfrac{1}{1^p} > \dfrac{1}{2^p} > \dfrac{1}{3^p} > \ldots.$

Thus $u_{n+1} < u_n$ for all n.

Also $$\lim u_n = \lim \frac{1}{n^p} = 0, \text{ since } p > 0.$$

Hence by alternating series test the given series is convergent for $p > 0$.

(ii) The given series is
$$\frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3} - \ldots, x \text{ is not a –ive integer.}$$

If $x > -1$, then the terms are alternately positive and negative from the beginning. If $x < -1$, excluding –ive integers, then the terms are *ultimately* alternating in sign.

Since the removal of a finite number of terms does not affect the convergence of the series, therefore we may assume the series to be alternating in sign in both the cases.

Obviously $u_1 > u_2 > u_3 > u_4 > \ldots$ *i.e.*, each term of the series is numerically less than the preceding term.

Also $$\lim u_n = \lim \frac{1}{x+n} = 0.$$

Hence by alternating series test, the given series is convergent.

Comprehensive Exercise 7

1. Examine the convergence of the series
$$\frac{1}{1.2} - \frac{1}{3.4} + \frac{1}{5.6} - \frac{1}{7.8} + \ldots.$$

2. Show that the series
$$\frac{\log 2}{2^2} - \frac{\log 3}{3^2} + \frac{\log 4}{4^2} - \ldots \text{ converges.}$$

3. Examine the convergence of the series
$$\sum_{n=1}^{\infty} (-1)^n \left[\frac{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}}{n} \right].$$

4. Test the convergence of the series
$$\sum_{n=1}^{\infty} \left[\frac{1}{n} + \frac{(-1)^{n+1}}{\sqrt{n}} \right].$$

Answers 7

1. Convergent 3. Convergent 4. Divergent

19 Absolute Convergence and Conditional Convergence

(Meerut 2012B)

Absolutely Convergent Series:

Definition: *A series Σu_n is said to be* **absolutely convergent** *if the series $\Sigma\,|u_n|$ is convergent.*

If $\Sigma\, u_n$ is a series of positive terms, then $\Sigma\, u_n$ and $\Sigma\,|u_n|$ are the same series and so if $\Sigma\, u_n$ is convergent, it is also absolutely convergent. Hence for a series of positive terms the concepts of convergence and absolute convergence are the same.

But if a series $\Sigma\, u_n$ contains an *infinite* number of positive and an infinite number of negative terms, then $\Sigma\, u_n$ is absolutely convergent only if the series $\Sigma\,|u_n|$ obtained from $\Sigma\, u_n$ by making all its terms positive is convergent.

For example the series

$$\Sigma\, u_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \dots$$

is absolutely convergent. Here we see that the series

$$\Sigma\,|u_n| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$$

is an infinite geometric series of positive terms with common ratio $\frac{1}{2}$ which is < 1 and so it is convergent. Hence the given series $\Sigma\, u_n$ is absolutely convergent.

Non-absolutely convergent or semi-convergent or conditionally convergent series:

Definition: *A series $\Sigma\, u_n$ is said to be* **semi-convergent** *or* **conditionally convergent** *or* **non-absolutely convergent** *if $\Sigma\, u_n$ is convergent but $\Sigma\,|u_n|$ is divergent.*

For example, consider the series

$$\Sigma\, u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots.$$

It is an alternating series in which each term is numerically less than the preceding term and $\lim u_n = \lim (1/n) = 0$. Hence by alternating series test, $\Sigma\, u_n$ is a convergent series.

But the series $\Sigma\,|u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is the series $\Sigma\,(1/n^p)$, for $p = 1$, and we know that it is divergent. Thus here $\Sigma\, u_n$ is convergent while $\Sigma\,|u_n|$ is divergent.

Hence $\Sigma\, u_n$ is a semi-convergent or conditionally convergent or non-absolutely convergent series.

Tests for absolute convergence: To test the absolute convergence of the series $\Sigma\, u_n$, we have to simply test the convergence of the series $\Sigma\,|\,u_n\,|$ which is a series of positive terms. Hence the various tests given for the series of positive terms are precisely the tests which we are to apply to check the absolute convergence of the series $\Sigma\, u_n$. We have to simply replace u_n by $|\,u_n\,|$ in these tests. For example by Cauchy's root test, the series $\Sigma\, u_n$ is absolutely convergent if $\lim |\,u_n\,|^{1/n} < 1\cdot$ Similarly by D'Alembert's ratio test the series $\Sigma\, u_n$ is absolutely convergent if

$$\lim \frac{|\,u_n|}{|\,u_{n+1}|} = \lim \left|\frac{u_n}{u_{n+1}}\right| > 1.$$

Similarly comparison test or other tests may be used.

However these tests cannot give any information about the conditional convergence of the series.

20 Some Important Theorems on Absolutely Convergent Series

Theorem 1: *Every absolutely convergent series is convergent but the converse is not necessarily true i.e., convergence need not imply absolute convergence.*

Theorem 2: *In an absolutely convergent series, the series formed by its positive terms alone is convergent and the series formed by its negative terms alone is convergent.*

Theorem 3: Re-arrangement of terms of an absolutely convergent series:

If the terms of an absolutely convergent series are re-arranged the series remains convergent and its sum unaltered.

Or The sum of an absolutely convergent series is independent of the order of terms.

Illustrative Examples

Example 37: *Show that the series*

$$\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$$ **(Meerut 2012)**

is conditionally convergent.

Solution: The given series is an alternating series

$$u_1 - u_2 + u_3 - \ldots + (-1)^{n-1}\, u_n + \ldots, (u_n > 0 \text{ for all } n).$$

Here $u_n = \dfrac{1}{\sqrt{n}} > 0$ for all n.

Also for all n, $\sqrt{(n+1)} > \sqrt{n}$

$\Rightarrow$ $$\frac{1}{\sqrt{(n+1)}} < \frac{1}{\sqrt{n}} \quad \Rightarrow \quad u_{n+1} < u_n \text{ for all } n.$$

Again $$\lim u_n = \lim \frac{1}{\sqrt{n}} = 0.$$

Hence by Leibnitz's test, the given series $\Sigma\,[(-1)^{n-1}/\sqrt{n}]$ is convergent.

Now $\Sigma\left|\dfrac{(-1)^{n-1}}{\sqrt{n}}\right| = \Sigma\,\dfrac{1}{\sqrt{n}}$ is divergent because $\Sigma\,(1/n^p)$ is divergent if $p \le 1$ and here $p = \dfrac{1}{2}\cdot$

Hence the given series is semi-convergent or conditionally convergent.

Example 38: *Examine the convergence and absolute convergence of the series*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, n}{n^2+1}.$$

(Kashi 2013)

Solution: Obviously the given series is an alternating series

$$u_1 - u_2 + u_3 - u_4 + \ldots,\ u_n > 0 \text{ for all } n.$$

Here $u_n = \dfrac{n}{n^2+1} > 0$ for all n.

Also $$u_{n+1} - u_n = \frac{n+1}{(n+1)^2+1} - \frac{n}{n^2+1} = \frac{-n^2-n+1}{(n^2+1)\,[(n+1)^2+1]} < 0 \text{ for all } n.$$

Thus $u_{n+1} < u_n$ for all n.

Again $$\lim u_n = \lim \frac{n}{n^2+1} = \lim \frac{1}{n\,[1+(1/n^2)]} = 0.$$

Hence by Leibnitz's test, the given series converges.

Now we shall test the given series for absolute convergence.

Consider the series $\Sigma\, u_n'$ of positive terms, where

$$u_n' = \left|\frac{(-1)^{n+1}\, n}{n^2+1}\right| = \frac{n}{n^2+1} = \frac{1}{n\,[1+(1/n^2)]}.$$

Take $v_n = \dfrac{1}{n}\cdot$ Then $\lim \dfrac{u_n'}{v_n} = \lim \dfrac{1}{1+(1/n^2)} = 1$ which is finite and non-zero. Hence by comparison test $\Sigma u_n'$ and $\Sigma\, v_n$ are either both convergent or both divergent. But for v_n, $p = 1$ so that $\Sigma\, v_n$ is divergent. Hence $\Sigma\, u_n'$ is divergent.

Hence the given series is not absolutely convergent *i.e.*, it is conditionally convergent.

Example 39: *Show that the series* $\Sigma\,(-1)^n\,[\sqrt{(n^2+1)} - n]$ *is conditionally convergent.*

Solution: The given series is an alternating series $\Sigma\,(-1)^n\, u_n$, $u_n > 0$ for all n.

Here $$u_n = \sqrt{(n^2+1)} - n = \frac{[\sqrt{(n^2+1)} - n]\,[\sqrt{(n^2+1)} + n]}{\sqrt{(n^2+1)} + n}$$

$$= \frac{n^2+1-n^2}{\sqrt{(n^2+1)}+n} = \frac{1}{\sqrt{(n^2+1)}+n}.$$

Obviously $u_n > 0$ for all n.

Also $u_{n+1} < u_n$ for all n.

Again $\qquad \lim u_n = \lim \dfrac{1}{\sqrt{(n^2+1)}+n} = \lim \dfrac{1}{n\,[(1+1/n^2)^{1/2}+1]} = 0\cdot$

Hence by Leibnitz's test, the given series is convergent.

Now let $\Sigma\, u_n'$ denote the series obtained from the given series by making all its terms positive *i.e.*,

$$u_n' = |(-1)^n \{\, |\sqrt{(n^2+1)} - n\,\}| = \sqrt{(n^2+1)} - n.$$

We shall apply comparison test to check the convergence of $\Sigma\, u_n'$.

We have $\qquad u_n' = \sqrt{(n^2+1)} - n = \dfrac{1}{\sqrt{(n^2+1)}+n} = \dfrac{1}{n\,[(1+1/n^2)^{1/2}+1]}\cdot$

Take $v_n = \dfrac{1}{n}\cdot$ Then $\dfrac{u_n'}{v_n} = \dfrac{1}{[(1+1/n^2)^{1/2}+1]}\cdot$

$\therefore \qquad \lim \dfrac{u_n'}{v_n} = \lim \dfrac{1}{[(1+1/n^2)^{1/2}+1]} = 1$ which is finite and non-zero.

Hence by comparison test $\Sigma\, u_n'$ and $\Sigma\, v_n$ are either both convergent or both divergent. But for v_n, $p=1$ and so $\Sigma\, v_n$ is divergent. Therefore $\Sigma\, u_n'$ is also divergent.

Hence the given series converges conditionally.

Example 40: *Show that the exponential series*

$$1 + \frac{x}{1\,!} + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \ldots$$

converges absolutely for all values of x.

Solution: Let the given series be $\Sigma\, u_n$, so that

$$u_n = \frac{x^{\,n-1}}{(n-1)\,!} \text{ and } u_{n+1} = \frac{x^{\,n}}{n\,!}\cdot$$

Now

$$\frac{|u_n|}{|u_{n+1}|} = \left|\frac{u_n}{u_{n+1}}\right| = \left|\frac{x^{\,n-1}}{(n-1)\,!}\cdot\frac{n\,!}{x^{\,n}}\right|$$

$$= \left|\frac{n}{x}\right| = \frac{n}{|x|}\ \cdot$$

$\therefore \qquad \lim \dfrac{|u_n|}{|u_{n+1}|} = \lim \dfrac{n}{|x|} = +\infty$, whatever $(x \neq 0)$ may be.

Therefore, by D'Alembert's ratio test, the series $\Sigma\,|u_n|$ converges for all x (except for $x = 0$) .

For $x = 0$, the series $\Sigma\,|u_n|$ obviously converges.

Hence the given series $\Sigma\, u_n$ converges absolutely for all values of x *i.e.*, for all $x \in \mathbf{R}$.

Note: The sum of this series is denoted by e^x.

Also since $\Sigma\, u_n$ converges, we must have

$$\lim u_n = 0 \quad i.e., \quad \lim_{n\to\infty} (x^n / n\,!) = 0.$$

Example 41: *Show that the series* $x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$ *converges if and only if* $-1 \le x \le 1$.

Solution: Let the given series be $\Sigma\, u_n$. Then

$$u_n = (-1)^{n-1} \frac{x^{2n-1}}{2n-1} \quad \text{and} \quad u_{n+1} = (-1)^n \frac{x^{2n+1}}{2n+1}.$$

$$\therefore \qquad \left| \frac{u_n}{u_{n+1}} \right| = \frac{2n+1}{2n-1} \cdot \frac{1}{x^2}.$$

Now
$$\lim_{n \to \infty} \left| \frac{u_n}{u_{n+1}} \right| = \lim \left[\frac{(2+1/n)}{(2-1/n)} \cdot \frac{1}{x^2} \right] = \frac{1}{x^2}.$$

$\therefore$ by D'Alembert's ratio test, $\Sigma |u_n|$ converges if $1/x^2 > 1$ *i.e.,* $x^2 < 1$ *i.e.,* $|x| < 1$ and diverges if $|x| > 1$. Since every absolutely convergent series is convergent, therefore the given series $\Sigma\, u_n$ converges when $|x| < 1$ *i.e.,* $-1 < x < 1$.

When $x = 1$, the series $\Sigma\, u_n$ becomes

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

which converges by Leibnitz's test for alternating series.

When $x = -1$, the series $\Sigma\, u_n$ becomes

$$-\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$

which is again convergent by Leibnitz's test.

When $x > 1$ or when $x < -1$, obviously u_n does not tend to zero as $n \to \infty$. So the series $\Sigma\, u_n$ does not converge when

$$|x| > 1.$$

Hence the given series converges iff $-1 \le x \le 1$.

Example 42: *Show that the binomial series*

$$1 + nx + \frac{n(n-1)}{2!} x^2 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!} x^r + \dots$$

is absolutely convergent when $|x| < 1$.

Solution: Omitting the first term of the given series, let u_r denote the r th term of the resulting series.

Then
$$\frac{u_r}{u_{r+1}} = \frac{r+1}{n-r} \cdot \frac{1}{x} = \frac{(1+1/r)}{(n/r-1)} \cdot \frac{1}{x}.$$

$$\therefore \qquad \left| \frac{u_r}{u_{r+1}} \right| = \frac{1+1/r}{|n/r-1|} \cdot \frac{1}{|x|}.$$

$$\therefore \qquad \lim_{n \to \infty} \left| \frac{u_r}{u_{r+1}} \right| = \frac{1}{|x|} \text{ which is} > 1 \text{ if } |x| < 1.$$

$\therefore$ The series $\Sigma |u_r|$ converges if $|x| < 1$ *i.e.,* the given series is absolutely convergent when $|x| < 1$.

Example 43: *Show that a series of positive terms, if convergent, is absolutely convergent. Prove that the series*

$$2\ sin\ \frac{x}{3} + 4\ sin\ \frac{x}{9} + 8\ sin\ \frac{x}{27} + \ldots$$

converges absolutely for all finite values of x.

Solution: **First part:** Let $\Sigma\, u_n$ be a convergent series of positive terms. Since $u_n > 0 \Rightarrow |u_n| = u_n$, therefore the series $\Sigma\, |u_n| = \Sigma\, u_n$ is also convergent and hence the series $\Sigma\, u_n$ is absolutely convergent.

Second part: Let the given series be denoted by $\Sigma\, u_n$. Then

$$u_n = 2^n \sin (x/3^n) \text{ and } u_{n+1} = 2^{n+1} \sin (x/3^{n+1}).$$

$$\therefore \qquad \frac{u_n}{u_{n+1}} = \frac{1}{2} \cdot \sin (x/3^n) \cdot \frac{1}{\sin (x/3^{n+1})}$$

$$= \frac{1}{2} \cdot \frac{\sin (x/3^n)}{x/3^n} \cdot \frac{x/3^{n+1}}{\sin (x/3^{n+1})} \cdot 3.$$

To test the convergence of the series $\Sigma\, |u_n|$, we have

$$\left|\frac{u_n}{u_{n+1}}\right| = \frac{3}{2} \cdot \left|\frac{\sin (x/3^n)}{x/3^n}\right| \cdot \left|\frac{x/3^{n+1}}{\sin (x/3^{n+1})}\right|.$$

$$\therefore \qquad \lim_{n \to \infty} \left|\frac{u_n}{u_{n+1}}\right| = \frac{3}{2} \text{ for all finite values of } x, \text{ because } \lim_{n \to \infty} \frac{\sin (x/3^n)}{x/3^n} = 1.$$

Since $\lim \left|\frac{u_n}{u_{n+1}}\right| > 1$ for all finite values of x, therefore by ratio test, the series $\Sigma\, |u_n|$ converges for all finite values of x. Hence the series $\Sigma\, u_n$ converges absolutely for all finite values of x.

Example 44: *Discuss the convergence of the logarithmic series*

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1} \frac{x^n}{n} + \ldots$$

Solution: Let $\quad \Sigma\, u_n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \cdot$

The series $\Sigma\, u_n$ is absolutely convergent if the series $\Sigma\, |\, u_n\, |$ is convergent. To discuss the convergence of $\Sigma\, |\, u_n\, |$ we shall apply ratio test.

We have
$$\left|\frac{u_n}{u_{n+1}}\right| = \left|\frac{x^n}{n} \cdot \frac{n+1}{x^{n+1}}\right| = \frac{n+1}{n} \cdot \frac{1}{|x|} = \left(1 + \frac{1}{n}\right) \cdot \frac{1}{|x|} \cdot$$

$$\therefore \qquad \lim \left|\frac{u_n}{u_{n+1}}\right| = \lim \left[\left(1 + \frac{1}{n}\right) \cdot \frac{1}{|x|}\right] = \frac{1}{|x|} \cdot$$

So by ratio test, the series $\Sigma\, |u_n|$ is convergent if $1/|x| > 1$ *i.e.*, $|x| < 1$ *i.e.* $-1 < x < 1$.

$\therefore$ the given series is absolutely convergent and hence also convergent if $-1 < x < 1$ *i.e.*, if $|x| < 1$.

When $x = 1$, the given series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

which converges by Leibnitz's test but converges conditionally.

When $x = -1$, the given series is $-\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\right)$ which diverges to $-\infty$.

When $x > 1$ or $x < -1$ *i.e.*, $|x| > 1$, obviously $\lim u_n \neq 0$ and so the series Σu_n does not converge.

Hence the given series is convergent if $-1 < x \leq 1$. For $|x| < 1$ *i.e.*, $-1 < x < 1$, it converges absolutely.

Comprehensive Exercise 8

1. Test for convergence the following series :

(i) $1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots$ (ii) $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \ldots$

(iii) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$ (iv) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{[n(n+1)(n+2)]}}$.

(v) $\Sigma \frac{(-1)^{n-1}(n+1)}{2n}$.

(vi) $\frac{1}{x} - \frac{1}{x+a} + \frac{1}{x+2a} - \frac{1}{x+3a} + \ldots, x > 0, a > 0$.

(vii) $\log\left(\frac{2}{1}\right) - \log\left(\frac{3}{2}\right) + \log\left(\frac{4}{3}\right) - \log\left(\frac{5}{4}\right) + \ldots$.

(viii) $\log\left(\frac{1}{2}\right) - \log\left(\frac{2}{3}\right) + \log\left(\frac{3}{4}\right) - \log\left(\frac{4}{5}\right) + \ldots$

2. Test the absolute convergence or conditional convergence of the following series :

(i) $1 - x + x^2 - x^3 + \ldots$ $(0 < x < 1)$

(ii) $\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots$ $(p > 0)$

(iii) $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \ldots$

(iv) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

(v) $\frac{1}{1.2} - \frac{1}{3.4} + \frac{1}{5.6} - \frac{1}{7.8} + \ldots$

(vi) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \ldots$ **(Meerut 2012B)**

(vii) $\Sigma\,(-1)^n \dfrac{\sin n\alpha}{n^3}$, $\alpha \in \mathbf{R}$.

(viii) $\Sigma\,(-1)^n \dfrac{\cos n\alpha}{n\sqrt{n}}$, $\alpha \in \mathbf{R}$.

(ix) $\Sigma \dfrac{(-1)^{n-1}}{2n+3}$.

(x) $\Sigma\,(-1)^{n-1} \dfrac{n^2}{(n+1)!}$.

(xi) $\sum_{n=1}^{\infty} (-1)^n \sin \dfrac{1}{n}$.

(xii) $\sum_{n=1}^{\infty} (-1)^{n-1} \left[\dfrac{1}{n^2} + \dfrac{1}{(n+1)^2}\right]$.

3. Show that the series $\sum_{n=1}^{\infty} (-1)^{n+1} [\sqrt{(n+1)} - \sqrt{n}]$ is semi-convergent.

4. Show that the series $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{(\sqrt{n} + \sqrt{a})^2}$ is semi-convergent.

5. Show that the series $\left(\dfrac{1}{2}\right)^2 - \left(\dfrac{1.3}{2.4}\right)^2 + \left(\dfrac{1.3.5}{2.4.6}\right)^2 - \dots$

is conditionally convergent.

6. Test the series

$$\frac{1}{2(\log 2)^p} - \frac{1}{3(\log 3)^p} + \frac{1}{4(\log 4)^p} - \dots,\ p > 0$$

for convergence and absolute convergence.

7. Test for convergence the following series :

(i) $\sum_{n=1}^{\infty} (-1)^n \dfrac{n+2}{2^n+5}$.

(ii) $\sum_{n=1}^{\infty} (-1)^{n-1} \left[\dfrac{1}{\sqrt{n^5}} + \dfrac{1}{\sqrt{(n+1)^5}}\right]$.

8. Show that the series $1 - 2 + 3 - 4 + 5 - 6 + \dots$ oscillates infinitely.

9. Discuss the convergence including absolute convergence of the series

$$1 - 2x + 3x^2 - 4x^3 + \dots$$

10. Show that the series $\dfrac{2}{1^2} - \dfrac{3}{2^2} + \dfrac{4}{3^2} - \dfrac{5}{4^2} + \dots$ converges conditionally.

11. Show that the series $\sum_{n=1}^{\infty} \dfrac{\sin(n\theta)}{n^2}$ is absolutely convergent.

12. Define absolute convergence. Show that the series

$$1 - \frac{1}{2^3} - \frac{1}{4^3} + \frac{1}{3^3} - \frac{1}{6^3} - \frac{1}{8^3} + \dots + \frac{1}{(2n-1)^3} - \frac{1}{(4n-2)^3} - \frac{1}{(4n)^3} + \dots$$

is absolutely convergent.

13. Prove that the series

$$z + \frac{a-b}{2!} z^2 + \frac{(a-b)(a-2b)}{3!} z^3 + \frac{(a-b)(a-2b)(a-3b)}{4!} z^4 + \dots$$

is absolutely convergent if $|z| < \dfrac{1}{|b|}$.

Answers 8

1. (i) Convergent (ii) Convergent (iii) Convergent
   (iv) Convergent (v) Oscillate (vi) Convergent
   (vii) Convergent (viii) Convergent
2. (i) Absolutely convergent
   (ii) Absolutely convergent if $p > 1$ and conditionally convergent if $0 < p \leq 1$
   (iii) Absolutely convergent
   (iv) Absolutely convergent for all $x \in \mathbf{R}$
   (v) Absolutely convergent
   (vi) Absolutely convergent
   (vii) Absolutely convergent
   (viii) Absolutely convergent
   (ix) Conditionally convergent
   (x) Absolutely convergent
   (xi) Semi-convergent
   (xii) Absolutely convergent
6. Absolutely convergent if $p > 1$ and semi-convergent if $0 < p \leq 1$
7. (i) Absolutely convergent
   (ii) Absolutely convergent
9. Absolutely convergent if $|x| < 1$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The series $\Sigma \dfrac{1}{n^p}$ is convergent if
   (a) $p < 1$ (b) $p = 1$
   (c) $p > 1$ (d) none of these
2. The series Σu_n of positive terms is convergent if
   (a) $\lim\limits_{n \to \infty} u_n^{1/n} < 1$ (b) $\lim\limits_{n \to \infty} u_n^{1/n} > 1$
   (c) $\lim\limits_{n \to \infty} \dfrac{u_n}{u_{n+1}} < 1$ (d) $\lim\limits_{n \to \infty} \left\{ n \left(\dfrac{u_n}{u_{n+1}} - 1 \right) \right\} < 1$

3. If u_n denotes the nth term of the series

$$\frac{1^2}{2^2}+\frac{1^2 . 3^2}{2^2 . 4^2}x+\frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2}x^2+\ldots, \text{ then}$$

(a) $\lim\limits_{n\to\infty}\frac{u_n}{u_{n+1}}=x$

(b) $\lim\limits_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{1}{x^2}$

(c) $\lim\limits_{n\to\infty}\frac{u_n}{u_{n+1}}=x^2$

(d) $\lim\limits_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{1}{x}$

4. The series $\Sigma\,\frac{1}{n^{2/3}}$ is

(a) convergent
(b) divergent
(c) oscillatory
(d) none of these

5. The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots$ is

(a) divergent
(b) oscillatory
(c) convergent
(d) absolutely convergent

6. The series $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$ is

(a) absolutely convergent
(b) oscillatory
(c) divergent
(d) semi-convergent

7. The series $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ is

(a) divergent
(b) absolutely convergent
(c) semi-convergent
(d) oscillatory

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. For every convergent series $\Sigma\, u_n$, we must have $\lim\limits_{n\to\infty} u_n=\ldots.$
2. The infinite geometric series $a+ax+ax^2+ax^3+\ldots$ is convergent if and only if $|x|<\ldots.$
3. The series $\Sigma\,\frac{1}{n^p}$ is divergent if $p\le\ldots.$
4. The nth term of the series $\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\ldots$, is $\ldots$.
5. Let $\Sigma\, u_n$ be a series of positive terms such that $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=l.$

   Then $\Sigma\, u_n$ converges if $\ldots$.
6. Let $\Sigma\, u_n$ be a series of positive terms such that $\lim\limits_{n\to\infty}\frac{u_n}{u_{n+1}}=l.$

   Then $\Sigma\, u_n$ diverges if $\ldots$.

7. If u_n denotes the nth term of the series
$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \ldots, \text{ then } u_n = \ldots.$$

8. If u_n denotes the nth term of the series
$$1 + 3x + 5x^2 + 7x^3 + \ldots \infty, \text{ then } \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \ldots.$$

9. The nth term of the series
$$\frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \frac{1.2.3.4}{3.5.7.9} + \ldots, \text{ is } \ldots.$$

10. If u_n denotes the nth term of the series
$$\frac{1^2}{2^2} + \frac{1^2.3^2}{2^2.4^2} + \frac{1^2.3^2.5^2}{2^2.4^2.6^2} + \ldots, \text{ then } \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \ldots.$$

11. A series $\Sigma\, u_n$ is said to be semi-convergent if ... .

12. A series $\Sigma\, u_n$ is said to be absolutely convergent if ... .

13. A series in which the terms are alternately positive and negative is called an ... .

14. An infinite series $\Sigma\,(-1)^{n-1} u_n$ in which the terms are alternately positive and negative is convergent if each term is numerically less than the preceding term and $\lim_{n \to \infty} u_n = \ldots.$

15. The series $\Sigma\, u_n$ of positive terms is divergent if $\lim_{n \to \infty} \left\{ n \left(\frac{u_n}{u_{n+1}} - 1 \right) \right\} < \ldots.$

True or False

Write 'T' for true and 'F' for false statement.

1. A series $\Sigma\, u_n$ of positive terms is divergent if $\lim_{n \to \infty} u_n > 0$.
2. A series $\Sigma\, u_n$ is convergent if $\lim_{n \to \infty} u_n = 0$.
3. A series $\Sigma\, u_n$ of positive terms is convergent if $\lim_{n \to \infty} u_n^{1/n} = 1$.
4. A series $\Sigma\, u_n$ of positive terms is divergent if $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} < 1$.
5. The series $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$ is divergent.
6. The series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is divergent.
7. The series $\sum_{n=1}^{\infty} \cos \frac{1}{n}$ is divergent.
8. The series $\sum_{n=1}^{\infty} \sin \frac{1}{n}$ is convergent.

9. The series whose nth term is $\dfrac{\sqrt{n}}{n^2+1}$ is convergent.

10. The series whose nth term is $\dfrac{1}{1+(1/n)}$ is convergent.

11. The series $\sum\limits_{n=1}^{\infty} \dfrac{x^{n-1}}{(n-1)!}$ is absolutely convergent.

12. The series $\dfrac{1}{\sqrt{1}} - \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{4}} + \dots$ is absolutely convergent.

13. The series $\sum\limits_{n=1}^{\infty} (-1)^n [\sqrt{(n^2+1)} - n]$ is semi-convergent.

14. The series $\dfrac{1}{1.2} - \dfrac{1}{3.4} + \dfrac{1}{5.6} - \dfrac{1}{7.8} + \dots$ is divergent. **(Purvanchal 2014)**

15. The series $1 - \dfrac{1}{1!} + \dfrac{1}{2!} - \dfrac{1}{3!} + \dots$ is semi-convergent.

16. The series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{(\sqrt{n}+\sqrt{a})^2}$ is absolutely convergent.

17. Every absolutely convergent series is convergent.

18. Every convergent series is absolutely convergent.

19. The series $\sum\limits_{n=1}^{\infty} (-1)^n \sin \dfrac{1}{n}$ is convergent but is not absolutely convergent.

20. The series $\dfrac{1}{\sqrt{1}+\sqrt{2}} + \dfrac{1}{\sqrt{2}+\sqrt{3}} + \dfrac{1}{\sqrt{3}+\sqrt{4}} + \dots$ is divergent.

21. The series $\Sigma [\sqrt{(n+1)} - \sqrt{n}]$ is convergent.

22. The series $\Sigma [\sqrt{(n^4+1)} - n^2]$ is convergent.

23. The series $\Sigma [\sqrt{(n^3+1)} - \sqrt{n^3}]$ is divergent.

24. The series $1 - \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} - \dfrac{1}{4\sqrt{4}} + \dots$ is absolutely convergent.

25. The series $\left(1+\dfrac{1}{1}\right)^1 + \left(1+\dfrac{1}{2}\right)^2 + \dots + \left(1+\dfrac{1}{n}\right)^n + \dots$ is convergent.

26. The infinite series of positive terms is always convergent or divergent and is never an oscillatory series.

27. The series $1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{4}} + \dots + \dfrac{1}{\sqrt{n}} + \dots$ is divergent.

28. Let Σu_n be an infinite series having all terms positive and let $\lim\limits_{n \to \infty} \dfrac{u_n}{u_{n+1}} = l.$

    If $l > 1$, then Σu_n is divergent.

Answers

Multiple Choice Questions

1. (c)	2. (a)	3. (d)	4. (b)	5. (c)
6. (a)	7. (c)			

Fill in the Blank(s)

1. 0	2. 1	3. 1	4. $\dfrac{n^n}{(n+1)^{n+1}}$	5. $l < 1$
6. $l < 1$	7. $\dfrac{x^{2n-2}}{(n+1)\sqrt{n}}$	8. $\dfrac{1}{x}$	9. $\dfrac{1.2.3.4\ldots n}{3.5.7.9\ldots(2n+1)}$	

10. 1

11. Σu_n is convergent but $\Sigma |u_n|$ is divergent

12. $\Sigma |u_n|$ is convergent

13. alternating series

14. 0

15. 1

True or False

1. *T*	2. *F*	3. *F*	4. *T*	5. *F*
6. *T*	7. *T*	8. *F*	9. *T*	10. *F*
11. *T*	12. *F*	13. *T*	14. *F*	15. *F*
16. *F*	17. *T*	18. *F*	19. *T*	20. *T*
21. *F*	22. *T*	23. *F*	24. *T*	25. *F*
26. *T*	27. *T*	28. *F*		

□

Chapter 6

The Riemann Integral

1 Introduction

In elementary treatments, the process of integration is generally introduced as the inverse of differentiation. If $F'(x) = f(x)$ for all x belonging to the domain of the function f, F is called an integral of the given function f.

Historically, however, the subject of integral arose in connection with the problem of finding areas of plane regions in which the area of a plane region is calculated as the limit of a sum. This notion of integral as summation is based on geometrical concepts.

A German mathematician G.F.B. Riemann gave the first rigorous arithmetic treatment of definite integral free from geometrical concepts. Riemann's definition covered only bounded functions. It was Cauchy who extended this definition to unbounded functions. Later on early in the twentieth century Lebesgue introduced the integral on a firm foundation with many refinements and generalisations.

In the present chapter we shall study the Riemann integral of real valued, bounded functions defined on some closed interval.

2 Partitions and Riemann Sums

Definition 1: *Let $I = [a, b]$ be a closed and bounded interval. Then by a* **partition** *(or a* **dissection** *or a* **net***) of I we mean a finite set of real numbers $P = \{x_0, x_1, \dots, x_{n-1}, x_n\}$ having the property that $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$.*

The closed sub-intervals

$$I_1 = [x_0, x_1], I_2 = [x_1, x_2], \dots, I_n = [x_{n-1}, x_n]$$

determined by P constitute the **segments** *of the partition.*

By changing the set of points, the partition can be changed. Thus there can be an infinite number of partitions of the interval $[a, b]$. The family of all partitions of $[a, b]$ shall be denoted by $\mathbf{P}\,[a, b]$.

We write $\Delta x_r = x_r - x_{r-1}$ for $r = 1, 2, \dots, n$ so that Δx_r is the length of the segment $[x_{r-1}, x_r]$. The **norm** (or **mesh**) of a partition P is the greatest of the lengths of the segments of a partition P and it is denoted by $\|P\|$. Thus

$$\|P\| = \max. (\Delta x_r : r = 1, 2, \dots, n).$$

Sometimes the norm of a partition P is also denoted by $\mu(P)$.

Definition 2: *A partition P^* is called a* **refinement** *of another partition P or we say that P^* is* **finer than** *P iff $P^* \supset P$, i.e., if every point of P is used in the construction of P^*.*

If $P^* = P_1 \cup P_2$, then P^* is called the **common refinement** *of the given two partitions P_1 and P_2.*

Definition 3: *Let f be a bounded function defined on a bounded interval $[a, b]$. Also let $P = \{a = x_0, x_1, \dots, x_n = b\}$ be any partition of $[a, b]$. Also let m_r and M_r be the infimum and supremum respectively of the function f on I_r, for $r = 1, 2, \dots, n$ i.e.,*

$$m_r = \inf \{ f(x) : x_{r-1} \le x \le x_r \}$$

and $$M_r = \sup \{ f(x) : x_{r-1} \le x \le x_r \}.$$

Let us now form two sums

$$L(P, f) = \sum_{r=1}^{n} m_r \,\Delta x_r \text{ and } U(P, f) = \sum_{r=1}^{n} M_r \,\Delta x_r .$$

Then $L(P, f)$ is called the **lower Riemann sum** *(or* **lower Darboux sum***) of f on $[a, b]$ with respect to the partition P and $U(P, f)$ is called the* **upper Riemann sum** *(or* **upper Darboux sum***) of f on $[a, b]$ with respect to the partition P.*

In brief, we shall refer to these sums as the *lower and upper R-sums* of f with respect to P. Obviously $L(P, f) \le U(P, f)$.

Theorem 1: *Let f be a bounded function defined on $[a, b]$ and let m and M be the infimum and supremum of $f(x)$ in $[a, b]$. Then for any partition $\mathbf{P}$ of $[a, b]$, we have*

$$m(b - a) \le L(P, f) \le U(P, f) \le M(b - a).$$ **(Garhwal 2008)**

Proof: Let $P = \{a = x_0, x_1, \dots, x_n = b\}$ be any partition of $[a, b]$. Then $I_r = [x_{r-1}, x_r], r = 1, 2, \dots, n$ are the subintervals of $[a, b]$. Let m_r and M_r be the infimum and supremum of $f(x)$ in $[x_{r-1}, x_r]$. Then for every value of r, we have

$$m \le m_r \le M_r \le M$$

$\Rightarrow$ $m\,\Delta x_r \le m_r\,\Delta x_r \le M_r\,\Delta x_r \le M\,\Delta x_r$ $[\because \Delta x_r > 0]$

$\Rightarrow$ $\sum_{r=1}^{n} m\,\Delta x_r \le \sum_{r=1}^{n} m_r\,\Delta x_r \le \sum_{r=1}^{n} M_r\,\Delta x_r \le \sum_{r=1}^{n} M\,\Delta x_r$. ...(1)

Now $\sum_{r=1}^{n} m\,\Delta x_r = m \sum_{r=1}^{n} \Delta x_r = m \sum_{r=1}^{n} (x_r - x_{r-1})$

$$= m\,(x_1 - x_0 + x_2 - x_1 + \ldots + x_n - x_{n-1})$$
$$= m\,(x_n - x_0) = m\,(b-a).$$

Similarly $\sum_{r=1}^{n} M\Delta\,x_r = M\,(b-a)$.

Also $\sum_{r=1}^{n} m_r\,\Delta x_r = L\,(P, f)$ and $\sum_{r=1}^{n} M_r\,\Delta x_r = U\,(P, f)$.

Hence, from (1), we conclude that

$$m\,(b-a) \le L\,(P, f) \le U\,(P, f) \le M\,(b-a)\ \forall\ P \in \boldsymbol{P}\,[a, b].$$

It follows that the sets of upper sums and lower sums are bounded.

Theorem 2: *If $f : [a, b] \to \boldsymbol{R}$ is a bounded function, then*

$$U\,(P, -f) = -L\,(P, f) \text{ and } L\,(P, -f) = -U\,(P, f).$$ **(Meerut 2012)**

Proof: Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be any partition of $[a, b]$. Let M_r and m_r be the supremum and infimum of f in I_r.

Now f is bounded on $[a, b] \Rightarrow -f$ is bounded on $[a, b]$.

Again M_r, m_r are supremum and infimum of f in I_r

$\Rightarrow$ $-m_r, -M_r$ are supremum and infimum of $-f$ in I_r.

We have $U\,(P, -f) = \sum_{r=1}^{n} (-m_r)\,\Delta x_r$, by definition of upper R-sum

$$= -\sum_{r=1}^{n} m_r\,\Delta x_r = -L\,(P, f).$$

Also $L\,(P, -f) = \sum_{r=1}^{n} (-M_r)\,\Delta x_r$, by definition of lower R-sum

$$= -\sum_{r=1}^{n} M_r\,\Delta x_r = -U\,(P, f).$$

Theorem 3: *Let f be a bounded function defined on $[a, b]$ and let P be a partition of $[a, b]$. If P^* is a refinement of P, then*

$$L\,(P^*, f) \ge L\,(P, f) \text{ and } U\,(P^*, f) \le U\,(P, f).$$

Proof: Let $P = \{a = x_0, x_1, x_2, \ldots, x_{r-1}, x_r, \ldots, x_n = b\}$

and $P^* = \{a = x_0, x_1, x_2, \ldots, x_{r-1}, y_1, x_r, \ldots, x_n = b\}$,

so that P^* has one more partition point y_1 than P.

Let m_r and M_r be the infimum and supremum of f in $[x_{r-1}, x_r]$. Let M_r', M_r'' be the suprema of f in $[x_{r-1}, y_1], [y_1, x_r]$ and m_r', m_r'' be the infima of f in $[x_{r-1}, y_1], [y_1, x_r]$ respectively.

Then $M_r \ge M_r', M_r''$ and $m_r \le m_r', m_r''$.

Since the rth subinterval only of P is split into two more subintervals of P^* and the remaining subintervals are identical in P and P^*, therefore, we have

$$U(P, f) - U(P^*, f)$$
$$= M_r (x_r - x_{r-1}) - \{M_r'(y_1 - x_{r-1}) + M_r''(x_r - y_1)\}.$$

But $\quad M_r'(y_1 - x_{r-1}) + M_r''(x_r - y_1) \le M_r (y_1 - x_{r-1}) + M_r (x_r - y_1)$
$$= M_r (x_r - x_{r-1}).$$

It gives $\quad U(P, f) - U(P^*, f) \ge M_r (x_r - x_{r-1}) - M_r (x_r - x_{r-1}) = 0$

i.e., $\quad U(P, f) \ge U(P^*, f)$ or $U(P^*, f) \le U(P, f)$.

If P^* contains p more partition points than P then repeating p times the above argument we can show that

$$U(P^*, f) \le U(P, f). \qquad ...(1)$$

In a similar manner we can show that

$$L(P^*, f) \ge L(P, f). \qquad ...(2)$$

Note: We know that for any partition P^*,

$$L(P^*, f) \le U(P^*, f). \qquad ...(3)$$

Thus from (1), (2) and (3), we get

$$L(P, f) \le L(P^*, f) \le U(P^*, f) \le U(P, f).$$

Theorem 4: *If P_1 and P_2 be any two partitions of $[a, b]$ then*

$$U(P_1, f) \ge L(P_2, f).$$

Proof: Let $P = P_1 \cup P_2$. Then P is the common refinement of both the partitions P_1 and P_2. Therefore by the theorem 3 above, we get

$$L(P_2, f) \le L(P, f) \text{ and } U(P, f) \le U(P_1, f).$$

But we have $\quad L(P, f) \le U(P, f)$.

Thus $\quad L(P_2, f) \le L(P, f) \le U(P, f) \le U(P_1, f)$.

It follows that $L(P_2, f) \le U(P_1, f)$ or $U(P_1, f) \ge L(P_2, f)$, *that is, every upper sum for f is greater than or equal to every lower sum for f.*

Theorem 5: *Let f, g be bounded functions defined on $[a, b]$ and let P be any partition of $[a, b]$. Then*

$$L(P, f + g) \ge L(P, f) + L(P, g)$$

and $$U(P, f + g) \le U(P, f) + U(P, g).$$

Proof: Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a, b]$. Since f and g are bounded functions on $[a, b]$, $f + g$ is also bounded on $[a, b]$.

Let m_r', M_r' be the infimum and supremum of f on I_r,

m_r'', M_r'' be the infimum and supremum of g on I_r

and m_r, M_r be the infimum and supremum of $f + g$ on I_r.

By definition of infimum we find that

$$f(x) \ge m_r',\ g(x) \ge m_r'' \ \forall\ x \in I_r$$

$$\Rightarrow \quad f(x) + g(x) \ge m_r' + m_r'' \ \forall\ x \in I_r$$

$\Rightarrow \qquad (f+g)(x) \geq m_r' + m_r'' \ \forall\, x \in I_r$

$\Rightarrow \qquad m_r' + m_r''$ is a lower bound of $f+g$ on I_r.

But m_r is the greatest lower bound of $f+g$ on I_r.

$\therefore \qquad m_r \geq m_r' + m_r'' \Rightarrow m_r\,\Delta x_r \geq m_r'\,\Delta x_r + m_r''\,\Delta x_r$

$$\Rightarrow \qquad \sum_{r=1}^{n} m_r\,\Delta x_r \geq \sum_{r=1}^{n} m_r'\,\Delta x_r + \sum_{r=1}^{n} m_r''\,\Delta x_r$$

$\Rightarrow \qquad L(P, f+g) \geq L(P, f) + L(P, g).$

Similarly we can prove the other result.

3 Lower and Upper Riemann Integrals

(Purvanchal 2009, 12)

Let f be a real valued bounded function defined on $[a, b]$. We know that the set of all numbers $L(P, f)$ with respect to all possible partitions P of $[a, b]$ is bounded above by $M(b-a)$ and hence there exists a supremum of $L(P, f)$. The **lower Riemann integral (lower *R*-integral)** of f over $[a, b]$ is the supremum of $L(P, f)$ over all partitions $P \in \mathbf{P}[a, b]$. It is denoted by

$$\underline{\int_a^b} f(x)\,dx.$$

Similarly the set of numbers $U(P, f)$ is bounded below by $m(b-a)$ and so it possesses an infimum. The **upper Riemann integral** (**upper *R*-integral**) of f over $[a, b]$ is the infimum of $U(P, f)$ over all partitions $P \in \mathbf{P}[a, b]$. It is denoted by

$$\overline{\int_a^b} f(x)\,dx.$$

Thus $\qquad \underline{\int_a^b} f(x)\,dx = \sup\{L(P, f) : P \text{ is a partition of } [a, b]\}$

and $\qquad \overline{\int_a^b} f(x)\,dx = \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}.$

We denote the lower and upper integrals of f simply by

$$\underline{\int_a^b} f \qquad \text{and} \qquad \overline{\int_a^b} f.$$

Since $L(P, -f) = -U(P, f)$ and $U(P, -f) = -L(P, f)$, it gives that

$$\underline{\int_a^b}(-f) = -\overline{\int_a^b} f \quad \text{and} \quad \overline{\int_a^b}(-f) = -\underline{\int_a^b} f.$$

Theorem 1: *The lower R-integral cannot exceed the upper R-integral, i.e.,*

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

(Purvanchal 2007, 09; Rohilkhand 11)

Proof: If P_1 and P_2 are any two partitions of $[a, b]$, then by theorem 4 of article 2, we have

$$L(P_1, f) \leq U(P_2, f). \qquad \ldots(1)$$

First, keeping P_2 fixed and taking the supremum over all partitions P_1, (1) gives

$$\underline{\int_a^b} f \leq U(P_2, f). \qquad \ldots(2)$$

Now taking infimum over all partitions P_2, (2) gives

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Theorem 2: (Darboux Theorem): *Let f be a bounded function defined on $[a, b]$. Then to every $\varepsilon > 0$, there corresponds $\delta > 0$ such that*

$$U(P, f) < \overline{\int_a^b} f + \varepsilon \quad \text{and} \quad L(P, f) > \underline{\int_a^b} f - \varepsilon$$

for all partitions P with $\|P\| \leq \delta$.

Proof: Let $\varepsilon > 0$ be given. Since $\overline{\int_a^b} f$ is the infimum of $U(P, f)$ and $\underline{\int_a^b} f$ is the supremum of $L(P, f)$ for all partitions P, therefore, for given $\varepsilon > 0$ there exist partitions P_1 and P_2 such that

$$U(P_1, f) < \overline{\int_a^b} f + \varepsilon \qquad \ldots(1)$$

and

$$L(P_2, f) > \underline{\int_a^b} f - \varepsilon. \qquad \ldots(2)$$

Let P_3 be the common refinement of P_1 and P_2. Then by theorem 3 of article 2, we get

$$U(P_3, f) \leq U(P_1, f) \text{ and } L(P_3, f) \geq L(P_2, f). \qquad \ldots(3)$$

Therefore, from (1), (2) and (3), we get

$$U(P, f) < \overline{\int_a^b} f + \varepsilon \quad \text{and} \quad L(P, f) > \underline{\int_a^b} f - \varepsilon$$

for all partitions P of $[a, b]$ with $\|P\| \leq \delta$, where $\delta = \|P_3\| > 0$.

Corollary: *If f is bounded on $[a, b]$ and P is a partition of $[a, b]$, then*

(i) $$\lim_{\|P\| \to 0} L(P, f) = \underline{\int_a^b} f$$

(ii) $$\lim_{\|P\| \to 0} U(P, f) = \overline{\int_a^b} f.$$

Proof: (i) Since $\underline{\int_a^b} f$ is the supremum of $L(P, f)$ for all partitions P, therefore we have

$$L(P, f) \leq \underline{\int_a^b} f. \qquad \ldots(1)$$

Using the above theorem, we see that for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$L(P, f) > \underline{\int_a^b} f - \varepsilon \qquad \ldots(2)$$

for all partitions P with $\|P\| \leq \delta$.

From (1) and (2), we get

$$\underline{\int_a^b} f - \varepsilon < L(P, f) \leq \underline{\int_a^b} f < \underline{\int_a^b} f + \varepsilon$$

or $$\underline{\int_a^b} f - \varepsilon < L(P, f) < \underline{\int_a^b} f + \varepsilon.$$

By definition of limit, this implies that

$$\lim_{\|P\| \to 0} L(P, f) = \underline{\int_a^b} f.$$

(ii) Since $\overline{\int_a^b} f$ is the infimum of $U(P, f)$ for all partitions P, therefore we have

$$U(P, f) \geq \overline{\int_a^b} f. \qquad ...(3)$$

Using the above theorem, we see that for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$U(P, f) < \overline{\int_a^b} f + \varepsilon \qquad ...(4)$$

for all partitions P with $\|P\| \leq \delta$.

From (3) and (4), we get

$$\overline{\int_a^b} f \leq U(P, f) < \overline{\int_a^b} f + \varepsilon$$

or $$\overline{\int_a^b} f - \varepsilon < U(P, f) < \overline{\int_a^b} f + \varepsilon.$$

By definition of limit, this implies that

$$\lim_{\|P\| \to 0} U(P, f) = \overline{\int_a^b} f.$$

4 *R*-Integrability

(Purvanchal 2012)

Definition: *Let f be a bounded function defined on the bounded interval $[a, b]$, then f is called* **Riemann integrable** *(or simply R-integrable) on $[a, b]$ iff*

$$\underline{\int_a^b} f = \overline{\int_a^b} f$$

and their common value is called the R-integral of f on $[a, b]$ and is denoted by $\int_a^b f$.

The class of all bounded functions f which are Riemann integrable on $[a, b]$ is denoted by $\mathbf{R}[a, b]$. The numbers a and b are called the **lower and upper limits of integration** respectively.

If $\underline{\int_a^b} f \neq \overline{\int_a^b} f$ then f is not Riemann integrable on $[a, b]$.

Note 1: The concept of integrability of a function over an interval as introduced here is subject to two very important limitations, *viz*.

(i) The function is bounded.

(ii) The interval of integration is finite *i.e.*, neither of the end points is infinite.

Note 2: It is not necessary that every bounded function is integrable *i.e.* there may exist a bounded function f for which

$$\underline{\int_a^b} f \neq \overline{\int_a^b} f .$$

Note 3: The statement that $\int_a^b f$ exists indicates that the function f is bounded and integrable over $[a, b]$.

5 Another Definition of Riemann Integral

A function f defined on $[a, b]$ is said to be Riemann integrable over $[a, b]$ iff for every $\varepsilon > 0$ there exists a $\delta > 0$ and a number I such that for every partition

$$P = \{a = x_0, x_1, \ldots, x_n = b\}$$

with $\|P\| \le \delta$ and for every choice of $\xi_r \in [x_{r-1}, x_r]$,

$$\left| \sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1}) - I \right| < \varepsilon .$$

In such a case I is said to be *Riemann integral* of f over $[a, b]$ *i.e.*,

$$I = \int_a^b f(x)\, dx.$$

Theorem: *Definitions of article 4 and article 5 are equivalent.*

Proof: (i) Let f be integrable according to definition of article 4. Then f is bounded and we have

$$\underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f . \qquad \ldots(1)$$

Let $\varepsilon > 0$ be given. Then by Darboux theorem, there exists $\delta > 0$ such that for every partition P with $\|P\| \le \delta$,

$$L(P, f) > \underline{\int_a^b} f - \varepsilon = \int_a^b f - \varepsilon. \qquad \ldots(2)$$

and $$U(P, f) < \overline{\int_a^b} f + \varepsilon = \int_a^b f + \varepsilon. \qquad \ldots(3)$$

If ξ_r is any point of the interval $[x_{r-1}, x_r]$, then

$$L(P, f) \le \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r \le U(P, f). \qquad \ldots(4)$$

From (2), (3) and (4), we conclude that for every partition P with $\|P\| \le \delta$,

$$\int_a^b f - \varepsilon < \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r < \int_a^b f + \varepsilon$$

i.e., $$\left| \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r - \int_a^b f \right| < \varepsilon$$

or $$\left| \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r - I \right| < \varepsilon, \text{ where } I = \int_a^b f.$$

$\therefore f$ is Riemann integrable according to the definition of article 5.

Thus the definition of article 4 $\Rightarrow$ the definition of article 5.

(ii) Let f be integrable according to the definition of article 5.

Then for $\varepsilon = 1 > 0$ there exists $\delta > 0$ and a number I such that for every partition P with $\|P\| \le \delta$ and for every choice of $\xi_r \in [x_{r-1}, x_r]$,

$$\left| \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r - I \right| < 1 \quad i.e., \quad \left| \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r \right| < |I| + 1. \qquad \ldots(1)$$

We have to show that f is bounded on $[a, b]$ and the lower and upper integrals of f over $[a, b]$ are equal.

Suppose f is not bounded on $[a, b]$, then f must not be bounded in at least one subinterval of P, say, $[x_{m-1}, x_m]$. Hence there exists a point $\xi_m \in [x_{m-1}, x_m]$ such that $f(\xi_m)$ is infinite.

Now, to form the sum $\sum_{r=1}^{n} f(\xi_r)\, \Delta x_r$ we have to choose points ξ_r in each subinterval $[x_{r-1}, x_r]$. Choose that ξ_m in the interval $[x_{m-1}, x_m]$ for which $f(\xi_m)$ is infinite.

In that case $\left| \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r \right| > |I| + 1,$

which is a contradiction to (1). Hence f is bounded on $[a, b]$.

Again, by definition of article 5, for any $\varepsilon > 0$ there exists $\delta > 0$ and a number $I \in \mathbf{R}$ such that for every partition P of $[a, b]$ with $\|P\| \le \delta$ and for every choice of $\xi_r \in [x_{r-1}, x_r]$,

$$\left| \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r - I \right| < \frac{1}{2}\varepsilon$$

$$i.e., \qquad I - \frac{1}{2}\varepsilon < \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r < I + \frac{1}{2}\varepsilon. \qquad \ldots(2)$$

Let m_r, M_r be the infimum and supremum of f on $[x_{r-1}, x_r]$, so that there exist points $\alpha_r, \beta_r \in [x_{r-1}, x_r]$ such that

$$f(\alpha_r) > M_r - \frac{\varepsilon}{2(b-a)} \quad \text{and} \quad f(\beta_r) < m_r + \frac{\varepsilon}{2(b-a)}.$$

$$\therefore \qquad \sum_{r=1}^{n} f(\alpha_r)\, \Delta x_r > \sum_{r=1}^{n} M_r\, \Delta x_r - \sum_{r=1}^{n} \frac{\varepsilon}{2(b-a)}\, \Delta x_r$$

$$\text{and} \qquad \sum_{r=1}^{n} f(\beta_r)\, \Delta x_r < \sum_{r=1}^{n} m_r\, \Delta x_r + \sum_{r=1}^{n} \frac{\varepsilon}{2(b-a)}\, \Delta x_r$$

$$i.e. \qquad \sum_{r=1}^{n} f(\alpha_r)\, \Delta x_r > U(P, f) - \frac{\varepsilon}{2} \qquad \ldots(3)$$

$$\text{and} \qquad \sum_{r=1}^{n} f(\beta_r)\, \Delta x_r < L(P, f) + \frac{\varepsilon}{2} \qquad \ldots(4)$$

Now from (2) and (3), we get

$$I + \frac{1}{2}\varepsilon > U(P, f) - \frac{1}{2}\varepsilon \qquad \ldots(5)$$

and from (2) and (4), we get

$$I - \frac{1}{2}\varepsilon < L(P, f) + \frac{1}{2}\varepsilon \qquad \text{...(6)}$$

i.e., $I + \varepsilon > U(P, f)$ and $I - \varepsilon < L(P, f)$ ...(7)

But $L(P, f) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le U(P, f)$. ...(8)

From (7) and (8), we get

$$I - \varepsilon < \underline{\int_a^b} f \le \overline{\int_a^b} f < I + \varepsilon \qquad \text{...(9)}$$

or $0 \le \overline{\int_a^b} f - \underline{\int_a^b} f < 2\varepsilon$ or $0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le 0$,

as $\varepsilon > 0$ is arbitrary

or $\overline{\int_a^b} f - \underline{\int_a^b} f = 0$ *i.e.*, $\overline{\int_a^b} f = \underline{\int_a^b} f$. ...(10)

$\therefore f$ is Riemann integrable according to the definition of article 4. Now from (9) and (10), we have

$$I - \varepsilon < \int_a^b f < I + \varepsilon.$$

Since ε is arbitrary, $I = \int_a^b f$.

Thus the definition of article 5 $\Rightarrow$ the definition of article 4.

Hence the two definitions of article 4 and article 5 are equivalent.

6 Riemann's Necessary and Sufficient Condition for *R*-Integrability

Oscillatory sum: With usual notations, we have

$$L(P, f) = \sum_{r=1}^{n} m_r \, \Delta x_r \,, U(P, f) = \sum_{r=1}^{n} M_r \, \Delta x_r \ .$$

Let $\omega_r = M_r - m_r$ so that ω_r is the oscillation of f on $[x_{r-1}, x_r]$.

$\therefore$ $U(P, f) - L(P, f) = \sum_{r=1}^{n} (M_r - m_r) \, \Delta x_r = \sum_{r=1}^{n} \omega_r \, \Delta x_r$.

The sum $\sum_{r=1}^{n} \omega_r \, \Delta x_r$ is called the **oscillatory sum** for the function f corresponding to the partition P and is denoted by $\omega(P, f)$.

Theorem: *A necessary and sufficient condition for R-integrability of a bounded function* $f : [a, b] \to \boldsymbol{R}$ *over* $[a, b]$ *is that for every* $\varepsilon > 0$ *, there exists a partition P of* $[a, b]$ *such that for P and all its refinements*

$$0 \le U(P, f) - L(P, f) < \varepsilon. \qquad \textbf{(Rohilkhand 2009; Gorakhpur 14, 15)}$$

Proof: The condition is necessary. Let $f \in \boldsymbol{R}[a, b]$ so that

$$\underline{\int_a^b} f = \overline{\int_a^b} f. \qquad \text{...(1)}$$

By Darboux theorem, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all partitions P of $[a, b]$ with $\| P \| \leq \delta$,

$$U(P, f) < \overline{\int_a^b} f + \frac{\varepsilon}{2} \qquad ...(2)$$

and $$L(P, f) > \underline{\int_a^b} f - \frac{\varepsilon}{2}. \qquad ...(3)$$

Adding the inequalities (2) and (3), we get

$$U(P, f) + \underline{\int_a^b} f - \frac{\varepsilon}{2} < L(P, f) + \overline{\int_a^b} f + \frac{\varepsilon}{2}.$$

In view of (1), this gives

$$U(P, f) < L(P, f) + \varepsilon \quad i.e., \; U(P, f) - L(P, f) < \varepsilon \qquad ...(4)$$

Since $U(P, f) \geq L(P, f)$, the inequality (4) can be written as

$$0 \leq U(P, f) - L(P, f) < \varepsilon.$$

Hence the condition is necessary.

The condition is sufficient: Let for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that for P and all its refinements,

$$0 \leq U(P, f) - L(P, f) < \varepsilon. \qquad ...(5)$$

By the definition of upper and lower integrals, we have

$$\overline{\int_a^b} f \leq U(P, f) \text{ and } \underline{\int_a^b} f \geq L(P, f)$$

or $$-\underline{\int_a^b} f \leq -L(P, f).$$

$\therefore$ $$\overline{\int_a^b} f - \underline{\int_a^b} f \leq U(P, f) - L(P, f) < \varepsilon \qquad ...(6)$$

or $$\overline{\int_a^b} f - \overline{\int_a^b} f < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\overline{\int_a^b} f - \underline{\int_a^b} f \leq 0. \qquad ...(7)$$

Also we know that the lower Riemann integral can never exceed the upper Riemann integral

i.e., $$\overline{\int_a^b} f - \underline{\int_a^b} f \geq 0. \qquad ...(8)$$

From (7) and (8), we get

$$\overline{\int_a^b} f - \underline{\int_a^b} f = 0 \quad \text{or} \quad \overline{\int_a^b} f = \underline{\int_a^b} f$$

i.e., the function f is Riemann integrable over $[a, b]$.

Note: **Another statement of the above theorem:** *A necessary and sufficient condition for a bounded function f to be integrable over $[a, b]$ is that for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that the oscillatory sum $\omega(P, f) < \varepsilon$*

or $\quad$ *lim $\omega(P, f) = 0$ as $\| P \|$ tends to zero.*

Illustrative Examples

Example 1: *Show that if f is defined on $[a, b]$ by*

$$f(x) = k \ \forall \ x \in [a, b]$$

where k is a constant, then $f \in \boldsymbol{R}[a, b]$ and $\int_a^b k = k(b - a)$. **(Gorakhpur 2015)**

Solution: Obviously the given function is bounded over $[a, b]$.

Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a, b]$. Then for any subinterval $[x_{r-1}, x_r]$, we have $m_r = k, M_r = k$.

Now, $$U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r = \sum_{r=1}^{n} k \Delta x_r = k \sum_{r=1}^{n} \Delta x_r$$

$$= k[\Delta x_1 + \Delta x_2 + \ldots + \Delta x_n]$$
$$= k[(x_1 - x_0) + (x_2 - x_1) + \ldots + (x_n - x_{n-1})]$$
$$= k(x_n - x_0) = k(b - a)$$

and $$L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n} k \Delta x_r = k(b - a).$$

Hence $$\overline{\int_a^b} f = \inf U(P, f) = \inf\{k(b - a)\} = k(b - a)$$

and $$\underline{\int_a^b} f = \sup L(P, f) = \sup\{k(b - a)\} = k(b - a).$$

Thus $$\overline{\int_a^b} f = \underline{\int_a^b} f = k(b - a).$$

Hence $$f \in R[a, b] \text{ and } \int_a^b f = k(b - a).$$

Example 2: *Let $f(x) = x$ on $[0, 1]$. Calculate $\underline{\int_0^1} x\,dx$ and $\overline{\int_0^1} x\,dx$ by dissecting $[0, 1]$ into n equal parts and hence show that $f \in \boldsymbol{R}[0, 1]$.*

(Purvanchal 2011; Garhwal 12; Meerut 12)

Solution: Let $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{r-1}{n}, \frac{r}{n}, \ldots, \frac{n}{n} = 1\right\}$.

Here $$m_r = \frac{r-1}{n}, M_r = \frac{r}{n} \text{ and } \Delta x_r = \frac{1}{n} \text{ for } r = 1, 2, \ldots, n.$$

Now, we have

$$L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n} \frac{r-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{r=1}^{n} (r - 1)$$

$$= \frac{1}{n^2}[1 + 2 + 3 + \ldots + (n - 1)] = \frac{(n-1) \cdot n}{2n^2} = \frac{n-1}{2n}$$

and $$U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r = \sum_{r=1}^{n} \frac{r}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{r=1}^{n} r$$

$$= \frac{1}{n^2}[1 + 2 + 3 + \ldots + n] = \frac{n(n+1)}{2n^2} = \frac{n+1}{2n}.$$

Hence by Corollary 1, theorem 2 of article 3, we get

$$\underline{\int_0^1} x\,dx = \lim_{\|P\|\to 0} L(P,f) = \lim_{n\to\infty} \frac{n-1}{2n} = \frac{1}{2}$$

and
$$\overline{\int_0^1} x\,dx = \lim_{\|P\|\to 0} U(P,f) = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}.$$

Since
$$\underline{\int_0^1} f = \overline{\int_0^1} f, f \in \boldsymbol{R}[0,1] \text{ and } \int_0^1 x\,dx = \frac{1}{2}.$$

Example 3: *Let $f(x) = x^2$ on $[0,a], a > 0$. Show that $f \in \boldsymbol{R}[0,a]$ and find $\int_0^a f$.*

(Rohilkhand 2010; Gorakhpur 14)

Solution: Let $P = \left\{0, \frac{a}{n}, \frac{2a}{n}, \ldots, \frac{(n-1)a}{n}, \frac{na}{n} = a\right\}$ be the partition of $[0,a]$ obtained by dissecting $[0,a]$ into n equal parts. Then

$$\Delta x_r = a/n, r = 1, 2, \ldots, n.$$

Also I_r = rth subinterval $= \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$.

Since $f(x) = x^2$ is an increasing function in $[0,a]$,

$$m_r = \frac{(r-1)^2 a^2}{n^2} \quad \text{and} \quad M_r = \frac{r^2 a^2}{n^2}, r = 1, 2, \ldots, n.$$

Now
$$L(P,f) = \sum_{r=1}^{n} m_r\,\Delta x_r = \sum_{r=1}^{n} \frac{(r-1)^2 a^2}{n^2} \cdot \frac{a}{n}$$

$$= \frac{a^3}{n^3} \sum_{r=1}^{n} (r-1)^2 = \frac{a^3}{n^3} \cdot \frac{(n-1)\,n\,(2n-1)}{6}.$$

$\therefore$
$$\underline{\int_0^a} x^2\,dx = \lim_{n\to\infty} L(P,f) = \lim_{n\to\infty} \frac{a^3}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right) = \frac{a^3}{3}.$$

Again
$$U(P,f) = \sum_{r=1}^{n} M_r\,\Delta x_r = \sum_{r=1}^{n} \frac{r^2 a^2}{n^2} \cdot \frac{a}{n}$$

$$= \frac{a^3}{n^3} \sum_{r=1}^{n} r^2 = \frac{a^3}{n^3} \cdot \frac{n\,(n+1)\,(2n+1)}{6}.$$

$\therefore$
$$\overline{\int_0^a} x^2\,dx = \lim_{n\to\infty} U(P,f) = \lim_{n\to\infty} \frac{a^3}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) = \frac{a^3}{3}.$$

Since
$$\overline{\int_0^a} f = \underline{\int_0^a} f, f \in \boldsymbol{R}[0,a] \text{ and } \int_0^a x^2\,dx = \frac{a^3}{3}.$$

Example 4: *If a function f is defined on $[0,a]$, $a > 0$ by $f(x) = x^3$, then show that f is Riemann integrable on $[0,a]$ and $\int_0^a f(x)\,dx = \frac{a^4}{4}$.*

Solution: Let $P = \left\{0, \frac{a}{n}, \frac{2a}{n}, \ldots, \frac{(n-1)a}{n}, \frac{na}{n} = a\right\}$ be the partition of $[0, a]$ obtained by dissecting $[0, a]$ into n equal parts. Then

$$I_r = r \text{ th sub-interval} = \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$$

and $$\Delta x_r = \text{length of } I_r = \frac{a}{n}, r = 1, 2, \ldots, n.$$

Let m_r and M_r be respectively the infimum and supremum of f in I_r.

Since $f(x) = x^3$ is an increasing function in $[0, a]$, therefore

$$m_r = \frac{(r-1)^3 a^3}{n^3} \quad \text{and} \quad M_r = \frac{r^3 a^3}{n^3}, r = 1, 2, \ldots, n.$$

Now $$L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n} \left[\frac{(r-1)^3 a^3}{n^3} \cdot \frac{a}{n}\right] = \frac{a^4}{n^4} \sum_{r=1}^{n} (r-1)^3$$

$$= \frac{a^4}{n^4} [1^3 + 2^3 + \ldots + (n-1)^3] = \frac{a^4}{n^4} \cdot \left[\frac{(n-1)n}{2}\right]^2 = \frac{a^4}{4}\left(1 - \frac{1}{n}\right)^2.$$

$\therefore$ $$\underline{\int_0^a} f(x)\, dx = \lim_{n \to \infty} L(P, f) = \lim_{n \to \infty} \frac{a^4}{4}\left(1 - \frac{1}{n}\right)^2 = \frac{a^4}{4}.$$

Again $$U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r = \sum_{r=1}^{n} \left[\frac{r^3 a^3}{n^3} \cdot \frac{a}{n}\right] = \frac{a^4}{n^4} \sum_{r=1}^{n} r^3$$

$$= \frac{a^4}{n^4} (1^3 + 2^3 + \ldots + n^3) = \frac{a^4}{n^4} \cdot \left[\frac{n(n+1)}{2}\right]^2 = \frac{a^4}{4}\left(1 + \frac{1}{n}\right)^2.$$

$\therefore$ $$\overline{\int_0^a} f(x)\, dx = \lim_{n \to \infty} U(P, f) = \lim_{n \to \infty} \frac{a^4}{4}\left(1 + \frac{1}{n}\right)^2 = \frac{a^4}{4}.$$

Since $\underline{\int_0^a} f = \overline{\int_0^a} f$, f is Riemann integrable on $[0, a]$ and

$$\int_0^a f(x)\, dx = \int_0^a x^3\, dx = \frac{a^4}{4}.$$

Example 5: *Let f be the function defined on $[0, 1]$ by*

$$f(x) = \begin{cases} 0 & \textit{when } x \textit{ is irrational,} \\ 1 & \textit{when } x \textit{ is rational.} \end{cases}$$

Calculate $\underline{\int_0^1} f$ and $\overline{\int_0^1} f$ and hence show that $f \notin \boldsymbol{R}[0, 1]$.

(Purvanchal 2008, 09; Garhwal 11; Rohilkhand 11)

Solution: First, we observe that f is bounded, for evidently

$$0 \le f(x) \le 1 \ \forall\ x \in [0, 1].$$

Let P be any partition of $[0, 1]$. Then for any subinterval $[x_{r-1}, x_r]$ of P, we have $m_r = 0$ and $M_r = 1$, because every subinterval will contain rational as well as irrational numbers. Note that rational as well as irrational points are everywhere dense.

It follows that

$$L(P, f) = \sum_{r=1}^{n} m_r \, \Delta x_r = \sum_{r=1}^{n} 0 \, . \, \Delta x_r = 0$$

and
$$U(P, f) = \sum_{r=1}^{n} M_r \, \Delta x_r = \sum_{r=1}^{n} 1 \, . \, \Delta x_r = \sum_{r=1}^{n} \Delta x_r = 1.$$

$\therefore$
$$\underline{\int_0^1} f = \lim_{n \to \infty} L(P, f) = 0 \text{ and } \overline{\int_0^1} f = \lim_{n \to \infty} U(P, f) = 1.$$

Since $\underline{\int_0^1} f \neq \overline{\int_0^a} f$, we have $f \notin \boldsymbol{R}[0, 1]$.

Example 6: *Give an example of a bounded function which is not R-integrable over* $[0, 1]$.

(Garhwal 2006, 09)

Solution: See example 5 above. As another example, consider the function f defined on $[0, 1]$ as follows :

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational.} \end{cases}$$

Evidently f is bounded on $[0, 1]$.

If P is any partition of $[0, 1]$, then for any subinterval $[x_{r-1}, x_r]$ of P, we have $m_r = -1$ and $M_r = 1, r = 1, 2, \ldots, n$.

$\therefore$
$$L(P, f) = \sum_{r=1}^{n} m_r \, \Delta x_r = \sum_{r=1}^{n} -1 \, . \, \Delta x_r = -1 \, . \sum_{r=1}^{n} \Delta x_r$$

$$= -1 \, . \, (1 - 0) = -1 \, . \, 1 = -1,$$

$$U(P, f) = \sum_{r=1}^{n} M_r \, \Delta x_r = 1. \sum_{r=1}^{n} \Delta x_r = 1 \, . \, (1 - 0) = 1.$$

Hence
$$\underline{\int_0^1} f = \lim_{n \to \infty} L(P, f) = -1, \overline{\int_0^1} f = \lim_{n \to \infty} U(P, f) = 1.$$

Thus $\underline{\int_0^1} f \neq \overline{\int_0^1} f$ and consequently $f \notin \boldsymbol{R}[0, 1]$.

Example 7: *Show that* $f(x) = \sin x$ *is integrable on* $\left[0, \frac{1}{2}\pi\right]$ *and* $\int_0^{\pi/2} \sin x \, dx = 1.$

Solution: Let $P = \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \ldots, \frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}, \ldots, \frac{n\pi}{2n} = \frac{\pi}{2}\right\}$

be the partition of $[0, \pi/2]$ obtained by dissecting $[0, \frac{1}{2}\pi]$ into n equal parts. The length of each subinterval $= \pi / 2n$ and the r th subinterval $I_r = \left[\frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}\right]$.

Since $f(x) = \sin x$ is increasing in $[0, \frac{1}{2}\pi]$, we have

$$m_r = \sin \frac{(r-1)\pi}{2n} \text{ and } M_r = \sin \frac{r\pi}{2n}, r = 1, 2, \ldots, n.$$

Now $$U(P, f) = \sum_{r=1}^{n} M_r \, \Delta x_r = \sum_{r=1}^{n} \left(\sin \frac{r\pi}{2n}\right) \cdot \frac{\pi}{2n}$$

$$= \frac{\pi}{2n}\left[\sin \frac{\pi}{2n} + \sin \frac{2\pi}{2n} + \ldots + \sin \frac{n\pi}{2n}\right].$$

We know that

$$\sin a + \sin (a + d) + \ldots + \sin \{a + (n-1) d\} = \frac{\sin\left(a + \frac{n-1}{2} d\right) \sin \frac{nd}{2}}{\sin (d/2)}.$$

$\therefore$ $$U(P, f) = \frac{\pi}{2n}\left[\frac{\sin\left(\frac{\pi}{2n} + \frac{n-1}{2} \cdot \frac{\pi}{2n}\right) \sin \frac{n\pi}{4n}}{\sin \frac{\pi}{4n}}\right] = \frac{\frac{\pi}{2n} \sin \frac{(n+1)\pi}{4n} \cdot \sin \frac{\pi}{4}}{\sin \frac{\pi}{4n}}$$

$$= \frac{\frac{\pi}{2n} \cdot \sin\left(\frac{\pi}{4} + \frac{\pi}{4n}\right) \cdot \frac{1}{\sqrt{2}}}{\sin \frac{\pi}{4n}} = \frac{\frac{\pi}{2\sqrt{2n}}\left\{\sin \frac{\pi}{4} \cos \frac{\pi}{4n} + \cos \frac{\pi}{4} \sin \frac{\pi}{4n}\right\}}{\sin\left(\frac{\pi}{4n}\right)}$$

$$= \frac{\pi}{2\sqrt{2n}} \cdot \frac{1}{\sqrt{2}}\left(\cot \frac{\pi}{4n} + 1\right) = \frac{\pi}{4n}\left(\cot \frac{\pi}{4n} + 1\right).$$

Similarly, we can find that

$$L(P, f) = \frac{\pi}{4n}\left(\cot \frac{\pi}{4n} - 1\right).$$

Now $$\underline{\int_0^{\pi/2}} f = \lim_{n \to \infty} L(P, f) = \lim_{n \to \infty} \frac{\pi}{4n}\left(\cot \frac{\pi}{4n} - 1\right)$$

$$= \lim_{n \to \infty} \frac{(\pi/4n)}{\tan (\pi/4n)} - \lim_{n \to \infty} \frac{\pi}{4n} = 1 - 0 = 1$$

$$\left[\because \lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1\right]$$

and $$\overline{\int_0^{\pi/2}} f = \lim_{n \to \infty} U(P, f) = \lim_{n \to \infty} \frac{\pi}{4n}\left(\cot \frac{\pi}{4n} + 1\right) = 1.$$

Since $$\underline{\int_0^{\pi/2}} f = \overline{\int_0^{\pi/2}} f, f \in \boldsymbol{R}\left[0, \frac{\pi}{2}\right] \text{ and } \int_0^{\pi/2} f = 1.$$

Example 8: *Let $f(x)$ be a function defined on $[0, \frac{1}{4}\pi]$ by*

$$f(x) = \begin{cases} \cos x, & \text{if } x \text{ is rational} \\ \sin x, & \text{if } x \text{ is irrational}. \end{cases}$$

Show that f is not Riemann integrable over $\left[0, \frac{1}{4}\pi\right]$.

Solution: Consider the partition

$$P = \left\{ 0, \frac{\pi}{4n}, \frac{2\pi}{4n}, \ldots, \frac{(r-1)\pi}{4n}, \frac{r\pi}{4n}, \ldots, \frac{n\pi}{4n} = \frac{\pi}{4} \right\}$$

obtained by dissecting $\left[0, \frac{1}{4}\pi\right]$ into n equal parts.

Then for any subinterval $\left[\frac{(r-1)\pi}{4n}, \frac{r\pi}{4n}\right], r = 1, 2, \ldots, n$, we have

$$m_r = \sin\frac{(r-1)\pi}{4n} \text{ and } M_r = \cos\frac{(r-1)\pi}{4n}, \Delta x_r = \frac{\pi}{4n}.$$

Hence $$L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n} \sin\frac{(r-1)\pi}{4n} \cdot \frac{\pi}{4n}$$

$$= \frac{\pi}{4n}\left[\sin\frac{\pi}{4n} + \ldots + \sin\frac{(n-1)\pi}{4n}\right]$$

$$= \frac{\pi}{4n} \cdot \frac{\sin\left(\frac{\pi}{4n} + \frac{n-2}{2} \cdot \frac{\pi}{4n}\right) \cdot \sin\frac{n\pi}{8n}}{\sin\frac{\pi}{8n}}$$ [summing up the series]

$$= \frac{(\pi/8n)}{\sin(\pi/8n)} \cdot 2\sin^2\frac{\pi}{8}$$

and $$U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r = \sum_{r=1}^{n} \cos\frac{(r-1)\pi}{4n} \cdot \frac{\pi}{4n}$$

$$= \frac{\pi}{4n}\left[\cos 0 + \cos\frac{\pi}{4n} + \ldots + \cos\frac{(n-1)\pi}{4n}\right]$$

$$= \frac{\pi}{4n} \cdot \frac{\cos\left(\frac{n-1}{2} \cdot \frac{\pi}{4n}\right) \sin\frac{n\pi}{8n}}{\sin\frac{\pi}{8n}}$$ [summing up the series]

$$= \frac{(\pi/8n)}{\sin(\pi/8n)} \cdot 2\cos\frac{(n-1)\pi}{8n} \cdot \sin\frac{\pi}{8}.$$

Hence $$\underline{\int_0^{\pi/4}} f = \lim_{||P|| \to 0} L(P, f) = \lim_{n \to \infty} L(P, f)$$

$$= \lim_{n \to \infty} \frac{(\pi/8n)}{\sin(\pi/8n)} \cdot 2\sin^2\frac{\pi}{8} = 2\sin^2\frac{\pi}{8} = \left(1 - \cos\frac{\pi}{4}\right)$$

$$= 1 - \frac{1}{\sqrt{2}}$$

and $$\overline{\int_0^{\pi/4}} f = \lim_{n \to \infty} U(P, f)$$

$$= \lim_{n \to \infty} \frac{(\pi/8n)}{\sin(\pi/8n)} \cdot 2\cos\frac{(n-1)\pi}{8n} \cdot \sin\frac{\pi}{8}$$

$$= \lim_{n \to \infty} \frac{(\pi/8n)}{\sin(\pi/8n)} \cdot 2\cos\frac{\pi}{8}\left(1 - \frac{1}{n}\right) \sin\frac{\pi}{8}$$

$$= 2\cos\frac{\pi}{8}\sin\frac{\pi}{8} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Since $\underline{\int_0^{\pi/4}} f \neq \overline{\int_0^{\pi/4}} f$, f is not Riemann integrable over $\left[0, \frac{\pi}{4}\right]$.

Example 9: *Let f be a function on $[0, 1]$ defined by*

$$f(x) = \begin{cases} 1 & \text{if } x \neq \frac{1}{2} \\ 0 & \text{if } x = \frac{1}{2}. \end{cases}$$

Show that $f \in \boldsymbol{R}[0, 1]$ and find $\int_0^1 f$.

Solution: Obviously $f(x)$ is bounded in $[0, 1]$ since

$$0 \leq f(x) \leq 1 \ \forall \ x \in [0, 1].$$

Let P be a partition of $[0, 1]$ such that the point $\frac{1}{2}$ belongs to the open interval $]x_{s-1}, x_s[$.

Then we have (with usual notations)

$$m_r = M_r = 1 \text{ for } r = 1, 2, \ldots, n$$

and $\quad r \neq s, m_s = 0, M_s = 1.$

Now $\quad U(P, f) - L(P, f)$

$$= \sum_{\substack{r=1 \\ r \neq s}}^{n} (M_r - m_r)\,\Delta x_r + (M_s - m_s)(x_s - x_{s-1})$$

$$= \sum_{r=1}^{n} (1 - 1)(x_r - x_{r-1}) + (1 - 0)(x_s - x_{s-1})$$

$$= x_s - x_{s-1}. \qquad \ldots(1)$$

Let $\varepsilon > 0$ be given. Then we choose a partition P such that the point $\frac{1}{2}$ is an interior point of one of the subintervals whose length is less than ε. Then, it follows from (1) that

$$U(P, f) - L(P, f) < \varepsilon.$$

Hence, by theorem of article 6, $f \in \boldsymbol{R}[0, 1]$.

Now to find $\int_0^1 f$, it is enough to find $\underline{\int_0^1} f$ or $\overline{\int_0^1} f$.

We calculate $\overline{\int_0^1} f$. For any partition P, we have

$$U(P, f) = \sum_{r=1}^{n} M_r\,\Delta x_r = \sum_{r=1}^{n} 1.\Delta x_r = \sum_{r=1}^{n} \Delta x_r$$

$$= \text{length of the interval } [0, 1] = 1.$$

Hence $\quad \overline{\int_0^1} f = \lim_{n \to \infty} U(P, f) = 1.$

Thus $\quad f \in \boldsymbol{R}[0, 1]$ and $\int_0^1 f = \overline{\int_0^1} f = 1.$

Example 10: *If* $f(x) = x + x^2$ *for rational values of* x *in the interval* $[0,2]$ *and* $f(x) = x^2 + x^3$ *for irrational values of x in the same interval, evaluate the upper and the lower Riemann integrals of* f *over* $[0,2]$. **(Purvanchal 2007)**

Solution: We have $(x + x^2) - (x^2 + x^3) = x - x^3 = x(1 - x^2)$

so that $$(x + x^2) - (x^2 + x^3) > 0 \quad \text{if } 0 < x < 1$$

and $$< 0 \quad \text{if } 1 < x < 2.$$

If P is any partition of $[0,2]$, then any subinterval of P, however small it may be, will contain rational as well as irrational points.

With usual notations, we have for all values of r

$$M_r = x + x^2, \quad \text{if} \quad 0 < x < 1$$
$$= x^2 + x^3, \quad \text{if} \quad 1 < x < 2$$

and $$m_r = x^2 + x^3, \quad \text{if} \quad 0 < x < 1$$
$$= x + x^2, \quad \text{if} \quad 1 < x < 2.$$

Hence $$\overline{\int_0^2} f(x)\, dx = \int_0^1 (x + x^2)\, dx + \int_1^2 (x^2 + x^3)\, dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} + \frac{x^4}{4}\right]_1^2$$

$$= \left(\frac{1}{2} + \frac{1}{3}\right) - 0 + \left(\frac{8}{3} + \frac{16}{4}\right) - \left(\frac{1}{3} + \frac{1}{4}\right) = \frac{83}{12} = 6\frac{11}{12}$$

and $$\underline{\int_0^2} f(x)\, dx = \int_0^1 (x^2 + x^3)\, dx + \int_1^2 (x + x^2)\, dx$$

$$= \left[\frac{x^3}{3} + \frac{x^4}{4}\right]_0^1 + \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{3} + \frac{1}{4} + \frac{4}{2} + \frac{8}{3} - \frac{1}{2} - \frac{1}{3} = \frac{53}{12} = 4\frac{5}{12}.$$

Example 11: *Find the upper and lower R-integrals for the function* f *defined on* $[0,1]$ *as follows:*

$$f(x) = \begin{cases} \sqrt{(1 - x^2)}, & \textit{if } x \textit{ is rational} \\ (1 - x), & \textit{if } x \textit{ is irrational}. \end{cases}$$

Solution: Here we have

$$(1 - x^2) - (1 - x)^2 = 2x - 2x^2 = 2x(1 - x) > 0 \quad \forall\, x \in \,]0,1[$$

i.e., $$\sqrt{(1 - x^2)} > (1 - x) \quad \forall\, x \in \,]0,1[.$$

With usual notations, we have for all values of r

$$M_r = \sqrt{(1 - x^2)} \text{ and } m_r = (1 - x).$$

Hence $$\underline{\int_0^1} f = \int_0^1 (1 - x)\, dx = \left[x - \frac{x^2}{2}\right]_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$$

and $\quad \overline{\int_0^1} f = \int_0^1 \sqrt{(1-x^2)}\, dx = \left[\frac{1}{2} x \sqrt{(1-x^2)} + \frac{1}{2}\sin^{-1} x\right]_0^1$

$$= \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}.$$

Thus $\quad \underline{\int_0^1} f \neq \overline{\int_0^1} f$. It follows that $f \notin \boldsymbol{R}\,[0,1]$.

7 Some Classes of Integrable Functions

Integrability of Continuous Functions: **(Garhwal 2007; Purvanchal 07; 08, 10; Rohilkhand 11; Gorakhpur 10, 13, 15)**

Theorem 1: *If f is continuous on $[a,b]$, then $f \in \boldsymbol{R}\,[a,b]$.*

Proof: Since f is continuous on $[a,b]$, f is bounded on $[a,b]$. Also since f is continuous on a closed and bounded interval $[a,b]$, f is uniformly continuous on $[a,b]$. Hence for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all points x', x'' of $[a,b]$

$$|f(x') - f(x'')| < \frac{\varepsilon}{b-a} \text{ whenever } |x' - x''| < \delta. \qquad \dots(1)$$

Let $\quad P = \{a = x_0, x_1, \dots, x_n = b\}$ be a partition of $[a,b]$ with

$$\|P\| < \delta.$$

Since f is continuous on $[x_{r-1}, x_r]$ it attains its infimum m_r and supremum M_r at some points c_r and d_r of $[x_{r-1}, x_r]$ respectively so that

$$m_r = f(c_r) \text{ and } M_r = f(d_r). \qquad \dots(2)$$

Since $|c_r - d_r| < \delta$, therefore from (1), we get

$$|f(c_r) - f(d_r)| < \varepsilon/(b-a).$$

But $\quad |f(c_r) - f(d_r)| = f(d_r) - f(c_r) \qquad [\because f(d_r) \geq f(c_r)]$

$$= M_r - m_r.$$

Thus $\quad M_r - m_r < \varepsilon/(b-a),\ r = 1, 2, \dots, n.$

Now for the partition P of $[a,b]$, we have

$$0 \leq U(P,f) - L(P,f) = \sum_{r=1}^{n} (M_r - m_r)\,\Delta x_r$$

$$< \sum_{r=1}^{n} \{\varepsilon/(b-a)\}\,\Delta x_r$$

$$= \frac{\varepsilon}{b-a} \sum_{r=1}^{n} \Delta x_r = \frac{\varepsilon}{b-a}\cdot(b-a) = \varepsilon.$$

Hence by theorem of article 6, f is R-integrable over $[a,b]$,

i.e., $\quad f \in \boldsymbol{R}\,[a,b]$.

Note: There exist functions which are integrable but not continuous. So, continuity is a sufficient condition but not necessary for integrability.

Integrability of Monotonic Functions:

Theorem 2: *If f is monotonic on $[a, b]$, then $f \in \mathbf{R}[a, b]$.*

(Garhwal 2007; Purvanchal 08, 12; Rohilkhand 10)

Proof: Suppose f is monotonic increasing on $[a, b]$.

Then $f(a) \le f(x) \le f(b) \;\forall\; x \in [a, b]$.

$\therefore f$ is bounded on $[a, b]$ and $\inf f = f(a)$, $\sup f = f(b)$.

Let $\varepsilon > 0$ be given and $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a, b]$ with $\|P\| \le \varepsilon / [f(b) - f(a) + 1]$.

If m_r and M_r be the inf and sup of f on I_r, then $m_r = f(x_{r-1})$ and $M_r = f(x_r)$ because f is monotonic increasing on $[a, b]$.

Hence
$$U(P, f) - L(P, f) = \sum_{r=1}^{n} (M_r - m_r)\,\Delta x_r$$
$$= \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})]\,\Delta x_r$$
$$\le \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})] \cdot \frac{\varepsilon}{f(b) - f(a) + 1}$$
$$= \frac{\varepsilon}{f(b) - f(a) + 1} \sum_{r=1}^{n} [f(x_r) - f(x_{r-1})]$$
$$= \frac{\varepsilon}{f(b) - f(a) + 1} [f(x_1) - f(x_0) + f(x_2) - f(x_1) + \ldots + f(x_n) - f(x_{n-1})]$$
$$= \frac{\varepsilon}{f(b) - f(a) + 1} [f(b) - f(a)] \qquad [\because f(x_0) = f(a), f(x_n) = f(b)]$$
$$< \varepsilon.$$

It follows from the theorem of article 6 that $f \in \mathbf{R}[a, b]$.

If f is monotonic decreasing on $[a, b]$, then $-f$ is monotonic increasing on $[a, b]$ and so $-f \in \mathbf{R}[a, b]$.

$$\therefore \qquad \underline{\int_a^b} - f(x)\,dx = \overline{\int_a^b} - f(x)\,dx$$

i.e.,
$$-\underline{\int_a^b} f(x)\,dx = -\overline{\int_a^b} f(x)\,dx$$

or
$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$$

i.e., f is R-integrable on $[a, b]$.

Hence if f is monotonic on $[a, b]$, $f \in \mathbf{R}[a, b]$.

Note: If we had taken $f(b) - f(a)$ instead of $f(b) - f(a) + 1$, the proof would not have been valid when $f(b) - f(a) = 0$ *i.e.,* when f is a constant function. It is to include this case that we have used this artifice.

Theorem 3: *If the set of points of discontinuity of a bounded function f defined on $[a, b]$ is finite, then $f \in \boldsymbol{R}[a, b]$.* **(Garhwal 2012)**

Proof: Since f is discontinuous on $[a, b]$, the supremum M and the infimum m of f in $[a, b]$ are not equal *i.e.*, $M - m \neq 0$.

Let $\{c_1, c_2, \ldots, c_p\}$ be the finite set of points of discontinuity of f in $[a, b]$, where $c_1 < c_2 < \ldots < c_p$. Let $\varepsilon > 0$ be given. We enclose the points $c_1, c_2, \ldots, c_p$ respectively in p mutually disjoint intervals

$$[a_1, b_1], [a_2, b_2], \ldots, [a_p, b_p] \qquad \ldots(1)$$

such that the sum of their lengths is $< \varepsilon / 2(M - m)$.

Now f is continuous on each of the subintervals

$$[a, a_1], [b_1, a_2], [b_2, a_3], \ldots, [b_p, b]. \qquad \ldots(2)$$

Consequently, there exist partitions P_r, $r = 1, 2, \ldots, p + 1$ respectively, of the subintervals in (2) such that, using theorem 1 of article 7,

$$\omega(P_r, f) < \frac{\varepsilon}{2(p+1)} \text{ for } r = 1, 2, \ldots, p + 1.$$

Now consider the partition P of $[a, b]$ defined by

$$P = \cup \{P_r : r = 1, 2, \ldots, p + 1\}.$$

The subintervals of P can be divided into two groups :

(i) all the subintervals of P_r, $r = 1, 2, \ldots, p + 1$

(ii) all the subintervals given in (1).

Corresponding to these two groups of subintervals we have two contributions to the oscillatory sum $\omega(P, f)$.

The total contribution to the oscillatory sum $\omega(P, f)$ of the subintervals in (i) is

$$< \frac{\varepsilon}{2(p+1)}(p+1) = \frac{\varepsilon}{2}\cdot$$

Also since the oscillation of f in each of the subintervals in (ii) is $\leq M - m$, the total contribution to the oscillatory sum $\omega(P, f)$ of the subintervals in (ii) is

$$< \frac{\varepsilon}{2(M-m)}\cdot(M - m) = \frac{\varepsilon}{2}\cdot$$

$\therefore$ for the partition P of $[a, b]$, we have

$$\omega(P, f) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus for each $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that

$$0 \leq \omega(P, f) < \varepsilon.$$

Hence $f \in \boldsymbol{R}[a, b]$, by theorem of article 6.

Theorem 4: *If the set of points of discontinuity of a bounded function f defined on $[a, b]$ has only a finite number of limit points then $f \in \boldsymbol{R}[a, b]$.*

Proof: Since f is discontinuous on $[a, b]$, the supremum M and the infimum m of f in $[a, b]$ are not equal *i.e.*, $M - m \neq 0$.

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ be the finite set of limit points of the set of the points of discontinuity of f in $[a, b]$, where $\alpha_1 < \alpha_2 < \ldots < \alpha_p$. Let $\varepsilon > 0$ be given. We enclose the points $\alpha_1, \alpha_2, \ldots, \alpha_p$ respectively in p mutually disjoint intervals

$$[a_1, b_1], [a_2, b_2], \ldots, [a_p, b_p] \qquad \ldots(1)$$

such that the sum of their lengths is $< \varepsilon / 2(M - m)$.

In each of the remaining $p + 1$ subintervals

$$[a, a_1], [b_1, a_2], [b_2, a_3], \ldots, [b_p, b] \qquad \ldots(2)$$

f has only a finite number of points of discontinuity because none of these $p + 1$ sub-intervals contains a limit point of the set of points of discontinuity of f.

Hence, by the previous theorem, there exist partitions P_r , $r = 1, 2, \ldots, p + 1$ respectively of the subintervals in (2) such that

$$\omega(P_r, f) < \frac{\varepsilon}{2(p+1)} \text{ for } r = 1, 2, \ldots, p + 1.$$

Now consider the partition P of $[a, b]$ determined by

$$P = \cup \{P_r : r = 1, 2, \ldots, p + 1\}.$$

The subintervals of P can be divided into two groups :

(i) all the subintervals of P_r , $r = 1, 2, \ldots, p + 1$.

(ii) all the subintervals given in (1).

The total contribution to the oscillatory sum $\omega(P, f)$ of the subintervals in (i) is

$$< \{\varepsilon / 2(p + 1)\}.(p + 1) = \varepsilon / 2.$$

Also since the oscillation of f in each of the subintervals in (ii) is $\leq M - m$, the total contribution to the oscillatory sum $\omega(P, f)$ of the subintervals in (ii) is

$$< \{\varepsilon / 2(M - m)\}.(M - m) = \varepsilon / 2.$$

Thus for any $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

$$\omega(P, f) < \varepsilon / 2 + \varepsilon / 2 = \varepsilon.$$

Hence, by theorem of article 6, $f \in \boldsymbol{R}[a, b]$.

Illustrative Examples

Example 12: *A function f is defined on $[0, 1]$ by*

$$f(x) = 1 / 2^n \textit{ for } 1 / 2^{n+1} < x \leq 1 / 2^n, n = 0, 1, 2, \ldots.$$

and $f(0) = 0$. Show that $f \in \boldsymbol{R}[0, 1]$ and calculate the value of

$$\int_0^x f(t)\, dt$$

where x lies between $1 / 2^m$ and $1 / 2^{m-1}$. **(Garhwal 2007)**

Solution: Here, for $n = 1, 2, 3, \ldots$, we have $f\left(\frac{1}{2^n} + 0\right) = \frac{1}{2^{n-1}}$ and $f\left(\frac{1}{2^n} - 0\right) = \frac{1}{2^n}$,

which shows that the function $f(x)$ is discontinuous at $x = 1 / 2^n$, $n = 1, 2, 3, \ldots$.

Also for $n = 0$, $f\left(\frac{1}{2^n}\right) = f(1) = 1$

and $\quad f\left(\frac{1}{2^n} - 0\right) = 1$ so that $f(x)$ is continuous at $x = \frac{1}{2^0} = 1$.

Thus the points of discontinuity of f are

$$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots$$

Since the set of these points of discontinuity of f has only one limiting point at $x = 0$, it follows from theorem 4 of article 7 that $f \in \boldsymbol{R}[0,1]$.

Now
$$\int_0^x f(t)\, dt = \int_{1/2^m}^x f + \int_{1/2^{m+1}}^{1/2^m} f + \int_{1/2^{m+2}}^{1/2^{m+1}} f + \ldots$$
$$= \int_{1/2^m}^x \frac{1}{2^{m-1}} + \int_{1/2^{m+1}}^{1/2^m} \frac{1}{2^m} + \int_{1/2^{m+2}}^{1/2^{m+1}} \frac{1}{2^{m+1}} + \ldots$$
$$= \frac{1}{2^{m-1}}\left[x - \frac{1}{2^m}\right] + \frac{1}{2^m}\left[\frac{1}{2^m} - \frac{1}{2^{m+1}}\right] + \frac{1}{2^{m+1}}\left[\frac{1}{2^{m+1}} - \frac{1}{2^{m+2}}\right] + \ldots$$
$$= \frac{1}{2^{m-1}}\left[x - \frac{1}{2^m}\right] + \frac{1}{2^{2m+1}} + \frac{1}{2^{2m+3}} + \frac{1}{2^{2m+5}} + \ldots$$
$$= \frac{x}{2^{m-1}} - \frac{1}{2^{2m-1}} + \frac{1/2^{2m+1}}{1 - \frac{1}{4}}$$

(Summing up the infinite G.P.)

$$= \frac{x}{2^{m-1}} - \frac{1}{2^{2m-1}} + \frac{1}{3.2^{2m-1}} = \frac{x}{2^{m-1}} - \frac{1}{3.2^{2m-2}}.$$

Example 13: *Let the function f be defined on $[0,1]$ as follows :*

$$f(x) = 2rx \quad \text{when} \quad \frac{1}{r+1} < x \le \frac{1}{r}, \ r = 1, 2, 3, \ldots$$

Prove that f is R-integrable in $[0,1]$ and evaluate $\int_0^1 f(x)\, dx$.

(Rohilkhand 2010; Gorakhpur 13)

Solution: The given function f is not defined for $x = 0$. We may, however, define f at this point in any manner we like provided f remains bounded.

Here
$$f(x) = 2x \quad \text{when} \quad \frac{1}{2} < x \le 1$$
$$f(x) = 4x \quad \text{when} \quad \frac{1}{3} < x \le \frac{1}{2}$$
$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$
$$f(x) = 2(r-1)x \quad \text{when} \quad \frac{1}{r} < x \le \frac{1}{r-1}$$
$$f(x) = 2rx \quad \text{when} \quad \frac{1}{r+1} < x \le \frac{1}{r}$$
$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

For $r = 2, 3, 4, \ldots$, we have

$$f\left(\frac{1}{r}+0\right)=\lim_{h\to 0} 2(r-1)\left(\frac{1}{r}+h\right)=2-\frac{2}{r}$$

and $$f\left(\frac{1}{r}-0\right)=\lim_{h\to 0} 2r\left(\frac{1}{r}-h\right)=2.$$

Since $$f\left(\frac{1}{r}+0\right)\neq f\left(\frac{1}{r}-0\right), f \text{ is not continuous at } x=1/r.$$

Also $f(1)=2$ and $f(1-0)=\lim_{h\to 0} 2(1-h)=2$, so that f is continuous at $x=1$.

Thus the given function f is not continuous at $x=1/r$, $r=2,3,4,\ldots$ and the set of points of discontinuity $\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots$ of f has only one limit point at $x=0$. Consequently, by theorem 4 of article 7, the given function is R-integrable.

Now $$\int_{1/(n+1)}^{1} f(x)\,dx=\int_{1/2}^{1} f+\int_{1/3}^{1/2} f+\int_{1/4}^{1/3} f+\ldots+\int_{1/(n+1)}^{1/n} f$$

$$=\sum_{r=1}^{n}\int_{1/(r+1)}^{1/r} f. \qquad \ldots(1)$$

We have $$\int_{1/(r+1)}^{1/r} f(x)\,dx=\int_{1/(r+1)}^{1/r} 2rx\,dx=[rx^2]_{1/(r+1)}^{1/r}$$

$$=r\left[\frac{1}{r^2}-\frac{1}{(r+1)^2}\right]. \qquad \ldots(2)$$

Putting $r=1,2,\ldots,n$ in (2) and then adding the partial integrals, in view of (1), we have

$$\int_{1/(n+1)}^{1} f(x)\,dx=1\left[\frac{1}{1^2}-\frac{1}{2^2}\right]+2\left[\frac{1}{2^2}-\frac{1}{3^2}\right]+3\left[\frac{1}{3^2}-\frac{1}{4^2}\right]+\ldots$$

$$+n\left[\frac{1}{n^2}-\frac{1}{(n+1)^2}\right]$$

$$=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2}-\frac{n}{(n+1)^2}$$

$$=\sum_{r=1}^{n}\frac{1}{r^2}-\frac{1/n}{\left(1+\frac{1}{n}\right)^2}.$$

Letting n tend to infinity, we get

$$\int_0^1 f(x)\,dx=\sum_1^{\infty}\frac{1}{r^2}-0=\sum_1^{\infty}\frac{1}{r^2}=\frac{\pi^2}{6}.$$

Example 14: *A function f is defined on $[0,1]$ by*

$$f(x)=1/n \text{ for } 1/(n+1)<x\leq 1/n,\ n=1,2,3,\ldots \text{ and } f(0)=0.$$

Prove that $f\in \boldsymbol{R}[0,1]$ and evaluate $\int_0^1 f(x)\,dx$.

Solution: It can be easily seen that the points of discontinuity of f are $\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots$.

Since the set of points of discontinuity of f has only one limit point at $x=0$, it follows from theorem 4 of article 7 that $f\in \boldsymbol{R}[0,1]$.

Now as in the previous example,

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} \int_{1/(r+1)}^{1/r} \frac{1}{r}\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{r}\left(\frac{1}{r}-\frac{1}{r+1}\right)$$

$$= \lim_{n\to\infty} \left[\left(1-\frac{1}{2}\right)+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\frac{1}{n}\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$= \lim_{n\to\infty} \left[\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}\right)-\left(\frac{1}{2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n.(n+1)}\right)\right]$$

$$= \lim_{n\to\infty} \left[\left(1+\frac{1}{2^2}+\ldots+\frac{1}{n^2}\right)-\left(\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{n}-\frac{1}{(n+1)}\right)\right]$$

$$= \lim_{n\to\infty} \left[\left(1+\frac{1}{2^2}+\ldots+\frac{1}{n^2}\right)-\left(1-\frac{1}{n+1}\right)\right] = \frac{\pi^2}{6}-1.$$

$$\left[\because \text{ the series } 1+\frac{1}{2^2}+\ldots+\frac{1}{n^2}+\ldots \text{ converges to } \frac{\pi^2}{6}\right]$$

Example 15: *Let a function f be defined on $[0,1]$ as follows :*

If x is irrational let $f(x)=0$, if x is a rational number p/q in its lowest terms, let $f(x)=1/q$, also let $f(0)=f(1)=0$. Show that f is integrable over $[0,1]$ and $\int_0^1 f(x)\,dx=0$.

Solution: Evidently, the function f is bounded on $[0,1]$. Let $\varepsilon>0$ be given. Then there exist only a finite number of fractions $\frac{1}{q}$ such that $\frac{1}{q}>\frac{\varepsilon}{2}$, for $\frac{1}{q}>\frac{\varepsilon}{2}$ iff $q<\frac{2}{\varepsilon}$ and ε being given, $\frac{2}{\varepsilon}$ is finite. We enclose these exceptional points, in order, in mutually disjoint closed intervals

$$[a_1,b_1],[a_2,b_2],\ldots,[a_m,b_m] \qquad \ldots(1)$$

such that the sum of their lengths is less than $\varepsilon/2$. On the remaining sub-intervals

$$[b_0=0,a_1],[b_1,a_2],[b_2,a_3],\ldots,[b_m,a_{m+1}=1] \qquad \ldots(2)$$

at each point the value of f is $<\varepsilon/2$.

It is obvious that the oscillation of f in each of the subintervals (1) cannot exceed 1 and the oscillation of f in each of the sub-intervals (2) is less than $\varepsilon/2$. Thus for the partition

$$P=\{0=b_0,a_1,b_1,a_2,b_2,\ldots,a_m,b_m,b_{m+1}=1\}$$

we have,

$$\omega(P,f)=U(P,f)-L(P,f)$$

$$< \sum_{r=1}^{m} 1.(b_r-a_r)+\sum_{r=0}^{m} \frac{\varepsilon}{2}(a_{r+1}-b_r)$$

$$< \frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon=\varepsilon.$$

$$\left[\because \sum_{r=1}^{m}(b_r-a_r)<\frac{\varepsilon}{2},\ \sum_{r=0}^{m}(a_{r+1}-b_r)<1\right]$$

Thus for a given $\varepsilon > 0$ there exists a partition P such that $\omega(P, f) < \varepsilon$.

Hence $\qquad f \in \mathbf{R}[0,1]$.

Also since for every partition $P, L(P, f) = 0$, we have

$$\int_0^1 f = \underline{\int_0^1} f = 0.$$

8 Algebra of Integrable Functions

Definition: *If $b < a$, we define $\int_a^b f$ to be $-\int_b^a f$ provided that $f \in \mathbf{R}[a,b]$.*

Also by definition, we write $\int_a^a f = 0$.

Theorem 1: *Let $f \in \mathbf{R}[a,b]$ and let m, M be bounds of f on $[a,b]$. Then*

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \text{ if } b \ge a,$$

$$m(b-a) \ge \int_a^b f(x)\,dx \ge M(b-a) \text{ if } b \le a.$$

(Purvanchal 2008)

Proof: If $a = b$, the result is trivial. Let $b > a$. Then by theorem 1 of article 2, we have for all partitions P of $[a,b]$,

$$m(b-a) \le L(P, f) \le U(P, f) \le M(b-a)$$

or $\qquad m(b-a) \le L(P, f) \le M(b-a)$

or $\qquad m(b-a) \le \sup L(P, f) \le M(b-a)$

or $\qquad m(b-a) \le \underline{\int_a^b} f \le M(b-a)$

or $\qquad m(b-a) \le \int_a^b f \le M(b-a). \qquad \left[\because \underline{\int_a^b} f = \int_a^b f\right]$

Now let $b < a$ so that $a > b$. Hence, as proved above we get

$$m(a-b) \le \int_b^a f \le M(a-b)$$

$\Rightarrow \qquad -m(b-a) \le -\int_a^b f \le -M(b-a)$, by the definition mentioned above

$\Rightarrow \qquad m(b-a) \ge \int_a^b f \ge M(b-a).$

Corollary 1: *Let $f \in \mathbf{R}[a,b]$. Then there exists a number μ lying between the bounds m and M of f such that $\int_a^b f(x)\,dx = \mu(b-a)$.*

(Purvanchal 2011)

This result is known as the **first mean value theorem of integral calculus.**

Corollary 2: *Let f be continuous on $[a,b]$. Then there exists a point $c \in [a,b]$ such that*

$$\int_a^b f(x)\,dx = (b-a) f(c).$$

Proof: Since f is continuous, $f \in \mathbf{R}[a,b]$. Therefore, by Corollary 1, there exists a number μ lying between m and M, such that

$$\int_a^b f(x)\,dx = \mu(b-a).$$

Now f being continuous on $[a, b]$, so it takes every value between its bounds m and M *i.e.*, in particular it takes the value μ lying between m and M. Consequently there is a point $c \in [a, b]$ such that $f(c) = \mu$ and hence

$$\int_a^b f(x)\, dx = (b - a) f(c).$$

Corollary 3: *Let $f \in \mathbf{R}[a, b]$ and let K be a number such that*

$$|f(x)| \le K \;\forall\; x \in [a, b].$$

Then $$\left| \int_a^b f(x)\, dx \right| \le K |b - a|.$$

(Purvanchal 2010)

Proof: The result is trivial when $a = b$. Let m, M be the bounds of f on $[a, b]$. Let $b > a$. We have, for all $x \in [a, b]$

$$|f(x)| \le K \Rightarrow -K \le f(x) \le K \quad \Rightarrow \quad -K \le m \le f(x) \le M \le K$$

$$\Rightarrow \quad -K(b - a) \le m(b - a) \le \int_a^b f(x)\, dx \le M(b - a) \le K(b - a)$$

$$\Rightarrow \quad \left| \int_a^b f(x)\, dx \right| \le K(b - a). \qquad \ldots(1)$$

Now let $b < a$ so that $a > b$.

Then we get from (1) that

$$\left| \int_b^a f(x)\, dx \right| \le K(a - b) \Rightarrow \left| -\int_a^b f(x)\, dx \right| \le K(a - b)$$

$$\Rightarrow \quad \left| \int_a^b f(x)\, dx \right| \le K(a - b). \qquad \ldots(2)$$

Hence, from (1) and (2), we get

$$\left| \int_a^b f(x)\, dx \right| \le K |b - a|.$$

Corollary 4: *Let $f \in \mathbf{R}[a, b]$ and let $f(x) \ge 0 \;\forall\; x \in [a, b]$. Then*

$$\int_a^b f(x)\, dx \ge 0 \text{ if } b \ge a \text{ and } \int_a^b f(x)\, dx \le 0 \text{ if } b \le a.$$

Proof: Since $f(x) \ge 0 \;\forall\; x \in [a, b]$, hence, $m \ge 0$.

If $b \ge a$, then from the first result of the above theorem, we get

$$\int_a^b f(x)\, dx \ge m(b - a) \ge 0$$

and if $b \le a$, then from the second result of the above theorem, we get

$$\int_a^b f(x)\, dx \le m(b - a) \le 0. \qquad [\because m \ge 0 \text{ and } b - a \le 0]$$

Corollary 5: *Let $f, g \in \mathbf{R}[a, b]$. Then*

$$f \ge g \Rightarrow \begin{cases} \int_a^b f(x)\, dx \ge \int_a^b g(x)\, dx \text{ if } b \ge a \\ \int_a^b f(x)\, dx \le \int_a^b g(x)\, dx \text{ if } b \le a. \end{cases}$$

Proof: We have

$$f \ge g \Rightarrow [f(x) - g(x)] \ge 0 \;\forall\; x \in [a, b]$$

$\Rightarrow \qquad \int_a^b [f(x) - g(x)]\, dx \geq 0 \;$ or $\; \leq 0$

according as $b \geq a$ or $b \leq a$ by Corollary 4,

$$\Rightarrow \qquad \left[\int_a^b f(x)\, dx - \int_a^b g(x)\, dx\right] \geq 0 \quad \text{or} \quad \leq 0$$

$$\Rightarrow \qquad \int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx \qquad \text{or} \qquad \int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$$

according as $b \geq a$ or $b \leq a$.

Theorem 2: *If $f \in \mathbf{R}[a, b]$ and $K \in \mathbf{R}$, then $Kf \in \mathbf{R}[a, b]$ and*

$$\int_a^b K f(x)\, dx = K \int_a^b f(x)\, dx.$$

Proof: If $K = 0$, the theorem is obvious. Suppose that $K \neq 0$. Since $f \in \mathbf{R}[a, b]$, f is bounded on $[a, b]$ and

$$\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx = \int_a^b f(x)\, dx. \qquad \ldots(1)$$

We know that $|K f| = |K| \cdot |f|$, so that Kf is bounded on $[a, b]$.

Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be any partition of $[a, b]$ and m_r and M_r be the infimum and supremum of f in $[x_{r-1}, x_r]$.

Case I: Let $K > 0$. Then $K m_r$ and $K M_r$ are the inf and sup of $K f$ in $[x_{r-1}, x_r]$. So we have

$$\underline{\int_a^b} Kf(x)\, dx = \sup\left[\sum_{r=1}^{n} K m_r \Delta x_r\right]$$

$$= K \sup\left[\sum_{r=1}^{n} m_r \Delta x_r\right] = K \underline{\int_a^b} f(x)\, dx \qquad \ldots(2)$$

$$= K \overline{\int_a^b} f(x)\, dx, \quad \text{using (1)}$$

$$= K.\inf\left[\sum_{r=1}^{n} M_r \Delta x_r\right] = \inf\left[\sum_{r=1}^{n} K M_r \Delta x_r\right]$$

$$= \underline{\int_a^b} K f(x)\, dx.$$

Thus $\qquad \underline{\int_a^b} K f(x)\, dx = \overline{\int_a^b} K f(x)\, dx.$

Hence $Kf \in \mathbf{R}[a, b]$.

Also from (2), we have

$$\underline{\int_a^b} K f(x)\, dx = K \underline{\int_a^b} f(x)\, dx$$

$$\Rightarrow \qquad \int_a^b K f(x)\, dx = K \int_a^b f(x)\, dx. \qquad [\because f \in \mathbf{R}[a, b] \text{ and } K f \in \mathbf{R}[a, b]]$$

Case II: Let $K < 0$. Then $K M_r$ and $K m_r$ respectively denote the inf and sup of $K f$ in $[x_{r-1}, x_r]$. We have

$$\underline{\int_a^b} Kf(x)\, dx = \sup\left[\sum_{r=1}^{n} K M_r \Delta x_r\right]$$

$$= K \inf \left[\sum_{r=1}^{n} M_r \, \Delta x_r \right], \text{ as } K < 0$$

$$= K \overline{\int_a^b} f(x)\, dx = K \underline{\int_a^b} f(x)\, dx \qquad \ldots(3)$$

$$= K \sup \left[\sum_{r=1}^{n} m_r \, \Delta x_r \right] = \inf \left[\sum_{r=1}^{n} K m_r \, \Delta x_r \right], \text{ as } K < 0$$

$$= \overline{\int_a^b} K f(x)\, dx.$$

Thus in this case, $\underline{\int_a^b} K f(x)\, dx = \overline{\int_a^b} K f(x)\, dx.$

Hence $Kf \in \boldsymbol{R}[a, b]$. Also from (3), we have

$$\underline{\int_a^b} K f(x)\, dx = K \underline{\int_a^b} f(x)\, dx$$

$\Rightarrow$ $$\int_a^b K f(x)\, dx = K \int_a^b f(x)\, dx. \qquad [\because f \in \boldsymbol{R}[a, b] \text{ and } Kf \in \boldsymbol{R}[a, b]]$$

Theorem 3: *Let $a < c < b$. Then $f \in \boldsymbol{R}[a, b]$ iff $f \in \boldsymbol{R}[a, c]$ and $f \in \boldsymbol{R}[c, b]$. In either case*

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

(Gorakhpur 2015)

Proof: Obviously f is bounded on $[a, c]$ and $[c, b]$ iff f is bounded on $[a, b]$.

Let $f \in \boldsymbol{R}[a, b]$. Then for a given $\varepsilon > 0$, there is a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon. \qquad \ldots(1)$$

Let $P^* = P \cup \{c\}$. Then P^* is also a partition of $[a, b]$ and it is a refinement of P so that

$$U(P^*, f) - L(P^*, f) \leq U(P, f) - L(P, f) < \varepsilon. \qquad \ldots(2)$$

Let P_1 and P_2 be the partitions of $[a, c]$ and $[c, b]$ respectively such that $P^* = P_1 \cup P_2$. Then

$$U(P^*, f) - L(P^*, f) = [U(P_1, f) + U(P_2, f)] - [L(P_1, f) + L(P_2, f)]$$

$$= [U(P_1, f) - L(P_1, f)] + [U(P_2, f) - L(P_2, f)]$$

$< \varepsilon$, by (2).

Since each of $[U(P_1, f) - L(P_1, f)]$ and $[U(P_2, f) - L(P_2, f)]$ is ≥ 0, each of them is less than ε.

Hence $f \in \boldsymbol{R}[a, c]$ and $f \in \boldsymbol{R}[c, b]$.

Also $U(P^*, f) = U(P_1, f) + U(P_2, f)$

$\Rightarrow$ $$\inf U(P^*, f) = \inf U(P_1, f) + \inf U(P_2, f)$$

$\Rightarrow$ $$\overline{\int_a^b} f(x)\, dx = \overline{\int_a^a} f(x)\, dx + \overline{\int_c^b} f(x)\, dx$$

$\Rightarrow$ $$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

$$[\because f \in \boldsymbol{R}[a, b], f \in \boldsymbol{R}[a, c], f \in \boldsymbol{R}[c, b]]$$

Conversely, let $f \in \boldsymbol{R}[a, c]$ and $f \in \boldsymbol{R}[c, b]$.

Then given $\varepsilon > 0$, there exist partitions P_1 and P_2 of $[a, c]$ and $[c, b]$ respectively such that

$$U(P_1, f) - L(P_1, f) < \varepsilon/2$$

and $$U(P_2, f) - L(P_2, f) < \varepsilon/2.$$

Let $P = P_1 \cup P_2$, then P is partition of $[a, b]$.

Now $$\begin{aligned} U(P, f) - L(P, f) &= [U(P_1, f) + U(P_2, f)] - [L(P_1, f) + L(P_2, f)] \\ &= [U(P_1, f) - L(P_1, f)] + [U(P_2, f) - L(P_2, f)] \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence $f \in \boldsymbol{R}[a, b]$. The remaining part of the theorem can be proved as before.

Theorem 4: *If $f, g \in \boldsymbol{R}[a, b]$ then $f \pm g \in \boldsymbol{R}[a, b]$*

and $$\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx.$$

Proof: Since $f, g \in \boldsymbol{R}[a, b]$, for a given $\varepsilon > 0$, there exist partitions P_1 and P_2 of $[a, b]$ such that

$$\left.\begin{aligned} U(P_1, f) - L(P_1, f) &< \varepsilon/2 \\ \text{and} \quad U(P_2, g) - L(P_2, g) &< \varepsilon/2 \end{aligned}\right\} \quad \text{....(1)}$$

Let $P = P_1 \cup P_2$, then P is a common refinement of P_1 and P_2 and it is a partition of $[a, b]$. We have

$$\begin{aligned} & U(P, f+g) - L(P, f+g) \\ & \le [U(P, f) + U(P, g)] - [L(P, f) + L(P, g)], \end{aligned}$$

[by theorem 5, article 2]

$$\begin{aligned} & = [U(P, f) - L(P, f)] + [U(P, g) - L(P, g)] \\ & < \varepsilon/2 + \varepsilon/2 = \varepsilon, \text{ by (1).} \end{aligned}$$

Since $U(P, f+g) - L(P, f+g) < \varepsilon$, $f + g \in \boldsymbol{R}[a, b]$.

Again, by the second definition of Riemann integrability, $f, g \in \boldsymbol{R}[a, b] \Rightarrow$ for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every partition P of $[a, b]$ with $\|P\| \le \delta$ and for every choice of $\xi_r \in [x_{r-1}, x_r]$,

$$\left| \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r - \int_a^b f(x)\, dx \right| < \varepsilon/2 \quad \text{...(2)}$$

and $$\left| \sum_{r=1}^{n} g(\xi_r)\, \Delta x_r - \int_a^b g(x)\, dx \right| < \varepsilon/2 \quad \text{...(3)}$$

From (2) and (3), we get

$$\left| \sum_{r=1}^{n} [f(\xi_r) + g(\xi_r)]\, \Delta x_r - \left[\int_a^b f(x)\, dx + \int_a^b g(x)\, dx\right] \right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx.$$

Similarly we can prove the result for the difference $f - g$ of the functions f and g.

Lemma: *Let f be a bounded function with bounds m and M defined on $[a, b]$. Then the oscillation $M - m$ of f on $[a, b]$ is the supremum of the set*

$$\{|f(x) - f(y)| : x, y \in [a, b]\}$$

of numbers.

Theorem 5: *If $f \in \boldsymbol{R}[a, b]$, $g \in \boldsymbol{R}[a, b]$, then $fg \in \boldsymbol{R}[a, b]$.*

(Garhwal 2010; Purvanchal 10; Bundelkhand 11)

Proof: Since $f, g \in \boldsymbol{R}[a, b]$, they are bounded on $[a, b]$.

So there exists a positive real number M such that

$$|f(x)| \leq M \text{ and } |g(x)| \leq M \quad \forall \ x \in [a, b].$$

It follows that $|f(x)\, g(x)| \leq M^2 \ \forall\, x \in [a, b]$.

Thus fg is bounded on $[a, b]$.

Again $f, g \in \cup [a, b] \Rightarrow$ for a given $\varepsilon > 0$, there exist partitions P_1 and P_2 of $[a, b]$ such that

$$U(P_1, f) - L(P_1, f) < \varepsilon / 2M$$

and $$U(P_2, g) - L(P_2, g) < \varepsilon / 2M.$$

Let $P = P_1 \cup P_2 = \{a = x_0, x_1, \dots, x_n = b\}$ so that P is refinement of both P_1 and P_2. It follows by theorem 3 of article 2 that

$$\left.\begin{aligned} &U(P, f) - L(P, f) < \varepsilon / 2M \\ \text{and} \quad &U(P, g) - L(P, g) < \varepsilon / 2M \end{aligned}\right\} \qquad \dots(1)$$

Let m_r, M_r ; m_r', M_r' and m_r'', M_r'' be respectively the bounds of fg, f and g in the r th interval $I_r = [x_{r-1}, x_r]$. Then for all

$x, y \in [x_{r-1}, x_r]$, we have

$$\begin{aligned} |(fg)(y) - (fg)(x)| &= |f(y)\, g(y) - f(x)\, g(x)|, \text{ by the def. of } fg \\ &= |g(y)[f(y) - f(x)] + f(x)[g(y) - g(x)]| \\ &\leq |g(y)||f(y) - f(x)| + |f(x)||g(y) - g(x)| \\ &\leq M|f(y) - f(x)| + M|g(y) - g(x)| \qquad \dots(2) \end{aligned}$$

Taking suprema of both sides of (2), we have by the lemma given above

$$M_r - m_r \leq M(M_r' - m_r') + M(M_r'' - m_r'')$$

$$\Rightarrow \quad \sum_{r=1}^{n} (M_r - m_r)\, \Delta x_r$$

$$\leq M \sum_{r=1}^{n} (M_r' - m_r')\, \Delta x_r + M \sum_{r=1}^{n} (M_r'' - m_r'')\, \Delta x_r$$

or $$U(P, fg) - L(P, fg)$$

$$\begin{aligned} &\leq M[U(P, f) - L(P, f)] + M[U(P, g) - L(P, g)] \\ &< M(\varepsilon / 2M) + M(\varepsilon / 2M), \text{ by (1)} \\ &= \varepsilon / 2 + \varepsilon / 2 = \varepsilon. \end{aligned}$$

Hence $fg \in \boldsymbol{R}[a, b]$ by theorem of article 6.

Corollary: *If $f \in \boldsymbol{R}[a, b]$ then $f^2 \in \boldsymbol{R}[a, b]$.*

Theorem 6: *If $f \in \mathbf{R}[a,b]$ then $|f| \in \mathbf{R}[a,b]$.* **(Purvanchal 2008)**

Show that $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$

Proof: Since $f \in \mathbf{R}[a,b]$, f is bounded on $[a,b]$ so that there exists a positive number k such that $|f(x)| \le k \ \forall\ x \in [a,b]$ *i.e.* $|f(x)|$ is bounded on $[a,b]$.

Again, since $f \in \mathbf{R}[a,b]$, for given $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that

$$U(P,f) - L(P,f) < \varepsilon \qquad \text{...(1)}$$

Let m_r, M_r and m_r', M_r' be the bounds of $|f|$ and f respectively in $[x_{r-1}, x_r]$. Then for all $x, y \in [x_{r-1}, x_r]$, we have

$$\Big|\,|f(y)| - |f(x)|\,\Big| \le |f(y) - f(x)| \qquad \text{...(2)}$$

Taking suprema of both sides of (2), we have by the lemma mentioned above

$$(M_r - m_r) \le (M_r' - m_r')$$

$$\Rightarrow \quad \sum_{r=1}^{n} (M_r - m_r)\,\Delta x_r \le \sum_{r=1}^{n} (M_r' - m_r')\,\Delta x_r$$

$$\Rightarrow \quad U(P,|f|) - L(P,|f|) \le U(P,f) - L(P,f) < \varepsilon, \text{ by (1).}$$

Hence $|f| \in \mathbf{R}[a,b]$.

Now, let $\quad f_1(x) = \frac{1}{2}\{|f(x)| + f(x)\}$

and $\quad f_2(x) = \frac{1}{2}\{|f(x)| - f(x)\}$.

Then $\quad |f(x)| = f_1(x) + f_2(x);\ f(x) = f_1(x) - f_2(x)$.

Since $f_1(x) \ge 0$ and $f_2(x) \ge 0\ \ \forall\ x \in [a,b]$, we have

$$\int_a^b f_1(x)\,dx \ge 0 \text{ and } \int_a^b f_2(x)\,dx \ge 0.$$

Hence, we get

$$\begin{aligned}
\left|\int_a^b f(x)\,dx\right| &= \left|\int_a^b f_1(x)\,dx - \int_a^b f_2(x)\,dx\right| \\
&\le \left|\int_a^b f_1(x)\,dx\right| + \left|\int_a^b f_2(x)\,dx\right| \\
&= \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx \\
&= \int_a^b [f_1(x) + f_2(x)]\,dx = \int_a^b |f(x)|\,dx.
\end{aligned}$$

Thus $\quad \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$

Note: The converse of the above theorem need not be true. Consider the function f defined on $[0,1]$ by

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational.} \end{cases}$$

We have seen earlier that $f \notin \boldsymbol{R}[0,1]$.

But $|f| \in \boldsymbol{R}[0,1]$. For, we have $|f(x)| = 1 \ \forall\ x \in [0,1]$ and we know that every constant function is R-integrable.

9 Fundamental Theorem of Integral Calculus

(Meerut 2012)

In this section we shall establish the close relationship between the derivative and the integral in a rigorous manner. In fact, we shall prove that integration and differentiation are, in a certain sense, inverse operations.

Integral function: Let $f \in \boldsymbol{R}[a,b]$. We define a new real valued function F with domain $[a,b]$ by setting

$$F(x) = \int_a^x f(t)\, dt, a < x \le b, F(a) = 0.$$

The function F is called an **integral function** or an **indefinite integral** of the integrable function f. The function F is well defined on $[a,b]$, as $f \in \boldsymbol{R}[a,x]$ where $a < x \le b$ and the condition $F(a) = 0$ is consistent with our previous definition that $\int_a^a f = 0$.

Primitive: Definition: *A differentiable function ϕ defined on $[a,b]$ such that its derivative ϕ' equals a given function f on $[a,b]$ is called a* **primitive** *or* **anti-derivative** *of f on $[a,b]$.*

If ϕ is a primitive of f then $\phi + c$ will also be a primitive of f where c is any constant. Hence the primitive of a function is not unique.

Theorem 1: *Let $f \in \boldsymbol{R}[a,b]$. Then the function F defined on $[a,b]$ by*

$$F(x) = \int_a^x f(t)\, dt$$

is continuous on $[a,b]$.

Proof: Since $f \in \boldsymbol{R}[a,b]$, f is bounded on $[a,b]$. It follows that there exists a positive number M such that

$$|f(t)| \le M \ \forall\ t \in [a,b].$$

Let $a \le x < y \le b$. Then we have

$$|F(y) - F(x)| = \left| \int_a^y f(t)\, dt - \int_a^x f(t)\, dt \right|$$

$$= \left| \int_a^y f(t)\, dt + \int_x^a f(t)\, dt \right| \qquad \left[\because \int_a^b f = -\int_b^a f \right]$$

$$= \left| \int_x^y f(t)\, dt \right| \qquad \text{[by theorem 3 of article 8]}$$

$$\le M\, |y - x| \qquad \text{[by Cor. 3 of theorem 1 of article 8]}$$

$$= M(y - x). \qquad \dots(1)$$

Let $\varepsilon > 0$ be given. Then, if $|y - x| < \varepsilon / M$, we conclude from (1) that

$$|F(y) - F(x)| < \varepsilon.$$

Thus given $\varepsilon > 0$, there exists $\delta\, (= \varepsilon / M) > 0$ such that

$$|F(y) - F(x)| < \varepsilon \text{ whenever } |y - x| < \delta \; \forall \; x, y \in [a, b].$$

Consequently the function F is uniformly continuous on $[a, b]$ and hence continuous on $[a, b]$.

Note: The above theorem can also be stated as follows :

The integral of an integrable function is continuous.

Theorem 2: *Let f be continuous on $[a, b]$ and let*

$$F(x) = \int_a^x f(t)\, dt \; \forall \; x \in [a, b].$$

Then $F'(x) = f(x) \;\; \forall \; x \in [a, b].$ **(Gorakhpur 2015)**

Proof: Let $x \in [a, b]$ be fixed. Choose $h \neq 0$ such that

$$x + h \in [a, b].$$

Then, we have

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt \\ &= \int_a^{x+h} f(t)\, dt + \int_x^a f(t)\, dt \\ &= \int_x^{x+h} f(t)\, dt. \end{aligned}$$

Since f is continuous on $[a, b]$ there exists a number c in the interval $[x, x+h]$, such that

$$\int_x^{x+h} f(t)\, dt = h\, f(c). \qquad \text{[By Cor. 2 of theorem 1, article 8]}$$

We see that if $h \to 0$, then $c \to x$.

Thus $$F(x+h) - F(x) = h\, f(c)$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} f(c)$$

$$= \lim_{c \to x} f(c) = f(x), \text{ as } f \text{ is continuous.}$$

Hence, we get $F'(x) = f(x) \;\; \forall \; x \in [a, b]$.

Note: The following theorem is an improvement on the above theorem for only the R-integrability of f and the continuity of f at the point x_0 is assumed.

Theorem 3: *Let $f \in \mathbf{R}[a, b]$ and let f be continuous at $x_0 \in [a, b]$.*

If $F(x) = \int_a^x f(t)\, dt, a \le x \le b,$ *then*

$$F'(x_0) = f(x_0).$$

Proof: Since f is continuous at x_0, for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x_0 + h) - f(x_0)| < \varepsilon \qquad \text{...(1)}$$

provided $|h| < \delta$ and $a \le x_0 + h \le b$.

We have $$F(x_0 + h) - F(x_0) = \int_a^{x_0 + h} f(t)\, dt - \int_a^{x_0} f(t)\, dt$$

$$= \int_a^{x_0 + h} f(t)\, dt$$

and $\int_{x_0}^{x_0+h} f(x_0)\,dt = f(x_0).[(x_0+h)-x_0] = h\,f(x_0).$

Now $\left|\dfrac{F(x_0+h)-F(x_0)}{h} - f(x_0)\right|$

$$= \left|\frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,dt - \frac{1}{h}\int_{x_0}^{x_0+h} f(x_0)\,dt\right|$$

$$= \left|\frac{1}{h}\int_{x_0}^{x_0+h} [f(t)-f(x_0)]\,dt\right|. \qquad ...(2)$$

By virtue of (1) and (2), we get on using Cor. 3 of theorem 1 of article 8

$$\left|\frac{F(x_0+h)-F(x_0)}{h} - f(x_0)\right| < \frac{1}{|h|}\cdot|h|\,\varepsilon = \varepsilon \text{ for all } h \text{ with } |h| < \delta.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\lim_{h\to 0} \frac{F(x_0+h)-F(x_0)}{h} = f(x_0)$$

i.e., $F'(x_0) = f(x_0)$.

Theorem 4: (Fundamental theorem of Integral Calculus): *Let f be a continuous function on $[a, b]$ and let ϕ be a differentiable function on $[a, b]$ such that $\phi'(x) = f(x)$ for all $x \in [a, b]$. Then*

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a).$$

Proof: Let $F(x) = \int_a^x f(t)\,dt$. Then by theorem 2 of article 9, we have

$$F'(x) = f(x) \;\forall\; x \in [a, b]. \qquad ...(1)$$

By hypothesis, $\phi'(x) = f(x) \;\forall\; x \in [a, b]$. ...(2)

From (1) and (2), we have $\forall\; x \in [a, b]$,

$F'(x) = \phi'(x)$ or $F'(x) - \phi'(x) = 0$

$\Rightarrow$ $(F-\phi)'(x) = 0 \Rightarrow (F-\phi)(x) = c$ for some $c \in \mathbf{R}$

$\Rightarrow$ $F(x) - \phi(x) = c$.

Thus $F(x) = \phi(x) + c \;\forall\; x \in [a, b]$.

Now $F(b) - F(a) = [\phi(b)+c] - [\phi(a)+c] = \phi(b) - \phi(a)$. ...(3)

Also $F(a) = \int_a^a f(t)\,dt = 0$ and $F(b) = \int_a^b f(t)\,dt$.

Putting these values in (3), we get

$$\int_a^b f(t)\,dt = \phi(b) - \phi(a)$$

or $$\int_a^b f(x)\,dx = \phi(b) - \phi(a).$$

Note: The following theorem is an improvement on the above theorem since only the R-integrability of f is assumed. In fact theorem 4 is a corollary of theorem 5.

Theorem 5: (Fundamental theorem of Integral Calculus): *Let $f \in \mathbf{R}[a, b]$ and let ϕ be a differentiable function on $[a, b]$ such that $\phi'(x) = f(x)$ for all x $[a, b]$. Then $\int_a^b f(x)\, dx = \phi(b) - \phi(a)$.*

(Garhwal 2006, 07, 09, 11; Rohilkhand 11, 12)

Proof: Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a, b]$. Now ϕ is differentiable on $[a, b]$ implies that ϕ is differentiable on each subinterval $[x_{r-1}, x_r]$.

Hence by the mean value theorem of differential calculus, we find that there exists ξ_r in $[x_{r-1}, x_r], r = 1, 2, \ldots, n$, such that

$$\phi(x_r) - \phi(x_{r-1}) = (x_r - x_{r-1})\, \phi'(\xi_r) = (\phi)'(\xi_r) \,.\, \Delta x_r$$

or $$\phi(x_r) - \phi(x_{r-1}) = f(\xi_r) \,.\, \Delta x_r \qquad [\because \phi'(\xi_r) = f(\xi_r)]$$

or $$\sum_{r=1}^{n} [\phi(x_r) - \phi(x_{r-1})] = \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r\,. \qquad \ldots(1)$$

Now $$\sum_{r=1}^{n} [\phi(x_r) - \phi(x_{r-1})] = \phi(x_1) - \phi(x_0) + \phi(x_2) - \phi(x_1) + \ldots$$
$$\ldots + \phi(x_n) - \phi(x_{n-1})$$
$$= \phi(x_n) - \phi(x_0) = \phi(b) - \phi(a).$$

It follows from (1) that

$$\phi(b) - \phi(a) = \sum_{r=1}^{n} f(\xi_r)\, \Delta x_r \qquad \ldots(2)$$

Taking limit as $|| P || \to 0$,

we get $$\phi(b) - \phi(a) = \int_a^b f(x)\, dx,$$

since $\sum_{r=1}^{n} f(\xi_r)\, \Delta x_r$ tends to $\int_a^b f(x)\, dx$ as $|| P || \to 0$.

The result of the above theorem is usually written in the form

$$\int_a^b \phi'(x)\, dx = \phi(b) - \phi(a).$$

Note: Some authors call theorem 2 or theorem 3 as the **first fundamental theorem** and the theorem 4 or theorem 5 as the **second fundamental theorem** of integral calculus.

10 Mean Value Theorems of Integral Calculus

Theorem 1: (First Mean Value Theorem): *Let $f \in \mathbf{R}[a, b]$. Then there exists a number μ lying between the bounds m and M of f such that*

$$\int_a^b f(x)\, dx = \mu\, (b - a).$$

Moreover if f is continuous, then

$$\int_a^b f(x)\, dx = (b - a)\, f(c), a \le c \le b.$$

(Purvanchal 2012)

Proof: See Cor. 1 and Cor. 2 of theorem 1 of article 8.

Theorem 2: **(Second Mean Value Theorem):** *Let*

$f \in \boldsymbol{R}[a,b]$ *and* $g \in \boldsymbol{R}[a,b]$,

and $g(x) \geq 0$ *or* $\leq 0 \;\forall\; x \in [a,b]$.

Then there exists a number μ *with* $m \leq \mu \leq M$ *such that*

$$\int_a^b f(x)\, g(x)\, dx = \mu \int_a^b g(x)\, dx$$

where m, M are the bounds of f on $[a, b]$.

Proof: First let $g(x) \geq 0 \;\forall\; x \in [a,b]$. **(Purvanchal 2012)**

Then $mg(x) \leq f(x)\, g(x) \leq Mg(x) \;\forall\; x \in [a,b]$.

If follows from Cor. 5 of theorem 1 of article 8 that

$$m \int_a^b g(x)\, dx \leq \int_a^b f(x)\, g(x)\, dx \leq M \int_a^b g(x)\, dx$$

or $$m \int_a^b g(x)\, dx \geq \int_a^b f(x)\, g(x)\, dx \geq M \int_a^b g(x)\, dx$$

according as $a \leq b$ or $a \geq b$.

Hence there exists a number μ with $m \leq \mu \leq M$ such that

$$\int_a^b f(x)\, g(x)\, dx = \mu \int_a^b g(x)\, dx.$$

Now let $g(x) \leq 0 \;\forall\; x \in [a,b]$.

Then $-g(x) \geq 0 \;\forall\; x \in [a,b]$.

Hence by the above result for some $\mu \in [m, M]$, we have

$$\int_a^b f(x)\, [-g(x)]\, dx = \mu \int_a^b [-g(x)]\, dx$$

or $$\int_a^b f(x)\, g(x)\, dx = \mu \int_a^b g(x)\, dx.$$

Corollary: *If, in addition to the conditions of the theorem, f is continuous as well, then there exists* $\xi \in [a,b]$ *such that*

$$\int_a^b f(x)\, g(x)\, dx = f(\xi) \int_a^b g(x)\, dx.$$

Proof: Since *f* is continuous on $[a, b]$, it takes every value between m and M. Hence by the above theorem there exists a point $\xi \in [a,b]$ such that

$$\int_a^b f(x)\, g(x)\, dx = f(\xi) \int_a^b g(x)\, dx.$$

Theorem 3: **(Bonnet's Mean Value Theorem):** *Let* $g \in \boldsymbol{R}[a,b]$ *and let f be monotonic and non-negative on* $[a,b]$. *Then for some* ξ *or* $\eta \in [a,b]$

$$\int_a^b f(x)\, g(x)\, dx = f(a) \int_a^{\xi} g(x)\, dx$$

or $$\int_a^b f(x)\, g(x)\, dx = f(b) \int_{\eta}^b g(x)\, dx$$

according as f is monotonically non-increasing or non-decreasing on $[a,b]$.

Proof: If $a = b$, the result is trivial. Let $b > a$ and let f be non-negative and monotonically non-increasing on $[a, b]$.

Let $\quad P = \{a = x_0, x_1, \ldots\ldots, x_n = b\}$

be any partition of $[a, b]$. Let m_r, M_r be the bounds of g on $[x_{r-1}, x_r]$ and ξ_r any point on $[x_{r-1}, x_r]$. Then

$$m_r \Delta x_r \le g(\xi_r)\,\Delta x_r \le M_r \Delta x_r$$

and
$$m_r \Delta x_r \le \int_{x_{r-1}}^{x_r} g \le M_r \Delta x_r .$$

[Theorem 1 of article 8]

On summing for each $r = 1, 2, \ldots, p \le n$, we get

$$\sum_1^p m_r\,\Delta x_r \le \sum_1^p g(\xi_r)\,\Delta x_r \le \sum_1^p M_r\,\Delta x_r \qquad \ldots(1)$$

and
$$\sum_1^p m_r\,\Delta x_r \le \int_a^{x_p} g \le \sum_1^p M_r \Delta x_r . \qquad \ldots(2)$$

Then (1) and (2) give

$$\left| \int_a^{x_p} g - \sum_1^p g(\xi_r)\,\Delta x_r \right| \le \sum_1^p (M_r - m_r)\,\Delta x_r$$

$$\le \sum_1^n (M_r - m_r)\,\Delta x_r = U(P, g) - L(P, g)$$

$$= \omega(P, g). \qquad \ldots(3)$$

[Note that if $b < a$, the inequalities (1) and (2) are reversed but (3) remains the same].

Now by theorem 1 of article 9, $\int_a^x g$ is continuous on $[a, b]$ and hence is bounded on $[a, b]$. Let m, M be its bounds on $[a, b]$. Then (3) gives

$$m - \omega(P, f) \le \sum_1^p g(\xi_r)\,\Delta x_r \le M + \omega(P, f).$$

Using Abel's lemma*, we get

$$f(a)\,[m - \omega(P, f)] \le \sum_1^p f(\xi_r)\, g(\xi_r)\,\Delta x_r$$

$$\le f(a)\,[M - \omega(P, f)]. \qquad \ldots(4)$$

Since f is monotonic, we have $f \in \boldsymbol{R}[a, b]$.

***Abel's Lemma.** If $v_1 \ge v_2 \ge \ldots \ge v_n \ge 0$ and numbers $u_1, u_2, \ldots, u_n$ and k_1, k_2 are such that

$$k_1 \le \sum_{r=1}^n u_r \le k_2 \text{ , then } k_1 v_1 \le \sum_{r=1}^n u_r\, v_r \le k_2 v_1 .$$

In our case, $\quad u_r = g(\xi_r)\,\Delta x_r$ and $v_r = f(\xi_r)$.

Also $\quad g \in \boldsymbol{R}[a, b]$.

Hence $\quad fg \in \boldsymbol{R}[a, b]$.

Now $\quad f \in \boldsymbol{R}[a, b] \Rightarrow \omega(P, f) \to 0$ as $\|P\| \to 0$.

Hence (4) gives $f(a)\, m \le \int_a^b fg \le f(a)\, M$.

Thus $\quad \int_a^b fg = f(a)\, \mu$

where μ lies between the bounds m, M of the continuous function

$$\int_a^x g \text{ on } [a, b].$$

Hence $\int_a^x g$ must take the value μ at some point $\xi \in [a, b]$

so that $\quad \mu = \int_a^\xi g(x)\, dx.$

Therefore $\quad \int_a^b f(x)\, g(x)\, dx = f(a) \int_a^\xi g(x)\, dx.$

If f is monotonically non-decreasing on $[a, b]$, then it is monotonically non-increasing on $[b, a]$ and so as above, we get

$$\int_a^b fg = f(b) \int_a^\eta g \text{ for some } \eta \in [a, b]$$

or $\quad \int_a^b fg = f(b) \int_\eta^b g$

i.e., $\quad \int_a^b f(x)\, g(x)\, dx = f(b) \int_\eta^b g(x)\, dx.$

Theorem 4: Weierstrass's (Second) Mean Value Theorem.

Let $g \in \boldsymbol{R}[a, b]$ and let f be bounded and monotonic on $[a, b]$.

Then $\quad \int_a^b fg = f(a) \int_a^\xi g + f(b) \int_\xi^b g.$

Proof: We assume that f is monotonically non-increasing on $[a, b]$. Then $f(x) - f(b)$ is monotonically non-increasing and non-negative on $[a, b]$. Hence by the Bonnet's theorem (theorem 3 of article 10), there exists some $\xi \in [a, b]$ such that

$$\int_a^b [f(x) - f(b)]\, g(x)\, dx = [f(a) - f(b)] \int_a^\xi g(x)\, dx$$

or

$$\int_a^b f(x)\, g(x)\, dx - f(b) \int_a^b g(x)\, dx = f(a) \int_a^\xi g(x)\, dx - f(b) \int_a^\xi g(x)\, dx$$

or

$$\int_a^b f(x)\, g(x)\, dx = f(a) \int_a^\xi g(x)\, dx + f(b) \left[\int_a^b g(x)\, dx - \int_a^\xi g(x)\, dx\right]$$

$$= f(a) \int_a^\xi g(x)\, dx + f(b) \int_\xi^b g(x)\, dx.$$

Now let f be monotonically non-decreasing on $[a, b]$. Then $-f$ is monotonically non-increasing on $[a, b]$. Hence by the above result, we have

$$\int_a^b [-f(x)]\, g(x)\, dx = [-f(a)] \int_a^{\xi} g(x)\, dx + [-f(b)] \int_{\xi}^{b} g(x)\, dx$$

or $$\int_a^b f(x)\, g(x)\, dx = f(a) \int_a^{\xi} g(x)\, dx + f(b) \int_{\xi}^{b} g(x)\, dx \text{ as before.}$$

Remark: Note that the condition that f is monotone on $[a, b]$ cannot be dropped in the above theorem. Consider, for example, the functions $f(x) = \cos x$ and $g(x) = x^2$ and the interval $[-\pi/2, \pi/2]$. In this case

$$f\left(-\frac{\pi}{2}\right) \int_{-\pi/2}^{\xi} g(x)\, dx + f\left(\frac{\pi}{2}\right) \int_{\xi}^{\pi/2} g(x)\, dx$$

$$= \cos\left(-\frac{\pi}{2}\right) \int_{-\pi/2}^{\xi} x^2\, dx + \cos\frac{\pi}{2} \int_{\xi}^{\pi 2} x^2\, dx = 0.$$

But $$\int_{-\pi/2}^{\pi/2} x^2 \cos x\, dx > 0.$$

Thus the theorem does not hold.

Illustrative Examples

Example 16: *Compute* $\int_1^2 x^3\, dx$.

Solution: Let $f(x) = x^3, 1 \le x \le 2$. Then f is continuous on $[1, 2]$. Moreover, if $\phi(x) = x^4/4\ (1 \le x \le 2)$, then $\phi'(x) = x^3 = f(x), (1 \le x \le 2)$.

Hence by the fundamental theorem of integral calculus, we have

$$\int_1^2 x^3\, dx = \phi(2) - \phi(1) = \frac{2^4}{4} - \frac{1^4}{4} = \frac{15}{4}.$$

Example 17: *(i) Taking $f(x) = x$, $g(x) = e^x$, verify the second mean value theorem in $[-1, 1]$. [Theorem 2 of article 10].*
(ii) Also verify Bonnet's mean value theorem in $[-1, 1]$ for the functions

$$f(x) = e^x \quad \text{and} \quad g(x) = x.$$

Solution: (i) Since f and g are continuous on $[-1, 1]$, we have

$$f, g \in \boldsymbol{R}[-1, 1].$$

Also $g(x) > 0$ for all $x \in [-1, 1]$. Hence the conditions of theorem 2 of article 10 are satisfied. Now

$$\int_{-1}^{1} f(x)\, g(x)\, dx = \int_{-1}^{1} xe^x\, dx = [xe^x - e^x]_{-1}^{1} = \frac{2}{e}. \qquad \text{...(1)}$$

and $$\int_{-1}^{1} g(x)\, dx = \int_{-1}^{1} e^x\, dx = [e^x]_{-1}^{1} = e - e^{-1} = \frac{e^2 - 1}{e}.$$

Since f is continuous on $[-1, 1]$, it takes every value between $f(-1) = -1$ and $f(1) = 1$. Let $\mu = 2/(e^2 - 1)$. Since $e > 2$, we have $e^2 > 4 \Rightarrow e^2 - 1 > 3$ so that $0 < \mu < 1$.

It follows that there is a point ξ in $[-1, 1]$ such that

$$f(\xi) = 2/(e^2 - 1).$$

Accordingly, we have

$$f(\xi) \int_{-1}^{1} g(x)\, dx = \frac{2}{e^2 - 1} \cdot \frac{e^2 - 1}{e} = \frac{2}{e}. \qquad \ldots(2)$$

From (1) and (2), we have

$$\int_{-1}^{1} f(x)\, g(x)\, dx = f(\xi) \int_{-1}^{1} g(x)\, dx.$$

Thus the second mean value theorem is verified.

(ii) Since $g(x) = x$ is continuous on $[-1, 1]$, we have

$$g \in \boldsymbol{R}[-1, 1].$$

Also $f(x) = e^x$ is monotonically non-decreasing and positive on $[-1, 1]$. Hence all the conditions of the Bonnet's mean value theorem are satisfied. As in (i), we have

$$\int_{-1}^{1} f(x)\, g(x)\, dx = \int_{-1}^{1} e^{x}\, x\, dx = \frac{2}{e}.$$

Now $$\int_{\eta}^{1} g(x)\, dx = \int_{\eta}^{1} x\, dx = \frac{1}{2}[1 - \eta^2].$$

$\therefore$ $$f(1) \int_{\eta}^{1} g(x)\, dx = \frac{e}{2}(1 - \eta^2).$$

We choose η such that

$$\frac{2}{e} = \frac{e}{2}(1 - \eta^2) \quad i.e., \quad \eta^2 = \frac{e^2 - 4}{e^2}.$$

Also it is easy to see that $0 < \eta < 1$, where $\eta = \dfrac{\sqrt{(e^2 - 4)}}{e}$.

For this value of η, we then have

$$\int_{-1}^{1} f(x)\, g(x)\, dx = f(1) \int_{\eta}^{1} g(x)\, dx.$$

Hence Bonnet's mean value theorem is verified.

Example 18: *Show that the Bonnet's mean value theorem does not hold on* $[-1, 1]$ *for* $f(x) = g(x) = x^2$.

Solution: The function $f(x) = x^2$ is not monotonic on $[-1, 1]$ since for the interval $[-1, 0]$ it is non-increasing and for $[0, 1]$ it is non-decreasing. Thus the conditions of the Bonnet's mean value theorem are not satisfied and hence the theorem does not hold in $[-1, 1]$.

Example 19: *Prove Bonnet's form of the second mean value theorem that if f' is continuous and of constant sign, and $f(b)$ has the same sign as $f(a) - f(b)$, then*

$$\int_a^b f(x)\, g(x)\, dx = f(a) \int_a^{\xi} g(x)\, dx$$

where ξ lies between a and b. Show that if $q > p > 0$, then

$$\left| \int_p^q \frac{\sin x}{x}\, dx \right| < \frac{2}{p}.$$

Solution: Let $\int_a^x g(t)\, dt = F(x)$

so that $F'(x) = g(x)$.

Then
$$\int_a^b f(x)\, g(x)\, dx = \int_a^b f(x)\, F'(x)\, dx$$

$$= [f(x)\, F(x)]_a^b - \int_a^b f'(x)\, F(x)\, dx$$

[Integrating by parts]

$$= f(b)\, F(b) - \int_a^b f'(x)\, F(x)\, dx \qquad [\because F(a) = 0]$$

$$= f(b)\, F(b) - F(x_0) \int_a^b f'(x)\, dx \quad [a \le x_0 \le b]$$

[by Cor. of theorem article 10]

$$= f(b)\, F(b) - F(x_0)\, [f(b) - f(a)] \qquad \ldots(1)$$

$= f(a).\mu$ where μ lies between $F(x_0)$ and $F(b)$.

Since F is continuous, there exists a point ξ between x_0 and b such that

$$\mu = F(\xi) = \int_a^{\xi} f(x)\, dx.$$

It follows that $\int_a^b f(x)\, g(x)\, dx = f(a) \int_a^{\xi} g(x)\, dx.$

Writing $f(x) = 1/x$, $g(x) = \sin x$, we find

$$\int_p^q \frac{\sin x}{x}\, dx = \frac{1}{p} \int_p^{\xi} \sin x\, dx, \ (p < \xi < q)$$

$$= \frac{1}{p} [-\cos x]_p^{\xi} = \frac{1}{p} [\cos p - \cos \xi].$$

Hence
$$\left| \int_p^q \frac{\sin x}{x}\, dx \right| = \left| \frac{1}{p} (\cos p - \cos \xi) \right| < \frac{2}{p}.$$

Remark: From (1), we get

$$\int_a^b f(x)\, g(x)\, dx = f(a)\, F(x_0) + f(b)\, [F(b) - F(x_0)]$$

$$= f(a) \int_a^{x_0} g(x)\, dx + f(b) \left[\int_a^b g(x)\, dx - \int_a^{x_0} g(x)\, dx \right]$$

$$= f(a) \int_a^{x_0} g(x)\, dx + f(b) \int_{x_0}^b g(x)\, dx$$

which is the theorem 4 of article 10.

Comprehensive Exercise 1

1. (i) Let $f(x) = x\ (0 \le x \le 1)$. Let P be the partition $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$ of $[0, 1]$. Compute $U(P, f)$ and $L(P, f)$.

   (ii) Let $f(x) = x\ (0 \le x \le 3)$. Let P be the partition $\{0, 1, 2, 3\}$ of $[0, 3]$. Compute $U(P, f)$ and $L(P, f)$.

2. Show by definition that $\int_0^1 x^4\, dx = \frac{1}{5}$.

3. Let $f(x) = x^{-1/2}$ on $[1, 4]$. Consider the partition obtained by dividing $[1, 4]$ into n equal parts and hence show that $\int_1^4 x^{-1/2}\, dx = 2$.

4. If $f(x) = \cos x\ \forall\ x \in [0, \pi/2]$, show that f is integrable on $[0, \pi/2]$ and $\int_0^{\pi/2} \cos x\, dx = 1$.

5. Show that $f(x) = 3x + 1$ is integrable on $[1, 2]$ and $\int_1^2 (3x + 1)\, dx = \frac{11}{2}$.

6. Give an example to prove that a bounded function need not be R-integrable.

7. Give an example of a discontinuous function which is R-integrable on $[0, 1]$.

8. Let f be defined on $[0, 1]$ by $f(x) = \frac{n}{n+1}$, when

$$\frac{1}{n+1} < x \le \frac{1}{n}, n = 1, 2, 3, \ldots$$

and $\quad f(x) = 1, x = 0$.

Then, show that f is Riemann integrable on $[0, 1]$ and $\int_0^1 f(x)\, dx = \frac{\pi^2}{6} - 1$.

9. A function $f(x)$ is defined on $[0, 1]$ as follows :

$$f(x) = \frac{n}{n+2}, \text{ when } \frac{1}{n+1} \le x \le \frac{1}{n}\ (n = 1, 2, 3, \ldots)$$

and $\quad f(x) = 1$, when $x = 0$.

Show that $f(x)$ is R-integrable on $[0, 1]$ and $\int_0^1 f(x)\, dx = \frac{1}{2}$. **(Garhwal 2008)**

10. Let a function $f(x)$ be defined on $[0, 1]$ as follows :

$$f(x) = \frac{1}{a^{r-1}}, \text{ when } \frac{1}{a^r} < x < \frac{1}{a^{r-1}},$$

for $r = 1, 2, 3, \ldots$ where a is an integer greater than 1. Show that $\int_0^1 f(x)\, dx$ exists and is equal to $\frac{a}{a+1}$.

11. Calculate the values of upper and lower integrals for the function f defined on $[0, 2]$ as follows :

$$f(x) = x^2 \quad \text{when } x \text{ is rational}$$

and $$f(x) = x^3 \quad \text{when } x \text{ is irrational.}$$

12. Let $f(x)$ be a function bounded on $[a, b]$ and let P_1 and P_2 be two partitions of $[a, b]$ such that $P_1 \subset P_2$. Then prove that

$$U(P_1, f) - L(P_1, f) \geq U(P_2, f) - L(P_2, f).$$

Answers 1

1. (i) $U(P, f) = \frac{2}{3}$, $L(P, f) = \frac{1}{3}$

(ii) $U(P, f) = 6$, $L(P, f) = 3$

11. Upper integral $= \frac{49}{12}$, lower integral $= \frac{31}{12}$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Let f be a bounded function defined on $[a, b]$ and let P be a partition of $[a, b]$. If P^* is a refinement of P, then

(a) $L(P^*, f) \leq L(P, f)$

(b) $U(P^*, f) \leq U(P, f)$

(c) $U(P^*, f) \geq U(P, f)$

(d) $L(P^*, f) = U(P, f)$.

2. Let f, g be bounded functions defined on $[a, b]$ and let P be any partition of $[a, b]$. Then

(a) $U(P, f + g) \leq U(P, f) + U(P, g)$

(b) $U(P, f + g) \geq U(P, f) + U(P, g)$

(c) $U(P, f + g) = U(P, f) + U(P, g)$

(d) $L(P, f + g) \leq L(P, f) + L(P, g)$.

3. Let f be a bounded function defined on the bounded interval $[a, b]$. Then f is Riemann integrable on $[a, b]$ if and only if

(a) $\underline{\int_a^b} f \leq \overline{\int_a^b} f$

(b) $\underline{\int_a^b} f \geq \overline{\int_a^b} f$

(c) $\underline{\int_a^b} f = \overline{\int_a^b} f$

(d) $\overline{\int_a^b} f + \underline{\int_a^b} f = 0$.

4. If f is Riemann integrable on $[a, b]$, then

(a) $\left|\int_a^b f(x)\,dx\right| = \int_a^b |f(x)|\,dx$

(b) $\left|\int_a^b f(x)\,dx\right| \geq \int_a^b |f(x)|\,dx$

(c) $\left|\int_a^b f(x)\,dx\right| = -\int_a^b |f(x)|\,dx$

(d) $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. Let $I = [a, b]$ be a closed and bounded interval. Then by a partition of I we mean a set of real numbers $P = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ having the property that ....
2. Let P and P^* be two partitions of a closed and bounded interval $[a, b]$. Then P^* is called a refinement of P if ....
3. Let P_1 and P_2 be two partitions of a closed and bounded interval $[a, b]$. If $P^* = P_1 \cup P_2$, then P^* is called the ...... of P_1 and P_2.
4. Let f be a bounded function defined on a bounded interval $[a, b]$. If corresponding to any partition P of $[a, b]$, $L(P, f)$ is the lower Riemann sum of f on $[a, b]$ and $U(P, f)$ is the upper Riemann sum of f on $[a, b]$, then $L(P, f) \ldots\ldots U(P, f)$.
5. Let f be a bounded function defined on $[a, b]$ and let P be a partition of $[a, b]$. If P^* is a refinement of P, then $L(P^*, f) \ldots\ldots L(P, f)$.
6. If P_1 and P_2 be any two partitions of $[a, b]$, then $U(P_1, f) \ldots\ldots L(P_2, f)$.
7. Let f be a real valued bounded function defined on $[a, b]$. The lower Riemann integral of f over $[a, b]$ is the ...... of $L(P, f)$ over all partitions $P \in \boldsymbol{P}[a, b]$.
8. Let f be a real valued bounded function defined on $[a, b]$. The upper Riemann integral of f over $[a, b]$ is the ...... of $U(P, f)$ over all partitions $P \in \boldsymbol{P}[a, b]$.
9. Let f be a real valued bounded function defined on $[a, b]$. Then the lower Riemann integral of f over $[a, b]$ cannot ...... the upper Riemann integral of f over $[a, b]$.
10. Let f be a bounded function defined on the bounded interval $[a, b]$. Then f is called Riemann integrable on $[a, b]$ if $\underline{\int_a^b} f = \ldots\ldots$
11. A necessary and sufficient condition for Riemann integrability of a bounded function $f : [a, b] \to \mathbf{R}$ over $[a, b]$ is that for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that for P and all its refinements
$$0 \leq U(P, f) - L(P, f) < \ldots\ldots$$
12. Let f be Riemann integrable on $[a, b]$ and let ϕ be a differentiable function on $[a, b]$ such that $\phi'(x) = f(x)$ for all $x \in [a, b]$. Then $\int_a^b f(x)\,dx = \ldots\ldots$

True or False

Write 'T' for true and 'F' for false statement.

1. Every continuous function defined on $[a, b]$ is Riemann integrable on $[a, b]$.
2. Every bounded function f defined on $[a, b]$ is Riemann integrable on $[a, b]$.
3. If a function f is discontinuous on $[a, b]$, then f cannot be Riemann integrable on $[a, b]$.
4. If a function f is monotonic on $[a, b]$, then f is Riemann integrable on $[a, b]$.
5. If f is Riemann integrable on $[a, b]$, then $|f|$ may or may not be Riemann integrable on $[a, b]$.
6. If a function f is Riemann integrable on $[a, b]$, then the function F defined on $[a, b]$ by

$$F(x) = \int_a^x f(t)\, dt$$

is uniformly continuous on $[a, b]$.

7. Let f be continuous on $[a, b]$ and let $F(x) = \int_a^x f(t)\, dt \ \forall\ x \in [a, b]$.

Then $\quad F'(x) = f(x) \ \forall\ x \in [a, b]$.

8. Let f be a continuous function on $[a, b]$ and let ϕ be a differentiable function on $[a, b]$ such that $\phi'(x) = f(x)$ for all $x \in [a, b]$. Then

$$\int_a^b f(x)\, dx = \phi(a) - \phi(b).$$

9. Let f be a bounded function defined on $[a, b]$, where $b \geq a$, and let m be the infimum of $f(x)$ in $[a, b]$. Then for any partition P of $[a, b]$, we have

$$m(b - a) \geq L(P, f).$$

10. If $f : [a, b] \to \mathbf{R}$ is a bounded function, then

$$L(P, -f) = -U(P, f).$$

11. Let f be a bounded function defined on $[a, b]$ and let P be a partition of $[a, b]$. If P^* is a refinement of P, then

$$U(P^*, f) \geq U(P, f).$$

12. Let f be a bounded function defined on $[a, b]$. Then

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

13. Let f be a continuous function defined on $[a, b]$. Then

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

14. Let f be a bounded function defined on $[a, b]$. If the set of points of discontinuity of f on $[a, b]$ is finite, then

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

Answers

Multiple Choice Questions

1. (b)	**2.** (a)	**3.** (c)	**4.** (d)

Fill in the Blank(s)

1. $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ **2.** $P^* \supset P$

2. common refinement **4.** $\leq$ **5.** $\geq$

6. $\geq$ **7.** supremum **8.** infimum

9. exceed **10.** $\overline{\int_a^b} f$ **11.** ε

12. $\phi(b) - \phi(a)$

True or False

1. *T*	**2.** *F*	**3.** *F*	**4.** *T*	**5.** *F*
6. *T*	**7.** *T*	**8.** *F*	**9.** *F*	**10.** *T*
11. *F*	**12.** *F*	**13.** *T*	**14.** *T*	

□

Chapter

7

Convergence of Improper Integrals

1 Some Definitions

1. Infinite Interval: The interval whose length (range) is infinite is said to be an *infinite interval*. Thus the intervals (a, ∞), $(-\infty, b)$ and $(-\infty, \infty)$ are infinite intervals.

2. Bounded Functions: A function $f(x)$ is said to be *bounded* over the interval I if there exist two real numbers a and b $(b > a)$ such that

$$a \leq f(x) \leq b \text{ for all } x \in I.$$

A function $f(x)$ is said to be unbounded at a point, if it becomes infinite at that point. Thus the function

$$f(x) = x / \{(x-1)(x-2)\}$$

is unbounded at each of the points $x = 1$ and $x = 2$.

3. Monotonic functions: A real valued function f defined on an interval I is said to be **monotonically** increasing if

$$x > y \Rightarrow f(x) > f(y) \;\forall\; x, y \in I$$

and **monotonically decreasing** if

$$x > y \Rightarrow f(x) < f(y) \;\forall\; x, y \in I.$$

A function f defined on an interval I is said to be a monotonic function if it is either monotonically decreasing or monotonically increasing on I.

For example the function f defined by $f(x) = \sin x$ is monotonically increasing in the interval $0 \le x \le \frac{1}{2}\pi$ and monotonically decreasing in the interval $\frac{1}{2}\pi \le x \le \pi$.

4. **Proper Integral:** The definite integral $\int_a^b f(x)\,dx$ is said to be a *proper integral* if the range of integration is finite and the integrand $f(x)$ is bounded. The integral $\int_0^{\pi/2} \sin x\,dx$ is a proper integral. Also $\int_0^1 \frac{\sin x}{x}\,dx$ is an example of a proper integral because $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

5. **Improper Integrals:** The definite integral $\int_a^b f(x)\,dx$ is said to be an *improper integral* if (*i*) the interval (a, b) is not finite (*i.e.*, is infinite) and the function $f(x)$ is bounded over this interval; or (*ii*) the interval (a, b) is finite and $f(x)$ is not bounded over this interval; or (*iii*) neither the interval (a, b) is finite nor $f(x)$ is bounded over it.

6. **Improper integrals of the first kind or infinite integrals:** A definite integral $\int_a^b f(x)\,dx$ in which the range of integration is infinite (*i.e.*, either $b = \infty$ or $a = -\infty$ or both) and the integrand $f(x)$ is bounded, is called an improper integral of the first kind or an infinite integral. Thus $\int_0^\infty \frac{dx}{1+x^2}$ is an improper integral of the first kind since the upper limit of integration is infinite and the integrand $1/(1+x^2)$ is bounded. Similarly $\int_{-\infty}^0 e^x\,dx$ is an example of an improper integral of the first kind because here the lower limit of integration is infinite. Also $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is an improper integral of the first kind.

In case the **interval (a, b) is infinite and the integrand $f(x)$ is bounded**, we define

(i) $\int_a^\infty f(x)\,dx = \lim_{X \to \infty} \int_a^X f(x)\,dx,$

provided that the limit exists finitely *i.e.*, the limit is equal to a definite real number.

(ii) $\int_{-\infty}^b f(x)\,dx = \lim_{X \to \infty} \int_{-X}^b f(x)\,dx,$

provided that the limit exists finitely.

(iii) $\int_{-\infty}^\infty f(x)\,dx = \lim_{x_1 \to \infty} \int_{-x_1}^c f(x)\,dx + \lim_{x_2 \to \infty} \int_c^{x_2} f(x)\,dx$

provided that both these limits exist finitely.

7. **Improper integrals of the second kind:** A definite integral $\int_a^b f(x)\,dx$ in which the range of integration is finite but the integrand $f(x)$ is unbounded at one or more points of the interval $a \le x \le b$, is called an improper integral of the second kind.

Thus $$\int_0^4 \frac{dx}{(x-2)(x-3)}$$

and $\int_0^1 \frac{1}{x^2}\,dx$ are improper integrals of the second kind.

In the case of the definite integral

$$\int_a^b f(x)\,dx,$$

if the range of integration (a, b) is finite and the integrand **$f(x)$ is unbounded at one or more points of the given interval,** we define the value of the integral as follows :

(i) If $f(x)$ is unbounded at $x = b$ only *i.e.*, if $f(x) \to \infty$ as $x \to b$ only, then we define

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x)\,dx,$$

provided that the limit exists finitely. Here ε is a small positive number.

(ii) If $f(x) \to \infty$ as $x \to a$ only, then we define

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b} f(x)\,dx,$$

provided that the limit exists finitely.

(iii) If $f(x) \to \infty$ as $x \to c$ only, where $a < c < b$, then we define

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)\,dx + \lim_{\varepsilon' \to 0} \int_{c+\varepsilon'}^{b} f(x)\,dx,$$

provided that both these limits exist finitely.

(iv) If $f(x)$ is unbounded at both the points a and b of the interval (a, b) and is bounded at each other point of this interval, we write

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

where $a < c < b$ and the value of the integral exists only if each of the integrals on the right hand side exists.

2 Convergence of Improper Integrals

When the limit of an improper integral as defined above, is a definite finite number, we say that the given integral is **convergent** and the value of the integral is equal to the value of that limit. When the limit is ∞ or $-\infty$, the integral is said to be **divergent** *i.e.*, the value of the integral does not exist.

In case the limit is neither a definite number nor ∞ or $-\infty$, the integral is said to be **oscillatory** and in this case also the value of the integral does not exist *i.e.*, the integral is not convergent. We can define the convergence of the infinite integral $\int_a^\infty f(x)\,dx$ as follows :

Definition: *The integral* $\int_a^\infty f(x)\,dx$ *is said to converge to the value I, if for any arbitrarily chosen positive number* ε, *however small but not zero, there exists a corresponding positive number N such that*

$$\left| \int_a^b f(x)\,dx - I \right| < \varepsilon \text{ for all values of } b \geq N.$$

Similarly we can define the convergence of an integral, when the lower limit is infinite, or when the integrand becomes infinite at the upper or lower limit.

Illustrative Examples

Example 1: *Discuss the convergence of the following integrals by evaluating them*

(*i*) $\int_1^\infty \frac{dx}{\sqrt{x}}$, (*ii*) $\int_1^\infty \frac{dx}{x^{3/2}}$.

Solution: (i) We have

$$\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{x \to \infty} \int_1^x \frac{dx}{\sqrt{x}}, \text{ (By def.)}$$

$$= \lim_{x \to \infty} \int_1^x x^{-1/2}\,dx = \lim_{x \to \infty} \left[\frac{x^{1/2}}{1/2} \right]_1^x$$

$$= \lim_{x \to \infty} [2\sqrt{x} - 2] = \infty.$$

Thus the limit does not exist finitely and therefore the given integral is divergent (*i.e.*, the integral does not exist).

(ii) We have

$$\int_1^\infty \frac{dx}{x^{3/2}} = \lim_{x \to \infty} \int_1^x \frac{dx}{x^{3/2}}, \quad \text{(By def.)}$$

$$= \lim_{x \to \infty} \int_1^x x^{-3/2}\,dx = \lim_{x \to \infty} \left[\frac{x^{-1/2}}{-1/2} \right]_1^x = \lim_{x \to \infty} \left[-\frac{2}{\sqrt{x}} \right]_1^x$$

$$= \lim_{x \to \infty} \left[-\frac{2}{\sqrt{x}} + 2 \right] = 2.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent and its value is 2.

Example 2: *Test the convergence of* $\int_0^\infty e^{-mx}\,dx$, $(m > 0)$.c

Solution: We have $\int_0^\infty e^{-mx}\,dx = \lim_{x \to \infty} \int_0^\infty e^{-mx}\,dx$, (by def.)

$$= \lim_{x \to \infty} \left[\frac{e^{-mx}}{-m} \right]_0^x = \lim_{x \to \infty} \left\{ -\frac{1}{m}(e^{-mx} - 1) \right\}$$

$$= -\frac{1}{m}[0 - 1] = \frac{1}{m}.$$

Thus the limit exists and is unique and finite, therefore the given integral is convergent.

Example 3: *Test the convergence of* $\int_0^\infty \frac{4a\,dx}{x^2+4a^2}$.

Solution: We have $\int_0^\infty \frac{4a\,dx}{x^2+4a^2} = \lim_{x\to\infty} \int_0^x \frac{4a\,dx}{x^2+(2a)^2}$, (By def.)

$$= \lim_{x\to\infty} \left[4a \cdot \frac{1}{2a} \tan^{-1} \frac{x}{2a}\right]_0^x = 2 \lim_{x\to\infty} \left[\tan^{-1} \frac{x}{2a}\right]_0^x$$

$$= 2 \,.\, \lim_{x\to\infty} \left[\tan^{-1} \frac{x}{2a} - 0\right] = 2\,.\,[\tan^{-1} \infty]$$

$$= 2\,.\,\frac{\pi}{2} = \pi.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent.

Example 4: *Test the convergence of* (*i*) $\int_{-\infty}^0 e^x\,dx$; (*ii*) $\int_{-\infty}^0 e^{-x}\,dx$.

Solution: (i) We have $\int_{-\infty}^0 e^x\,dx = \lim_{x\to\infty} \int_{-x}^0 e^x\,dx$, (By def.)

$$= \lim_{x\to\infty} [e^x]_{-x}^0 = \lim_{x\to\infty} [1 - e^{-x}] = [1-0] = 1.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent.

(ii) We have $\int_{-\infty}^0 e^{-x}\,dx = \lim_{x\to\infty} \int_{-x}^0 e^{-x}\,dx$, (By def.)

$$= \lim_{x\to\infty} \left[\frac{e^{-x}}{-1}\right]_{-x}^0 = -\lim_{x\to\infty} [e^0 - e^x] = \infty.$$

Thus the limit does not exist finitely and therefore the given integral is divergent(*i.e.*, the integral does not exist).

Example 5: *Test the convergence of* $\int_{-\infty}^\infty \frac{dx}{1+x^2}$.

(Kanpur 2008; Gorakhpur 11)

Solution: We have

$$\int_{-\infty}^\infty \frac{dx}{1+x^2} = \int_{-\infty}^0 \frac{dx}{1+x^2} + \int_0^\infty \frac{dx}{1+x^2}$$

$$= \lim_{x\to\infty} \int_{-x}^0 \frac{dx}{1+x^2} + \lim_{x\to\infty} \int_0^x \frac{dx}{1+x^2}$$

$$= \lim_{x\to\infty} [\tan^{-1} x]_{-x}^0 + \lim_{x\to\infty} [\tan^{-1} x]_0^x$$

$$= \lim_{x\to\infty} [0 - \tan^{-1}(-x)] + \lim_{x\to\infty} [\tan^{-1} x - 0]$$

$$= -(-\pi/2) + \pi/2 = \pi.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent.

Example 6: *Evaluate* $\int_0^1 \frac{dx}{\sqrt{x}}$.

(Gorakhpur 2010)

Solution: In the given integral, the integrand $1/\sqrt{x}$ becomes infinite at the lower limit $x = 0$. Therefore we have

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0} \int_{0+\varepsilon}^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0} [2\sqrt{x}]_\varepsilon^1$$

$$= \lim_{\varepsilon \to 0} [2 - 2\sqrt{\varepsilon}] = 2.$$

Hence the given integral is convergent and its value is 2.

Example 7: *Evaluate* $\int_0^1 \frac{dx}{\sqrt{(1-x)}}$.

Solution: Here the integrand *i.e.*, $1/\sqrt{(1-x)}$ becomes unbounded *i.e.*, infinite at the upper limit (*i.e.*, $x = 1$).

$$\therefore \quad \int_0^1 \frac{dx}{\sqrt{(1-x)}} = \lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} \frac{dx}{\sqrt{(1-x)}}$$

$$= \lim_{\varepsilon \to 0} [-2\sqrt{(1-x)}]_0^{1-\varepsilon} = \lim_{\varepsilon \to 0} [-2\sqrt{\varepsilon} + 2] = 2,$$

which is a definite real number. Hence the given integral is convergent and its value is 2.

Example 8: *Evaluate* $\int_{-1}^1 \frac{dx}{x^2}$.

Solution: Here the integrand becomes infinite at $x = 0$ and $-1 < 0 < 1$.

$$\therefore \quad \int_{-1}^1 \frac{dx}{x^2} = \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} \frac{dx}{x^2} + \lim_{\varepsilon' \to 0} \int_{\varepsilon'}^1 \frac{dx}{x^2}$$

$$= \lim_{\varepsilon \to 0} \left[-\frac{1}{x}\right]_{-1}^{-\varepsilon} + \lim_{\varepsilon' \to 0} \left[-\frac{1}{x}\right]_{\varepsilon'}^1$$

$$= \lim_{\varepsilon \to 0} \left[\frac{1}{\varepsilon} - 1\right] + \lim_{\varepsilon' \to 0} \left[-1 + \frac{1}{\varepsilon'}\right].$$

Since both the limits do not exist finitely, therefore the integral does not exist and is divergent.

Comprehensive Exercise 1

Evaluate the following integrals and discuss their convergence :

1. $\int_1^\infty \frac{dx}{x}$.
2. $\int_3^\infty \frac{dx}{(x-2)^2}$.
3. $\int_0^\infty e^{2x}\, dx$.
4. $\int_0^\infty \frac{dx}{(1+x)^{2/3}}$
5. $\int_{-\infty}^0 \sinh x\, dx$;
6. $\int_{-\infty}^0 \cosh x\, dx$.
7. $\int_0^\infty \cos x\, dx$
8. $\int_{-\infty}^\infty e^{-x}\, dx$.
9. $\int_{-\infty}^\infty \frac{dx}{x^2 + 2x + 2}$.

10. $\int_0^1 \frac{dx}{x^3}$. **11.** $\int_0^1 \frac{dx}{1-x}$. **12.** $\int_{-1}^1 \frac{dx}{x^{2/3}}$.

(Gorakhpur 2011)

Answers 1

1.	∞, divergent	2.	1, convergent	3.	∞, divergent
4.	∞, divergent	5.	$-\infty$, divergent	6.	∞, divergent
7.	Oscillates and so not convergent				
8.	∞, divergent	9.	π, convergent	10.	∞, divergent
11.	∞, divergent	12.	6, convergent		

3 Tests for Convergence of Improper Integrals of the First Kind

To test the convergence of improper integrals in which the range of integration is infinite and the integrand is bounded.

If an integral of the form $\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^b f(x)\,dx$ cannot be actually integrated, its convergence is determined with the help of the following tests :

4 Comparison Test

(Meerut 2012)

Let $f(x)$ and $g(x)$ be two functions which are bounded and integrable in the interval (a, ∞). Also let $g(x)$ be positive and $|f(x)| \le g(x)$ when $x \ge a$. Then, if $\int_a^\infty g(x)\,dx$ is **convergent**, *$\int_a^\infty f(x)\,dx$ is also* **convergent**.

Similarly if $|f(x)| \ge g(x)$ for all values of x greater than some number $x_0 > a$ and $\int_a^\infty g(x)\,dx$ is **divergent**, *then $\int_a^\infty f(x)\,dx$ is also* **divergent**.

Alternative form of the above comparison test:

If $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ is a definite number, other than zero, the integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both converge or both diverge.

Note: While applying comparison test, we generally take $g(x) = \frac{1}{x^n}$ *i.e.,* $\int_a^\infty \frac{dx}{x^n}$ is generally taken as the comparison integral.

Theorem: *The comparison integral* $\int_a^\infty \frac{dx}{x^n}$, *where* $a > 0$, *is convergent when* $n > 1$ *and divergent when* $n \le 1$.

Proof: By the definition of an improper integral, we have

$$\int_a^\infty \frac{dx}{x^n} = \lim_{x \to \infty} \int_a^x \frac{dx}{x^n} = \lim_{x \to \infty} \int_a^x x^{-n}\, dx$$

$$= \lim_{x \to \infty} \left[\frac{x^{1-n}}{1-n}\right]_a^x, \text{ if } n \ne 1$$

$$= \lim_{x \to \infty} \left[\frac{x^{1-n}}{1-n} - \frac{a^{1-n}}{1-n}\right]. \quad \text{...(1)}$$

If $n > 1$, then $1 - n$ is negative and so $n - 1$ is positive.
Therefore in this case

$$\lim_{x \to \infty} x^{1-n} = \lim_{x \to \infty} \frac{1}{x^{n-1}} = \frac{1}{\infty} = 0.$$

Hence from (1), we have

$$\int_a^\infty \frac{dx}{x^n} = \frac{a^{1-n}}{n-1}, \text{ if } n > 1.$$

Hence the given integral is convergent when $n > 1$.

If $n < 1$, then $1 - n$ is positive and so $\lim_{x \to \infty} x^{1-n} = \infty$.

$\therefore$ from (1), we have $\int_a^\infty \frac{dx}{x^n} = \infty$.

Hence the given integral is divergent when $n < 1$.

When $n = 1$, we have

$$\int_a^\infty \frac{dx}{x^n} = \int_a^\infty \frac{dx}{x} = \lim_{x \to \infty} \int_a^x \frac{dx}{x} = \lim_{x \to \infty} [\log x]_a^x$$

$$= \lim_{x \to \infty} [\log x - \log a] = \infty - \log a = \infty.$$

Hence the given integral is divergent when $n = 1$.

$\therefore$ $\int_a^\infty \frac{dx}{x^n}$ converges when $n > 1$ and diverges when $n \le 1$.

In other words $\int_a^\infty \frac{dx}{x^n}$ converges if and only if $n > 1$.

Illustrative Examples

Example 9: *Test the convergence of the integral*

$$\int_0^\infty \frac{\cos mx}{x^2 + a^2}\, dx.$$

(Rohilkhand 2011)

Solution: Here $f(x) = \dfrac{\cos mx}{x^2 + a^2}\cdot$ Let $g(x) = \dfrac{1}{x^2 + a^2}\cdot$

Obviously $g(x)$ is positive in the interval $(0, \infty)$.

We have $|f(x)| = \left|\dfrac{\cos mx}{x^2 + a^2}\right| = \dfrac{|\cos mx|}{x^2 + a^2} \leq \dfrac{1}{x^2 + a^2}$, since $|\cos mx| \leq 1$.

Thus $|f(x)| \leq g(x)$ when $x \geq 0$.

$\therefore$ by comparison test, $\int_0^\infty \dfrac{\cos mx}{x^2 + a^2}\, dx$ is convergent if $\int_0^\infty \dfrac{dx}{x^2 + a^2}$ is convergent.

But
$$\int_0^\infty \frac{dx}{x^2 + a^2} = \lim_{x \to \infty} \int_0^\infty \frac{dx}{x^2 + a^2} = \lim_{x \to \infty} \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^x$$
$$= \lim_{x \to \infty}\left[\frac{1}{a}\tan^{-1}\frac{x}{a} - 0\right] = \frac{1}{a}\cdot\frac{\pi}{2}$$
$$= \text{a definite real number.}$$

$\therefore$
$$\int_0^\infty \frac{dx}{x^2 + a^2} \text{ is convergent.}$$

Hence $\int_0^\infty \dfrac{\cos mx}{x^2 + a^2}\, dx$ is also convergent.

Example 10: *Test the convergence of the integral*
$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx.$$

Solution: Let $a > 0$. Then we can write
$$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \int_0^a \frac{\sin^2 x}{x^2}\, dx + \int_a^\infty \frac{\sin^2 x}{x^2}\, dx.$$

Since $\lim\limits_{x \to 0} \dfrac{\sin^2 x}{x^2} = 1$, therefore the integrand $\dfrac{\sin^2 x}{x^2}$ is bounded throughout the finite interval $(0, a)$.

So $\int_0^a \dfrac{\sin^2 x}{x^2}\, dx$ is a proper integral and we need to check the convergence of the integral $\int_a^\infty \dfrac{\sin^2 x}{x^2}\, dx$ only.

Here $f(x) = \dfrac{\sin^2 x}{x^2}\cdot$ Take $g(x) = \dfrac{1}{x^2}\cdot$

Obviously $g(x)$ is positive in the interval (a, ∞).

We have $|f(x)| = \left|\dfrac{\sin^2 x}{x^2}\right| = \dfrac{\sin^2 x}{x^2} \leq \dfrac{1}{x^2}$, since $\sin^2 x \leq 1$.

$\therefore$ by comparison test, $\int_a^\infty \dfrac{\sin^2 x}{x^2}\, dx$ is convergent if $\int_a^\infty \dfrac{dx}{x^2}$ is convergent.

But the comparison integral $\int_a^\infty \frac{dx}{x^2}$ is convergent because here $n = 2$ which is > 1.

$\therefore$ $$\int_a^\infty \frac{\sin^2 x}{x^2}\, dx \text{ is convergent.}$$

Hence $\int_0^\infty \frac{\sin^2 x}{x^2}\, dx$ is convergent.

Example 11: *Show that the integral* $\int_a^\infty \frac{dx}{x\sqrt{(1+x^2)}}$ *converges, where* $a > 0$.

Solution: Let $f(x) = \frac{1}{x\sqrt{(1+x^2)}}$.

Then $f(x)$ is bounded in the interval (a, ∞). Take $g(x) = 1/x^2$. Then $g(x)$ is positive in the interval (a, ∞). We have

$$|f(x)| = \left|\frac{1}{x\sqrt{(1+x^2)}}\right| = \frac{1}{x^2\sqrt{\{1+(1/x^2)\}}}$$
$$< \frac{1}{x^2}, \text{ since } \frac{1}{\sqrt{\{1+(1/x^2)\}}} < 1.$$

$\therefore$ by comparison test, $\int_a^\infty \frac{dx}{x\sqrt{(1+x^2)}}$ is convergent if $\int_a^\infty \frac{dx}{x^2}$ is convergent.

But the comparison integral $\int_a^\infty \frac{dx}{x^2}$ is convergent because here $n = 2$ which is > 1.

Hence $\int_a^\infty \frac{dx}{x\sqrt{(1+x^2)}}$ is also convergent.

Alternative Method: Here $f(x) = \frac{1}{x\sqrt{(1+x^2)}} = \frac{1}{x^2\sqrt{\{1+(1/x^2)\}}}$.

Take $g(x) = \frac{1}{x^2}$.

We have $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{1}{\sqrt{\{1+(1/x^2)\}}} = 1$, which is finite and non-zero.

Therefore $\int_a^\infty f(x)\, dx$ and $\int_a^\infty g(x)\, dx$ either both converge or both diverge. But $\int_a^\infty g(x)\, dx = \int_a^\infty \frac{dx}{x^2}$ is convergent because here $n = 2$ which is > 1.

Hence $\int_a^\infty f(x)\, dx$ *i.e.*, $\int_a^\infty \frac{1}{x\sqrt{(1+x^2)}}\, dx$ is also convergent.

Example 12: *Test the convergence of* $\int_0^\infty e^{-x} \frac{\sin x}{x}\, dx$. **(Kanpur 2011)**

Solution: We can write

$$\int_0^\infty e^{-x} \frac{\sin x}{x}\, dx = \int_0^1 e^{-x} \frac{\sin x}{x}\, dx + \int_1^\infty e^{-x} \frac{\sin x}{x}\, dx.$$

Since $\lim_{x \to 0} e^{-x} \dfrac{\sin x}{x} = 1$, therefore the integrand $e^{-x} \dfrac{\sin x}{x}$ is bounded throughout the finite interval $(0, 1)$. So $\int_0^1 e^{-x} \dfrac{\sin x}{x} dx$ is a proper integral and therefore it is convergent. Thus we need to check the convergence of $\int_1^\infty e^{-x} \dfrac{\sin x}{x} dx$ only.

Let $f(x) = e^{-x} \dfrac{\sin x}{x} \cdot$ Then $f(x)$ is bounded in the interval $(1, \infty)$.

Take $g(x) = e^{-x}$. Then $g(x)$ is positive in the interval $(1, \infty)$.

We have

$$|f(x)| = \left| e^{-x} \frac{\sin x}{x} \right| = e^{-x} . |\sin x| \cdot \frac{1}{x}$$

$$\leq e^{-x}, \text{since} \, |\sin x| \leq 1 \, \text{and} \, \frac{1}{x} \leq 1.$$

Thus $|f(x)| \leq g(x)$ throughout the interval $(1, \infty)$.

$\therefore$ by comparison test $\int_1^\infty f(x) \, dx$ is convergent if $\int_1^\infty g(x) \, dx$ is convergent.

Now
$$\int_1^\infty g(x) \, dx = \int_1^\infty e^{-x} \, dx = \lim_{x \to \infty} \int_1^\infty e^{-x} \, dx = \lim_{x \to \infty} [-e^{-x}]_1^x$$

$$= \lim_{x \to \infty} [-e^{-x} + e^{-1}] = 0 + e^{-1} = 1/e,$$

which is a definite finite number. Hence $\int_1^\infty g(x) \, dx$ is convergent.

$\therefore$ $\int_1^\infty f(x) \, dx$ is also convergent.

Hence $\int_0^\infty e^{-x} \dfrac{\sin x}{x} dx$ is convergent because the sum of two convergent integrals is also convergent.

Example 13: *Show that the integral* $\int_0^\infty e^{-x^2} dx$ *is convergent.*

(Rohilkhand 2010; Meerut 12)

Solution: We have

$$\int_0^\infty e^{-x^2} dx = \int_0^1 e^{-x^2} dx + \int_1^\infty e^{-x^2} dx.$$

Obviously $\int_0^1 e^{-x^2} dx$ is a proper integral because here the interval of integration $(0, 1)$ is finite and the integrand e^{-x^2} is bounded throughout this interval. Therefore this integral is convergent. So we need to check the convergence of $\int_1^\infty e^{-x^2} dx$ only.

Let $f(x) = e^{-x^2}$. Take $g(x) = xe^{-x^2}$ so that $g(x)$ is positive throughout the interval $(1, \infty)$. We have

$$|f(x)| = e^{-x^2} \le xe^{-x^2}, \text{ since } x \ge 1.$$

Thus $|f(x)| \le g(x)$ throughout the interval $(1, \infty)$.

$\therefore$ by comparison test $\int_1^\infty e^{-x^2}\, dx$ is convergent if $\int_1^\infty xe^{-x^2}\, dx$ is convergent.

Now
$$\int_1^\infty xe^{-x^2}\, dx = \lim_{x \to \infty} \int_1^x xe^{-x^2}\, dx$$
$$= \lim_{x \to \infty} \left(-\frac{1}{2} e^{-x^2}\right)_1^x$$
$$= \lim_{x \to \infty} \left(-\frac{1}{2} e^{-x^2} + \frac{1}{2} e^{-1}\right)$$
$$= \frac{1}{2} e^{-1}, \text{ which is a definite number.}$$

$\therefore$ $\int_1^\infty xe^{-x^2}\, dx$ is convergent and so $\int_1^\infty e^{-x^2} dx$ is also convergent.

Hence the given integral $\int_0^\infty e^{-x^2} dx$ is also convergent as it is the sum of two convergent integrals.

5 The μ-Test

(Gorakhpur 2012)

Let $f(x)$ be bounded and integrable in the interval (a, ∞) where $a > 0$.

If there is a number $\mu > 1$, such that $\lim_{x \to \infty} x^\mu f(x)$ exists, then $\int_a^\infty f(x)\, dx$ is convergent.

If there is a number $\mu \le 1$, such that $\lim_{x \to \infty} x^\mu f(x)$ exists and is non-zero, then $\int_a^\infty f(x)\, dx$ is divergent and the same is true if $\lim_{x \to \infty} x^\mu f(x)$ is $+\infty$ or $-\infty$.

While applying the μ-test, the value of μ is usually taken to be equal to the highest power of x in the denominator of the integrand minus the highest power of x in the numerator of the integrand.

Illustrative Examples

Example 14: *Examine the convergence of* $\int_1^\infty \frac{dx}{x^{1/3}(1 + x^{1/2})}$. **(Gorakhpur 2012)**

Solution: Let
$$f(x) = \frac{1}{x^{1/3}(1 + x^{1/2})} = \frac{1}{x^{1/3}\, x^{1/2}\, \{1 + (1/x^{1/2})\}}$$
$$= \frac{1}{x^{5/6}\, \{1 + (1/x^{1/2})\}}.$$

Obviously $f(x)$ is bounded in the interval $(1, \infty)$.

Take $\mu = \frac{5}{6} - 0 = \frac{5}{6} \cdot$ We have

$$\lim_{x \to \infty} x^{\mu} f(x) = \lim_{x \to \infty} x^{5/6} \cdot \frac{1}{x^{5/6}\{1 + (1/x^{1/2})\}}$$

$$= \lim_{x \to \infty} \frac{1}{1 + (1/x^{1/2})} = 1,$$

which is finite and non-zero. Since $\mu = \frac{5}{6}$ *i.e.*, < 1, it follows from the μ-test that the given integral is divergent.

Example 15: *Examine the convergence of* $\int_0^{\infty} \frac{x\,dx}{(1+x)^3} \cdot$

(Rohilkhand 2011; Purvanchal 11)

Solution: Let $a > 0$. Then we have

$$\int_0^{\infty} \frac{x\,dx}{(1+x)^3} = \int_0^{a} \frac{x\,dx}{(1+x)^3} + \int_a^{\infty} \frac{x\,dx}{(1+x)^3} \cdot$$

The first integral on the right hand side is convergent because it is a proper integral. We observe that in this integral the range of integration $(0, a)$ is finite and the integrand $x/(1+x)^3$ is bounded throughout the interval $(0, a)$. So we need to check the convergence of $\int_a^{\infty} \frac{x\,dx}{(1+x)^3}$ only.

Let $f(x) = \frac{x}{(1+x)^3} \cdot$ Then $f(x)$ is bounded in the interval (a, ∞).

Take $\mu = 3 - 1 = 2$. Then

$$\lim_{x \to \infty} x^{\mu} f(x) = \lim_{x \to \infty} x^2 \cdot \frac{x}{(1+x)^3} = \lim_{x \to \infty} \frac{1}{\{1 + (1/x)\}^3} = 1,$$

which exists *i.e.*, is equal to a definite real number.

Since $\mu = 2$ *i.e.*, > 1, therefore by μ-test the integral $\int_a^{\infty} \frac{x\,dx}{(1+x)^3}$ is convergent.

Hence $\int_0^{\infty} \frac{x\,dx}{(1+x)^3}$ is also convergent because it is the sum of two convergent integrals.

Example 16: *Examine the convergence of* $\int_a^{\infty} \frac{dx}{x(\log x)^{n+1}}$, *where* $a > 1$.

Solution: Let $\log x = t$ so that $(1/x)\,dx = dt$.

$$\therefore \quad \int_a^{\infty} \frac{dx}{x(\log x)^{n+1}} = \int_{\log a}^{\infty} \frac{dt}{t^{n+1}} \cdot$$

Let $\quad f(t) = 1/t^{n+1}$.

Then $f(t)$ is bounded in the interval $(\log a, \infty)$.

Take $\mu = (n+1) - 0 = n + 1$. Then

$$\lim_{t \to \infty} t^{\mu} f(t) = \lim_{t \to \infty} \frac{t^{n+1}}{t^{n+1}} = \lim_{t \to \infty} 1 = 1,$$

which is finite and non-zero.

Therefore by μ-test, the given integral is convergent if

$$\mu > 1 \text{ i.e., } n + 1 > 1 \text{ i.e., } n > 0$$

and divergent if $\mu \le 1$ *i.e.*, $n + 1 \le 1$ *i.e.*, $n \le 0$.

Example 17: *Show that the integral* $\int_1^{\infty} x^{n-1} e^{-x} \, dx$ *is convergent.*

Solution: Let $f(x) = x^{n-1} e^{-x}$. Then $f(x)$ is bounded in the interval $(1, \infty)$. We have

$$\lim_{x \to \infty} x^{\mu} f(x) = \lim_{x \to \infty} \frac{x^{\mu} \cdot x^{n-1}}{e^x} = \lim_{x \to \infty} \frac{x^{\mu + n - 1}}{1 + x + \frac{x^2}{2!} + \dots}$$

$$= 0 \qquad \text{for all values of } \mu \text{ and } n.$$

Taking $\mu > 1$, we see by μ-test that the integral $\int_1^{\infty} x^{n-1} e^{-x} \, dx$

is convergent for all values of n.

Example 18: *Test the convergence of the integral* $\int_0^{\infty} \frac{x^{2m}}{1 + x^{2n}} \, dx$,

where m and n are positive integers.

(Purvanchal 2007; Rohilkhand 12; Gorakhpur 13, 15)

Solution: Let $a > 0$. We have

$$\int_0^{\infty} \frac{x^{2m}}{1 + x^{2n}} \, dx = \int_0^{a} \frac{x^{2m}}{1 + x^{2n}} \, dx + \int_a^{\infty} \frac{x^{2m}}{1 + x^{2n}} \, dx.$$

The first integral on the right hand side is a proper integral and so it is convergent. Therefore the given integral is convergent or divergent according as $\int_a^{\infty} \frac{x^{2m}}{1 + x^{2n}} \, dx$ is convergent or divergent.

To test the convergence of $\int_a^{\infty} \frac{x^{2m}}{1 + x^{2n}} \, dx$, let us take $\mu = 2n - 2m$.

We have

$$\lim_{x \to \infty} x^{\mu} \cdot \frac{x^{2m}}{1 + x^{2n}} = \lim_{x \to \infty} x^{2n - 2m} \cdot \frac{x^{2m}}{x^{2n} \{1 + (1 / x^{2n})\}}$$

$$= \lim_{x \to \infty} \frac{1}{1 + (1 / x^{2n})} = 1,$$

which is finite and non-zero.

$\therefore$ by μ-test, the given integral is convergent if $\mu > 1$ *i.e.*, if $2n - 2m > 1$ which is possible if $n > m$ since m and n are positive integers. Also by μ-test, the given integral is divergent if $\mu \le 1$ *i.e.*, if $2n - 2m \le 1$ *i.e.*, if $n \le m$ since n and m are positive integers.

6 Abel's Test for the Convergence of Integral of a Product

If $\int_a^\infty f(x)\, dx$ *converges and* $\phi(x)$ *is bounded and monotonic for* $x > a$, *then* $\int_a^\infty f(x)\,\phi(x)\, dx$ *is convergent.* **(Rohilkhand 2008, 11; Purvanchal 07, 10, 11; Kanpur 12; Gorakhpur 14, 15)**

Illustrative Examples

Example 19: *Test the convergence of* $\int_a^\infty (1 - e^{-x}) \frac{\cos x}{x^2}\, dx$, *when* $a > 0$.

(Rohilkhand 2009; Gorakhpur 15)

Solution: Let $f(x) = \frac{\cos x}{x^2}$ and $\phi(x) = 1 - e^{-x}$.

We have $\left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2}$ as $|\cos x| \le 1$.

Since $\int_a^\infty \frac{1}{x^2}\, dx$ is convergent, therefore by comparison test $\int_a^\infty \frac{\cos x}{x^2}\, dx$ is also convergent.

Again $\phi(x) = 1 - e^{-x}$ is monotonic increasing and bounded function for $x > a$.

Hence by Abel's test $\int_a^\infty (1 - e^{-x}) \frac{\cos x}{x^2}\, dx$ is convergent.

Example 20: *Test the convergence of* $\int_a^\infty e^{-x} \frac{\sin x}{x^2}\, dx$, *where* $a > 0$.

Solution: Let $f(x) = \frac{\sin x}{x^2}$ and $\phi(x) = e^{-x}$.

Since $\left|\frac{\sin x}{x^2}\right| \le \frac{1}{x^2}$ and $\int_a^\infty \frac{1}{x^2}\, dx$ is convergent, therefore by comparison test $\int_a^\infty \frac{\sin x}{x^2}\, dx$ is also convergent.

Again e^{-x} is monotonic decreasing and bounded function for $x > a$.

Hence by Abel's test $\int_a^\infty e^{-x} \frac{\sin x}{x^2}\, dx$ is convergent.

7 Dirichlet's Test for the Convergence of Integral of a Product

If $f(x)$ *be bounded and monotonic in the interval* $a \le x < \infty$ *and if* $\lim_{x \to \infty} f(x) = 0$, *then the integral* $\int_a^\infty f(x)\,\phi(x)\, dx$ *converges provided* $\left|\int_a^x \phi(x)\, dx\right|$ *is bounded as* x *takes all finite values.* **(Purvanchal 2012)**

Illustrative Examples

Example 21: *Test the convergence of the integral*

$$\int_a^\infty \frac{\sin x}{\sqrt{x}}\, dx, \text{ where } a > 0.$$

(Garhwal 2006, 09)

Solution: Let $f(x) = \dfrac{1}{\sqrt{x}}$ and $\phi(x) = \sin x$.

Now $\dfrac{1}{\sqrt{x}}$ is bounded and monotonic decreasing for all $x \geq a$ and $\lim_{x \to \infty} \dfrac{1}{\sqrt{x}} = 0$.

Also $\left| \int_a^x \phi(x)\, dx \right| = \left| \int_a^x \sin x\, dx \right| = |\cos a - \cos x| \leq 2$, for all finite values of x.

[Note that the value of $\cos x$ lies between -1 and 1].

$\therefore$ $\left| \int_a^x \phi(x)\, dx \right|$ is bounded for all finite values of x.

Hence by Dirichlet's test the integral $\int_a^\infty \dfrac{\sin x}{\sqrt{x}}\, dx$ is convergent.

Example 22: *Show that* $\int_0^\infty \sin x^2\, dx$ *is convergent.*

(Agra 2012)

Solution: We have $\int_0^\infty \sin x^2\, dx = \int_0^1 \sin x^2\, dx + \int_1^\infty \sin x^2\, dx$.

But $\int_0^1 \sin x^2\, dx$ is a proper integral and hence convergent.

Now it remains to test the convergence of $\int_1^\infty \sin x^2\, dx$. We can write

$$\int_1^\infty \sin x^2\, dx = \int_1^\infty 2x \,.\, (\sin x^2) \,.\, \frac{1}{2x}\, dx.$$

Let $f(x) = \dfrac{1}{2x}$ and $\phi(x) = 2x \sin x^2$.

The function $f(x) = \dfrac{1}{2x}$ is bounded and monotonic decreasing for all $x \geq 1$ and

$\lim_{x \to \infty} \dfrac{1}{2x} = 0$.

Also $\left| \int_1^x \phi(x)\, dx \right| = \int_1^x 2x \sin x^2\, dx$

$= |\cos 1^2 - \cos x^2| \leq 2$, for all finite values of x.

$\therefore$ $\left| \int_1^x \phi(x)\, dx \right|$ is bounded for all finite values of x.

Hence by Dirichlet's test

$$\int_1^\infty \frac{1}{2x} \,.\, (\sin x^2)\, 2x\, dx \text{ i.e., } \int_1^\infty \sin x^2\, dx \text{ is convergent.}$$

Since the sum of two convergent integrals is convergent, therefore the integral $\int_0^\infty \sin x^2\, dx$ is convergent.

Example 23: *Show that the integral* $\int_0^{\infty} \frac{\sin x}{x}\, dx$ *is convergent.*

(Garhwal 2007; Purvanchal 10, 12)

Solution: We have $\int_0^{\infty} \frac{\sin x}{x}\, dx = \int_0^{a} \frac{\sin x}{x}\, dx + \int_a^{\infty} \frac{\sin x}{x}\, dx$, where $a > 0$.

Since $\lim_{x \to 0} \frac{\sin x}{x} = 1$, the integral $\int_0^{a} \frac{\sin x}{x}\, dx$ is a proper integral and hence convergent.

Now to test the convergence of $\int_a^{\infty} \frac{\sin x}{x}\, dx$.

Let $f(x) = 1/x$ and $\phi(x) = \sin x$.

The function $f(x) = 1/x$ is bounded and monotonic decreasing for all $x \ge a$ and $\lim_{x \to \infty} \frac{1}{x} = 0$.

Also $\left| \int_a^{x} \phi(x)\, dx \right| = \left| \int_a^{x} \sin x\, dx \right| = |\cos a - \cos x| \le 2$, for all finite values of x.

$\therefore$ $\left| \int_a^{x} \phi(x)\, dx \right|$ is bounded for all finite values of x.

Hence by Dirichlet's test the integral $\int_a^{\infty} \frac{\sin x}{x}\, dx$ is convergent.

Since the sum of two convergent integrals is convergent, therefore $\int_0^{\infty} \frac{\sin x}{x}\, dx$ is convergent.

Example 24: *Prove that* $\int_a^{\infty} \frac{\cos \alpha x - \cos \beta x}{x}\, dx$ *is convergent where* $a > 0$.

(Rohilkhand 2011)

Solution: We have

$$\int_a^{\infty} \frac{\cos \alpha x - \cos \beta x}{x}\, dx = \int_a^{\infty} \frac{\cos \alpha x}{x}\, dx - \int_a^{\infty} \frac{\cos \beta x}{x}\, dx.$$

The function $f(x) = 1/x$ is bounded and monotonic decreasing for all $x \ge a$ and $\lim_{x \to \infty} \frac{1}{x} = 0$.

Also $$\left| \int_a^{x} \cos \alpha x\, dx \right| = \left| \frac{1}{\alpha} (\sin \alpha x - \sin \alpha a) \right| \le \frac{2}{|\alpha|}.$$

$\therefore$ $\left| \int_a^{x} \cos \alpha x\, dx \right|$ is bounded for all finite values of x.

Similarly $\left| \int_a^{x} \cos \beta x\, dx \right|$ is bounded for all finite values of x.

$\therefore$ by Dirichlet's test both the integrals

$$\int_a^{\infty} \frac{\cos \alpha x}{x}\, dx \text{ and } \int_a^{\infty} \frac{\cos \beta x}{x}\, dx \text{ are convergent.}$$

Hence the given integral is convergent.

Example 25: *Show that the integral*

$$\int_0^{\infty} e^{-ax} \frac{\sin x}{x}\, dx,\ a \geq 0 \text{ is convergent.}$$

(Kanpur 2009; Garhwal 10)

Solution: We have

$$\int_0^{\infty} e^{-ax} \frac{\sin x}{x}\, dx = \int_0^{\alpha} e^{-ax} \frac{\sin x}{x}\, dx + \int_{\alpha}^{\infty} e^{-ax} \frac{\sin x}{x}\, dx, \text{ where } \alpha > 0.$$

Since $\lim_{x \to 0} e^{-ax} \frac{\sin x}{x} = 1$, the integral $\int_0^{\alpha} e^{-ax} \frac{\sin x}{x}\, dx$ is a proper integral and hence convergent.

Now it remains to test the convergence of

$$\int_{\alpha}^{\infty} e^{-ax} \frac{\sin x}{x}\, dx. \quad \text{Let } f(x) = \frac{e^{-ax}}{x} \text{ and } \phi(x) = \sin x.$$

Obviously the function $f(x) = \frac{1}{x e^{ax}}$ is bounded and monotonic decreasing for all $x \geq \alpha$ and $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{x e^{ax}} = 0.$

Moreover $\left| \int_{\alpha}^{x} \phi(x)\, dx \right| = \left| \int_{\alpha}^{x} \sin x\, dx \right| = |\cos \alpha - \cos x| \leq 2$, for all finite values of x.

$\therefore$ $\left| \int_{\alpha}^{x} \phi(x)\, dx \right|$ is bounded for all finite values of x.

$\therefore$ by Dirichlet's test $\int_{\alpha}^{\infty} e^{-ax} \frac{\sin x}{x}\, dx$ is convergent.

Since the sum of two convergent integrals is convergent, therefore $\int_0^{\infty} e^{-ax} \frac{\sin x}{x}\, dx$ is convergent.

8 Absolute Convergence

The infinite integral $\int_a^{\infty} f(x)\, dx$ *is said to be absolutely convergent if the integral* $\int_a^{\infty} |f(x)|\, dx$ *is convergent.*

If the integral $\int_a^{\infty} f(x)\, dx$ is absolutely convergent, it is necessarily convergent. But if the integral $\int_a^{\infty} f(x)\, dx$ is convergent, it is not necessarily absolutely convergent. Thus absolute convergence gives a sufficient but not a necessary condition for the convergence of an infinite integral.

Illustrative Examples

Example 26: *Show that* $\int_1^\infty \frac{\sin x}{x^4}\, dx$ *is absolutely convergent.*

Solution: The integral $\int_1^\infty \frac{\sin x}{x^4}\, dx$ will be absolutely convergent if $\int_1^\infty \left|\frac{\sin x}{x^4}\right| dx$ is convergent.

Let $f(x) = \left|\frac{\sin x}{x^4}\right|$. Then $f(x)$ is bounded in the interval $(1, \infty)$. We have

$$f(x) = \left|\frac{\sin x}{x^4}\right| = \frac{|\sin x|}{x^4} \le \frac{1}{x^4}, \text{ since } |\sin x| \le 1.$$

$\therefore$ by comparison test, $\int_0^\infty f(x)\, dx$ is convergent if $\int_1^\infty \frac{1}{x^4}\, dx$ is convergent. But the comparison integral $\int_1^\infty \frac{1}{x^4}\, dx$ is convergent because here $n = 4$ which is > 1.

Hence $\int_1^\infty f(x)\, dx$ is convergent and so the given integral is absolutely convergent.

Example 27: *Show that* $\int_0^\infty \frac{\sin mx}{a^2 + x^2}\, dx$ *converges absolutely.*

(Garhwal 2008)

Solution: The integral $\int_0^\infty \frac{\sin mx}{a^2 + x^2}\, dx$ will be absolutely convergent if

$$\int_0^\infty \left|\frac{\sin mx}{a^2 + x^2}\right| dx \text{ is convergent.}$$

Let $f(x) = \left|\frac{\sin mx}{a^2 + x^2}\right|$. Then $f(x)$ is bounded in the interval $(0, \infty)$. We have

$$f(x) = \left|\frac{\sin mx}{a^2 + x^2}\right| = \frac{|\sin mx|}{a^2 + x^2} \le \frac{1}{a^2 + x^2}, \text{ since } |\sin mx| \le 1.$$

$\therefore$ by comparison test, $\int_0^\infty f(x)\, dx$ is convergent if $\int_0^\infty \frac{1}{a^2 + x^2}\, dx$ is convergent.

But $$\int_0^\infty \frac{dx}{a^2 + x^2} = \lim_{x \to \infty} \int_0^x \frac{dx}{a^2 + x^2} = \lim_{x \to \infty} \left[\frac{1}{a} \tan^{-1} \frac{x}{a}\right]_0^x$$

$$= \lim_{x \to \infty} \left[\frac{1}{a} \tan^{-1} \frac{x}{a} - 0\right] = \frac{1}{a} \cdot \frac{\pi}{2},$$

which is a definite real number.

$\therefore$ $\int_0^\infty \frac{dx}{a^2 + x^2}$ is convergent. Hence $\int_0^\infty f(x)\, dx$ is also convergent and so the given integral is absolutely convergent.

Example 28: *Show that the integral* $\int_0^{\infty} e^{-x} \cos mx\, dx$ *converges absolutely.*

Solution: The integral $\int_0^{\infty} e^{-x} \cos mx\, dx$ will be absolutely convergent if $\int_0^{\infty} |e^{-x} \cos mx|\, dx$ is convergent.

Let $f(x) = |e^{-x} \cos mx|$. Then $f(x)$ is bounded in the interval $(0, \infty)$. We have

$$f(x) = |e^{-x} \cos mx| = e^{-x} |\cos mx|$$

$$\leq e^{-x}, \quad \text{since } |\cos mx| \leq 1.$$

$\therefore$ by comparison test, $\int_0^{\infty} f(x)\, dx$ is convergent if $\int_0^{\infty} e^{-x}\, dx$ is convergent.

But $$\int_0^{\infty} e^{-x}\, dx = \lim_{x \to \infty} \int_0^{x} e^{-x}\, dx = \lim_{x \to \infty} [-e^{-x}]_0^{\infty}$$

$$= \lim_{x \to \infty} [-e^{-x} + 1] = 1, \text{ which is a definite real number.}$$

$\therefore$ $\int_0^{\infty} e^{-x}\, dx$ is convergent.

Hence $\int_0^{\infty} f(x)\, dx$ is convergent and so the given integral is absolutely convergent.

9 Tests for Convergence of Improper Integrals of The Second Kind

Now we shall make a study of the tests for the convergence of a definite integral of the type $\int_a^b f(x)\, dx$ in which the range of integration is finite and the integrand $f(x)$ is unbounded at one or more points of the given interval $[a, b]$. It is sufficient to consider the case when $f(x)$ becomes unbounded at $x = a$ and bounded for all other values of x in the interval $[a, b]$. In this case we have

$$\int_a^b f(x)\, dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b} f(x)\, dx.$$

In the articles to follow we give a few important tests for the convergence of the above integral.

10 Comparison Test

Consider the improper integral $\int_a^b f(x)\, dx$, *where the range of integration* (a, b) *is finite and* $f(x)$ *is unbounded only at* $x = a$. *Let* $g(x)$ *be positive in the interval* $(a+\varepsilon, b)$ *and* $|f(x)| \leq g(x)$ *in the interval* $(a+\varepsilon, b)$. *Then* $\int_a^b f(x)\, dx$ *is convergent if* $\int_a^b g(x)\, dx$ *is convergent.*

Similarly if $|f(x)| \geq g(x)$ *for all values of* x *in the interval* $(a+\varepsilon, b)$, *then* $\int_a^b f(x)\,dx$ *is divergent provided* $\int_a^b g(x)\,dx$ *is divergent.*

Alternative form of the above comparison test:

If $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ is a definite number, other than zero, the integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ either both converge or both diverge.

Note: While applying the above comparison test, we generally take $g(x) = \dfrac{1}{(x-a)^n}$ *i.e.,* $\int_a^b \dfrac{dx}{(x-a)^n}$ is generally taken as the comparison integral.

Theorem: *The comparison integral* $\int_a^b \dfrac{dx}{(x-a)^n}$ *is convergent when* $n < 1$ *and divergent when* $n \geq 1$. **(Purvanchal 2011; Gorakhpur 13)**

Proof: We have

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b (x-a)^{-n}\,dx$$

$$= \lim_{\varepsilon \to 0} \left[\frac{(x-a)^{-n+1}}{1-n}\right]_{a+\varepsilon}^b, \text{ if } n \neq 1$$

$$= \lim_{\varepsilon \to 0} \left[\frac{(b-a)^{1-n}}{1-n} - \frac{\varepsilon^{1-n}}{1-n}\right]. \qquad \ldots(1)$$

If $n < 1$, then $1-n$ is positive and so $\lim_{\varepsilon \to 0} \varepsilon^{1-n} = 0$. Therefore from (1), we have

$$\int_a^b \frac{dx}{(x-a)^n} = \frac{(b-a)^{1-n}}{1-n}, \text{ if } n < 1.$$

Hence the given integral converges when $n < 1$.

If $n > 1$, then $1-n$ is negative and so $n-1$ is positive. Therefore in this case, from (1), we have

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0} \left[\frac{(b-a)^{1-n}}{1-n} + \frac{1}{(n-1)\,\varepsilon^{n-1}}\right] = \infty.$$

Hence the given integral diverges when $n > 1$.

When $n = 1$, we have

$$\int_a^b \frac{dx}{(x-a)^n} = \int_a^b \frac{dx}{(x-a)} = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b \frac{dx}{x-a}$$

$$= \lim_{\varepsilon \to 0} \left[\log(x-a)\right]_{a+\varepsilon}^b = \lim_{\varepsilon \to 0} [\log(b-a) - \log \varepsilon]$$

$$= \infty \qquad [\because \log 0 = -\infty]$$

Hence the given integral diverges when $n = 1$.

Illustrative Examples

Example 29: *Show that the integral* $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ *is convergent.*

Solution: In the given integral, the integrand $f(x) = \frac{1}{x^{1/3}(1+x^2)}$ is unbounded at the lower limit of integration $x = 0$.

Take $g(x) = 1/x^{1/3}$.

Then $$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{1}{1+x^2} = 1, \text{ which is finite and non-zero.}$$

$\therefore$ by comparison test

$$\int_0^1 f(x)\, dx \text{ and } \int_0^1 g(x)\, dx$$

either both converge or both diverge. But the comparison integral $\int_0^1 \frac{dx}{x^{1/3}}$ is convergent because here $n = 1/3$ which is less than 1. Hence the integral $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ is also convergent.

Example 30: *Test the convergence of the integral* $\int_1^2 \frac{dx}{\sqrt{(x^4-1)}}$.

Solution: In the given integral the integrand $f(x) = 1/\sqrt{(x^4-1)}$ is unbounded at the lower limit of integration $x = 1$.

Take $$g(x) = 1/\sqrt{(x^2-1)}.$$

Then $$\lim_{x \to 1} \frac{f(x)}{g(x)} = \lim_{x \to 1} \left\{ \frac{1}{\sqrt{(x^4-1)}} \cdot \sqrt{(x^2-1)} \right\} = \lim_{x \to 1} \frac{1}{\sqrt{(x^2+1)}}$$

$$= 1/\sqrt{2}, \text{ which is finite and non-zero.}$$

Therefore by comparison test,

$$\int_1^2 f(x)\, dx \text{ and } \int_1^2 g(x)\, dx$$

are either both convergent or both divergent.

But $$\int_1^2 g(x)\, dx = \int_1^2 \frac{dx}{\sqrt{(x^2-1)}} = \lim_{\varepsilon \to 0} \int_{1+\varepsilon}^2 \frac{dx}{\sqrt{(x^2-1)}}$$

$$= \lim_{\varepsilon \to 0} \left[\log \{x + \sqrt{(x^2-1)}\}\right]_{1+\varepsilon}^2$$

$$= \lim_{\varepsilon \to 0} [\log(2+\sqrt{3}) - \log\{1 + \varepsilon + \sqrt{(\varepsilon^2+\varepsilon)}\}]$$

$$= \log(2+\sqrt{3}), \quad \text{which is a definite real number.}$$

$\therefore \quad \int_1^2 g(x)\,dx$ is convergent.

Hence $\int_1^2 \dfrac{1}{\sqrt{(x^4-1)}}\,dx$ is also convergent.

Example 31: *Show that the integral* $\int_0^1 \dfrac{\sec x}{x}\,dx$ *is divergent.*

Solution: In the given integral the integrand $f(x) = \dfrac{\sec x}{x}$ is unbounded at the lower limit of integration $x = 0$. Take $g(x) = 1/x$.

Then $$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \left\{ \frac{\sec x}{x} \cdot x \right\} = \lim_{x \to 0} \sec x = 1,$$

which is finite and non-zero.

Therefore, by comparison test, $\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ either both converge or both diverge. But the comparison integral $\int_0^1 \dfrac{1}{x}\,dx$ is divergent because here $n = 1$.

Hence the given integral $\int_0^1 \dfrac{\sec x}{x}\,dx$ is also divergent.

Example 32: *Show that* $\int_0^1 x^{n-1} e^{-x}\,dx$ *is convergent if* $n > 0$.

Solution: If $n \geq 1$, then $\int_0^1 x^{n-1} e^{-x}\,dx$ is a proper integral because the integrand $f(x) = x^{n-1} e^{-x}$ is bounded in the interval $(0, 1)$. So the given integral is convergent when $n \geq 1$.

If $0 < n < 1$, the integrand $f(x) = x^{n-1} e^{-x}$ is unbounded at $x = 0$. Take $g(x) = x^{n-1}$.

Then $\displaystyle \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} e^{-x} = 1$, which is finite and non-zero.

$\therefore$ by comparison test, $\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ either both converge or both diverge.

But $$\int_0^1 g(x)\,dx = \int_0^1 x^{n-1}\,dx = \lim_{\varepsilon \to 0} \int_\varepsilon^1 x^{n-1}\,dx = \lim_{\varepsilon \to 0} \left[\frac{x^n}{n} \right]_\varepsilon^1$$

$$= \lim_{\varepsilon \to 0} \left[\frac{1}{n} - \frac{\varepsilon^n}{n} \right] = \frac{1}{n},$$ which is a definite real number.

$\therefore \quad \int_0^1 g(x)\,dx$ is convergent.

Hence $\int_0^1 x^{n-1} e^{-x}\,dx$ is also convergent.

Example 33: *Show that the integral* $\int_0^{\infty} x^{n-1} e^{-x}\, dx$ *is convergent if* $n > 0$.

(Garhwal 2006)

Solution: We have

$$\int_0^{\infty} x^{n-1} e^{-x}\, dx = \int_0^1 x^{n-1} e^{-x}\, dx + \int_1^{\infty} x^{n-1} e^{-x}\, dx.$$

Let $$I_1 = \int_0^1 x^{n-1} e^{-x}\, dx \text{ and } I_2 = \int_1^{\infty} x^{n-1} e^{-x}\, dx.$$

The integral I_2 is convergent for all values of n.

[For proof see Ex. 17 after article 5]

Also the integral I_1 is convergent if $n > 0$. [For proof see Ex. 32 above]

Hence the given integral is convergent if $n > 0$ because then it is the sum of two convergent integrals.

11 The μ-Test

Let $f(x)$ *be unbounded at* $x = a$ *and be bounded and integrable in the arbitrary interval* $(a + \varepsilon, b)$, *where* $0 < \varepsilon < b - a$. *If there is a number* μ *between* 0 *and* 1 *such that*

$$\lim_{x \to a+0} (x - a)^{\mu} f(x) \text{ exists, then } \int_a^b f(x)\, dx$$

is convergent.

If there is a number $\mu \geq 1$ *such that* $\lim_{x \to a+0} (x - a)^{\mu} f(x)$ *exists and is non-zero, then* $\int_a^b f(x)\, dx$ *is divergent and the same is true if*

$$\lim_{x \to a+0} (x - a)^{\mu} f(x) = +\infty \quad \text{or} \quad -\infty.$$

In case $f(x)$ is unbounded at $x = b$, we should find

$$\lim_{x \to b-0} (b - x)^{\mu} \cdot f(x),$$

the other conditions of the test remaining the same.

Illustrative Examples

Example 34: *Prove that the integral* $\int_0^1 \frac{dx}{\sqrt{\{x(1-x)\}}}$ *converges.*

(Garhwal 2009, 12)

Solution: In the given integral the integrand $f(x) = 1/\sqrt{\{x(1-x)\}}$ is unbounded both at $x = 0$ and at $x = 1$. If $0 < a < 1$, we can write

$$\int_0^1 \frac{dx}{\sqrt{\{x(1-x)\}}} = \int_0^a \frac{dx}{\sqrt{\{x(1-x)\}}} + \int_a^1 \frac{dx}{\sqrt{\{x(1-x)\}}} = I_1 + I_2, \text{ say.}$$

In the integral I_1 the integrand $f(x)$ is unbounded at the lower limit of integration $x = 0$ and in the integral I_2 the integrand $f(x)$ is unbounded at the upper limit of integration $x = 1$.

To test the convergence of I_1. Take $\mu = \frac{1}{2} \cdot$ We have

$$\lim_{x \to 0} x^{\mu} f(x) = \lim_{x \to 0} x^{1/2} \cdot \frac{1}{\sqrt{\{x(1-x)\}}} = \lim_{x \to 0} \frac{1}{\sqrt{(1-x)}}$$

$$= 1 \text{ } i.e., \text{ the limit exists.}$$

Since $0 < \mu < \frac{1}{2}$, therefore by μ-test I_1 is convergent.

To test the convergence of I_2. Take $\mu = \frac{1}{2} \cdot$ We have

$$\lim_{x \to 1-0} (1-x)^{\mu} \cdot f(x) = \lim_{x \to 1-0} (1-x)^{1/2} \cdot \frac{1}{\sqrt{\{x(1-x)\}}}$$

$$= \lim_{x \to 1-0} \frac{1}{\sqrt{x}} = \lim_{\varepsilon \to 0} \frac{1}{\sqrt{(1-\varepsilon)}} = 1.$$

Hence by μ-test I_2 is convergent since $0 < \mu < 1$.

Thus the given integral is the sum of two convergent integrals. Hence the given integral itself is convergent.

Example 35: *Test the convergence of* $\int_0^1 \frac{\log x}{\sqrt{(2-x)}} dx$.

(Agra 2012)

Solution: Let $f(x) = \frac{\log x}{\sqrt{(2-x)}} \cdot$ Then $f(x)$ is unbounded both at $x = 0$ and $x = 2$. If $0 < a < 2$, we can write

$$\int_0^2 \frac{\log x}{\sqrt{(2-x)}} dx = \int_0^a \frac{\log x}{\sqrt{(2-x)}} dx + \int_a^2 \frac{\log x}{\sqrt{(2-x)}} dx = I_1 + I_2, \text{ say.}$$

To test the convergence of I_1. We have

$$\lim_{x \to 0} x^{\mu} f(x) = \lim_{x \to 0} \left\{ x^{\mu} \cdot \frac{\log x}{\sqrt{(x-2)}} \right\} = 0 \text{ if } \mu > 0.$$

Therefore taking μ between 0 and 1, it follows by μ-test that I_1 is convergent.

To test the convergence of I_2.

Take $\mu = \frac{1}{2} \cdot$ We have

$$\lim_{x \to 2-0} (2-x)^{\mu} \cdot f(x) = \lim_{x \to 2-0} (2-x)^{1/2} \cdot \frac{\log x}{\sqrt{(2-x)}}$$

$$= \lim_{x \to 2-0} \log x = \lim_{\varepsilon \to 0} \log(2-\varepsilon)$$

$$= \log 2.$$

$\therefore$ by μ-test I_2 is convergent because $0 < \mu < 1$.

Hence the given integral is also convergent, it being the sum of two convergent integrals.

Example 36: *Test the convergence of* $\int_0^1 x^{p-1} e^{-x}\, dx$.

Solution: Let $f(x) = x^{p-1} e^{-x}$

and $$I = \int_0^1 x^{p-1} e^{-x}\, dx.$$

If $p \geq 1$, $f(x)$ is bounded throughout the interval $(0, 1)$ and so I is a proper integral and hence it is convergent if $p \geq 1$.

If $p < 1$, $f(x)$ is unbounded at $x = 0$. In this case, we have

$$\lim_{x \to 0} x^{\mu} f(x) = \lim_{x \to 0} x^{\mu} . x^{p-1} e^{-x} = \lim_{x \to 0} x^{\mu + p - 1} e^{-x}$$

$$= 1 \text{ if } \mu + p - 1 = 0 \text{ i.e., } \mu = 1 - p.$$

So by μ-test when $0 < \mu < 1$ *i.e.*, $0 < p < 1$, the given integral is convergent and when $\mu \geq 1$ *i.e.*, $p \leq 0$, the given integral is divergent.

Hence I is convergent if $p > 0$ and is divergent if $p \leq 0$.

12 Abel's Test

If $\int_a^b f(x)\, dx$ *converges and* $\phi(x)$ *is bounded and monotonic for* $a \leq x \leq b$, *then* $\int_a^b f(x)\, \phi(x)\, dx$ *converges.*

13 Dirichlet's Test

If $\int_{a+\varepsilon}^b f(x)\, dx$ *be bounded and* $\phi(x)$ *be bounded and monotonic on the interval* $a \leq x \leq b$, *converging to zero as x tends to a, then* $\int_a^b f(x)\, \phi(x)\, dx$ *converges.*

Illustrative Examples

Example 37: *Test the convergence of* $\int_0^{\pi/2} \frac{\cos x}{x^n}\, dx$.

Solution: When $n \leq 0$, the given integral is a proper integral and hence convergent.

When $n > 0$, the integrand becomes unbounded at $x = 0$.

Let $$f(x) = \frac{\cos x}{x^n}.$$

Then $$\lim_{x \to 0} x^{\mu} f(x) = \lim_{x \to 0} x^{\mu - n} \cos x = 1, \text{ if } \mu = n.$$

Hence by μ-test it follows that the given integral is convergent when $0 < n < 1$, and divergent when $n \geq 1$.

From the above discussion we conclude that the given integral is convergent when $n < 1$, and divergent when $n \geq 1$.

Example 38: *Show that the integral* $\int_0^{\pi/2}$ *log sin x dx converges.*

(Meerut 2012; Rohilkhand 12)

Solution: The only point of infinite discontinuity of the integrand is $x = 0$.

Now $\lim_{x \to 0} x^{\mu} \log \sin x$, when $\mu > 0$

$$= \lim_{x \to 0} \frac{\log \sin x}{x^{-\mu}}, \qquad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x \to 0} \frac{\cot x}{-\mu\, x^{-\mu - 1}}$$

$$= \lim_{x \to 0} -\frac{1}{\mu} \cdot \frac{x^{\mu + 1}}{\tan x} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 0} -\frac{1}{\mu} \cdot \frac{(\mu + 1)\, x^{\mu}}{\sec^2 x}, \qquad \text{[by L'Hospital's rule]}$$

$$= 0, \text{ if } \mu > 0.$$

Taking μ between 0 and 1, it follows from μ-test that the given integral is convergent.

Example 39: *Discuss the convergence of the integral*

$$\int_0^1 x^{n-1} \log x\, dx.$$

(Purvanchal 2009; Garhwal 12)

Solution: (i) Since $\lim_{x \to 0} x^r \log x = 0$ where $r > 0$, the integral is a proper integral, when $n > 1$.

(ii) When $n = 1$, we have

$$\int_0^1 \log x\, dx = \lim_{\varepsilon \to 0} \int_\varepsilon^1 \log x\, dx = \lim_{\varepsilon \to 0} [x \log x - x]_\varepsilon^1$$

$$= \lim_{\varepsilon \to 0} [-1 - \varepsilon \log \varepsilon + \varepsilon] = -1.$$

$\therefore$ the integral is convergent if $n = 1$.

(iii) Let $n < 1$ and $f(x) = x^{n-1} \log x$.

Then
$$\lim_{x \to 0} x^{\mu} f(x) = \lim_{x \to 0} x^{\mu + n - 1} \log x$$

$$= 0 \qquad \text{if } \mu > 1 - n \qquad \text{...(1)}$$

and
$$= -\infty \qquad \text{if } \mu \le 1 - n. \qquad \text{...(2)}$$

Hence when $0 < n < 1$, we can choose μ between 0 and 1 and satisfying (1). The integral is therefore convergent by μ-test when $0 < n < 1$.

Again when $n \le 0$, we can take $\mu = 1$ and satisfying (2). Hence by μ-test the integral is divergent when $n \le 0$.

Therefore from (i), (ii) and (iii), we conclude that the given integral is convergent when $n > 0$ and divergent when $n \le 0$.

Example 40: *Discuss the convergence or divergence of the integral*

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx.$$

(Garhwal 2006, 11)

Solution: Let $f(x)=\dfrac{x^{a-1}}{1+x}\cdot$ If $b>0$, we can write

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx=\int_0^b \frac{x^{a-1}}{1+x}\,dx+\int_b^\infty \frac{x^{a-1}}{1+x}\,dx$$

$$=I_1+I_2\,,\ \text{say}.$$

Let $a\geq 1$. Then $f(x)$ is bounded throughout the interval $(0,b)$ and so the integral I_1 is a proper integral and hence it is convergent. To test the convergence of the infinite integral I_2 in this case, we have

$$\lim_{x\to\infty} x^\mu f(x)=\lim_{x\to\infty} x^\mu\cdot\frac{x^{a-1}}{1+x}=\lim_{x\to\infty}\frac{x^{\mu+a-1}}{x+1}$$

$$=1,\ \text{if } \mu+a-1=1\quad i.e.,\quad \text{if } \mu=2-a$$

which is ≤ 1 since $a\geq 1$.

Hence by μ-test I_2 is divergent.

$\therefore$ the given integral is divergent if $a\geq 1$.

Let $a<1$. Then in the interval $(0,b)$, $f(x)$ is unbounded only at $x=0$. Also $f(x)$ is bounded throughout the interval (b,∞). Therefore in this case I_1 is an improper integral of the second kind and I_2 is an improper integral of the first kind. To test the convergence of I_1, we have

$$\lim_{x\to 0} x^\mu\cdot\frac{x^{a-1}}{x+1}=\lim_{x\to 0}\frac{x^{\mu+a-1}}{x+1}=1,$$

if $\mu+a-1=0$ *i.e.*,

if $\mu=1-a$.

If we take $0<a<1$, then we have $0<\mu<1$ and so by μ-test I_1 is convergent. If we take $a\leq 0$, then $\mu\geq 1$ and so by μ-test I_1 is divergent.

To test the convergence of I_2 when $a<1$, we have

$$\lim_{x\to\infty} x^\mu\cdot\frac{x^{a-1}}{x+1}=\lim_{x\to\infty}\frac{x^{\mu+a+1}}{x+1}=1,\ \text{if } \mu+a-1=1\ i.e.,$$

if $\mu=2-a$ which is >1 since $a<1$.

Hence by μ-test I_2 is convergent if $a<1$.

Thus I_2 is convergent if $a<1$. But I_1 is convergent if $0<a<1$ and is divergent if $a\leq 0$.

$\therefore$ the given integral is convergent if $0<a<1$ and is divergent if $a\leq 0$.

Hence the given integral is convergent if $0<a<1$ and is divergent if $a\geq 1$ or if $a\leq 0$.

Example 41: *Discuss the convergence of the Beta function*

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

(Purvanchal 2007, 12; Kanpur 12)

Solution: Let $f(x) = x^{m-1}(1-x)^{n-1}$.

The following different cases arise :

(i) When m and n are both ≥ 1, the integrand $f(x)$ is bounded throughout the interval $(0, 1)$ and so the given integral is a proper integral and is convergent.

(ii) When m and n are both < 1, the integrand $f(x)$ becomes infinite both at $x = 0$ and at $x = 1$. In this case we take $0 < a < 1$ and we write

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^a x^{m-1}(1-x)^{n-1}\,dx + \int_a^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= I_1 + I_2, \text{ say.}$$

In the case of the integral I_1, the interval of integration is $(0, a)$ and so the integrand is unbounded at $x = 0$ only. To test the convergence of I_1, we have

$$\lim_{x \to 0} x^{\mu} . f(x) = \lim_{x \to 0} x^{\mu} . x^{m-1}(1-x)^{n-1}$$

$$= \lim_{x \to 0} x^{\mu + m - 1}(1-x)^{n-1}$$

$$= 1, \text{ if } \mu + m - 1 = 0 \text{ } i.e., \text{ if } \mu = 1 - m.$$

If we take $0 < m < 1$, we have $0 < \mu < 1$ and so by μ-test I_1 is convergent. If we take $m \leq 0$, we have $\mu \geq 1$ and so by μ-test I_1 is divergent.

Again in the case of the integral I_2, the interval of integration is $(a, 1)$ and so the integrand is unbounded at $x = 1$ only. To test the convergence of I_2, we have

$$\lim_{x \to 1-0} (1-x)^{\mu} . f(x) = \lim_{x \to 1-0} (1-x)^{\mu} x^{m-1}(1-x)^{n-1}$$

$$= \lim_{x \to 1-0} (1-x)^{\mu + n - 1} x^{m-1}$$

$$= \lim_{\varepsilon \to 0} \{1 - (1-\varepsilon)\}^{\mu + n - 1}(1-\varepsilon)^{m-1}$$

$$= \lim_{\varepsilon \to 0} \varepsilon^{\mu + n - 1}(1-\varepsilon)^{m-1}$$

$$= 1, \text{ if } \mu + n - 1 = 0 \text{ } i.e., \text{ if } \mu = 1 - n.$$

If we take $0 < n < 1$, we have $0 < \mu < 1$ and so by μ-test I_2 is convergent. If we take $n \leq 0$, we have $\mu \geq 1$ and so by μ-test I_2 is divergent.

Thus if m and n are both < 1, the given integral is convergent only if $0 < m < 1$ and $0 < n < 1$.

(iii) When $m < 1$ and $n \geq 1$, the integrand $f(x)$ is unbounded only at $x = 0$. In this case by μ-test, the given integral is convergent if $0 < m < 1$ and is divergent if $m \leq 0$.

Again if $m \geq 1$ and $n < 1$, the integrand $f(x)$ is unbounded only at $x = 1$. In this case by μ-test, the given integral is convergent if $0 < n < 1$ and is divergent if $n \leq 0$.

Hence from (i), (ii) and (iii) it follows that the given integral is convergent if both m and n are > 0 and divergent otherwise.

Example 42: *Discuss the convergence of the Gamma function*

$$\int_0^\infty x^{n-1} e^{-x} \, dx.$$

(Kanpur 2008; Garhwal 10; Purvanchal 08, 11, 12; Rohilkhand 10, 11)

Solution: We can write

$$\int_0^\infty x^{n-1} e^{-x} \, dx = \int_0^1 x^{n-1} e^{-x} \, dx + \int_1^\infty x^{n-1} e^{-x} \, dx$$

$$= I_1 + I_2 \text{, say.}$$

Let us first discuss the convergence of I_1.

Let $\quad f(x) = x^{n-1} e^{-x}$.

If $n \geq 1$, $f(x)$ is bounded throughout the interval $[0, 1]$ and so I_1 is a proper integral and hence it is convergent if $n \geq 1$.

If $n < 1$, $f(x)$ is unbounded at $x = 0$. In this case we have

$$\lim_{x \to 0} x^\mu f(x) = \lim_{x \to 0} x^\mu \cdot x^{n-1} e^{-x} = \lim_{x \to 0} x^{\mu + n - 1} e^{-x}$$

$$= 1 \text{, if } \mu + n - 1 = 0 \text{ } i.e., \mu = 1 - n.$$

So by μ-test when $0 < \mu < 1$ *i.e.*, $0 < n < 1$, the integral I_1 is convergent and when $\mu \geq 1$ *i.e.* $n \leq 0$, the integral I_1 is divergent.

$\therefore$ I_1 is convergent if $n > 0$ and is divergent if $n \leq 0$.

Now let us discuss the convergence of the integral I_2. The function $f(x) = x^{n-1} e^{-x}$ is bounded for all values of x in the interval $(1, \infty)$. We have

$$\lim_{x \to \infty} x^\mu f(x) = \lim_{x \to \infty} \frac{x^\mu \cdot x^{n-1}}{e^x} = \lim_{x \to \infty} \frac{x^{\mu + n - 1}}{1 + x + \frac{x^2}{2!} + \dots}$$

$$= 0 \text{ for all values of } \mu \text{ and } n.$$

Taking $\mu > 1$, we see by μ-test that the integral

$$I_2 = \int_1^\infty x^{n-1} e^{-x} \, dx \text{ is convergent for all values of } n.$$

Hence the given integral is convergent if $n > 0$ and is divergent if $n \leq 0$.

14 Infinite Integrals Depending on a Parameter

Uniform convergence of improper integrals when range is infinite.

The improper integral $\phi(\alpha) = \int_a^\infty f(x, \alpha) \, dx$

which is an infinite series of functions of parameter α, is said to be uniformly convergent in the interval $[\alpha_1, \alpha_2]$ if for every $\varepsilon > 0$, there exists a positive number δ depending on ε but not on α such that

$$\left| \phi(\alpha) - \int_a^t f(x, \alpha) \, dx \right| = \left| \int_t^\infty f(x, \alpha) \, dx \right| < \varepsilon$$

$$\forall \, t \geq \delta \text{ and } \alpha \in [\alpha_1, \alpha_2].$$

Uniform convergence of improper integrals when the integrand is unbounded.

The improper integral $\phi(\alpha) = \int_a^b f(x, \alpha)\, dx$

such that $f(x, \alpha) \to \infty$, when $x \to a$ for some $\alpha \in [\alpha_1, \alpha_2]$ is said to be uniformly convergent in the interval $[\alpha_1, \alpha_2]$ if for every $\varepsilon > 0$,

there exists a positive number δ depending on ε but not on α such that

$$|\phi(\alpha) - \int_{a+t}^b f(x, \alpha)\, dx| < \varepsilon, \forall 0 \le t \le \delta.$$

Note: Weierstrass's '*M*' test can be applied to test uniform convergence.

Thus, if there exists a function $M(x) > 0$ for all x, such that $\int_a^\infty M(x)\, dx$

converges and for all α and x

$$|f(x, \alpha)| \le M(x),$$

then the improper integral

$$\phi(\alpha) = \int_a^\infty f(x, \alpha)\, dx$$

is uniformly convergent.

Continuity of Improper Integrals as a Function of a Parameter

The improper integral

$$\phi(\alpha) = \int_a^\infty f(x, \alpha)\, dx$$

is continuous in $[\alpha_1, \alpha_2]$ if

(i) $f(x, \alpha)$ is continuous for $x \ge \alpha$ and $\alpha \in [\alpha_1, \alpha_2]$ and

(ii) $\int_a^\infty f(x, \alpha)\, dx$ is uniformly convergent for $\alpha \in [\alpha_1, \alpha_2]$.

Integrability of Improper Integrals as a Function of a Parameter

If (i) $f(x, \alpha)$ is continuous for $x \ge \alpha$ and $\alpha \in [\alpha_1, \alpha_2]$ and

(ii) $\int_a^\infty f(x, \alpha)\, dx$ is uniformly convergent for $\alpha \in [\alpha_1, \alpha_2]$,

then the improper integral $\phi(\alpha) = \int_a^\infty f(x, \alpha)\, dx$

can be integrated under the integral sign and we have

$$\int_{\alpha_1}^{\alpha_2} \left\{ \int_a^\infty f(x, \alpha)\, dx \right\} d\alpha = \int_{\alpha_1}^{\alpha_2} \phi(\alpha)\, d\alpha = \int_a^\infty \{ \int_{\alpha_1}^{\alpha_2} f(x, \alpha)\, d\alpha \}\, dx$$

Derivability of Improper Integrals as a Function of a Parameter

If (i) $f(x, \alpha)$ is continuous and has a continuous partial derivative with respect to α for $x \ge \alpha$ and $\alpha \in [\alpha_1, \alpha_2]$ and (ii) $\int_a^\infty f_\alpha(x, \alpha)\, dx$ converges uniformly in $\alpha \in [\alpha_1, \alpha_2]$, then

$\phi(\alpha)$ is differentiable and

$$\phi'(\alpha) = \int_a^\infty f_\alpha(x, \alpha)\, dx.$$

Illustrative Examples

Example 43: *Evaluate* $\int_0^\infty e^{-x^2}$ *cos* αx *dx with differentiation w.r.t. parameter.*

Solution: Let $\phi(\alpha) = \int_0^\infty f(x,\alpha)\,dx$ ...(1)

where $f(x,\alpha) = e^{-x^2} \cos \alpha x$

Then $f_\alpha(x,\alpha) = -x\,e^{-x^2} \sin \alpha x$

We have $|xe^{-x^2} \sin \alpha x| \le x\,e^{-x^2}$

$\Rightarrow$ $|e^{-x^2} \cos \alpha x| \le e^{-x^2}$ [$\because$ $|\cos \alpha\, x| \le 1$]

Both the integrals $\int_0^\infty e^{-x^2} dx$ and $\int_0^\infty xe^{-x^2} dx$

are convergent, therefore by Weierstrass M test

$\int_0^\infty f(x,\alpha)\,dx$ and $\int_0^\infty f_\alpha(x,\alpha)\,dx$ are uniformly convergent,

$\Rightarrow$ $\phi'(\alpha)$ exists.

Now $\phi'(\alpha) = \int_0^\infty f_\alpha(x,\alpha)\,dx = \int_0^\infty -x\,e^{-x^2} \sin \alpha x\,dx$

$$= \left[\frac{1}{2} e^{-x^2} \sin x\right]_0^\infty - \frac{\alpha}{2}\int_0^\infty e^{-x^2} \cos\, \alpha x\,dx,$$ [Integrating by parts]

$$= -\frac{\alpha}{2}\phi(\alpha) \quad \text{or} \quad \frac{\phi'(\alpha)}{\phi(\alpha)} = -\frac{\alpha}{2}.$$

Integrating w.r.t. α, we get

$$\log \phi(\alpha) = -\frac{\alpha^2}{4} + c_1,$$

where c_1 is an arbitrary constant.

or $\phi(\alpha) = c\, e^{-\alpha^2/4}$, ...(2)

where c is another constant

Putting $\alpha = 0$ in (1) and (2), we get

$$\phi(0) = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}. \quad \text{and} \quad \phi(0) = c\,e^0 = c \;\therefore\; c = \frac{\sqrt{\pi}}{2}.$$

Substituting this value of c in (2), we get

$$\phi(\alpha) = \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}$$

Hence $\int_0^\infty e^{-x^2} \cos \alpha x\,dx = \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}$ [From (1)]

Example 44: *Evaluate* $f(\alpha,\beta)=\int_0^\infty e^{-\alpha x}\frac{\sin\beta x}{x}\,dx$*, where* $\alpha\ge 0$*. Hence deduce that*

$$\int_0^\infty \frac{\sin\beta x}{x}\,dx=\begin{cases}\pi/2, & \text{if } \beta>0\\ 0, & \text{if } \beta=0.\\ -\pi/2, & \text{if } \beta<0\end{cases}$$

Solution: We have

$$\left|e^{-\alpha x}\frac{\sin\beta x}{x}\right|\le\frac{e^{-\alpha x}}{x}\quad\text{for } x>0.$$

$$\int_0^\infty \frac{e^{-\alpha x}}{x}\,dx \text{ is convergent at } \infty \text{ if } \alpha>0\ ;$$

therefore by Weierstrass M test,

$$\int_0^\infty e^{-\alpha x}\frac{\sin\beta x}{x}\,dx \text{ is uniformly convergent.}$$

Now the derivative of $\frac{e^{-\alpha x}\sin\beta x}{x}$ with respect to β is $e^{-\alpha x}\cos\beta x$. For a fixed $\alpha>0$, we have

$$|e^{-\alpha x}\cos\beta x|\le e^{-\alpha x}\qquad [\because |\cos\beta x|\le 1]$$

But $\int_0^\infty e^{-\alpha x}dx$ is convergent over $[0,\infty)$ therefore for a fixed α we have

$$\int_0^\infty e^{-\alpha x}\cos\beta x\,dx \text{ is uniformly convergent}$$

$\therefore$ $\quad f_\beta(\alpha,\beta)$ exists.

Thus $f_\beta(\alpha,\beta)=\int_0^\infty e^{-\alpha x}\cos\beta x\,dx$

$$=[e^{-\alpha x}\{\beta\sin\beta x-\alpha\cos\beta x\}]_0^\infty=\frac{\alpha}{(\alpha^2+\beta^2)}.$$

Integrating with respect to β, we get

$$f(\alpha,\beta)=\tan^{-1}\frac{\beta}{\alpha}+c,\qquad \ldots(1)$$

where c is an arbitrary constant.

Putting $\beta=0$ in (1), we get

$$0=f(\alpha,0)=0+c\quad\text{or}\quad c=0\quad\therefore\quad f(\alpha,\beta)=\tan^{-1}\frac{\beta}{\alpha}.$$

Deduction: For $\alpha>0$ and for a fixed β, we have

$$\left|\int_{t_1}^t e^{-\alpha x}\frac{\sin\beta x}{x}\,dx\right|=\left|e^{-\alpha t}\int_{t_1}^\xi \frac{\sin\beta x}{x}\,dx\right|\le\left|\int_{t_1}^\xi\frac{\sin\beta x}{x}\,dx\right|,$$

[Applying the Bonnett's form of second mean value theorem]

Since $\int_0^\infty \frac{\sin\beta x}{x}\,dx$ is convergent, we have $\int_0^\infty e^{-\alpha x}\frac{\sin\beta x}{x}\,dx$ is uniformly convergent with α as a parameter.

$\therefore \quad \lim\limits_{x \to 0} f(\alpha, \beta) = f(0, \beta)$ ...(2)

But when $\alpha \to 0$ from positive side and $\beta > 0$, we have

$$\lim_{\alpha \to 0} f(\alpha, \beta) = \lim_{\alpha \to 0} \int_0^\infty e^{-\alpha x} \frac{\sin \beta x}{x} dx = \frac{\pi}{2} \qquad \text{...(3)}$$

From (2) and (3), we get

$$f(0, \beta) = \frac{\pi}{2}, \text{ if } \beta > 0$$

or $$\int_0^\infty \frac{\sin \beta x}{x} dx = \frac{\pi}{2}, \text{ if } \beta > 0.$$

Similarly $$\int_0^\infty \frac{\sin \beta x}{x} dx = 0, \text{ if } \beta = 0$$

and $$\int_0^\infty \frac{\sin \beta x}{x} dx = -\frac{\pi}{2}, \text{ if } \beta < 0.$$

Example 45: *Evaluate* $\int_0^\infty \frac{1 - e^{-\alpha x}}{x e^x} dx$ *if* $a > -1$, *with the help of differentiation w.r.t. parameter.*

Solution: Let $\phi(\alpha) = \int_0^\infty \frac{1 - e^{-\alpha x}}{x e^x} dx = \int_0^\infty f(x, \alpha) \, dx$ ...(1)

Then $$f_\alpha(x, \alpha) = \frac{e^{-\alpha x}}{e^x} = e^{-(\alpha + 1) x} \qquad \text{...(2)}$$

Now $\int_0^\infty f(x, \alpha) \, dx$ and $\int_0^\infty f_\alpha(x, \alpha) \, dx$ are uniformly convergent.

$\Rightarrow \quad \phi'(\alpha)$ exists.

$$\therefore \quad \phi'(\alpha) = \int_0^\infty f_\alpha(x, \alpha) \, dx = \int_0^\infty e^{-(\alpha + 1) x} dx$$

$$= \left[\frac{e^{-(\alpha + 1) x}}{-(\alpha + 1)} \right]_0^\infty = \frac{1}{\alpha + 1}, \text{ if } \alpha > -1.$$

Integrating w.r.t. α, we get

$$\phi(\alpha) = \log(1 + \alpha) + c, \quad \text{where } c \text{ is an arbitrary constant} \qquad \text{....(3)}$$

$$\Rightarrow \quad \phi(0) = c \qquad \text{...(4)}$$

Putting $\alpha = 0$ in (1), we have

$$\phi(0) = 0.$$

$$\therefore \quad c = 0.$$

Hence $$\phi(\alpha) = \log(1 + \alpha)$$

or $$\int_0^\infty \frac{1 - e^{-\alpha x}}{x e^x} dx = \log(1 + \alpha); \text{ if } \alpha > -1.$$

Comprehensive Exercise 2

1. Show that the integral $\int_{\pi}^{\infty} \frac{\sin^2 x}{x^2}\, dx$ is convergent.

2. Test the convergence of the following integrals:

(i) $\int_0^{\infty} \frac{\cos x}{1+x^2}\, dx$

(Purvanchal 2010; Bundelkhand 11; Rohilkhand 12; Gorakhpur 12)

(ii) $\int_{\pi}^{\infty} \frac{\sin x}{x^2}\, dx$ (iii) $\int_0^{\infty} \frac{\sin mx}{x^2+a^2}\, dx$ **(Garhwal 2008)**

(iv) $\int_0^{\infty} \frac{x^3}{(x^2+a^2)^2}\, dx$ (v) $\int_1^{\infty} \frac{dx}{\sqrt{(x^3+1)}}$ **(Gorakhpur 2012, 15)**

(vi) $\int_0^{\infty} \frac{1-\cos x}{x^2}\, dx$ (vii) $\int_2^{\infty} \frac{dx}{\sqrt{(x^2-x-1)}}$

(viii) $\int_2^{\infty} \frac{dx}{\sqrt{(x^2-1)}}$ (ix) $\int_0^{\infty} \frac{x^2 dx}{(1+x)^3}$

(x) $\int_0^{\infty} \frac{x^{3/2}}{(b^2x^2+c)}\, dx.$

3. Show that the following integrals are convergent:

(i) $\int_0^{\infty} \frac{x^2}{(a^2+x^2)^2}\, dx$ (ii) $\int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}$.

4. Test the convergence of $\int_b^{\infty} \frac{x^{3/2}\, dx}{\sqrt{(x^4-a^4)}}$, where $b > a$.

5. Show that the integral $\int_a^{\infty} x^{n-1} e^{-x}\, dx$ is convergent, where $a > 0$.

6. Test the convergence of the following integrals:

(i) $\int_0^{1} \frac{dx}{x^3(1+x^2)}$ (ii) $\int_0^{1} \frac{dx}{(x+1)\sqrt{(1-x^2)}}$

(iii) $\int_0^{\pi/2} \frac{\cos x}{x^2}\, dx$ (iv) $\int_0^{\pi/4} \frac{1}{\sqrt{(\tan x)}}\, dx$ **(Agra 2012)**

(v) $\int_0^{\pi/2} \frac{\sin x}{x^{1+n}}\, dx.$

7. Test the convergence of the following integrals :

(i) $\int_0^{\infty} \frac{dx}{x^{1/3}(1+x^{1/2})}$ (ii) $\int_0^{\infty} \frac{dx}{x\sqrt{(1+x^2)}}$

(iii) $\int_0^{\infty} \frac{x^{1/2}}{x^2+4}\, dx$ (iv) $\int_0^{\infty} \frac{x}{1+x^2} \sin x\, dx.$

8. Examine the convergence of the integral $\int_0^\infty \frac{\sin x}{x^{3/2}}\,dx.$

9. Show that the integral $\int_0^\infty e^{-a^2x^2}\cos bx\,dx$ is absolutely convergent.

10. Evaluate $\int_0^\infty \frac{\cos xy}{1+x^2}\,dx$ with the help of differentiation w.r.t., parameter.

11. Evaluate $\int_0^\infty e^{-x^2}\cos \alpha x\,dx$ using differentiation with respect to parameter.

Answers 2

2. (i) Convergent (ii) Convergent
(iii) Convergent (iv) Divergent
(v) Convergent (vi) Convergent
(vii) Divergent (viii) Divergent
(ix) Divergent (x) Divergent

4. Divergent

6. (i) Divergent (ii) Convergent
(iii) Divergent (iv) Convergent
(v) Convergent if $n < 1$ and divergent if $n \geq 1$

7. (i) Divergent (ii) Divergent
(iii) Divergent (iv) Convergent

8. Convergent

10. $\frac{\pi}{2} e^{-\alpha}$ 11. $\frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The integral $\int_a^\infty \frac{dx}{x^n}$, where $a > 0$, is convergent when

(a) $n = 1$ (b) $n < 1$
(c) $n \leq 1$ (d) $n > 1$

(Garhwal 2009; Rohilkhand 11)

2. The integral $\int_a^b \frac{dx}{(x-a)^n}$ is convergent when

(a) $n < 1$ (b) $n > 1$
(c) $n = 1$ (d) $n \geq 1$ **(Garhwal 2006, 10, 11)**

3. The integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is convergent when

(a) $m > 0$
(b) $n > 0$
(c) $m > 0, n > 0$
(d) $m = 0, n > 1$

4. The integral $\int_0^\infty x^{n-1} e^{-x}\,dx$ is divergent when

(a) $n > 0$
(b) $n > 1$
(c) $n \le 0$
(d) $n = \frac{1}{2}$

5. The integral $\int_a^\infty \frac{\sin^2 x}{x^2}\,dx, a > 0$

(a) convergent
(b) divergent
(c) uniformly convergent
(d) none of these

(Rohilkhand 2012)

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. The definite integral $\int_a^b f(x)\,dx$ is said to be a ...... if the range of integration (a, b) is finite and the integrand $f(x)$ is bounded over (a, b).
2. The definite integral $\int_a^b f(x)\,dx$ is said to be an improper integral if the interval (a, b) is finite and $f(x)$ is not ...... over this interval.
3. The definite integral $\int_a^b f(x)\,dx$ is said to be an ...... if the interval (a, b) is not finite and $f(x)$ is bounded over (a, b).
4. A definite integral $\int_a^b f(x)\,dx$ in which the range of integration (a, b) is finite but the integrand $f(x)$ is unbounded at one or more points of the interval $a \le x \le b$, is called an improper integral of the ...... kind.
5. The integral $\int_0^\infty \frac{dx}{1+x^2}$ is an improper integral of the ...... kind.
6. The integral $\int_0^4 \frac{dx}{(x-2)(x-3)}$ is an improper integral of the ...... kind.
7. The integral $\int_a^\infty \frac{dx}{x^n}$, where $a > 0$, is convergent when ....
8. The integral $\int_a^b \frac{dx}{(x-a)^n}$ is divergent when ....
9. The integral $\int_0^\infty e^{-x} x^{n-1}\,dx$ is convergent if ....
10. The integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is convergent if both m and n are > ......

True or False

Write 'T' for true and 'F' for false statement.

1. The integral $\int_3^\infty \dfrac{dx}{(x-2)^2}$ is divergent.
2. The integral $\int_0^\infty \dfrac{\cos x}{1+x^2}\,dx$ is convergent. **(Garhwal 2012)**
3. The integral $\int_a^\infty \dfrac{dx}{x^n}$, where $a > 0$, is convergent when $n \le 1$.
4. The integral $\int_a^b \dfrac{dx}{(x-a)^n}$ is divergent when $n < 1$.
5. The integral $\int_0^1 \dfrac{dx}{x^3(1+x^2)}$ is convergent.
6. The integral $\int_0^1 \dfrac{dx}{x^{1/3}(1+x^2)}$ is convergent.
7. The integral $\int_1^\infty \dfrac{dx}{\sqrt{(x^3+1)}}$ is convergent.
8. The integral $\int_2^\infty \dfrac{dx}{\sqrt{(x^2-1)}}$ is convergent.

Answers

Multiple Choice Questions

1. (d) 2. (a) 3. (c) 4. (c) 5. (a)

Fill in the Blank(s)

1. proper integral 2. bounded
3. improper integral of the first kind 4. second 5. first
6. second 7. $n > 1$ 8. $n \ge 1$ 9. $n > 0$ 10. 0

True or False

1. *F* 2. *T* 3. *F* 4. *F* 5. *F*
6. *T* 7. *T* 8. *F*

□