Krishna's

Question Bank

Integral Calculus

(*For B.A. and B.Sc. I year students of All Colleges affiliated to Allahabad State University*)

As per Allahabad State University Syllabus

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By

A. R. Vasishtha

Retired Head, Dep't. of Mathematics

Meerut College, Meerut (U.P.)

Hemlata Vasishtha

M.Sc. (Gold Medalist), Ph.D.

C.C.S. University, Meerut (U.P.)

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INTEGRAL CALCULUS

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Definite Integrals

Comprehensive Problems 1

Problem 1: *Evaluate (i)* $\int_0^{\pi} \cos^6 x\, dx$. *(ii)* $\int_0^{\pi} \sin^3 x\, dx$.

Solution: **(i)** Let $f(x) = \cos^6 x$. Then

$$f(\pi - x) = \cos^6(\pi - x) = (-\cos x)^6 = \cos^6 x = f(x).$$

Now $\int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx$, if $f(2a - x) = f(x)$. [Refer property 6]

$\therefore$ $\int_0^{\pi} \cos^6 x\, dx = 2\int_0^{\pi/2} \cos^6 x\, dx$

$$= 2\,\frac{5.3.1}{6.4.2}\cdot\frac{\pi}{2},$$ [By Walli's formula]

$$= 5\pi/16.$$

(ii) Let $f(x) = \sin^3 x$.

Then $f(\pi - x) = \sin^3(\pi - x) = \sin^3 x = f(x)$.

Now $\int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx$, if $f(2a - x) = f(x)$.

$\therefore$ $\int_0^{\pi} \sin^3 x\, dx = 2\int_0^{\pi/2} \sin^3 x\, dx$

$$= 2\cdot\frac{2}{3.1}\cdot 1 = \frac{4}{3}.$$ [By Walli's formula]

Problem 2: *Evaluate (i)* $\int_{-1}^{1} \frac{x^2 \sin^{-1} x}{\sqrt{(1 - x^2)}}\, dx$. *(ii)* $\int_{-a}^{a} x\sqrt{(a^2 - x^2)}\, dx$.

(iii) $\int_{-1}^{1} \frac{x \sin^{-1} x}{\sqrt{(1 - x^2)}}\, dx$.

Solution: **(i)** Let $I = \int_{-1}^{1} \frac{x^2 \sin^{-1} x}{\sqrt{(1 - x^2)}}\, dx$.

Put $\sin^{-1} x = t$, so that $\{1/\sqrt{(1 - x^2)}\}\, dx = dt$ and $x = \sin t$.

When $x = -1$, $t = \sin^{-1}(-1) = -\pi/2$ and when $x = 1$, $t = \sin^{-1} 1 = \pi/2$.

$\therefore$ $I = \int_{-\pi/2}^{\pi/2} t \sin^2 t\, dt.$

Now let $f(t) = t \sin^2 t$. Then

$$f(-t) = (-t)\sin^2(-t)$$

$$= -t(-\sin t)^2 = -t\sin^2 t = -f(t).$$

Therefore $f(t)$ is an odd function of t.

$\therefore$ $$I = \int_{-\pi/2}^{\pi/2} t\sin^2 t\,dt = 0.$$ [Refer property 5]

(ii) Let $$I = \int_{-a}^{a} x\sqrt{(a^2 - x^2)}\,dx.$$

Let $f(x) = x\sqrt{(a^2 - x^2)}$.

Then $$f(-x) = -x\sqrt{\{a^2 - (-x)^2\}}$$
$$= -x\sqrt{(a^2 - x^2)} = -f(x).$$

Therefore $f(x)$ is an odd function of x.

$\therefore$ $$I = \int_{-a}^{a} x\sqrt{(a^2 - x^2)}\,dx = 0.$$ [Refer property 5]

(iii) Let $$I = \int_{-1}^{1} \frac{x\sin^{-1} x}{\sqrt{(1 - x^2)}}\,dx.$$

Proceeding as in part (i) of this question, we have

$$I = \int_{-\pi/2}^{\pi/2} t\sin t\,dt.$$

Let $f(t) = t\sin t$. Then

$$f(-t) = (-t)\sin(-t)$$
$= t\sin t$, so that $f(t)$ is an even function of t.

$\therefore$ $$I = 2\int_0^{\pi/2} t\sin t\,dt,$$ [Refer property 5]

$$= 2\,[t(-\cos t)]_0^{\pi/2} - 2\int_0^{\pi/2} 1.(-\cos t)\,dt$$

$$= 2 \times 0 + 2\int_0^{\pi/2} \cos t\,dt = 2\,[\sin t]_0^{\pi/2} = 2\,[1 - 0] = 2.$$

Problem 3: *Evaluate* (i) $\displaystyle\int_0^{\pi} \frac{dx}{a + b\cos x}$. (ii) $\displaystyle\int_0^{2\pi} \frac{dx}{a + b\cos x + c\sin x}$.

Solution: **(i)** Let $$I = \int_0^{\pi} \frac{dx}{a + b\cos x}. \quad ...(1)$$

Then $$I = \int_0^{\pi} \frac{dx}{a + b\cos(\pi - x)}$$ [Refer property 4]

$$= \int_0^{\pi} \frac{dx}{a - b\cos x}. \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{2a}{a^2 - b^2\cos^2 x}\,dx$$

$$= 2a\,.2\int_0^{\pi/2} \frac{dx}{a^2 - b^2\cos^2 x}.$$ [Refer property 6]

$\therefore \quad I = 2a \int_0^{\pi/2} \frac{dx}{a^2 - b^2 \cos^2 x} = 2a \int_0^{\pi/2} \frac{\sec^2 x \, dx}{a^2 \sec^2 x - b^2}$

$$= 2a \int_0^{\pi/2} \frac{\sec^2 x \, dx}{a^2 (1 + \tan^2 x) - b^2} = 2a \int_0^{\pi/2} \frac{\sec^2 x \, dx}{a^2 \tan^2 x + a^2 - b^2}.$$

Now let $a > b > 0$.

Put $a \tan x = t$, so that $a \sec^2 x \, dx = dt$. The new limits for t are 0 to ∞.

$\therefore \quad I = 2 \int_0^{\infty} \frac{dt}{t^2 + \{\sqrt{(a^2 - b^2)}\}^2}$

$$= 2 \frac{1}{\sqrt{(a^2 - b^2)}} \left[\tan^{-1} \frac{t}{\sqrt{(a^2 - b^2)}} \right]_0^{\infty}$$

$$= \frac{2}{\sqrt{(a^2 - b^2)}} [\tan^{-1} \infty - \tan^{-1} 0]$$

$$= \frac{2}{\sqrt{(a^2 - b^2)}} \cdot \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{\sqrt{(a^2 - b^2)}}.$$

(ii) Let $\quad I = \int_0^{2\pi} \frac{dx}{a + b \cos x + c \sin x}.$

Let $b = r \cos \alpha$ and $c = r \sin \alpha$, so that $r = \sqrt{(b^2 + c^2)}$ and $\tan \alpha = c/b$.

Then $\quad b \cos x + c \sin x = r (\cos \alpha \cos x + \sin \alpha \sin x) = r \cos (x - \alpha)$.

$\therefore \quad I = \int_0^{2\pi} \frac{dx}{a + r \cos (x - \alpha)}.$

Put $x - \alpha = t$, so that $dx = dt$. When $x = 0$, $t = -\alpha$ and when $x = 2\pi$, $t = 2\pi - \alpha$.

$\therefore \quad I = \int_{-\alpha}^{2\pi - \alpha} \frac{dt}{a + r \cos t}$

$$= \int_{-\alpha}^{0} \frac{dt}{a + r \cos t} + \int_0^{2\pi - \alpha} \frac{dt}{a + r \cos t}. \quad \text{...(1)}$$

[Refer property 3]

Now put $t = z - 2\pi$ in the first integral on the R.H.S. of (1). Then $dt = dz$ and the limits for z are $2\pi - \alpha$ to 2π.

$\therefore \quad I = \int_{2\pi - \alpha}^{2\pi} \frac{dz}{a + r \cos (z - 2\pi)} + \int_0^{2\pi - \alpha} \frac{dt}{a + r \cos t}$

$$= \int_0^{2\pi - \alpha} \frac{dt}{a + r \cos t} + \int_{2\pi - \alpha}^{2\pi} \frac{dz}{a + r \cos z}$$

$= \int_0^{2\pi - \alpha} \frac{dt}{a + r \cos t} + \int_{2\pi - \alpha}^{2\pi} \frac{dt}{a + r \cos t}$, because a definite integral does not change by changing the variable

$$= \int_0^{2\pi} \frac{dt}{a + r \cos t},$$ [Refer property 3]

$$= 2 \int_0^{\pi} \frac{dt}{a + r \cos t}.$$ [Refer property 6]

Now proceeding as in part (i) of this question, we get

$$\int_0^{\pi} \frac{dt}{a + r \cos t} = \frac{\pi}{\sqrt{(a^2 - r^2)}}, \text{ provided } a > r > 0 .$$

$\therefore$ $$I = \frac{2\pi}{\sqrt{[a^2 - (b^2 + c^2)]}}, \text{ provided } a > \sqrt{(b^2 + c^2)} > 0$$

$$= \frac{2\pi}{\sqrt{(a^2 - b^2 - c^2)}}.$$

Problem 4: *(i) Show that* $\displaystyle\int_0^{\pi} \frac{x\, dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} = \frac{\pi^2 (a^2 + b^2)}{4a^3 b^3}.$

(Kumaun 2007, 09)

(ii) Show that $\displaystyle\int_0^{\pi} \frac{x\, dx}{a^2 - \cos^2 x} = \frac{\pi^2}{2a \sqrt{(a^2 - 1)}}, (a > 1).$

(iii) Evaluate $\displaystyle\int_0^{\pi} \frac{x\, dx}{1 + \cos^2 x}.$

Solution: **(i)** Let $\displaystyle I = \int_0^{\pi} \frac{x\, dx}{[a^2 \cos^2 x + b^2 \sin^2 x]^2}$

$$= \int_0^{\pi} \frac{(\pi - x)\, dx}{[a^2 \cos^2 (\pi - x) + b^2 \sin^2 (\pi - x)]^2},$$ [Refer property 4]

$$= \int_0^{\pi} \frac{(\pi - x)\, dx}{[a^2 \cos^2 x + b^2 \sin^2 x]^2} = \int_0^{\pi} \frac{\pi\, dx}{[a^2 \cos^2 x + b^2 \sin^2 x]^2} - I.$$

(Note)

$\therefore$ $$2I = \int_0^{\pi} \frac{\pi\, dx}{[a^2 \cos^2 x + b^2 \sin^2 x]^2}$$

$$= 2\int_0^{\pi/2} \frac{\pi\, dx}{[a^2 \cos^2 x + b^2 \sin^2 x]^2},$$ [Refer property 6]

or $$I = \pi \int_0^{\pi/2} \frac{\sec^4 x\, dx}{(a^2 + b^2 \tan^2 x)^2},$$

[Dividing the Nr. and the Dr. by $\cos^4 x$]

$$= \pi \int_0^{\pi/2} \frac{(1 + \tan^2 x) \sec^2 x}{(a^2 + b^2 \tan^2 x)^2}\, dx.$$

Now put $b \tan x = a \tan \theta$, so that $b \sec^2 x\, dx = a \sec^2 \theta\, d\theta$. Also when $x = 0$, $\theta = 0$ and when $x = \frac{1}{2}\pi$, $\theta = \frac{1}{2}\pi$.

$\therefore$ $$I = \pi \int_0^{\pi/2} \frac{\{1 + (a^2/b^2) \tan^2 \theta\}.\, (a/b) \sec^2 \theta\, d\theta}{a^4 \sec^4 \theta}$$

$$= \frac{\pi}{a^3 b^3} \int_0^{\pi/2} (b^2 \cos^2 \theta + a^2 \sin^2 \theta)\, d\theta$$

$$= \frac{\pi}{a^3 b^3}\left[b^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} + a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right], \quad \text{[By Walli's formula]}$$

$$= \frac{\pi^2}{4a^3 b^3}(a^2 + b^2).$$

(ii) Let $$I = \int_0^{\pi} \frac{x\,dx}{a^2 - \cos^2 x} = \int_0^{\pi} \frac{(\pi - x)\,dx}{a^2 - \cos^2(\pi - x)}, \quad \text{[Refer prop. 4]}$$

$$= \pi \int_0^{\pi} \frac{dx}{a^2 - \cos^2 x} - I.$$

$\therefore$ $$2I = \int_0^{\pi} \frac{\pi\,dx}{a^2 - \cos^2 x} = 2\int_0^{\pi/2} \frac{\pi\,dx}{a^2 - \cos^2 x}, \quad \text{[Refer prop. 6]}$$

or $$I = \pi \int_0^{\pi/2} \frac{\sec^2 x\,dx}{a^2 \sec^2 x - 1},$$ dividing the Nr. and the Dr. by $\cos^2 x$

$$= \pi \int_0^{\pi/2} \frac{\sec^2 x\,dx}{a^2(1 + \tan^2 x) - 1} = \pi \int_0^{\pi/2} \frac{\sec^2 x\,dx}{(a^2 - 1) + a^2 \tan^2 x}.$$

Now put $a \tan x = t$, so that $a \sec^2 x\,dx = dt$. Also $t = 0$ when $x = 0$ and $t \to \infty$ when $x \to \frac{1}{2}\pi$.

$\therefore$ $$I = \frac{\pi}{a}\int_0^{\infty} \frac{dt}{(a^2 - 1) + t^2} = \frac{\pi}{a} \cdot \frac{1}{\sqrt{(a^2 - 1)}}\left[\tan^{-1}\left\{\frac{t}{\sqrt{(a^2 - 1)}}\right\}\right]_0^{\infty}$$

$$= \frac{\pi}{a\sqrt{(a^2 - 1)}}[\tan^{-1}\infty - \tan^{-1} 0]$$

$$= \frac{\pi}{a\sqrt{(a^2 - 1)}}\left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{2a\sqrt{(a^2 - 1)}}.$$

(iii) Let $$I = \int_0^{\pi} \frac{x\,dx}{1 + \cos^2 x} = \int_0^{\pi} \frac{(\pi - x)\,dx}{1 + \cos^2(\pi - x)}, \quad \text{[Refer prop. 4]}$$

$$= \int_0^{\pi} \frac{(\pi - x)\,dx}{1 + \cos^2 x} = \int_0^{\pi} \frac{\pi\,dx}{1 + \cos^2 x} - \int_0^{\pi} \frac{x\,dx}{1 + \cos^2 x}$$

$$= \pi \int_0^{\pi} \frac{dx}{1 + \cos^2 x} - I.$$

$\therefore$ $$2I = \pi \int_0^{\pi} \frac{dx}{1 + \cos^2 x} = 2\pi \int_0^{\pi/2} \frac{dx}{1 + \cos^2 x}, \quad \text{[Refer prop. 6]}$$

or $$I = \pi \int_0^{\pi/2} \frac{\sec^2 x\,dx}{\sec^2 x + 1},$$ dividing the Nr. and the Dr. by $\cos^2 x$

$$= \pi \int_0^{\pi/2} \frac{\sec^2 x\,dx}{2 + \tan^2 x}.$$

Now put $\tan x = t$, so that $\sec^2 x\,dx = dt$. Also $t = 0$ when $x = 0$ and $t \to \infty$ when $x \to \frac{1}{2}\pi$.

$\therefore \qquad I = \pi \int_0^{\infty} \frac{dt}{2 + t^2} = \pi \cdot \frac{1}{\sqrt{2}} \left[\tan^{-1} \frac{t}{\sqrt{2}} \right]_0^{\infty}$

$$= \frac{\pi}{\sqrt{2}} [\tan^{-1} \infty - \tan^{-1} 0] = \frac{\pi}{\sqrt{2}} \left[\frac{\pi}{2} - 0 \right] = \frac{\pi^2}{2\sqrt{2}} = \frac{\pi^2 \sqrt{2}}{4}.$$

Problem 5: *(i) Evaluate* $\int_0^{\pi/2} (\sin x - \cos x) \log (\sin x + \cos x)\, dx.$

(ii) Evaluate $\int_0^{\pi/2} \sin 2x \log \tan x\, dx.$

Solution: **(i)** Let $I = \int_0^{\pi/2} (\sin x - \cos x) \log (\sin x + \cos x)\, dx$.

Then $\qquad I = \int_0^{\pi/2} (\cos x - \sin x) \log (\sin x + \cos x)\, dx$

$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx. \text{ Note that } \sin (\tfrac{1}{2} \pi - x) = \cos x \right.$

$\left. \text{and } \cos \left(\frac{1}{2} \pi - x \right) = \sin x \right]$

$$= - \int_0^{\pi/2} (\sin x - \cos x) \log (\sin x + \cos x)\, dx = - I.$$

$\therefore \qquad 2I = 0 \quad \text{or} \quad I = 0.$

(ii) Let $\qquad I = \int_0^{\pi/2} \sin 2x \log \tan x\, dx$. ...(1)

Then $\qquad I = \int_0^{\pi/2} \sin 2 \left(\frac{1}{2} \pi - x \right) \log \tan (\tfrac{1}{2} \pi - x)$, [Refer prop. 4]

$$= \int_0^{\pi/2} \sin (\pi - 2x) \log \cot x\, dx$$

$$= \int_0^{\pi/2} \sin 2x \log \cot x\, dx. \qquad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \sin 2x (\log \tan x + \log \cot x)\, dx$$

$$= \int_0^{\pi/2} \sin 2x \log (\tan x \cot x)\, dx = \int_0^{\pi/2} (\sin 2x) . \log 1\, dx$$

$$= \int_0^{\pi/2} 0 . \sin 2x\, dx = 0 \times \int_0^{\pi/2} \sin 2x\, dx = 0 .$$

$\therefore \qquad I = 0.$

Problem 6: *(i) Evaluate* $\int_0^{\pi} \frac{x \sin x}{(1 + \cos^2 x)} dx.$ **(Kumaun 2007)**

(ii) Evaluate $\int_0^{\pi} x \sin^6 x \cos^4 x\, dx.$

Solution: **(i)** Let $I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$. ...(1)

Then $I = \int_0^{\pi} \frac{(\pi - x) \sin(\pi - x)}{1 + \cos^2(\pi - x)}\, dx$, $\left[\because \int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx\right]$

$= \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x}\, dx$. ...(2)

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x}\, dx = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}\, dx$$

$= 2\pi \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x}\, dx,$ [Refer prop. 6]

or $I = \pi \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x}\, dx.$

Now put $\cos x = t$, so that $-\sin x\, dx = dt$. Also $t = 1$ when $x = 0$ and $t = 0$ when $x = \frac{1}{2}\pi$.

$$\therefore \quad I = \pi \int_1^0 \frac{-dt}{1 + t^2} = \pi \int_0^1 \frac{dt}{1 + t^2} = \pi \left[\tan^{-1} t\right]_0^1$$

$$= \pi(\tan^{-1} 1 - \tan^{-1} 0) = \pi\left(\frac{1}{4}\pi - 0\right) = \frac{1}{4}\pi^2.$$

(ii) Let $I = \int_0^{\pi} x \sin^6 x \cos^4 x\, dx$

$= \int_0^{\pi} (\pi - x) \sin^6(\pi - x) \cos^4(\pi - x)\, dx,$ [Refer prop. 4]

$$= \int_0^{\pi} (\pi - x) \sin^6 x \cos^4 x\, dx$$

$$= \int_0^{\pi} \pi \sin^6 x \cos^4 x\, dx - \int_0^{\pi} x \sin^6 x \cos^4 x\, dx$$

$$= \pi \int_0^{\pi} \sin^6 x \cos^4 x\, dx - I.$$

$\therefore \quad 2I = \pi \int_0^{\pi} \sin^6 x \cos^4 x\, dx = 2\pi \int_0^{\pi/2} \sin^6 x \cos^4 x\, dx,$

[Refer prop. 6]

or $I = \pi \int_0^{\pi/2} \sin^6 x \cos^4 x\, dx = \pi \frac{5.3.1.3.1}{10.8.6.4.2} \cdot \frac{\pi}{2} = \frac{3\pi^2}{512},$

by Walli's formula.

Problem 7: *(i) Prove that* $\int_0^{\pi} \frac{x \sin x}{1 + \sin x}\, dx = \pi\left(\frac{\pi}{2} - 1\right)$.

(Kumaun 2011)

(ii) Show that $\int_0^{\pi} \frac{x \tan x}{\sec x + \cos x}\, dx = \frac{1}{4}\pi^2$.

(iii) Show that $\int_0^{\pi} \frac{x \tan x\, dx}{\sec x + \tan x} = \pi\left(\frac{1}{2}\pi - 1\right)$.

Solution: **(i)** Let $I = \int_0^{\pi} \frac{x \sin x}{1 + \sin x}\, dx = \int_0^{\pi} \frac{(\pi - x) \sin (\pi - x)}{1 + \sin (\pi - x)}\, dx,$ [Refer prop. 4]

$$= \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \sin x}\, dx = \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x}\, dx - \int_0^{\pi} \frac{x \sin x}{1 + \sin x}\, dx$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x}\, dx - I.$$

$\therefore$ $$2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x}\, dx = 2\pi \int_0^{\pi/2} \frac{\sin x}{1 + \sin x}\, dx,$$ [Refer prop. 6]

or $$I = \pi \int_0^{\pi/2} \frac{(1 + \sin x) - 1}{1 + \sin x}\, dx = \pi \int_0^{\pi/2} \left[1 - \frac{1}{1 + \sin x}\right] dx$$

$$= \pi \int_0^{\pi/2} dx - \pi \int_0^{\pi/2} \frac{dx}{1 + \sin x}$$

$$= \pi\, [x]_0^{\pi/2} - \pi \int_0^{\pi/2} \frac{dx}{1 + \sin\left(\frac{1}{2}\pi - x\right)}$$ [Refer prop. 4]

$$= \pi \left(\frac{\pi}{2} - 0\right) - \pi \int_0^{\pi/2} \frac{dx}{1 + \cos x} = \frac{\pi^2}{2} - \pi \int_0^{\pi/2} \frac{dx}{2 \cos^2 \frac{1}{2} x}$$

$$= \frac{\pi^2}{2} - \pi \int_0^{\pi/2} \frac{1}{2} \sec^2 \frac{1}{2} x\, dx = \frac{\pi^2}{2} - \pi \left[\tan \frac{1}{2} x\right]_0^{\pi/2}$$

$$= \frac{\pi^2}{2} - \pi \left[\tan \frac{1}{4} \pi - \tan 0\right] = \frac{\pi^2}{2} - \pi = \pi \left(\frac{\pi}{2} - 1\right).$$

(ii) The given integral $I = \int_0^{\pi} \frac{x\,.(\sin x / \cos x)}{(1/\cos x) + \cos x}\, dx = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx = \frac{\pi^2}{4}.$

[Proceed as in problem 6(i)]

(iii) Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx = \int_0^{\pi} \frac{x\,.(\sin x/\cos x)}{(1/\cos x) + (\sin x/\cos x)}\, dx = \int_0^{\pi} \frac{x \sin x}{1 + \sin x}\, dx.$

[Proceed as in problem 7(i)]

Problem 8: *(i) Evaluate* $\int_0^{\pi} \sin^3 \theta\, (1 + 2 \cos \theta)\, (1 + \cos \theta)^2\, d\theta.$

(ii) Evaluate $\int_0^{\pi} \sin^5 x\, (1 - \cos x)^3\, dx.$

Solution: **(i)** Let $I = \int_0^{\pi} \sin^3 \theta\, (1 + 2 \cos \theta)\, (1 + \cos \theta)^2\, d\theta$

$$= \int_0^{\pi} \sin^3 \theta\, (1 + 2 \cos \theta)\, (1 + 2 \cos \theta + \cos^2 \theta)\, d\theta$$

$$= \int_0^{\pi} \sin^3 \theta\, (1 + 4 \cos \theta + 5 \cos^2 \theta + 2 \cos^3 \theta)\, d\theta$$

$$= \int_0^{\pi} (\sin^3 \theta + 4 \sin^3 \theta \cos \theta + 5 \sin^3 \theta \cos^2 \theta + 2 \sin^3 \theta \cos^3 \theta)\, d\theta.$$

Now $\int_0^{\pi} \sin^m \theta \cos^n \theta \, d\theta = 2 \int_0^{\pi/2} \sin^m \theta \cos^n \theta \, d\theta$ or $= 0$,

according as n is an even or an odd integer. [Refer prop. 6]

$$\therefore \quad I = 2 \int_0^{\pi/2} \sin^3 \theta \, d\theta + 5 \times 2 \int_0^{\pi/2} \sin^3 \theta \cos^2 \theta \, d\theta,$$

because the integrals containing odd powers of $\cos \theta$ vanish

$$= 2 \cdot \frac{2}{3 \,.\, 1} + 10 \cdot \frac{2 \,.\, 1}{5 \,.\, 3 \,.\, 1} = \frac{4}{3} + \frac{4}{3} = \frac{8}{3} \cdot$$

(ii) Let $I = \int_0^{\pi} \sin^5 x \, (1 - \cos x)^3 \, dx$

$$= \int_0^{\pi} \left(2 \sin \frac{x}{2} \cos \frac{x}{2}\right)^5 \left(2 \sin^2 \frac{x}{2}\right)^3 dx$$

$$= 2^8 \int_0^{\pi} \sin^{11} \frac{x}{2} \cos^5 \frac{x}{2} \, dx \,.$$

Now put $x/2 = t$, so that $dx = 2 \, dt$.

When $x = 0$, $t = 0$ and when $x = \pi$, $t = \frac{1}{2} \pi$.

$$\therefore \quad I = 2 \times 2^8 \int_0^{\pi/2} \sin^{11} t \cos^5 t \, dt$$

$$= 2 \times 2^8 \cdot \frac{10 \,.\, 8 \,.\, 6 \,.\, 4 \,.\, 2 \,.\, 4 \,.\, 2}{16 \,.\, 14 \,.\, 12 \,.\, 10 \,.\, 8 \,.\, 6 \,.\, 4 \,.\, 2 \,.\, 4 \,.\, 2} = \frac{32}{21} \cdot$$

Problem 9: *(i) Show that* $\int_0^{\pi/2} \log (\tan x) \, dx = 0.$ **(Kumaun 2010)**

(ii) Prove that $\int_0^{1} \log \sin \left(\frac{\pi}{2} y\right) dy = \log \frac{1}{2} \cdot$

Solution: **(i)** Let

$$I = \int_0^{\pi/2} \log (\tan x) \, dx = \int_0^{\pi/2} \log \tan \left(\frac{1}{2} \pi - x\right) dx, \quad \text{[Refer prop. 4]}$$

$$= \int_0^{\pi/2} \log \cot x \, dx = \int_0^{\pi/2} \log (\tan x)^{-1} \, dx$$

$$= - \int_0^{\pi/2} \log \tan x \, dx = - I.$$

$\therefore \quad 2I = 0$ *i.e.*, $I = 0$.

(ii) Let $u = \int_0^{1} \log \sin \left(\frac{\pi}{2} y\right) dy.$

Put $\frac{1}{2} \pi y = x$, so that $\frac{1}{2} \pi \, dy = dx.$

When $y = 0$, $x = 0$ and when $y = 1$, $x = \frac{1}{2} \pi.$

$$\therefore \quad u = \int_0^{\pi/2} (\log \sin x) \cdot \frac{2}{\pi} \, dx = \frac{2}{\pi} \int_0^{\pi/2} \log \sin x \, dx \,.$$

Now let $I = \int_0^{\pi/2} \log \sin x \, dx.$

Then proceeding as in example7, we get

$$I = \frac{1}{2} \pi \log \frac{1}{2}.$$

$$\therefore \quad u = \frac{2}{\pi} I = \frac{2}{\pi} \cdot \frac{1}{2} \pi \log \frac{1}{2} = \log \frac{1}{2}.$$

Problem 10: *(i) Evaluate* $\int_0^{\pi} x \log \sin x \, dx.$

(ii) Evaluate $\int_0^{\pi/2} \log \cos x \, dx.$

(iii) Evaluate $\int_0^{\pi/2} \log \sin 2x \, dx.$

(iv) Evaluate $\int_0^{\infty} \frac{\tan^{-1} x \, dx}{x(1+x^2)}.$

Solution: **(i)** Let $I = \int_0^{\pi} x \log \sin x \, dx.$

Then $I = \int_0^{\pi} (\pi - x) \log \sin (\pi - x) \, dx,$ [Refer prop. 4]

$$= \int_0^{\pi} (\pi - x) \log \sin x \, dx$$

$$= \int_0^{\pi} \pi \log \sin x \, dx - \int_0^{\pi} x \log \sin x \, dx$$

$$= \pi \int_0^{\pi} \log \sin x \, dx - I.$$

$\therefore \quad 2I = \pi \int_0^{\pi} \log \sin x \, dx = 2\pi \int_0^{\pi/2} \log \sin x \, dx,$ [Refer prop. 6]

or $I = \pi \int_0^{\pi/2} \log \sin x \, dx.$

Now let $u = \int_0^{\pi/2} \log \sin x \, dx.$ Then proceeding as in example 7, we have $u = \frac{1}{2} \pi \log \frac{1}{2}.$

$$\therefore \quad I = \pi u = \pi \cdot \frac{1}{2} \pi \log \frac{1}{2} = \frac{1}{2} \pi^2 \log \frac{1}{2}.$$

(ii) Let $I = \int_0^{\pi/2} \log \cos x \, dx.$ ...(1)

Then $I = \int_0^{\pi/2} \log \cos \left(\frac{1}{2} \pi - x \right) dx$ [Refer prop. 4]

$= \int_0^{\pi/2} \log \sin x \, dx.$ ...(2)

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} (\log \cos x + \log \sin x)\, dx = \int_0^{\pi/2} \log (\sin x \cos x)\, dx .$$

Now proceed as in Example 7 and get $I = \frac{1}{2}\pi \log \frac{1}{2}\cdot$

(iii) Let $\quad I = \int_0^{\pi/2} \log \sin 2x\, dx.$

Put $2x = t$, so that $2\, dx = dt$. Also $t = 0$ when $x = 0$ and $t = \pi$ when $x = \frac{1}{2}\pi$.

$$\therefore \quad I = \frac{1}{2}\int_0^{\pi} \log \sin t\, dt = \frac{1}{2} . 2 \int_0^{\pi/2} \log \sin t\, dt, \qquad \text{[Refer prop. 6]}$$

$$= \int_0^{\pi/2} \log \sin t\, dt.$$

Now proceeding as in example 7, we get $I = \frac{1}{2}\pi \log \frac{1}{2}\cdot$

(iv) Let $\quad I = \int_0^{\infty} \frac{\tan^{-1} x}{x(1+x^2)}\, dx.$

Put $\tan^{-1} x = t$, so that $\{1/(1+x^2)\}\, dx = dt$ and $x = \tan t$. Also when $x = 0$, $t = 0$ and when $x = \infty$, $t = \pi/2$.

$$\therefore \quad I = \int_0^{\pi/2} \frac{t\, dt}{\tan t} = \int_0^{\pi/2} t \cot t\, dt.$$

Now proceeding as in example 8, we get $I = \frac{1}{2}\pi \log 2.$

Problem 11: *(i) Show that* $\int_0^{\pi/2} \left(\frac{\theta}{\sin\theta}\right)^2 d\theta = \pi \log 2.$

(ii) Show that $\int_0^{\infty} (\cot^{-1} x)^2\, dx = \pi \log 2.$

(iii) Show that $\int_0^{1} \frac{\sin^{-1} x}{x}\, dx = \frac{1}{2}\pi \log 2.$

(iv) Show that $\int_0^{\pi} \log (1 + \cos x)\, dx = \pi \log \frac{1}{2}\cdot$

Solution: **(i)** Let $I = \int_0^{\pi/2} \left(\frac{\theta}{\sin\theta}\right)^2 d\theta = \int_0^{\pi/2} \theta^2 \operatorname{cosec}^2 \theta\, d\theta.$

Integrating by parts taking $\operatorname{cosec}^2\theta$ as the second function, we get

$$I = [\theta^2(-\cot\theta)]_0^{\pi/2} - \int_0^{\pi/2} 2\theta.(-\cot\theta)\, d\theta$$

$$= -\left(\frac{\pi}{2}\right)^2 \cot\frac{\pi}{2} + \lim_{\theta\to 0} \theta^2 \cot\theta + 2\int_0^{\pi/2} \theta\cot\theta\, d\theta$$

$$= 0 + \lim_{\theta\to 0} \theta^2 \cot\theta + 2\int_0^{\pi/2} \theta\cot\theta\, d\theta.$$

Now $\lim_{\theta \to 0} \theta^2 \cot\theta = \lim_{\theta \to 0} \frac{\theta^2}{\tan\theta}$, $\left[\text{form} \frac{0}{0}\right]$

$$= \lim_{\theta \to 0} \frac{2\theta}{\sec^2 \theta} = \frac{0}{1} = 0.$$

$\therefore$ $I = 2 \int_0^{\pi/2} \theta \cot \theta \, d\theta.$

Now proceed as in Example 8.

(ii) Let $I = \int_0^{\infty} (\cot^{-1} x)^2 \, dx.$

Put $\cot^{-1} x = \theta$ *i.e.*, $x = \cot \theta$, so that $dx = -\operatorname{cosec}^2 \theta \, d\theta$. The new limits for θ are $\frac{1}{2}\pi$ to 0.

$\therefore$ $I = \int_{\pi/2}^{0} \theta^2 . (-\operatorname{cosec}^2 \theta) \, d\theta = \int_0^{\pi/2} \theta^2 \operatorname{cosec}^2 \theta \, d\theta.$

Now proceed as in Problem 11(i).

(iii) Let $I = \int_0^1 \frac{\sin^{-1} x}{x} \, dx.$

Put $x = \sin t$, so that $dx = \cos t \, dt$. Also $t = 0$ when $x = 0$ and $t = \frac{1}{2}\pi$ when $x = 1$.

$\therefore$ $I = \int_0^{\pi/2} \frac{t}{\sin t} \cos t \, dt = \int_0^{\pi/2} t \cot t \, dt.$

Now proceed as in Example 8.

(iv) Let $I = \int_0^{\pi} \log (1 + \cos x) \, dx$. ...(1)

Then $I = \int_0^{\pi} \log \{1 + \cos (\pi - x)\} \, dx,$ [Refer prop. 4]

$$= \int_0^{\pi} \log (1 - \cos x) \, dx. \quad \text{...(2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} [\log (1 + \cos x) + \log (1 - \cos x)] \, dx$$

$$= \int_0^{\pi} \log \{(1 + \cos x)(1 - \cos x)\} \, dx$$

$$= \int_0^{\pi} \log (1 - \cos^2 x) \, dx = \int_0^{\pi} \log \sin^2 x \, dx = 2 \int_0^{\pi} \log \sin x \, dx.$$

$\therefore$ $I = \int_0^{\pi} \log \sin x \, dx = 2 \int_0^{\pi/2} \log \sin x \, dx.$ [Refer prop. 6]

Now let $u = \int_0^{\pi/2} \log \sin x \, dx$. Then proceeding as in example 7, we have

$$u = \frac{1}{2} \pi \log \frac{1}{2} \cdot$$

$\therefore$ $I = 2u = 2 . \frac{1}{2} \pi \log \frac{1}{2} = \pi \log \frac{1}{2} \cdot$

Problem 12: *(i) Show that* $\int_0^{\infty} \log\left(x + \frac{1}{x}\right) \frac{dx}{1+x^2} = \pi \log 2.$

(ii) Show that $\int_0^{\infty} \frac{\log(1+x^2)\, dx}{(1+x^2)} = \pi \log 2.$

(iii) Show that $\int_0^{1} \frac{\log(1+x)}{1+x^2}\, dx = \frac{1}{8}\pi \log 2.$

(Kumaun 2008)

Solution: **(i)** Put $x = \tan\theta$, so that $dx = \sec^2\theta\, d\theta$.

Also when $x = 0$, $\theta = 0$ and when $x \to \infty$, $\theta \to \pi/2$.

$$\therefore \quad I = \int_0^{\infty} \log\left(x + \frac{1}{x}\right)\frac{dx}{1+x^2}$$

$$= \int_0^{\pi/2} \log\left(\tan\theta + \frac{1}{\tan\theta}\right)\frac{\sec^2\theta}{\sec^2\theta}\, d\theta$$

$$= \int_0^{\pi/2} \log\left(\frac{1+\tan^2\theta}{\tan\theta}\right) d\theta = \int_0^{\pi/2} \log \frac{\sec^2\theta}{\tan\theta}\, d\theta$$

$$= \int_0^{\pi/2} \log\left(\frac{1}{\sin\theta\cos\theta}\right) d\theta = \int_0^{\pi/2} \log(\sin\theta\cos\theta)^{-1}\, d\theta$$

$$= -\int_0^{\pi/2} \log\sin\theta\, d\theta - \int_0^{\pi/2} \log\cos\theta\, d\theta.$$

Now let $u = \int_0^{\pi/2} \log\sin\theta\, d\theta$. Then proceeding as in Example 7, we have

$$u = \int_0^{\pi/2} \log\cos\theta\, d\theta = -\frac{\pi}{2}\log 2.$$

$$\therefore \quad I = -u - u = -2u = -2\left(-\frac{1}{2}\pi\log 2\right) = \pi\log 2.$$

(ii) Put $x = \tan\theta$, so that $dx = \sec^2\theta\, d\theta$. The new limits for θ are 0 to $\frac{1}{2}\pi$.

$$\therefore \quad I = \int_0^{\infty} \frac{\log(1+x^2)}{(1+x^2)}\, dx = \int_0^{\pi/2} \frac{\log(1+\tan^2\theta)\sec^2\theta\, d\theta}{\sec^2\theta}$$

$$= \int_0^{\pi/2} \log\sec^2\theta\, d\theta, \qquad [\because 1+\tan^2\theta = \sec^2\theta]$$

$$= 2\int_0^{\pi/2} \log\sec\theta\, d\theta = 2\int_0^{\pi/2} \log(\cos\theta)^{-1}\, d\theta$$

$$= -2\int_0^{\pi/2} \log\cos\theta\, d\theta.$$

Now let $u = \int_0^{\pi/2} \log\cos\theta\, d\theta$. Then proceeding as in Problem 10(ii), we have

$$u = -\frac{1}{2}\pi\log 2.$$

$\therefore \qquad I = -2u = -2 \cdot \left(-\frac{1}{2} \pi \log 2\right) = \pi \log 2.$

(iii) Put $x = \tan\theta$, so that $dx = \sec^2\theta\, d\theta$.

And the new limits are $\theta = 0$ to $\theta = \pi/4$.

$$\therefore \qquad I = \int_0^1 \frac{\log(1+x)}{1+x^2}\, dx = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta} \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta = \frac{1}{8}\pi \log 2.$$

[Proceeding as in Example 9]

Problem 13: *(i) Evaluate* $\int_0^{\pi/2} \frac{dx}{1+\tan x}$.

(ii) Evaluate $\int_0^{\pi/2} \frac{dx}{1+\cot x}$.

Solution: **(i)** We have $I = \int_0^{\pi/2} \frac{dx}{1+\tan x} = \int_0^{\pi/2} \frac{dx}{1+(\sin x/\cos x)}$

$$= \int_0^{\pi/2} \frac{\cos x\, dx}{\cos x + \sin x}$$

$$= \frac{\pi}{4}.$$ [Proceeding as in Example 10]

(ii) We have $\int_0^{\pi/2} \frac{dx}{1+\cot x} = \int_0^{\pi/2} \frac{dx}{1+(\cos x/\sin x)} = \int_0^{\pi/2} \frac{\sin x\, dx}{\sin x + \cos x}$

$$= \frac{\pi}{4}.$$ [Proceeding as in Example 10]

Problem 14: *(i) Show that* $\int_0^{\infty} \frac{x\, dx}{(1+x)(1+x^2)} = \frac{\pi}{4}$.

(ii) Show that $\int_0^{a} \frac{dx}{x + \sqrt{(a^2 - x^2)}} = \frac{\pi}{4}$.

Solution: **(i)** Put $x = \tan\theta$, so that $dx = \sec^2\theta\, d\theta$.

Also when $x = 0$, $\theta = 0$ and when $x \to \infty$, $\theta \to \pi/2$.

$\therefore$ The given integral $I = \int_0^{\pi/2} \frac{\tan\theta \sec^2\theta\, d\theta}{(1+\tan\theta)(1+\tan^2\theta)}$

$$= \int_0^{\pi/2} \frac{\tan\theta\, d\theta}{1+\tan\theta} = \int_0^{\pi/2} \frac{\sin\theta/\cos\theta}{1+(\sin\theta/\cos\theta)}\, d\theta$$

$$= \int_0^{\pi/2} \frac{\sin\theta\, d\theta}{\cos\theta + \sin\theta} = \frac{\pi}{4}.$$ [See Example 10]

(ii) Put $x = a\sin\theta$, so that $dx = a\cos\theta\, d\theta$.

Also the new limits are $\theta = 0$ to $\theta = \pi/2$.

$\therefore$ The given integral $I = \int_0^{\pi/2} \frac{a \cos\theta \, d\theta}{a \sin\theta + a \cos\theta}$

$$= \int_0^{\pi/2} \frac{\cos\theta \, d\theta}{\sin\theta + \cos\theta} = \frac{\pi}{4}. \quad \text{[See Example 10]}$$

Problem 15: *(i) Show that* $\int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}} dx = \frac{\pi}{4}.$

(Lucknow 2007)

(ii) Show that $\int_0^{\pi/2} \frac{\tan x}{\tan x + \cot x} dx = \frac{\pi}{4}.$

(iii) Show that $\int_0^{\pi/2} \frac{dx}{1 + \sqrt{(\tan x)}} = \frac{\pi}{4}.$

(iv) Show that $\int_0^{\pi/2} \frac{\sqrt{(\tan x)}\, dx}{1 + \sqrt{(\tan x)}} = \frac{\pi}{4}.$

Solution: **(i)** Let $I = \int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}} dx.$ ...(1)

Then
$$I = \int_0^{\pi/2} \frac{\sqrt{\left[\sin\left(\frac{1}{2}\pi - x\right)\right]} dx}{\sqrt{\left[\sin\left(\frac{1}{2}\pi - x\right)\right]} + \sqrt{\left[\cos\left(\frac{1}{2}\pi - x\right)\right]}}, \quad \text{[Refer prop. 4]}$$

$$= \int_0^{\pi/2} \frac{\sqrt{(\cos x)}\, dx}{\sqrt{(\cos x)} + \sqrt{(\sin x)}}. \quad \text{...(2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \left[\frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}} + \frac{\sqrt{(\cos x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}\right] dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{(\sin x)} + \sqrt{(\cos x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}} dx = \int_0^{\pi/2} 1 \,.\, dx = \frac{\pi}{2}.$$

$\therefore$ $I = \frac{1}{4}\pi.$

(ii) Let $I = \int_0^{\pi/2} \frac{\tan x \, dx}{\tan x + \cot x}$ ...(1)

then
$$I = \int_0^{\pi/2} \frac{\tan\left(\frac{\pi}{2} - x\right) dx}{\tan\left(\frac{\pi}{2} - x\right) + \cot\left(\frac{\pi}{2} - x\right)}$$

$$= \int_0^{\pi/2} \frac{\cot x \, dx}{\cot x - \tan x} \quad \text{...(2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\tan x \, dx}{\tan x + \cot x} + \int_0^{\pi/2} \frac{\cot x \, dx}{\cot x + \tan x}$$

$$= \int_0^{\pi/2} \frac{\tan x + \cot x}{\tan x + \cot x} dx$$

$$= \int_0^{\pi/2} 1 \cdot dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

$\therefore \qquad I = \frac{\pi}{4}.$

(iii) We have $\quad I = \int_0^{\pi/2} \frac{dx}{1 + \sqrt{(\tan x)}} = \int_0^{\pi/2} \frac{dx}{1 + \sqrt{(\sin x / \cos x)}}$

$$= \int_0^{\pi/2} \frac{\sqrt{(\cos x)}\, dx}{\sqrt{(\cos x)} + \sqrt{(\sin x)}}.$$

Now proceeding exactly as in Problem 15(i) we get the result.

(iv) We have $\quad I = \int_0^{\pi/2} \frac{\sqrt{(\tan x)}\, dx}{1 + \sqrt{(\tan x)}} = \int_0^{\pi/2} \frac{\sqrt{(\sin x)}\, dx}{\sqrt{(\cos x)} + \sqrt{(\sin x)}}. \quad \left[\because \tan x = \frac{\sin x}{\cos x}\right]$

$= \frac{\pi}{4}.$ [Proceed as in Problem 15(i)]

Problem 16: *(i) Prove that* $\int_0^{\pi/2} \frac{\sqrt{(\tan x)}}{\sqrt{(\tan x)} + \sqrt{(\cot x)}} dx = \frac{\pi}{4}.$

(ii) Show that $\int_0^{\pi/2} \frac{\sin^2 x\, dx}{(\sin x + \cos x)} = \frac{1}{\sqrt{2}} \log(\sqrt{2} + 1).$

Solution: **(i)** Here $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$, changing to $\sin x$ and $\cos x$. Now proceed as in Example 10.

(ii) Let $\quad I = \int_0^{\pi/2} \frac{\sin^2 x\, dx}{(\sin x + \cos x)}. \qquad \ldots(1)$

Then $\quad I = \int_0^{\pi/2} \frac{\left[\sin\left(\frac{1}{2}\pi - x\right)\right]^2 dx}{\sin\left(\frac{1}{2}\pi - x\right) + \cos\left(\frac{1}{2}\pi - x\right)}, \qquad$ [Refer prop. 4]

or $\quad I = \int_0^{\pi/2} \frac{\cos^2 x\, dx}{\cos x + \sin x}. \qquad \ldots(2)$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sin^2 x\, dx}{\sin x + \cos x} + \int_0^{\pi/2} \frac{\cos^2 x\, dx}{\cos x + \sin x}$$

$$= \int_0^{\pi/2} \frac{(\sin^2 x + \cos^2 x)}{(\sin x + \cos x)} dx = \int_0^{\pi/2} \frac{dx}{(\sin x + \cos x)}$$

$$= \int_0^{\pi/2} \frac{(1/\sqrt{2})\, dx}{(1/\sqrt{2}) \sin x + (1/\sqrt{2}) \cos x} \qquad \textbf{(Note)}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\cos\left(x - \frac{1}{4}\pi\right)}, \qquad \left[\because \cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \sec\left(x - \frac{1}{4}\pi\right) dx$$

$$= \frac{1}{\sqrt{2}} \log\left[\sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right)\right]_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}}\left[\log\left(\sec\frac{1}{4}\pi + \tan\frac{1}{4}\pi\right) - \log\left\{\sec\left(-\frac{1}{4}\pi\right) + \tan\left(-\frac{1}{4}\pi\right)\right\}\right]$$

$$= \frac{1}{\sqrt{2}} \log\left[\frac{\sec\frac{1}{4}\pi + \tan\frac{1}{4}\pi}{\sec\left(-\frac{1}{4}\pi\right) + \tan\left(-\frac{1}{4}\pi\right)}\right] = \frac{1}{\sqrt{2}} \log\left[\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]$$

$$= \frac{1}{\sqrt{2}} \log\left[\frac{(\sqrt{2}+1)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}\right] = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)^2$$

$$= (1/\sqrt{2}) . 2 \log(\sqrt{2}+1).$$

$\therefore$ $$I = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1).$$

Problem 17: *(i) Evaluate* $\int_0^{\pi/2} \frac{\cos^2 x}{(\sin x + \cos x)} dx.$

(ii) Evaluate $\int_0^{a} \frac{a\, dx}{\{x + \sqrt{(a^2 - x^2)}\}^2}.$

(iii) Evaluate $\int_0^{\pi/2} \frac{x\, dx}{\sin x + \cos x}.$

Solution: **(i)** Proceed exactly as in problem 16(ii). The answer is the same as problem 16(ii).

(ii) Put $x = a \sin\theta$, so that $dx = a\cos\theta\, d\theta$.

When $x = 0, \theta = 0$ and when $x = a, \theta = \pi/2$.

$\therefore$ The given integral $I = \int_0^{\pi/2} \frac{a \,.\, a\cos\theta\, d\theta}{a^2(\sin\theta + \cos\theta)^2}$

$$= \int_0^{\pi/2} \frac{\cos\theta\, d\theta}{(\sin\theta + \cos\theta)^2}. \quad ...(1)$$

Also $$I = \int_0^{\pi/2} \frac{\cos\left(\frac{1}{2}\pi - \theta\right) d\theta}{\left[\sin\left(\frac{1}{2}\pi - \theta\right) + \cos\left(\frac{1}{2}\pi - \theta\right)\right]^2}, \quad \text{[Refer prop. 4]}$$

$$= \int_0^{\pi/2} \frac{\sin\theta\, d\theta}{(\cos\theta + \sin\theta)^2}. \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\cos\theta + \sin\theta}{(\sin\theta + \cos\theta)^2} d\theta = \int_0^{\pi/2} \frac{d\theta}{\sin\theta + \cos\theta}$$

$= (1/\sqrt{2}) .2 \log (\sqrt{2}+1)$, [Proceeding as in problem 16(ii)]

$\therefore \qquad I = (1/\sqrt{2}) \log (\sqrt{2}+1)$.

(iii) Let $\qquad I = \int_0^{\pi/2} \frac{x\, dx}{\sin x + \cos x}$. ...(1)

Then $\qquad I = \int_0^{\pi/2} \frac{\left(\frac{1}{2}\pi - x\right) dx}{\sin\left(\frac{1}{2}\pi - x\right) + \cos\left(\frac{1}{2}\pi - x\right)}$, [Refer prop. 4]

$$= \int_0^{\pi/2} \frac{\left(\frac{1}{2}\pi - x\right) dx}{\sin x + \cos x}. \qquad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\frac{1}{2}\pi\, dx}{\sin x + \cos x} = \frac{\pi}{2} \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}.$$

$$\therefore \qquad I = \frac{\pi}{4} \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}.$$

Now proceeding as in problem 16(ii), we have

$$\int_0^{\pi/2} \frac{dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}} \cdot 2 \log (\sqrt{2}+1).$$

$$\therefore \qquad I = \frac{\pi}{4} \cdot \frac{1}{\sqrt{2}} \cdot 2 \log (\sqrt{2}+1) = \frac{\pi}{2\sqrt{2}} \log (\sqrt{2}+1).$$

Problem 18: *(i) Show that* $\int_0^{\pi/2} \phi(\sin 2x) \sin x\, dx = \int_0^{\pi/2} \phi(\sin 2x) \cos x\, dx$

$$= \sqrt{2} \int_0^{\pi/2} \phi(\cos 2x) \cos x\, dx.$$

(Kumaun 2007)

(ii) Show that $\int_0^{\pi} \frac{x^2 \sin 2x \sin\left(\frac{1}{2}\pi \cos x\right)}{2x - \pi} dx = \frac{8}{\pi}$.

Solution: **(i)** We have $\int_0^{\pi/2} \phi(\sin 2x) . \sin x\, dx$

$$= \int_0^{\pi/2} \phi\left[\sin 2\left(\frac{\pi}{2} - x\right)\right] \cdot \sin\left(\frac{\pi}{2} - x\right) dx, \qquad \text{[Refer prop. 4]}$$

$$= \int_0^{\pi/2} \phi[\sin(\pi - 2x)] \cos x\, dx$$

$$= \int_0^{\pi/2} \phi(\sin 2x) \cos x\, dx.$$

The first part proved.

Now to prove the second part, let

$$I = \int_0^{\pi/2} \phi(\sin 2x) \sin x\, dx.$$

Put $x = \frac{1}{4}\pi + t$, so that $dx = dt$.

When $x = 0$, $t = -\pi/4$ and when $x = \pi/2$, $t = \pi/4$.

$$\therefore \quad I = \int_{-\pi/4}^{\pi/4} \phi\left[\sin 2\left(\frac{1}{4}\pi + t\right)\right]\sin\left(\frac{1}{4}\pi + t\right) dt$$

$$= \int_{-\pi/4}^{\pi/4} \phi\left[\sin\left(\frac{1}{2}\pi + 2t\right)\right]\sin\left(\frac{1}{4}\pi + t\right) dt$$

$$= \int_{-\pi/4}^{\pi/4} \phi(\cos 2t)\left(\sin\frac{1}{4}\pi\cos t + \cos\frac{1}{4}\pi\sin t\right) dt$$

$$= \frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} \phi(\cos 2t)\cos t\, dt + \frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} \phi(\cos 2t)\sin t\, dt.$$

Now $\int_{-\pi/4}^{\pi/4} \phi(\cos 2t)\sin t\, dt = 0,$

because $\phi(\cos 2t)\sin t$ is an odd function of t

and $\int_{-\pi/4}^{\pi/4} \phi(\cos 2t)\cos t\, dt = 2\int_{0}^{\pi/4} \phi(\cos 2t)\cos t\, dt,$

because $\phi(\cos 2t)\cos t$ is an even function of t.

$$\therefore \quad I = \frac{2}{\sqrt{2}}\int_{0}^{\pi/4} \phi(\cos 2t)\cos t\, dt = \sqrt{2}\int_{0}^{\pi/4} \phi(\cos 2x)\cos x\, dx,$$

because a definite integral does not change by changing the variable.

(ii) Let $I = \int_0^{\pi} \dfrac{x^2 \sin 2x . \sin\left(\frac{1}{2}\pi\cos x\right)}{2x - \pi}\, dx.$

Put $x = \frac{1}{2}\pi - t$, so that $dx = -dt$.

Also $t = \frac{1}{2}\pi$ when $x = 0$ and $t = -\frac{1}{2}\pi$ when $x = \pi$.

$$\therefore \quad I = \int_{\pi/2}^{-\pi/2} \frac{\left(\frac{1}{2}\pi - t\right)^2 \sin 2\left(\frac{1}{2}\pi - t\right).\sin\left\{\frac{1}{2}\pi\cos\left(\frac{1}{2}\pi - t\right)\right\}}{2\left(\frac{1}{2}\pi - t\right) - \pi}(-dt)$$

$$= \int_{-\pi/2}^{\pi/2} \frac{\left(\frac{1}{2}\pi - t\right)^2 \sin 2t . \sin\left(\frac{1}{2}\pi\sin t\right)}{-2t}\, dt$$

$$= -\frac{1}{2}\int_{-\pi/2}^{\pi/2} \frac{\left(\frac{1}{4}\pi^2 - \pi t + t^2\right)\sin 2t . \sin\left(\frac{1}{2}\pi\sin t\right)}{t}\, dt.$$

Now $\dfrac{\frac{1}{4}\pi^2 \sin 2t . \sin\left(\frac{1}{2}\pi\sin t\right)}{t}$ and $t\sin 2t . \sin\left(\frac{1}{2}\pi\sin t\right)$ are both odd functions of t

while $\sin 2t . \sin\left(\frac{1}{2}\pi\sin t\right)$ is an even function of t.

$\therefore \quad I = -\frac{1}{2} . 2 \int_0^{\pi/2} (-\pi) \sin 2t . \sin \left(\frac{1}{2} \pi \sin t \right) dt,$ [Refer prop. 5]

$$= \pi \int_0^{\pi/2} 2 \sin t \cos t . \sin \left(\frac{1}{2} \pi \sin t \right) dt.$$

Now put $\frac{1}{2} \pi \sin t = z$, so that $\frac{1}{2} \pi \cos t \, dt = dz$.

Also $z = 0$ when $t = 0$ and $z = \frac{1}{2} \pi$ when $t = \frac{1}{2} \pi$.

$$\therefore \quad I = \pi \int_0^{\pi/2} \frac{2 . 2z}{\pi} \cdot \sin z \cdot \frac{2}{\pi} dz = \frac{8}{\pi} \int_0^{\pi/2} z \sin z \, dz$$

$$= \frac{8}{\pi} \left[\{ z (-\cos z) \}_0^{\pi/2} - \int_0^{\pi/2} 1 . (-\cos z) \, dz \right]$$

$$= \frac{8}{\pi} \left[0 + \int_0^{\pi/2} \cos z \, dz \right] = \frac{8}{\pi} [\sin z]_0^{\pi/2} = \frac{8}{\pi} (1 - 0) = \frac{8}{\pi} \cdot$$

Comprehensive Problems 2

Problem 1: *Find by summation the value of* $\int_a^b x \, dx$.

Solution: Here $f(x) = x$;

$\therefore \quad f(a) = a, \; f(a + h) = a + h,$

$f(a + 2h) = a + 2h$, etc.

Now $\int_a^b f(x) \, dx = \lim_{h \to 0} h [f(a) + f(a + h) + \ldots + f\{a + (n - 1) h\}]$,

where $n \to \infty$ and $nh \to b - a$ as $h \to 0$.

$$\therefore \quad \int_a^b x \, dx = \lim_{h \to 0} h [a + (a + h) + (a + 2h) + \ldots + \{a + (n - 1) h\}]$$

$= \lim_{h \to 0} h \left[\frac{n}{2} \{2a + (n - 1) h\} \right]$, summing the A .P.

$$= \lim_{h \to 0} \frac{nh}{2} [2a + nh - h]$$

$= \lim_{h \to 0} \frac{b - a}{2} [2a + (b - a) - h],$ $[\because nh = b - a]$

$$= \frac{1}{2} (b - a) [2a + (b - a)] = \frac{1}{2} (b - a)(b + a) = \frac{1}{2} (b^2 - a^2).$$

Problem 2: *Evaluate by summation* $\int_1^2 x \, dx$.

Solution: Proceed as problem 1. Here $b = 2, a = 1$.

Thus proceeding as above, we get $\int_1^2 x \, dx = \frac{1}{2} (4 - 1) = \frac{3}{2}$.

Problem 3: *Evaluate by summation* $\int_0^2 x^3\, dx$.

Solution: Here $f(x) = x^3$ and $a = 0, b = 2$; therefore $nh = 2 - 0 = 2$.

$$\therefore \int_0^2 x^3\, dx = \lim_{h\to 0} h[0^3 + h^3 + 2^3h^3 + 3^3h^3 + \ldots + (n-1)^3h^3]$$

$$= \lim_{h\to 0} h^4 [1^3 + 2^3 + 3^3 + \ldots + (n-1)^3], \text{ where } nh = 2$$

$$= \lim_{h\to 0} h^4 \left[\frac{(n-1)^2 \{(n-1)+1\}^2}{4}\right], \text{ summing up the series}$$

using the formula $\Sigma n^3 = \left[\frac{n(n+1)}{2}\right]^2$

$$= \lim_{h\to 0} \frac{1}{4} h^4 (n-1)^2 n^2, \text{ where } nh = 2$$

$$= \lim_{h\to 0} \frac{1}{4} (nh - h)^2 (nh)^2 = \frac{1}{4}(2-0)^2.2^2 = 4.$$

Problem 4: *Using the definition of integral as the limit of a sum, show that*

$$\int_a^b \cos x\, dx = \sin b - \sin a.$$

Solution: Proceed exactly as in example 14.

Problem 5: *Evaluate by summation* $\int_0^{\pi/2} \sin x\, dx$.

Solution: Here $f(x) = \sin x;\ a = 0$ and $b = \pi/2$,

$$nh = b - a = \frac{1}{2}\pi - 0 = \frac{1}{2}\pi.$$

Proceeding exactly as in example 14, we get

$$\int_0^{\pi/2} \sin x\, dx = \cos 0 - \cos \frac{1}{2}\pi = 1 - 0 = 1.$$

Problem 6: *Evaluate by summation* $\int_0^{\pi/2} \cos x\, dx$.

Solution: Here $f(x) = \cos x; a = 0$ and $b = \pi/2,\ nh = b - a = \frac{1}{2}\pi - 0 = \frac{1}{2}\pi.$

Proceeding exactly as in example 14, we get

$$\int_0^{\pi/2} \cos x\, dx = \sin \frac{1}{2}\pi - \sin 0 = 1 - 0 = 1.$$

Problem 7: *Evaluate by summation* $\int_a^b \frac{1}{x^2}\, dx$.

Solution: Proceed as in Example 12. Here $m = -2$.

Comprehensive Problems 3

Problem 1: *Evaluate* $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}\right]$.

Solution: Here the general term (*i.e.*, the rth term) $=\frac{1}{n+r}$ and r varies from 1 to n. Thus we have to find $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r}$.

We have $$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n+r}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n\{1+(r/n)\}}=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+(r/n)}.$$

The limits of r in this summation are 1 to n.

Therefore the lower limit of integration $=\lim_{n\to\infty}\frac{1}{n}=0.$

and the upper limit of integration $=\lim_{n\to\infty}\frac{n}{n}=\lim_{n\to\infty}1=1.$

Hence the required limit $=\int_0^1\frac{1}{1+x}\,dx=[\log(1+x)]_0^1=\log 2.$

Problem 2: *Evaluate* $\lim_{n\to\infty}\left[\frac{1}{n+m}+\frac{1}{n+2m}+\ldots+\frac{1}{n+nm}\right]$.

Solution: Here the rth term $=\frac{1}{n+rm}=\frac{1}{n}\left\{\frac{1}{1+(r/n)\,m}\right\}$ and r varies from 1 to n.

$\therefore$ The given limit $=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\cdot\left\{\frac{1}{1+(r/n)m}\right\}=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+(r/n)m}.$

Also the lower limit of integration

$=\lim_{n\to\infty}(1/n)=0,$ [$\because$ $r=1$ for the first term]

and the upper limit of integration $=\lim_{n\to\infty}(n/n)=1.$ [$\because$ $r=n$ for the last term]

$\therefore$ The required limit $=\int_0^1\frac{1}{1+mx}\,dx=\left[\frac{1}{m}\log(1+mx)\right]_0^1=(1/m)\log(1+m).$

Problem 3: *Evaluate* $\lim_{n\to\infty}\left[\frac{n}{(n+1)^2}+\frac{n}{(n+2)^2}+\ldots+\frac{n}{(n+n)^2}\right]$.

Solution: Here the rth term $=\frac{n}{(n+r)^2}=\frac{n}{n^2\{1+(r/n)\}^2}$

$=\frac{1}{n}\cdot\frac{1}{\{1+(r/n)\}^2}$, and r varies from 1 to n.

$\therefore$ We have to find $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \frac{1}{n} \cdot \frac{1}{\{1+(r/n)\}^2}$.

The lower limit of integration $= \lim\limits_{n\to\infty} \left(\frac{1}{n}\right) = 0$, [$\because$ $r = 1$ for the 1st term]

and the upper limit $= \lim\limits_{n\to\infty} \left(\frac{n}{n}\right) = 1$. [$\because$ $r = n$ for the last term]

$\therefore$ The required limit $= \int_0^1 \frac{1}{(1+x)^2}\, dx = \left[-\frac{1}{(1+x)}\right]_0^1 = -\frac{1}{2} - (-1) = -\frac{1}{2} + 1 = \frac{1}{2}\cdot$

Problem 4: *Evaluate* $\lim\limits_{n\to\infty} [\,\{\sqrt{(n+1)} + \sqrt{(n+2)} + \ldots + \sqrt{(2n)}\}/n\sqrt{n}\,]$.

Solution: The given limit

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{\sqrt{(n+r)}}{n\sqrt{n}} = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \sqrt{\left(1+\frac{r}{n}\right)}$$

$$= \int_0^1 \sqrt{(1+x)}\, dx = \left[\frac{(1+x)^{3/2}}{3/2}\right]_0^1$$

$$= \frac{2}{3}[(1+x)^{3/2}]_0^1 = \frac{2}{3}[2^{3/2} - 1] = \frac{2}{3}[2\sqrt{2} - 1]\cdot$$

Problem 5: *Evaluate*

$$\lim_{n\to\infty} n\left[\frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+4)} + \frac{1}{(n+3)(n+6)} + \ldots + \frac{1}{6n^2}\right].$$

Solution: The given limit

$$= \lim_{n\to\infty} n \sum_{r=1}^{n} \frac{1}{(n+r)(n+2r)} = \lim_{n\to\infty} \frac{n}{n^2} \sum_{r=1}^{n} \frac{1}{(1+r/n)(1+2r/n)}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{(1+r/n)(1+2r/n)} = \int_0^1 \frac{1}{(1+x)(1+2x)} dx.$$

Let $\frac{1}{(1+x)(1+2x)} \equiv \frac{A}{1+x} + \frac{B}{1+2x}$.

Then $A = -1, B = 2$.

$\therefore$ The required limit $= \int_0^1 \left[\frac{-1}{1+x} + \frac{2}{1+2x}\right] dx = [-\log(1+x) + \log(1+2x)]_0^1$

$$= \left[\log\left(\frac{1+2x}{1+x}\right)\right]_0^1 = \log\frac{3}{2} - \log 1 = \log\frac{3}{2}\cdot$$

Problem 6: *Evaluate* $\lim\limits_{n\to\infty}\left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \ldots + \frac{1}{2n}\right]$.

Solution: Here the rth term $= \dfrac{n}{n^2 + r^2}$, and r varies from 0 to n. Proceeding as in example 16, we get the required limit $= \pi/4$.

Problem 7: *Evaluate* $\lim_{n\to\infty}\left[\dfrac{n+1}{n^2+1^2} + \dfrac{n+2}{n^2+2^2} + \ldots + \dfrac{1}{n}\right]$.

Solution: Here the rth term

$$= \frac{n+r}{n^2+r^2} = \frac{1+(r/n)}{n\{1+(r/n)^2\}} = \frac{1}{n}\cdot\left\{\frac{1+(r/n)}{1+(r/n)^2}\right\},$$

and r varies from 1 to n.

$\therefore$ The given limit $= \lim_{n\to\infty}\sum_{r=1}^{n}\dfrac{1}{n}\cdot\left\{\dfrac{1+(r/n)}{1+(r/n)^2}\right\}$

$$= \int_0^1 \frac{x+1}{x^2+1}\,dx = \int_0^1\left[\frac{x}{x^2+1} + \frac{1}{x^2+1}\right]dx$$

$$= \left[\frac{1}{2}\log(x^2+1) + \tan^{-1}x\right]_0^1 = \frac{1}{2}\log 2 + \frac{\pi}{4}.$$

Problem 8: *Evaluate* $\lim_{n\to\infty}\left[\dfrac{1}{n^3}(1+4+9+16+\ldots+n^2)\right]$.

Solution: Here the rth term $= \dfrac{1}{n^3}(r^2) = \dfrac{1}{n}\cdot\left(\dfrac{r}{n}\right)^2$, and r varies from 1 to n.

$\therefore$ The given limit $= \lim_{n\to\infty}\sum_{r=1}^{n}\dfrac{1}{n}\cdot\left(\dfrac{r}{n}\right)^2 = \int_0^1 x^2\,dx = \left[\dfrac{x^3}{3}\right]_0^1 = \dfrac{1}{3}$.

Problem 9: *Evaluate* $\lim_{n\to\infty}\left[\dfrac{1}{n} + \dfrac{n^2}{(n+1)^3} + \dfrac{n^2}{(n+2)^3} + \ldots + \dfrac{1}{8n}\right]$.

Solution: Here the general term

$$= \frac{n^2}{(n+r)^3} = \frac{n^2}{n^3\{1+(r/n)\}^3} = \frac{1}{n}\cdot\frac{1}{\{1+(r/n)\}^3},$$

and r varies from 0 to n.

$\therefore$ The given limit $= \lim_{n\to\infty}\sum_{r=0}^{n}\dfrac{1}{n}\cdot\dfrac{1}{\{1+(r/n)\}^3}$

$$= \int_0^1 \frac{1}{(1+x)^3}\,dx = \left[-\frac{1}{2(1+x)^2}\right]_0^1 = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}.$$

Problem 10: *Evaluate* $\lim_{n\to\infty}\left[\dfrac{n^{1/2}}{n^{3/2}} + \dfrac{n^{1/2}}{(n+3)^{3/2}} + \dfrac{n^{1/2}}{(n+6)^{3/2}} + \ldots + \dfrac{n^{1/2}}{\{n+3(n-1)\}^{3/2}}\right]$.

(Lucknow 2006)

Solution: Here the general term $= \dfrac{n^{1/2}}{(n+3r)^{3/2}} = \dfrac{1}{n\{1+(3r/n)\}^{3/2}}$,

and r varies from 0 to $n-1$.

$\therefore$ The given limit $= \lim\limits_{n\to\infty} \sum\limits_{r=0}^{n-1} \dfrac{1}{n} \cdot \dfrac{1}{\{1+(3r/n)\}^{3/2}}$

$$= \int_0^1 \frac{dx}{(1+3x)^{3/2}} = -\frac{2}{3} \cdot \left[\frac{1}{(1+3x)^{1/2}}\right]_0^1 = -\frac{2}{3}\left[\frac{1}{2}-1\right] = \frac{1}{3}.$$

Problem 11: *Evaluate* $\lim\limits_{n\to\infty}\left[\dfrac{1}{n} + \dfrac{1}{\sqrt{(n^2-1^2)}} + \dfrac{1}{\sqrt{(n^2-2^2)}} + \ldots + \dfrac{1}{\sqrt{\{n^2-(n-1)^2\}}}\right]$.

Solution: Here the general term

$= \dfrac{1}{\sqrt{(n^2-r^2)}} = \dfrac{1}{n} \cdot \dfrac{1}{\sqrt{\{1-(r/n)^2\}}}$, and r varies from 0 to $(n-1)$.

$\therefore$ The given limit $= \lim\limits_{n\to\infty} \sum\limits_{r=0}^{n-1} \dfrac{1}{n} \cdot \dfrac{1}{\sqrt{\{1-(r/n)^2\}}} = \displaystyle\int_0^1 \frac{1}{\sqrt{(1-x^2)}}\,dx = [\sin^{-1} x]_0^1$

$$= \sin^{-1} 1 - \sin^{-1} 0 = \frac{1}{2}\pi.$$

Problem 12: *Evaluate* $\lim\limits_{n\to\infty}\left[\dfrac{1}{n^2} \sec^2 \dfrac{1}{n^2} + \dfrac{2}{n^2} \sec^2 \dfrac{4}{n^2} + \dfrac{3}{n^2} \sec^2 \dfrac{9}{n^2} + \ldots + \dfrac{1}{n} \sec^2 1\right]$.

(Lucknow 2010)

Solution: Here the rth term

$= \dfrac{r}{n^2} \sec^2 \dfrac{r^2}{n^2} = \dfrac{1}{n} \cdot \left\{\dfrac{r}{n} \sec^2 \dfrac{r^2}{n^2}\right\}$, and r varies from 1 to n.

$\therefore$ The given limit $= \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \dfrac{1}{n} \cdot \left\{\dfrac{r}{n}\sec^2 \dfrac{r^2}{n^2}\right\}$

$= \displaystyle\int_0^1 x \sec^2 x^2\, dx = \frac{1}{2}\int_0^1 \sec^2 t\, dt$, putting $x^2 = t$ so that

$2x\,dx = dt$ and the limits for t are 0 to 1

$$= \frac{1}{2}[\tan t]_0^1 = \frac{1}{2}(\tan 1 - \tan 0) = \frac{1}{2}\tan 1.$$

Problem 13: *Evaluate* $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \dfrac{r^3}{r^4+m^4}$.

Solution: Here $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \dfrac{r^3}{r^4+n^4} = \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \dfrac{1}{n^4}\left\{\dfrac{r^3}{(r/n)^4+1}\right\} = \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \dfrac{1}{n} \cdot \left\{\dfrac{(r/n)^3}{(r/n)^4+1}\right\}$

(Note)

$$= \int_0^1 \frac{x^3}{x^4+1}\,dx = \frac{1}{4}[\log(1+x^4)]_0^1 = \frac{1}{4}\log 2.$$

Problem 14: *Evaluate* $\lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{n} \cdot \sqrt{\left(\frac{n+r}{n-r}\right)}.$

(Lucknow 2014)

Solution: Here $\lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{n} \cdot \sqrt{\left(\frac{n+r}{n-r}\right)} = \lim_{n\to\infty} \sum_{r=0}^{n-1} \frac{1}{n} \cdot \sqrt{\left\{\frac{1+(r/n)}{1-(r/n)}\right\}}$

$$= \int_0^1 \sqrt{\left(\frac{1+x}{1-x}\right)}\,dx = \int_0^1 \frac{(1+x)}{\sqrt{(1-x^2)}}\,dx. \qquad \textbf{(Note)}$$

Now put $x = \sin\theta$ so that $dx = \cos\theta\,d\theta$.
Also $\theta = 0$ when $x = 0$ and $\theta = \pi/2$ when $x = 1$.

$\therefore$ The required limit $= \int_0^{\pi/2} \frac{1+\sin\theta}{\cos\theta}\cos\theta\,d\theta = \int_0^{\pi/2} (1+\sin\theta)\,d\theta = [\theta - \cos\theta]_0^{\pi/2}$

$$= \left(\frac{\pi}{2} - 0\right) - (0-1) = \frac{\pi}{2} + 1.$$

Problem 15: *Evaluate* $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{\sqrt{n}}{\sqrt{r}\cdot(3\sqrt{r}+4\sqrt{n})^2}.$

Solution: The given limit $= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \frac{1}{\sqrt{(r/n)}.\{3\sqrt{(r/n)}+4\}^2} = \int_0^1 \frac{dx}{\sqrt{x}.(3\sqrt{x}+4)^2}.$

Now put $3\sqrt{x} + 4 = t$, so that $3.(1/2\sqrt{x})\,dx = dt$.
The limits for t are 4 to 7.

$\therefore$ The required limit $= \frac{2}{3}\int_4^7 \frac{dt}{t^2} = -\frac{2}{3}\left[\frac{1}{t}\right]_4^7 = -\frac{2}{3}\left[\frac{1}{7} - \frac{1}{4}\right] = \frac{1}{14}.$

Problem 16: *Evaluate* $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{n^2}{(n^2+r^2)^{3/2}}.$

Solution: The given limit $= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \frac{1}{[1+(r/n)^2]^{3/2}}$

$= \int_0^1 \frac{dx}{(1+x^2)^{3/2}} = \int_0^{\pi/4} \frac{\sec^2\theta\,d\theta}{\sec^3\theta}$, putting $x = \tan\theta$ so that $dx = \sec^2\theta\,d\theta$ and the limits for θ are 0 to $\pi/4$

$$= \int_0^{\pi/4} \cos\theta\,d\theta = [\sin\theta]_0^{\pi/4} = \sin\frac{\pi}{4} - \sin 0 = \frac{1}{\sqrt{2}}.$$

Problem 17: *Evaluate* $\lim_{n\to\infty}\left[\frac{1}{\sqrt{(2n-1^2)}} + \frac{1}{\sqrt{(4n-2^2)}} + \ldots + \frac{1}{n}\right].$

Solution: Here the rth term

$$= \frac{1}{\sqrt{(2nr - r^2)}} = \frac{1}{n} \cdot \frac{1}{\sqrt{\{2(r/n) - (r/n)^2\}}}, \text{ and } r \text{ varies from } 1 \text{ to } n.$$

$\therefore$ The given limit $= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \frac{1}{\sqrt{\{2(r/n) - (r/n)^2\}}}$

$$= \int_0^1 \frac{dx}{\sqrt{(2x - x^2)}} = \int_0^1 \frac{dx}{\sqrt{\{1 - (x-1)^2\}}} = [\sin^{-1}(x-1)]_0^1$$

$$= \sin^{-1} 0 - \sin^{-1}(-1) = 0 + \sin^{-1} 1 = \pi/2.$$

Problem 18: *Evaluate*

$$\lim_{n \to \infty} \left[\frac{n}{(n+1)\sqrt{(2n+1)}} + \frac{n}{(n+2)\sqrt{\{2(2n+2)\}}} + \ldots + \frac{n}{2n\sqrt{(n.3n)}} \right].$$

Solution: Here the rth term $= \frac{n}{(n+r)\sqrt{\{r(2n+r)\}}} = \frac{n}{(n+r)\, n \sqrt{[(r/n)\{2 + (r/n)\}]}}$

$$= \frac{1}{n} \cdot \frac{1}{\{1 + (r/n)\}\sqrt{[(r/n)\{2 + (r/n)\}]}}.$$

Also r varies from 1 to n.

$\therefore$ The given limit $= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \frac{1}{\{1 + (r/n)\}\sqrt{[(r/n)\{2 + (r/n)\}]}}$

$$= \int_0^1 \frac{dx}{(1+x)\sqrt{\{x(2+x)\}}} = \int_0^1 \frac{dx}{(1+x)\sqrt{\{(1+x)^2 - 1\}}}$$

$$= [\sec^{-1}(1+x)]_0^1 = \sec^{-1} 2 - \sec^{-1} 1 = \frac{\pi}{3} - 0 = \frac{\pi}{3}.$$

Problem 19: *Evaluate*

$$\lim_{n \to \infty} \left[\frac{(n-m)^{1/3}}{n} + \frac{(2^2 n - m)^{1/3}}{2n} + \frac{(3^2 n - m)^{1/3}}{3n} + \ldots + \frac{(n^3 - m)^{1/3}}{n^2} \right].$$

Solution: Here the rth term

$$= \frac{(r^2 n - m)^{1/3}}{r\, n} = \frac{n \left[\frac{r^2}{n^2} - \frac{m}{n^3} \right]^{1/3}}{r \cdot n} = \frac{1}{n} \cdot \frac{1}{(r/n)} \cdot \left[\frac{r^2}{n^2} - \frac{m}{n^3} \right]^{1/3}.$$

$\therefore$ The given limit $= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \frac{1}{(r/n)} \cdot \left[\frac{r^2}{n^2} - \frac{m}{n^3} \right]^{1/3}$

$$= \int_0^1 \frac{(x^2 - 0)^{1/3}}{x}\, dx = \int_0^1 x^{-1/3}\, dx = \left[\frac{3}{2} x^{2/3} \right]_0^1 = \frac{3}{2}.$$

Problem 20: *Evaluate* $\lim_{n \to \infty} \left[\frac{1}{na} + \frac{1}{na+1} + \frac{1}{na+2} + \ldots + \frac{1}{nb} \right].$

(Kumaun 2011)

Solution: Here the last term $= \dfrac{1}{nb} = \dfrac{1}{na + n(b-a)}$. **(Note)**

Now the rth term $= \dfrac{1}{na + r}$, and r varies from 0 to $n(b-a)$.

$\therefore$ The given limit $= \lim\limits_{n \to \infty} \sum\limits_{r=0}^{n(b-a)} \dfrac{1}{na + r} = \lim\limits_{n \to \infty} \sum\limits_{r=0}^{n(b-a)} \dfrac{1}{n} \cdot \left\{ \dfrac{1}{a + (r/n)} \right\}$.

Also the lower limit of integration

$$= \lim_{n \to \infty} \left(\frac{r}{n} \right), \text{ for the 1st term}$$

$$= 0, \qquad [\because r = 0 \text{ for the 1st term}]$$

and upper limit $= \lim\limits_{n \to \infty} \left(\dfrac{r}{n} \right)$, for the last term

$$= \lim_{n \to \infty} \frac{n(b-a)}{n}, \qquad [\because r = n(b-a) \text{ for the last term}]$$

$$= (b-a).$$

$\therefore$ The required limit

$$= \int_0^{(b-a)} \frac{1}{a+x}\, dx = [\log(a+x)]_0^{(b-a)} = \log b - \log a = \log(b/a).$$

Problem 21: *Evaluate* $\lim\limits_{n \to \infty} \dfrac{1 + 2^{10} + 3^{10} + \ldots + n^{10}}{n^{11}}$.

Solution: The given limit may be written as

$$\lim_{n \to \infty} \left[\frac{1}{n} \cdot \left\{ \left(\frac{1}{n} \right)^{10} + \left(\frac{2}{n} \right)^{10} + \left(\frac{3}{n} \right)^{10} + \ldots + \left(\frac{n}{n} \right)^{10} \right\} \right]. \qquad \textbf{(Note)}$$

Now the rth term $= \dfrac{1}{n} \cdot \left(\dfrac{r}{n} \right)^{10}$, and r varies from 1 to n.

$\therefore$ The given limit $= \lim\limits_{n \to \infty} \sum\limits_{r=1}^{n} \dfrac{1}{n} \cdot \left(\dfrac{r}{n} \right)^{10} = \int_0^1 x^{10}\, dx = \left[\dfrac{x^{11}}{11} \right]_0^1 = \dfrac{1}{11}$.

Problem 22: *Prove that* $\lim\limits_{n \to \infty} \dfrac{1^m + 2^m + 3^m + \ldots + n^m}{n^{m+1}} = \dfrac{1}{m+1}, (m > 1)$.

Solution: The given limit $= \lim\limits_{n \to \infty} \sum\limits_{r=1}^{n} \dfrac{r^m}{n^{m+1}} = \lim\limits_{n \to \infty} \dfrac{1}{n} \sum\limits_{r=1}^{n} \left(\dfrac{r}{n} \right)^m$

$$= \int_0^1 x^m\, dx = \left[\frac{x^{m+1}}{m+1} \right]_0^1 = \frac{1}{m+1}.$$

Problem 23: *Evaluate* $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right]^{1/n}$. **(Lucknow 2009)**

Solution: Let $$P=\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right]^{1/n}.$$

Then $$\log P=\lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+\ldots+\log\left(1+\frac{n}{n}\right)\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\log\left(1+\frac{r}{n}\right)=\int_0^1\log(1+x)\,dx$$

$$=\int_0^1\{\log(1+x)\}.1\,dx=[\{\log(1+x)\}.x]_0^1-\int_0^1\frac{x}{1+x}\,dx,$$

integrating by parts taking 1 as the 2nd function

$$=\log 2-\int_0^1\frac{(1+x)-1}{1+x}\,dx=\log 2-\int_0^1 dx+\int_0^1\frac{1}{1+x}\,dx$$

$$=\log 2-[x]_0^1+[\log(1+x)]_0^1=\log 2-1+\log 2$$

$$=2\log 2-1=\log 2^2-\log e=\log(4/e).$$

$\therefore$ $$P=4/e.$$

Problem 24: *Evaluate* $\lim_{n\to\infty}\left[\frac{(n+1)(n+2)(n+3)\ldots(n+n)}{n^n}\right]^{1/n}$.

Solution: The given limit

$$=\lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right)\ldots\left(\frac{n+n}{n}\right)\right]^{1/n}$$

$$=\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right]^{1/n}.$$

Now proceed as in problem 23.

Problem 25: *Evaluate* $\lim_{n\to\infty}\left[\left(1+\frac{1}{n^4}\right)\left(1+\frac{2^4}{n^4}\right)^{1/2}\left(1+\frac{3^4}{n^4}\right)^{1/3}\ldots\left(1+\frac{n^4}{n^4}\right)^{1/n}\right]$.

Solution: Let $$P=\lim_{n\to\infty}\left[\left(1+\frac{1}{n^4}\right)\left(1+\frac{2^4}{n^4}\right)^{1/2}\left(1+\frac{3^4}{n^4}\right)^{1/3}\ldots\left(1+\frac{n^4}{n^4}\right)\right]^{1/n}$$

$\therefore$ $$\log P=\lim_{n\to\infty}\left[\log\left(1+\frac{1}{n^4}\right)+\frac{1}{2}\log\left(1+\frac{2^4}{n^4}\right)+\frac{1}{3}\log\left(1+\frac{3^4}{n^4}\right)+\ldots+\frac{1}{n}\log\left(1+\frac{n^4}{n^4}\right)\right]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{r} \log\left(1+\frac{r^4}{n^4}\right) = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n}\cdot\frac{n}{r}\left(1+\frac{r^4}{n^4}\right)$$

$$= \int_0^1 \frac{1}{x}\log(1+x^4)\,dx = \int_0^1 \frac{1}{x}\left[x^4 - \frac{x^8}{2} + \frac{x^{12}}{3} - \ldots \infty\right] dx$$

$$= \int_0^1 \left[x^3 - \frac{x^7}{2} + \frac{x^{11}}{3} - \ldots \infty\right] dx = \left[\frac{x^4}{4} - \frac{x^8}{16} + \frac{x^{12}}{36} - \ldots \infty\right]_0^1$$

$$= \frac{1}{4} - \frac{1}{16} + \frac{1}{36} - \ldots \infty = \frac{1}{4}\left[1 - \frac{1}{4} + \frac{1}{9} - \ldots \infty\right]$$

$$= \frac{1}{4}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \ldots \infty\right] = \frac{1}{4}\cdot\frac{\pi^2}{12}, \text{ from trigonometry.}$$

Thus $\log P = \dfrac{\pi^2}{48}\cdot$

$\therefore$ $P = e^{\pi^2/48}.$

Problem 26: *Evaluate* $\displaystyle\lim_{n\to\infty}\left[\sin\frac{\pi}{2n}\sin\frac{2\pi}{2n}\sin\frac{3\pi}{2n}\ldots\sin\frac{n\pi}{2n}\right]^{1/n}.$

Solution: Let $\displaystyle P = \lim_{n\to\infty}\left[\sin\frac{\pi}{2n}\sin\frac{2\pi}{2n}\ldots\sin\frac{n\pi}{2n}\right]^{1/n}.$

$\therefore$ $$\log P = \lim_{n\to\infty}\frac{1}{n}\left[\log\left(\sin\frac{\pi}{2n}\right)+\ldots+\log\left(\frac{\sin n\pi}{2n}\right)\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\log\left(\sin\frac{r\pi}{2n}\right) = \int_0^1 \log\left(\sin\frac{\pi}{2}x\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\log(\sin\theta)\,d\theta,$$

putting $\frac{\pi}{2}x = \theta$ and changing the limits

$$= \frac{2}{\pi}\cdot\left(-\frac{\pi}{2}\log 2\right),$$

$$\left[\because \int_0^{\pi/2}\log\sin\theta\,d\theta = -\frac{\pi}{2}\log 2. \text{ (See example 7)}\right]$$

$$= -\log 2 = \log(2^{-1}) = \log(1/2).$$

Thus $\log P = \log\left(\dfrac{1}{2}\right)$ or $P = \dfrac{1}{2}\cdot$

Problem 27: *Evaluate* $\displaystyle\lim_{n\to\infty}\left[\tan\frac{\pi}{2n}\tan\frac{2\pi}{2n}\tan\frac{3\pi}{2n}\ldots\tan\frac{n\pi}{2n}\right]^{1/n}.$ **(Kumaun 2012)**

Solution: Let $\displaystyle P = \lim_{n\to\infty}\left[\tan\frac{\pi}{2n}\tan\frac{2\pi}{2n}\ldots\tan\frac{n\pi}{2n}\right]^{1/n}.$

Then $$\log P = \lim_{n\to\infty} \frac{1}{n}\left[\log\left(\tan\frac{\pi}{2n}\right) + \log\left(\tan\frac{2\pi}{2n}\right) + \ldots + \log\left(\tan\frac{n\pi}{2n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \log\left(\tan\frac{r\pi}{2n}\right) = \int_0^1 \log\left\{\tan\frac{\pi}{2}x\right\} dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \log(\tan\theta)\, d\theta,$$

putting $(\pi x/2) = \theta$ and changing the limits accordingly

$$= \frac{2}{\pi}\int_0^{\pi/2} \log\left(\frac{\sin\theta}{\cos\theta}\right) d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \log\sin\theta\, d\theta - \frac{2}{\pi}\int_0^{\pi/2} \log\cos\theta\, d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \log\sin\left(\frac{\pi}{2} - \theta\right) d\theta - \frac{2}{\pi}\int_0^{\pi/2} \log\cos\theta\, d\theta$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \log\cos\theta\, d\theta - \frac{2}{\pi}\int_0^{\pi/2} \log\cos\theta\, d\theta = 0.$$

$\therefore$ $P = e^0 = 1$.

Problem 28: *Evaluate the limit*

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{2/n^2}\left(1+\frac{2^2}{n^2}\right)^{4/n^2}\left(1+\frac{3^2}{n^2}\right)^{6/n^2}\ldots\left(1+\frac{n^2}{n^2}\right)^{2n/n^2}.$$

(Kumaun 2009)

Solution: Let the required limit be P. Then

$$\log P = \lim_{n\to\infty}\left[\frac{2}{n^2}\log\left(1+\frac{1}{n^2}\right) + \frac{4}{n^2}\log\left(1+\frac{2^2}{n^2}\right) + \ldots + \frac{2n}{n^2}\log\left(1+\frac{n^2}{n^2}\right)\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{2r}{n^2}\log\left(1+\frac{r^2}{n^2}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} 2\left(\frac{r}{n}\right)\log\left\{1+\left(\frac{r}{n}\right)^2\right\}$$

$$= \int_0^1 2x\log(1+x^2)\,dx.$$

Now put $1 + x^2 = t$, so that $2x\,dx = dt$.

When $x = 0$, $t = 1$ and when $x = 1$, $t = 2$.

$\therefore$ $$\log P = \int_0^2 \log t\,dt = [t\log t]_1^2 - \int_1^2 t\cdot\frac{1}{t}\,dt,$$

integrating by parts taking 1 as the second function

$$= (2\log 2 - \log 1) - \int_1^2 dt = 2\log 2 - [t]_1^2$$

$$= \log 2^2 - (2-1) = \log 4 - 1 = \log 4 - \log e = \log(4/e).$$

$\therefore$ $P = 4/e$.

Hints to Objective Type Questions

Multiple Choice Questions

1. We have $\int_{-\pi/2}^{\pi/2} \sin^2 x \, dx = 2 \int_0^{\pi/2} \sin^2 x \, dx = 2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{2} \cdot$

2. We have $\lim_{n \to \infty} \sum_{r=1}^{n} \frac{r^3}{r^4 + n^4} = \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \frac{(r/n)^3}{1 + (r/n)^4} = \int_0^1 \frac{x^3}{1 + x^4} dx$

$$= \frac{1}{4} \int_0^1 \frac{4x^3}{1 + x^4} dx = \frac{1}{4} [\log (1 + x^4)]_0^1 = \frac{1}{4} [\log 2 - \log 1] = \frac{1}{4} \log 2.$$

3. If $f(x) = \sin^3 x$, we have

$$f(-x) = \sin^3 (-x) = -\sin^3 x = -f(x).$$

$\therefore \quad \int_{-\pi/2}^{\pi/2} \sin^3 x \, dx = 0.$

4. We have $f[\sin (\pi - x)] = f(\sin x)$.

Now we know that $\int_0^{2a} f(x) \, dx = 2 \int_0^a f(x) \, dx$, if $f(2a - x) = f(x)$.

$\therefore \quad \int_0^{\pi} f(\sin x) \, dx = 2 \int_0^{\pi/2} f(\sin x) \, dx.$

Fill in the Blank(s)

1. If $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x) \, dx = 0$.

See article 2, Property 5.

2. If $f(2a - x) = -f(x)$, then $\int_0^{2a} f(x) \, dx = 0$.

See article 2, Property 6.

3. If $f(-x) = f(x)$, then $\int_{-a}^{a} f(x) \, dx = 2 \int_0^a f(x) \, dx$.

See article 2, Property 5.

4. If $f(2a - x) = f(x)$, then $\int_0^{2a} f(x) \, dx = 2 \int_0^a f(x) \, dx$.

See article 2, Property 6.

5. If $f(x) = \sin^3 x \cos^2 x$, then

$$f(-x) = [\sin (-x)]^3 [\cos (-x)]^2 = (-\sin x)^3 (\cos x)^2$$
$$= -\sin^3 x \cos^2 x = -f(x).$$

$\therefore \quad \int_{-\pi/2}^{\pi/2} \sin^3 x \cos^2 x \, dx = 0.$

6. If $f(x) = \dfrac{x^2 \sin^{-1} x}{\sqrt{(1-x^2)}}$, then

$$f(-x) = \frac{(-x)^2 \sin^{-1}(-x)}{\sqrt{[1-(-x)^2]}} = -\frac{x^2 \sin^{-1} x}{\sqrt{(1-x^2)}} = -f(x).$$

$$\therefore \quad \int_{-1}^{1} \frac{x^2 \sin^{-1} x}{\sqrt{(1-x^2)}}\, dx = 0.$$

7. Let $I = \displaystyle\int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x}\, dx.$

Then $I = \displaystyle\int_0^{\pi/2} \frac{\cos(\pi/2 - x) - \sin(\pi/2 - x)}{1 + \sin(\pi/2 - x)\cos(\pi/2 - x)}\, dx$

$$= \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \cos x \sin x}\, dx = -\int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x}\, dx = -I.$$

$\therefore \quad 2I = 0 \quad \text{or} \quad I = 0.$

8. Let $I = \displaystyle\int_0^{\pi/2} \frac{dx}{1 + \sqrt{(\tan x)}} = \int_0^{\pi/2} \frac{\sqrt{(\cos x)}}{\sqrt{(\cos x)} + \sqrt{(\sin x)}}\, dx.$ ...(1)

Then $I = \displaystyle\int_0^{\pi/2} \frac{\sqrt{[\cos(\pi/2 - x)]}}{\sqrt{[\cos(\pi/2 - x)]} + \sqrt{[\sin(\pi/2 - x)]}}\, dx$

$= \displaystyle\int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}}\, dx.$ ...(2)

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{(\cos x)} + \sqrt{(\sin x)}}{\sqrt{(\cos x)} + \sqrt{(\sin x)}}\, dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}.$$

$\therefore \quad I = \dfrac{\pi}{4}.$

9. Proceed as in Example 10.

10. We have $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1 + (r/n)} = \int_0^1 \frac{1}{1+x}\, dx = [\log(1+x)]_0^1$

$$= \log 2 - \log 1 = \log 2 - 0 = \log 2.$$

True or False

1. We have $\displaystyle\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx.$

2. See article 2, Property 4.

3. If $f(x) = \sin^m x \cos^{2m+1} x$, then

$$f(\pi - x) = \sin^m(\pi - x)\cos^{2m+1}(\pi - x) = \sin^m x \cdot (-\cos x)^{2m+1}$$
$$= (-1)^{2m+1} \sin^m x \cos^{2m+1} x = -\sin^m x \cos^{2m+1} x = -f(x).$$

$\therefore \quad \int_0^{\pi} \sin^m x \cos^{2m+1} x \, dx = 0.$ [See article 2, Property 6]

4. Let $I = \int_0^{\pi/2} \frac{x \, dx}{\sin x + \cos x}$. ...(1)

Then $I = \int_0^{\pi/2} \frac{(\pi/2 - x) \, dx}{\sin(\pi/2 - x) + \cos(\pi/2 - x)} dx$

$= \int_0^{\pi/2} \frac{(\pi/2 - x) \, dx}{\sin x + \cos x}$. ...(2)

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\pi/2}{\sin x + \cos x} dx \quad \text{or} \quad I = \frac{\pi}{4} \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}.$$

5. If $f(x) = \cos^3 x$, then

$$f(-x) = \cos^3(-x) = \cos^3 x = f(x).$$

$$\therefore \quad \int_{-\pi/2}^{\pi/2} \cos^3 x \, dx = 2 \int_0^{\pi/2} \cos^3 x \, dx = 2 \cdot \frac{2}{3 \cdot 1} = \frac{4}{3}.$$

6. We have

$$\lim_{n \to \infty} \sum_{r=0}^{2n} \frac{1}{n+r} = \lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{2n} \frac{1}{1 + (r/n)} = \int_0^2 \frac{1}{1+x} dx.$$

7. We have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{n+1} \frac{1}{1 + (r/n)^2} = \int_0^1 \frac{1}{1+x^2} dx$$

$$= [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

8. We have

$$\lim_{n \to \infty} \left[\frac{1}{n^3} (1 + 4 + 9 + 16 + \ldots + n^2) \right]$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{r^2}{n^3} = \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} (r/n)^2 = \int_0^1 x^2 \, dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3}.$$

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Chapter-2

Reduction Formulae

(For Trigonometric Functions)

Comprehensive Problems 1

Problem 1: *Evaluate* $\int \sin^6 x \, dx.$

Solution: We know that

$$\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{(n-1)}{n} \int \sin^{n-2} x \, dx.$$

[Establish the formula here]

Taking $n = 6$ and applying the above formula successively, we have

$$\int \sin^6 x \, dx = -\frac{\sin^{6-1} x \cos x}{6} + \frac{6-1}{6} \int \sin^{6-2} x \, dx$$

$$= -\frac{1}{6} \sin^5 x \cos x + \frac{5}{6} \int \sin^4 x \, dx$$

$$= -\frac{1}{6} \sin^5 x \cos x + \frac{5}{6} \left[-\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x \, dx \right]$$

$$= -\frac{1}{6} \sin^5 x \cos x - \frac{5}{24} \sin^3 x \cos x + \frac{5}{8} \int \sin^2 x \, dx$$

$$= -\frac{1}{6} \sin^5 x \cos x - \frac{5}{24} \sin^3 x \cos x + \frac{5}{8} \int \frac{1}{2} (1 - \cos 2x) \, dx$$

$$= -\frac{1}{6} \sin^5 x \cos x - \frac{5}{24} \sin^3 x \cos x + \frac{5}{8} \left(\frac{1}{2} x - \frac{1}{2} \cdot \frac{1}{2} \sin 2x\right)$$

$$= -\frac{1}{6} \sin^5 x \cos x - \frac{5}{24} \sin^3 x \cos x - \frac{5}{16} \sin x \cos x + \frac{5}{16} x .$$

Problem 2: *Evaluate* $\int_0^{\pi/2} \sin^6 x \, dx$. **(Kanpur 2001, 05, 06)**

Solution: Here $n = 6$ (even). Hence from article 3 (case II), we get

$$\int_0^{\pi/2} \sin^6 x \, dx = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{5\pi}{32} \cdot$$

Problem 3: *Evaluate* $\int_0^{\pi/2} \cos^9 x \, dx.$

Solution: Here $n = 9$ (odd). Hence from article 3 (Case I), we get

$$\int_0^{\pi/2} \cos^9 x\, dx = \frac{8.6.4.2}{9.7.5.3.1} \cdot 1 = \frac{128}{315}.$$

Problem 4: *Evaluate* $\int_0^{\pi/2} \sin^{10} x\, dx$ *or* $\int_0^{\pi/2} \cos^{10} x\, dx$.

Solution: Here $n = 10$ (even) in both the cases.

$$\therefore \quad \int_0^{\pi/2} \sin^{10} x\, dx = \int_0^{\pi/2} \cos^{10} x\, dx = \frac{9.7.5.3.1}{10.8.6.4.2} \cdot \frac{\pi}{2} = \frac{63\pi}{512}.$$

Problem 5: *Evaluate* $\int_0^{\pi/4} \tan^5\theta\, d\theta$.

Solution: Proceeding as in article 4 (a), we have

$$\int \tan^n\theta\, d\theta = \frac{\tan^{n-1}\theta}{n-1} - \int \tan^{n-2}\theta\, d\theta. \qquad ...(1)$$

Putting $n = 5$ in (1), we have

$$\int \tan^5\theta\, d\theta = \frac{1}{4}\tan^4\theta - \int \tan^3\theta\, d\theta$$

$$= \frac{1}{4}\tan^4\theta - [\frac{1}{2}\tan^2\theta - \int \tan\theta\, d\theta], \quad \text{putting } n = 3 \text{ in (1)}$$

$$= \frac{1}{4}\tan^4\theta - \frac{1}{2}\tan^2\theta - \log\cos\theta.$$

$$\therefore \quad \int_0^{\pi/4} \tan^5\theta\, d\theta = \left[\frac{1}{4}\tan^4\theta - \frac{1}{2}\tan^2\theta - \log\cos\theta\right]_0^{\pi/4}$$

$$= [\frac{1}{4} - \frac{1}{2} - \log\cos\frac{1}{4}\pi] - [0 - \log\cos 0] = [-\frac{1}{4} - \log(1/\sqrt{2})]$$

$$= [-\frac{1}{4} + \frac{1}{2}\log 2] = \frac{1}{2}[\log 2 - \frac{1}{2}].$$

Problem 6: *Evaluate* $\int_0^a x^5 (2a^2 - x^2)^{-3} dx$.

Solution: Put $x = \sqrt{2}\,.\,a\sin\theta$, so that $dx = \sqrt{2}\,.\,a\cos\theta\, d\theta$.

Also when $x = 0$, $\sin\theta = 0$ or $\theta = 0$ and when $x = a$, $\sin\theta = 1/\sqrt{2}$ or $\theta = \pi/4$.

Making these substitutions the given integral

$$= \int_0^{\pi/4} (\sqrt{2}\,.a\sin\theta)^5 . (2a^2 - 2a^2\sin^2\theta)^{-3} . \sqrt{2}\,.a\cos\theta\, d\theta$$

$$= \int_0^{\pi/4} \left[\frac{2^{5/2} a^5 \sin^5\theta \,.\, 2^{1/2} a\cos\theta}{2^3 a^6 \cos^6\theta}\right] d\theta$$

$$= \int_0^{\pi/4} \tan^5\theta\, d\theta = \frac{1}{2}[\log 2 - \frac{1}{2}]. \qquad \text{[See Problem 5]}$$

Problem 7: *Evaluate* $\int \sec^3 x\, dx$.

Solution: We have $\int \sec^3 x\, dx = \int \sec x . \sec^2 x\, dx$

$= \sec x \tan x - \int \sec x \tan x \tan x\, dx,$

(integrating by parts taking $\sec^2 x$ as the 2nd function)

` $= \sec x \tan x - \int \sec x \tan^2 x\, dx = \sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx$

$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx.$

Transposing the term $-\int \sec^3 x\, dx$ to the left, we have

$$2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$$

or $$\int \sec^3 x\, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \log(\sec x + \tan x).$$

Problem 8: *Evaluate* $\int_0^{\pi/4} \sec^3 x\, dx.$

Solution: Let $$I = \int_0^{\pi/4} \sec^3 x\, dx = \int_0^{\pi/4} \sec x . \sec^2 x\, dx$$

$$= \int_0^{\pi/4} \sqrt{(1 + \tan^2 x)} . \sec^2 x\, dx.$$

Now put $\tan x = t$, so that $\sec^2 x\, dx = dt$. When $x = 0$, $t = 0$ and when $x = \frac{1}{4}\pi$, $t = 1$.

$\therefore$ $$I = \int_0^1 \sqrt{(1 + t^2)}\, dt = \left[\frac{1}{2} t \sqrt{(1 + t^2)} + \frac{1}{2} \log\{t + \sqrt{(1 + t^2)}\}\right]_0^1$$

$$= [\frac{1}{2}\sqrt{2} + \frac{1}{2}\log(1 + \sqrt{2})] - [0 + \frac{1}{2}\log 1] = \frac{1}{2}\sqrt{2} + \frac{1}{2}\log(\sqrt{2} + 1).$$

Problem 9: *Evaluate* $\int_0^a (a^2 + x^2)^{5/2}\, dx$.

Solution: Put $x = a \tan\theta$, so that $dx = a \sec^2\theta\, d\theta$.

Then $$I = \int_0^a (a^2 + x^2)^{5/2}\, dx$$

$$= \int_0^{\pi/4} a^5 \sec^5\theta\; a \sec^2\theta\, d\theta = a^6 \int_0^{\pi/4} \sec^7\theta\, d\theta.$$

Now form a reduction formula for $\int \sec^n\theta\, d\theta$. By repeated application of this formula, we get

$$I = a^6 \left[\left(\frac{1}{6} \sec^5 \theta \tan \theta \right)_0^{\pi/4} + \frac{5}{6} \int_0^{\pi/4} \sec^5 \theta \, d\theta \right]$$

$$= a^6 \left[\frac{4\sqrt{2}}{6} + \frac{5}{6} \left(\frac{\sec^3 \theta \tan \theta}{4} \right)_0^{\pi/4} + \frac{5}{6} \cdot \frac{3}{4} \int_0^{\pi/4} \sec^3 \theta \, d\theta \right]$$

$$= a^6 \left[\frac{2\sqrt{2}}{3} + \frac{5\sqrt{2}}{12} + \frac{5}{8} \cdot \frac{1}{2} (\sec \theta \tan \theta)_0^{\pi/4} + \frac{5}{8} \cdot \frac{1}{2} \int_0^{\pi/4} \sec \theta \, d\theta \right]$$

$$= a^6 \left[\frac{2\sqrt{2}}{3} + \frac{5\sqrt{2}}{12} + \frac{5\sqrt{2}}{16} + \frac{5}{16} \log \tan \left(\frac{1}{4}\pi + \frac{1}{2}\theta \right)_0^{\pi/4} \right]$$

$$= a^6 \left[\frac{2\sqrt{2}}{3} + \frac{5\sqrt{2}}{12} + \frac{5\sqrt{2}}{16} + \frac{5}{16} \log \tan \left(\frac{3}{8}\pi \right) \right]$$

$$= a^6 \left[\frac{67\sqrt{2}}{48} + \frac{5}{16} \log \tan \left(\frac{3}{8}\pi \right) \right]$$

$$= \frac{a^6}{48} \left[67\sqrt{2} + 15 \log \tan \left(\frac{3}{8}\pi \right) \right].$$

Problem 10: *Evaluate* $\int_0^{\pi/4} \sin^2 \theta \cos^4 \theta \, d\theta$.

Solution: From article 6, we have the reduction formula

$$\int \sin^m x \cos^n x \, dx = \frac{\sin^{m+1} x \,.\, \cos^{n-1} x}{m+n} + \frac{n-1}{m+1} \int \sin^m x \cos^{n-2} x \, dx.$$

[Derive it here]

Here $m = 2$ and $n = 4$; hence we have

$$\int_0^{\pi/4} \sin^2 \theta \cos^4 \theta \, d\theta = \left[\frac{\sin^3 \theta \cos^3 \theta}{4} \right]_0^{\pi/4} + \frac{3}{6} \int_0^{\pi/4} \sin^2 \theta \cos^2 \theta \, d\theta$$

$$= \frac{1}{48} + \frac{1}{2} \left[\left(\frac{\sin^3 \theta \cos \theta}{4} \right)_0^{\pi/4} + \frac{1}{4} \int_0^{\pi/4} \sin^2 \theta \, d\theta \right],$$

[Putting $m = 2$ and $n = 2$ in the above reduction formula]

$$= \frac{1}{48} + \frac{1}{32} + \frac{1}{8} \cdot \frac{1}{2} \int_0^{\pi/4} (1 - \cos 2\theta) \, d\theta$$

$$= \frac{1}{48} + \frac{1}{32} + \frac{1}{16} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/4}$$

$$= \frac{1}{48} + \frac{1}{32} + \frac{1}{16} \left[\frac{\pi}{4} - \frac{1}{2} \right] = \frac{1}{48} + \frac{\pi}{64}.$$

Problem 11: *Evaluate* $\int \tan^6 x \, dx$.

Solution: We have $\int \tan^6 x \, dx = \frac{1}{5}\tan^5 x - \int \tan^4 x \, dx$

$$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \int \tan^2 x \, dx$$

$$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \int (\sec^2 x - 1) \, dx$$

$$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x$$

Problem 12: *Show that* $\int_0^a \frac{x^4}{\sqrt{(a^2 - x^2)}} dx = \frac{3a^4 \pi}{16}$.

Solution: Put $x = a \sin \theta$, so that $dx = a \cos \theta \, d\theta$.

Also when $x = 0$, $\sin \theta = 0$ *i.e.*, $\theta = 0$

and when $x = a$, $\sin \theta = 1$ *i.e.*, $\theta = \pi/2$.

Then
$$\int_0^a \frac{x^4 \, dx}{\sqrt{(a^2 - x^2)}} = \int_0^{\pi/2} \frac{a^4 \sin^4 \theta \, a \cos \theta \, d\theta}{\sqrt{(a^2 - a^2 \sin^2 \theta)}}$$

$$= a^4 \int_0^{\pi/2} \sin^4 \theta \, d\theta = a^4 . \frac{3 . 1}{4 . 2} \cdot \frac{\pi}{2} = \frac{3\pi a^4}{16}.$$

Problem 13: *If* $I_n = \int_0^{\pi/4} \tan^n x \, dx$, *show that* $I_n + I_{n-2} = \frac{1}{n-1}$,

and deduce the value of I_5.

(Kanpur 2005, 12; Bundelkhand 06; Avadh 06, 11; Purvanchal 14)

Solution: We know by article 4 (a) that

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx \qquad \text{(derive it here)}$$

$\therefore$
$$I_n = \int_0^{\pi/4} \tan^n x \, dx = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2} x \, dx$$

$$= \frac{1}{n-1} - I_{n-2}, \qquad \left[\because I_{n-2} = \int_0^{\pi/4} \tan^{n-2} x \, dx\right]$$

or
$$I_n + I_{n-2} = \frac{1}{n-1}.$$

Putting $n = 5$ in the reduction formula

$$I_n = \frac{1}{n-1} - I_{n-2},$$

we get
$$I_5 = \frac{1}{4} - I_3 = \frac{1}{4} - [\frac{1}{2} - I_1] = \frac{1}{4} - \frac{1}{2} + I_1$$

$$= -\frac{1}{4} + \int_0^{\pi/4} \tan x \, dx = -\frac{1}{4} + [\log \sec x]_0^{\pi/4}$$

$$= -\frac{1}{4} + \left[\log \sec \frac{\pi}{4} - \log \sec 0\right] = -\frac{1}{4} + [\log \sqrt{2} - \log 1]$$

$$= \frac{1}{2} \log 2 - \frac{1}{4} = \frac{1}{2}\left(\log 2 - \frac{1}{2}\right).$$

Problem 14: *If* $I_n = \int_0^{\pi/4} \tan^n x \, dx$, *prove that*

$$n(I_{n-1} + I_{n+1}) = 1.$$ **(Kanpur 2005, 12; Avadh 06)**

Solution: Proceeding as in article 4 (1), we have

$$I_{n+1} = \int_0^{\pi/4} \tan^{n+1} x \, dx = \left[\frac{\tan^n x}{n}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-1} x \, dx$$

$$= [1/n] - I_{n-1}$$

or $[I_{n+1} + I_{n-1}] = 1/n$ or $n(I_{n+1} + I_{n-1}) = 1.$

Similarly, $I_n + I_{n-2} = 1/(n-1).$

Comprehensive Problems 2

Problem 1: *Evaluate* $\int_0^{\pi/2} \sin^2 x \cos^3 x \, dx.$

(Bundelkhand 2009, 10)

Solution: Here $m = 2, n = 3$; using the Gamma function, we have the the given integral

$$= \frac{\Gamma\left(\frac{2+1}{2}\right) . \Gamma\left(\frac{3+1}{2}\right)}{2\,\Gamma\left(\frac{2+3+2}{2}\right)} = \frac{\Gamma \frac{3}{2} . \Gamma 2}{2\,\Gamma \frac{7}{2}} = \frac{\frac{1}{2}\sqrt{\pi} . 1}{2 \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}} = \frac{2}{15}.$$

Aliter: By Walli's formula, the given integral

$$= \frac{1 . 2}{5 . 3 . 1} \times 1 = \frac{2}{15}.$$

Problem 2: *Evaluate* $\int_0^{\pi/2} \sin^4 x \cos^6 x \, dx.$

Solution: Here $m = 4, n = 6$; using the Gamma function, we have the given integral

$$= \frac{\Gamma\left(\frac{4+1}{2}\right) . \Gamma\left(\frac{6+1}{2}\right)}{2\,\Gamma\left(\frac{4+6+2}{2}\right)} = \frac{\Gamma \frac{5}{2} . \Gamma \frac{7}{2}}{2\,\Gamma 6}$$

$$= \frac{\frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}}{2 . 5 . 4 . 3 . 2 . 1} = \frac{3\pi}{512}.$$

Aliter: By Walli's formula, the given integral $= \dfrac{3.1.5.3.1}{10.8.6.4.2} \cdot \dfrac{\pi}{2} = \dfrac{3\pi}{512}$.

Problem 3: *Evaluate* $\int_0^{\pi/2} \sin^5 x \cos^8 x \, dx$.

Solution: Here $m = 5$, $n = 8$; using the Gamma function, we have the given integral

$$= \frac{\Gamma\left(\frac{5+1}{2}\right)\Gamma\left(\frac{8+1}{2}\right)}{2\,\Gamma\left(\frac{5+8+2}{2}\right)} = \frac{\Gamma(3).\Gamma(\frac{9}{2})}{2\,\Gamma(\frac{15}{2})} = \frac{2.1.\Gamma\frac{9}{2}}{2\cdot\frac{13}{2}\cdot\frac{11}{2}\cdot\frac{9}{2}\Gamma\frac{9}{2}}$$

$$= \frac{2.2.2.2}{2.13.11.9} = \frac{8}{1287}.$$

Problem 4: *Evaluate* $\int_0^{\pi/2} \sin^{12} x \cos^{18} x \, dx$.

Solution: Using the Gamma function, we have the given integral

$$= \frac{\Gamma\left(\frac{12+1}{2}\right).\Gamma\left(\frac{18+1}{2}\right)}{2\,\Gamma\left(\frac{12+18+2}{2}\right)} = \frac{\Gamma\left(\frac{13}{2}\right)\Gamma\left(\frac{19}{2}\right)}{2\,\Gamma(16)}.$$

Problem 5: *Evaluate* $\int_0^{\pi/8} \cos^3 4x \, dx$.

Solution: To bring the given integral into the form of Gamma function, put $4x = \theta$, so that $4\,dx = d\theta$. Also for limits, $\theta = 0$ at $x = 0$ and $\theta = \frac{1}{2}\pi$ at $x = \frac{1}{8}\pi$.

$\therefore$ The given integral $= \dfrac{1}{4}\displaystyle\int_0^{\pi/2} \cos^3\theta\, d\theta = \dfrac{1}{4}\int_0^{\pi/2} \sin^0\theta.\cos^3\theta\, d\theta$

$$= \frac{1}{4}\cdot\frac{\Gamma\frac{1}{2}\cdot\Gamma 2}{2\,\Gamma\frac{5}{2}} = \frac{\Gamma\frac{1}{2}.1}{4.2.\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}} = \frac{1}{6}.$$

Problem 6: *Evaluate* $\int_0^{\pi} \sin^6 \frac{x}{2} \cos^8 \frac{x}{2} \, dx$.

Solution: Putting $x/2 = \theta$, so that $dx = 2\,d\theta$, the given integral

$$= 2\int_0^{\pi/2} \sin^6\theta\cos^8\theta\, d\theta = 2\cdot\frac{5.3.1.7.5.3.1}{14.12.10.8.6.4.2}\cdot\frac{\pi}{2} = \frac{5\pi}{2048}.$$

Problem 7: *Evaluate* $\int_0^{\pi/2} \cos^5 x \sin 3x \, dx$.

Solution: The given integral $I = \displaystyle\int_0^{\pi/2} \cos^5 x\,(3\sin x - 4\sin^3 x)\,dx$

$$= 3\int_0^{\pi/2} \cos^5 x \sin x\, dx - 4\int_0^{\pi/2} \cos^5 x \sin^3 x\, dx$$

$$= 3 \cdot \frac{4.2}{6.4.2} - 4 \cdot \frac{4.2.2}{8.6.4.2} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}.$$

Problem 8: *Evaluate* $\int_0^{\pi/2} \sin^3 x \cos^4 x \cos 2x\, dx$.

Solution: The given integral $I = \int_0^{\pi/2} \sin^3 x \cos^4 x (\cos^2 x - \sin^2 x)\, dx$

$$= \int_0^{\pi/2} \sin^3 x \cos^6 x\, dx - \int_0^{\pi/2} \sin^5 x \cos^4 x\, dx$$

$$= \frac{2.5.3.1}{9.7.5.3.1} - \frac{4.2.3.1}{9.7.5.3.1} = \frac{2}{63} - \frac{8}{315} = \frac{10-8}{315} = \frac{2}{315}.$$

Problem 9: *Evaluate* $\int_0^{\pi/6} \cos^4 3\phi \sin^3 6\phi\, d\phi$.

Solution: Proceed as in Example 8. Ans. $\frac{1}{15}$.

Problem 10: *Evaluate* $\int_0^1 x^2 (1 - x^2)^{3/2} dx$.

Solution: With the substitutions of Example 10 (*i.e.*, $x = \sin\theta$), we have

$$\text{the given integral} = \int_0^{\pi/2} \sin^2\theta \cos^3\theta . \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta \cos^4\theta\, d\theta = \frac{1.3.1}{6.4.2} \cdot \frac{\pi}{2}, \quad \text{[By Walli's formula]}$$

$$= \frac{\pi}{32}.$$

Problem 11: *Evaluate* $\int_0^1 x^4 (1 - x^2)^{3/2}\, dx$.

Solution: As in Example 10, putting $x = \sin\theta$ etc., we get the given integral

$$= \int_0^{\pi/2} \sin^4\theta . \cos^3\theta . \cos\theta\, d\theta = \int_0^{\pi/2} \sin^4\theta \cos^4\theta\, d\theta$$

$$= \frac{3.1.3.1}{8.6.4.2} \cdot \frac{\pi}{2}, \quad \text{[By Walli's formula]}$$

$$= 3\pi/256.$$

Problem 12: *Evaluate* $\int_0^1 x^6 (1 - x^2)^{1/2} dx$.

Solution: As in Example 10, putting $x = \sin\theta$ etc., we get the given integral

$$= \int_0^{\pi/2} \sin^6\theta \sqrt{(1-\sin^2\theta)}\cos\theta\, d\theta = \int_0^{\pi/2} \sin^6\theta \cos^2\theta\, d\theta$$

$$= \frac{5.3.1.1}{8.6.4.2} \cdot \frac{\pi}{2},$$ [By Walli's formula]

$$= \frac{5\pi}{256}.$$

Problem 13: *Evaluate* $\int_0^a x^2(a^2 - x^2)^{3/2}\, dx$.

Solution: Putting $x = a\sin\theta$ and proceeding as in Example 11, we get the given integral

$$= a^6 \int_0^{\pi/2} \sin^2\theta\cos^4\theta\, d\theta = \frac{\pi a^6}{32}.$$

Problem 14: *Evaluate* $\int_0^1 x^m(1-x)^n\, dx$.

Solution: Here we put $x = \sin^2\theta$, so that $dx = 2\sin\theta\cos\theta\, d\theta$.

Also when $x = 0, \theta = 0$ and when $x = 1, \theta = \frac{1}{2}\pi$.

$\therefore$ The given integral $= \int_0^{\pi/2} \sin^{2m}\theta \cdot \cos^{2n}\theta \cdot 2\sin\theta\cos\theta\, d\theta$

$$= 2\int_0^{\pi/2} \sin^{2m+1}\theta\cos^{2n+1}\theta\, d\theta$$

$$= 2 \cdot \frac{\Gamma\left\{\frac{1}{2}(2m+2)\right\}\Gamma\left\{\frac{1}{2}(2n+2)\right\}}{2\,\Gamma\left\{\frac{1}{2}(2m+2n+4)\right\}} = \frac{\Gamma(m+1)\,\Gamma(n+1)}{\Gamma(m+n+2)}.$$

Problem 15: *Evaluate* $\int_0^1 x^{3/2}(1-x)^{3/2}\, dx$.

Solution: Put $x = \sin^2\theta$, so that $dx = 2\sin\theta\cos\theta\, d\theta$.

Also when $x = 0, \theta = 0$ and when $x = 1, \theta = \frac{1}{2}\pi$.

$\therefore$ The given integral $= \int_0^{\pi/2} \sin^3\theta\cos^3\theta \,.\, 2\sin\theta\cos\theta\, d\theta$

$$= 2\int_0^{\pi/2} \sin^4\theta\cos^4\theta\, d\theta = 2\,\frac{3.1.3.1}{8.6.4.2}\cdot\frac{\pi}{2},$$

[By Walli's formula]

$$= \frac{3\pi}{128}.$$

Problem 16: *Evaluate* $\int_0^1 x^{3/2} \sqrt{(1-x)}\, dx$.

Solution: Making substitutions as in Problem 15, the given integral reduces to

$$2 \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta \, d\theta = 2 \cdot \frac{3.1.1}{6.4.2} \cdot \frac{\pi}{2} = \frac{\pi}{16}.$$

Problem 17: *Evaluate* $\int_0^{2a} x^m \sqrt{(2ax - x^2)}\, dx$, *m being a positive integer.*

(Kanpur 2007; Bundelkhand 07)

Solution: We have $\int_0^{2a} x^m \sqrt{(2ax - x^2)}\, dx = \int_0^{2a} x^m . x^{1/2} \sqrt{(2a - x)}\, dx$.

Now put $x = 2a \sin^2 \theta$, so that $dx = 4a \sin \theta \cos \theta \, d\theta$.

Also when $x = 0, \theta = 0$ and when $x = 2a, \theta = \frac{1}{2}\pi$.

∴ The given integral

$$= \int_0^{\pi/2} (2a \sin^2 \theta)^m (2a \sin^2 \theta)^{1/2} . \sqrt{(2a \cos^2 \theta)} . 4a \sin \theta \cos \theta \, d\theta$$

$$= 2^{m+3} a^{m+2} \int_0^{\pi/2} \sin^{2m+2} \theta \cos^2 \theta \, d\theta$$

$$= 2^{m+3} a^{m+2} \frac{\Gamma\left(\frac{2m+3}{2}\right) \Gamma \frac{3}{2}}{2 \Gamma\left(\frac{2m+2+2+2}{2}\right)}$$

$$= 2^{m+2} a^{m+2} \frac{\frac{2m+1}{2} \cdot \frac{2m-1}{2} \ldots \frac{3}{2} \cdot \frac{1}{2} . \sqrt{\pi} \cdot \frac{1}{2} \sqrt{\pi}}{(m+2)(m+1) m (m-1) \ldots 2 \cdot 1}$$

$$= a^{m+2} \frac{(2m+1)(2m-1) \ldots 3.1}{(m+2)(m+1) m (m-1) \ldots 2 \cdot 1}.$$

Problem 18: *Evaluate* $\int_0^{2a} x^5 \sqrt{(2ax - x^2)}\, dx$.

Solution: Proceeding as in Problem 17, the given integral

$$= 2^8 a^7 \int_0^{\pi/2} \sin^{12} \theta \cos^2 \theta \, d\theta$$

$$= 2^8 a^7 \frac{11.9.7.5.3.1.1}{14.12.10.8.6.4.2} \cdot \frac{\pi}{2} = \frac{33}{16} \pi a^7.$$

Problem 19: *Evaluate* $\int_0^a x^3 (2ax - x^2)^{3/2}\, dx$.

Solution: We have

$$\int_0^a x^3 (2ax - x^2)^{3/2} dx = \int_0^a x^3 . x^{3/2} (2a - x)^{3/2} dx = \int_0^a x^{9/2} (2a - x)^{3/2} dx.$$

Now put $x = 2a \sin^2 \theta$, so that $dx = 4a \sin\theta \cos\theta \, d\theta$.

Also when $x = 0$, $2a \sin^2 \theta = 0$ or $\sin\theta = 0$ *i.e.*, $\theta = 0$

and when $x = a$, $2a \sin^2 \theta = a$ or $\sin\theta = 1/\sqrt{2}$ *i.e.*, $\theta = \pi/4$.

∴ The given integral

$$= \int_0^{\pi/4} (2a \sin^2 \theta)^{9/2} (2a)^{3/2} (\cos^2 \theta)^{3/2} . 4a \sin\theta \cos\theta \, d\theta$$

$$= \int_0^{\pi/4} (2a)^{9/2} \sin^9 \theta \, (2a)^{3/2} . \cos^3 \theta . 4a \sin\theta \cos\theta \, d\theta$$

$$= (2a)^6 . 4a \int_0^{\pi/4} \sin^{10} \theta . \cos^4 \theta \, d\theta .$$

This is not Gamma function as the limits are from 0 to $\pi/4$. We shall reduce it to the form of Gamma function by suitable trigonometrical adjustment. Thus the given integral

$$= a^7 . 2^8 \int_0^{\pi/4} (\sin^2 \theta)^3 (\sin^4 \theta \cos^4 \theta) \, d\theta , \qquad \textbf{(Note)}$$

$$= 2a^7 \int_0^{\pi/4} (2 \sin^2 \theta)^3 . (2 \sin\theta \cos\theta)^4 \, d\theta$$

$$= 2a^7 \int_0^{\pi/4} (1 - \cos 2\theta)^3 \sin^4 2\theta \, d\theta .$$

Now put $2\theta = \alpha$, so that $2 \, d\theta = d\alpha$ and the new limits are $\alpha = 0$ to $\alpha = \pi/2$.

∴ The given integral

$$= 2a^7 \int_0^{\pi/2} (1 - \cos\alpha)^3 \sin^4 \alpha . \frac{1}{2} d\alpha = a^7 \int_0^{\pi/2} (1 - \cos\alpha)^3 \sin^4 \alpha \, d\alpha$$

$$= a^7 \int_0^{\pi/2} (1 - 3 \cos\alpha + 3 \cos^2 \alpha - \cos^3 \alpha) \sin^4 \alpha \, d\alpha$$

$$= a^7 \int_0^{\pi/2} (\sin^4 \alpha - 3 \sin^4 \alpha \cos\alpha + 3 \sin^4 \alpha \cos^2 \alpha - \cos^3 \alpha \sin^4 \alpha) \, d\alpha$$

$$= a^7 \left[\frac{3.1}{4.2} \cdot \frac{\pi}{2} - 3 . \frac{3.1.1}{5.3.1} . 1 + 3 . \frac{3.1.1}{6.4.2} \cdot \frac{\pi}{2} - \frac{2.3.1}{7.5.3.1} \cdot 1 \right],$$

by Walli's formula

$$= a^7 \left(\frac{3\pi}{16} + \frac{3\pi}{32} - \frac{3}{5} - \frac{2}{35} \right) = a^7 \left(\frac{9\pi}{32} - \frac{23}{35} \right).$$

Problem 20: *Evaluate* $\int_0^a x^2 (2ax - x^2)^{5/2} \, dx$.

Solution: Putting $x = 2a \sin^2 \theta$ and proceeding exactly as in Problem 19, we get the given integral

$$= \int_0^{\pi/4} (2a \sin^2 \theta)^{9/2} (2a)^{5/2} (\cos^2 \theta)^{5/2} 4a \sin\theta \cos\theta \, d\theta$$

$$= 2^9 a^8 \int_0^{\pi/4} \sin^{10}\theta \cos^6\theta \, d\theta$$

$$= 2a^8 \int_0^{\pi/4} (2\sin^2\theta)^2 (2\sin\theta\cos\theta)^6 \, d\theta \qquad \textbf{(Note)}$$

$$= 2a^8 \int_0^{\pi/4} (1-\cos 2\theta)^2 \sin^6 2\theta \, d\theta$$

$$= a^8 \int_0^{\pi/2} (1-\cos\alpha)^2 . \sin^6\alpha \, d\alpha, \text{ putting } 2\theta = \alpha$$

$$= a^8 \int_0^{\pi/2} (\sin^6\alpha - 2\cos\alpha . \sin^6\alpha + \cos^2\alpha \sin^6\alpha) \, d\alpha$$

$$= a^8 \left[\frac{\Gamma\frac{7}{2} . \Gamma\frac{1}{2}}{2\,\Gamma 4} - 2\frac{\Gamma 1 . \Gamma\frac{7}{2}}{2\,\Gamma\frac{9}{2}} + \frac{\Gamma\frac{3}{2} . \Gamma\frac{7}{2}}{2\,\Gamma 5} \right] = a^8 \left[\frac{45\pi}{256} - \frac{2}{7} \right].$$

Problem 21: *Evaluate* $\int_0^a \frac{x^4}{(x^2+a^2)^4} dx$.

Solution: Put $x = a\tan\theta$, so that $dx = a\sec^2\theta \, d\theta$.

When $x = 0$, $\theta = 0$ and when $x = a$, $\tan\theta = 1$ *i.e.*, $\theta = \frac{1}{4}\pi$.

$\therefore$ The given integral $I = \int_0^{\pi/4} \frac{a^4\tan^4\theta}{(a^2\tan^2\theta + a^2)^4} a\sec^2\theta \, d\theta$

$$= \frac{1}{a^3} \int_0^{\pi/4} \frac{\tan^4\theta}{\sec^8\theta} \sec^2\theta \, d\theta = \frac{1}{a^3} \int_0^{\pi/4} \sin^4\theta\cos^2\theta \, d\theta.$$

Now proceeding as in Example 12, we get

$$I = \frac{1}{a^3} \cdot \frac{1}{16} \left[\frac{\pi}{4} - \frac{1}{3} \right].$$

Problem 22: *Evaluate the integral* $\int_0^a x^2 \sqrt{\left(\frac{a-x}{a+x}\right)} dx$.

Solution: Let $I = \int_0^a x^2 \sqrt{\left(\frac{a-x}{a+x}\right)} dx = \int_0^a \frac{x^2(a-x)}{\sqrt{(a^2-x^2)}} dx$.

Now put $x = a\sin\theta$ so that $dx = a\cos\theta \, d\theta$.

Also $\theta = 0$ when $x = 0$ and $\theta = \frac{1}{2}\pi$ when $x = a$.

$$\therefore \quad I = \int_0^{\pi/2} \frac{a^2\sin^2\theta\,(a - a\sin\theta)}{a\cos\theta} a\cos\theta \, d\theta$$

$$= a^3 \int_0^{\pi/2} (\sin^2\theta - \sin^3\theta) \, d\theta$$

$$= a^3\left[\frac{1}{2}\cdot\frac{\pi}{2}-\frac{2}{3.1}.1\right],$$ [By Walli's formula]

$$= a^3\left(\frac{1}{4}\pi-\frac{2}{3}\right).$$

Problem 23: *Evaluate the integral* $\int_0^a x\sqrt{\left(\frac{a^2-x^2}{a^2+x^2}\right)}\,dx$.

Solution: Let $I = \int_0^a x\sqrt{\left(\frac{a^2-x^2}{a^2+x^2}\right)}\,dx = \int_0^a \frac{x(a^2-x^2)}{\sqrt{(a^4-x^4)}}\,dx$.

Now put $x^2 = a^2\sin\theta$ so that $2x\,dx = a^2\cos\theta\,d\theta$.

Also $\theta = 0$ when $x = 0$ and $\theta = \frac{1}{2}\pi$ when $x = a$.

$$\therefore \qquad I = \int_0^{\pi/2}\frac{(a^2-a^2\sin\theta)}{a^2\cos\theta}\cdot\frac{a^2}{2}\cos\theta\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}(1-\sin\theta)\,d\theta = \frac{a^2}{2}\Big[\theta+\cos\theta\Big]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left[\left(\frac{\pi}{2}+0\right)-(0+1)\right] = \frac{1}{2}a^2\left(\frac{1}{2}\pi-1\right) = \frac{1}{4}a^2(\pi-2).$$

Problem 24: *Evaluate* $\int_a^b (x-a)^m(b-x)^n\,dx$.

Solution: Here we put $x = b\sin^2\theta + a\cos^2\theta$,

so that $dx = 2(b-a)\sin\theta\cos\theta\,d\theta$.

Also when $x = a$, $(b-a)\sin^2\theta = 0$ *i.e.*, $\theta = 0$

and when $x = b$, $(a-b)\cos^2\theta = 0$ *i.e.*, $\theta = \pi/2$.

Also $(x-a) = (b-a)\sin^2\theta$ and $(b-x) = (b-a)\cos^2\theta$.

Thus the given integral

$$= \int_0^{\pi/2}(b-a)^m\sin^{2m}\theta.(b-a)^n\cos^{2n}\theta.2(b-a)\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2}(b-a)^{m+n+1}\sin^{2m+1}\theta\cos^{2n+1}\theta\,d\theta$$

$$= (b-a)^{m+n+1}\left[\frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}\right],$$ applying Gamma function.

Problem 25: *Prove that* $\int_0^1\frac{dx}{\sqrt{(1-x^n)}} = \frac{\sqrt{\pi}\,\Gamma(1/n)}{n\,\Gamma\left\{\frac{1}{2}+(1/n)\right\}}$.

Solution: Put $x^n = \sin^2\theta$ *i.e.,* $x = (\sin\theta)^{2/n}$

so that $dx = (2/n)(\sin\theta)^{(2/n)-1} . \cos\theta\, d\theta$.

Also $\theta = 0$ when $x = 0$ and $\theta = \frac{1}{2}\pi$ when $x = 1$.

$\therefore$ The given integral $= \dfrac{2}{n}\displaystyle\int_0^{\pi/2} \dfrac{(\sin\theta)^{(2/n)-1} . \cos\theta\, d\theta}{\cos\theta}$

$$= \frac{2}{n}\int_0^{\pi/2} (\sin\theta)^{(2/n)-1} d\theta = \frac{2}{n}\int_0^{\pi/2} (\sin\theta)^{(2/n)-1}\cos^0\theta\, d\theta$$

$$= \frac{2\,\Gamma\frac{1}{2}\,\Gamma(1/n)}{n\,2\,\Gamma\left\{\frac{1}{2}+(1/n)\right\}} = \frac{\sqrt{\pi}\,.\,\Gamma(1/n)}{n\,\Gamma\left\{\frac{1}{2}+(1/n)\right\}}.$$

Problem 26: *Show that* $\displaystyle\int_0^{\infty} \frac{x^4\, dx}{(1+x^2)^4} = \frac{\pi}{32}$.

Solution: Here we put $x = \tan\theta$ so that $dx = \sec^2\theta\, d\theta$.

Also when $x = 0$, $\theta = 0$ and when $x = \infty$, $\theta = \pi/2$.

Hence the given integral

$$= \int_0^{\pi/2} \frac{\tan^4\theta\,\sec^2\theta\, d\theta}{(1+\tan^2\theta)^4} = \int_0^{\pi/2} \frac{\tan^4\theta\,\sec^2\theta\, d\theta}{\sec^8\theta}$$

$$= \int_0^{\pi/2} \sin^4\theta\cos^2\theta\, d\theta \qquad \textbf{(Note)}$$

$$= \frac{\Gamma\frac{5}{2}\cdot\Gamma\frac{3}{2}}{2\,\Gamma 4} = \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\,.\,\frac{1}{2}\sqrt{\pi}}{2\,.\,3\,.\,2\,.\,1} = \frac{\pi}{32}.$$

Problem 27: *If m, n are positive integers, then prove that*

$$\int_0^1 x^{m-1}(1-x)^{n-1}\, dx = \int_0^1 x^{n-1}(1-x)^{m-1}\, dx = \frac{1\,.\,2\,.\,3\,......(m-1)}{n\,(n+1)\,......\,(n+m-1)}.$$

Solution: Put $x = \sin^2\theta$, so that $dx = 2\sin\theta\cos\theta\, d\theta$.

Also when $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \pi/2$.

$$\therefore \int_0^1 x^{m-1}(1-x)^{n-1} dx = \int_0^{\pi/2} \sin^{2m-2}\theta\,(1-\sin^2\theta)^{n-1} . \,2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$$

$$= 2\,\frac{\Gamma\left\{\frac{2m-1+1}{2}\right\}.\,\Gamma\left\{\frac{2n-1+1}{2}\right\}}{2\,\Gamma\left\{\frac{2m-1+2n-1+2}{2}\right\}} = \frac{\Gamma m\,.\,\Gamma n}{\Gamma(m+n)}.$$

Thus $$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}. \quad ...(1)$$

Interchanging m and n in (1), we get

$$\int_0^1 x^{n-1}(1-x)^{m-1}dx = \frac{\Gamma n.\Gamma m}{\Gamma(n+m)}. \quad ...(2)$$

From (1) and (2), we have

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$$= \frac{\Gamma m.\Gamma n}{\Gamma(m+n)} = \frac{(m-1)!(n-1)!}{(m+n-1)!}, \quad [\because \ \Gamma r = (r-1)!]$$

$$= \frac{(m-1)!(n-1)!}{(n+m-1)(n+m-2)\ldots n.(n-1)(n-2)\ldots 2.1} \quad \textbf{(Note)}$$

$$= \frac{(m-1)!(n-1)!}{(n+m-1)(n+m-2)\ldots n.(n-1)!}$$

$$= \frac{(m-1)(m-2)....3.2.1}{(n+m-1)(n+m-2)...(n+1)n}.$$

Comprehensive Problems 3

Problem 1: *Evaluate* $\int_0^{\pi/2} x^3 \sin 3x\,dx$.

Solution: Integrating by parts taking $\sin 3x$ as the 2nd function, we have the given integral

$$= \left[x^3\cdot\frac{-\cos 3x}{3}\right]_0^{\pi/2} + \int_0^{\pi/2} 3x^2\cdot\frac{\cos 3x}{3}\,dx$$

$$= 0 + \int_0^{\pi/2} x^2\cos 3x\,dx = \left[x^2\,\frac{\sin 3x}{3}\right]_0^{\pi/2} - \int_0^{\pi/2} 2x\cdot\frac{\sin 3x}{3}\,dx,$$

again integrating by parts regarding $\sin 3x$ as the 2nd function

$$= \frac{\pi^2}{4}\cdot\left(\frac{-1}{3}\right) - \frac{2}{3}\int_0^{\pi/2} x\sin 3x\,dx$$

$$= -\frac{\pi^2}{12} - \frac{2}{3}\left[\left(-x\,\frac{\cos 3x}{3}\right)_0^{\pi/2} + \int_0^{\pi/2}\frac{1}{3}\cos 3x\,dx\right]$$

$$= -\frac{\pi^2}{12} - \frac{2}{9}\int_0^{\pi/2}\cos 3x\,dx = -\frac{\pi^2}{12} - \frac{2}{9}\left[\frac{\sin 3x}{3}\right]_0^{\pi/2} = -\frac{\pi^2}{12} + \frac{2}{27}.$$

Problem 2: *Evaluate* $\int_0^{\pi} x \sin^2 x \cos x\,dx$.

Solution: Let $I = \int_0^{\pi} x\sin^2 x\cos x\,dx$. Integrating by parts taking $(\sin^2 x\cos x)$ as the 2nd function, we have

$$I = \left[\frac{x \sin^3 x}{3}\right]_0^{\pi} - \int_0^{\pi} 1 \cdot \frac{\sin^3 x}{3}\, dx = -\frac{1}{3}\int_0^{\pi} \sin^3 x\, dx$$

$$= -\frac{1}{3}\int_0^{\pi} (1 - \cos^2 x) \sin x\, dx = -\frac{1}{3}\left[-\cos x + \frac{1}{3}\cos^3 x\right]_0^{\pi}$$

$$= -\frac{1}{3}\left[(1 - \frac{1}{3}) - (-1 + \frac{1}{3})\right] = -\frac{4}{9}.$$

Problem 3: *Evaluate* $\int_0^1 x^6 \sin^{-1} x\, dx.$ **(Kanpur 2008)**

Solution: Let $I = \int_0^1 x^6 \sin^{-1} x\, dx.$

Put $\sin^{-1} x = t$ or $x = \sin t$, so that $dx = \cos t\, dt$.

$\therefore$ $$I = \int_0^{\pi/2} t \sin^6 t \cos t\, dt.$$

Integrating by parts taking $\sin^6 t \cos t$ as the second function and t as the first function, we have

$$I = \left[t \cdot \frac{\sin^7 t}{7}\right]_0^{\pi/2} - \int_0^{\pi/2} 1 \cdot \frac{\sin^7 t}{7}\, dt = \frac{\pi}{14} - \frac{1}{7} \cdot \frac{6.4.2}{7.5.3.1} = \frac{\pi}{14} - \frac{16}{245}.$$

Problem 4: *Evaluate* $\int_0^a \sqrt{(a^2 - x^2)} \left\{\cos^{-1}\left(\frac{x}{a}\right)\right\}^2 dx.$

Solution: Here put $x = a\cos\theta$, so that $dx = -a\sin\theta\, d\theta$.

Also when $x = 0, \theta = \pi/2$ and when $x = a, \theta = 0$.

$\therefore$ The given integral $= -\int_{\pi/2}^{0} (a\sin\theta)(\theta^2) \,.\, a\sin\theta\, d\theta$

$$= -a^2\int_{\pi/2}^{0} \theta^2 \sin^2\theta\, d\theta = -a^2\int_{\pi/2}^{0} \theta^2 \,.\, \frac{1}{2} \,.\, 2\sin^2\theta\, d\theta$$

$$= -\frac{a^2}{2}\int_{\pi/2}^{0} \theta^2 (1 - \cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\int_{\pi/2}^{0} \theta^2 \cos 2\theta\, d\theta - \frac{a^2}{2}\int_{\pi/2}^{0} \theta^2\, d\theta$$

$$= \frac{a^2}{2}\int_{\pi/2}^{0} \theta^2 \cos 2\theta\, d\theta - \frac{a^2}{2}\left\{\frac{1}{3}\theta^3\right\}_{\pi/2}^{0}$$

$$= \frac{\pi^3 a^2}{48} + \frac{a^2}{2}\int_{\pi/2}^{0} \theta^2 \cos 2\theta\, d\theta.$$

Now to evaluate $\int \theta^2 \cos 2\theta\, d\theta$, applying integration by parts taking $\cos 2\theta$ as the 2nd function, we have

$$\int \theta^2 \cos 2\theta \, d\theta = \theta^2 . \frac{1}{2} \sin 2\theta - \int 2\theta . \frac{1}{2} \sin 2\theta \, d\theta$$

$$= \frac{1}{2} \theta^2 \sin 2\theta - \int \theta \sin 2\theta \, d\theta$$

$$= \frac{1}{2} \theta^2 \sin 2\theta - [\theta . (-\frac{1}{2} \cos 2\theta) - \int 1 .(-\frac{1}{2} \cos 2\theta) \, d\theta]$$

$$= \frac{1}{2} \theta^2 \sin 2\theta + \frac{1}{2} \theta \cos 2\theta - \frac{1}{4} \sin 2\theta .$$

$$\therefore \qquad \int_{\pi/2}^{0} \theta^2 \cos 2\theta \, d\theta = \left[\frac{\theta^2 \sin 2\theta}{2} + \frac{\theta \cos 2\theta}{2} - \frac{\sin 2\theta}{4} \right]_{\pi/2}^{0} = \frac{\pi}{4} .$$

$\therefore$ The required integral $= \dfrac{\pi^3 a^2}{48} + \dfrac{1}{2} a^2 \cdot \dfrac{\pi}{4} = \dfrac{\pi a^2}{8} (1 + \dfrac{1}{6} \pi^2)$.

Problem 5: *Evaluate* $\int_0^{\pi} x \sin^3 x \, dx$.

Solution: From article 10, we have

$$\int x \sin^n x \, dx = -\frac{x}{n} \cdot \sin^{n-1} x \cos x + \frac{\sin^n x}{n^2} + \frac{(n-1)}{n} \int x \sin^{n-2} x \, dx \qquad ...(1)$$

Putting $n = 3$ in (1), we have

$$\int x \sin^3 x \, dx = -\frac{1}{3} x \sin^2 x \cos x + \frac{1}{9} \sin^3 x + \frac{2}{3} \int x \sin x \, dx.$$

Now $\quad \int x \sin x \, dx = x .(-\cos x) - \int 1 (-\cos x) \, dx = -x \cos x + \sin x .$

$$\therefore \qquad \int_0^{\pi} x \sin^3 x \, dx = \left[-\frac{1}{3} x \sin^2 x \cos x + \frac{1}{9} \sin^3 x - \frac{2}{3} x \cos x + \frac{2}{3} \sin x \right]_0^{\pi}$$

$$= -\frac{2}{3} \pi \cos \pi = -\frac{2}{3} \pi (-1) = \frac{2}{3} \pi .$$

Problem 6: *Evaluate* $\int x \sin^4 x \, dx$.

Solution: Here putting $n = 4$ in the reduction formula (1) and proceeding as in part (i), we have

$$\int x \sin^4 x \, dx = -\frac{x \sin^3 x \cos x}{4} + \frac{\sin^4 x}{16} + \frac{3}{16} x^2 - \frac{3}{16} x \sin 2x - \frac{3}{32} \cos 2x .$$

Problem 7: *Evaluate* $\int_0^{\pi/2} x \cos^3 x \, dx$.

Solution: We know from article 10 that

$$\int x \cos^n x \, dx = \frac{x \cos^{n-2} x \sin x}{n} + \frac{\cos^n x}{n^2} + \frac{n-1}{n} I_{n-2}.$$

[Derive it here.]

Putting $n = 3$, we have

$$\int x\cos^3 x\,dx = \frac{1}{3}x\cos^2 x\sin x + \frac{1}{9}\cos^3 x + \frac{2}{3}\int x\cos x\,dx.$$

Now $\int x\cos x\,dx = x\,.\sin x - \int 1\,.\sin x\,dx = x\sin x + \cos x.$

$\therefore$ $$\int_0^{\pi/2} x\cos^3 x\,dx = \left[\frac{1}{3}x\cos^2 x\sin x + \frac{1}{9}\cos^3 x + \frac{2}{3}(x\sin x + \cos x)\right]_0^{\pi/2}$$

$$= \frac{1}{3}\pi - \frac{1}{9} - \frac{2}{3} = \frac{1}{3}\pi - \frac{7}{9} = \frac{1}{3}[\pi - \frac{7}{3}].$$

Problem 8: *Evaluate* $\int e^x (x\cos x + \sin x)\,dx.$

Solution: The given integral $= \int xe^x \cos x\,dx + \int e^x \sin x\,dx$

$= I_1 + I_2$, say.

Now $$I_1 = \int x\,.\,e^x\cos x\,dx = x\,.\,\frac{1}{2}e^x(\cos x + \sin x) - \int \frac{1}{2}e^x(\cos x + \sin x)\,dx,$$

integrating by parts taking $(e^x \cos x)$ as 2nd function

$$= \frac{1}{2}xe^x(\cos x + \sin x) - \frac{1}{2}\int e^x\cos x\,dx - \frac{1}{2}\int e^x\sin x\,dx$$

$$= \frac{1}{2}xe^x(\cos x + \sin x) - \frac{1}{2}\cdot\frac{1}{2}e^x(\cos x + \sin x) - \frac{1}{2}\cdot\frac{1}{2}e^x(\sin x - \cos x)$$

$$= \frac{1}{2}xe^x(\cos x + \sin x) - \frac{1}{2}e^x\sin x.$$

And $$I_2 = \int e^x\sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x).$$

$\therefore$ The required integral $= I_1 + I_2$

$$= \frac{1}{2}xe^x(\cos x + \sin x) - \frac{1}{2}e^x\sin x + \frac{1}{2}e^x(\sin x - \cos x)$$

$$= \frac{1}{2}e^x[x(\cos x + \sin x) - \cos x].$$

Remember: $$\int e^x\cos x\,dx = \frac{1}{2}e^x(\cos x + \sin x),$$

and $$\int e^x\sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x).$$

Problem 9: *Evaluate* $\int x^2 e^{2x\cos\alpha}\sin(2x\sin\alpha)\,dx.$

Solution: We know that

$$\int e^{ax}\sin bx\,dx = \frac{1}{r}e^{ax}\sin(bx - \phi),$$

where $r = \sqrt{(a^2 + b^2)}$ and $\phi = \tan^{-1}(b/a)$.

If we take $a = 2\cos\alpha$ and $b = 2\sin\alpha$, we have

$$r = \sqrt{(a^2 + b^2)} = 2$$

and $\phi = \tan^{-1}(b/a) = \tan^{-1}(\tan\alpha) = \alpha$.

Now the given integral $= \int x^2 . e^{ax} \sin bx\, dx$

$$= x^2 \left[\frac{e^{ax}}{r}\sin(bx - \phi)\right] - \int 2x . \left[\frac{e^{ax}}{r}\sin(bx-\phi)\right] dx,$$

integrating by parts taking $(e^{ax}\sin bx)$ as 2nd function

$$= \frac{1}{2}x^2 . e^{ax}\sin(bx-\phi) - \int x e^{ax}\sin(bx-\phi)\, dx \qquad ...(1)$$

$[\because\ r = 2]$

Also $\int x . e^{ax}\sin(bx-\phi)\, dx$

$$= [x.(1/r).e^{ax}\sin(bx-\phi) - \int (1/r).e^{ax}\sin(bx-2\phi)\, dx],$$

again integrating by parts

$$= \frac{1}{2}xe^{ax}\sin(bx-2\phi) - \frac{1}{2}\int e^{ax}\sin(bx-2\phi), \qquad [\because\ r=2]$$

$$= \frac{1}{2}xe^{ax}\sin(bx-2\phi) - \frac{1}{2}\cdot\frac{1}{2}e^{ax}\sin(bx-3\phi) \qquad ...(2)$$

Hence from (1) and (2), the given integral

$$= \frac{1}{2}x^2 e^{ax}\sin(bx-\phi) - \frac{1}{2}xe^{ax}\sin(bx-2\phi) + \frac{1}{4}e^{ax}\sin(bx-3\phi),$$

where $a = 2\cos\alpha$, $b = 2\sin\alpha$ and $\phi = \alpha$.

Problem 10: *Evaluate* $\int_0^1 (\sin^{-1}x)^4\, dx$.

Solution: Put $\sin^{-1}x = t$ *i.e.*, $x = \sin t$, so that $dx = \cos t\, dt$.

When $x = 0$, $t = \sin^{-1}0 = 0$ and when $x = 1$, $t = \sin^{-1}1 = \frac{1}{2}\pi$.

$\therefore$ The given integral $I = \int_0^{\pi/2} t^4\cos t\, dt = \left[t^4\sin t\right]_0^{\pi/2} - \int_0^{\pi/2} 4t^3\sin t\, dt,$

integrating by parts taking $\cos t$ as the second function

$$= \frac{\pi^4}{16} - 4\int_0^{\pi/2} t^3\sin t\, dt$$

$$= \frac{\pi^4}{16} - 4\left[\left\{t^3\left(-\cos t\right)\right\}_0^{\pi/2} + \int_0^{\pi/2} 3t^2\cos t\, dt\right],$$

again integrating by parts

$$= \frac{\pi^4}{16} - 4\times 0 - 12\int_0^{\pi/2} t^2\cos t\, dt$$

$$= \frac{\pi^4}{16} - 12\left[\{t^2 \sin t\}_0^{\pi/2} - \int_0^{\pi/2} 2t \sin t \, dt\right]$$

$$= \frac{\pi^4}{16} - 12\left(\frac{\pi^2}{4}\right) + 24\int_0^{\pi/2} t \sin t \, dt$$

$$= \frac{\pi^4}{16} - 3\pi^2 + 24\left[\left\{t\left(-\cos t\right)\right\}_0^{\pi/.2} + \int_0^{\pi/2} \cos t \, dt\right]$$

$$= \frac{\pi^4}{16} - 3\pi^2 + 24 \times 0 + 24\left[\sin t\right]_0^{\pi/2}$$

$$= \frac{1}{16}\pi^4 - 3\pi^2 + 24.$$

Problem 11: *Evaluate the integral* $\int_1^\infty \frac{x^4 + 1}{x^2 (x^2 + 1)^2} dx$,

Solution: Let $I = \int_1^\infty \frac{x^4 + 1}{x^2 (x^2 + 1)^2} dx = \int_1^\infty \frac{(x^2 + 1)^2 - 2x^2}{x^2 (x^2 + 1)^2} dx$

$$= \int_1^\infty \frac{1}{x^2} dx - 2\int_1^\infty \frac{1}{(x^2 + 1)^2} dx$$

$$= \left[-\frac{1}{x}\right]_1^\infty - 2\int_1^\infty \frac{dx}{(x^2 + 1)^2} = 1 - 2\int_1^\infty \frac{dx}{(x^2 + 1)^2}.$$

Now put $x = \tan t$, so that $dx = \sec^2 t \, dt$. When $x = 1$, $t = \frac{1}{4}\pi$ and when $x = \infty$, $t = \frac{1}{2}\pi$.

$\therefore$ $$I = 1 - 2\int_{\pi/4}^{\pi/2} \frac{\sec^2 t \, dt}{(1 + \tan^2 t)^2} = 1 - 2\int_{\pi/4}^{\pi/2} \frac{\sec^2 t}{\sec^4 t} dt$$

$$= 1 - 2\int_{\pi/4}^{\pi/2} \cos^2 t \, dt = 1 - \int_{\pi/4}^{\pi/2} (1 + \cos 2t) \, dt$$

$$= 1 - \left[t + \frac{1}{2}\sin 2t\right]_{\pi/4}^{\pi/2} = 1 - \left[\left(\frac{1}{2}\pi + \frac{1}{2}\sin \pi\right) - \left(\frac{1}{4}\pi + \frac{1}{2}\sin \frac{1}{2}\pi\right)\right]$$

$$= 1 - [\frac{1}{2}\pi - \frac{1}{4}\pi - \frac{1}{2}] = 1 - [\frac{1}{4}\pi - \frac{1}{2}] = \frac{3}{2} - \frac{1}{4}\pi.$$

Problem 12: *Evaluate the integral* $\int_1^\infty \frac{x^2 + 3}{x^6 (x^2 + 1)} dx$.

Solution : The given integral $I = \int_1^\infty \frac{(x^2 + 1) + 2}{x^6 (x^2 + 1)} dx$

$$= \int_1^\infty \frac{1}{x^6} dx + 2\int_1^\infty \frac{1}{x^6 (x^2 + 1)} dx = \left[-\frac{1}{5x^5}\right]_1^\infty + 2\int_1^\infty \frac{1}{x^6 (x^2 + 1)} dx$$

$$= \frac{1}{5} + 2\int_1^{\infty} \frac{dx}{x^6 (x^2 + 1)}.$$

Now put $x = \tan t$, so that $dx = \sec^2 t\, dt$. When $x = 1$, $t = \pi/4$ and when $x = \infty$, $t = \pi/2$.

$$\therefore \qquad I = \frac{1}{5} + 2\int_{\pi/4}^{\pi/2} \cot^6 t\, dt.$$

Now we know that $\int \cot^n t\, dt = -\dfrac{\cot^{n-1} t}{n-1} - \int \cot^{n-2} t\, dt.$

[See article 4]

$$\therefore \qquad \int \cot^6 t\, dt = -\frac{\cot^5 t}{5} - \int \cot^4 t\, dt$$

$$= -\frac{\cot^5 t}{5} - \left[-\frac{\cot^3 t}{3} - \int \cot^2 t\, dt\right]$$

$$= -\frac{1}{5}\cot^5 t + \frac{1}{3}\cot^3 t + \int (\operatorname{cosec}^2 t - 1)\, dt$$

$$= -\frac{1}{5}\cot^5 t + \frac{1}{3}\cot^3 t - \cot t - t.$$

$$\therefore \qquad I = \frac{1}{5} + 2\left[-\frac{1}{5}\cot^5 t + \frac{1}{3}\cot^3 t - \cot t - t\right]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{5} + 2\,[-\frac{1}{2}\pi - (-\frac{1}{5} + \frac{1}{3} - 1 - \frac{1}{4}\pi)]$$

$$= \frac{1}{5} + \frac{2}{5} - \frac{2}{3} + 2 + \left(-\pi + \frac{1}{2}\pi\right) = \frac{29}{15} - \frac{1}{2}\pi = \frac{1}{30}(58 - 15\pi).$$

Problem 13: *If* $u_n = \int_0^{\pi/2} x^n \sin mx\, dx$*, prove that*

$$u_n = \frac{n\pi^{n-1}}{m^2 . 2^{n-1}} - \frac{n(n-1)}{m^2} u_{n-2}, \text{ if } m \text{ is of the form } 4r + 1.$$

(Kanpur 2011)

Solution: We have $\int_0^{\pi/2} x^n \sin mx\, dx$

$$= \left[x^n \cdot \left(-\frac{\cos mx}{m}\right)\right]_0^{\pi/2} - \int_0^{\pi/2} nx^{n-1} \cdot \left(-\frac{\cos mx}{m}\right) dx$$

$$= 0 + \frac{n}{m}\int_0^{\pi/2} x^{n-1} \cos mx\, dx,$$

[$\because$ If m is of the form $4r + 1$, then
$\cos\{(4r + 1)\pi/2\} = \cos(2r\pi + \frac{1}{2}\pi) = \cos\frac{1}{2}\pi = 0$]

$$= \frac{n}{m}\left[\left\{x^{n-1} \cdot \frac{\sin mx}{m}\right\}_0^{\pi/2} - \int_0^{\pi/2} (n-1)\, x^{n-2} \cdot \left(\frac{\sin mx}{m}\right) dx\right],$$

again integrating by parts taking $\cos mx$ as 2nd function

$$= \frac{n}{m}\left[\left\{\left(\frac{\pi}{2}\right)^{n-1} \cdot \frac{1}{m}\right\} - \frac{(n-1)}{m}\int_0^{\pi/2} x^{n-2} \sin mx\, dx\right],$$

$$\left[\because \sin\frac{m\pi}{2} = \sin(4r+1)\frac{\pi}{2} = 1\right]$$

$$= \frac{n\pi^{n-1}}{m^2\, 2^{n-1}} - \frac{n(n-1)}{m^2} u_{n-2}\,.$$

Problem 14: *If* $I_n = \int_0^{\pi/2} x^n \sin(2p+1)\,x\,dx$, *prove that*

$$I_n + \frac{n(n-1)}{(p+1)^2} I_{n-2} = (-1)^p \frac{n}{(2p+1)^2}\left(\frac{\pi}{2}\right)^{n-1},$$

where n and p are positive integers. Hence deduce that $\int_0^{\pi/2} x^3 \sin 3x\,dx = \frac{2}{27} - \frac{\pi^2}{12}$.

Solution: We have $I_n = \int_0^{\pi/2} x^n \sin(2p+1)x\,dx$

$$= \left[\left\{-x^n \cdot \frac{\cos(2p+1)x}{2p+1}\right\}_0^{\pi/2} + \frac{n}{(2p+1)}\int_0^{\pi/2} x^{n-1}\cos(2p+1)\,x\,dx\right],$$

integrating by parts taking $\sin(2p+1)\,x$ as the 2nd function

$$= 0 + \frac{n}{(2p+1)}\left[x^{n-1}\cdot\frac{\sin(2p+1)\,x}{(2p+1)}\right]_0^{\pi/2} - \frac{n(n-1)}{(2p+1)^2} I_{n-2},$$

again integrating by parts

$$= \frac{n}{(2p+1)^2}\left(\frac{\pi}{2}\right)^{n-1}\sin(2p+1)\frac{\pi}{2} - \frac{n(n-1)}{(2p+1)^2} I_{n-2}.$$

$$\therefore \quad I_n + \frac{n(n-1)}{(2p+1)^2} I_{n-2} = (-1)^p \frac{n}{(2p+1)^2}\left(\frac{\pi}{2}\right)^{n-1} \quad ...(1)$$ **Proved**

Now to evaluate $\int_0^{\pi/2} x^3 \sin 3x\,dx$, put $n = 3$, $p = 1$ in (1).

$$\therefore \quad I_3 = -1\cdot\frac{3}{3^2}\left(\frac{\pi}{2}\right)^{3-1} - \frac{3\,.\,2}{3^2} I_1 = -\frac{\pi^2}{12} - \frac{2}{3}\int_0^{\pi/2} x\sin 3x\,dx$$

$$= -\frac{\pi^2}{12} - \frac{2}{3}\left[\left\{-x\frac{\cos 3x}{3}\right\}_0^{\pi/2} + \frac{1}{3}\int_0^{\pi/2}\cos 3x\,dx\right]$$

$$= -\frac{\pi^2}{12} - \frac{2}{3^3}\left[\sin 3x\right]_0^{\pi/2}$$

$$= -\frac{\pi^2}{12} + \frac{2}{27} = \frac{2}{27} - \frac{\pi^2}{12}.$$

Problem 15: *If* $u_n = \int_0^{\pi/2} \theta \sin^n\theta \, d\theta$ *and* $n > 1$, *prove that*

$$u_n = \frac{(n-1)}{n} u_{n-2} + \frac{1}{n^2} \cdot \textit{Hence deduce that } u_5 = \frac{149}{225}.$$

(Lucknow 2005; Gorakhpur 06; Rohilkhand 07)

Solution: We know that

$$\int \theta \sin^n \theta \, d\theta = -\frac{\theta \sin^{n-1}\theta \cos\theta}{n} + \frac{\sin^n\theta}{n^2} + \frac{(n-1)}{n} \int \theta \sin^{n-2}\theta \, d\theta.$$

[Derive it here]

$$\therefore \qquad u_n = \int_0^{\pi/2} \theta \sin^n\theta \, d\theta$$

$$= \left[-\frac{\theta \sin^{n-1}\theta \cos\theta}{n} + \frac{\sin^n\theta}{n^2}\right]_0^{\pi/2} + \frac{(n-1)}{n} \int_0^{\pi/2} \theta \sin^{n-2}\theta \, d\theta$$

$$= \left[\frac{1}{n^2} - 0\right] + \frac{(n-1)}{n} u_{n-2} = \frac{1}{n^2} + \frac{(n-1)}{n} u_{n-2}. \qquad \text{...(1) } \textbf{Proved.}$$

Putting $n = 5$ in the above reduction formula, we have

$$u_5 = \frac{1}{25} + \frac{4}{5} u_3 = \frac{1}{25} + \frac{4}{5} \left[\frac{1}{9} + \frac{2}{3} u_1\right], \qquad \text{[Putting } n = 3 \text{ in (1)]}$$

$$= \frac{1}{25} + \frac{4}{45} + \frac{8}{15} u_1 = \left(\frac{1}{25} + \frac{4}{45}\right) + \frac{8}{15} \int_0^{\pi/2} \theta \sin\theta \, d\theta$$

$$= \left(\frac{29}{225}\right) + \frac{8}{15} \left[\left(-\theta \cos\theta\right)_0^{\pi/2} + \int_0^{\pi/2} \cos\theta \, d\theta\right]$$

$$= \frac{29}{225} + \frac{8}{15} \left[0 + \sin\theta\right]_0^{\pi/2} = \frac{29}{225} + \frac{8}{15} = \frac{149}{225}.$$

Problem 16: *Prove that if n be a positive integer greater than unity, then*

$$\int_0^{\pi/2} \cos^{n-2} x \sin nx \, dx = \frac{1}{n-1}.$$

(Avadh 2004; Kanpur 10)

Solution: Taking $m = n - 2$ and proceeding as in article 13, we first establish the reduction formula

$$\int_0^{\pi/2} \cos^{n-2} x \sin nx \, dx = \frac{1}{2n-2} + \frac{n-2}{2n-2} \int_0^{\pi/2} \cos^{n-3} x \sin(n-1) x \, dx.$$

Applying this formula repeatedly, we have

$$\int_0^{\pi/2} \cos^{n-2} x \sin nx \, dx = \frac{1}{2(n-1)} + \frac{n-2}{2(n-1)} \left\{\frac{1}{2n-4} + \frac{n-3}{2n-4} \int_0^{\pi/2} \cos^{n-4} x \sin(n-2) x \, dx\right\}$$

$$= \frac{1}{2(n-1)} + \frac{1}{2^2(n-1)} + \frac{n-3}{2^2(n-1)} \cdot \int_0^{\pi/2} \cos^{n-4} x \sin(n-2)x\, dx$$

$$= \frac{1}{2(n-1)} + \frac{1}{2^2(n-1)} + \frac{n-3}{2^2(n-1)} \left\{\frac{1}{2n-6} + \frac{n-4}{2n-6} \int_0^{\pi/2} \cos^{n-5} x \sin(n-3)x\, dx\right\}$$

and finally $= \frac{1}{(n-1)}\left[\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{n-2}}\right]$

$$+ \frac{1}{2^{n-2}(n-1)} \int_0^{\pi/2} (\cos x)^0 \sin 2x\, dx$$

$$= \frac{1}{(n-1)} \cdot \frac{\frac{1}{2}\{1-(\frac{1}{2})^{n-2}\}}{1-\frac{1}{2}} + \frac{1}{2^{n-2}(n-1)}\left[-\frac{1}{2}\cos 2x\right]_0^{\pi/2}$$

$$= \frac{1}{(n-1)}\left[1 - \frac{1}{2^{n-2}}\right] + \frac{1}{2^{n-2}\cdot(n-1)} \cdot 1$$

$$= \frac{1}{(n-1)}\left[1 - \frac{1}{2^{n-2}} + \frac{1}{2^{n-2}}\right] = \frac{1}{n-1}.$$

Problem 17: *If* $I_{(m,\,n)} = \int_0^{\pi/2} \cos^m x \cos nx\, dx$, *prove that*

$$I_{(m,\,n)} = \left\{\frac{m(m-1)}{m^2-n^2}\right\} I_{(m-2\,,\,n)}.$$

(Purvanchal 2014)

Solution: We have $I_{m,\,n} = \int_0^{\pi/2} \cos^m x \cos nx\, dx$

$$= \left[\cos^m x \frac{\sin nx}{n}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin nx}{n}\, m \cos^{m-1} x\,(-\sin x)\, dx,$$

integrating by parts taking $\cos nx$ as the 2nd function

$$= 0 + \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x \sin nx \sin x\, dx.$$

Again integrating by parts taking $\sin nx$ as the 2nd function, we have

$$I_{m,\,n} = \frac{m}{n}\left[(\cos^{m-1} x \sin x)\left(\frac{-\cos nx}{n}\right)\right]_0^{\pi/2}$$

$$- \frac{m}{n}\int_0^{\pi/2}\left(-\frac{\cos nx}{n}\right)[\cos^{m-1} x \cos x - (m-1)\cos^{m-2} x \sin^2 x]\, dx$$

$$= 0 + \frac{m}{n^2}\int_0^{\pi/2} \cos nx\,\{\cos^m x - (m-1)\cos^{m-2} x\,.(1-\cos^2 x)\}\, dx$$

$$= \frac{m}{n^2} \int_0^{\pi/2} \cos nx \{m \cos^m x - (m-1) \cos^{m-2} x\} \, dx$$

$$= \frac{m}{n^2} \{ I_{m,n} - (m-1) I_{m-2,n} \}.$$

$\therefore$ $$\left(1 - \frac{m^2}{n^2}\right) I_{m,n} = -\frac{m(m-1)}{n^2} I_{m-2,n}$$

or $$I_{m,n} = \frac{m(m-1)}{m^2 - n^2} I_{m-2,n} \cdot$$

Problem 18: *Prove that* $\int_0^{\pi} \left(\frac{\sin n\theta}{\sin \theta}\right)^2 d\theta = n\pi$.

Solution: Let $I_n = \int_0^{\pi} \left(\frac{\sin n\theta}{\sin \theta}\right)^2 d\theta$, then

$$I_{n-1} = \int_0^{\pi} \left(\frac{\sin (n-1)\theta}{\sin \theta}\right)^2 d\theta .$$

$\therefore$ $$I_n - I_{n-1} = \int_0^{\pi} \frac{\sin^2 n\theta - \sin^2 (n-1)\theta}{\sin^2 \theta} d\theta$$

$$= \int_0^{\pi} \frac{\sin (2n-1)\theta \,.\, \sin \theta}{\sin^2 \theta} d\theta = \int_0^{\pi} \frac{\sin (2n-1)\theta}{\sin \theta} d\theta \quad \textbf{(Note)}$$

$$= \pi .$$

$\left[\because \text{From Example 20, } \int_0^{\pi} \frac{\sin n\theta}{\sin\theta} d\theta = \pi \text{ if } n \text{ is odd, here } (2n-1) \text{ is odd.}\right]$

Hence $I_n = I_{n-1} + \pi$ ...(1)

$= I_{n-2} + 2\pi$, [$\because$ From (1), $I_{n-1} = I_{n-2} + \pi$]

$= I_{n-3} + 3\pi$, and so on.

Thus $I_n = (n-1)\pi + I_1$

$$= (n-1)\pi + \int_0^{\pi} \left(\frac{\sin \theta}{\sin \theta}\right)^2 d\theta = (n-1)\pi + \pi = n\pi .$$

Problem 19: *If* $S_n = \int_0^{\pi/2} \frac{\sin (2n-1) x}{\sin x} dx$,

$V_n = \int_0^{\pi/2} \left(\frac{\sin nx}{\sin x}\right)^2 dx$, (*n is an integer*), *show that*

$$S_{n+1} - S_n = 0 , V_{n+1} - V_n = S_{n+1} .$$

Solution: S_{n+1} and V_{n+1} will be obtained by writing $(n+1)$ in place of n in S_n and V_n respectively. Thus

$$S_{n+1} = \int_0^{\pi/2} \frac{\sin(2n+1)x\,dx}{\sin x}$$

and $$V_{n+1} = \int_0^{\pi/2} \left\{\frac{\sin(n+1)x}{\sin x}\right\}^2 dx.$$

$\therefore$ $$S_{n+1} - S_n = \int_0^{\pi/2} \frac{[\sin(2n+1)x - \sin(2n-1)x]\,dx}{\sin x}$$

$$= \int_0^{\pi/2} \frac{\cos 2nx \sin x\,dx}{\sin x}$$

$$= \int_0^{\pi/2} 2\cos 2nx\,dx = \left[\frac{2\sin 2nx}{2n}\right]_0^{\pi/2} = 0,$$

$[\because \sin n\pi = 0$ when n is an integer and $\sin 0 = 0]$

Also $$V_{n+1} - V_n = \int_0^{\pi/2} \left[\left(\frac{\sin(n+1)x}{\sin x}\right)^2 - \left(\frac{\sin nx}{\sin x}\right)^2\right] dx$$

$$= \int_0^{\pi/2} \frac{[\sin^2(n+1)x - \sin^2(nx)]\,dx}{\sin^2 x}$$

$$= \int_0^{\pi/2} \frac{\{2\sin^2(n+1)x - 2\sin^2(nx)\}\,dx}{2\sin^2 x}$$

$$= \int_0^{\pi/2} \frac{\{1 - \cos 2(n+1)x - 1 + \cos 2nx\}\,dx}{2\sin^2 x}$$

$$= \int_0^{\pi/2} \frac{\{\cos 2nx - \cos 2(n+1)x\}\,dx}{2\sin^2 x}$$

$$= \int_0^{\pi/2} \frac{2\sin(2n+1)x \sin x\,dx}{2\sin^2 x} = \int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x}\,dx$$

$$= S_{n+1}.$$

Hints to Objective Type Questions

Multiple Choice Questions

1. See Problem 5 of Comprehensive Problems 1.
2. See article 4, part (b).
3. See Example 8.
4. See Example 11.
5. See Problem 15 of Comprehensive Problems 2.
6. See Example 15.
7. See Example 19.

8. See Problem 3 of Comprehensive Problems 1.
9. See Problem 6 of Comprehensive Problems 1.
10. See Problem 10 of Comprehensive Problems 1.
11. See Problem 12 of Comprehensive Problems 1.
12. See Problem 14 of Comprehensive Problems 1.
13. See Problem 3 of Comprehensive Problems 2.
14. See Problem 7 of Comprehensive Problems 2.
15. See Problem 13 of Comprehensive Problems 2.
16. See Problem 19 of Comprehensive Problems 2.
17. See Problem 22 of Comprehensive Problems 2.
18. See Problem 1 of Comprehensive Problems 3.
19. See Problem 4 of Comprehensive Problems 3.
20. See Problem 5 of Comprehensive Problems 3.
21. See Problem 7 of Comprehensive Problems 3.
22. See Problem 11 of Comprehensive Problems 3.

Fill in the Blanks

1. See article 2, part (1).
2. See article 4, part (1).
3. See article 5, part (1).
4. See Problem 2 of Comprehensive Problems 1.
5. See Problem 2 of Comprehensive Problems 2.
6. See Problem 13 of Comprehensive Problems 3.

True or False

1. See article 5, part (2).
2. Like Example 7.
3. See Example 15.
4. See Problem 1 of Comprehensive Problems 2.

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Chapter-3

Reduction Formulae Continued

(For Irrational Algebraic and Transcendental Functions)

Comprehensive Problems 1

Problem 1: *Prove the reduction formula*

$$\int (a^2 + x^2)^{n/2}\, dx = \frac{x\,(a^2 + x^2)^{n/2}}{(n+1)} + \frac{na^2}{(n+1)} \int (a^2 + x^2)^{(n/2)-1}\, dx.$$

Hence evaluate $\int (x^2 + a^2)^{5/2}\, dx$.

(Bundelkhand 2005; Kanpur 05)

Solution: We have $\int (a^2 + x^2)^{n/2}\, dx = \int (a^2 + x^2)^{n/2} . 1\, dx$ **(Note)**

$$= (a^2 + x^2)^{n/2}\, x - \int \frac{1}{2} n\,(a^2 + x^2)^{(n/2)-1}\, 2x \,.\, x\, dx$$

$$= x\,(a^2 + x^2)^{n/2} - n \int (a^2 + x^2)^{(n/2)-1} \{(a^2 + x^2) - a^2\}\, dx \quad \textbf{(Note)}$$

$$= x\,(a^2 + x^2)^{n/2} - n \int (a^2 + x^2)^{n/2}\, dx + na^2 \int (a^2 + x^2)^{(n/2)-1}\, dx.$$

$\therefore$ $$(1+n) \int (a^2 + x^2)^{n/2}\, dx = x\,(a^2 + x^2)^{n/2} + na^2 \int (a^2 + x^2)^{(n/2)-1}\, dx$$

or $$\int (a^2 + x^2)^{n/2}\, dx = \frac{x\,(a^2 + x^2)^{n/2}}{(n+1)} + \frac{na^2}{(n+1)} \int (a^2 + x^2)^{(n/2)-1}\, dx \quad \ldots(1)$$

is the required reduction formula.

Now to evaluate $\int (a^2 + x^2)^{5/2}\, dx$, putting $n = 5$ in (1), we get

$$\int (a^2 + x^2)^{5/2}\, dx = \frac{x\,(a^2 + x^2)^{5/2}}{6} + \frac{5a^2}{6} \int (a^2 + x^2)^{3/2}\, dx$$

$$= \frac{x\,(a^2 + x^2)^{5/2}}{6} + \frac{5a^2}{6} \left[\frac{x\,(a^2 + x^2)^{3/2}}{4} + \frac{3a^2}{4} \int (a^2 + x^2)^{1/2}\, dx \right]$$

$$= \frac{x\,(a^2 + x^2)^{5/2}}{6} + \frac{5a^2}{24} x\,(a^2 + x^2)^{3/2}$$

$$+ \frac{5a^4}{16} \left[x \sqrt{(a^2 + x^2)} + a^2 \sin^{-1} \frac{x}{a} \right].$$

Problem 2:. *Find a reduction formula for* $\int x^m (1+x^2)^{n/2}\, dx$, *where m and n are positive integers.*

Hence evaluate $\int x^5 (1+x^2)^{7/2}\, dx$.

Solution: We have $\int x^m (1+x^2)^{n/2}\, dx = \frac{1}{2}\int x^{m-1} (1+x^2)^{n/2} . 2x\, dx$ **(Note)**

$$= \frac{1}{2}\left[x^{m-1} \frac{(1+x^2)^{(n/2)+1}}{\frac{1}{2}n+1} - \frac{m-1}{(\frac{1}{2}n+1)} \int x^{m-2} (1+x^2)^{(n/2)+1}\, dx \right],$$

integrating by parts taking x^{m-1} as first function

$$= x^{m-1} \frac{(1+x^2)^{(n+2)/2}}{(n+2)} - \frac{m-1}{(n+2)} \int x^{m-2} (1+x^2)^{(n+2)/2}\, dx, \quad ...(1)$$

which is the required reduction formula.

Now to evaluate $\int x^5 (1+x^2)^{7/2}\, dx$, put $m = 5$, $n = 7$ in (1).

Then $$\int x^5 (1+x^2)^{7/2}\, dx = \frac{x^4 (1+x^2)^{9/2}}{9} - \frac{4}{9}\int x^2 (1+x^2)^{9/2}\, dx$$

$$= \frac{x^4 (1+x^2)^{9/2}}{9} - \frac{4}{9}\left[\frac{x^2 (1+x^2)^{11/2}}{11} - \frac{2}{11}\int x (1+x^2)^{11/2}\, dx \right],$$

putting $m = 3$, $n = 9$ in (1)

$$= \frac{x^4 (1+x^2)^{9/2}}{9} - \frac{4x^2 (1+x^2)^{11/2}}{99} + \frac{4}{99}\int (1+x^2)^{11/2} . 2x\, dx$$

$$= \frac{x^4 (1+x^2)^{9/2}}{9} - \frac{4x^2 (1+x^2)^{11/2}}{99} + \frac{4}{99} \frac{(1+x^2)^{13/2}}{(13/2)}$$

$$= \frac{1}{9}(1+x^2)^{9/2} \left[x^4 - \frac{4}{11}x^2 (1+x^2) + \frac{8}{143}(1+x^2)^2\right].$$

Problem 3: *If* $I_{m,n} = \int \frac{x^m\, dx}{(1+x^2)^n}$, *prove that*

$$2(n-1) I_{m,n} = -x^{m-1} (x^2+1)^{1-n} + (m-1) I_{m-2,n-1} .$$

Solution: We have $I_{m,n} = \int \frac{x^m}{(1+x^2)^n}\, dx$

$$= \frac{1}{2}\int x^{m-1} \{(1+x^2)^{-n} . 2x\}\, dx$$

$$= \frac{1}{2} x^{m-1} \frac{(1+x^2)^{-n+1}}{(-n+1)} - \frac{1}{2}\int (m-1) x^{m-2} \frac{(1+x^2)^{-n+1}}{(-n+1)}\, dx,$$

integrating by parts taking $\{(1+x^2)^{-n} . 2x\}$ as the 2nd function

$$= \frac{-1}{2(n-1)} x^{m-1} (x^2+1)^{1/n} + \frac{(m-1)}{2(n-1)} \int \frac{x^{m-2}}{(1+x^2)^{n-1}} dx .$$

Multiplying both sides by $2(n-1)$, we get

$$2(n-1) I_{m,n} = -x^{m-1} (1+x^2)^{-n+1} + (m-1) I_{m-2,n-1},$$

which is the required reduction formula.

Problem 4: *If* $\phi(n) = \int_0^x \frac{x^n\, dx}{\sqrt{(x-1)}}$, *prove that* $(2n+1)\phi(n) = 2x^n \sqrt{(x-1)} + 2n\phi(n-1)$.

Solution: We have $\phi(n) = \int_0^x \frac{x^n}{\sqrt{(x-1)}} dx = \int_0^x x^n (x-1)^{-1/2} dx$

$$= \left[x^n \frac{(x-1)^{1/2}}{1/2} \right]_0^x - \int_0^x \frac{nx^{n-1} (x-1)^{1/2}}{1/2} dx,$$

integrating by parts taking $(x-1)^{-1/2}$ as the 2nd function

$$= 2x^n \sqrt{(x-1)} - 2n \int_0^x \frac{x^{n-1}(x-1)}{\sqrt{(x-1)}} dx$$

$$= 2x^n \sqrt{(x-1)} - 2n \int_0^x \frac{x^n}{\sqrt{(x-1)}} dx + 2n \int_0^x \frac{x^{n-1}}{\sqrt{(x-1)}} dx$$

$$= 2x^n \sqrt{(x-1)} - 2n\phi(n) + 2n\phi(n-1).$$

Transposing the middle term to the left, we get

$$(2n+1)\phi(n) = 2x^n (x-1)^{1/2} + 2n\phi(n-1).$$

Problem 5: *If* I_n *denotes* $\int_0^\infty \frac{1}{(a^2+x^2)^n} dx$, *where n is a positive integer* ≥ 2, *prove that*

$I_n = \frac{2n-3}{2a^2(n-1)} I_{n-1}$. *Hence or otherwise evaluate* $\int_0^\infty \frac{1}{(a^2+x^2)^4} dx.$

Solution: Proceed as in article 2.

Let $I_n = \int_0^\infty \frac{1}{(a^2+x^2)^n} dx$, where n is a +ive integer ≥ 2.

To form a reduction formula for I_n, we shall integrate by parts $\int_0^\infty \frac{1}{(a^2+x^2)^{n-1}} dx$,

taking 1 as the 2nd function. Thus

$$\int_0^\infty \frac{1}{(a^2+x^2)^{n-1}} \cdot 1\, dx = \left[\frac{x}{(a^2+x^2)^{n-1}} \right]_0^\infty - \int_0^\infty x \frac{-(n-1)}{(a^2+x^2)^n} \cdot 2x\, dx$$

or $$I_{n-1} = \left[\lim_{x\to\infty} \frac{x}{(a^2+x^2)^{n-1}} - 0 \right] + 2(n-1) \int_0^\infty \frac{x^2}{(a^2+x^2)^n} dx$$

$$= 0 + 2(n-1)\int_0^\infty \frac{(a^2+x^2)-a^2}{(a^2+x^2)^n}\,dx,$$

$$\left[\because \lim_{x\to\infty} \frac{x}{(a^2+x^2)^{n-1}} = 0 \text{ if } n \geq 2\right]$$

$$= 2(n-1)\int_0^\infty \frac{dx}{(a^2+x^2)^{n-1}} - 2(n-1)\,a^2 \int_0^\infty \frac{1}{(a^2+x^2)^n}\,dx$$

$$= 2(n-1)\,I_{n-1} - 2(n-1)\,a^2\, I_n .$$

$\therefore$ $$2(n-1)\,a^2\, I_n = \{2(n-1)-1\}\, I_{n-1}$$

or $$I_n = \frac{2n-3}{2a^2(n-1)}\, I_{n-1}, \qquad \text{...(1)}$$

is the reduction formula for I_n.

Now putting $n = 4$ in (1), we get

$$I_4 = \frac{5}{2a^2 . 3}\, I_3 = \frac{5}{6a^2} \cdot \frac{3}{2a^2 . 2}\, I_2,$$

putting $n = 3$ in (1) to get I_3 in terms of I_2

$$= \frac{5}{8a^4} \cdot \frac{1}{2a^2 . 1}\, I_1 = \frac{5}{16a^6}\int_0^\infty \frac{1}{(a^2+x^2)}\,dx = \frac{5}{16a^6}\,\frac{1}{a}\left[\tan^{-1}\frac{x}{a}\right]_0^\infty$$

$$= \frac{5}{16a^7}\,[\tan^{-1}\infty - \tan^{-1} 0] = \frac{5}{16a^7} \cdot \frac{\pi}{2} = \frac{5\pi}{32a^7}.$$

Aliter: Let $I = \int_0^\infty \frac{1}{(a^2+x^2)^4}\,dx$.

Put $x = a\tan\theta$, so that $dx = a\sec^2\theta\, d\theta$. The limits for θ are from 0 to $\pi/2$.

$\therefore$ $$I = \int_0^{\pi/2} \frac{1}{(a^2\sec^2\theta)^4} \cdot a\sec^2\theta\, d\theta = \frac{1}{a^7}\int_0^{\pi/2} \cos^6\theta\, d\theta$$

$$= \frac{1}{a^7} \cdot \frac{5 . 3 . 1}{6 . 4 . 2} \cdot \frac{\pi}{2}, \text{ by Walli's formula}$$

$$= \frac{5\pi}{32a^7}.$$

Problem 6: *If* $I_m = \int_0^{2a} x^m \sqrt{(2ax - x^2)}\,dx$, *prove that*

$$2^m\, m!\,.\,(m+2)!\, I_m = a^{m+2}\,(2m+1)!\,\pi .$$

Hence or otherwise evaluate $\int_0^{2a} x^3 \sqrt{(2ax - x^2)}\,dx$.

(Bundelkhand 2007, 10)

Solution: Proceeding as in article 3 and taking limits from 0 to $2a$, we get

$$I_m = -\left[\frac{x^{m-1}(2ax - x^2)^{3/2}}{m+2}\right]_0^{2a} + \frac{(2m+1)\,a}{m+2} I_{m-1}$$

or $$I_m = 0 + \frac{(2m+1)\,a}{m+2} I_{m-1} = \frac{(2m+1)\,a}{m+2} I_{m-1}. \qquad ...(1)$$

$\therefore$ $$I_m = \frac{(2m+1)\,a}{m+2} \cdot \frac{(2m-1)\,a}{m+1} I_{m-2},$$

replacing m by $m-1$ in (1) to get I_{m-1} in terms of I_{m-2}

$$= \frac{(2m+1)\,a}{m+2} \cdot \frac{(2m-1)\,a}{m+1} \cdot \frac{(2m-3)\,a}{m} I_{m-3}, \text{ and so on.}$$

Thus applying the reduction formula (1) successively, we have finally

$$I_m = \frac{(2m+1)\,a}{m+2} \cdot \frac{(2m-1)\,a}{m+1} \cdot \frac{(2m-3)\,a}{m} \cdots \frac{5a}{4} \cdot \frac{3a}{3} I_0,$$

where $$I_0 = \int_0^{2a} x^0 \sqrt{(2ax - x^2)}\, dx = \int_0^{2a} \sqrt{x} \,.\, \sqrt{(2a - x)}\, dx.$$

Put $x = 2a \sin^2 \theta$, so that $dx = 4a \sin\theta \cos\theta \, d\theta$, and the new limits are from $\theta = 0$ to $\theta = \pi/2$.

$\therefore$ $$I_0 = \int_0^{\pi/2} \sqrt{(2a)} \sin\theta \,.\, \sqrt{(2a)} \cos\theta \,.\, 4a \sin\theta \cos\theta \, d\theta$$

$$= 8a^2 \int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta = 8a^2 \cdot \frac{\Gamma\frac{3}{2} \,.\, \Gamma\frac{3}{2}}{2\,\Gamma 3}$$

$$= 8a^2 \frac{\frac{1}{2}\sqrt{\pi} \,.\, \frac{1}{2}\sqrt{\pi}}{2.2.1} = \frac{\pi a^2}{2}.$$

$\therefore$ $$I_m = \frac{a^m\,(2m+1)(2m-1)(2m-3)\ldots 5.3}{(m+2)(m+1)\,m \ldots 4.3} \cdot \frac{\pi a^2}{2}$$

$$= \frac{\pi a^{m+2}}{(m+2)!} \cdot (2m+1)(2m-1)(2m-3)\ldots 5.3.$$

Multiplying the numerator and the denominator by $2m.(2m-2)\ldots 4.2$, we get

$$I_m = \frac{\pi\, a^{m+2}}{(m+2)!} \cdot \frac{(2m+1)\,2m\,(2m-1)(2m-2)\ldots 5.4.3.2.1}{2m.(2m-2)\ldots 4.2}$$

$$= \frac{\pi a^{m+2}}{(m+2)!} \cdot \frac{(2m+1)!}{2^m\,[m.(m-1)\ldots 2.1]} = \frac{\pi\, a^{m+2}.(2m+1)!}{2^m.(m)!.(m+2)!}$$

or $$2^m\,(m)!\,(m+2)!\,I_m = a^{m+2}.(2m+1)!\,\pi.$$ **Proved.**

Now let $$I = \int_0^{2a} x^3 \sqrt{(2ax - x^2)}\, dx = \int_0^{2a} x^{3+1/2} \sqrt{(2a - x)}\, dx$$

$$= \int_0^{2a} x^{7/2} \sqrt{(2a-x)}\, dx\,.$$

Put $x = 2a \sin^2 \theta$, so that $dx = 4a \sin\theta \cos\theta\, d\theta$, and the new limits are $\theta = 0$ to $\theta = \pi/2$.

$\therefore$
$$I = \int_0^{\pi/2} (2a)^4 \sin^7 \theta . \cos\theta . 4a \sin\theta \cos\theta\, d\theta$$

$$= 64\, a^5 \int_0^{\pi/2} \sin^8 \theta \cos^2 \theta\, d\theta = 64\, a^5 \frac{\Gamma \frac{9}{2} \Gamma \frac{3}{2}}{2\,\Gamma\, 6} = \frac{7\pi a^5}{8}\,.$$

Problem 7: *If* $I_n = \int x^n (a-x)^{1/2}\, dx$, *prove that*

$$(2n+3)\, I_n = 2an\, I_{n-1} - 2x^n (a-x)^{3/2}\,.$$ **(Bundelkhand 2011)**

Hence evaluate $\int_0^a x^2 \sqrt{(ax - x^2)}\, dx$.

Solution: We have $I_n = \int x^n (a-x)^{1/2}\, dx$

$$= x^n . \{-\frac{2}{3}(a-x)^{3/2}\} - \int nx^{n-1} . \{-\frac{2}{3}(a-x)^{3/2}\}\, dx,$$

integrating by parts taking $(a-x)^{1/2}$ as the 2nd function

$$= -\frac{2}{3} x^n (a-x)^{3/2} + \frac{2}{3} n \int x^{n-1} (a-x)(a-x)^{1/2}\, dx$$ **(Note)**

$$= -\frac{2}{3} x^n (a-x)^{3/2} + \frac{2}{3} na \int x^{n-1} (a-x)^{1/2}\, dx$$

$$- \frac{2}{3} n \int x^n (a-x)^{1/2}\, dx]$$

$$= -\frac{2}{3} x^n (a-x)^{3/2} + \frac{2}{3} na\, I_{n-1} - \frac{2}{3} nI_n\,.$$

Transposing the last term to the left, we get

$$(1 + \frac{2}{3} n)\, I_n = \frac{2}{3} na\, I_{n-1} - \frac{2}{3} x^n (a-x)^{3/2}$$

or
$$(2n+3)\, I_n = 2na\, I_{n-1} - 2\, x^n (a-x)^{3/2}$$ **Proved**

or
$$I_n = \frac{2na}{(2n+3)} I_{n-1} - \frac{2x^n (a-x)^{3/2}}{(2n+3)}\,. \qquad ...(1)$$

Taking limits $x = 0$ to $x = a$ in (1), we have

$$\int_0^a x^n (a-x)^{1/2}\, dx$$

$$= \frac{2na}{(2n+3)} \int_0^a x^{n-1} (a-x)^{1/2}\, dx - \left[\frac{2x^n (a-x)^{3/2}}{(2n+3)}\right]_0^a$$

$$= \frac{2na}{(2n+3)} \int_0^a x^{n-1} (a-x)^{1/2}\, dx - 0\,. \qquad ...(2)$$

To evaluate $\int_0^a x^2 \sqrt{(ax - x^2)}\, dx$,

we have $\int_0^a x^2 \sqrt{(ax - x^2)}\, dx = \int_0^a x^2 . x^{1/2} . \sqrt{(a - x)}\, dx$

$$= \int_0^a x^{5/2} (a - x)^{1/2}\, dx = \frac{2a . \frac{5}{2}}{2 . \frac{5}{2} + 3} \int_0^a x^{3/2} (a - x)^{1/2}\, dx,$$

applying the reduction formula (2) by taking $n = 5/2$

$$= \frac{5a}{8} \cdot \frac{2a . \frac{3}{2}}{2 . \frac{3}{2} + 3} \int_0^a x^{1/2} (a - x)^{1/2}\, dx,$$

again applying (2) by taking $n = 3/2$

$$= \frac{5a}{8} \cdot \frac{a}{2} \int_0^a x^{1/2} (a - x)^{1/2}\, dx.$$

Now put $x = a \sin^2 \theta$, so that $dx = 2a \sin\theta \cos\theta\, d\theta$, and the new limits are $\theta = 0$ to $\theta = \pi/2$.

$$\therefore \quad \int_0^a x^2 \sqrt{(ax - x^2)}\, dx = \frac{5a^2}{16} \int_0^{\pi/2} a^{1/2} \sin\theta . a^{1/2} \cos\theta . 2a \sin\theta \cos\theta\, d\theta$$

$$= \frac{5a^2}{16} . 2a^2 \int_0^{\pi/2} \sin^2\theta \cos^2\theta\, d\theta$$

$$= \frac{5a^4}{8} \cdot \frac{\Gamma \frac{3}{2} . \Gamma \frac{3}{2}}{2\, \Gamma 3}, \qquad \text{[By Gamma function]}$$

$$= \frac{5a^4}{8} \frac{\frac{1}{2} \sqrt{\pi} . \frac{1}{2} \sqrt{\pi}}{2 . 2 . 1} = \frac{5 \pi a^4}{128}.$$

Problem 8: *If* $u_n = \int x^n (a^2 - x^2)^{1/2}\, dx$, *prove that*

$$u_n = - \frac{x^{n-1} (a^2 - x^2)^{3/2}}{n + 2} + \frac{n - 1}{n + 2} a^2 u_{n-2}.$$

Hence evaluate $\int_0^a x^4 \sqrt{(a^2 - x^2)}\, dx$.

Solution: We have $u_n = \int x^n (a^2 - x^2)^{1/2}\, dx$

$$= - \frac{1}{2} \int x^{n-1} . \{(a^2 - x^2)^{1/2} . (-2x)\}\, dx$$

$$= - \frac{1}{2} x^{n-1} [\frac{2}{3} (a^2 - x^2)^{3/2}] + \frac{1}{3} (n - 1) \int x^{n-2} (a^2 - x^2)^{3/2}\, dx,$$

integrating by parts taking $(a^2 - x^2)^{1/2} (-2x)$ as the second function

$$= -\frac{1}{3} x^{n-1} (a^2 - x^2)^{3/2} + \frac{1}{3}(n-1) \int x^{n-2} (a^2 - x^2)^{1/2} (a^2 - x^2)\, dx$$

(Note)

$$= -\frac{1}{3} x^{n-1} (a^2 - x^2)^{3/2} + \frac{1}{3}(n-1)\, u_{n-2} - \frac{1}{3}(n-1)\, u_n .$$

Transposing the last term to the left, we have

$$\{1 + \frac{1}{3}(n-1)\}\, u_n = -\frac{1}{3} x^{n-1} (a^2 - x^2)^{3/2} + \frac{1}{3}(n-1)\, a^2\, u_{n-2}$$

or $$u_n = -\frac{x^{n-1} (a^2 - x^2)^{3/2}}{(n+2)} + \frac{n-1}{n+2}\, a\, u_{n-2} . \quad ...(1)$$

Proved

Taking limits $x = 0$ to $x = a$ in (1), we have

$$\int_0^a x^n \sqrt{(a^2 - x^2)}\, dx$$

$$= -\left[\frac{x^{n-1} (a^2 - x^2)^{3/2}}{(n+2)}\right]_0^a + \frac{(n-1)\, a^2}{(n+2)} \int_0^a x^{n-2} \sqrt{(a^2 - x^2)}\, dx$$

$$= 0 + \frac{n-1}{n+2} a^2 \int_0^a x^{n-2} (a^2 - x^2)^{1/2}\, dx . \quad ...(2)$$

Putting $n = 4$ in (2), we have

$$\int_0^a x^4 (a^2 - x^2)^{1/2}\, dx = \frac{4-1}{4+2} a^2 \int_0^a x^2 (a^2 - x^2)^{1/2}\, dx$$

$$= \frac{3a^2}{6} \cdot \frac{2-1}{2+2} a^2 \int_0^a x^0 (a^2 - x^2)^{1/2}\, dx ,$$

again applying (2) by taking $n = 2$

$$= \frac{1}{8} a^4 \left[\frac{1}{2} x \sqrt{(a^2 - x^2)} + \frac{1}{2} a^2 \sin^{-1} \frac{x}{a}\right]_0^a$$

$$= \frac{1}{8} a^4 \cdot \frac{1}{2} a^2 \cdot \frac{1}{2} \pi = \frac{\pi a^6}{32} .$$

Problem 9: *Show that* $\int_0^\infty e^{-ax} x^n\, dx = \frac{n!}{a^{n+1}}$*, where a is a positive quantity and n is a positive integer.*

Solution: Integrating by parts regarding e^{-ax} as the 2nd function, we get

$$\int_0^\infty e^{-ax} x^n\, dx = \left[x^n \left(\frac{e^{-ax}}{-a}\right)\right]_0^\infty - \int_0^\infty n x^{n-1} \left(\frac{e^{-ax}}{-a}\right) dx$$

$$= 0 + \frac{n}{a} \int_0^\infty e^{-ax} x^{n-1}\, dx .$$

$$\left[\because \left\{ -x^n e^{-ax} \right\}_0^\infty = 0 \text{, as shown in Example 3}\right]$$

Thus $$\int_0^{\infty} e^{-ax} x^n \, dx = \frac{n}{a} \int_0^{\infty} e^{-ax} x^{n-1} \, dx. \quad ...(1)$$

Now by the repeated application of the reduction formula (1), we have ultimately

$$\int_0^{\infty} e^{-ax} x^n \, dx = \frac{n!}{a^n} \int_0^{\infty} e^{-ax} x^0 \, dx = \frac{n!}{a^n} \left[- \frac{1}{ae^{ax}} \right]_0^{\infty}$$

$$= \frac{n!}{a^n} \cdot \frac{1}{a} = \frac{n!}{a^{n+1}}.$$

Problem 10: *Evaluate* $\int_0^1 (\log x)^4 \, x^m \, dx$.

Solution: Let $I_{m,\,n} = \int_0^1 x^m (\log x)^n \, dx$, where n is an integer ≥ 0. Then proceeding as in Example 4, we have

$$I_{m,\,n} = -\frac{n}{(m+1)} I_{m,\,n-1}. \quad ...(1)$$

Putting $m = 4$ in (1), we get

$$I_{m,\,4} = \int_0^1 (\log x)^4 \, x^m \, dx = \frac{-4}{m+1} \int_0^1 x^m (\log x)^{4-1} \, dx$$

$$= -\frac{4}{m+1} I_{m,\,3}.$$

Now by repeated application of (1), we have

$$I_{m,\,4} = \left(\frac{-4}{m+1}\right) \cdot \left(\frac{-3}{m+1}\right) \left(\frac{-2}{m+1}\right) \left(\frac{-1}{m+1}\right) \int_0^1 x^m (\log x)^0 \, dx$$

$$= \left(\frac{-4}{m+1}\right) \cdot \left(\frac{-3}{m+1}\right) \cdot \left(\frac{-2}{m+1}\right) \left(\frac{-1}{m+1}\right) \int_0^1 x^m \, dx$$

$$= \frac{24}{(m+1)^4} \int_0^1 x^m \, dx = \frac{24}{(m+1)^4} \cdot \left(\frac{x^{m+1}}{m+1} \right)_0^1 = \frac{24}{(m+1)^5}.$$

Problem 11: *If m and n are positive integers, and* $f(m, n) = \int_0^1 x^{n-1} (\log x)^m \, dx$, *prove that* $f(m, n) = -(m/n) \, f(m-1, n)$. *Deduce that* $f(m, n) = (-1)^m . \, m!/n^{m+1}$.

Solution: Integrating by parts regarding x^{n-1} as the 2nd function, we have

$$f(m, n) = \left[(\log x)^m . \frac{x^n}{n} \right]_0^1 - \frac{m}{n} \int_0^1 (\log x)^{m-1} \cdot \frac{1}{x} \cdot x^n \, dx$$

$$= 0 - \frac{m}{n} \int_0^1 x^{n-1} (\log x)^{m-1} \, dx,$$

$$\left[\lim_{x \to 0} x^n (\log x)^m = 0, \text{ as shown Example 4} \right]$$

or $$f(m, n) = -(m/n)\, f(m-1, n) \qquad ...(1)$$

Proved

$$= (-1)^2 \left(\frac{m}{n}\right) \cdot \left(\frac{m-1}{n}\right) f(m-2, n), \text{ applying (1)}$$

$$= (-1)^3 \left(\frac{m}{n}\right)\left(\frac{m-1}{n}\right)\left(\frac{m-2}{n}\right) f(m-3, n), \text{ again applying (1)}$$

$$= (-1)^3 \frac{m(m-1)(m-2)}{n^3} f(m-3, n).$$

Proceeding similarly by successive application of (1), ultimately we have

$$f(m, n) = (-1)^m \frac{m(m-1)(m-2)\ldots 2.1}{n^m} f(0, n).$$

But $$f(0, n) = \int_0^1 x^{n-1} (\log x)^0 \, dx = \int_0^1 x^{n-1} \, dx = \left[\frac{x^n}{n}\right]_0^1 = \frac{1}{n}.$$

$\therefore$ $$f(m, n) = (-1)^m \frac{m!}{n^m} \cdot \frac{1}{n} = (-1)^m \frac{m!}{n^{m+1}}.$$

Problem 12: *Evaluate* $\int_0^\infty \frac{x}{(1+e^x)} dx$.

Solution: We have

$$I = \int_0^\infty \frac{x\,dx}{(1+e^x)} = \int_0^\infty \frac{x\,dx}{e^x(1+e^{-x})}$$

$$= \int_0^\infty xe^{-x}(1+e^{-x})^{-1} dx. \qquad \textbf{(Note)}$$

On expanding $(1+e^{-x})^{-1}$ by binomial theorem, we have

$$(1+e^{-x})^{-1} = 1 - e^{-x} + e^{-2x} - e^{-3x} + e^{-4x} - \ldots$$

$\therefore$ $$I = \int_0^\infty x\,e^{-x} [1 - e^{-x} + e^{-2x} - e^{-3x} + \ldots]\, dx$$

$$= \int_0^\infty x [e^{-x} - e^{-2x} + e^{-3x} - e^{-4x} + \ldots]\, dx. \qquad ...(1)$$

Also $$\int_0^\infty xe^{-nx} dx = \left[-\frac{xe^{-nx}}{n}\right]_0^\infty \int_0^\infty \frac{e^{-nx}}{n} dx = 0 + \left[\frac{e^{-nx}}{-n^2}\right]_0^\infty = \frac{1}{n^2}.$$

...(2)

Now applying (2) to each term of the R.H.S. of (1), we get

$$I = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots \infty = \frac{\pi^2}{12}, \text{ from trigonometry.}$$

Hints to Objective Type Questions

Multiple Choice Questions

1. See Example 2.
2. See Example 3.
3. See Problem 5 of Comprehensive Problems 1.
4. See Problem 7 of Comprehensive Problems 1.
5. See Problem 10 of Comprehensive Problems 1.

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Chapter-4

Beta and Gamma Functions

Comprehensive Problems 1

Problem 1(i): *Prove that* $\int_0^a \frac{dx}{(a^n - x^n)^{1/n}} = \frac{1}{n} \cdot \frac{\pi}{\sin(\pi/n)}$. **(Kanpur 2010)**

Solution: In the given integral put $x^n = a^n \sin^2 \theta$

i.e., $x = a \sin^{2/n} \theta \, d\theta$ so that $dx = (2a/n) \sin^{(2/n)-1} \theta \cos \theta \, d\theta$.

Also when $x = 0$, $\theta = 0$ and when $x = a$, $\theta = \pi/2$.

$$\therefore \quad \int_0^a \frac{dx}{(a^n - x^n)^{1/n}} = \frac{2a}{n} \int_0^{\pi/2} \frac{\sin^{(2/n)-1} \theta \cos \theta \, d\theta}{a \cos^{2/n} \theta}$$

$$= \frac{2}{n} \int_0^{\pi/2} \sin^{(2/n)-1} \theta \cos^{1-(2/n)} \theta \, d\theta = \frac{\frac{2}{n} \Gamma \frac{1}{n} \Gamma \left(1 - \frac{1}{n}\right)}{2 \Gamma 1}$$

$$= \frac{1}{n} \cdot \Gamma \frac{1}{n} \Gamma \left(1 - \frac{1}{n}\right) = \frac{1}{n} \cdot \frac{\pi}{\sin(\pi/n)}. \qquad \left[\because \ \Gamma(n) \Gamma(1-n) = \frac{\pi}{\sin n \pi}\right]$$

Problem 1(ii): *Prove that* $\int_0^2 (8 - x^3)^{-1/3} dx = \frac{2\pi}{3\sqrt{3}}$. **(Kumaun 2008)**

Solution: Let $I = \int_0^2 (8 - x^3)^{-1/3} \, dx = \int_0^2 x^{-2} (8 - x^3)^{-1/3} x^2 \, dx$

Put $x^3 = 8y$ so that $3x^2 \, dx = 8 \, dy$.

When $x = 0$, $y = 0$ and when $x = 2$, $y = 1$.

$$\therefore \quad I = \int_0^1 (8y)^{-2/3} (8 - 8y)^{-1/3} . \frac{8}{3} dy$$

$$= \int_0^1 8^{-2/3} y^{-2/3} . 8^{-1/3} (1-y)^{-1/3} . \frac{8}{3} \, dy = \frac{1}{3} \int_0^1 y^{-2/3} (1-y)^{-1/3} \, dy$$

$$= \frac{1}{3} \int_0^1 y^{(1/3)-1} (1-y)^{(2/3)-1} dy = \frac{1}{3} B\left(\frac{1}{3}, \frac{2}{3}\right)$$

$$= \frac{1}{3} \frac{\Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3} + \frac{2}{3}\right)} = \frac{1}{3} \frac{\Gamma\left(\frac{1}{3}\right) \Gamma\left(1 - \frac{1}{3}\right)}{\Gamma 1} = \frac{1}{3} \cdot \frac{\pi}{\sin \frac{1}{3} \pi} = \frac{\pi}{3} \cdot \frac{2}{\sqrt{3}} = \frac{2\pi}{3\sqrt{3}}.$$

Problem 2: *Show that* $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\,dx = \frac{\Gamma(m)\,\Gamma(n)}{a^n\,(1+a)^m\,\Gamma(m+n)}$.

Solution: Proceed exactly as in Example 4. Here we have $b=1$.

Problem 3: *Show that* $\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{1}{2}\,\mathbf{B}\left(\frac{p+1}{2},\frac{q+1}{2}\right)$, $p>-1,\ q>-1$.

Deduce that $\int_0^2 x^4(8-x^3)^{-1/3}\,dx = \frac{16}{3}\,\mathbf{B}\left(\frac{5}{3},\frac{2}{3}\right)$.

Solution: We have

$$\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta = \int_0^{\pi/2}\sin^{p-1}\theta\cos^{q-1}\theta\sin\theta\cos\theta\,d\theta \quad (\textbf{Note})$$

$$= \int_0^{\pi/2}\sin^{p-1}\theta.(1-\sin^2\theta)^{\frac{1}{2}(q-1)}\sin\theta\cos\theta\,d\theta.$$

Put $\sin^2\theta = x$ so that $2\sin\theta\cos\theta\,d\theta = dx$.

Also when $\theta=0$, $x=0$ and when $\theta=\frac{1}{2}\pi$, $x=1$.

$\therefore$ The given integral $= \int_0^1 x^{(p-1)/2}.(1-x)^{(q-1)/2}\cdot\frac{1}{2}\,dx$

$$= \frac{1}{2}\int_0^1 x^{\{(p+1)/2\}-1}(1-x)^{\{(q+1)/2\}-1}\,dx$$

$$= \frac{1}{2}\,\mathbf{B}\left(\frac{p+1}{2},\frac{q+1}{2}\right), \qquad \text{...(1)}$$

where $\frac{p+1}{2}>0$ *i.e.*, $p>-1$ and $\frac{q+1}{2}>0$ *i.e.*, $q>-1$.

Second part: We have $I = \int_0^2 x^4(8-x^3)^{-1/3}\,dx = \int_0^2 x^2(8-x^3)^{-1/3}.x^2\,dx$. **(Note)**

Put $x^3 = 8\sin^2\theta$ so that $3x^2\,dx = 16\sin\theta\cos\theta\,d\theta$.

Also when $x=0$, $\theta=0$ and when $x=2$, $\theta=\frac{1}{2}\pi$.

$$\therefore \quad I = \int_0^{\pi/2}(8\sin^2\theta)^{2/3}(8-8\sin^2\theta)^{-1/3}\cdot\frac{16}{3}\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} 4\sin^{4/3}\theta\cdot\frac{1}{2}\cos^{-2/3}\theta\cdot\frac{16}{3}\sin\theta\cos\theta\,d\theta$$

$$= \frac{32}{3}\int_0^{\pi/2}\sin^{7/3}\theta\cos^{1/3}\theta\,d\theta$$

$$= \frac{32}{3}\cdot\frac{1}{2}\,\mathbf{B}\left(\frac{\frac{7}{3}+1}{2},\frac{\frac{1}{3}+1}{2}\right) \qquad \left[\text{From (1); here } p=\frac{7}{3},\ q=\frac{1}{3}\right]$$

$$= \frac{16}{3}\,\mathbf{B}\left(\frac{5}{3},\frac{2}{3}\right).$$

Problem 4: *Show that* **B** *(m, n) =* **B** *(m + 1, n) +* **B** *(m, n + 1), for m>0, n>0.*

(Kanpur 2005; Gorakhpur 05; Bundelkhand 11; Avadh 06, 11, 14)

Solution: We know that $\mathbf{B}\,(m, n) = \dfrac{\Gamma\,(m)\,\Gamma\,(n)}{\Gamma\,(m+n)}$.

$$\therefore \qquad \mathbf{B}\,(m+1, n) = \frac{\Gamma\,(m+1)\,\Gamma\,(n)}{\Gamma\,(m+1+n)}$$

$$\text{and} \qquad \mathbf{B}\,(m, n+1) = \frac{\Gamma\,(m)\,\Gamma\,(n+1)}{\Gamma\,(m+n+1)}.$$

Adding, we get $\mathbf{B}\,(\boldsymbol{m}+\mathbf{1}, \boldsymbol{n}) + \mathbf{B}\,(m, n+1)$

$$= \frac{\Gamma\,(m+1)\,\Gamma\,(n) + \Gamma\,(m)\,\Gamma\,(n+1)}{\Gamma\,(m+n+1)} = \frac{m\,\Gamma\,(m)\,\Gamma\,(n) + \Gamma\,(m).n\,\Gamma\,(n)}{(m+n)\,\Gamma\,(m+n)}$$

$$= \frac{(m+n)\,\Gamma\,(m)\,\Gamma\,(n)}{(m+n)\,\Gamma\,(m+n)} = \frac{\Gamma\,(m)\,\Gamma\,(n)}{\Gamma\,(m+n)} = \mathbf{B}\,(m, n),$$

provided $m > 0$ and $n > 0$.

Problem 5(i): *Prove that* $\displaystyle\int_0^\infty e^{-ax}\, x^{n-1}\, dx = \frac{\Gamma(n)}{a^n}$.

Solution: Let $I = \displaystyle\int_0^\infty e^{-ax}\, x^{n-1}\, dx$

Put $ax = y$ so that $a\,dx = dy$. When $x = 0, y = 0$ and when $x \to \infty$, $y \to \infty$.

$$\therefore \qquad I = \int_0^\infty e^{-y} \left(\frac{y}{a}\right)^{n-1} \cdot \frac{1}{a}\, dy = \frac{1}{a^n} \int_0^\infty e^{-y}\, y^{n-1}\, dy$$

$$= \frac{1}{a^n}\,\Gamma(n), \text{ by the definition of Gamma function.}$$

Problem 5(ii): *Prove that* $\displaystyle\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx = \Gamma\,(n)$.

(Kumaun 2000, 07; Garhwal 2000)

Solution: Let $\quad I = \displaystyle\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx$.

Put $\qquad \log\,(1/x) = y$ *i.e.*, $1/x = e^{\,y}$ *i.e.*, $x = e^{-\,y}$ so that $dx = -\,e^{-\,y}\,dy$.

Also when $x \to 0$, $y \to \infty$ and when $x = 1$, $y = 0$. [Note that $\log \infty = \infty$]

$$\therefore \qquad I = -\int_\infty^0 y^{n-1}\, e^{-\,y}\, dy = \int_0^\infty e^{-y}\, y^{n-1}\, dy = \Gamma\,(n).$$

Problem 6: *Show that, if* $m > -1$, *then* $\displaystyle\int_0^\infty x^m e^{-n^2 x^2} dx = \frac{1}{2n^{m+1}}\,\Gamma\left(\frac{m+1}{2}\right)$.

(Kumaun 2009)

Solution: Let $\quad I = \displaystyle\int_0^\infty x^m e^{-\,n^2 x^2} dx = \int_0^\infty x^{m-1} e^{-n^2 x^2}\, x dx.$

Put $n^2 x^2 = t$, so that $2n^2 x\,dx = dt$. Also $x = t^{1/2} / n$.

When $x = 0, t = 0$ and when $x \to \infty, t \to \infty$.

$$\therefore \quad I = \int_0^\infty \left(\frac{t^{1/2}}{n}\right)^{m-1} e^{-t} . \frac{1}{2n^2} \, dt = \frac{1}{2n^{m-1}.n^2} \int_0^\infty e^{-t} \, t^{(m-1)/2} dt$$

$$= \frac{1}{2n^{m+1}} \int_0^\infty e^{-t} t^{\{(m+1/2)\}-1} dt = \frac{1}{2n^{m+1}} \Gamma\left(\frac{m+1}{2}\right),$$

by definition of Gamma function, provided $m + 1 > 0$ *i.e.*, $m > -1$.

Problem 7: *Prove that* $\int_0^1 x^m (1 - x^n)^p \, dx = \frac{1}{n} B\left(\frac{m+1}{n}, p+1\right)$. **(Lucknow 2010)**

Solution: Let $I = \int_0^1 x^m (1 - x^n)^p \, dx.$

Put $x^n = y$ or $x = y^{1/n}$.

Then $dx = \frac{1}{n} y^{(1/n)} - 1 \, dy.$

When $x = 0,\ y = 0$ and when $x = 1,\ y = 1$.

$$\therefore \quad I = \int_0^1 (y^{1/n})^m (1 - y)^p \cdot \frac{1}{n} y^{(1/n)-1} dy$$

$$= \frac{1}{n} \int_0^1 y^{(m/n)+(1/n)-1} (1 - y)^{(p+1)-1} dy$$

$$= \frac{1}{n} \int_0^1 y^{\{(m+1)/n\}-1} (1 - y)^{(p+1)-1} \, dy = \frac{1}{n} B\left(\frac{m+1}{n}, p+1\right).$$

Problem 8: *Prove that* $\int_0^1 (1 - x^n)^{1/n} \, dx = \frac{1}{n} \frac{[\Gamma(1/n)]^2}{2\,\Gamma(2/n)}$.

Solution: Let $I = \int_0^1 (1 - x^n)^{1/n} dx.$

Put $x^n = y$ or $x = y^{1/n}$.

Then $dx = (1/n)\, y^{(1/n)-1} dy.$

When $x = 0,\ y = 0$ and when $x = 1,\ y = 1$.

$$\therefore \quad I = \int_0^1 (1 - y)^{1/n} \cdot \frac{1}{n} y^{(1/n)-1} dy$$

$$= \frac{1}{n} \int_0^1 y^{(1/n)-1} (1 - y)^{\{(1/n)+1\}-1} dy$$

$$= \frac{1}{n} B\left(\frac{1}{n}, \frac{1}{n} + 1\right) = \frac{1}{n} \frac{\Gamma(1/n)\, \Gamma\{(1/n)+1\}}{\Gamma\{(2/n)+1\}}$$

$$= \frac{1}{n} \cdot \frac{\Gamma(1/n) . (1/n)\, \Gamma(1/n)}{(2/n) . \Gamma(2/n)} = \frac{1}{n} \frac{[\Gamma(1/n)]^2}{2\,\Gamma(2/n)}.$$

Problem 9: *Show that* $\Gamma(0.1)\,\Gamma(0.2)\,\Gamma(0.3)\ldots\Gamma(0.9)=\dfrac{(2\pi)^{9/2}}{\sqrt{10}}$.

Solution: Proceed as in Example 14. Put $n=10$.

Problem 10: *Show that* $\int_0^{\pi/2}\sqrt{(\sin\theta)}\,d\theta\times\int_0^{\pi/2}\dfrac{d\theta}{\sqrt{(\sin\theta)}}=\pi$.

(Garhwal 2002; Lucknow 09)

Solution: The given integral $=\int_0^{\pi/2}\sin^{-1/2}\theta\,d\theta\times\int_0^{\pi/2}\sin^{1/2}\theta\,d\theta$

$$=\int_0^{\pi/2}\sin^{-1/2}\theta\,\cos^0\theta\,d\theta\times\int_0^{\pi/2}\sin^{1/2}\theta\cos^0\theta\,d\theta \qquad \textbf{(Note)}$$

$$=\frac{\Gamma\left\{\frac{1}{2}\left(-\frac{1}{2}+1\right)\right\}\Gamma\left\{\frac{1}{2}(0+1)\right\}}{2\Gamma\left\{\frac{1}{2}\left(-\frac{1}{2}+0+2\right)\right\}}\times\frac{\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+1\right)\right\}\Gamma\left\{\frac{1}{2}(0+1)\right\}}{2\,\Gamma\left\{\frac{1}{2}\left(\frac{1}{2}+0+2\right)\right\}}$$

$$=\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{4}\right)}\times\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\frac{1}{2}}{2\Gamma\left(\frac{5}{4}\right)}=\frac{\Gamma\frac{1}{4}\cdot\sqrt{\pi}\cdot\sqrt{\pi}}{2\cdot2\cdot\frac{1}{4}\,\Gamma\left(\frac{1}{4}\right)}=\pi.$$

Problem 11: *Show that* $\int_0^1\dfrac{x^2\,dx}{(1-x^4)^{1/2}}\times\int_0^1\dfrac{dx}{(1+x^4)^{1/2}}=\dfrac{\pi}{4\sqrt{2}}$.

(Garhwal 2001, 03; Lucknow 06, 11)

Solution: In the first integral put $x^2=\sin\theta$ so that $2x\,dx=\cos\theta\,d\theta$ and the corresponding limits for θ are 0 to $\pi/2$.

$$\therefore \quad \int_0^1\frac{x^2\,dx}{(1-x^4)^{1/2}}=\int_0^1\frac{\frac{1}{2}\,x\,.\,2x\,dx}{(1-x^4)^{1/2}}=\frac{1}{2}\int_0^{\pi/2}\frac{\sin^{1/2}\theta\,.\cos\theta\,d\theta}{(1-\sin^2\theta)^{1/2}}$$

$$=\frac{1}{2}\int_0^{\pi/2}\frac{\sqrt{(\sin\theta)}\,.\cos\theta\,d\theta}{\cos\theta}=\frac{1}{2}\int_0^{\pi/2}\sqrt{(\sin\theta)}\,d\theta.$$

Again, in 2nd integral put $x^2=\tan\theta$ so that $2x\,dx=\sec^2\theta\,d\theta$ and the corresponding limits for θ are 0 to $\pi/4$.

$$\therefore \quad \int_0^1\frac{dx}{(1+x^4)^{1/2}}=\frac{1}{2}\int_0^{\pi/4}\frac{\sec^2\theta\,d\theta}{\sqrt{(\tan\theta)}\sec\theta}=\frac{1}{2}\int_0^{\pi/4}\frac{d\theta}{\sqrt{(\sin\theta\cos\theta)}}$$

$$=\frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{d\theta}{\sqrt{(\sin 2\theta)}}=\frac{1}{2\sqrt{2}}\int_0^{\pi/2}\frac{1}{\sqrt{(\sin\phi)}}\,d\phi,$$

where $2\theta=\phi$.

Hence the given integral becomes

$$\int_0^1 \frac{x^2\,dx}{(1-x^4)^{1/2}} \times \int_0^1 \frac{dx}{(1+x^4)^{1/2}}$$

$$= \frac{1}{2}\int_0^{\pi/2} \sqrt{(\sin\theta)}\,d\theta \times \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \frac{d\phi}{\sqrt{(\sin\phi)}}$$

$$= \frac{1}{4\sqrt{2}}\cdot\int_0^{\pi/2} \sqrt{(\sin\theta)}\,d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{(\sin\theta)}} = \frac{1}{4\sqrt{2}}\cdot\pi.$$

[As proved in problem 10; prove it here.]

Problem 12: *Show that* $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = 2\int_0^{\pi/2} \sqrt{(\tan\theta)}d\theta = 4\int_0^{\infty} \frac{x^2 dx}{1+x^4} = \pi\sqrt{2}.$

(Lucknow 2008, 11)

Solution: Let $I = 4\int_0^{\infty} \frac{x^2\,dx}{1+x^4} = 4\int_0^{\infty} \frac{x\,.\,x\,dx}{1+x^4}$.

Put $x^2 = \tan\theta$ so that $2x\,dx = \sec^2\theta\,d\theta$. Also when $x=0, \theta=0$ and when $x\to\infty, \theta\to\pi/2$.

Then
$$I = 4\int_0^{\pi/2} \frac{\sqrt{(\tan\theta)}\cdot\frac{1}{2}\sec^2\theta\,d\theta}{(1+\tan^2\theta)} = 2\int_0^{\pi/2} \sqrt{(\tan\theta)}\,d\theta \quad \text{...(1)}$$

$$= 2\int_0^{\pi/2} \sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta = 2\,\frac{\Gamma\{\frac{1}{2}(\frac{1}{2}+1)\}\,\Gamma\{\frac{1}{2}(-\frac{1}{2}+1)\}}{2\,\Gamma\{\frac{1}{2}(\frac{1}{2}-\frac{1}{2}+2)\}}$$

$$= 2\frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{4})}{2\,\Gamma 1} = \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) \quad \text{...(2)}$$

$$= \Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\sin\frac{1}{4}\pi} \qquad \left[\because\ \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{\pi}{1/\sqrt{2}} = \pi\sqrt{2}. \quad \text{...(3)}$$

From (1), (2) and (3), the required result follows.

Problem 13: *Show that the perimeter of a loop of the curve* $r^n = a^n\cos n\theta$ *is*

$$\frac{a}{n}\cdot 2^{(1/n)-1}\cdot\frac{[\Gamma(1/2n)]^2}{\Gamma(1/n)}.$$

Solution: The given curve is

$$r^n = a^n\cos n\theta. \quad \text{...(1)}$$

The curve (1) is symmetrical about the initial line. We have $r=0$ when $\cos n\theta = 0$ *i.e.*, when $n\theta = -\pi/2, \pi/2$ or $\theta = -\pi/2n\ ,\ \pi/2n$. Therefore one loop of the curve lies between the lines $\theta = -\pi/2n$ and $\theta = \pi/2n$. This loop is symmetrical about the initial line and for the portion of this loop lying above the initial line θ varies from 0 to $\pi/2n$.

Taking log of both sides of (1), we have

$$n \log r = n \log a + \log \cos n\theta.$$

Differentiating w.r.t θ, we get

$$\frac{n}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta}(-n\sin n\theta) \qquad \text{or} \qquad \frac{dr}{d\theta} = -r\tan n\theta.$$

$$\therefore \qquad \frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]} = \sqrt{(r^2 + r^2\tan^2 n\theta)} = r\sec n\theta$$

or $$ds = r\sec n\theta\, d\theta = a\cos^{1/n} n\theta \sec n\theta\, d\theta = a\,(\cos n\theta)^{(1/n)-1} d\theta.$$

$\therefore$ The perimeter of a loop of the curve (1)

$$= 2\int_0^{\pi/2n} a(\cos n\theta)^{(1/n)-1} d\theta$$

$$= 2a\int_0^{\pi/2} (\cos t)^{(1/n)-1}\frac{dt}{n}, \text{ putting } n\theta = t \text{ so that } n\,d\theta = dt$$

$$= \frac{2a}{n}\int_0^{\pi/2} (\cos t)^{(1/n)-1}\sin^0 t\, dt = \frac{2a}{n}\frac{\Gamma(1/2n)\,\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left\{(1/2n)+\frac{1}{2}\right\}}$$

$$= \frac{a}{n}\cdot\frac{\sqrt{\pi}.\,[\Gamma(1/2n)]^2}{\Gamma(1/2n)\,\Gamma\{(1/2n)+\frac{1}{2}\}} = \frac{a}{n}\cdot\sqrt{\pi} = \frac{[\Gamma(1/2n)]^2}{\frac{\sqrt{\pi}}{2^{(1/n)-1}}\Gamma(1/n)}$$

$$\left[\because\ \Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m), \text{ by article 11}\right]$$

$$= \frac{a}{n}\cdot 2^{(1/n)-1}\frac{[\Gamma(1/2n)]^2}{\Gamma(1/n)}.$$

Problem 14: *Prove that* $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^4)}} = \frac{\sqrt{2}}{8\sqrt{\pi}}\left[\Gamma\left(\frac{1}{4}\right)\right]^2.$

Solution: Let $\displaystyle I = \int_0^1 \frac{dx}{\sqrt{(1-x^4)}}.$

Put $x^4 = \sin^2\theta$ *i.e.*, $x = \sin^{1/2}\theta$ so that $dx = \frac{1}{2}\sin^{-1/2}\theta\cos\theta\, d\theta.$

When $x = 0, \theta = 0$ and when $x = 1, \theta = \pi/2.$

$$\therefore \qquad I = \int_0^{x/2}\frac{\frac{1}{2}\sin^{-1/2}\theta\cos\theta\, d\theta}{\cos\theta} = \frac{1}{2}\int_0^{\pi/2}\sin^{-1/2}\theta\cos^0\theta\, d\theta$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{3}{4}\right)} = \frac{1}{4} \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2 . \sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{4} \cdot \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)}$$

$$= \frac{\sqrt{\pi}}{4} \cdot \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{\pi / \left(\sin\frac{1}{4}\pi\right)} = \frac{\sqrt{\pi}}{4} . \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{\pi . \sqrt{2}} = \frac{\sqrt{2}}{8\sqrt{\pi}} \left[\Gamma\left(\frac{1}{4}\right)\right]^2.$$

Problem 15: *Show that* $\int_0^{\pi/2} \sin^p \theta \, d\theta = \frac{1}{2} B\left(\frac{1}{2}, \frac{p+1}{2}\right)$.

Solution: We have

$$I = \int_0^{\pi/2} \sin^p \theta \, d\theta = \int_0^{\pi/2} \sin^p \theta \cos^0 \theta \, d\theta$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{p+2}{2}\right)} = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{1}{2}\right) = \frac{1}{2} B\left(\frac{1}{2}, \frac{p+1}{2}\right).$$

Aliter: Put $\sin^2 \theta = y$ or $\sin\theta = y^{1/2}$

Then $\cos\theta \, d\theta = \frac{1}{2} y^{-1/2} dy$

or $d\theta = \frac{1}{2} \frac{y^{-1/2} dy}{\sqrt{(1-\sin^2\theta)}} = \frac{1}{2}(1-y)^{-1/2} y^{-1/2} dy.$

When $\theta = 0$, $y = 0$ and when $\theta = \pi/2$, $y = 1$.

$\therefore$
$$I = \int_0^1 y^{p/2} . \frac{1}{2}(1-y)^{-1/2} y^{-1/2} dy = \frac{1}{2}\int_0^1 y^{(p-1)/2}(1-y)^{-1/2} dy$$

$$= \frac{1}{2}\int_0^1 y^{\{(p+1)/2\}-1}(1-y)^{(1/2)-1} dy$$

$$= \frac{1}{2} B\left(\frac{p+1}{2}, \frac{1}{2}\right) = \frac{1}{2} B\left(\frac{1}{2}, \frac{p+1}{2}\right).$$

Problem 16: *Show that*

(i) $\int_0^\infty x e^{-\alpha x} \cos\beta x \, dx = \frac{(\alpha^2 - \beta^2)}{(\alpha^2 + \beta^2)^2}$ *(ii)* $\int_0^\infty x e^{-\alpha x} \sin\beta x \, dx = \frac{2\alpha\beta}{(\alpha^2 - \beta^2)^2}.$

Solution: We have

$$\int_0^\infty x e^{-\alpha x} \cos\beta x \, dx + i \int_0^\infty x e^{-\alpha x} \sin\beta x \, dx$$

$$= \int_0^\infty x e^{-\alpha x}(\cos\beta x + i \sin\beta x) dx = \int_0^\infty x e^{-\alpha x} e^{i\beta x} \, dx$$

$$= \int_0^\infty x e^{-(\alpha - i\beta)x} dx = \int_0^\infty e^{-(\alpha - i\beta)x} \, x^{2-1} \, dx$$

$$= \frac{\Gamma(2)}{(\alpha - i\beta)^2} = \frac{1}{(\alpha - i\beta)^2} = \frac{(\alpha + i\beta)^2}{[(\alpha - i\beta)(\alpha + i\beta)]^2} = \frac{(\alpha^2 - \beta^2) + 2\, i\alpha\beta}{(\alpha^2 + \beta^2)^2}$$

$$= \frac{\alpha^2 - \beta^2}{(\alpha^2 + \beta^2)^2} + i \frac{2\,\alpha\beta}{(\alpha^2 + \beta^2)^2}. \qquad \ldots(1)$$

Equating real and imaginary parts in (1), we get

$$\int_0^\infty x e^{-\alpha x} \cos\beta x \, dx = \frac{\alpha^2 - \beta^2}{(\alpha^2 + \beta^2)^2}$$

and
$$\int_0^\infty x e^{-\alpha x} \sin\beta x \, dx = \frac{2\alpha\beta}{(\alpha^2 + \beta^2)^2}.$$

Problem 17: *Prove that* $\int_{-\infty}^{\infty} \cos\left(\frac{1}{2}\pi x^2\right) dx = 1.$

Solution: We have $I = \int_{-\infty}^{\infty} \cos\left(\frac{1}{2}\pi x^2\right) dx = 2\int_0^\infty \cos\left(\frac{1}{2}\pi x^2\right) dx.$

Put $\frac{1}{2}\pi x^2 = t$ *i.e.*, $x = \sqrt{(2/\pi)}\, t^{1/2}$, so that $dx = \frac{1}{2}\sqrt{(2/\pi)}\, t^{-1/2} dt.$

$\therefore$
$$I = 2 \cdot \frac{1}{2} \cdot \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty t^{-1/2} \cos t \, dt = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty t^{-1/2} \cos t \, dt$$

$$= \text{real part in } \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty t^{-1/2} e^{-it} \, dt, \qquad [\because e^{-it} = \cos t - i \sin t]$$

$$= \text{real part in } \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty t^{(1/2)-1} e^{-it} \, dt$$

$$= \text{real part in } \sqrt{\left(\frac{2}{\pi}\right)} \frac{\Gamma\left(\frac{1}{2}\right)}{(i)^{1/2}}$$

$$= \text{real part in } \sqrt{\left(\frac{2}{\pi}\right)} \frac{\sqrt{\pi}}{\left(\cos\frac{1}{2}\pi + i \sin\frac{1}{2}\pi\right)^{1/2}}$$

$$= \text{real part in } \sqrt{2}. \left(\cos\frac{1}{2}\pi + i \sin\frac{1}{2}\pi\right)^{-1/2}$$

$$= \text{real part in } \sqrt{2}. \left(\cos\frac{1}{4}\pi - i \sin\frac{1}{4}\pi\right) = \sqrt{2}.\cos\frac{\pi}{4} = \sqrt{2}.\frac{1}{\sqrt{2}} = 1.$$

Problem 18: *Prove that* $\mathbf{B}\,(m, m).\ \mathbf{B}\left(m + \frac{1}{2}, m + \frac{1}{2}\right) = \frac{\pi\, m^{-1}}{2^{4m-1}}.$ **(Rohilkhand 2005)**

Solution: We have

$$\mathbf{B}(m, m).\ \mathbf{B}\left(m+\frac{1}{2}, m+\frac{1}{2}\right)$$

$$= \frac{\Gamma(m).\Gamma(m)}{\Gamma(m+m)} \cdot \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}+m+\frac{1}{2}\right)}, \qquad \left[\because B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{\left[\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)\right]^2}{\Gamma(2m).\,\Gamma(2m+1)} = \frac{\left[\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)\right]^2}{\Gamma(2m).\,2m\,\Gamma(2m)}, \qquad [\because \Gamma(p+1) = p\,\Gamma(p)]$$

$$= \frac{1}{2m}\left[\frac{\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(2m)}\right]^2 = \frac{1}{2m}\cdot\left[\frac{\sqrt{\pi}}{2^{2m-1}}\right]^2 \qquad \textbf{(Note)}$$

$$\left[\because \text{By Duplication formula, (article 11)}, \frac{\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(2m)} = \frac{\sqrt{\pi}}{2^{2m-1}}\right]$$

$$= \frac{1}{2m}\cdot\frac{\pi}{2^{4m-2}} = \frac{\pi\, m^{-1}}{2^{4m-1}}.$$

Problem 19: *Prove that* $\int_0^{\pi} \frac{\sin^{n-1} x\, dx}{(a+b\cos x)^n} = \frac{2^{n-1}}{(a^2-b^2)^{n/2}} \cdot \mathbf{B}\left(\frac{n}{2}, \frac{n}{2}\right), a > b$.

(Kumaun 2008)

Solution: Let $I = \int_0^{\pi} \frac{\sin^{n-1} x}{(a+b\cos x)^n}\, dx$

$$= \int_0^{\pi} \frac{\left(2\sin\frac{1}{2}x\cos\frac{1}{2}x\right)^{n-1}}{\left[a\left(\cos^2\frac{1}{2}x+\sin^2\frac{1}{2}x\right)+b\left(\cos^2\frac{1}{2}x-\sin^2\frac{1}{2}x\right)\right]^n}\, dx$$

$$= \int_0^{\pi} \frac{2^{n-1}\left(\sin\frac{1}{2}x\right)^{n-1}\left(\cos\frac{1}{2}x\right)^{n-1}}{\left[(a+b)\cos^2\frac{1}{2}x+(a-b)\sin^2\frac{1}{2}x\right]^n}\, dx$$

$$= \int_0^{\pi} \frac{2^{n-1}\left(\tan\frac{1}{2}x\right)^{n-1}\sec^2\frac{1}{2}x\, dx}{\left[(a+b)+(a-b)\tan^2\frac{1}{2}x\right]^n},$$

dividing the Nr. and Dr. by $\left(\cos^2\frac{1}{2}x\right)^n$ *i.e.*, $\left(\cos\frac{1}{2}x\right)^{2n}$

$$= 2^{n-1}\int_0^{\pi} \frac{\left(\tan\frac{1}{2}x\right)^{n-2}\tan\frac{1}{2}x\sec^2\frac{1}{2}x\, dx}{\left[(a+b)+(a-b)\tan^2\frac{1}{2}x\right]^n}.$$

Put $(a-b)\tan^2 \frac{1}{2}x = (a+b)\,y$ so that

$$(a-b).(2\tan\frac{1}{2}x\sec^2\frac{1}{2}x)\cdot\frac{1}{2}dx = (a+b)\,dy.$$

When $x = 0$, $y = 0$ and when $x \to \pi$, $y \to \infty$.

$$\therefore \quad I = 2^{n-1}\int_0^\infty \frac{\{(a+b)\,y/(a-b)\}^{(n-2)/2}}{[(a+b)+(a+b)\,y]^n}\,\frac{a+b}{a-b}\,dy$$

$$= 2^{n-1}\int_0^\infty \frac{(a+b)^{(n-2)/2}\,(a+b)}{(a+b)^n\,(a-b)^{(n-2)/2}\,(a-b)}\cdot\frac{y^{(n-2)/2}}{(1+y)^n}\,dy$$

$$= \frac{2^{n-1}}{(a+b)^{n/2}\,(a-b)^{n/2}}\int_0^\infty \frac{y^{(n/2)-1}}{(1+y)^{(n/2)+(n/2)}}\,dy$$

$$= \frac{2^{n-1}}{(a^2-b^2)^{n/2}}\,\mathbf{B}\left(\frac{n}{2},\frac{n}{2}\right).$$

Hints to Objective Type Questions

Multiple Choice Questions

1. See article 7.
2. We have $\int_0^\infty e^{-x}\,x^{-1/2}\,dx = \int_0^\infty e^{-x}\,x^{1/2-1}\,dx = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$
3. See article 10, part (i).
4. If $a > 0$ and $n > 0$, then $\int_0^\infty e^{-ax}\,x^{n-1}\,dx = \frac{\Gamma(n)}{a^n}\cdot$
5. We have $\int_0^1 x^4(1-x)^3\,dx = \int_0^1 x^{5-1}(1-x)^{4-1}\,dx = B(5,4) = \frac{\Gamma(5)\cdot\Gamma(4)}{\Gamma(5+4)} = \frac{1}{280}\cdot$
6. We have $\int_0^{\pi/2}\sin^4 x\cos^2 x\,dx = \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{4+2+2}{2}\right)} = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma\,(4)} = \frac{\pi}{32}.$
7. See article 3 and article 7.
8. Let $I = \int_0^1 x^{n-1}\left(\log\frac{1}{x}\right)^{m-1}dx.$

   Put $\log(1/x) = y$ *i.e.*, $1/x = e^y$ *i.e.*, $x = e^{-y}$ so that $dx = -e^{-y}\,dy$.

   Also when $x \to 0$, $y \to \infty$ and when $x = 1$, $y = 0$. [Note that $\log\infty = \infty$]

$\therefore$ $$I = -\int_{\infty}^{0} (e^{-y})^{n-1}\, y^{m-1} e^{-y}\, dy = \int_{0}^{\infty} (e^{-y})^{n-1+1}\, y^{m-1}\, dy$$

$$= \int_{0}^{\infty} e^{-ny}\, y^{m-1}\, dy = \frac{\Gamma m}{n^m}, \text{ provided } m > 0 \text{ and } n > 0.$$

[See article 6 part (i)]

9. See Problem 12 of Comprehensive Problems 1.
10. See Example 12(iii).
11. See article 2.
12. See article 9.
13. See Problem 5(ii) of Comprehensive Problems 1.
14. See Example 14.
15. See article 3.
16. See article 9.
17. See article 4.
18. We have $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right) = \frac{\pi}{\sin\frac{\pi}{3}}$ $\left[\because\ \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$

$$= \frac{2\pi}{\sqrt{3}}\cdot$$

19. If n is a positive integer, then $\Gamma(n) = (n-1)\,!$. See article 5.
20. If $0 < n < 1$, then $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\cdot$

See article 7, corollary.

21. $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$. See article 8, important deduction.
22. For $m > 0$, $\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\cdot\Gamma(2m)$. See article 11.
23. See Problem 5(i) of Comprehensive Problems 1.
24. We have $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \Gamma\left(\frac{1}{3}\right)\cdot\Gamma\left(1-\frac{1}{3}\right) = \frac{\pi}{\sin \pi/3} = \frac{2\pi}{\sqrt{3}}\cdot$

See article 7 corollary.

25. See Example 2.
26. See article 7, Corollary.
27. See article 8, Important Deduction.
28. See article 13, Deductions (i).
29. See article 13, Deduction (ii).
30. See Example 8(iii).
31. See Example 11.

Fill in the Blanks

1. See article 1, definition of Beta function.
2. See article 4, definition of Gamma function.
3. See article 7.
4. We have $\dfrac{B(m+1, n)}{B(m, n)} = \dfrac{\Gamma(m+1)\,\Gamma(n)}{\Gamma(m+1+n)} \div \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

$$= \frac{m\,\Gamma(m)\,\Gamma(n)}{(m+n)\,\Gamma\,(m+n)} \cdot \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)} \qquad [\because \Gamma(n+1) = n\,\Gamma(n)]$$

$$= \frac{m}{m+n}.$$

5. See Example 6.
6. See article 5
7. See article 8.
8. See article 9.

True or False

1. We have $\int_0^\infty e^{-x} x^{1/2}\, dx = \int_0^\infty e^{-x} x^{3/2-1}\, dx = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\cdot\sqrt{\pi}.$
2. See Example 8, part (iii).
3. We have $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\, dx = 0.$ See Example 8, part (i).
4. See article 3.

   We have $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \mathbf{B}\,(m, n)$ and $\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx = \mathbf{B}\,(m, n).$
5. We have $\mathbf{B}\,(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$ and $\mathbf{B}\,(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$.
6. We have $\Gamma(6) = \Gamma(5+1) = 5! = 120.$
7. We have $\mathbf{B}\,(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}.$
8. We have $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$ See article 8, Important deduction.
9. See Problem 4, of Comprehensive Problems 1.

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Chapter-5

Dirichlet's and Liouville's Integrals

Comprehensive Problems 1

Problem 1: *Show that the integral* $\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$ *integrated over the region in the first octant below the surface* $(x/a)^p + (y/b)^q + (z/c)^r = 1$ *is*

$$\frac{a^l b^m c^n}{pqr} \cdot \frac{\Gamma(l/p)\,\Gamma(m/q)\,\Gamma(n/r)}{\Gamma(l/p + m/q + n/r + 1)}.$$

(Garhwal 2002; Avadh 11)

Or

Evaluate the integral $\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$, *where* x, y, z *are all positive but limited by the condition* $(x/a)^p + (y/b)^q + (z/c)^r \le 1$.

Solution: The required integral $= \iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$, where the integral is extended to all positive values of the variables x, y and z subject to the condition

$$(x/a)^p + (y/b)^q + (z/c)^r \le 1.$$

Put $(x/a)^p = u$ i.e., $x = au^{1/p}$ so that $dx = (a/p)\, u^{(1/p)-1}\, du$,

$(y/b)^q = v$ i.e., $y = bv^{1/q}$ so that $dy = (b/q)\, v^{(1/q)-1}\, dv$,

and $(z/c)^r = w$ i.e., $z = cw^{1/r}$ so that $dz = (c/r)\, w^{(1/r)-1}\, dw$.

Then the required integral

$$= \iiint (a^{l-1} u^{(l-1)/p})(b^{m-1} v^{(m-1)/q})(c^{n-1} w^{(n-1)/r}) \cdot \frac{a}{p} u^{(1/p)-1} \cdot \frac{b}{q} v^{(1/q)-1} \cdot \frac{c}{r} w^{(1/r)-1}\, du\, dv\, dw$$

$$= \frac{a^l b^m c^n}{pqr} \iiint u^{(l/p)-1} v^{(m/q)-1} w^{(n/r)-1}\, du\, dv\, dw,$$

where $u + v + w \le 1$

$$= \frac{a^l b^m c^n}{pqr} \cdot \frac{\Gamma(l/p)\,\Gamma(m/q)\,\Gamma(n/r)}{\Gamma(l/p + m/q + n/r + 1)},$$ by Dirichlet's integral.

Problem 2: *Show that if* l, m, n *are all positive*

$$\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{a^l b^m c^n}{8} \cdot \frac{\Gamma(l/2)\,\Gamma(m/2)\,\Gamma(n/2)}{\Gamma(l/2 + m/2 + n/2 + 1)},$$

where the triple integral is taken throughout the part of the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$, *which lies in the positive octant.*

Solution: The required integral $= \iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$,

where the integral is extended to all positive values of the variables x, y, z subject to the condition

$$x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1.$$

Put $x^2/a^2 = u$ *i.e.*, $x = au^{1/2}$ so that $dx = \frac{1}{2} au^{-1/2}\, du$,

$y^2/b^2 = v$ *i.e.*, $y = bv^{1/2}$ so that $dy = \frac{1}{2} bv^{-1/2}\, dv$,

and $z^2/c^2 = w$ *i.e.*, $z = cw^{1/2}$ so that $dz = \frac{1}{2} cw^{-1/2}\, dw$.

Then the required integral

$$= \iiint a^{l-1} u^{(l-1)/2} . b^{m-1} v^{(m-1)/2} . c^{n-1} w^{(n-1)/2} . \frac{1}{8} abc\, u^{-1/2} v^{-1/2} w^{-1/2}\, du\, dv\, dw$$

$$= \frac{a^l b^m c^n}{8} \iiint u^{(l/2)-1} v^{(m/2)-1} w^{(n/2)-1}\, du\, dv\, dw,$$

where $u + v + w \le 1$

$$= \frac{a^l b^m c^n}{8} \frac{\Gamma(l/2)\,\Gamma(m/2)\,\Gamma(n/2)}{\Gamma(l/2 + m/2 + n/2 + 1)}, \text{ by Dirichlet's integral.}$$

Problem 3: *Prove that the area in the positive quadrant between the curve* $x^n + y^n = a^n$ *and the coordinate axes is* $\dfrac{a^2\,[\Gamma(1/n)]^2}{2n\,\Gamma(2/n)}$.

(Kanpur 2009)

Solution: Given curve $x^n + y^n = a^n$ ...(1)

or $\left(\frac{x}{a}\right)^n + \left(\frac{y}{a}\right)^n = 1$

The area of positive quadrant between the curve $x^n + y^n = a^n$ and the co-ordinate axes is

$$I = \iint dx\, dy$$

put $\left(\frac{x}{a}\right)^n = u$, $\left(\frac{y}{a}\right)^n = v$ or $x = au^{1/n}$, $y = av^{1/n}$

$$dx = \frac{a}{n} u^{\frac{1}{n}-1}\, du,\; dy = \frac{a}{n} v^{\frac{1}{n}-1}\, dv$$

Then the required integral is

$$I = \iint \frac{a}{n} u^{\frac{1}{n}-1} \cdot \frac{a}{n} v^{\frac{1}{n}-1}\, du\, dv = \frac{a^2}{n^2} \iint u^{\frac{1}{n}-1} v^{\frac{1}{n}-1}\, du\, dv,$$

where u, v take all +ive values subject to the condition $u + v \le 1$

$$= \frac{a^2}{n^2} \frac{\Gamma(1/n)\,\Gamma(1/n)}{\Gamma\left(\frac{1}{n} + \frac{1}{n} + 1\right)}, \text{ by Dirichlet's theorem}$$

$$= \frac{a^2}{n^2} \frac{[\Gamma(1/n)]^2}{\frac{2}{n}\Gamma\left(\frac{2}{n}\right)} = \frac{a^2}{2n} \frac{[\Gamma(1/n)]^2}{\Gamma\left(\frac{2}{n}\right)}.$$

Problem 4(i): *Evaluate* $\iiint dx\, dy\, dz$, where $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \le 1$.

(Kumaun 2007)

Solution: Here we are to evaluate the given integral over the whole volume of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ which is symmetrical in all the eight octants.

Obviously the required integral $= 8 \iiint dx\, dy\, dz$,

where the integral is to be evaluated throughout the volume of the ellipsoid which lies in the positive octant.

Let $I = \iiint dx\, dy\, dz$, where the integral is extended to all positive values of the variables x, y, z subject to the condition

$$x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1.$$

Put $x^2/a^2 = u$ *i.e.,* $x = au^{1/2}$ so that $dx = \frac{1}{2} au^{-1/2}\, du$,

$y^2/b^2 = v$ *i.e.,* $y = bv^{1/2}$ so that $dy = \frac{1}{2} bv^{-1/2}\, dv$,

and $z^2/c^2 = w$ *i.e.,* $z = cw^{1/2}$ so that $dz = \frac{1}{2} cw^{-1/2}\, dw$.

Then
$$I = \iiint \frac{1}{8} abc\, u^{-1/2} v^{-1/2} w^{-1/2}\, du\, dv\, dw$$

$$= \iiint \frac{1}{8} abc\, u^{(1/2)-1} v^{(1/2)-1} w^{(1/2)-1}\, du\, dv\, dw,$$

where $u + v + w \le 1$

$$= \frac{abc}{8} \frac{\Gamma\left(\frac{1}{2}\right).\Gamma\left(\frac{1}{2}\right).\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+1\right)}, \qquad \text{[By Dirichlet's integral]}$$

$$= \frac{abc}{8} \frac{(\sqrt{\pi})^3}{\Gamma\left(\frac{5}{2}\right)} = \frac{abc}{8} \frac{\pi . \sqrt{\pi}}{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}} = \frac{\pi\, abc}{6}.$$

Hence the required integral $= 8 \cdot \dfrac{\pi\, abc}{6} = \dfrac{4}{3} \pi\, abc$.

Problem 4(ii): *Find the volume in the positive octant of the ellipsoid*

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$$

Solution: Proceed as in problem 4 part (iii). The answer is $\pi\, abc/6$.

Problem 4(iii): *Find the volume of the ellipsoid* $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$.

(Kumaun 2015)

Solution: The given ellipsoid is symmetrical in all the eight octants. Therefore the volume of the given ellipsoid = 8 × the volume of the part of the ellipsoid lying in the positive octant.

Now the volume of a small element situated at any point $(x, y, z) = dx\, dy\, dz$.

$\therefore$ The volume of the given ellipsoid $= 8 \iiint dx\, dy\, dz$,

where the integral is extended to all positive values of the variables x, y, z subject to the condition

$$x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1.$$

Now to evaluate this integral proceed as in Problem 4 part (i).

Hence the required volume $= 8 \cdot \dfrac{\pi\, abc}{6} = \dfrac{4}{3} \pi\, abc$.

Problem 5: *Evaluate* $\iiint xyz\, dx\, dy\, dz$ *for all positive values of the variables throughout the ellipsoid* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. **(Kanpur 2011)**

Solution: We are to evaluate the integral $\iiint xyz\, dx\, dy\, dz$, for all positive values of the variables x, y, z subject to the condition

$$x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1$$

Put $\quad x^2/a^2 + u, y^2/b^2 + v, z^2/c^2 = w$ so that

$$x\, dx = \frac{1}{2} a^2 du,\ y\, dy = \frac{1}{2} b^2 dv,\ z\, dz = \frac{1}{2} c^2 dw$$

Then the required integral $= \dfrac{1}{8} a^2 b^2 c^2 \iiint du\, dv\, dw$, where $u + v + w \le 1$

$$= \frac{1}{8} a^2 b^2 c^2 \iiint u^{1-1}\, v^{1-1}\, w^{1-1}\, du\, dv\, dw, \text{ where } u + v + w + \le 1$$

$$= \frac{1}{8} a^2 b^2 c^2 \frac{\Gamma(1)\, \Gamma(1)\, \Gamma(1)}{\Gamma(1+1+1+1)}, \qquad \text{[By Dirichlet's integral]}$$

$$= \frac{a^2 b^2 c^2}{8} = \frac{1}{\Gamma(4)} = \frac{a^2 b^2 c^2}{8} \times \frac{1}{3\,.\,2\,.\,1} = \frac{a^2 b^2 c^2}{48}.$$

Problem 6: *Find the volume of the tetrahedron bounded by the plane* $x/a + y/b + z/c = 1$ *and the coordinate planes.*

Solution: The volume of a small element situated at any point $(x, y, z) = dx\, dy\, dz$.

$\therefore$ the volume of the given tetrahedron

$= \iiint dx\, dy\, dz$, where the integral is extended throughout the volume enclosed by the coordinate planes and the plane $x/a + y/b + z/c = 1$

$= \iiint dx\, dy\, dz$, where the integral is extended to all positive values of the variables x, y, z subject to the condition $x/a + y/b + z/c \le 1$.

Put $x/a = u$, $y/b = v$, $z/c = w$ so that $dx = a\, du$, $dy = b\, dv$, $dz = c\, dw$.

Then the required volume

$$= \iiint abc\ du\ dv\ dw, \text{ where } u+v+w \le 1$$

$$= abc \iiint u^{1-1}\, v^{1-1}\, w^{1-1}\, du\ dv\ dw$$

$$= abc\, \frac{[\Gamma(1)]^3}{\Gamma(1+1+1+1)} \qquad \text{[By Dirichlet's integral]}$$

$$= abc \cdot \frac{1}{\Gamma(4)} = \frac{abc}{3.2.1} = \frac{abc}{6}.$$

Problem 7: *The plane $x/a + y/b + z/c = 1$ meets the coordinate axes in the points A, B, C. Use Dirichlet's integral to evaluate the mass of the tetrahedron $OABC$, the density at any point (x, y, z) being kxyz.* **(Garhwal 2003)**

Solution: The mass of a small element of volume $dx\, dy\, dz$ situated at any point (x, y, z)

$$= \rho\,.\, dx\, dy\, dz, \quad \text{where } \rho \text{ is the density per unit volume}$$

$$= k\ x\ y\ z\ dx\ dy\ dz.$$

∴ The mass of the tetrahedron $OABC$

$$= \iiint k\ xyz\ dx\ dy\ dz,$$ where the integral is extended to all positive values of the variables x, y, z subject to the condition $(x/a) + (y/b) + (z/c) \le 1$.

Put $x/a = u$, $y/b = v$, $z/c = w$ so that $dx = a\ du$, $dy = b\ dv$, $dz = c\ dw$.

Then the required mass

$$= \iiint k\ .au\,.\ bv.\ cw\ .abc\ du\ dv\ dw, \text{ where } u+v+w \le 1$$

$$= k\ a^2\ b^2\ c^2 \iiint u^{2-1}\, v^{2-1}\, w^{2-1}\, du\ dv\ dw$$

$$= k\ a^2\ b^2\ c^2\, \frac{[\Gamma(2)]^3}{\Gamma(2+2+2+1)}, \qquad \text{[By Dirichlet's integral]}$$

$$= k\ a^2\ b^2\ c^2 \cdot \frac{1}{\Gamma(7)} \qquad [\because\ \Gamma(2) = 1\,\Gamma(1) = 1]$$

$$= \frac{k\ a^2\ b^2\ c^2}{6\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1} = \frac{k\ a^2\ b^2\ c^2}{720}.$$

Problem 8: *Evaluate the integral $\iiint x^2\ y\ z\ dx\ dy\ dz$ over the volume enclosed by the region $x, y, z, \ge 0$ and $x + y + z \le 1$.* **(Agra 2003)**

Solution: The required integral $\iiint x^2 y\ z\ dx\ dy\ dz$, over the region $x, y, z, \ge 0$ and $x + y + z \le 1$

$$= \iiint x^{3-1}\ y^{2-1}\ z^{2-1}\ dx\ dy\ dz,$$ where the integral is extended to all positive values of the variables x, y, z subject to the condition $x + y + z \le 1$

$$= \iiint x^{3-1} \cdot y^{2-1} \cdot z^{2-1}\, dx\, dy\, dz$$

$$= \frac{(\Gamma 3)(\Gamma 2)(\Gamma 2)}{\Gamma(3+2+2+1)} \qquad \text{[By Dirichlet's integral]}$$

$$= \frac{(\Gamma 3)(\Gamma 2)(\Gamma 2)}{\Gamma(8)} = \frac{(2\,!)(1\,!)(1\,!)}{(7\,!)}$$

$$= \frac{2}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{1}{2520}.$$

Problem 9: *Evaluate the double integral* $\iint_D x^{1/2}\, y^{1/2}\,(1-x-y)^{2/3}\, dx\, dy$ *over the domain D bounded by the lines* $x = 0,\ y = 0,\ x + y = 1.$

Solution: Here the region of integration is bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$.

So the variable x, y take all positive values subject to the condition $0 < x + y < 1$.

Hence the given integral

$$\iint x^{\left(\frac{1}{2}+1\right)-1}\, y^{\left(\frac{1}{2}+1\right)-1}\, \{1-(x+y)\}^{2/3}\, dx\, dy$$

$$= \frac{\Gamma\left(\frac{1}{2}+1\right)\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{1}{2}+1+\frac{1}{2}+1\right)} \int_0^1 u^{\frac{1}{2}+1+\frac{1}{2}+1-1}\,(1-u)^{2/3}\, du,$$

[By Liouville's extension of Dirichlet's theorem]

$$= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)} \int_0^1 u^{3-1}\,(1-u)^{\left(\frac{2}{3}+1\right)-1}\, du$$

$$= \frac{\frac{1}{2}\sqrt{\pi}\,\frac{1}{2}\sqrt{\pi}}{2}\, B\left(3, \frac{2}{3}+1\right) = \frac{\pi}{8}\, \frac{\Gamma 3\, \Gamma\left(\frac{2}{3}+1\right)}{\Gamma\left(3+\frac{2}{3}+1\right)}$$

$$= \frac{\pi}{8} \cdot \frac{2 \cdot \Gamma(5/2)}{\Gamma(14/3)} = \frac{\pi}{4} \cdot \frac{\Gamma(5/3)}{\frac{11}{3} \cdot \frac{8}{3} \cdot \frac{5}{3} \cdot \Gamma(5/3)}$$

$$= \frac{27\pi}{1760}.$$

Problem 10: *Evaluate* $\iint_T x^{1/2}\, y^{1/2}\,(1-x-y)^{3/2}\, dx\, dy$*, where T is the region bounded by* $x \geq 0,\ y \geq 0,\ x + y \leq 1.$

Solution: Here the region of integration is bounded by the lines $x \geq 0$, $y \geq 0$ and $x + y \leq 1$.

So the variable x, y take all positive values subject to the condition $0 \leq x + y \leq 1$.

Hence the given integral

$$\iint x^{\left(\frac{1}{2}+1\right)-1} y^{\left(\frac{1}{2}+1\right)-1} \{1-(x+y)\}^{3/2}\, dx\, dy$$

$$= \frac{\Gamma\left(\frac{1}{2}+1\right)\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{1}{2}+1+\frac{1}{2}+1\right)} \int_0^1 u^{\frac{1}{2}+\frac{1}{2}+1-1} (1-u)^{3/2}\, du,$$

[By Liouville's extension of Dirichlet's theorem]

$$= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)} \int_0^1 u^{3-1} (1-u)^{\frac{5}{2}-1} du = \frac{\frac{1}{2}\sqrt{\pi}\,\frac{1}{2}\sqrt{\pi}}{2} B\left(3, \frac{5}{2}\right)$$

$$= \frac{\pi}{8} \cdot \frac{\Gamma(3)\,\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(3+\frac{5}{2}\right)} = \frac{\pi}{8} \frac{2 \cdot \frac{3}{2}\frac{1}{2}\sqrt{\pi}}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}} = \frac{2\pi}{315}.$$

Problem 11: *Find the value of* $\iint x^{l-1}\, y^{-l}\, e^{x+y}\, dx\, dy,$

extended to all positive values of x and y subject to $x + y < h$.

Solution: The given integral

$$I = \iint x^{l-1}\, y^{-l}\, e^{x+y}\, dx\, dy, \text{ where } 0 < x + y < h$$

$$= \iint x^{l-1}\, y^{(1-l)-1}\, e^{x+y}\, dx\, dy$$

$$= \frac{\Gamma(l)\,\Gamma(1-l)}{\Gamma(l+1-l)} \int_0^h e^u\, u^{l+(1-l)-1}\, du,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{\Gamma(l)\,\Gamma(1-l)}{\Gamma(1)} \int_0^h e^u\, du = \frac{\pi}{\sin \pi l}[e^u]_0^h = \frac{\pi}{\sin \pi l}(e^h - 1).$$

Problem 12: *Evaluate* $\iiint e^{x+y+z}\, dx\, dy\, dz$ *taken over the positive octant such that* $x + y + z \le 1$. **(Kanpur 2005, 10)**

Solution: Here the integral is to be extended for all positive values of x, y and z such that $0 \le x + y + z \le 1$.

$\therefore$ The required integral

$$\iiint e^{x+y+z}\, dx\, dy\, dz = \iiint e^{x+y+z}\, x^{1-1}\, y^{1-1}\, z^{1-1}\, dx\, dy\, dz,$$

where $0 \le x + y + z \le 1$

$$= \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1)} \int_0^1 e^u\, u^{3-1}\, du,$$

[By Liouville's extension of Dirichlet's integral]

$$= \frac{1.1.1}{2.1}\left[\{u^2 . e^u\}_0^1 - \int_0^1 2\,u.e^u\,du\right],$$

[Integrating by parts taking e^u as the second function]

$$= \frac{1}{2}\left[e - 2\left\{(e^u . u)_0^1 - \int_0^1 1.e^u\,du\right\}\right]$$

$$= \frac{1}{2}[e - 2e + 2\{e^u\}_0^1] = \frac{1}{2}[e - 2e + 2e - 2] = \frac{(e-2)}{2}.$$

Problem 13: *Evaluate* $\iiint x^{-1/2}\,y^{-1/2}\,z^{-1/2}\,(1 - x - y - z)^{1/2}\,dx\,dy\,dz$ *extended to all positive values of the variables subject to the condition* $x + y + z < 1$. **(Kanpur 2007)**

Solution: The given condition is $0 < x + y + z < 1$.

$\therefore$ The required integral $\iiint x^{-1/2} y^{-1/2}\,z^{-1/2}\,(1 - x - y - z)^{1/2}\,dx\,dy\,dz$

$$= \iiint x^{1/2-1}\,y^{1/2-1}\,z^{1/2-1}\,\{1 - (x + y + z)\}^{1/2}\,dx\,dy\,dz,$$

where $0 < x + y + z < 1$

$$= \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2}+\frac{1}{2})}\int_0^1 (1-u)^{1/2}\,u^{1/2+1/2+1/2-1}\,du,$$

[By Liouville's extension of Dirichlet's theorem]

$$= \frac{[\Gamma(\frac{1}{2})]^3}{\Gamma(3/2)}\int_0^1 (1-u)^{3/2-1}\,u^{(3/2)-1}\,du = \frac{(\sqrt{\pi})^3}{\frac{1}{2}\cdot\sqrt{\pi}}\cdot\frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma(\frac{3}{2}+\frac{3}{2})}$$

$$= 2\,\pi\cdot\frac{\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{2.1} = \frac{\pi^2}{4}.$$

Problem 14: *Prove that* $\iiint \frac{dx\,dy\,dz}{\sqrt{(1 - x^2 - y^2 - z^2)}} = \frac{\pi^2}{8}$, *the integral being extended to all positive values of the variables for which the expression is real.* **(Lucknow 2006)**

Solution: The given expression $1/\sqrt{(1 - x^2 - y^2 - z^2)}$ is real if $x^2 + y^2 + z^2 < 1$. Therefore the given integral is to be extended for all positive values of the variables x, y and z such that

$$0 < x^2 + y^2 + z^2 < 1.$$

Now put $x^2 = u_1$ *i.e.*, $x = u_1^{1/2}$, so that $dx = \frac{1}{2}u_1^{-1/2}\,du_1$,

$y^2 = u_2$ *i.e.*, $y = u_2^{1/2}$, so that $dy = \frac{1}{2}u_2^{-1/2}\,du_2$,

and $z^2 = u_3$ *i.e.*, $z = u_3^{1/2}$, so that $dz = \frac{1}{2}u_3^{-1/2}\,du_3$.

With these substitutions the given condition reduces to

$$0 < u_1 + u_2 + u_3 < 1$$

and the required integral becomes

$$= \iiint \frac{(\frac{1}{2})^3 \, u_1^{-1/2} \, u_2^{-1/2} \, u_3^{-1/2} \, du_1 \, du_2 \, du_3}{\sqrt{(1 - u_1 - u_2 - u_3)}}, \text{ for } 0 < u_1 + u_2 + u_3 < 1$$

$$= \frac{1}{8} \iiint \frac{u_1^{1/2 - 1} \, u_2^{1/2 - 1} \, u_3^{1/2 - 1} \, du_1 \, du_2 \, du_3}{\sqrt{\{1 - (u_1 + u_2 + u_3)\}}}$$

$$= \frac{1}{8} \frac{[\Gamma(\frac{1}{2})]^3}{\Gamma(\frac{1}{2} + \frac{1}{2} + \frac{1}{2})} \int_0^1 u^{3/2 - 1} \cdot \frac{1}{\sqrt{(1 - u)}} \, du,$$

[By Liouville's Extension of Dirichlet's Theorem]

$$= \frac{1}{8} \cdot \frac{[\sqrt{\pi}]^3}{\frac{1}{2}\sqrt{\pi}} \int_0^{\pi/2} \frac{\sin\theta \, . \, 2 \sin\theta \cos\theta \, d\theta}{\sqrt{(1 - \sin^2\theta)}}, \text{ putting } u = \sin^2\theta \text{ etc.}$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \sin^2\theta \, d\theta = \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{8}.$$

Problem 15: *Show that* $\iint \ldots \int \frac{dx_1 \, dx_2 \, \ldots \, dx_n}{\sqrt{(1 - x_1^2 - x_2^2 - \ldots - x_n^2)}} = \frac{\pi^{(n+1)/2}}{2^n \, \Gamma\left(\frac{n+1}{2}\right)}$

the integral being extended to all positive values of the variables for which the expression is real.

Solution: Proceed as in Example 2.

Problem 16: *If S is a unit sphere with its centre at the origin, then prove that*

$$\iiint_S \frac{dx \, dy \, dz}{\sqrt{(1 - x^2 - y^2 - z^2)}} = \pi^2.$$

Solution: We have for a unit sphere the condition of positive quadrant for x, y, z in $x^2 + y^2 + z^2 < 1$

put $\quad x^2 = 4, \quad y^2 = v, \quad z^2 = w \quad \text{or} \quad x = \sqrt{a}, \quad y = \sqrt{v}, \quad z = \sqrt{w}$

$\therefore \quad dx = \frac{1}{2} u^{-1/2} du, \quad dy = \frac{1}{2} v^{-1/2} dv, \quad dz = \frac{1}{2} w^{-1/2} dw$

Then integration is

$$\iiint \frac{1}{8} \frac{u^{-\frac{1}{2}} \, v^{-\frac{1}{2}} \, w^{-\frac{1}{2}} \, du \, dv \, dw}{\sqrt{1 - u - v - w}} = \frac{1}{8} \iiint \frac{u^{\frac{1}{2} - 1} \, v^{\frac{1}{2} - 1} \, w^{\frac{1}{2} - 1} \, du \, dv \, dw}{\sqrt{1 - (u + v + w)}}$$

$$= \frac{1}{8} \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \int_0^1 \frac{h^{1/2} dh}{\sqrt{1 - h}}$$

[By Liouville's theorem]

$$= \frac{1}{8} \frac{(\sqrt{\pi})^3}{\frac{1}{2}\sqrt{\pi}} \int_0^{\pi/2} \frac{\sin\theta \cdot 2\sin\theta\cos\theta \, d\theta}{\sqrt{1-\sin^2\theta}}, \text{ where } h = \sin^2\theta$$

$$= \frac{\pi}{4} \cdot \int_0^{\pi/2} 2\sin^2\theta \, d\theta = \frac{\pi}{4} \cdot 2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{8}$$

Then the required integral is $\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = 8 \cdot \frac{\pi^2}{8} = \pi^2.$

Problem 17: *Evaluate* $\iiint_R (x+y+z+1)^2 \, dx\,dy\,dz$,

where R is defined by $x \geq 0$, $y \geq 0$, $z \geq 0$, $x+y+z \leq 1$.

Solution: As given x, y, z are all positive such that $0 \leq x+y+z \leq 1$.

$$\therefore \quad \iiint (x+y+z+1)^2 \, dx\,dy\,dz$$

$$= \iiint x^{1-1} y^{1-1} z^{1-1} \{(x+y+z)+1\}^2 \, dx\,dy\,dz$$

$$= \frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1)} \int_0^1 (u+1)^2 \, . \, u^{1+1+1-1} \, du,$$

[By Liouville's extension of Dirichlet's theorem]

$$= \frac{1}{2} \int_0^1 (u^2+2u+1) \, u^2 \, du = \frac{1}{2}\left[\frac{u^5}{5} + \frac{2u^4}{4} + \frac{u^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{5} + \frac{1}{2} + \frac{1}{3}\right] = \frac{1}{2} \frac{(6+15+10)}{5 \times 2 \times 3} = \frac{1}{2} \cdot \frac{31}{30} = \frac{31}{60}.$$

Problem 18: *Show that* $\iint \left(\frac{1-x^2-y^2}{1+x^2+y^2}\right)^{1/2} dx\,dy = \frac{\pi}{8}(\pi - 2)$

over the positive quadrant of the circle $x^2 + y^2 = 1$.

Solution: Here the given integral is to be extended to all positive values of x and y such that

$$0 \leq x^2 + y^2 \leq 1. \qquad \ldots(1)$$

Put $x^2 = u$, $y^2 = v$ *i.e.*, $x = u^{1/2}$, $y = v^{1/2}$ so that

$$dx = \frac{1}{2} u^{-1/2} \, du, \; dy = \frac{1}{2} v^{-1/2} \, dv.$$

With these substitutions the condition (1) becomes $0 \leq u+v \leq 1$.

Hence the required integral

$$= \iint \left[\frac{1-(u+v)}{1+(u+v)}\right]^{1/2} \frac{1}{4} u^{-1/2} v^{-1/2} \, du\,dv$$

$$= \frac{1}{4} \iint \left[\frac{1-(u+v)}{1+(u+v)}\right]^{1/2} u^{(1/2)-1} v^{(1/2)-1} \, du\,dv, \text{ where } 0 \leq u+v \leq 1$$

$$= \frac{1}{4} \frac{\Gamma(1/2)\,\Gamma(1/2)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} \int_0^1 \left[\frac{1-h}{1+h}\right]^{1/2} h^{(1/2)+(1/2)-1}\, dh,$$

[By Liouville's extension of Dirichlet's theorem]

$$= \frac{1}{4} \frac{\sqrt{\pi}\,.\sqrt{\pi}}{\Gamma(1)} \int_0^1 \frac{1-h}{\sqrt{(1-h^2)}}\, dh$$

$$= \frac{\pi}{4} \int_0^{\pi/2} \frac{(1-\sin\theta)}{\cos\theta} \cos\theta\, d\theta, \text{ putting } h = \sin\theta \text{ so that } dh = \cos\theta\, d\theta$$

$$= \frac{\pi}{4} [\theta + \cos\theta]_0^{\pi/2} = \frac{\pi}{4}\left[\frac{\pi}{2} - 1\right] = \frac{\pi}{8}(\pi - 2).$$

Problem 19: *Find the value of* $\iiint xyz \sin(x+y+z)\, dx\, dy\, dz$, *the integral being extended to all positive values of the variables subject to the condition* $x + y + z \le \pi/2$.

Solution: Here $0 < x + y + z \le \pi/2$.

$\therefore$ The required integral

$$I = \iiint \sin(x+y+z)\, x^{2-1} y^{2-1} z^{2-1}\, dx\, dy\, dz,$$

where $0 < x + y + z \le \pi/2$

$$= \frac{\Gamma(2)\,\Gamma(2)\,\Gamma(2)}{\Gamma(2+2+2)} \int_0^{\pi/2} (\sin u)\, u^{2+2+2-1}\, du,$$

[By Liouville's theorem]

$$= \frac{1}{\Gamma(6)} \int_0^{\pi/2} u^5 \sin u\, du.$$

Applying successive integration by parts, we have

$$I = \frac{1}{5!}\Big[u^5(-\cos u) - (5u^4)(-\sin u) + (20u^3)(\cos u)$$

$$- (60u^2)(\sin u) + (120u)(-\cos u) - 120(-\sin u) \Big]_0^{\pi/2}.$$

In the above expression all the terms vanish for $u = 0$ and all those which involve $\cos u$ vanish for $u = \pi/2$.

$$\therefore \qquad I = \frac{1}{120}[-5(\pi/2)^4.(-1) - 60(\pi/2)^2.(1) - 120.(-1)]$$

$$= \frac{1}{120}\left[\frac{5\pi^4}{16} - 15\pi^2 + 120\right] = \frac{1}{384}[\pi^4 - 48\pi^2 + 384].$$

Problem 20: *Evaluate* $\iiint \sqrt{\left(\frac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2}\right)}\, dx\, dy\, dz$

integral being taken over all positive values of x, y, z such that $x^2 + y^2 + z^2 \le 1$.

Solution: Put $x^2 = u$, $y^2 = v$, $z^2 = w$.

Then $\quad x = u^{1/2}, dx = \frac{1}{2} u^{-1/2}\, du$; $y = v^{1/2}$, $dy = \frac{1}{2} v^{-1/2}\, dv$;

$$z = w^{1/2}, dz = \frac{1}{2} w^{-1/2}\, dw .$$

$\therefore$ The given integral

$$= \frac{1}{8} \iiint \sqrt{\left\{\frac{1-(u+v+w)}{1+(u+v+w)}\right\}}\, u^{1/2-1}\, v^{1/2-1}\, w^{1/2-1}\, du\, dv\, dw ,$$

where u, v, w are all +ive and $0 < u + v + w \le 1$

$$= \frac{1}{8} \cdot \frac{[\Gamma(\frac{1}{2})]^3}{\Gamma(\frac{3}{2})} \int_0^1 \sqrt{\left(\frac{1-t}{1+t}\right)}\, t^{1/2+1/2+1/2-1}\, dt$$

$$= \frac{\pi}{4} \int_0^1 \frac{1-t}{\sqrt{(1-t^2)}}\, t^{1/2}\, dt .$$

Put $t^2 = z$ or $t = z^{1/2}$ so that $dt = \frac{1}{2} z^{-1/2}\, dz$.

$$\therefore \quad I = \frac{\pi}{4} \int_0^1 (1-z)^{-1/2}\, [1 - z^{1/2}]\, z^{1/4} . \frac{1}{2} z^{-1/2}\, dz$$

$$= \frac{\pi}{8} \int_0^1 (1-z)^{-1/2}\, (z^{-1/4} - z^{1/4})\, dz$$

$$= \frac{\pi}{8} \int_0^1 [z^{(3/4)-1} (1-z)^{(1/2)-1} - z^{(5/4)-1} (1-z)^{(1/2)-1}]\, dz$$

$$= \frac{\pi}{8} \left[B\left(\frac{3}{4}, \frac{1}{2}\right) - B\left(\frac{5}{4}, \frac{1}{2}\right) \right].$$

Problem 21: *Prove that* $\iint_D e^{-x^2-y^2}\, dx\, dy = \frac{\pi}{4} (1 - e^{-R^2})$, *where* D *is the region defined by* $x \ge 0,\ y \ge 0,\ x^2 + y^2 \le R^2$. **(Lucknow 2009)**

Solution: Put $\frac{x^2}{R^2} = u$, $\frac{y^2}{R^2} = v$.

Then $\quad x = R\sqrt{u},\ y = R\sqrt{v}$

and $\quad dx = \frac{R}{2} u^{-1/2}\, du,\ dy = \frac{R}{2} v^{-1/2}\, dv.$

$$\therefore \quad \iint_D e^{-(x^2+y^2)}\, dx\, dy = \iint e^{-R^2(u+v)}\, \frac{R}{2} u^{-1/2}\, \frac{R}{2} v^{-1/2}\, du\, dv$$

$$= \frac{R^2}{4} \iint e^{-R^2(u+v)}\, u^{-1/2}\, v^{-1/2}\, du\, dv$$

$$= \frac{R^2}{4} \iint e^{-R^2(u+v)}\, u^{\frac{1}{2}-1}\, v^{\frac{1}{2}-1}\, du\, dv$$

$$= \frac{R^2}{4} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} \int_0^1 e^{-R^2 h}\, h^{\frac{1}{2}+\frac{1}{2}-1}\, dh$$

$$= \frac{R^2}{4} \sqrt{\pi}\,\sqrt{\pi} \left[\frac{e^{-R^2 h}}{-R^2}\right]_0^1 = \frac{R^2 \pi}{4} \frac{e^{-R^2} - e^0}{-R^2} = \frac{\pi}{4}(1 - e^{-R^2}).$$

Problem 22: *(i) Evaluate* $\iint_R \sqrt{(x^2 + y^2)}\, dx\, dy$

where R is the region in the xy-plane bounded by $x^2 + y^2 = 4$ *and* $x^2 + y^2 = 9$.

(ii) Evaluate $\iint_R \sqrt{(x^2 + y^2)}\, dx\, dy$ *where R is the region* $x^2 + y^2 \le a^2$.

Solution: (i) $\iint_R \sqrt{x^2 + y^2}\, dx\,dy$

where R is the region in the xy-plane bounded by

$$x^2 + y^2 = 4 \text{ and } x^2 + y^2 = 9.$$

$\therefore$
$$\iint_R \sqrt{x^2 + y^2}\, dx\,dy = 4\iint_R \sqrt{x^2 + y^2}\, dx\,dy \quad \{\text{For } x^2 + y^2 = 9\}$$
$$-4\iint_R \sqrt{x^2 + y^2}\, dx\,dy \ \{\text{For } x^2 + y^2 = 4\} \quad \ldots(1)$$

Now to solve $\iint_R \sqrt{x^2 + y^2}\, dx\,dy$ for $x^2 + y^2 = 9$ or $\frac{x^2}{9} + \frac{y^2}{9} = 1$.

Let $\frac{x^2}{9} = u, \frac{y^2}{9} = v$ or $x = 3\sqrt{u}, y = 3\sqrt{v}$

or $dx = \frac{3}{2} u^{-1/2} du, dy = \frac{3}{2} v^{-1/2} dv$ and $u + v = 1$.

So
$$\iint_R \sqrt{x^2 + y^2}\, dx\, dy = \iint \sqrt{9u + 9v}\, \frac{3}{2} u^{-1/2}\, du\, \frac{3}{2} v^{-1/2} dv$$

$$= \frac{3\cdot 3\cdot 3}{2\cdot 2} \iint u^{1/2-1} v^{1/2-1} \sqrt{u+v}\, du\,dv$$

$$= \frac{27}{4} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} \int_0^1 h^{\frac{1}{2}+\frac{1}{2}-1} \sqrt{h}\, dh, \text{ by Liouville's Theorem}$$

$$= \frac{27}{4} \frac{\sqrt{\pi}\,\sqrt{\pi}}{1} \cdot \frac{2}{3} = \frac{9}{2}\pi.$$

Similarly $\iint_R \sqrt{x^2 + y^2}\, dx\,dy$, for $x^2 + y^2 = 4 = \frac{4}{3}\pi$.

From equation (1)

$$\iint_R \sqrt{x^2 + y^2}\, dx\,dy = 4\left(\frac{9}{2}\pi\right) - 4\left(\frac{y}{3}\pi\right) = 4\pi\left[\frac{9}{2} - \frac{4}{3}\right] = 4\pi \cdot \frac{19}{6} = \frac{38\pi}{3}.$$

(ii) R is region $x^2 + y^2 \le a^2$ or $\frac{x^2}{a^2} + \frac{y^2}{a^2} \le 1$.

Let $\frac{x^2}{a^2} = u, \quad \frac{y^2}{a^2} = v.$

Then $x = a\sqrt{u}, \ y = a\sqrt{v}.$

$\therefore$ $dx = \frac{a}{2} u^{-1/2} \ du, \ dy = \frac{a}{2} v^{-1/2} \ dv.$

$\therefore$ $\iint_R \sqrt{x^2 + y^2} \ dx \, dy = 4 \iint \sqrt{a^2 u + a^2 v} \cdot \frac{a}{2} u^{-1/2} \frac{a}{2} v^{-1/2} \ du dv$ (For R)

$$= 4 \cdot \frac{a^3}{4} \iint u^{\frac{1}{2}-1} v^{\frac{1}{2}-1} \sqrt{u + v} \ du dv$$

$$= \frac{a^3}{4} \cdot 4 \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} \int_0^1 h^{\frac{1}{2}+\frac{1}{2}-1} \sqrt{h} \ dh,$$

[By Liouville's Theorem]

$$= 4 \cdot \frac{a^3}{4} \cdot \frac{\pi}{1} \cdot \frac{2}{3} = 4 \cdot \frac{\pi a^3}{6} = \frac{2\pi a^3}{3}.$$

Problem 23: *Evaluate the integral* $\iiint_R \sqrt{(1 - x^2 - y^2 - z^2)} \ dx \, dy \, dz$ *where R is the region interior to the sphere* $x^2 + y^2 + z^2 = 1$.

Solution: Here the region is the region interior to the sphere $x^2 + y^2 + z^2 = 1$.

Put $x^2 = u, \quad y^2 = v, \quad z^2 = w$

$x = \sqrt{u}, \quad y = \sqrt{v}, \quad z = \sqrt{w}.$

$\therefore$ $dx = \frac{1}{2} u^{-1/2} du, \ dy = \frac{1}{2} v^{-1/2} dv, \quad dz = \frac{1}{2} w^{-1/2} dw.$

Then integral is $\iiint \frac{1}{8} u^{-1/2} \ v^{-1/2} w^{-1/2} \sqrt{1 - (u + v + w)} \ du dv \ dw$

$$= \frac{1}{8} \iiint u^{\frac{1}{2}-1} v^{\frac{1}{2}-1} w^{\frac{1}{2}-1} \{1 - 4\}^{\frac{1}{2}} \ du \ dv \ dw$$

$$= \frac{1}{8} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \int_0^h h^{\frac{3}{2}-1} (1 - h)^{\frac{3}{2}-1} \ dh$$

[By Liouville's theorem]

$$= \frac{1}{8} 2\pi \, B\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{\pi}{4} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{3}{2}\right)} = \frac{\pi}{4} \cdot \frac{\frac{1}{2}\sqrt{\pi} \, \frac{1}{2}\sqrt{\pi}}{2} = \frac{\pi^2}{32}.$$

$\therefore$ The required integral is $= 8 \cdot \frac{\pi^2}{32} = \frac{\pi^2}{4}$.

Problem 24: *Find the mass of the region bounded by the ellipsoid*

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

if the density varies as the square of the distance from its centre.

Solution: Region bounded by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

and density varies as the square of the distance from its centre

i.e., density $\propto (x^2 + y^2 + z^2)$ or density $= K(x^2 + y^2 + z^2)$.

We have mass = volume × density $= \iiint dx\,dy\,dz \cdot K(x^2 + y^2 + z^2)$

$$= K\iiint (x^2 + y^2 + z^2)\,dx\,dy\,dz = I \text{ (Say)}$$

Put $\dfrac{x^2}{a^2} = u, \quad \dfrac{y^2}{b^2} = v, \quad \dfrac{z^2}{c^2} = w$ or $x = a\sqrt{u}, \quad y = b\sqrt{v}, \quad z = c\sqrt{w}$

$\therefore$ $dx = \dfrac{a}{2} u^{-1/2}\,du,\ dy = \dfrac{b}{2} v^{-1/2}\,dv,\ dw = \dfrac{c}{2} w^{-1/2}\,dw$

$\therefore$ $I = K \iiint (a^2u + b^2v + c^2w)\,\dfrac{a}{2}u^{-1/2}\,du\,\dfrac{b}{2}v^{-1/2}dv\,\dfrac{c}{2}w^{-1/2}dw$

$$= \frac{K\,abc}{8} \iiint \{a^2u^{1/2}\,v^{-1/2}w^{-1/2} + b^2u^{-1/2}v^{1/2}w^{-1/2}$$

$$+ c^2u^{-1/2}\,v^{-1/2}w^{1/2}\}\,du\,dv\,dw$$

$$= \frac{Kabc}{8}\left[\frac{a^2\,\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{2}+\frac{1}{2}+1\right)} + b^2\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{3}{2}+\frac{1}{2}+1\right)} + c^2\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{3}{2}+1\right)}\right]$$

by Dirichlet's theorem

$$= \frac{Kabc}{8}\,\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}\right)}(a^2 + b^2 + c^2)$$

$$= \frac{Kabc}{8}\cdot\frac{\frac{1}{2}\sqrt{\pi}\sqrt{\pi}\sqrt{\pi}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}\cdot(a^2 + b^2 + c^2) = \frac{Kabc\ \pi(a^2 + b^2 + c^2)}{30}.$$

The required mass of given region is $8I = 8\cdot\dfrac{K\,abc\ \pi\,(a^2 + b^2 + c^2)}{30}$.

Problem 25: *Prove that* $\displaystyle\iiint \frac{dx\,dy\,dz}{(x + y + z + 1)^3} = \frac{1}{2}\left[\log 2 - \frac{5}{8}\right]$

throughout the volume bounded by the coordinate planes and the plane $x + y + z = 1$.

(Rohilkhand 2013)

Solution: The region is bounded by the co-ordinate planes and the plane $x + y + z = 1$.

The integral $I = \iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$

$$= \iiint \frac{x^{1-1} y^{1-1} z^{1-1}\,dx\,dy\,dz}{\{(x+y+z)+1\}^3}$$

$$= \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(3)} \int_0^1 u^{1+1+1-1} \frac{1}{(u+1)^3}\,du \text{, by Liouville's theorem}$$

$$= \frac{1}{2}\int_0^1 \frac{u^2}{(u+1)^3}\,du.$$

Put $u + 1 = t$, $du = dt$, limit 1 to 2.

$$\therefore \quad I = \frac{1}{2}\int_1^2 \frac{(t-1)^2}{t^3}\,dt = \frac{1}{2}\int_1^2 \left(\frac{1}{t} + \frac{1}{t^3} - \frac{2}{t^2}\right) dt = \frac{1}{2}\left[\log t - \frac{1}{2t^2} + \frac{2}{t}\right]_1^2$$

$$= \frac{1}{2}\left[\log 2 - \frac{1}{8} + 1 + \frac{1}{2} - 2\right] = \frac{1}{2}\left(\log 2 - \frac{5}{8}\right).$$

Problem 26: *Evaluate the integral* $\iiint_R (ax^2 + by^2 + cz^2)\,dx\,dy\,dz$ *where R is the region given by* $x^2 + y^2 + z^2 \le d^2$.

Solution: Proceed as in Problem 24. Ans: $\frac{4}{15} d^5 \pi (a + b + c)$.

Problem 27: *Evaluate the following integrals :*

(i) $\int_0^a \int_0^{\sqrt{(a^2 - y^2)}} (a^2 - x^2 - y^2)\,dx\,dy$ (ii) $\int_0^2 \int_0^{\sqrt{(4 - x^2)}} (x^2 + y^2)\,dx\,dy$.

Solution: **(i)** The given integral

$$I = \int_{y=0}^{a} \int_{x=0}^{\sqrt{(a^2 - y^2)}} [(a^2 - y^2) - x^2]\,dy\,dx$$

$$= \int_0^a \left[(a^2 - y^2)x - \frac{1}{3}x^3\right]_{x=0}^{\sqrt{(a^2-y^2)}} dy,$$

integrating w.r.t. x treating y as constant

$$= \int_0^a [(a^2 - y^2)^{3/2} - \frac{1}{3}(a^2 - y^2)^{3/2}]\,dy = \frac{2}{3}\int_0^a (a^2 - y^2)^{3/2}\,dy$$

$$= \frac{2}{3}\int_0^{\pi/2} a^3 \cos^3\theta.\, a\cos\theta\,d\theta,$$

putting $y = a\sin\theta$ so that $dy = a\cos\theta\,d\theta$

$$= \frac{2}{3}a^4 \int_0^{\pi/2} \cos^4\theta\,d\theta = \frac{2}{3}a^4 \cdot \frac{3.1}{4.2}\cdot\frac{\pi}{2} \quad \text{[By Walli's formula]}$$

$$= \pi a^4/8.$$

(ii) Proceed as in part (i). Ans: 2π.

Hints to Objective Type Questions

Multiple Choice Questions

1. See article 3.
2. See article 2.
3. See article 1.

Fill in the Blank(s)

1. See article 1.
2. See article 3.

True or False

1. The given triple integral $= \iiint x^{1-1} \, y^{1-1} \, z^{1-1} \dfrac{dx\, dy\, dz}{(x+y+z+1)^3}$,

   where x, y, z are all positive such that $0 \le x+y+z \le 1$

$$= \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1)} \int_0^1 \frac{1}{(u+1)^3} \cdot u^{1+1+1-1}\, du, \text{ by Liouville's theorem}$$

$$= \frac{1}{\Gamma(3)} \int_0^1 \frac{u^2}{(u+1)^3}\, du = \frac{1}{2\,!} \int_0^1 \frac{u^2\, du}{(u+1)^3} = \frac{1}{2} \int_0^1 \frac{u^2\, du}{(u+1)^3}.$$

2. The given triple integral $= \iiint x^{1-1} \, y^{1-1} \, z^{1-1} (x+y+z+1)^2 \, dx\, dy\, dz$,

   where x, y, z are all positive such that $0 \le x+y+z \le 1$

$$= \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1)} \int_0^1 (u+1)^2 \; u^{1+1+1-1}\, du, \text{ by Liouville's theorem}$$

$$= \frac{1}{\Gamma(3)} \int_0^1 u^2 (u+1)^2\, du = \frac{1}{2} \int_0^1 u^2 (u+1)^2\, du.$$

3. Under the given conditions, by Dirichlet's integral, we have

$$\iint x^{l-1} \, y^{m-1} dx\, dy = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}.$$

OOO

Chapter-6

Double and Triple Integrals

(Multiple Integrals, Change of Order of Integration)

Comprehensive Problems 1

Problem 1(i): *Evaluate* $\int_0^2 \int_0^{\sqrt{(4+x^2)}} \frac{dx\,dy}{4+x^2+y^2}$.

(Rohilkhand 2005)

Solution: The given integral

$$I = \int_{x=0}^{2} \int_{y=0}^{\sqrt{(4+x^2)}} \frac{dx\,dy}{(4+x^2)+y^2}$$

$$= \int_0^2 \frac{1}{\sqrt{(4+x^2)}} \left[\tan^{-1} \frac{y}{\sqrt{(4+x^2)}}\right]_{y=0}^{\sqrt{(4+x^2)}} dx,$$

integrating w.r.t. y treating x as constant

$$= \int_0^2 \frac{1}{\sqrt{(4+x^2)}} [\tan^{-1} 1 - \tan^{-1} 0]\, dx = \frac{\pi}{4} \int_0^2 \frac{dx}{\sqrt{(4+x^2)}}$$

$$= \frac{\pi}{4} [\log \{x + \sqrt{(4+x^2)}\}]_0^2 = \frac{\pi}{4} [\log (2 + 2\sqrt{2}) - \log 2]$$

$$= \frac{\pi}{4} \log \frac{2+2\sqrt{2}}{2} = \frac{\pi}{4} \log (1+\sqrt{2}).$$

Problem 1(ii): *Evaluate double integral* $\int_1^a \int_1^b \frac{dx\,dy}{xy}$.

Solution: We have $\int_1^a \int_1^b \frac{dx\,dy}{xy} = \int_1^a \frac{1}{x} [\log y]_{y=1}^{b}\, dx,$

(integrating w.r.t. y treating x as constant)

$$= \int_1^a \frac{(\log b - \log 1)}{x}\, dx,$$

$$= \log b \int_1^a \frac{1}{x}\, dx = (\log b) \Big[\log x\Big]_1^a = (\log b)(\log a - \log 1)$$

$$= (\log b) . (\log a).$$

Problem 1(iii): *Evaluate double integral* $\int_0^{\pi/2} \int_{\pi/2}^{\pi} \cos(x+y)\, dy\, dx$.

(Kanpur 2007, 11)

Solution: We have $\int_0^{\pi/2} \int_{\pi/2}^{\pi} \cos(x+y)\, dy\, dx = \int_0^{\pi/2} \left[\int_{\pi/2}^{\pi} \cos(x+y)\, dx \right] dy$

$$= \int_0^{\pi/2} [\sin(x+y)]_{x=\pi/2}^{\pi}\, dy,$$

[Integrating w.r.t. x treating y as constant]

$$= \int_0^{\pi/2} [\sin(\pi+y) - \sin(\tfrac{1}{2}\pi + y)]\, dy$$

$$= \int_0^{\pi/2} (-\sin y - \cos y)\, dy$$

$$= [\cos y - \sin y]_0^{\pi/2} = (0-1) - (1-0) = -2.$$

Problem 1(iv): *Evaluate double integral* $\int_0^1 \int_0^{x^2} e^{y/x}\, dx\, dy.$

Solution: We have $\int_0^1 \int_0^{x^2} e^{y/x}\, dx\, dy = \int_0^1 [xe^{y}/x]_{y=0}^{x^2}\, dx,$

[Integrating w.r.t. y treating x as constant]

$$= \int_0^1 [xe^{x^2/x} - xe^{0/x}]\, dx = \int_0^1 (xe^x - x)\, dx$$

$$= \left[xe^x \right]_0^1 - \int_0^1 e^x\, dx - \left[\frac{x^2}{2} \right]_0^1$$

$$= e - \left[e^x \right]_0^1 - \frac{1}{2} = e - (e-1) - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}.$$

Problem 1(v): *Evaluate double integral* $\int_1^2 \int_0^{3y} y\, dy\, dx.$

Solution: We have $\int_1^2 \int_0^{3y} y\, dy\, dx = \int_1^2 y\, [x]_0^{3y}\, dy,$

[Integrating w.r.t. x regarding y as a constant]

$$= \int_1^2 y\, [3y - 0]\, dy = 3 \int_1^2 y^2\, dy$$

$$= 3 \left[\frac{y^3}{3} \right]_1^2 = \left[y^3 \right]_1^2 = 8 - 1 = 7.$$

Problem 1(vi): *Evaluate* $\int_0^2 \int_0^{\sqrt{(2x - x^2)}} x\, dx\, dy.$ **(Lucknow 2006; Kanpur 08)**

Solution: Here the variable limits are those of y and so the first integration must be performed w.r.t. y regarding x as constant.

$\therefore \quad \int_0^2 \int_0^{\sqrt{(2x-x^2)}} x\,dx\,dy = \int_0^2 x\,[y]_0^{\sqrt{(2x-x^2)}}\,dx$

$= \int_0^2 x\sqrt{(2x-x^2)}\,dx = \int_0^2 x\sqrt{\{1-(1-x)^2\}}\,dx.$ **(Note)**

Now put $(1-x) = t$ so that $-dx = dt$.

Also when $x = 0$, $t = 1$ and when $x = 2$, $t = -1$.

$\therefore$ the required integral $= \int_{-1}^{1} (1-t)\sqrt{(1-t^2)}\,dt$

$= \int_{-1}^{1} \sqrt{(1-t^2)}\,dt - \int_{-1}^{1} t\sqrt{(1-t^2)}\,dt$

$= 2\int_0^1 \sqrt{(1-t^2)}\,dt - 0,$ the second integral vanishes because the integrand is an odd function of t

$= 2\left[\frac{t}{2}\sqrt{(1-t^2)} + \frac{1}{2}\sin^{-1} t\right]_0^1 = 2\left[0 + \frac{1}{2}\cdot\frac{1}{2}\pi\right] = \frac{\pi}{2}.$

Problem 2(i): *Evaluate* $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{\{(1-x^2)(1-y^2)\}}}.$

Solution: We have

$$\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{\{(1-x^2)(1-y^2)\}}} = \int_{y=0}^{1} \frac{1}{\sqrt{(1-y^2)}}\left[\int_{x=0}^{1} \frac{1}{\sqrt{(1-x^2)}}\,dx\right] dy$$

$$= \int_0^1 \frac{1}{\sqrt{(1-y^2)}}\left[\sin^{-1} x\right]_0^1 dy,$$

[Integrating w.r.t. x treating y as constant]

$$= \int_0^1 \frac{\pi}{2\sqrt{(1-y^2)}}\,dy = \frac{\pi}{2}\left[\sin^{-1} y\right]_0^1 = \frac{\pi}{2}\cdot\frac{\pi}{2}\cdot = \frac{\pi^2}{4}.$$

Problem 2(ii): *Evaluate* $\int_0^1 \int_0^{\sqrt{(1-y^2)}} 4y\,dy\,dx.$

(Lucknow 2008)

Solution: The given integral

$$I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{(1-y^2)}} 4y\,dy\,dx$$

$= \int_0^1 4y\,[x]_{x=0}^{\sqrt{(1-y^2)}}\,dy,$ integrating w.r.t. x treating y as constant

$$= 4\int_0^1 y\sqrt{(1-y^2)}\,dy = 4\int_0^1 \left(-\frac{1}{2}\right).(1-y^2)^{1/2}(-2y)\,dy$$

$= -2\cdot\frac{2}{3}\left[(1-y^2)^{3/2}\right]_0^1,$ by power formula

$$= -\frac{4}{3}[0-1] = \frac{4}{3}.$$

Problem 2(iii): *Evaluate* $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dx\, dy.$

Solution: The given integral $I = \int_{x=0}^1 \int_{y=x}^{\sqrt{x}} (x^2 + y^2)\, dx\, dy$

$$= \int_0^1 \left[x^2 y + \frac{1}{3} y^3 \right]_{y=x}^{\sqrt{x}} dx, \text{ integrating w.r.t. } y \text{ treating } x \text{ as constant}$$

$$= \int_0^1 [x^2 \sqrt{x} + \frac{1}{3} x \sqrt{x} - x^3 - \frac{1}{3} x^3]\, dx$$

$$= \int_0^1 \left[x^{5/2} + \frac{1}{3} x^{3/2} - \frac{4}{3} x^3 \right] dx = \left[\frac{2}{7} x^{7/2} + \frac{1}{3} \cdot \frac{2}{5} x^{5/2} - \frac{1}{3} x^4 \right]_0^1$$

$$= \frac{2}{7} + \frac{2}{15} - \frac{1}{3} = \frac{30 + 14 - 35}{105} = \frac{3}{35}.$$

Problem 2(iv): *Evaluate* $\int_2^3 \int_0^{y-1} \frac{dy\, dx}{y}.$

Solution: The given integral $I = \int_{y=2}^3 \int_{x=0}^{y-1} \frac{dy\, dx}{y}$

$$= \int_2^3 \frac{1}{y} [x]_{x=0}^{y-1} dy, \text{ integrating w.r.t. } x \text{ treating } y \text{ as constant}$$

$$= \int_2^3 \frac{y-1}{y} dy = \int_2^3 \left(1 - \frac{1}{y}\right) dy = \left[y - \log y \right]_2^3$$

$$= 3 - \log 3 - 2 + \log 2 = 1 - \log \frac{3}{2}.$$

Problem 2(v): *Evaluate* $\int_0^a \int_0^{\sqrt{(a^2 - y^2)}} (a^2 - x^2 - y^2)\, dy\, dx.$

Solution: The given integral $I = \int_{y=0}^a \int_{x=0}^{\sqrt{(a^2 - y^2)}} [(a^2 - y^2) - x^2]\, dy\, dx$

$$= \int_0^a \left[(a^2 - y^2) x - \frac{1}{3} x^3 \right]_{x=0}^{\sqrt{(a^2 - y^2)}} dy,$$

integrating w.r.t. x treating y as constant

$$= \int_0^a [(a^2 - y^2)^{3/2} - \frac{1}{3}(a^2 - y^2)^{3/2}]\, dy = \frac{2}{3} \int_0^a (a^2 - y^2)^{3/2}\, dy$$

$$= \frac{2}{3} \int_0^{\pi/2} a^3 \cos^3 \theta . \, a \cos\theta\, d\theta,$$

putting $y = a \sin\theta$ so that $dy = a \cos\theta\, d\theta$

$$= \frac{2}{3} a^4 \int_0^{\pi/2} \cos^4 \theta\, d\theta = \frac{2}{3} a^4 \cdot \frac{3.1}{4.2} \cdot \frac{\pi}{2}$$ [By Walli's formula]

$$= \pi a^4 / 8.$$

Problem 2(vi): *Evaluate* $\int_0^a \int_0^{\sqrt{(a^2-x^2)}} (x+y)\,dx\,dy$.

Solution: The given integral $I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{(a^2-x^2)}} (x+y)\,dx\,dy$

$$= \int_0^a \left[xy + \frac{1}{2}y^2\right]_{y=0}^{\sqrt{(a^2-x^2)}} dx,$$

integrating w.r.t. y treating x as constant

$$= \int_0^a [x\sqrt{(a^2-x^2)} + \frac{1}{2}(a^2-x^2)]\,dx$$

$$= \int_0^a [-\frac{1}{2}(a^2-x^2)^{1/2}(-2x) + \frac{1}{2}(a^2-x^2)]\,dx$$

$$= \left[-\frac{1}{2}\cdot\frac{2}{3}(a^2-x^2)^{3/2}\right]_0^a + \frac{1}{2}\left[a^2 x - \frac{1}{3}x^3\right]_0^a, \text{ by power formula}$$

$$= 0 + \frac{1}{3}a^3 + \frac{1}{2}[a^3 - \frac{1}{3}a^3] = \frac{2}{3}a^3.$$

Problem 3(i): *Show that* $\int_1^2 \int_3^4 (xy + e^y)\,dx\,dy = \int_3^4 \int_1^2 (xy + e^y)\,dy\,dx$.

(Kumaun 2015)

Solution: Integral on the L.H.S.

$$= \int_1^2 \left[\int_3^4 (xy + e^y)\,dy\right] dx = \int_1^2 \left[\frac{xy^2}{2} + e^y\right]_3^4 dx$$

$$= \int_1^2 \left[8x + e^4 - \frac{9}{2}x - e^3\right] dx$$

$$= \int_1^2 \left[\frac{7}{2}x + e^4 - e^3\right] dx = \left[\frac{7x^2}{4} + (e^4 - e^3)\,x\right]_1^2$$

$$= 7 + 2(e^4 - e^3) - \frac{7}{4} - (e^4 - e^3) = \frac{21}{4} + e^4 - e^3.$$

And the integral on the R.H.S.

$$= \int_3^4 \left[\int_1^2 (xy + e^y)\,dx\right] dy = \int_3^4 \left[\frac{yx^2}{2} + xe^y\right]_1^2 dy$$

$$= \int_3^4 \left[2y + 2e^y - \frac{y}{2} - e^y\right] dy = \int_3^4 \left[\frac{3y}{2} + e^y\right] dy$$

$$= \left[\frac{3y^2}{4} + e^y\right]_3^4 = 12 + e^4 - \frac{27}{4} - e^3 = \frac{21}{4} + e^4 - e^3.$$

Hence the result.

Problem 3(ii): *Show that* $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\, dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\, dx$.

Find the values of the two integrals. **(Garhwal 2002)**

Solution: The integral on the L.H.S.

$$= \int_0^1 dx \int_0^1 \frac{2x-(x+y)}{(x+y)^3}\, dy = \int_0^1 dx \int_0^1 \left\{ \frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2} \right\} dy$$

$$= \int_0^1 \left[\frac{-x}{(x+y)^2} + \frac{1}{x+y} \right]_0^1 dx,$$

[Integrating w.r.t. y regarding x as constant]

$$= \int_0^1 \left[-\frac{x}{(1+x)^2} + \frac{1}{x} + \frac{1}{1+x} - \frac{1}{x} \right] dx = \int_0^1 \frac{dx}{(1+x)^2} = \left[\frac{-1}{1+x} \right]_0^1$$

$$= -\frac{1}{2} + 1 = \frac{1}{2}.$$

And the integral on the R.H.S.

$$= \int_0^1 dy \int_0^1 \frac{(x+y)-2y}{(x+y)^3}\, dx$$

$$= \int_0^1 dy \int_0^1 \left\{ \frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3} \right\} dx = \int_0^1 \left[\frac{-1}{x+y} + \frac{y}{(x+y)^2} \right]_0^1 dy$$

$$= \int_0^1 \left[\frac{-1}{1+y} + \frac{1}{y} + \frac{y}{(1+y)^2} - \frac{1}{y} \right] dy = -\int_0^1 \frac{dy}{(1+y)^2} = \left[\frac{1}{1+y} \right]_0^1$$

$$= \frac{1}{2} - 1 = -\frac{1}{2}.$$

Thus the two integrals are not equal.

Problem 4(i): *Evaluate the double integral* $\int_0^a \int_0^{\sqrt{(a^2-x^2)}} x^2 y\, dx\, dy$.

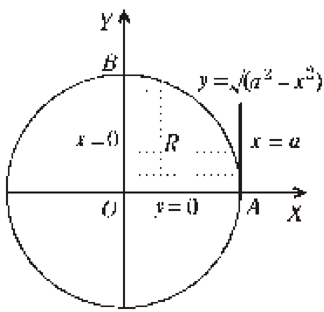
Mention the region of integration involved in this double integral.

Solution: The given integral

$$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{(a^2-x^2)}} x^2 y\, dx\, dy$$

$$= \int_0^a x^2 \left[\frac{y^2}{2} \right]_{y=0}^{\sqrt{(a^2-x^2)}} dx,$$

integrating w.r.t. y treating x as constant



$$= \frac{1}{2} \int_0^a x^2 (a^2 - x^2)\, dx$$

$$= \frac{1}{2}\int_0^a (a^2x^2 - x^4)\, dx = \frac{1}{2}\left[a^2\frac{x^3}{3} - \frac{x^5}{5}\right]_0^a = \frac{1}{2}\left[\frac{a^5}{3} - \frac{a^5}{5}\right] = \frac{1}{15}a^5 .$$

From the limits of integration it is obvious that the region of integration R is bounded by $y = 0$, $y = \sqrt{(a^2 - x^2)}$ and $x = 0$, $x = a$ *i.e.*, the region of integration is the area of the circle $x^2 + y^2 = a^2$ between the lines $x = 0$, $x = a$ and lying above the line $y = 0$ *i.e.*, the axis of x. Thus the region of integration is the area OAB of the circle $x^2 + y^2 = a^2$ lying in the positive quadrant.

Problem 4(ii): *Evaluate* $\iint x^2 y^3\, dx\, dy$ *over the circle* $x^2 + y^2 = a^2$.

(Rohilkhand 2013B)

Solution: If the first integration is to be performed w.r.t. y regarding x as constant, then the region of integration R can be expressed as

$$-a \le x \le a, -\sqrt{(a^2 - x^2)} \le y \le \sqrt{(a^2 - x^2)}.$$

$$\therefore \qquad \iint_R x^2 y^3\, dx\, dy = \int_{x=-a}^{a} \int_{y=-\sqrt{(a^2 - x^2)}}^{\sqrt{(a^2 - x^2)}} x^2 y^3\, dx\, dy = 0 .$$

[$\because y^3$ is an odd function of y] **(Note)**

Problem 5: *Evaluate* $\iint (x + y + a)\, dx\, dy$ *over the circular area* $x^2 + y^2 \le a^2$.

Solution: Here the region of integration R can be expressed as

$$-a \le y \le a, -\sqrt{(a^2 - y^2)} \le x \le \sqrt{(a^2 - y^2)},$$

where the first integration is to be performed w.r.t. x regarding y as constant.

$$\therefore \qquad \iint_R (x + y + a)\, dx\, dy = \int_{y=-a}^{a} \int_{x=-\sqrt{(a^2 - y^2)}}^{\sqrt{(a^2 - y^2)}} (x + y + a)\, dx\, dy$$

$$= \int_{-a}^{a} \left[\frac{x^2}{2} + (y + a)\, x\right]_{-\sqrt{(a^2 - y^2)}}^{\sqrt{(a^2 - y^2)}} dy ,$$

[Integrating w.r.t. x treating y as a constant]

$$= \int_{-a}^{a} \left[\frac{a^2 - y^2}{2} + (y + a)\sqrt{(a^2 - y^2)} - \frac{a^2 - y^2}{2} + (y + a)\sqrt{(a^2 - y^2)}\right] dy$$

$$= \int_{-a}^{a} 2\,(y + a)\sqrt{(a^2 - y^2)}\, dy$$

$$= \int_{-a}^{a} 2y \,.\sqrt{(a^2 - y^2)}\, dy + 2a \int_{-a}^{a} \sqrt{(a^2 - y^2)}\, dy$$

$$= 0 + 2a.2\int_0^a \sqrt{(a^2 - y^2)}\, dy ,$$ the first integral vanishes because the integrand is an odd function of y

$$= 4a\left[\frac{y\sqrt{(a^2-y^2)}}{2}+\frac{a^2}{2}\sin^{-1}\frac{y}{a}\right]_0^a = 4a\left[0+\frac{1}{2}a^2\sin^{-1}1-0\right]$$

$$= 4a.\frac{1}{2}a^2.\frac{1}{2}\pi = \pi a^3.$$

Problem 6: *Evaluate* $\iint x^2\, y^2\, dx\, dy$ *over the region bounded by* $x=0$, $y=0$ *and* $x^2+y^2=1$.

(Avadh 2012)

Solution: The given region for integration is the area of the positive quadrant of the circle $x^2+y^2=1$ in the xy-plane. This region R can be expressed either as

$$0\le x\le\sqrt{(1-y^2)},\, 0\le y\le 1$$

or as $0\le y\le\sqrt{(1-x^2)},\, 0\le x\le 1$.

$\therefore$ $\iint_R x^2\, y^2\, dx\, dy = \int_{y=0}^{1}\int_{x=0}^{\sqrt{(1-y^2)}} x^2 y^2\, dx\, dy$, the first integration to be performed w.r.t. x regarding y as constant

$$= \int_0^1 y^2\left[\frac{x^3}{3}\right]_0^{\sqrt{(1-y^2)}} dy = \int_0^1 \frac{1}{3}y^2(1-y^2)^{3/2}\, dy.$$

Put $y=\sin\theta$ so that $dy=\cos\theta\, d\theta$.

When $y=0,\theta=0$ and when $y=1,\theta=\pi/2$.

$\therefore$ $$\iint_R x^2y^2\, dx\, dy = \int_0^{\pi/2}\frac{1}{3}\sin^2\theta\,(1-\sin^2\theta)^{3/2}.\cos\theta\, d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2}\sin^2\theta\cos^4\theta\, d\theta = \frac{1}{3}\cdot\frac{1.3.1}{6.4.2}\cdot\frac{\pi}{2} = \frac{\pi}{96}.$$

Problem 7: *Evaluate* $\iint xy\, dx\, dy$ *over the region in the positive quadrant for which* $x+y\le 1$.

Solution: The region of integration is the area bounded by the lines $x=0$, $y=0$ and $x+y=1$.

To cover this region of integration R, x varies from 0 to 1 and y varies from 0 to $1-x$.

$\therefore$ $$\iint_R xy\, dx\, dy = \int_{x=0}^{1}\int_{y=0}^{1-x} xy\, dx\, dy = \int_0^1 x\left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \frac{1}{2}\int_0^1 x(1-x)^2\, dx = \frac{1}{2}\int_0^1 x\,(1-2x+x^2)\, dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}-2\cdot\frac{x^3}{3}+\frac{x^4}{4}\right]_0^1 = \frac{1}{2}\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right] = \frac{1}{24}.$$

Problem 8: *Evaluate* $\iint e^{2x+3y}\, dx\, dy$ *over the triangle bounded by* $x=0$, $y=0$ *and* $x+y=1$.

Solution: The given region of integration R can be expressed as

$$0 \le x \le 1, 0 \le y \le 1 - x,$$

where the first integration is to be performed w.r.t. y regarding x as a constant.

$$\therefore \quad \iint_R e^{2x+3y} \, dx \, dy = \int_0^1 \int_0^{1-x} e^{2x+3y} \, dx \, dy$$

$$= \int_0^1 \left[\frac{e^{2x+3y}}{3} \right]_0^{1-x} dx = \frac{1}{3} \int_0^1 [e^{3-x} - e^{2x}] \, dx$$

$$= \frac{1}{3} \left[- e^{3-x} - \frac{e^{2x}}{2} \right]_0^1 = - \frac{1}{3} [(e^2 - e^3) + \frac{1}{2} (e^2 - e^0)]$$

$$= - \frac{1}{3} [- e^2 (e-1) + \frac{1}{2} (e+1)(e-1)] = \frac{1}{3} (e-1) [e^2 - \frac{1}{2} (e+1)]$$

$$= \frac{1}{6} (e-1)(2e^2 - e - 1) = \frac{1}{6} (e-1) \{(e-1)(2e+1)\}$$

$$= \frac{1}{6} (e-1)^2 (2e+1).$$

Problem 9: *Evaluate* $\iint \frac{xy}{\sqrt{(1-y^2)}} \, dx \, dy$ *over the positive quadrant of the circle* $x^2 + y^2 = 1$.

Solution: Here the region of integration R is the area of the circle $x^2 + y^2 = 1$ lying in the positive quadrant. This region of integration R can be expressed as

$$0 \le x \le \sqrt{(1-y^2)}, 0 \le y \le 1.$$

$$\therefore \quad \iint_R \frac{xy}{\sqrt{(1-y^2)}} \, dx \, dy = \int_{y=0}^1 \int_{x=0}^{\sqrt{(1-y^2)}} \frac{xy}{\sqrt{(1-y^2)}} \, dx \, dy$$

$$= \int_0^1 \frac{y}{\sqrt{(1-y^2)}} \left[\frac{x^2}{2} \right]_{x=0}^{\sqrt{(1-y^2)}} dy,$$

integrating w.r.t. x treating y as constant

$$= \frac{1}{2} \int_0^1 y \sqrt{(1-y^2)} \, dy = \frac{1}{2} \int_0^1 - \frac{1}{2} \cdot (1-y^2)^{1/2} (-2y) \, dy$$

$$= - \frac{1}{4} \cdot \frac{2}{3} \left[(1-y^2)^{3/2} \right]_0^1, \text{ by power formula}$$

$$= \frac{1}{6}.$$

Problem 10: *Find the area of the ellipse* $x^2/a^2 + y^2/b^2 = 1$, *by double integration.*

Solution: From the equation of the ellipse, we have $\frac{y}{b} = \pm \sqrt{\left\{ 1 - \frac{x^2}{a^2} \right\}}$.

So the region of integration R to cover the area of the ellipse can be considered as bounded by

$$y = -b\sqrt{(1 - x^2/a^2)},\ y = b\sqrt{(1 - x^2/a^2)},\ x = -a \text{ and } x = a.$$

Therefore the required area of the ellipse

$$= \iint_R dx\,dy = \int_{x=-a}^{a} \int_{y=-b\sqrt{(1-x^2/a^2)}}^{b\sqrt{(1-x^2/a^2)}} 1.\,dx\,dy$$

$$= \int_{-a}^{a} \left[2\int_0^{b\sqrt{(1-x^2/a^2)}} 1 \cdot dy \right] dx = 2\int_{-a}^{a} [y]_0^{b\sqrt{(1-x^2/a^2)}}\,dx$$

$$= 2\int_{-a}^{a} b\sqrt{\left(1 - \frac{x^2}{a^2}\right)}\,dx = 2.2\int_0^{a} b\sqrt{\left(1 - \frac{x^2}{a^2}\right)}\,dx$$

$$= \frac{4b}{a}\int_0^{a} \sqrt{(a^2 - x^2)}\,dx = \frac{4b}{a}\left[\frac{x\sqrt{(a^2 - x^2)}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left[0 + \frac{a^2}{2}\{\sin^{-1} 1 - \sin^{-1} 0\}\right] = \frac{4b}{a}\cdot\frac{a^2}{2}\cdot\frac{\pi}{b} = \pi ab.$$

Problem 11: *Compute the value of* $\iint_R y\,dx\,dy$, *where R is the region in the first quadrant bounded by the ellipse* $x^2/a^2 + y^2/b^2 = 1$.

Solution : If the first integration is to be performed w.r.t. y regarding x as a constant, then the given region of integration can be expressed as

$$0 \le x \le a,\ 0 \le y \le b\sqrt{(1 - x^2/a^2)}.$$

$$\therefore \quad \iint_R y\,dx\,dy = \int_{x=0}^{a} \int_{y=0}^{b\sqrt{(1-x^2/a^2)}} y\,dx\,dy$$

$$= \int_0^a \left[\frac{y^2}{2}\right]_0^{b\sqrt{(1-x^2/a^2)}} dx = \frac{1}{2}\int_0^a b^2\left(1 - \frac{x^2}{a^2}\right)dx$$

$$= \frac{b^2}{2a^2}\int_0^a (a^2 - x^2)\,dx = \frac{b^2}{2a^2}\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{b^2}{2a^2}\cdot\frac{2a^3}{3} = \frac{ab^2}{3}.$$

Problem 12: *Find the mass of a plate in the form of a quadrant of an ellipse* $x^2/a^2 + y^2/b^2 = 1$ *whose density per unit area is given by* $\rho = kxy$.

Solution: Proceed as in problem 11. Ans. $ka^2b^2/8$.

Problem 13: *Prove by the method of double integration that the area lying between the parabolas* $y^2 = 4ax$ *and* $x^2 = 4ay$ *is* $\frac{16}{3}a^2$.

Solution: Draw the two parabolas in the same figure. The two parabolas intersect at the points whose abscissae are given by $(x^2/4a)^2 = 4ax$ *i.e.*, $x(x^3 - 64a^3) = 0$ *i.e.*, $x = 0$ and $x^3 = 64a^3$. Thus the two parabolas intersect at the points where $x = 0$ and $x = 4a$.

Now the area of a small element situated at any point $(x, y) = dx\,dy$.

$\therefore$ The required area

$$= \int_{x=0}^{4a} \int_{y=x^2/4a}^{\sqrt{(4ax)}} dx\,dy = \int_0^{4a} [y]_{x^2/4a}^{\sqrt{(4ax)}}\, dx$$

$$= \int_0^{4a} \left[2\sqrt{a}.\, x^{1/2} - \frac{1}{4a} x^2\right] dx = \left[2\sqrt{a}\,.\, x^{3/2}\,.\,\frac{2}{3} - \frac{1}{4a}\cdot\frac{x^3}{3}\right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}.\,(4a)^{3/2} - \frac{1}{12a}.\,64a^3 = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2\,.$$

Problem 14: *Find by double integration the area lying between the parabola $y = 4x - x^2$ and the line $y = x$.*

Solution: Solving $y = 4x - x^2$ and $y = x$ for x, we have

$$4x - x^2 = x \quad \text{or} \quad x^2 - 3x = 0 \quad \text{or} \quad x(x-3) = 0 \text{ } i.e,\ x = 0 \text{ or } 3.$$

Thus the curves $y = 4x - x^2$ and $y = x$ intersect at the points where $x = 0$ and $x = 3$. When $0 < x < 3$, we have $4x - x^2 > x$.

So the required area can be considered as lying between the curves $y = x$, $y = 4x - x^2$, $x = 0$ and $x = 3$.

$\therefore$ The required area $= \displaystyle\int_{x=0}^{3} \int_{y=x}^{4x-x^2} dx\,dy$

$$= \int_0^3 [y]_x^{4x-x^2}\, dx = \int_0^3 (4x - x^2 - x)\, dx = \int_0^3 (3x - x^2)\, dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - \frac{27}{3} = 27\cdot\frac{1}{6} = \frac{9}{2}\cdot$$

Problem 15: *Evaluate $\iint y\,dx\,dy$ over the area between the parabolas $y^2 = 4x$ and $x^2 = 4y$.*

Solution: The two parabolas intersect at the points whose abscissae are given by $(\frac{1}{4}x^2)^2 = 4x$ or $x(x^3 - 64) = 0$ *i.e.*, $x = 0$ or 4. When $0 < x < 4$, we have $2\sqrt{x} > \frac{1}{4}x^2$.

Therefore the given region of integration can be expressed as

$$0 \le x \le 4,\ \frac{1}{4}x^2 \le y \le 2\sqrt{x}\,.$$

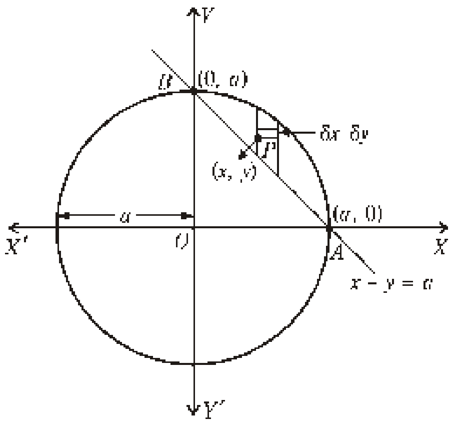
$\therefore$ The required integral

$$= \int_{x=0}^{4} \int_{y=x^2/4}^{2\sqrt{x}} y\,dx\,dy = \int_0^4 \left[\frac{y^2}{2}\right]_{x^2/4}^{2\sqrt{x}} dx$$

$$= \int_0^4 \left[2x - \frac{x^4}{32}\right] dx = \left[\frac{2x^2}{2} - \frac{x^5}{32\times 5}\right]_0^4 = 16 - \frac{32}{5} = \frac{48}{5}\cdot$$

Problem 16: *Find by double integration the area of the region enclosed by the curves* $x^2 + y^2 = a^2$, $x + y = a$ *(in the first quadrant).*

Solution: The given equations of the circle $x^2 + y^2 = a^2$ [centre $(0, 0)$ and radius a] and of the straight line $x + y = a$ (with equal intercepts a on both the axes) can be easily traced as shown in the figure.



The required area is the area bounded by the arc AB and the line AB. To find it with the help of double integration take any point $P(x, y)$ in this portion and consider an elementary area $\delta x\, \delta y$ at P. The required area can now be covered by first moving y from the straight line $x + y = a$ to the arc of the circle $x^2 + y^2 = a^2$ and then moving x from 0 to a.

$\therefore$ The required area $= \int_{x=0}^{a} \int_{y=(a-x)}^{\sqrt{(a^2 - x^2)}} dx\, dy$, the first integration to be performed w.r.t. y whose limits are variable

$$= \int_0^a [y]_{(a-x)}^{\sqrt{(a^2-x^2)}} dx = \int_0^a [\sqrt{(a^2 - x^2)} - (a - x)]\, dx$$

$$= \left[\left\{\frac{1}{2} x \sqrt{(a^2 - x^2)} + \frac{1}{2} a^2 \sin^{-1}(x/a)\right\} - ax + \frac{1}{2} x^2\right]_0^a$$

$$= \frac{1}{2} a^2 . (\frac{1}{2}\pi) - a^2 + \frac{1}{2} a^2 = \frac{1}{2} a^2 (\frac{1}{2}\pi - 1) = \frac{1}{4} a^2 (\pi - 2).$$

Note: The required area can also be covered by first moving x from the st. line $x + y = a$ to the arc of the circle $x^2 + y^2 = a^2$ and then moving y from 0 to a.

Comprehensive Problems 2

Problem 1(i): *Evaluate* $\int_0^{\pi} \int_0^{a \sin \theta} r\, d\theta\, dr$.

(Kashi 2013)

Solution: Here the limits of r are variable and those of θ are constant. Therefore first integration shall be performed w.r.t. r regarding θ as a constant. We have

$$\int_0^{\pi} \int_0^{a \sin \theta} r\, d\theta\, dr = \int_0^{\pi} \left[\frac{r^2}{2}\right]_0^{a \sin \theta} d\theta = \frac{1}{2} \int_0^{\pi} a^2 \sin^2 \theta\, d\theta$$

$$= \frac{a^2}{2} . 2 \int_0^{\pi/2} \sin^2 \theta\, d\theta = \frac{a^2}{2} . 2 . \frac{1}{2} . \frac{\pi}{2} = \frac{\pi a^2}{4}.$$

Problem 1(ii): *Evaluate* $\int_0^{\pi/2}\int_0^{a\cos\theta} r\sin\theta\, d\theta\, dr$.

Solution: We have $\int_0^{\pi/2}\int_0^{a\cos\theta} r\sin\theta\, d\theta\, dr = \int_0^{\pi/2}\sin\theta\left[\frac{r^2}{2}\right]_0^{a\cos\theta} d\theta,$

integrating first w.r.t. r regarding θ as a constant

$$= \frac{1}{2}\int_0^{\pi/2}\sin\theta\cdot a^2\cos^2\theta\, d\theta = \frac{a^2}{2}\int_0^{\pi/2}\sin\theta\cos^2\theta\, d\theta = \frac{1}{2}a^2\cdot\frac{1\cdot 1}{3\cdot 1} = \frac{1}{6}a^2.$$

Problem 1(iii): *Evaluate* $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3\sin\theta\cos\theta\, d\theta\, dr.$ **(Agra 2003; Kumaun 09)**

Solution: We have

$$I = \int_0^{\pi}\int_0^{a(1+\cos\theta)} r^3\sin\theta\cos\theta\, d\theta\, dr$$

$$= \int_0^{\pi}\sin\theta\cos\theta\left[\int_0^{a(1+\cos\theta)} r^3\, dr\right] d\theta$$

$$= \int_0^{\pi}\sin\theta\cos\theta\left[\frac{r^4}{4}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{4}\int_0^{\pi}\sin\theta\cos\theta\, a^4\,(1+\cos\theta)^4\, d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi}\sin\theta\cos\theta\,(1+4\cos\theta+6\cos^2\theta+4\cos^3\theta+\cos^4\theta)\, d\theta$$

$$= \frac{a^4}{4}\int_0^{\pi}(\sin\theta\cos\theta+4\sin\theta\cos^2\theta+6\sin\theta\cos^3\theta+4\sin\theta\cos^4\theta + \sin\theta\cos^5\theta)\, d\theta$$

Now $\int_0^{\pi}\sin^m\theta\cos^n\theta\, d\theta = 2\int_0^{\pi/2}\sin^m\theta\cos^n\theta\, d\theta$ or $=0,$

according as n is an even or an odd integer.

$\therefore$ $$I = \frac{a^4}{2}\int_0^{\pi/2}(4\sin\theta\cos^2\theta+4\sin\theta\cos^4\theta)\, d\theta$$

because the integrals containing odd powers of $\cos\theta$ vanish

$$= \frac{a^4}{2}\left[4\cdot\frac{1}{3.1}+4\cdot\frac{3.1}{5.3.1}\right] = \frac{16}{15}a^4.$$

Problem 2: *Evaluate* $\iint r^2\, d\theta\, dr$ *over the area of the circle* $r = a\cos\theta$. **(Kanpur 2010)**

Solution: The circle $r = a\cos\theta$ passes through the pole and the diameter through the pole is initial line. The region of integration can be covered by radial strips originating from $r=0$ and terminating at $r = a\cos\theta$. From the equation of the circle, we have $r=0$ when $\cos\theta = 0$ *i.e.*, $\theta = \pm\,\pi/2$. Therefore for the given area θ varies from $-\pi/2$ to $\pi/2$. Therefore the required integral

$$= \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{a\cos\theta} r^2 \, d\theta \, dr = \int_{-\pi/2}^{\pi/2} \left[\int_0^{a\cos\theta} r^2 \, dr \right] d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^3}{3} \right]_0^{a\cos\theta} d\theta = \int_{-\pi/2}^{\pi/2} \frac{a^3 \cos^3 \theta}{3} \, d\theta$$

$$= \frac{2a^3}{3} \int_0^{\pi/2} \cos^3 \theta \, d\theta = \frac{2a^3}{3} \cdot \frac{2}{3.1} = \frac{4a^3}{9} \cdot$$

Problem 3: *Integrate $r \sin \theta$ over the area of the cardioid $r = a(1 + \cos \theta)$ lying above the initial line.* **(Kanpur 2010)**

Solution: For the area of the cardioid $r = a(1 + \cos \theta)$ above the initial line θ varies from 0 to π. Also for the required area r varies from $r = 0$ to $r = a(1 + \cos \theta)$. If A denotes the region consisting of the area of the cardioid lying above the initial line, then the required integral

$$= \iint_A r \sin \theta \, dA = \int_0^{\pi} \int_0^{a(1+\cos\theta)} r \sin \theta \, r \, d\theta \, dr$$

$$= \int_0^{\pi} \sin \theta \left[\frac{r^3}{3} \right]_0^{a(1+\cos\theta)} d\theta = \frac{a^3}{3} \int_0^{\pi} \sin \theta (1 + \cos \theta)^3 \, d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi} 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \left(2 \cos^2 \frac{\theta}{2} \right)^3 d\theta$$

$$= \frac{16a^3}{3} \int_0^{\pi/2} \sin \phi \cos^7 \phi \, . \, 2 \, d\phi, \text{ putting } \frac{\theta}{2} = \phi \text{ so that } d\theta = 2 \, d\phi$$

$$= 32 \, . \frac{a^3}{3} \left[- \frac{\cos^8 \phi}{8} \right]_0^{\pi/2} = \frac{32a^3}{3} \left[0 + \frac{1}{8} \right] = \frac{4a^3}{3} \cdot$$

Problem 4: *Find the mass of a loop of the lemniscate $r^2 = a^2 \sin 2\theta$ if density $\rho = kr^2$.*

Solution: In the equation of the lemniscate $r^2 = a^2 \sin 2\theta$, putting $r = 0$, we get $\sin 2\theta = 0$ *i.e.*, $2\theta = 0, \pi$ *i.e.*, $\theta = 0, \frac{1}{2}\pi$. Therefore for one loop of the given lemniscate θ varies from 0 to $\pi/2$ and r varies from 0 to $a\sqrt{(\sin 2\theta)}$.

$\therefore$ Mass of a loop of the lemniscate

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\sqrt{(\sin 2\theta)}} \rho r \, d\theta \, dr = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\sqrt{(\sin 2\theta)}} k \, r^2 \, . r \, d\theta \, dr$$

$$= k \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\sqrt{(\sin \theta)}} r^3 \, d\theta \, dr = k . \int_0^{\pi/2} \left[\frac{r^4}{4} \right]_{r=0}^{a\sqrt{(\sin 2\theta)}} d\theta$$

$$= \frac{ka^4}{4} \int_0^{\pi/2} \sin^2 2\theta \, d\theta = \frac{ka^4}{8} \int_0^{\pi/2} (1 - \cos 4\theta) \, d\theta$$

$$= \frac{ka^4}{8} \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/2} = \frac{ka^4}{8} \cdot \frac{\pi}{2} = \frac{\pi ka^4}{16} \cdot$$

Problem 5 : *Find by double integration the area lying inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = a$.*

Solution: Eliminating r between the given equations of the cardioid $r = a(1 + \cos\theta)$ and the circle $r = a$, we have

$$a = a(1 + \cos\theta) \quad \text{or} \quad \cos\theta = 0 \text{ i.e., } \theta = \pm\,\pi/2.$$

Thus the region of integration A is enclosed by

$$r = a,\ r = a(1 + \cos\theta),\ \theta = -\,\pi/2,\ \theta = \pi/2.$$

$$\therefore \text{ The required area } = \iint_A r d\theta\, dr = \int_{-\pi/2}^{\pi/2} \int_{a}^{a(1+\cos\theta)} r\, d\theta\, dr$$

$$= \int_{-\pi/2}^{\pi/2} \left[\frac{r^2}{2}\right]_a^{a(1+\cos\theta)} d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} [a^2(1+\cos\theta)^2 - a^2]\, d\theta$$

$$= \frac{a^2}{2}\int_{-\pi/2}^{\pi/2} (1 + \cos^2\theta + 2\cos\theta - 1)\, d\theta$$

$$= \frac{a^2}{2} . 2\int_0^{\pi/2} [\cos^2\theta + 2\cos\theta]\, d\theta$$

$$= a^2\left[\frac{1}{2}\cdot\frac{1}{2}\,\pi + 2\,\{\sin\theta\}_0^{\pi/2}\right] = a^2\,[\frac{1}{4}\,\pi + 2] = \frac{a^2}{4}(\pi + 8).$$

Problem 6: *Find by double integration the area lying inside the cardioid $r = 1 + \cos\theta$ and outside the parabola $r(1 + \cos\theta) = 1$.*

Solution: Eliminating r between the given equations of the cardioid and the parabola, we have

$$(1 + \cos\theta) = 1/(1 + \cos\theta) \quad \text{or} \quad (1 + \cos\theta)^2 = 1$$

or $\quad \cos^2\theta + 2\cos\theta = 0 \quad$ or $\quad \cos\theta\,(2 + \cos\theta) = 0$

or $\quad \cos\theta = 0$, because $\cos\theta$ cannot be equal to -2

or $\quad \theta = \pm\,\pi/2$.

Thus the two curves intersect at the points where $\theta = -\,\pi/2$ and $\theta = \pi/2$.

Therefore the required area is enclosed by $r = 1/(1 + \cos\theta)$, $r = (1 + \cos\theta)$, $\theta = -\,\pi/2$, $\theta = \pi/2$.

Hence the required area

$$= \int_{-\pi/2}^{\pi/2} \int_{1/(1+\cos\theta)}^{(1+\cos\theta)} r\, d\theta\, dr = \int_{-\pi/2}^{\pi/2} \left[\frac{1}{2}\,r^2\right]_{1/(1+\cos\theta)}^{(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left[(1 + \cos\theta)^2 - \frac{1}{(1+\cos\theta)^2}\right] d\theta$$

$$= 2\cdot\frac{1}{2}\int_0^{\pi/2} \left[(1 + 2\cos\theta + \cos^2\theta) - \frac{1}{(2\cos^2\frac{1}{2}\theta)^2}\right] d\theta$$

$$= \int_0^{\pi/2} (1 + 2\cos\theta)\, d\theta + \int_0^{\pi/2} \cos^2\theta\, d\theta - \frac{1}{4}\int_0^{\pi/2} \sec^4 \frac{1}{2}\theta\, d\theta$$

$$= [\theta + 2\sin\theta]_\theta^{\pi/2} + \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{4}\int_0^{\pi/2}\left(1 + \tan^2\frac{1}{2}\theta\right)\sec^2\frac{1}{2}\theta\, d\theta$$

$$= \frac{\pi}{2} + 2 + \frac{\pi}{4} - \frac{1}{4}\int_0^{\pi/2}\left[\sec^2\frac{1}{2}\theta + 2\left(\tan^2\frac{1}{2}\theta\right)\left(\frac{1}{2}\sec^2\frac{1}{2}\theta\right)\right] d\theta$$

$$= \frac{3\pi}{4} + 2 - \frac{1}{4}\left[2\tan\frac{1}{2}\theta + \frac{2}{3}\tan^3\frac{1}{2}\theta\right]_0^{\pi/2}$$

$$= \frac{3\pi}{4} + 2 - \frac{1}{4}\left[2 + \frac{2}{3}\right] = \frac{3\pi}{4} + 2 - \frac{2}{3} = \frac{3\pi}{4} + \frac{4}{3} = \frac{(9\pi + 16)}{12}.$$

Problem 7 (i): *Transform the following double integral to polar coordinates and hence evaluate them:*

$$\int_{y=0}^{a}\int_{x=0}^{\sqrt{(a^2 - y^2)}} (a^2 - x^2 - y^2)\, dx\, dy.$$

Solution: The given double integral

$$I = \int_{y=0}^{a}\int_{x=0}^{\sqrt{(a^2 - y^2)}} [a^2 - (x^2 + y^2)]\, dx\, dy.$$

From the limits of integration it is obvious that the region of integration R is bounded by

$$x = 0,\ x = \sqrt{(a^2 - y^2)} \text{ and } y = 0,\ y = a.$$

Thus the region of integration is the area OAB of the circle $x^2 + y^2 = a^2$ lying in the positive quadrant.

Putting $x = r\cos\theta$, $y = r\sin\theta$ the corresponding polar equation of the circle is $r = a$. From the figure it is obvious that for the area OAB, r varies from 0 to a and θ varies from 0 to $\pi/2$. Also the polar equivalent of $dx\, dy$ is $r\, d\theta\, dr$.

$$\therefore \quad I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} (a^2 - r^2)\, r\, d\theta\, dr, \qquad [\because\ x^2 + y^2 = r^2]$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} [a^2 r - r^3]\, d\theta\, dr = \int_0^{\pi/2}\left[\frac{a^2 r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{a} d\theta$$

$$= \int_0^{\pi/2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] d\theta = \frac{a^4}{4}\int_0^{\pi/2} d\theta = \frac{a^4}{4}[\theta]_0^{\pi/2} = \frac{\pi a^4}{8}.$$

Problem 7(ii): *Transform the following double integral to polar coordinates and hence evaluate them*

$$\int_0^1\int_x^{\sqrt{(2x - x^2)}} (x^2 + y^2)\, dx\, dy.$$

Solution: The given double integral $I = \int_{x=0}^{1}\int_{y=x}^{\sqrt{(2x - x^2)}} (x^2 + y^2)\, dx\, dy$.

Here the region of integration R is bounded by

$$y = x \, , \, y = \sqrt{(2\,x - x^2)} \text{ and } x = 0 \, , x = 1$$

i.e., the region of integration is the area $OBCO$ of the circle $x^2 + y^2 - 2\,x = 0$ bounded by the lines $y = x \, , x = 0$ and $x = 1$.

Putting $x = r\cos\theta$, $y = r\sin\theta$ the corresponding polar equation of the circle is

$$r^2(\cos^2\theta + \sin^2\theta) - 2\,r\cos\theta = 0$$

or $$r = 2\cos\theta.$$

The point B is on the line $y = x$ which makes an angle $\pi/4$ with OX and so, at B, $\theta = \pi/4$. At the point O of the circle $r = 2\cos\theta$, we have $r = 0$ and so $\theta = \pi/2$. Thus for the region R, r varies from 0 to $2\cos\theta$ and θ varies from $\pi/4$ to $\pi/2$. Also the polar equivalent of $dx\,dy$ is $r\,d\theta\,dr$.

Hence transforming to polar coordinates, we have

$$I = \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{2\cos\theta} r^2 . r\,d\theta\,dr = \int_{\theta=\pi/4}^{\pi/2} \int_{r=0}^{2\cos\theta} r^3\,d\theta\,dr$$

$$= \int_{\pi/4}^{\pi/2} \left[\frac{r^4}{4}\right]_{r=0}^{2\cos\theta} d\theta = \frac{16}{4}\int_{\pi/4}^{\pi/2} \cos^4\theta\,d\theta$$

$$= 4\int_{\pi/4}^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \int_{\pi/4}^{\pi/2} (1 + 2\cos 2\theta + \cos^2 2\theta)\,d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left[1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right] d\theta$$

$$= \int_{\pi/4}^{\pi/2} \left[\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right] d\theta$$

$$= \left[\frac{3}{2}\theta + 2 . \frac{\sin 2\theta}{2} + \frac{1}{2}\cdot\frac{\sin 4\theta}{4}\right]_{\pi/4}^{\pi/2} = \left[\frac{3\pi}{4} - \frac{3\pi}{8} - 1\right] = \frac{3\pi}{8} - 1.$$

Problem 7(iii): *Transform the following double integral to polar coordinates and hence evaluate them:*

$$\int_0^a \int_0^{\sqrt{(a^2 - x^2)}} y^2 \sqrt{(x^2 + y^2)}\,dx\,dy.$$

Solution: The given double integral $I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{(a^2-x^2)}} y^2 \sqrt{(x^2+y^2)}\,dx\,dy$.

Here the region of integration R is bounded by $y = 0$, $y = \sqrt{(a^2 - x^2)}$ and $x = 0, x = a$. Thus the region of integration R is the area of the circle $x^2 + y^2 = a^2$ lying in the positive quadrant. The polar equation of this circle is $r = a$ and for the region R, r varies from 0 to a and θ varies from 0 to $\pi/2$. Putting $x = r\cos\theta$, $y = r\sin\theta$ and replacing $dx\,dy$ by $r\,d\theta\,dr$, we have

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2 \sin^2\theta . r . r\,d\theta\,dr = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^4 \sin^2\theta\,d\theta\,dr$$

$$= \int_0^{\pi/2} \left[\frac{r^5}{5}\right]_0^a \sin^2\theta\,d\theta = \frac{a^5}{5}\int_0^{\pi/2} \sin^2\theta\,d\theta = \frac{a^5}{5}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi a^5}{20}.$$

Comprehensive Problems 3

Problem 1(i): *Evaluate* $\int_{x=0}^{1}\int_{y=0}^{2}\int_{z=1}^{2} x^2 yz\, dz\, dy\, dx.$

Solution: We have $\int_{x=0}^{1}\int_{y=0}^{2}\int_{z=1}^{2} x^2 yz\, dz\, dy\, dx$

$$= \int_{x=0}^{1}\int_{y=0}^{2}\left\{\int_{1}^{2} x^2\ yz\ dz\right\} dy\, dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{2}\left[x^2\ y\cdot\frac{z^2}{2}\right]_1^2 dy\, dx = \frac{1}{2}\int_0^1\left[\int_0^2 (3x^2\ \ y)\, dy\right] dx$$

$$= \frac{3}{2}\int_0^1\left[x^2\cdot\frac{y^2}{2}\right]_0^2 dx = \frac{3}{4}\int_0^1 4x^2\, dx = 3\left[\frac{x^3}{3}\right]_0^1 = 3\cdot\frac{1}{3} = 1\,.$$

Problem 1(ii): *Evaluate* $\int_0^1\int_0^1\int_0^1 e^{\,x+y+z}\, dx\, dy\, dz\,.$

Solution: $\int_0^1\int_0^1\int_0^1 e^{\,x+y+z}\, dx\, dy\, dz = \int_0^1\int_0^1\left[\int_0^1 e^{\,x+y+z}\, dx\right] dy\, dz$

$$= \int_0^1\int_0^1\left[e^{\,x+y+z}\right]_0^1 dy\, dz = \int_0^1\left[\int_0^1 \{e^{1+y+z} - e^{\,y+z}\}\, dy\right] dz$$

$$= \int_0^1\left[e^{1+y+z} - e^{\,y+z}\right]_0^1 dz = \int_0^1 \{(e^{2+z} - e^{1+z}) - (e^{1+z} - e^{\,z})\}\, dz$$

$$= \int_0^1 (e^{2+z} - 2\,e^{1+z} + e^{\,z})\, dz = \int_0^1 (e^2 - 2e + 1)\, e^{\,z}\, dz$$

$$= (e^2 - 2e + 1)\int_0^1 e^{\,z}\, dz = (e-1)^2\,[\,e^{\,z}\,]_0^1 = (e-1)^2\,(e - e^0\ )$$

$$= (e-1)^2\,(e-1) = (e-1)^3\,.$$

Problem 1(iii): *Evaluate* $\int_{-1}^{1}\int_0^z\int_{x-z}^{x+z} (\,x + \ y + z\,)\, dy\, dx\, dz\,.$

Solution: Here $x - z$ to $x + z$ are the limits of integration of y, 0 to z are those of x and -1 to 1 are those of z. The given triple integral is

$$= \int_{-1}^{1}\int_0^z\left[\int_{x-z}^{x+z} (\,x + \ y + z\,)\, dy\right] dx\, dz$$

$$= \int_{-1}^{1}\int_0^z\left[xy + \frac{y^2}{2} + zy\right]_{x-z}^{x+z} dx\, dz$$

$$= \int_{-1}^{1}\int_0^z\left[x\,(x+z) + \frac{(x+z)^2}{2} + z\,(x+z) - x\,(x-z)\right.$$

$$\left. - \frac{(x-z)^2}{2} - z\,(x-z)\right]\ dx\, dz$$

$$= \int_{-1}^{1}\left[\int_{0}^{z} (4xz + 2z^2)\, dx\right] dz = \int_{-1}^{1} [2zx^2 + 2z^2x]_0^z \, dz$$

$$= \int_{-1}^{1} (2z.z^2 + 2z^2.z)\, dz = 4\int_{-1}^{1} z^3 dz = 4\left[\frac{z^4}{4}\right]_{-1}^{1} = 1.[1-1] = 0 .$$

Problem 1(iv): *Evaluate* $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z}\, dx\, dy\, dz$

Solution: We have $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z}\, dx\, dy\, dz$

$$= \int_0^{\log 2} \int_0^x \left[e^{x+y+z}\right]_0^{x+\log y} dx\, dy,$$

integrating w.r.t. z regarding x and y as constants

$$= \int_0^{\log 2} \int_0^x [e^{x+y+x+\log y} - e^{x+y}]\, dx\, dy$$

$$= \int_0^{\log 2} \int_0^x [e^{2x}e^y\ e^{\log y} - e^x e^y]\, dx\, dy$$

$$= \int_0^{\log 2} \int_0^x [e^{2x}\ ye^y - e^x e^y]\, dx\, dy. \qquad [\because e^{\log y} = y]$$

$$= \int_0^{\log 2} \left[\int_0^x e^{2x} ye^y\, dy - \int_0^x e^x e^y\, dy\right] dx$$

$$= \int_0^{\log 2} \left[e^{2x}\{ye^y\}_0^x - e^{2x}\int_0^x e^y\, dy - e^x\{e^y\}_0^x\right] dx$$

integrating w.r.t. y regarding x as a constant; to integrate ye^y we have applied integration by parts

$$= \int_0^{\log 2} \left[e^{2x} \cdot xe^x - e^{2x}\{e^y\}_0^x - e^x(e^x - 1)\right] dx$$

$$= \int_0^{\log 2} [xe^{3x} - e^{2x}(e^x - 1) - e^{2x} + e^x]\, dx$$

$$= \int_0^{\log 2} [xe^{3x} - e^{3x} + e^x]\, dx = \int_0^{\log 2} xe^{3x} dx - \int_0^{\log 2} e^{3x} dx + \int_0^{\log 2} e^x dx$$

$$= \frac{1}{3}[xe^{3x}]_0^{\log 2} - \frac{1}{3}\int_0^{\log 2} e^{3x} dx - \int_0^{\log 2} e^{3x} dx + \int_0^{\log 2} e^x dx$$

$$= \frac{1}{3}(\log 2)e^{3\log 2} - \frac{4}{3}\left[\frac{e^{3x}}{3}\right]_0^{\log 2} + [e^x]_0^{\log 2}$$

$$= \frac{1}{3}(\log 2)\, e^{\log 8} - \frac{4}{9}(e^{3\log 2} - 1) + (e^{\log 2} - 1)$$

$$= \frac{8}{2}\log 2 - \frac{4}{9}(8-1) + (2-1) = \frac{8}{3}\log 2 - \frac{28}{9} + 1 = \frac{8}{3}\log 2 - \frac{19}{9}\cdot$$

Problem 2(i): *Evaluate* $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dy\, dx\, dz$

Solution: We have $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dy\, dx\, dz = \int_0^1 \int_{y^2}^1 x\, [z]_0^{1-x}\, dy\, dx$

integrating w.r.t. z regarding x and y as constants

$$= \int_0^1 \int_{y^2}^1 x(1-x)\, dy\, dx = \int_0^1 \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{y^2}^1 dy$$

integrating w.r.t x regarding y as constant

$$= \int_0^1 \left[\frac{1}{2} - \frac{1}{3} - \frac{y^4}{2} + \frac{y^6}{3}\right] dy = \int_0^1 \left[\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3}\right] dy$$

$$= \left[\frac{1}{6} y - \frac{y^5}{10} + \frac{y^6}{21}\right]_0^1 = \frac{1}{6} - \frac{1}{10} + \frac{1}{21} = \frac{35 - 21 + 10}{210} = \frac{24}{210} = \frac{4}{35}.$$

Problem 2(ii): *Evaluate* $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dx\, dy\, dz}{(1 + x + y + z)^3}$.

(Kanpur 2008; Avadh 13)

Solution: We have $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dx\, dy\, dz}{(1 + x + y + z)^3}$

$$= \int_0^1 \int_0^{1-x} \left[-\frac{1}{2(1 + x + y + z)^2}\right]_0^{1-x-y} dx\, dy$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} \left[-\frac{1}{4} + \frac{1}{(1 + x + y)^2}\right] dx\, dy$$

$$= \frac{1}{2} \int_0^1 \left[-\frac{1}{4} y - \frac{1}{(1 + x + y)}\right]_0^{1-x} dx = \frac{1}{2} \int_0^1 \left[-\frac{1}{4} (1-x) - \frac{1}{2} + \frac{1}{(1+x)}\right] dx$$

$$= \frac{1}{2} \int_0^1 \left[-\frac{3}{4} + \frac{1}{4} x + \frac{1}{(1+x)}\right] dx = \frac{1}{2} \left[-\frac{3}{4} x + \frac{1}{4} \cdot \frac{x^2}{2} + \log (1+x)\right]_0^1$$

$$= \frac{1}{2} \left[\left(-\frac{3}{4} + \frac{1}{4} \cdot \frac{1}{2} + \log 2\right) - \log 1\right] = \frac{1}{2}\left(\log 2 - \frac{5}{8}\right).$$

Problem 2(iii): *Evaluate* $\int_1^3 \int_{1/x}^1 \int_0^{\sqrt{(xy)}} xyz\, dx\, dy\, dz$.

Solution: The given triple integral is

$$\int_1^3 \int_{1/x}^1 \left[\int_0^{\sqrt{(xy)}} xyz\, dz\right] dx\, dy = \int_1^3 \int_{1/x}^1 \left[x\, y \cdot \frac{z^2}{2}\right]_0^{\sqrt{(xy)}} dx\, dy$$

$$= \frac{1}{2} \int_1^3 \left[\int_{1/x}^1 x^2\, y^2\, dy\right] dx = \frac{1}{2} \int_1^3 \left[x^2 \cdot \frac{y^3}{3}\right]_{1/x}^1 dx$$

$$= \frac{1}{6}\int_1^3 \left[x^2 - \frac{1}{x}\right] dx = \frac{1}{6}\left[\frac{x^3}{3} - \log x\right]_1^3$$

$$= \frac{1}{6}\,[(9 - \log 3) - (\frac{1}{3} - \log 1)] = \frac{1}{6}\,[(9 - \frac{1}{3}) - \log 3] = \frac{1}{6}\,[\frac{26}{3} - \log 3].$$

Problem 2(iv): *Evaluate* $\int_0^{\pi/2} d\theta \int_0^{a\sin\theta} dr \int_0^{(a^2 - r^2)/a} r\, dz$.

Solution: The given triple integral is

$$= \int_0^{\pi/2} d\theta \int_0^{a\sin\theta} dr\, [r\,z]_0^{(a^2 - r^2)/a} = \int_0^{\pi/2} d\theta \int_0^{a\sin\theta} \frac{r\,(a^2 - r^2)}{a}\, dr$$

$$= \frac{1}{a}\int_0^{\pi/2} \left[\frac{a^2 r^2}{2} - \frac{r^4}{4}\right]_0^{a\sin\theta} d\theta = \frac{a^3}{4}\int_0^{\pi/2} (2\sin^2\theta - \sin^4\theta)\, d\theta$$

$$= \frac{a^3}{4}\left[2\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{5a^3\,\pi}{64}.$$

Problem 3(i): *Evaluate* $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dx\, dy\, dz$.

Solution: The given triple integral is

$$= \int_0^a\int_0^x \left[\int_0^{x+y} e^{x+y+z}\, dz\right] dx\, dy = \int_0^a\int_0^x \left[e^{\,x+y+z}\right]_{z=0}^{x+y} dx\, dy$$

$$= \int_0^a\int_0^x [e^{2(x+y)} - e^{(x+y)}]\, dx\, dy = \int_0^a \left[\frac{1}{2}e^{2(x+y)} - e^{(x+y)}\right]_0^x dx$$

$$= \int_0^a [\frac{1}{2}(e^{4x} - e^{2x}) - (e^{2x} - e^x)]\, dx = \int_0^a (\frac{1}{2}e^{4x} - \frac{3}{2}e^{2x} + e^x)\, dx$$

$$= \left[\frac{1}{2}\cdot\frac{1}{4}e^{4x} - \frac{3}{4}\cdot\frac{1}{2}e^{2x} + e^x\right]_0^a$$

$$= [(\frac{1}{8}e^{4a} - \frac{3}{4}e^{2a} + e^a) - (\frac{1}{8}e^0 - \frac{3}{4}e^0 + e^0)]$$

$$= \frac{1}{8}e^{4a} - \frac{3}{4}e^{2a} + e^a - (\frac{1}{8} - \frac{3}{4} + 1) = \frac{1}{8}(e^{4a} - 6e^{2a} + 8\,e^a - 3).$$

Problem 3(ii): *Evaluate* $\int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2 dx\, dy\, dz$.

Solution: The given triple integral

$$= \int_0^a \int_0^{a-x} \int_0^{a-x-y} x^2\, dx\, dy\, dz = \int_0^a \int_0^{a-x} x^2\, [z]_0^{a-x-y}\, dx\, dy,$$

integrating w.r.t. z regarding x and y as constants

$$= \int_0^a \int_0^{a-x} x^2\,[a - x - y]\, dx\, dy = \int_0^a\int_0^{a-x} x^2\,[(a - x) - y]\, dx\, dy$$

$$= \int_0^a x^2 \left[(a - x) y - \frac{1}{2} y^2 \right]_0^{a - x} dx,$$

integrating w.r.t. y regarding x as constant

$$= \int_0^a x^2 \left[(a - x)^2 - \frac{1}{2} (a - x)^2 \right] dx$$

$$= \int_0^a x^2 \cdot \frac{1}{2} (a - x)^2 \, dx = \frac{1}{2} \int_0^a x^2 (a^2 - 2ax + x^2) \, dx$$

$$= \frac{1}{2} \int_0^a (x^2 a^2 - 2 a x^3 + x^4) \, dx = \frac{1}{2} \left[a^2 \frac{x^3}{3} - 2 a \frac{x^4}{4} + \frac{x^5}{5} \right]_0^a$$

$$= \frac{1}{2} \left[\frac{1}{3} a^5 - \frac{1}{2} a^5 + \frac{1}{5} a^5 \right] = \frac{1}{2} \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right) a^5 = \frac{1}{60} a^5.$$

Problem 4: *Evaluate the triple integral of the function* $f(x, y, z) = x^2$ *over the region* V *enclosed by the planes* $x = 0$, $y = 0$, $z = 0$ *and* $x + y + z = a$.

(Rohilkhand 2012; Avadh 12)

Solution: The given region V is bounded by the co-ordinate planes $x = 0$, $y = 0$, $z = 0$ and the plane $x + y + z = a$. To cover the region V, let the values of x, y lie within the triangle bounded by x-axis, the y-axis and the line $(x + y = a, z = 0)$. Then for any point $(x, y, 0)$ within this triangle, z varies from $z = 0$ to $z = a - x - y$ in the region V. But the values of x and y vary within the triangle formed in the xy-plane. Therefore x varies from 0 to a and for any intermediary value of x, y varies from 0 to $a - x$. Therefore the region of integration V can be expressed as

$$0 \le x \le a, 0 \le y \le a - x, 0 \le z \le a - x - y.$$

Hence the required triple integral $= \int_0^a \int_0^{a - x} \int_0^{a - x - y} x^2 \, dx \, dy \, dz$.

Now proceed as in problem 3 (ii).

Problem 5: *Find the volume of the tetrahedron bounded by the plane* $x/a + y/b + z/c = 1$ *and the coordinate planes.*

Solution: Here the region of integration V to cover the volume of the given tetrahedron can be expressed as

$$0 \le x \le a, 0 \le y \le b(1 - x/a), 0 \le z \le c(1 - x/a - y/b).$$

Therefore the required volume of the tetrahedron

$$= \iiint_V dx \, dy \, dz = \int_0^a \int_0^{b(1 - x/a)} \int_0^{c(1 - x/a - y/b)} dx \, dy \, dz.$$

Now proceed as in Example 18. The required volume $= \dfrac{abc}{6}$.

Problem 6(i): *Evaluate* $\iiint \dfrac{dx \, dy \, dz}{(x + y + z + 1)^3}$ *over the region*

$x \ge 0, y \ge 0, z \ge 0, x + y + z \le 1.$ **(Avadh 2013; Kanpur 15)**

Solution: The given region of integration R can be expressed as

$$0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1 - x - y.$$

Hence the required triple integral $= \iiint_R \dfrac{dx\,dy\,dz}{(x + y + z + 1)^3}$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(x + y + z + 1)^3}\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^{1-x} \left[\int_0^{1-x-y} (x + y + z + 1)^{-3}\,dz \right] dx\,dy$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{(x + y + z + 1)^{-2}}{-2} \right]_0^{1-x-y} dx\,dy$$

$$= -\frac{1}{2} \int_0^1 \int_0^{1-x} \left[\frac{1}{4} - \frac{1}{(x + y + 1)^2} \right] dx\,dy$$

$$= -\frac{1}{2} \int_0^1 \left[\frac{y}{4} + \frac{1}{(x + y + 1)} \right]_0^{1-x} dx = -\frac{1}{2} \int_0^1 \left[\frac{1 - x}{4} + \frac{1}{2} - \frac{1}{(x + 1)} \right] dx$$

$$= -\frac{1}{2} \left[\frac{(1 - x)^2}{2 \times 4 \times (-1)} + \frac{1}{2} x - \log(x + 1) \right]_0^1$$

$$= -\frac{1}{2} [\{0 + \frac{1}{2} - \log 2\} - \{-\frac{1}{8} + 0 - 0\}] = -\frac{1}{2} [\frac{1}{2} - \log 2 + \frac{1}{8}]$$

$$= -\frac{1}{2} [\frac{5}{8} - \log 2] = \frac{1}{2} [\log 2 - \frac{5}{8}].$$

Problem 6(ii): *Evaluate* $\iiint xyz\,dx\,dy\,dz$

over the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. **(Kanpur 2011)**

Solution: Here the region of integration can be expressed as

$$-a \le x \le a, -b\sqrt{\{1 - (x^2/a^2)\}} \le y \le b\sqrt{\{1 - (x^2/a^2)\}}$$

and $\quad -c\sqrt{\{1 - (x^2/a^2) - (y^2/b^2)\}} \le z \le c\sqrt{\{1 - (x^2/a^2) - (y^2/b^2)\}}$.

∴ The required triple integral

$$= \int_{-a}^{a} \int_{-b\sqrt{\{1-(x^2/a^2)\}}}^{b\sqrt{\{1-(x^2/a^2)\}}} \left[\int_{-c\sqrt{\{1-(x^2/a^2)-(y^2/b^2)\}}}^{c\sqrt{\{1-(x^2/a^2)-(y^2/b^2)\}}} (xy) . z\,dz \right] dx\,dy$$

$= 0$. [∵ z is an odd function of z and xy is treated as constant while integrating w.r.t. z]

Problem 6(iii): *Evaluate* $\iiint (z^5 + z)\,dx\,dy\,dz$ *over the sphere* $x^2 + y^2 + z^2 = 1$.

Solution: The given region of integration can be expressed as

$$-1 \le x \le 1, -\sqrt{(1 - x^2)} \le y \le \sqrt{(1 - x^2)}, -\sqrt{(1 - x^2 - y^2)} \le z \le \sqrt{(1 - x^2 - y^2)}.$$

Hence the required triple integral

$$= \int_{-1}^{1} \int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} \left[\int_{-\sqrt{(1-x^2-y^2)}}^{\sqrt{(1-x^2-y^2)}} (z^5 + z)\, dz \right] dx\, dy$$

$$= 0. \qquad [\because (z^5 + z) \text{ is an odd function of } z]$$

Problem 6(iv): *Evaluate* $\iiint_R u^2 v^2 w \, du\, dv\, dw$, *where R is the region* $u^2 + v^2 \le 1, 0 \le w \le 1$.

Solution: Here the limits of integration to cover the region R can be taken as

$$-1 \le u \le 1, -\sqrt{(1-u^2)} \le v \le \sqrt{(1-u^2)}, 0 \le w \le 1,$$

where the first integration is to be performed with respect to v.

$$\therefore \quad \iiint_R u^2 v^2 w\, du\, dv\, dw = \int_0^1 \int_{-1}^1 \int_{-\sqrt{(1-u^2)}}^{\sqrt{(1-u^2)}} u^2 v^2 w\, dw\, du\, dv$$

$$= \int_0^1 \int_{-1}^1 u^2 w \left[\int_{-\sqrt{(1-u^2)}}^{\sqrt{(1-u^2)}} v^2 dv \right] dw\, du,$$

because the first integration is to be performed w.r.t. v regarding u and w as constants

$$= \int_0^1 \int_{-1}^1 \left[2u^2 w \int_0^{\sqrt{(1-u^2)}} v^2 dv \right] dw\, du,$$

because v^2 is an even function of v

$$= \int_0^1 \int_{-1}^1 2u^2 w \left[\frac{v^3}{3} \right]_0^{\sqrt{(1-u^2)}} dw\, du$$

$$= \frac{2}{3} \int_0^1 \int_{-1}^1 w u^2 (1-u^2)^{3/2} dw\, du$$

$$= \frac{2}{3} \int_0^1 \left[w . 2 \int_0^1 u^2 (1-u^2)^{3/2} du \right] dw$$

$$= \frac{4}{3} \int_0^1 w \left[\int_0^{\pi/2} \sin^2 \theta . \cos^3 \theta . \cos \theta\, d\theta \right] dw, \text{ putting } u = \sin \theta$$

$$= \frac{4}{3} \int_0^1 w \left[\int_0^{\pi/2} \sin^2 \theta \cos^4 \theta\, d\theta \right] dw = \frac{4}{3} \int_0^1 w . \frac{1.3.1}{6.4.2} \cdot \frac{\pi}{2} dw$$

$$= \frac{\pi}{24} \int_0^1 w\, dw = \frac{\pi}{24} \left[\frac{w^2}{2} \right]_0^1 = \frac{\pi}{48} [1 - 0] = \frac{\pi}{48}.$$

Comprehensive Problems 4

Problem 1: *Change the order of integration in* $\int_0^1 \int_x^{x(2-x)} f(x, y)\, dx\, dy$.

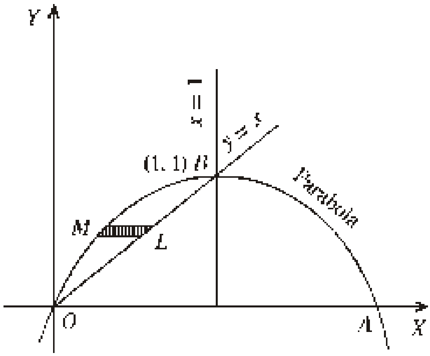
Solution: In the given integral the limits of integration of y are given by $y = x$, which is a straight line passing through the origin, and

$$y = x(2 - x) \text{ or } y = 2x - x^2 \quad \text{or} \quad (x-1)^2 = -(y-1)$$

which is a parabola with vertex (1, 1) and passing through the origin.

Again the limits of integration of x are given by $x = 0$ *i.e.*, the y-axis and $x = 1$ which is a straight line parallel to the y-axis at a distance 1 from the origin.

We draw the curves $y = x$, $(x-1)^2 = -(y-1)$, $x = 0$ and $x = 1$, giving the limits of integration, in the same figure. We observe that the region of integration is the area $OLBMO$.



In the given integral, the limits of integration of y being variable, we are required to integrate first w.r.t. y regarding x as a constant and then w.r.t. x.

If we want to reverse the order of integration, we have to first integrate w.r.t. x regarding y as a constant and then we integrate w.r.t. y. This is done by covering the area of integration $OLBMO$ by drawing the straight lines $y =$ constant *i.e.*, by dividing this area into strips parallel to the x-axis.

So divide the region $OLBMO$ into strips parallel to the x-axis starting from the arc OMB of the parabola and terminating on the line OLB.

For the point B, $x = 1$. Putting $x = 1$ in the equation of the line $y = x$, we get $y = 1$. So the y-coordinate of the point B is also 1.

For the region $OMBLO$, the lower limit of x is the value of x found in terms of y from the equation $(x-1)^2 = 1 - y$ and the upper limit of x is the value of x found in terms of y from the equation $y = x$. From the equation $(x-1)^2 = 1 - y$, we get $x - 1 = \pm\sqrt{(1-y)}$ or $x = 1 \pm \sqrt{(1-y)}$. Since in the region $OMBLO$, x takes values less than 1, therefore we take $x = 1 - \sqrt{(1-y)}$.

Thus in the region $OMBLO$, x varies from $1 - \sqrt{(1-y)}$ to y and y varies from 0 to a.

Hence by changing the order of integration, we have the given integral

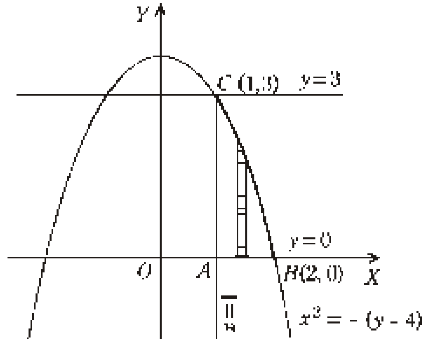
$$= \int_0^1 \int_{1-\sqrt{(1-y)}}^{y} f(x, y)\, dy\, dx.$$

Problem 2: *Change the order of integration in the integral*

$$\int_0^3 \int_1^{\sqrt{(4-y)}} (x + y)\, dy\, dx.$$

Solution: In the given integral the limits of integration of x are given by the straight line $x = 1$ and the curve $x = \sqrt{(4-y)}$ *i.e.*, $x^2 = 4 - y$ *i.e.*, $x^2 = -(y-4)$ which is a parabola, symmetrical about the y-axis, with vertex at the point (0, 4) and existing in the region $y \le 4$. Again the limits of integration of y are given by the straight lines $y = 0$ (*i.e.*, the x-axis) and $y = 3$.

We draw the curves $x=1$, $x^2=-(y-4)$, $y=0$ and $y=3$, giving the limits of integration in the same figure. Putting $x=1$ in the equation $x^2=-(y-4)$, we get $y=3$. Thus the straight line $y=3$ passes through the point of intersection C of $x=1$ and $x^2=-(y-4)$. Also at the point of intersection B of the parabola $x^2=-(y-4)$ and the x-axis (*i.e.*, the line $y=0$), we have $x=2$. We observe that the region of integration is the area $ABCA$.



In the given integral the limits of integration of x are variable while those of y are constant. Thus we have to first integrate w.r.t. x regarding y as a constant and then we integrate w.r.t. y.

If we want to change the order of integration, we have to first integrate w.r.t. y regarding x as a constant and then we integrate w.r.t. x. This is done by covering the area $ABCA$ by strips drawn parallel to the y-axis. These strips start from the line AB (*i.e.*, $y=0$) and terminate on the arc BC of the parabola $x^2=4-y$. Therefore for the region $ABCA$, y varies from 0 to $4-x^2$ and x varies from 1 to 2. Hence by changing the order of integration, we have the given integral

$$=\int_1^2 \int_0^{4-x^2} (x+y)\, dx\, dy.$$

Problem 3: *Change the order of integration in the integral*

$$\int_0^{a\cos\alpha} \int_{x\tan\alpha}^{\sqrt{(a^2-x^2)}} f(x, y)\, dx\, dy.$$

(Kumaun 2002, 10; Kanpur 05; Avadh 11)

Solution: In the given integral the limits of integration of y are given by $y = x\tan\alpha$ which is a straight line passing through the origin and

$$y=\sqrt{(a^2-x^2)} \text{ i.e., } y^2=a^2-x^2 \quad \text{i.e., } \quad x^2+y^2=a^2$$

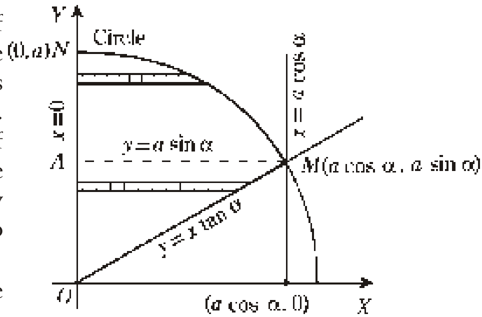
which is a circle of radius a with centre at the origin $(0,0)$.

Again the limits of integration of x are given by $x=0$ *i.e.*, the y-axis and $x=a\cos\alpha$ which is a straight line parallel to the y-axis at a distance $a\cos\alpha$ from the origin.

We draw the curves $y=x\tan\alpha$, $x^2+y^2=a^2$, $x=0$ and $x=a\cos\alpha$, giving the limits of integration, in the same figure. We observe that the region of integration is the area $OMNO$.

In the given integral the limits of integration of y are variable while those of x are constant. Thus we have to first integrate with respect to y regarding x as constant and then we integrate w.r.t. x. This is done by covering the area of integration $OMNO$ by drawing the straight lines $x=$ constant *i.e.*, by dividing this area into strips parallel to the y-axis.

If we want to reverse the order of integration, we have to first integrate with respect to x regarding y as constant and then we integrate w.r.t. y. This is done by covering the area of integration $OMNO$ by drawing the straight lines $y =$ constant *i.e.*, by dividing this area into strips parallel to the x-axis.



Now if we take strips parallel to the x-axis starting from the line $x = 0$, some of these strips end on the line OM while the others end on the arc MN of the circle $x^2 + y^2 = a^2$. So we draw the line of demarcation MA dividing the area $OMNO$ into two portions OMA and AMN.

For the point M, $x = a\cos\alpha$. Putting $x = a\cos\alpha$ in the equation of the line $y = x\tan\alpha$, we get $y = a\sin\alpha$. So the y-coordinate of the point M is $a\sin\alpha$ and the equation of the line of demarcation MA is $y = a\sin\alpha$.

For the region OMA, x varies from 0 to $y\cot\alpha$ and y varies from 0 to $a\sin\alpha$.

For the region AMN, x varies from 0 to $(a^2 - y^2)$ and y varies from $a\sin\alpha$ to a.

Therefore, changing the order of integration, the given double integral transforms to

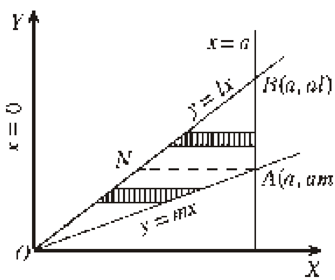
$$\int_0^{a\sin\alpha}\int_0^{y\cot\alpha} f(x, y)\,dy\,dx + \int_{a\sin\alpha}^{a}\int_0^{\sqrt{(a^2 - y^2)}} f(x, y)\,dy\,dx.$$

Problem 4: *Change the order of integration in* $\int_0^a \int_{mx}^{lx} f(x, y)\,dx\,dy$.

(Lucknow 2010)

Solution: Here the area of integration is bounded by the straight lines $y = mx$, $y = l\,x$, $x = 0$ and $x = a$. Drawing all these lines in one figure, we observe that the area of integration is $OABO$.

To reverse the order of integration, cover this area $OABO$ by strips parallel to the axis of x. Draw the straight line AN parallel to the x-axis and thus divide the area $OABO$ into two portions OAN and NBA according to the character of the strips.



For the point A, $x = a$. Putting $x = a$ in the equation of the line $y = mx$, we get $y = ma$. Also for the point B, $x = a$; therefore putting $x = a$ in the equation of the line $y = l\,x$, we get $y = la$.

Now for the area ONA, x varies from the line $y = l\,x$ to $y = mx$ *i.e.*, x varies from y/l to y/m and y varies from 0 to am. Again for the area NBA, x varies from the line NB ($y = lx$) to the line $x = a$ *i.e.*, x varies from y/l to a and y varies from am to al.

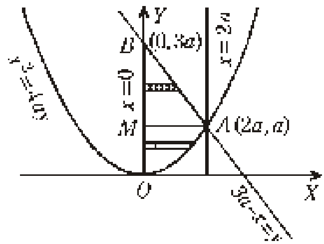
Therefore, by changing the order of integration the given integral transforms to

$$\int_0^{am} \int_{y/l}^{y/m} f(x, y)\, dy\, dx + \int_{am}^{al} \int_{y/l}^{a} f(x, y)\, dy\, dx .$$

Problem 5: *Change the order of integration in* $\int_0^{2a} \int_{x^2/4a}^{3a - x} f(x, y)\, dx\, dy$.

(Agra 2001)

Solution: In the given integral the limits of integration are given by $x^2/4a = y$ *i.e.*, $x^2 = 4ay$, (which is a parabola passing through the origin), and the lines $y = 3a - x$, $x = 0$, and $x = 2a$. Drawing these curves in one figure we observe that the region of integration is the area $OABMO$.



To change the order of integration, first we divide the region of integration into two portions OAM and MAB, by drawing the line AM parallel to the x-axis. Now to reverse the order of integration, cover the whole region $OABMO$ by strips parallel to the x-axis starting from the line $x = 0$. Some of these strips end on the arc OA while others end on the line AB.

For the point A, we have $x = 2a$. Putting $x = 2a$ in the equation of the line $y = 3a - x$, we get $y = a$.

For the region OAM, x varies from 0 to $\sqrt{(4ay)}$ and y varies from 0 to a. Again for the region MAB, x varies from 0 to $3a - y$ and y varies from a to $3a$.

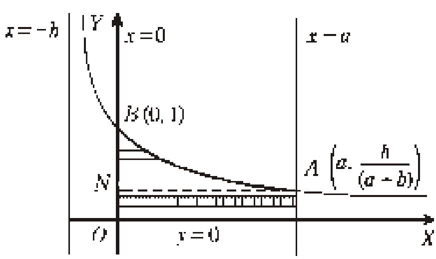
Hence the transformed integral is given by

$$\int_0^{a} \int_0^{\sqrt{(4ay)}} f(x, y)\, dy\, dx + \int_a^{3a} \int_0^{3a-y} f(x, y)\, dy\, dx .$$

Problem 6: *Change the order of integration in the double integral*

$$\int_0^{a} \int_0^{b/(b+x)} f(x, y)\, dx\, dy .$$

Solution: In the given integral the limits of integration of y are given by $y = 0$ (*i.e.*, the x-axis) and $y = b/(b + x)$ *i.e.*, $y(b + x) = b$ which is a rectangular hyperbola having for its asymptotes the straight lines $y = 0$ and $x = -b$. Again the limits of integration of x are given by the straight lines $x = 0$ (*i.e.*, the y-axis) and $x = a$. We draw the curves $y(b + x) = b$, $y = 0$, $x = 0$ and $x = a$, giving the limits of integration, in the same figure. We observe that the region of integration is the area $OMABO$.



In the given integral we are required to integrate first w.r.t. y and then w.r.t. x. To change the order of integration, we have to first integrate w.r.t. x regarding y as constant and then we integrate w.r.t. y. This is done by covering the area of integration $OMABO$ by drawing the straight lines $y =$ constant *i.e.*, by dividing this area into strips parallel to the x-axis.

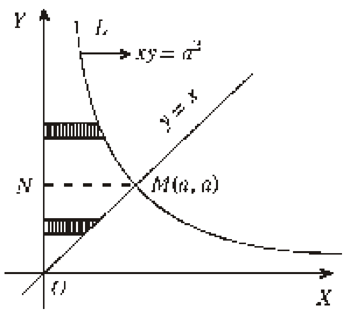
Now if we take strips parallel to the x-axis originating from the line $x = 0$, some of these strips terminate on the line AM while the others terminate on the arc AB. So according to the character of the strips we divide the region of integration into two portions namely $NOMA$ and NAB, by drawing the line AN parallel to the axis of x.
For the point B, $x = 0$. Putting $x = 0$ in the equation $y(b + x) = b$, we get $y = 1$. So the coordinates of the point B are $(1, 1)$.
Similarly putting $x = a$ in the equation $y(b + x) = b$, we get $y = b/(a + b)$ and thus the coordinates of the point A are $(a, b/(a + b))$.
For the area $NOMA$, x varies from 0 to a and y varies from 0 to $b/(a + b)$.
For the area NBA, x varies from 0 to $b(1 - y)/y$ and y varies from $b/(a + b)$ to 1.
Therefore, changing the order of integration, the given double integral transforms to

$$\int_0^{b/(a+b)} \int_0^a f(x, y)\, dy\, dx + \int_{b/(a+b)}^1 \int_0^{b(1-y)/y} f(x, y)\, dy\, dx.$$

Problem 7: *Change the order of integration in* $\int_0^a \int_x^{a^2/x} f(x, y)\, dx\, dy$.

(Lucknow 2009; Kanpur 10; Kumaun 12)

Solution: In the given integral the limits of integration of y are given by $y = x$ which is a straight line passing through the origin equally inclined to both the axes and $y = a^2/x$ or $xy = a^2$ which is a rectangular hyperbola. Again the limits of integration of x are given by the straight lines $x = 0$ (*i.e.*, the y-axis) and $x = a$.



We draw the curves $y = x$, $xy = a^2$, $x = 0$ and $x = a$, giving the limits of integration, in the same figure. We observe that the region of integration is the area $LMOY$ … extended upto infinity on the above side.
In the given integral we are required to integrate first w.r.t. y and then w.r.t. x. If we want to change the order of integration, we have to first integrate w.r.t. x regarding y as constant and then we integrate w.r.t. y. This is done by covering the area of integration by strips parallel to the x-axis.
Now if we take strips parallel to the x-axis starting from the line $x = 0$, some of these strips end on the line OM while the others end on the arc ML of the rectangular hyperbola. So we divide the region of integration into two portions, the triangle OMN and the area $YNML$ which extends upto infinity, by drawing the line MN parallel to the axis of x.
For the point M, $x = a$. Putting $x = a$ in the equation of the line $y = x$ or the rectangular hyperbola $xy = a^2$, we get $y = a$.
So the y-coordinate of the point M is a and the equation of the line of demarcation MN is $y = a$.
For the area OMN, x varies from 0 to y and y varies from 0 to a.

For the area $YMNL$ ..., x varies from 0 to a^2/y and y varies from a to ∞.

Hence by changing the order of integration, we have the given integral

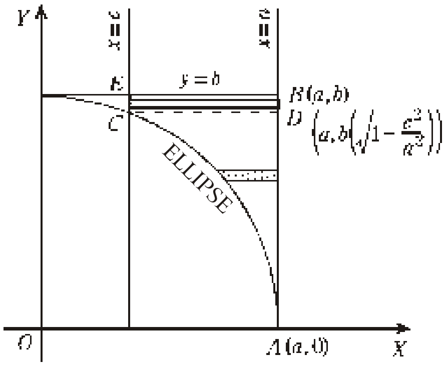
$$= \int_0^a \int_0^y f(x, y)\, dy\, dx + \int_a^\infty \int_0^{a^2/y} f(x, y)\, dy\, dx.$$

Problem 8: *Change the order of integration in* $\int_c^a \int_{(b/a)\sqrt{(a^2 - b^2)}}^b f(x, y)\, dx\, dy,$ *where* $c < a$.

Solution: In the given integral the limits of integration of y are given by

$$y = \frac{b}{a}\sqrt{(a^2 - x^2)} \text{ i.e., } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is an ellipse with centre $(0, 0)$ and the straight line $y = b$.



Again the limits of integration of x are given by the straight lines $x = c$ and $x = a$. Draw the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the straight lines $y = b$, $x = c$ and $x = a$, bounding the region of integration, in the same figure. We observe that the region of integration is the area $ABECA$. In the given integral, the limits of integration of y being variable, we are required to integrate first w.r.t. y and then w.r.t. x.

In order to integrate in the reverse order, divide the whole area into strips parallel to the x-axis originating either from the line EC (*i.e.*, $x = c$) or from the arc AC of the ellipse and terminating on the line BA (*i.e.*, $x = a$). While integrating we must first obviously divide the region of integration $ABECA$ into two portions CAD and $ECDB$ according to the character of the strips. For the point C, $x = c$. Putting $x = c$ in the equation of the ellipse $x^2/a^2 + y^2/b^2 = 1$, we get $y = b\sqrt{\{1 - (c^2/a^2)\}}$ which is the y-coordinate of the point C. The equation of the line of demarcation CD is thus $y = b\sqrt{\{1 - (c^2/a^2)\}}$.

For the area CAD, x varies from $a\sqrt{\{1 - (y^2/b^2)\}}$ to a and y varies from 0 to $b\sqrt{\{1 - (c^2/a^2)\}}$.

For the area $ECDB$, x varies from c to a and y varies from $b\sqrt{\{1 - (c^2/a^2)\}}$ to b.

Therefore, changing the order of integration, the given double integral transforms to

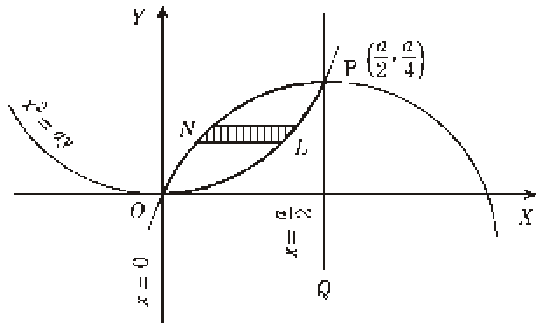
$$\int_0^{b\sqrt{\{1 - (c^2/a^2)\}}} \int_{a\sqrt{\{1 - (y^2/b^2)\}}}^a f(x, y)\, dy\, dx$$

$$+ \int_{b\sqrt{\{1 - (c^2/a^2)\}}}^a \int_c^a f(x, y)\, dy\, dx.$$

Problem 9: *Change the order of integration in* $\int_0^{a/2}\int_{x^2/a}^{x-(x^2/a)} f(x, y)\, dx\, dy$.

Solution: In the given integral the limits of integration of y are given by $y = x^2/a$ *i.e.*, $x^2 = ay$ which is a parabola with vertex $(0,0)$ and $x - x^2/a = y$ *i.e.*, $ax - x^2 = ay$ *i.e.*, $(x-\frac{1}{2}a)^2 = -a(y - \frac{1}{4}a)$ which is also a parabola with vertex $(\frac{1}{2}a, \frac{1}{4}a)$.

The points of intersection of the two parabolas are $(0,0)$ and $(\frac{1}{2}a, \frac{1}{4}a)$.



Again the limits of integration of x are given by $x = 0$ *i.e.*, the y-axis and $x = a/2$ which is a straight line parallel to the y-axis at a distance $a/2$ from the origin.

Draw the two parabolas $x^2 = ay$ and $(x - \frac{1}{2}a)^2 = -a(y - \frac{1}{4}a)$ intersecting at $O(0,0)$ and $P(\frac{1}{2}a, \frac{1}{4}a)$ along with the lines $x = 0$ and $x = a/2$ in the same figure. We observe that the region of integration is $ONPLO$. In the given integral we are required to integrate first w.r.t. y ($\because$ the limits of integration of y are variable) and then w.r.t. x. To reverse the order of integration, draw strips parallel to the x-axis originating from the arc ONP of the parabola $ax - x^2 = ay$ and terminating on the arc OLP of the parabola $x^2 = ay$. Then for the region $ONPLO$, the limits of integration for x are given by $ax - x^2 = ay$ and $x^2 = ay$.

Solving $ay = ax - x^2$ *i.e.*, $x^2 - ax + ay = 0$ for x, we get

$$x = \frac{1}{2}[a \pm \sqrt{(a^2 - 4ay)}] \quad \text{or} \quad x = \frac{1}{2}[a - \sqrt{(a^2 - 4ay)}],$$

rejecting the +ive sign since x cannot be greater than $\frac{1}{2}a$ in the region $ONPLO$.

Thus the limits of x are $x = \frac{1}{2}[a - \sqrt{(a^2 - 4ay)}]$ and $x = \sqrt{(ay)}$.

Clearly for this region y varies from 0 to $\frac{1}{4}a$.

Hence by changing the order of integration, we have

$$\int_0^{a/2}\int_{x^2/a}^{x-x^2/a} f(x, y)\, dx\, dy = \int_0^{a/4}\int_{\frac{1}{2}[a-\sqrt{(a^2-4ay)}]}^{\sqrt{(ay)}} f(x, y)\, dy\, dx.$$

Problem 10: *Change the order of integration in*

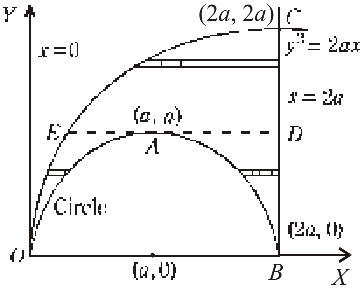
$$\int_0^{2a}\int_{\sqrt{(2ax-x^2)}}^{\sqrt{(2ax)}} f(x, y)\, dx\, dy.$$

(Kumaun 2013)

Solution: In the given integral the limits of integration of y are given by

$$y = \sqrt{(2ax - x^2)} \quad i.e., \quad y^2 = 2ax - x^2 \quad i.e., \quad (x - a)^2 + y^2 = a^2$$

which is a circle with centre $(a, 0)$ and radius a and $y = \sqrt{(2ax)}$ *i.e.*, $y^2 = 2ax$ which is a parabola with vertex $(0, 0)$ and the x-axis as its axis. Again the limits of integration of x are given by $x = 0$ *i.e.*, the y-axis and $x = 2a$, a line parallel to the y-axis at a distance $2a$ from the origin.



We draw the curves $(x - a)^2 + y^2 = a^2$, $y^2 = 2ax$, $x = 0$ and $x = 2a$, giving the limits of integration, in the same figure. We observe that the region of integration is the area $OABCO$.

To reverse the order of integration, cover this area of integration $OABCO$ by strips parallel to the x-axis. Through A, draw the line EAD parallel to the x-axis (*i.e.*, tangent to the circle at A) so that the region of integration is divided into three portions OEA, ABD and ECD.

For the point A, $x = a$. Putting $x = a$ in $(x - a)^2 + y^2 = a^2$, we get $y = a$ as the y-coordinate of A.

For the point C, $x = 2a$; therefore from $y^2 = 2ax$, we get $y = 2a$ at C.

Now from the equation of the circle $(x - a)^2 + y^2 = a^2$, we have $x = a \pm \sqrt{(a^2 - y^2)}$ *i.e.*, x for the arc OA is given by $a - \sqrt{(a^2 - y^2)}$ and for the arc AB, x is given by $a + \sqrt{(a^2 - y^2)}$.

Now for the region OEA, x varies from $y^2/2a$ (which is the value of x on the arc OE of the parabola $y^2 = 2ax$) to $a - \sqrt{(a^2 - y^2)}$ which is the value of x on the arc OA of the circle and y varies from 0 to a. For the region ABD, x varies from the arc AB of the circle to the straight line BD (*i.e.*, x varies from $a + \sqrt{(a^2 - y^2)}$ to $2a$ and y varies from 0 to a.

And for the region ECD, x varies from the arc EC of the parabola to the straight line $x = 2a$ *i.e.*, x varies from $y^2/2a$ to $2a$ and y varies from a to $2a$.

Hence the transformed integral is

$$= \int_0^a \int_{y^2/2a}^{a - \sqrt{(a^2 - y^2)}} f(x, y)\, dy\, dx + \int_0^a \int_{a + \sqrt{(a^2 - y^2)}}^{2a} f(x, y)\, dy\, dx$$

$$+ \int_0^{2a} \int_{y^2/2a}^{2a} f(x, y)\, dy\, dx.$$

Problem 11: *Change the order of integration in the double integral*

$$\int_0^{ab/\sqrt{(a^2 + b^2)}} \int_0^{(a/b)\sqrt{(b^2 - y^2)}} f(x, y)\, dy\, dx.$$

Solution: In the given integral the limits of integration of x are given by $x = 0$ *i.e.*, the y-axis and $x = (a/b)\sqrt{(b^2 - y^2)}$ *i.e.*, $x^2/a^2 + y^2/b^2 = 1$

which is an ellipse with centre as origin. **(Note)**

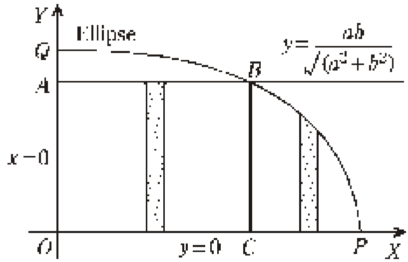
Again the limits of integration of y are given by $y = 0$ *i.e.*, the x-axis and $y = ab/\sqrt{(a^2 + b^2)}$ which is a straight line parallel to the x-axis at a distance $ab/\sqrt{(a^2 + b^2)}$ from the origin.

We draw the curves

$$x = 0 \;, x^2/a^2 + y^2/b^2 = 1 \;, y = 0$$

and $\quad y = ab/\sqrt{(a^2 + b^2)}$,

giving the limits of integration, in the same figure. We observe that the region of integration is the area $OPBAO$.



In the given integral the limits of integration of x are variable while those of y are constant. Thus we have to first integrate w.r.t. x regarding y as a constant and then we integrate w.r.t. y.

If we want to reverse the order of integration, we have to first integrate w.r.t. y regarding x as constant and then we integrate w.r.t. x. This is done by covering the area of integration $OPBAO$ by strips parallel to the y-axis. Now if we take strips parallel to the y-axis starting from the line $y = 0$, some of these strips end on the line AB while the others end on the arc BP of the ellipse. So we draw the line of demarcation BC dividing the area $OPBAO$ into two portions $OCBA$ and BCP. For the point B, $y = ab/\sqrt{(a^2 + b^2)}$. Putting this value of y in the equation of the ellipse $x^2/a^2 + y^2/b^2 = 1$, we get $x = ab/\sqrt{(a^2 + b^2)}$. For the region $OCBA$, y varies from 0 to $ab/\sqrt{(a^2 + b^2)}$ and x varies from 0 to $ab/\sqrt{(a^2 + b^2)}$.

For the region BCP, y varies from 0 to $(b/a)\sqrt{(a^2 - x^2)}$ and x varies from $ab/\sqrt{(a^2 + b^2)}$ to a.

Hence the given integral transforms to

$$\int_0^{ab/\sqrt{(a^2+b^2)}} \int_0^{ab/\sqrt{(a^2+b^2)}} f(x, y)\, dx\, dy + \int_{ab/\sqrt{(a^2+b^2)}}^{a} \int_0^{(b/a)\sqrt{(a^2 - x^2)}} f(x, y)\, dx\, dy \,.$$

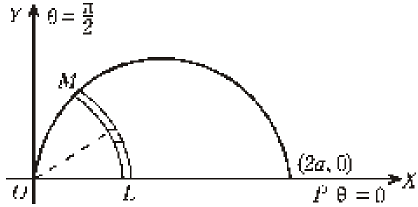
Problem 12: *Change the order of integration in* $\int_0^{\pi/2} \int_0^{2a\cos\theta} f(r, \theta)\, d\theta\, dr$.

(Kumaun 2009; Kumaun 11)

Solution: Here the region of integration is bounded by the polar curves $r = 0$ (the pole), $r = 2a\cos\theta$ (a circle of diameter $2a$ passing through the pole), $\theta = 0$ (the initial line) and $\theta = \pi/2$ (a line through the pole perpendicular to initial line).

We draw the curves $r = 0$, $r = 2a\cos\theta$, $\theta = 0$ and $\theta = \pi/2$, giving the limits of integration, in the same figure.

We observe that the region of integration is the area of the semi-circle *OMPO*.



In the given integral the limits of integration of r are variable while those of θ are constant. Thus we have to first integrate with respect to r regarding θ as a constant and then we integrate w.r.t. θ.

If we want to reverse the order of integration, we have to first integrate with respect to θ regarding r as constant and then we integrate w.r.t. r. This is done by covering the area of integration *OMPO* by circular arcs with centre as pole. On these arcs θ varies and r remains constant. Thus for the area *OMPO*, for a fixed value of r, θ varies from the initial line (*i.e.*, $\theta = 0$ a point on the arc *OMP* of the circle $r = 2a\cos\theta$ *i.e.*, to a point for which $\theta = \cos^{-1}(r/2a)$ and r varies from 0 to $2a$.

Hence by changing the order of integration, we have

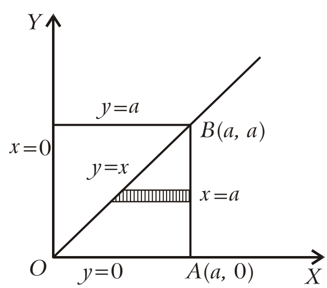
$$\int_0^{\pi/2} \int_0^{2a\cos\theta} f(r, \theta)\, d\theta\, dr = \int_0^{2a} \int_0^{\cos^{-1}(r/2a)} f(r, \theta)\, d\theta\, dr.$$

Problem 13: *Change the order of integration in the double integral* $\int_0^a \int_0^x \frac{\phi'(y)\, dx\, dy}{\sqrt{\{(a-x)(x-y)\}}}$ *and hence find its value.*

Solution: In the given integral the limits of integration are given by the lines $y = 0$, $y = x$, $x = 0$ and $x = a$. We observe that the region of integration is the area *OAB*. Now proceed as in Example 21. By changing the order of integration, we have

$$\int_0^a \int_0^x \frac{\phi'(y)\, dx\, dy}{\sqrt{\{(a-x)(x-y)\}}} = \int_0^a \int_y^a \frac{\phi'(y)\, dx\, dy}{\sqrt{\{(a-x)(x-y)\}}}$$

$$= \int_0^a \phi'(y)\, dy \int_y^a \frac{dx}{\sqrt{\{(a-x)(x-y)\}}} \quad \ldots(1)$$



Put $\quad x = a\cos^2\theta + y\sin^2\theta.$

Then $\quad a - x = a - a\cos^2\theta - y\sin^2\theta$

$$= a(1 - \cos^2\theta) - y\sin^2\theta$$

$$= (a - y)\sin^2\theta$$

and $\quad x - y = a\cos^2\theta + y\sin^2\theta - y$

$$= a\cos^2\theta + y(\sin^2\theta - 1)$$

$$= (a - y)\cos^2\theta$$

When $x = y$ then $x - y = 0$ *i.e.*, $0 = (a - y)\cos^2\theta$

or $\quad \cos^2\theta = 0$ or $\theta = \pi$.

When $x = a$ then $a - x = 0$ *i.e.*, $0 = (a - y)\sin^2\theta$ or $\sin^2\theta = 0$ or $\theta = 0$.

Now, $$\int_y^a \frac{dx}{\sqrt{\{(a-x)(x-y)\}}} = \int_{\pi/2}^{a} \frac{-2(a-y)\sin\theta\cos\theta\, d\theta}{(a-y)\sin\theta\cos\theta}$$
$$= 2\int_0^{\pi/2} d\theta = 2\,[\theta]_0^{\pi/2} = \pi .$$

Thus, $$\int_0^a \pi\, \phi'(y)\, dy = \pi\,[\phi(y)]_0^a = \pi\,[\phi(a) - \phi(0)].$$

Comprehensive Problems 5

Problem 1: *Transform* $\int_0^a \int_0^{a-x} f(x, y)\, dx\, dy$ *, by the substitution* $x + y = u,\ y = uv$.

Solution: As shown in Example 27.
$$\iint f(x, y)\, dx\, dy = \iint F(u, v)\, u\, du\, dv .$$ (Prove it here.)

Now in the given integral, the region of integration is bounded by the lines
$y = 0$, $y = a - x$, $x = 0$ and $x = a$.
Put $x = u - y = u - uv = u(1 - v)$ and $y = uv$.
Then in the uv-plane the four straight lines become
$uv = 0$, $uv = a - u(1 - v)$, $u(1 - v) = 0$ and $u(1 - v) = a$, giving
$v = 0$, $v = 1$, $u = 0$ and $u = a$.
Hence for the given region, v varies from 0 to 1 and u varies from 0 to a .
Therefore, by changing the variables, the given double integral transforms to
$$\int_0^a \int_0^1 F(u, v)\, u\, du\, dv .$$

Problem 2: *By using the transformation* $x + y = u$, $y = uv$ *, show that*
$$\int_0^1 \int_0^{1-x} e^{y/(x+y)}\, dx\, dy = \frac{1}{2}(e - 1) .$$

Solution: As proved in Example 27, we have $dx\, dy = u\, du\, dv$. (Prove it here.)

Here the region of integration is bounded by the lines
$y = 0$, $y = 1 - x$, $x = 0$ and $x = 1$.
Changing these equations to new variables u and v by using the relations
$x = u - y = u - uv = u(1 - v)$ and $y = uv$, we have
$uv = 0$, $uv = 1 - u(1 - v)$, $u(1 - v) = 0$ and $u(1 - v) = 1$,
giving $v = 0$, $v = 1$, $u = 0$ and $u = 1$.
Hence for the given region v varies from 0 to 1 and u varies from 0 to 1.
Further $e^{y/(x+y)} = e^{uv/u} = e^v$. [$\because\ x + y = u$, $y = uv$]
Therefore, changing the variables to u, v , the given integral becomes
$$= \int_0^1 \int_0^1 e^v .\, u\, du\, dv = \int_0^1 [e^v]_0^1\, u\, du = \int_0^1 (e^1 - e^0)\, u\, du$$
$$= (e - 1)\int_0^1 u\, du = (e - 1) . \left[\frac{u^2}{2}\right]_0^1 = \frac{1}{2}(e - 1) .$$

Problem 3: *By using the transformation* $x + y = u$, $y = uv$, *prove that*

$$\iint \{xy(1-x-y)\}^{1/2}\, dx\, dy$$

taken over the area of the triangle bounded by the lines $x = 0$, $y = 0$, $x + y = 1$ *is* $2\pi/105$.

Solution: As proved in problem 2, we have $dx\, dy = u\, du\, dv$; u varies from 0 to 1 and also v varies from 0 to 1.

Now $\{xy(1-x-y)\}^{1/2} = [xy\{1-(x+y)\}]^{1/2}$

$= [u(1-v).uv.(1-u)]^{1/2}$ [$\because$ $x = u(1-v)$, $y = uv$]

$= u(1-u)^{1/2}.v^{1/2}(1-v)^{1/2}$.

Hence the given double integral transforms to

$$\int_0^1 \int_0^1 u(1-u)^{1/2}.v^{1/2}(1-v)^{1/2}.u\, du\, dv$$

$$= \left[\int_0^1 u^2(1-u)^{1/2}\, du\right] \cdot \left[\int_0^1 v^{1/2}(1-v)^{1/2}\, dv\right]$$

$$= \left[\int_0^1 u^{3-1}(1-u)^{3/2-1}\, du\right] \cdot \left[\int_0^1 v^{3/2-1}(1-v)^{3/2-1} dv\right]$$

$= B(3, \frac{3}{2}).B(\frac{3}{2}, \frac{3}{2})$ [By the def. of Beta function]

$$= \frac{\Gamma(3)\,\Gamma(\frac{3}{2})}{\Gamma(3+\frac{3}{2})} \cdot \frac{\Gamma\frac{3}{2}.\Gamma\frac{3}{2}}{\Gamma(\frac{3}{2}+\frac{3}{2})} = \frac{2.[\frac{1}{2}\sqrt{\pi}]^3}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}.2} = \frac{2\pi}{105}.$$

Problem 4: *Evaluate* $\iint (x^2 + y^2)^{7/2}\, dx\, dy$ *over the circle* $x^2 + y^2 = 1$.

Solution: Here the region of integration is a circle. Therefore we shall change the given double integral to polar coordinates by putting $x = r\cos\theta$ and $y = r\sin\theta$. We have

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

$\therefore$ $dx\, dy = J\, d\theta\, dr = r\, d\theta\, dr$.

Clearly the region of integration is the circle $x^2 + y^2 = 1$ *i.e.*, the circle with centre $(0, 0)$ and radius 1.

Changing to polar coordinates, the region of integration is covered when r varies from 0 to 1 and θ varies from 0 to 2π.

$$\therefore \quad \iint_{x^2+y^2\le 1} (x^2 + y^2)^{7/2}\, dx\, dy = \int_0^{2\pi} \int_0^1 (r^2)^{7/2}\, J\, d\theta\, dr$$

$$= \int_0^{2\pi} \int_0^1 r^7.r.\, d\theta\, dr = \int_0^{2\pi} \int_0^1 r^8\, d\theta\, dr = \int_0^{2\pi} \left[\frac{r^9}{9}\right]_0^1 d\theta$$

$$= \frac{1}{9}\int_0^{2\pi} d\theta = \frac{1}{9}[\theta]_0^{2\pi} = \frac{2}{9}\pi.$$

Problem 5: *Evaluate* $\iint xy\,(x^2 + y^2)^{3/2}\,dx\,dy$ *over the positive quadrant of the circle* $x^2 + y^2 = 1$.

Solution: Changing to polars by putting $x = r\cos\theta$, $y = r\sin\theta$, we have $J = r$ so that

$$dx\,dy = J\,d\theta\,dr = r\,d\theta\,dr.$$

The given region of integration is the area lying in the positive quadrant of the circle $x^2 + y^2 = 1$.

Changing to polar coordinates, this region of integration is covered when r varies 0 to 1 and θ varies from 0 to $\pi/2$.

∴ The required integral

$$\iint xy\,(x^2 + y^2)^{3/2}\,dx\,dy = \int_0^{\pi/2}\int_0^1 r\cos\theta.\,r\sin\theta.\,(r^2)^{3/2}.\,r\,d\theta\,dr$$

$$= \int_0^{\pi/2}\int_0^1 r^6 \sin\theta\cos\theta\,d\theta\,dr = \int_0^{\pi/2}\left[\frac{r^7}{7}\right]_0^1 \sin\theta\cos\theta\,d\theta$$

$$= \frac{1}{7}\int_0^{\pi/2}\frac{1}{2}\sin 2\theta\,d\theta = \frac{1}{14}\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi/2} = -\frac{1}{28}[-1-1] = \frac{1}{14}.$$

Problem 6: *Evaluate* $\iint e^{-(x^2+y^2)}\,dx\,dy$ *over the circle* $x^2 + y^2 = a^2$.

Solution: Changing to polar coordinates, the equation $x^2 + y^2 = a^2$ transforms to

$$r^2\cos^2\theta + r^2\sin^2\theta = a^2 \text{ i.e., } r = a.$$

Hence for the given region r varies from 0 to a and θ varies from 0 to 2π.

Also $$dx\,dy = r\,d\theta\,dr.$$

∴ The required integral

$$= \iint e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\int_0^a e^{-r^2}\,r\,d\theta\,dr$$

$$= \int_0^{2\pi}\int_0^{a^2} e^{-t}.\frac{1}{2}\,d\theta\,dt, \quad \text{putting } r^2 = t \text{ so that } 2r\,dr = dt$$

$$= \frac{1}{2}\int_0^{2\pi}\left[\frac{e^{-t}}{-1}\right]_0^{a^2} d\theta = -\frac{1}{2}\int_0^{2\pi}(e^{-a^2} - 1)\,d\theta$$

$$= -\frac{1}{2}(e^{-a^2} - 1)[\theta]_0^{2\pi} = \frac{1}{2}(1 - e^{-a^2})\cdot 2\pi$$

$$= \pi(1 - e^{-a^2}).$$

Hints to Objective Type Questions

Multiple Choice Questions

1. We have $\int_{\theta=0}^{2\pi}\int_{r=0}^{a} r\,d\theta\,dr = \int_{\theta=0}^{2\pi}\left[\frac{r^2}{2}\right]_{r=0}^{a} d\theta = \frac{a^2}{2}\int_0^{2\pi} d\theta$

$$= \frac{a^2}{2}[\theta]_0^{2\pi} = \frac{a^2}{2}\cdot 2\pi = \pi a^2 .$$

2. We have $\int_0^1\int_0^1\int_0^1 xyz\,dx\,dy\,dz = \int_{x=0}^{1}\int_{y=0}^{1} xy\left[\frac{z^2}{2}\right]_{z=0}^{1} dx\,dy$

$$= \frac{1}{2}\int_0^1 x\left[\frac{y^2}{2}\right]_{y=0}^{1} dx = \frac{1}{2}\cdot\frac{1}{2}\int_0^1 x\,dx$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot\left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}.$$

3. We have $\int_0^a\int_0^{\sqrt{(a^2-y^2)}} dy\,dx = \int_{y=0}^{a}\left[\int_{x=0}^{\sqrt{(a^2-y^2)}} dx\right] dy = \int_0^a [x]_{x=0}^{\sqrt{(a^2-y^2)}}\,dy$

$$= \int_0^a \sqrt{(a^2-y^2)}\,dy$$

$$= \int_0^{\pi/2} a\cos t\cdot a\cos t\,dt, \quad \text{putting } y = a\sin t \text{ so that } dy = a\cos t\,dt$$

$$= a^2\int_0^{\pi/2}\cos^2 t\,dt = a^2\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi a^2}{4}.$$

4. We have $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2}\cos^2 x\cos^2 y\cos^2 z\,dx\,dy\,dz$

$$= \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2}\cos^2 x\cos^2 y\left[\int_0^{\pi/2}\cos^2 z\,dz\right]dx\,dy$$

$$= \int_{x=0}^{\pi/2}\int_{y=0}^{\pi/2}\cos^2 x\cos^2 y\cdot\left[\frac{1}{2}\cdot\frac{\pi}{2}\right]dx\,dy$$

$$= \frac{\pi}{4}\int_{x=0}^{\pi/2}\cos^2 x\left[\int_0^{\pi/2}\cos^2 y\,dy\right]dx$$

$$= \frac{\pi}{4}\int_0^{\pi/2}\cos^2 x\cdot\left[\frac{1}{2}\cdot\frac{\pi}{2}\right]dx$$

$$= \frac{\pi}{4}\cdot\frac{\pi}{4}\int_0^{\pi/2}\cos^2 x\,dx = \frac{\pi}{4}\cdot\frac{\pi}{4}\cdot\frac{\pi}{4} = \frac{\pi^3}{64}.$$

5. We have $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z}\, dx\, dy\, dz = \int_{x=0}^1 \int_{y=0}^1 e^x \cdot e^y \left[\int_{z=0}^1 e^z\, dz\right] dx\, dy$

$$= \int_{x=0}^1 \int_{y=0}^1 \int_0^1 e^x \cdot e^y\, [e^z]_{z=0}^1\, dx\, dy$$

$$= \int_{x=0}^1 \int_{y=0}^1 e^x \cdot e^y\, (e-1)\, dx\, dy$$

$$= (e-1) \int_{x=0}^1 e^x [e^y]_{y=0}^1\, dx$$

$$= (e-1)^2 \int_0^1 e^x\, dx = (e-1)^2\, [e^x]_0^1 = (e-1)^3.$$

6. We have $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \cos x \cos y \cos z\, dx\, dy\, dz$

$$= \left[\int_0^{\pi/2} \cos x\, dx\right]\left[\int_0^{\pi/2} \cos y\, dy\right]\left[\int_0^{\pi/2} \cos z\, dz\right] = 1 \cdot 1 \cdot 1 = 1.$$

7. See Problem 1(i), of Comprehensive Problems 2.

8. We have $\int_0^2 \int_0^2 (x^2 + y^2)\, dx\, dy = \int_0^2 \left[x^2 y + \frac{y^3}{3}\right]_0^2 dx$

$$= \int_0^2 \left[2x^2 + \frac{8}{3}\right] dx = \left[\frac{2x^3}{3} + \frac{8x}{3}\right]_0^2 = \frac{16}{3} + \frac{16}{3} = \frac{32}{3}.$$

9. See Problem 1(ii), of Comprehensive Problems 1.

10. See Problem 1(ii), of Comprehensive Problems 2.

11. We have $\int_0^{\pi} \int_0^{x} \sin y\, dy\, dx = \int_0^{\pi} (-\cos y)_0^x\, dx = \int_0^{\pi} (-\cos x + \cos 0)\, dx$

$$= \int_0^{\pi} (1 - \cos x)\, dx = (x - \sin x)_0^{\pi} = \pi.$$

12. See Example 1(ii).
13. See Example 4.
14. See Example 11.
15. See Problem 7(i) of Comprehensive Problems 2.
16. See Example 16(i).

Fill in the Blanks

1. We have $\int_0^3 \int_1^2 dx\, dy = \int_0^3 \left[\int_1^2 dy\right] dx = \int_0^3 [y]_1^2\, dx$

$$= \int_0^3 dx = [x]_0^3 = 3.$$

2. We have $\int_0^1 \int_0^1 xy\, dx\, dy = \int_{x=0}^1 x \left[\int_{y=0}^1 y\, dy\right] dx$

$$= \int_0^1 x \left[\frac{y^2}{2} \right]_{y=0}^{1} dx = \frac{1}{2} \int_0^1 x \, dx = \frac{1}{2} \cdot \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

3. We have $\int_0^1 \int_0^x xy \, dx \, dy = \int_{x=0}^1 x \left[\int_{y=0}^1 y \, dy \right] dx$

$$= \int_0^1 x \left[\frac{y^2}{2} \right]_{y=0}^{x} dx = \int_{x=0}^1 \frac{x^3}{2} dx = \frac{1}{2} \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$

4. We have $\int_0^{\pi/2} \int_0^{2a\cos\theta} r \, d\theta \, dr = \int_{\theta=0}^{\pi/2} \left[\int_{r=0}^{2a\cos\theta} r \, dr \right] d\theta$

$$= \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_{r=0}^{2a\cos\theta} d\theta = \int_0^{\pi/2} 2a^2 \cos^2\theta \, d\theta$$

$$= 2a^2 \int_{\theta=0}^{\pi/2} \cos^2\theta \, d\theta = 2a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi a^2}{2}.$$

5. We have $\int_0^2 \int_0^2 \int_0^2 xyz \, dx \, dy \, dz = \int_{x=0}^2 \int_{y=0}^2 xy \left[\int_{z=0}^2 z \, dz \right] dx \, dy$

$$= \int_{x=0}^2 \int_{y=0}^2 xy \cdot \left[\frac{z^2}{2} \right]_{z=0}^2 dx \, dy = 2 \int_{x=0}^2 x \left[\int_{y=0}^2 y \, dy \right] dx$$

$$= 2 \int_0^2 x \left[\frac{y^2}{2} \right]_{y=0}^2 dx = 4 \int_0^2 x \, dx = 4 \left[\frac{x^2}{2} \right]_0^2 = 8.$$

6. We have $\int_1^2 \int_1^2 \int_1^3 dx \, dy \, dz = \int_{x=1}^2 \int_{y=1}^2 \left[\int_{z=1}^3 dz \right] dx \, dy$

$$= \int_{x=1}^2 \int_{y=1}^2 [z]_{z=1}^3 \, dx \, dy = 2 \int_{x=1}^2 \left[\int_{y=1}^2 dy \right] dx$$

$$= 2 \int_1^2 [y]_{y=1}^2 \, dx = 2 \int_1^2 dx = 2 [x]_1^2 = 2 \cdot (2-1) = 2.$$

7. We have $\int_{-a}^a \int_0^{\sqrt{(a^2-x^2)}} dx \, dy = \int_{x=-a}^a \left[\int_{y=0}^{\sqrt{(a^2-x^2)}} dy \right] dx$

$$= \int_{-a}^a [y]_{y=0}^{\sqrt{(a^2-x^2)}} dx = \int_{-a}^a \sqrt{(a^2-x^2)} \, dx$$

$$= 2\int_0^a \sqrt{(a^2-x^2)} \, dx = 2 \left[\frac{x}{2} \sqrt{(a^2-x^2)} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a$$

$$= 2 \cdot \frac{a^2}{2} \cdot \frac{\pi}{2} = \pi \frac{a^2}{2}.$$

True or False

1. We have $\int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{a} r\, d\theta\, dr = \int_{-\pi/2}^{\pi/2} \left[\frac{r^2}{2}\right]_{r=0}^{a} d\theta$

$$= \frac{a^2}{2} \int_{-\pi/2}^{\pi/2} d\theta = \frac{a^2}{2} [\theta]_{-\pi/2}^{\pi/2} = \frac{a^2}{2} \cdot \left[\frac{\pi}{2} + \frac{\pi}{2}\right] = \frac{\pi a^2}{2}.$$

2. We have $\int_{-a}^{a} \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} dx\, dy = \int_{-a}^{a} [y]_{y=-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} dx$

$$= 2 \int_{-a}^{a} \sqrt{(a^2 - x^2)}\, dx = 2 \,.\, 2 \int_{0}^{a} \sqrt{(a^2 - x^2)}\, dx$$

$$= 4 \left[\frac{x}{2} \sqrt{(a^2 - x^2)} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}\right]_0^a = 4 \cdot \frac{a^2}{2} \cdot \frac{\pi}{2} = \pi a^2.$$

3. We have $\int_{-a}^{a} \int_{0}^{\sqrt{(a^2-x^2)}} x\, dx\, dy = \int_{x=-a}^{a} x\, [y]_{y=0}^{\sqrt{(a^2-x^2)}} dx$

$$= \int_{-a}^{a} x\sqrt{(a^2 - x^2)}\, dx = 0 \qquad [\because x\sqrt{(a^2 - x^2)} \text{ is an odd function of } x]$$

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Chapter-7

Areas of Curves

Comprehensive Problems 1

Problem 1: *Find the area bounded by the axis of* x, *and the following curves and the given ordinates :*

(i) $y = \log x\,;\, x = a\,,\, x = b\ (b > a > 1)$. **(Kanpur 2015)**

(ii) $xy = c^2\,;\, x = a\,,\, x = b\,,(a > b > 0)$. **(Kashi 2012)**

Solution: **(i)** The required area $= \int_a^b y\,dx = \int_a^b \log x\,dx$

$$= \Big[(\log x)\,.\,x\Big]_a^b - \int_a^b \frac{1}{x}\,.\,x\,dx, \text{ integrating by parts}$$

$$= [b\log b - a\log a] - [b - a] = (b\log b - b) - (a\log a - a)$$

$$= b\log(b/e) - a\log(a/e). \qquad [\because \log e = 1]$$

(ii) The required area $= \int_b^a y\,dx = \int_b^a (c^2/x)\,dx$, $\qquad [\because y = c^2/x]$

$$= c^2.\Big[\log x\Big]_b^a = c^2(\log a - \log b) = c^2\log(a/b).$$

Problem 2(i): *Find the area bounded by the curve* $y = x^3$, *the* y*-axis and the lines* $y = 1$ *and* $y = 8$.

Solution: Here the curve is bounded between the axis of y and the lines parallel to x-axis.

Hence the required area $= \int_{y=1}^{8} x\,dy$ **(Note)**

$$= \int_1^8 y^{1/3}\,dy, \qquad [\because x^3 = y \text{ or } x = y^{1/3}]$$

$$= \left[\frac{3}{4}y^{4/3}\right]_1^8 = \frac{3}{4}[8^{4/3} - 1^{4/3}] = \frac{3}{4}[2^4 - 1] = \frac{45}{4}.$$

Problem 2(ii): *Show that the area cut off a parabola by any double ordinate is two third of the corresponding rectangle contained by that double ordinate and its distance from the vertex.*

Solution: Let the parabola be $y^2 = 4ax$. Also let $x = b$ be any double ordinate. Since the curve is symmetrical about x-axis, therefore the area cut off the parabola $y^2 = 4ax$ by the double ordinate $x = b$ is $2 \times$ (area included between the x-axis, $x = b$ and curve in the +ive quadrant).

$\therefore$ The required area $= 2 \int_{x=0}^{b} y \, dx = 2 \int_0^b \sqrt{(4ax)} \, dx$, $\quad [\because y^2 = 4ax]$

$$= 4\sqrt{(a)} \left[\frac{2}{3} x^{3/2}\right]_0^b = \frac{8}{3} \sqrt{(a)} . b^{3/2}. \qquad ...(1)$$

Again, at $x = b$, from $y^2 = 4ax$, we have

$$y^2 = 4ab \quad \text{or} \quad y = 2\sqrt{(ab)}.$$

$\therefore$ Length of the double ordinate $= 2y = 2.2\sqrt{(ab)} = 4\sqrt{(ab)}$.

Now area of the rectangle contained by the double ordinate and its distance from the vertex

$$= 2y \times x = 2y . b = 4\sqrt{(ab)} . b = 4a^{1/2} b^{3/2}.$$

Its two thirds $= \frac{2}{3} . 4a^{1/2} b^{3/2} = \frac{8}{3} a^{1/2} b^{3/2}$

= area cut off a parabola by the double ordinate, from (1).

Problem 3(i): *Find the area of the quadrant of an ellipse* $(x^2/a^2) + (y^2/b^2) = 1$.

(Agra 2000; Bundelkhand 10; Kanpur 11)

Solution: See Fig. of Ex. 1 after article 2. Here the required area (*i.e.*, the area of a quadrant) lies between the limits $x = 0$ and $x = a$.

$\therefore$ Area of the quadrant $= \int_0^a y \, dx = \frac{b}{a} \int_0^a \sqrt{(a^2 - x^2)} \, dx$, from (1) of Ex. 1

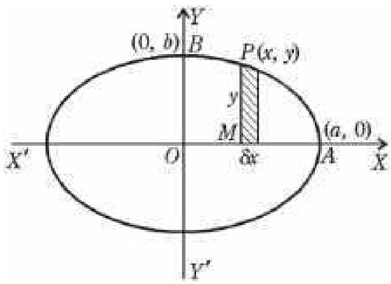
$$= \frac{b}{a} \left[\frac{1}{2} x\sqrt{(a^2 - x^2)} + \frac{1}{2} a^2 \sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{b}{a} \left[\frac{1}{2}(0 - 0) + \frac{1}{2} a^2 \left(\frac{1}{2}\pi - 0\right)\right] = \frac{\pi ab}{4}.$$

Problem 3(ii): *Find the whole area of the ellipse* $(x^2/a^2) + (y^2/b^2) = 1$.

(Avadh 2010; Rohilkhand 10B; Kumaun 12)

Solution: Clearly the area of an ellipse is 4 times the area of a quadrant.



$\therefore$ The required area of the ellipse

$$= 4 \int_0^a y \, dx = 4 . \frac{\pi ab}{4},$$

from problem 3(i)

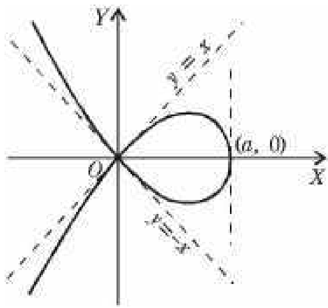
$$= \pi ab.$$

Problem 4(i): *Trace the curve $ay^2 = x^2 (a - x)$ and show that the area of its loop is $8a^2/15$.*

(Avadh 2008)

Solution: Tracing. (i) The curve is symmetrical about x-axis and passes through the origin.

(ii) Equating to zero the lowest degree terms in the equation of the curve, we get $y^2 - x^2 = 0$ *i.e.*, $y = \pm x$ as the tangents at origin and these being real and distinct the node is expected at the origin.



(iii) At $y = 0$, we get $x = 0$ and $x = a$ *i.e.*, the curve crosses the x-axis at $(0, 0)$ and $(a, 0)$. Also when $x > a$, y^2 is negative, *i.e.*, y is imaginary. Hence the curve does not exist for values of $x > a$. Also as x decreases from 0 to $-\infty$, y increases from 0 to ∞.

(iv) No asymptotes. Thus the shape of the curve is as shown in the figure. Clearly the loop is formed between $x = 0$ and $x = a$.

$\therefore$ Required area of the loop $= 2\int_0^a y\,dx$, [$\because$ Curve is symmetrical about x-axis]

$$= 2\int_0^a \frac{x\sqrt{(a-x)}}{\sqrt{a}}\,dx, \text{ putting for } y \text{ from the equation of the curve}$$

$$= 2\int_0^{\pi/2} \frac{a\sin^2\theta\sqrt{(a)}.\cos\theta}{\sqrt{a}}\,2a\sin\theta\cos\theta\,d\theta,$$

[Putting $x = a\sin^2\theta$, $dx = 2a\sin\theta\cos\theta\,d\theta$]

$$= 4a^2\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta = 4a^2\,\frac{\Gamma 2.\Gamma\frac{3}{2}}{2\,\Gamma\frac{7}{2}} = 4a^2.\frac{\frac{1}{2}\sqrt{\pi}}{2.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}} = \frac{8}{15}a^2.$$

Problem 4(ii): *Find the area of the loop of the curve $3ay^2 = x(x - a)^2$.*

Solution: The curve is symmetrical about x-axis. Putting $y = 0$, we get $x = 0$ and $x = a$, *i.e.*, the loop is formed between $x = 0$ and $x = a$.

$\therefore$ Required area of the loop $= 2\int_0^a y\,dx$, [$\because$ Curve is symmetrical about x-axis]

Now for the portion of the loop lying in the first quadrant, y is +ive and x lies between 0 and a. Therefore for this portion of the loop, we have $y = \{1/\sqrt{(3a)}\}.\sqrt{(x)}.(a - x)$.

$\therefore$ Required area of the loop

$$= 2\int_0^a \frac{(a-x).\sqrt{x}}{\sqrt{(3a)}}, \quad \text{putting for } y \text{ from the given equation of the curve}$$

$$= \frac{2}{\sqrt{(3a)}}\int_0^a (ax^{1/2} - x^{3/2})\,dx$$

$$= \frac{2}{\sqrt{(3a)}}\left[\frac{2}{3}ax^{3/2} - \frac{2}{5}x^{5/2}\right]_0^a = \frac{8a^2}{15/\sqrt{3}}.$$

Problem 4(iii): *Find the area of the loop of the curve* $y^2 = x(x-1)^2$.

Solution: Here also the curve is symmetrical about x-axis. Putting $y = 0$, we get $x = 0$ and $x = 1$ *i.e.*, the loop is formed between $x = 0$ and $x = 1$.

$\therefore$ Required area of the loop $= 2\int_0^1 y\, dx$

$$= 2\int_0^1 (1-x).\sqrt{x}\, dx, \text{ putting for } y \text{ from the equation of the curve.}$$

[Note that we have taken, $y = \sqrt{(x)}.(1-x)$]

$$= 2\int_0^1 (x^{1/2} - x^{3/2})\, dx = 2\left[\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^1 = 2\left[\frac{2}{3} - \frac{2}{5}\right] = \frac{8}{15}.$$

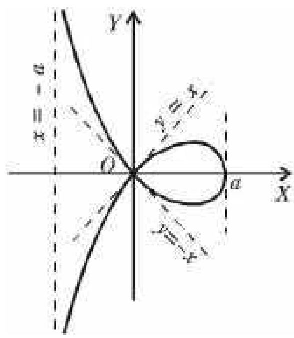
Problem 5: *Find the area (i) of the loop of the curve*

$$x(x^2 + y^2) = a(x^2 - y^2) \text{ or } y^2(a + x) = x^2(a - x).$$

(ii) of the portion bounded by the curve and its asymptotes.

(Garhwal 2000, 02; Meerut 04)

Solution: The curve is symmetrical about x-axis. The tangents at origin are $a(y^2 - x^2) = 0$ *i.e.*, $y = \pm x$. Since there are two real and distinct tangents at the origin, therefore the origin is a node on the curve. Putting $y = 0$, we get $x = 0$ and $x = a$ *i.e.*, the loop is formed between $x = 0$ and $x = a$.



$\therefore$ Required area of the loop

$$= 2\int_0^a y\, dx, \text{ by symmetry}$$

$$= 2\int_0^a x\sqrt{\left(\frac{a-x}{a+x}\right)}\, dx, \text{ putting for } y \text{ from the given equation of the curve}$$

$$= 2\int_0^a \frac{x(a-x)}{\sqrt{(a^2-x^2)}}\, dx, \text{ multiplying the numerator and the denominator by } \sqrt{(a-x)}$$

$$= 2\int_0^{\pi/2} \frac{a\sin\theta\,(a - a\sin\theta)}{\sqrt{(a^2 - a^2\sin^2\theta)}}.\, a\cos\theta\, d\theta,$$

putting $x = a\sin\theta$ so that $dx = a\cos\theta\, d\theta$

$$= 2a^2\int_0^{\pi/2} \sin\theta\,(1-\sin\theta)\, d\theta = 2a^2\left[\int_0^{\pi/2} \sin\theta\, d\theta - \int_0^{\pi/2} \sin^2\theta\, d\theta\right]$$

$$= 2a^2\left[1 - \frac{1}{2}.\frac{1}{2}\pi\right], \text{ by Walli's formula}$$

$$= 2a^2\left(1 - \frac{1}{4}\pi\right) = \frac{1}{2}a^2(4-\pi).$$

(ii) The line $x = -a$ is the asymptote of the curve.

Now the area lying between the curve and its asymptote

$= 2 \int_{-a}^{0} y \, dx,$

the value of y to be put from the equation of the curve

$= 2 \int_{-a}^{0} - x \sqrt{\left(\frac{a - x}{a + x}\right)} \, dx,$ [Note that for the arc of the curve lying in the second quadrant x is –ve and y is +ive so that $y = - x \sqrt{\{(a - x)/(a + x)\}}$ for this arc.]

$= 2 \int_{-a}^{0} \frac{- x (a - x)}{\sqrt{(a^2 - x^2)}} \, dx$

$= 2 \int_{\pi/2}^{0} \frac{- (- a \sin \theta)(a + a \sin \theta)}{\sqrt{(a^2 - a^2 \sin^2 \theta)}} \cdot (- a \cos \theta) \, d\theta,$

putting $x = - a \sin \theta$ so that $dx = - a \cos \theta \, d\theta$

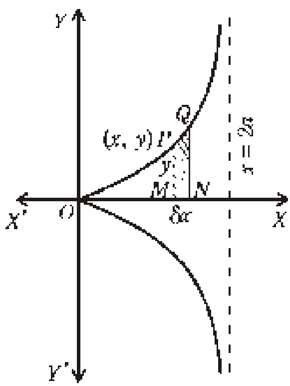
$= - 2a^2 \int_{\pi/2}^{0} \sin \theta (1 + \sin \theta) \, d\theta = 2a^2 \int_{0}^{\pi/2} (\sin \theta + \sin^2 \theta) \, d\theta$

$= 2a^2 \left[1 + \frac{1}{2} \cdot \frac{1}{2} \pi\right]$, by Walli's formula.

$= 2a^2 \left(1 + \frac{1}{4} \pi\right) = \frac{1}{2} a^2 (4 + \pi).$

Problem 6(i): *Trace the curve $y^2 (2a - x) = x^3$ and find the entire area between the curve and its asymptotes.* **(Avadh 2011)**

Solution: **Tracing of the curve $y^2 (2a - x) = x^3$.**



(i) Since in the equation of the curve the powers of y that occur are all even, therefore the curve is symmetrical about the axis of x.

(ii) The curve passes through the origin. Equating to zero the lowest degree terms in the equation of the curve, we get the tangents at the origin as $2ay^2 = 0$ *i.e.*, $y = 0$, $y = 0$ are two coincident tangents at the origin. Therefore the origin may be a cusp.

(iii) The curve cuts the coordinate axes only at the origin.

(iv) Solving the equation of the curve for y, we get

$$y^2 = \frac{x^3}{2a - x}.$$

When $x = 0$, $y^2 = 0$.

When $x \to 2a$, $y^2 \to \infty$. Therefore $x = 2a$ is an asymptote of the curve.

When $0 < x < 2a$, y^2 is +ive *i.e.*, y is real. Therefore the curve exists in this region.

When $x > 2a$, y^2 is –ive *i.e.*, y is imaginary. Therefore the curve does not exist in the region $x > 2a$.

When $x < 0$, y^2 is –ive. Therefore the curve does not exist in the region $x < 0$.
Combining all these facts, we see that the shape of the curve is as shown in the figure.
Now the required area = 2 × area in the first quadrant

$$= 2\int_0^{2a} y\,dx = 2\int_0^{2a} \frac{x^{3/2}}{\sqrt{(2a-x)}}\,dx. \qquad \left[\because\ y^2 = \frac{x^3}{2a-x}\right]$$

Now put $x = 2a\sin^2\theta$, so that $dx = 4a\sin\theta\cos\theta\,d\theta$.

$\therefore$ The required area $= 2\int_0^{\pi/2} \frac{(2a\sin^2\theta)^{3/2}}{\sqrt{(2a-2a\sin^2\theta)}}\cdot 4a\sin\theta\cos\theta\,d\theta$

$$= 16a^2\int_0^{\pi/2}\frac{\sin^3\theta}{\cos\theta}\sin\theta\cos\theta\,d\theta = 16a^2\int_0^{\pi/2}\sin^4\theta\,d\theta$$

$$= 16a^2 \cdot \frac{3\,.\,1}{4\,.\,2}\cdot\frac{\pi}{2}, \text{ by Walli's formula} = 3\,\pi a^2.$$

Problem 6(ii): *Find the area between the curve $y^2(4-x) = x^2$ and its asymptote.*

(Avadh 2012; Kanpur 14; Bundelkhand 14)

Solution: The curve is symmetrical about the x-axis. It cuts the x-axis at $x = 0$ *i.e.*, at the origin. The straight line $x = 4$ is the asymptote of the curve.

$\therefore$ Required area $= 2\int_0^4 y\,dx = 2\int_0^4 \sqrt{\left(\frac{x^2}{4-x}\right)}\,dx,$

putting for y from the equation of the curve

$$= 2\int_0^4 \frac{x}{\sqrt{(4-x)}}\,dx = 2\int_0^{\pi/2}\frac{4\sin^2\theta}{\sqrt{(4-4\sin^2\theta)}}\cdot 8\sin\theta\cos\theta\,d\theta,$$

putting $x = 4\sin^2\theta$ so that $dx = 8\sin\theta\cos\theta\,d\theta$

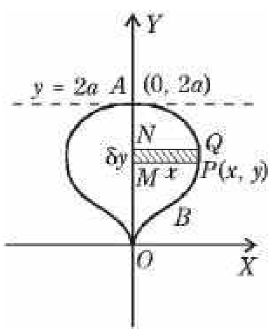
$$= 32\int_0^{\pi/2}\sin^3\theta\,d\theta = 32\cdot\frac{2}{3.1} \qquad \text{[By Walli's formula]}$$

$$= \frac{64}{3} \text{ units of area.}$$

Problem 6(iii): *Find the whole area of the curve $a^2x^2 = y^3(2a-y)$.*

Solution: The given curve is $a^2x^2 = y^3(2a-y)$. ...(1)

It is symmetrical about y-axis and it cuts the y-axis at the points $(0, 0)$ and $(0, 2a)$. The curve does not exist for $y > 2a$ and $y < 0$.



$\therefore$ The required area = 2 × area OBA

$$= 2\int_0^{2a} x\,dy = 2\int_0^{2a}\frac{y^{3/2}\sqrt{(2a-y)}}{a}\,dy,$$

from (1).

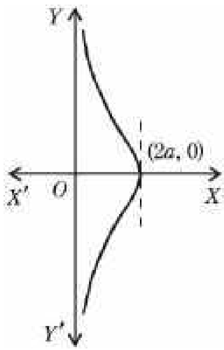
Putting $y = 2a\sin^2\theta$ and proceeding as in Example 4, after article 2, we get the required area $= \pi a^2$.

(Note that this is also the area of a circle of radius a).

Problem 7(i): *Find the area bounded by the curve $xy^2 = 4a^2 (2a - x)$ and its asymptote.*

(Rohilkhand 2009B)

Solution: The given curve is symmetrical about the x-axis and cuts the x-axis at the point $(2a, 0)$.



Equating to zero the coefficient of highest power of y in the equation of the curve we get $x = 0$ *i.e.*, the y-axis as the asymptote of the curve parallel to y-axis.

Hence the required area

$$= 2 \int_0^{2a} y \, dx = 2 \int_0^{2a} \frac{2a \sqrt{(2a - x)}}{\sqrt{x}} dx ,$$

[$\because$ from the given equation of the curve, $y^2 = 4a^2 (2a - x)/x$].

Putting $x = 2a \sin^2 \theta$ so that $dx = 4a \sin \theta \cos \theta \, d\theta$, we get the required area

$$= 4a \int_0^{\pi/2} \frac{\sqrt{(2a)} \cos \theta . 4a \sin \theta \cos \theta \, d\theta}{\sqrt{(2a)} . \sin \theta}$$

$$= 16a^2 \int_0^{\pi/2} \cos^2 \theta \, d\theta$$

$$= 16a^2 . \frac{1}{2} . \frac{1}{2} \pi \text{, by Walli's formula} = 4\pi a^2.$$

Problem 7(ii): *Find the area enclosed by the curve $xy^2 = a^2 (a - x)$ and y-axis.*

Solution: Proceed exactly as in Problem 7(i). Here also y-axis is the asymptote and in place of $2a$ we have a *i.e.*, replace $a/2$ in place of a.

The required area $= 4\pi . (a/2)^2 = \pi a^2$.

Problem 7(iii): *Trace the curve $a^2 y^2 = a^2 x^2 - x^4$ and find the whole area within it.*

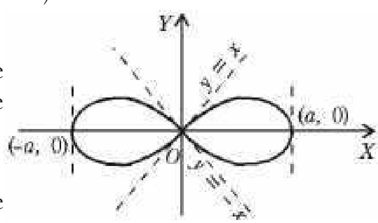
(Rohilkhand 2012; Avadh 12; Bundelkhand 14)

Solution: The given curve is $a^2 y^2 = x^2 (a^2 - x^2)$. Since in the equation of the curve, the powers of x and y are all even, therefore the curve is symmetrical about both the axes.

The curve passes through the origin and the tangents at the origin are

$$a^2 y^2 - a^2 x^2 = 0 \quad i.e., \quad a^2 (y^2 - x^2) = 0$$

i.e., $\quad y^2 - x^2 = 0 \quad i.e., \quad y = \pm x$.



The curve cuts the axis of x where $y = 0$ *i.e.*, where $x^2 (a^2 - x^2) = 0$ or $x = 0, \pm a$. Therefore the curve cuts the x-axis at $(0, 0)$, $(a, 0)$ and $(-a, 0)$.

The curve intersects the y-axis only at the origin.

Tangent at $(a, 0)$. Shifting the origin to the point $(a, 0)$ the equation of the curve becomes

$$a^2 y^2 = (x + a)^2 \{a^2 - (x + a)^2\} = (x + a)^2 (-x^2 - 2ax).$$

Equating to zero the lowest degree terms, we get $x = 0$ as the tangent at the new origin. Thus new y-axis is tangent at the new origin. Solving the equation of the curve for y ,we get

$$y^2 = \frac{x^2 (a^2 - x^2)}{a^2}.$$

When $\quad x = 0, \quad y^2 = 0$.

When $\quad x = a, \quad y^2 = 0$.

When $\quad 0 < x < a, \quad y^2$ is +ive.

Therefore the curve exists in the region $0 < x < a$.

When $x > a$, y^2 is – ive. Therefore the curve does not exist in the region $x > a$.

Hence the curve is as shown in the figure and it consists of two equal loops.

By symmetry, the whole area within the curve

$$= 4 \times \text{area of half a loop} = 4 \int_0^a y \, dx = 4 \int_0^a \frac{x \sqrt{(a^2 - x^2)}}{a} dx,$$

putting for y from the given equation of the curve

$$= 4 \int_0^{\pi/2} \sin\theta \,.\, a \cos\theta \,.\, a \cos\theta \, d\theta,$$

putting $x = a \sin\theta$ so that $dx = a \cos\theta \, d\theta$

$$= 4a^2 \int_0^{\pi/2} \cos^2\theta \sin\theta \, d\theta = 4a^2 \cdot \frac{1}{3.1} \cdot 1, \text{ by Walli's formula} = \frac{4}{3} a^2.$$

Problem 8(i): *Prove that the area of a loop of the curve* $a^4 y^2 = x^4 (a^2 - x^2)$ *is* $\pi a^2/8$.

Solution: The curve is symmetrical about both the axes. Putting $y = 0$ in the given equation of the curve, we get $x^4 (a^2 - x^2) = 0$ *i.e.*, $x = 0$, $x = \pm a$. Thus the above curve will have a loop between $x = 0$ and $x = a$. By symmetry, the area of a loop

$$= 2 \int_0^a y \, dx = 2 \int_0^a \frac{x^2 \sqrt{(a^2 - x^2)}}{a^2} dx,$$

putting for y from the given equation of the curve

$$= \frac{2}{a^2} \int_0^{\pi/2} a^2 \sin^2\theta \sqrt{(a^2 - a^2 \sin^2\theta)} \,.\, a \cos\theta \, d\theta,$$

putting $x = a \sin\theta$ so that $dx = a \cos\theta \, d\theta$

$$= 2a^2 \int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta$$

$$= 2a^2 \cdot \frac{1.1}{4.2} \cdot \frac{1}{2} \pi \qquad \text{[By Walli's formula]}$$

$$= \pi a^2/8.$$

Problem 8(ii): *Show that the whole area of the curve* $a^4 y^2 = x^5 (2a - x)$ *is to that of the circle whose radius is a, as 5 to 4.* **(Kanpur 2010)**

Solution: The given curve is symmetrical about x-axis. It passes through the origin and the tangents at the origin are $a^4 \; y^2 = 0$ *i.e.*, $y^2 = 0$ *i.e.*, $y = 0$, $y = 0$.

The curve cuts the x-axis at the points $(0, 0)$ and $(2a, 0)$. It intersects the y-axis only at the origin. When $0 < x < 2a$, y^2 is +ive so that the curve exists in this region. When $x > 2a$, y^2 is –ive so that the curve does not exist in this region. When $x < 0$, y^2 is –ive so that the curve does not exist in this region.

Thus the given curve consists of a loop lying between $x = 0$ and $x = 2a$. Hence the whole area of this curve

$$= 2 \int_0^{2a} y \, dx = 2 \int_0^{2a} \frac{x^{5/2} \, (2a - x)^{1/2}}{a^2} \, dx ,$$

putting for y from the given equation of the curve

$$= 2 \int_0^{\pi/2} \frac{(2a)^{5/2} \sin^5 \theta \, . (2a)^{1/2} \cos \theta \, . 4a \sin \theta \cos \theta \, d\theta}{a^2} ,$$

putting $x = 2a \sin^2 \theta$ so that $dx = 2a \, . 2 \sin \theta \cos \theta \, d\theta$

$$= 64a^2 \int_0^{\pi/2} \sin^6 \theta \cos^2 \theta \, d\theta = 64a^2 \, \frac{\Gamma\left(\frac{7}{2}\right) \Gamma\left(\frac{3}{2}\right)}{2\Gamma(5)}$$

$$= 64a^2 \, \frac{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} \, . \frac{1}{2} \sqrt{\pi}}{2 \, . 4 \, . 3 \, . 2 \, . 1} = \frac{5 \pi a^2}{4} .$$

Also the area of the circle of radius a is πa^2.

$$\therefore \qquad \frac{\text{Area of the curve}}{\text{Area of the circle}} = \frac{5 \pi a^2 / 4}{\pi a^2} = \frac{5}{4} .$$

Problem 9(i): *Find the area between the curve $y^2 (a - x) = x^3$ (**cissoid**) and its asymptotes. Also find the ratio in which the ordinate $x = a/2$ divides the area.* **(Agra 2001, 03)**

Solution: The figure of the curve is similar to problem 6(i). Equating to zero the coefficient of the highest power of y , we get $a - x = 0$ *i.e.*, $x = a$ as an asymptote of the curve parallel to the y-axis.

Let A be the whole area between the curve and its asymptote. Then

$$A = 2 \times (\text{area in the first quadrant}) = 2 \int_0^a y \, dx = 2 \int_0^a \frac{x^{3/2}}{\sqrt{(a - x)}} \, dx ,$$

putting for y from the given equation of the curve

$$= 2 \int_0^{\pi/2} \frac{a^{3/2} \sin^3 \theta \, . 2a \sin \theta \cos \theta}{\sqrt{(a)} \, . \cos \theta} \, d\theta ,$$

putting $x = a \sin^2 \theta$ so that $dx = a \, . 2 \sin \theta \cos \theta \, d\theta$

$$= 4a^2 \int_0^{\pi/2} \sin^4 \theta \, d\theta = 4a^2 \, . \frac{3 \, . 1}{4 \, . 2} \cdot \frac{\pi}{2} = \frac{3 \pi a^2}{4} . \qquad ...(1)$$

Now let A_1 be the area of the portion between $x = 0$ and $x = \frac{1}{2} a$. Then

$$A_1 = 2\int_0^{a/2} y\,dx = 2\int_0^{a/2} \frac{x^{3/2}}{\sqrt{(a-x)}}\,dx,$$

putting for y from the given equation of the curve

$$= 4a^2 \int_0^{\pi/4} \sin^4\theta\,d\theta, \text{ putting } x = a\sin^2\theta, \text{ etc.}$$

$$= a^2\int_0^{\pi/4} (2\sin^2\theta)^2\,d\theta = a^2\int_0^{\pi/4} (1-\cos 2\theta)^2\,d\theta \qquad \textbf{(Note)}$$

$$= \frac{1}{2}a^2\int_0^{\pi/2} (1-\cos\phi)^2\,d\phi,$$

[putting $2\theta = \phi$ so that $2\,d\theta = d\phi$ and adjusting the limits]

$$= \frac{1}{2}a^2\int_0^{\pi/2} (1 - 2\cos\phi + \cos^2\phi)\,d\phi$$

$$= \frac{1}{2}a^2\left[\int_0^{\pi/2} d\phi - 2\int_0^{\pi/2}\cos\phi\,d\phi + \int_0^{\pi/2}\cos^2\phi\,d\phi\right]$$

$$= \frac{1}{2}a^2\left[\{\phi\}_0^{\pi/2} - 2\{\sin\phi\}_0^{\pi/2} + \frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$= \frac{1}{2}a^2\left[\frac{1}{2}\pi - 2.1 + \frac{1}{4}\pi\right]$$

$$= \frac{1}{2}a^2\left[\frac{3\pi}{4} - 2\right] = \frac{1}{8}a^2(3\pi - 8).$$

Now let A_2 be the area of the portion of the curve lying between $x = \frac{1}{2}a$ and $x = a$.

Then $A_2 = A - A_1 = \frac{3}{4}\pi a^2 - \frac{1}{8}a^2(3\pi - 8) = \frac{1}{8}a^2(3\pi + 8)$.

$\therefore$ Required ratio $= \dfrac{A_1}{A_2} = \dfrac{(a^2/8)(3\pi-8)}{(a^2/8)(3\pi+8)} = \dfrac{3\pi-8}{3\pi+8}$.

Problem 9(ii): *Find the area of the loop of the curve* $y^2(a-x) = x^2(a+x)$.

(Purvanchal 2011)

Solution: The given curve is symmetrical about the x-axis and cuts the x-axis at the points $(0,0)$ and $(-a,0)$.

The tangents at $(0,0)$ are $y^2 = x^2$ *i.e.*, $y = \pm x$.

Clearly there is a loop which lies between $x = -a$ and $x = 0$.

$\therefore$ The required area of the loop = 2 × area of the upper half of the loop

$$= 2\int_{-a}^{0} y\,dx = 2\int_{-a}^{0} -x\cdot\frac{\sqrt{(a+x)}}{\sqrt{(a-x)}}\,dx,$$

putting for y for the upper half of the loop from the given equation of the curve

$$= 2\int_{-a}^{0} -x\,\frac{(a+x)}{\sqrt{(a^2-x^2)}}\,dx,$$

multiplying the numerator and the denominator by $\sqrt{(a+x)}$.

Now put $x = -a \sin\theta$ so that $dx = -a \cos\theta \, d\theta$.
When $x = -a$, $\theta = \pi/2$ and when $x = 0$, $\theta = 0$.

$\therefore$ The required area $= 2 \int_{\pi/2}^{0} -(-a \sin\theta) \dfrac{(a - a \sin\theta)}{\sqrt{(a^2 - a^2 \sin^2\theta)}} \cdot (-a \cos\theta) \, d\theta$

$$= -2a^2 \int_{\pi/2}^{0} \sin\theta \, (1 - \sin\theta) \, d\theta$$

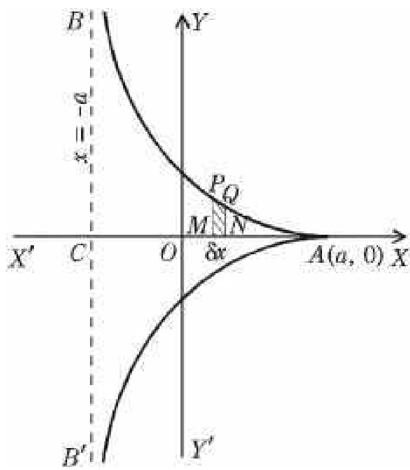
$$= 2a^2 \int_{0}^{\pi/2} (\sin\theta - \sin^2\theta) \, d\theta$$

$$= 2a^2 \left[1 - \frac{1}{2} \cdot \frac{1}{2} \pi\right] \qquad \text{[By Walli's formula]}$$

$$= 2a^2 \left(1 - \frac{1}{4}\pi\right) = \frac{1}{2} a^2 (4 - \pi).$$

Problem 10: *Trace the curve $y^2 (a + x) = (a - x)^3$ and find the area between the curve and its asymptotes.* **(Purvanchal 2007)**

Solution: The curve is symmetrical about x-axis. It does not pass through the origin. Putting $x = 0$ in the equation of the curve, we get $y = \pm a$ and putting $y = 0$ in it we get $x = a$. Thus the curve cuts the y-axis at the points $(0, \pm a)$ and it cuts the x-axis at the point $(a, 0)$.



Equating to zero the coefficient of the highest power of y the asymptote parallel to y-axis is $a + x = 0$ *i.e.*, $x = -a$.

The equation of the curve can be written as $y^2 = (a - x)^3 / \{(a + x)\}$ which shows that for $x > a$, y is imaginary *i.e.*, the curve does not exist for $x > a$. Thus the shape of the curve is as shown in the figure. Now the required area $= 2 \times$ area lying above the x-axis

$$= 2 \int_{-a}^{a} y \, dx = 2 \int_{-a}^{a} \frac{(a - x)^{3/2}}{\sqrt{(a + x)}} \, dx,$$

putting for y from the equation of the curve

$$= 2 \int_{-a}^{a} \frac{(a - x)^2}{\sqrt{(a^2 - x^2)}} \, dx, \text{ multiplying the Nr. and the Dr. by } \sqrt{(a - x)}$$

$$= 2 \int_{-\pi/2}^{\pi/2} \frac{(a - a \sin\theta)^2}{\sqrt{(a^2 - a^2 \sin^2\theta)}} \cdot a \cos\theta \, d\theta,$$

putting $x = a \sin\theta$ so that $dx = a \cos\theta \, d\theta$

$$= 2a^2 \int_{-\pi/2}^{\pi/2} (1 - \sin\theta)^2 \, d\theta = 2a^2 \int_{-\pi/2}^{\pi/2} (1 - 2\sin\theta + \sin^2\theta)\, d\theta$$

$$= 2a^2 \int_{-\pi/2}^{\pi/2} \{1 - 2\sin\theta + \frac{1}{2}(1 - \cos 2\theta)\}\, d\theta$$

$$= 2a^2 \left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 2a^2 \left(\frac{3\pi}{2}\right) = 3\pi a^2.$$

Comprehensive Problems 2

Problem 1: *Find the common area between the curves* $y^2 = 4ax$ *and* $x^2 = 4ay$.

(Agra 2002, 14; Meerut 04B, 08)

Solution: Proceed exactly as in the Example 6. Here $b = a$. Solving the given equations, we get the points of intersection as $(0, 0)$ and $(4a, 4a)$ and hence the required area

$$= \frac{16}{3} a^2.$$

By double integration. The required area

$$= \int_{x=0}^{4a} \int_{y=(x^2/4a)}^{2\sqrt{(ax)}} dx\, dy = \int_0^{4a} \left[\, y \,\right]_{(x^2/4a)}^{2\sqrt{(ax)}} dx$$

$$= \int_0^{4a} \left[2\sqrt{(ax)} - \frac{x^2}{4a}\right] dx = 2\sqrt{(a)} \left[\frac{2}{3} x^{3/2}\right]_0^{4a} - \frac{1}{12a} [\, x^3 \,]_0^{4a}$$

$$= \frac{4}{3} \sqrt{(a)} \,.(4a)^{3/2} - \left[\frac{(4a)^3}{12a}\right] = \frac{32}{3} a^2 - \frac{16}{3} a^2 = \frac{16}{3} a^2.$$

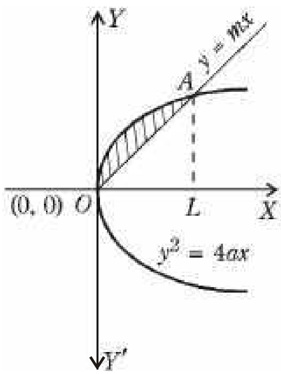
Problem 2(i): *Find the area included between* $y^2 = 4ax$ *and* $y = mx$.

Solution: Solving the equation of the parabola $y^2 = 4ax$ and the equation of the line $y = mx$ for x, we get

$$m^2 x^2 = 4ax \quad \text{or} \quad x(m^2 x - 4a) = 0.$$

This gives $x = 0$ or $x = 4a/m^2$.

Thus the two curves cut at the points where $x = 0$ and $x = 4a/m^2$.



$\therefore$ The required area

$$= \int_0^{4a/m^2} y\, dx \text{, from the curve } y^2 = 4ax$$

$$- \int_0^{4a/m^2} y\, dx \text{, from the st. line } y = mx$$

$$= \int_0^{4a/m^2} \sqrt{(4ax)}\, dx - \int_0^{4a/m^2} mx\, dx$$

$$= 2\sqrt{(a)}\left[\frac{2}{3}x^{3/2}\right]_0^{4a/m^2} - m\left[\frac{1}{2}x^2\right]_0^{4a/m^2}$$

$$= \frac{4\sqrt{a}}{3}\left(\frac{4a}{m^2}\right)^{3/2} - \frac{1}{2}m\left(\frac{4a}{m^2}\right)^2 = \frac{32a^2}{3m^3} - \frac{8a^2}{m^3} = \frac{8a^2}{3m^3}.$$

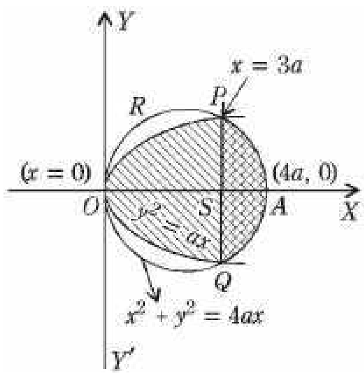
Problem 2(ii): *Find the area of the segment cut off from the parabola* $y^2 = 4x$ *by the line* $y = 8x - 1$.

Solution: Proceed exactly as in Example 7. Here the points of intersection are $\left(\frac{1}{16}, -\frac{1}{2}\right)$ and $\left(\frac{1}{4}, 1\right)$ hence the required area $= \frac{9}{64}$.

Problem 3(i): *Find the area common to the two curves* $y^2 = ax$, $x^2 + y^2 = 4ax$.

(Meerut 2005B, 06, 09B)

Solution: $y^2 = ax$ is a parabola with vertex at the origin and axis along x-axis and latus rectum a, and $x^2 + y^2 = 4ax$ is a circle with centre $(2a, 0)$ and radius $2a$.



Both these curves are symmetrical about x-axis. Solving the equations of the two curves for x, we have

$$x^2 + ax = 4ax \quad \text{or} \quad x^2 - 3ax = 0$$

or $\quad x(x - 3a) = 0$.

Therefore $x = 0, 3a$.

Thus the two curves intersect at the points where $x = 0$ and $x = 3a$.

Also A is the point $(4a, 0)$.

The area common to the parabola and the circle (*i.e.*, the shaded area)

$= 2\,[\text{Area } OPS + \text{Area } PSA]$, [By symmetry]

$$= 2\left[\left(\int_0^{3a} y\,dx, \text{ from the parabola } y^2 = ax\right) + \left(\int_{3a}^{4a} y\,dx, \text{ from the circle } x^2 + y^2 = 4ax\right)\right]$$

$$= 2\left[\int_0^{3a}\sqrt{(ax)}\,dx + \int_{3a}^{4a}\sqrt{(4ax - x^2)}\,dx\right]$$

$$= 2\sqrt{a}\int_0^{3a} x^{1/2}\,dx + 2\int_{3a}^{4a}\sqrt{[4a^2 - (x - 2a)^2]}\,dx$$

$$= 2\sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_0^{3a} + 2\left[\frac{1}{2}(x - 2a)\sqrt{\{4a^2 - (x - 2a)^2\}} + \frac{4a^2}{2}\sin^{-1}\frac{x - 2a}{2a}\right]_{3a}^{4a}$$

$$= 4a^2 \sqrt{3} + 2\left[\left\{0 - \frac{1}{2} a\sqrt{3}\,.a\right\} + 2a^2 \left\{(\pi/2) - (\pi/6)\right\}\right]$$

$$= 4\sqrt{3}\, a^2 - a^2\sqrt{3} + \frac{4}{3}\pi a^2 = 3\sqrt{3}a^2 + \frac{4}{3}\pi a^2 = a^2\left(3\sqrt{3} + \frac{4}{3}\pi\right).$$

Problem 3(ii): *Find the area lying above x-axis and included between the circle* $x^2 + y^2 = 2ax$ *and the parabola* $y^2 = ax$. **(Bundelkhand 2007)**

Solution: The figure is similar to that of Problem 3 part (i). Here we are required to find the unshaded area *ORP* in the figure of part (i). Solving the given equations $y^2 = ax$ and $x^2 + y^2 = 2ax$ for x, we get $x = 0$ and $x = a$.

$\therefore$ The required area $= \left[\int_0^a y\, dx, \text{ from the curve } x^2 + y^2 = 2ax\right]$

$$- \left[\int_0^a y\, dx, \text{ from the curve } y^2 = ax\right]$$

$$= \int_0^a \sqrt{(2ax - x^2)}\, dx - \int_0^a \sqrt{(ax)}\, dx$$

$$= \int_0^a \{a^2 - (x-a)^2\}\, dx - \sqrt{a}\,.\left[\frac{2}{3} x^{3/2}\right]_0^a$$

$$= \left[\frac{1}{2}(x-a)\sqrt{(2ax - x^2)} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)\right]_0^a - \frac{2}{3}a^2$$

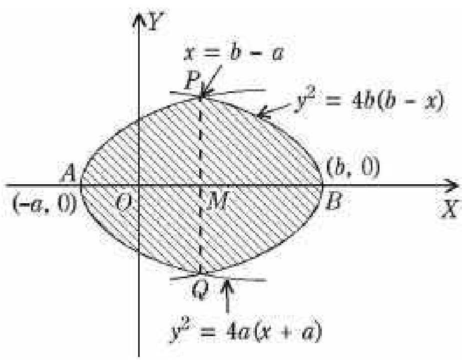
$$= \left[0 - \frac{a^2}{2}\sin^{-1}(-1)\right] - \frac{2}{3}a^2 = \frac{a^2}{2}\cdot\frac{\pi}{2} - \frac{2}{3}a^2 = a^2\left[\frac{\pi}{4} - \frac{2}{3}\right].$$

Problem 4(i): *Show that the area included between the parabolas*

$$y^2 = 4a(x+a),\ y^2 = 4b(b-x) \text{ is } \frac{8}{3}(a+b)\sqrt{(ab)}.$$

(Rohilkhand 2013)

Solution: $y^2 = 4a(x+a)$ represents a parabola whose vertex is $(-a, 0)$ and latus rectum is $4a$. Also $y^2 = 4b(b-x)$ represents a parabola whose vertex is $(b, 0)$ and latus rectum $4b$. Both the curves have been shown in the figure. Equating the values of y^2 from the two given equations of parabolas, we get $4a(x+a) = 4b(b-x)$ or $x = b - a$ *i.e.*, the abscissa of the point of intersection P is $b - a$.



Now both the curves are symmetrical about x-axis.

$\therefore$ The required area

$$= 2\,[\text{Area } APM + \text{Area } PMB], \text{ by symmetry}$$

$$= 2\left[\left(\int_{-a}^{b-a} y\,dx, \text{ for the parabola } y^2 = 4a\,(x+a)\right.\right.$$

$$\left.\left. + \left(\int_{b-a}^{b} y\,dx, \text{ from the parabola } y^2 = 4b\,(b-x)\right)\right)\right]$$

$$= 2\left[\int_{-a}^{b-a} \sqrt{\{4a\,(x+a)\}}\,dx + \int_{b-a}^{b} \sqrt{\{4b\,(b-x)\}}\,dx\right]$$

$$= 4\sqrt{a}\int_{-a}^{b-a}(x+a)^{1/2}\,dx + 4\sqrt{b}\int_{b-a}^{b}\sqrt{(b-x)}\,dx$$

$$= 4\sqrt{a}\left[\frac{2}{3}(x+a)^{3/2}\right]_{-a}^{b-a} - 4\sqrt{b}\left[\frac{2}{3}(b-x)^{3/2}\right]_{b-a}^{b}$$

$$= \frac{1}{3}[8\sqrt{a}\,.\,b^{3/2}] + \frac{1}{3}[8\sqrt{b}\,.a^{3/2}]$$

$$= \frac{8}{3}\sqrt{(ab)}\,.(b+a) = \frac{8}{3}(a+b)\sqrt{(ab)}.$$

Problem 4(ii): *Show that the area common to the ellipses* $a^2x^2 + b^2y^2 = 1$, $b^2x^2 + a^2y^2 = 1$, *where* $0 < a < b$, *is* $4\,(ab)^{-1}\tan^{-1}(a/b)$.

Solution: The given equations of the two ellipses are

$$a^2x^2 + b^2y^2 = 1 \qquad ...(1)$$

and $$b^2x^2 + a^2y^2 = 1. \qquad ...(2)$$

Since $0 < a < b$, therefore $(1/a) > (1/b)$.

The ellipse (1) cuts the x-axis on the positive side at the point $(1/a, 0)$ and it cuts the y-axis on the positive side at the point $B\,(0, 1/b)$. The ellipse (2) cuts the x-axis on the positive side at the point $A\,(1/b, 0)$. Both the ellipses are symmetric about both the axes. Solving (1) and (2) we have the coordinates of the point of intersection P in the first quadrant as

$$(1/\sqrt{\{a^2+b^2\}}, 1/\sqrt{\{a^2+b^2\}}).$$

Draw PM and PN perpendiculars to the axis of x and the axis of y respectively.

Now the area common to the two ellipses (*i.e.*, the shaded area)

$$= 4 \times (\text{common area in the Ist quadrant}) = 4 \text{ area } OAPB$$

$$= 4\,[\text{area } OMPB + \text{area } APM]$$

$$= 4\,[\{\text{area of the square } OMPN + \text{area } BPN\} + \text{area } APM]$$

$$= 4\,[\text{area of the square } OMPN + 2 \text{ area } APM] \qquad ...(3)$$

[$\because$ area BPN = area APM on account of the symmetrical situation of the area $OAPB$ about OP]

Now the area of the square $OMPN = OM\,.ON$

$$= \frac{1}{\sqrt{(a^2+b^2)}} \cdot \frac{1}{\sqrt{(a^2+b^2)}} = \frac{1}{(a^2+b^2)}.$$

Also the area $APM = \int_{1/\sqrt{(a^2+b^2)}}^{a/b} y\, dx$, from the ellipse $b^2x^2 + a^2y^2 = 1$

$$= \int_{1/\sqrt{(a^2+b^2)}}^{1/b} \frac{(1-b^2x^2)^{1/2}}{a}\, dx = \frac{b}{a}\int_{1/\sqrt{(a^2+b^2)}}^{1/b} \sqrt{\left\{\frac{1}{b^2} - x^2\right\}}\, dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{\left(\frac{1}{b^2} - x^2\right)} + \frac{1}{2}\cdot\frac{1}{b^2}\sin^{-1}\left(\frac{x}{1/b}\right)\right]_{1/\sqrt{(a^2+b^2)}}^{1/b}$$

$$= \frac{b}{2a}\left[0 + \frac{1}{b^2}\sin^{-1}1 - \frac{1}{\sqrt{(a^2+b^2)}}\cdot\frac{a}{b\sqrt{(a^2+b^2)}} - \frac{1}{b^2}\sin^{-1}\frac{b}{\sqrt{(a^2+b^2)}}\right]$$

$$= \frac{1}{2ab}\left[\frac{\pi}{2} - \frac{ab}{a^2+b^2} - \sin^{-1}\frac{b}{\sqrt{(a^2+b^2)}}\right]$$

$$= \frac{1}{2ab}\left[\left\{\frac{\pi}{2} - \sin^{-1}\frac{b}{\sqrt{(a^2+b^2)}}\right\} - \frac{ab}{a^2+b^2}\right]$$

$$= \frac{1}{2ab}\left[\cos^{-1}\left\{\frac{b}{\sqrt{(a^2+b^2)}}\right\} - \frac{ab}{a^2+b^2}\right]$$

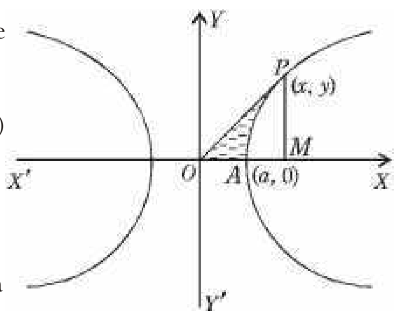
$$= \frac{1}{2ab}\left[\tan^{-1}\left(\frac{a}{b}\right) - \frac{ab}{a^2+b^2}\right].$$

Hence from (3), the required area

$$= 4\left[\frac{1}{a^2+b^2} + 2\left\{\frac{1}{2ab}\tan^{-1}\frac{a}{b} - \frac{1}{2(a^2+b^2)}\right\}\right] = \frac{4}{ab}\tan^{-1}\left(\frac{a}{b}\right).$$

Problem 5: *If A is the vertex, O the centre and P any point (x, y) on the hyperbola $x^2/a^2 - y^2/b^2 = 1$, show that $x = a\cosh(2S/ab)$, $y = b\sinh(2S/ab)$, where S is the sectorial area OPA.*

Solution: The given hyperbola is shown in the figure. We have



S = the sectorial area OAP

(*i.e.*, the dotted area)

= the area of the $\Delta\, OMP$ – the area PAM

$$= \frac{1}{2} OM \cdot MP - \int_a^x y\, dx, \text{ for the hyperbola}$$

$$= \frac{1}{2}xy - \int_a^x \frac{b}{a}\sqrt{(x^2 - a^2)}\, dx,$$

[$\because$ from the equation of the hyperbola $y = (b/a)\sqrt{(x^2 - a^2)}$]

$$= \frac{1}{2} x \cdot \frac{b}{a} \sqrt{(x^2 - a^2)} - \frac{b}{a} \left[\frac{x}{2} \sqrt{(x^2 - a^2)} - \frac{1}{2} a^2 \cosh^{-1} \frac{x}{a} \right]_a^x$$

$$= \frac{bx}{2a} \sqrt{(x^2 - a^2)} - \frac{b}{a} \left[\frac{x}{2} \sqrt{(x^2 - a^2)} - \frac{1}{2} a^2 \cosh^{-1} \frac{x}{a} - (0 - 0) \right]$$

$$= \frac{bx}{2a} \sqrt{(x^2 - a^2)} - \frac{bx}{2a} \sqrt{(x^2 - a^2)} + \frac{ab}{2} \cosh^{-1} \frac{x}{a} = \frac{ab}{2} \cosh^{-1} \frac{x}{a}.$$

Thus $S = \frac{ab}{2} \cosh^{-1} \frac{x}{a}$;

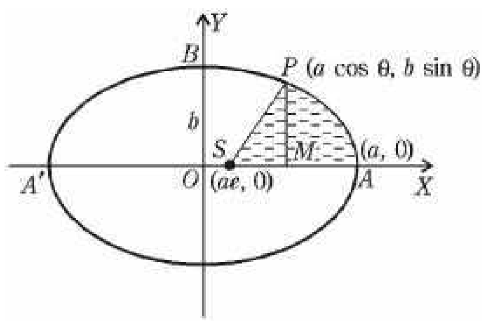
$\therefore$ $\cosh^{-1} \frac{x}{a} = \frac{2S}{ab}$ or $x = a \cosh \frac{2S}{ab}$.

Also $y = \frac{b}{a} \sqrt{(x^2 - a^2)} = \frac{b}{a} \sqrt{\left\{ a^2 \cosh^2 \frac{2S}{ab} - a^2 \right\}}$

$$= \frac{b}{a} \cdot a \sqrt{\left\{ \cosh^2 \frac{2S}{ab} - 1 \right\}} = b \sinh \frac{2S}{ab}.$$

Problem 6: *Prove that the area of a sector of the ellipse of semi-axes a and b between the major axis and a radius vector from the focus is* $\frac{1}{2} ab (\theta - e \sin \theta)$, *where* θ *is the eccentric angle of the point to which the radius vector is drawn.*

Solution: Let the equation of the ellipse be $x^2/a^2 + y^2/b^2 = 1$.



Let O be its centre and S be its focus $(ae, 0)$. Let θ be the eccentric angle of any point $P(x, y)$ on the ellipse. Then

$x = a \cos \theta$,

$y = b \sin \theta$.

Now SP is the radius vector of P drawn through the focus S and SA is the radius vector along the major axis. At the point A, $x = a$ and $\theta = 0$.

Draw PM perpendicular to the x-axis.

The required area of the sector SAP (*i.e.*, the dotted area)

$= \text{area of the } \Delta\, SMP + \text{area } PMA$

$= \frac{1}{2} SM \cdot MP + \int_{a \cos \theta}^{a} y\, dx$, for the ellipse

$$= \frac{1}{2} (OM - OS) \cdot MP + \int_{\theta}^{0} y \frac{dx}{d\theta} d\theta$$

$$= \frac{1}{2} (a \cos \theta - ae)\, b \sin \theta + \int_{\theta}^{0} b \sin \theta . (-a \sin \theta)\, d\theta,$$

[$\because x = a \cos \theta$ and $y = b \sin \theta$]

$$= \frac{1}{2} ab (\cos \theta - e) \sin \theta + \int_{0}^{\theta} ab \cdot \frac{1}{2} (1 - \cos 2\theta)\, d\theta$$

$$= \frac{1}{2} ab (\cos \theta - e) \sin \theta + \frac{1}{2} ab \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\theta}$$

$$= \frac{1}{2} ab (\cos \theta - e) \sin \theta + \frac{1}{2} ab (\theta - \frac{1}{2} . 2 \sin \theta \cos \theta)$$

$$= \frac{1}{2} ab [\cos \theta \sin \theta - e \sin \theta + \theta - \sin \theta \cos \theta] = \frac{1}{2} ab (\theta - e \sin \theta).$$

Problem 7: *Find the area common to the circle* $x^2 + y^2 = 4$ *and the ellipse* $x^2 + 4y^2 = 9$.

(Garhwal 2002; Purvanchal 09)

Solution: The equation of the circle is $x^2 + y^2 = 4$, ...(1)

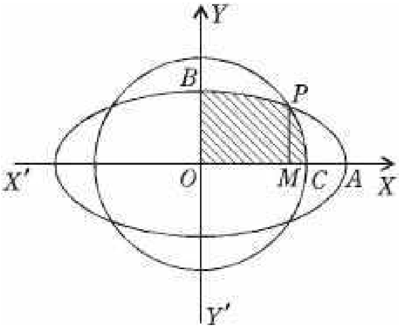
and the equation of the ellipse is $x^2 + 4y^2 = 9$. ...(2)

Both the curves (1) and (2) are symmetrical about both the axes and have been shown in the figure. Solving (1) and (2) for x, we have

$$x^2 + 4(4 - x^2) = 9 \quad \text{or} \quad 3x^2 = 7 \quad \text{or} \quad x^2 = 7/3.$$

∴ The x-coordinate of the point of intersection P lying in the first quadrant is $\sqrt{(7/3)}$.

Also putting $y = 0$ in $x^2 + y^2 = 4$, we get $x = 2$ at C.



Now the required area is symmetrical about both the axes.

∴ The required area (*i.e.*, the area common to the circle and the ellipse)

$= 4 \times$ (common area lying in the first quadrant)

$= 4 \times$ area $OCPB$

$= 4$ [area $OBPM$ + area CPM]

$$= 4 \left[\int_0^{\sqrt{(7/3)}} y \, dx, \text{ for the ellipse} + \int_{\sqrt{(7/3)}}^2 y \, dx, \text{ for the circle} \right]$$

$$= 4 \left[\int_0^{\sqrt{(7/3)}} \frac{1}{2} \sqrt{(9 - x^2)} \, dx + \int_{\sqrt{(7/3)}}^2 \sqrt{(4 - x^2)} \, dx \right],$$

[∵ for the ellipse, $y = \frac{1}{2} \sqrt{(9 - x^2)}$ and for the circle, $y = \sqrt{(4 - x^2)}$]

$$= 2 \left[\frac{x \sqrt{(9 - x^2)}}{2} + \frac{9}{2} \sin^{-1} \left(\frac{x}{3} \right) \right]_0^{\sqrt{(7/3)}}$$

$$+ 4 \left[\frac{x \sqrt{(4 - x^2)}}{2} + 2 \sin^{-1} \left(\frac{x}{2} \right) \right]_{\sqrt{(7/3)}}^2$$

$$= 2\left[\frac{1}{2}\cdot\sqrt{\left(\frac{7}{3}\right)}\sqrt{\left(\frac{20}{3}\right)} + \frac{9}{2}\sin^{-1}\left\{\frac{1}{3}\sqrt{\left(\frac{7}{3}\right)}\right\}\right]$$

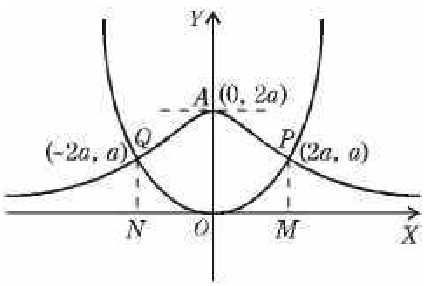
$$+ 4\left[2\sin^{-1}(1) - \frac{1}{2}\sqrt{(\frac{7}{3})}\sqrt{(\frac{5}{3})} - 2\sin^{-1}\left\{\frac{1}{2}\sqrt{(\frac{7}{3})}\right\}\right]$$

$$= \frac{2\sqrt{(35)}}{3} + 9\sin^{-1}\frac{\sqrt{7}}{3\sqrt{3}} + 4\pi - \frac{2}{3}\sqrt{(35)} - 8\sin^{-1}\frac{\sqrt{7}}{2\sqrt{3}}$$

$$= 4\pi + 9\sin^{-1}\left\{\frac{1}{3}.\sqrt{(7/3)}\right\} - 8\sin^{-1}\left\{\frac{1}{2}.\sqrt{(7/3)}\right\}.$$

Problem 8: *Find the area included between the parabola $x^2 = 4ay$ and the curve $y = 8a^3/(x^2 + 4a^2)$.* **(Rohilkhand 2008B)**

Solution: The curve $y(x^2 + 4a^2) = 8a^3$ is symmetrical about y-axis. Equating to zero the coefficient of the highest power of x in the given equation, we find that $y = 0$ *i.e.*, x-axis is an asymptote of the curve parallel to x-axis. Also this curve cuts the y-axis at $(0, 2a)$.



Solving the two given equations $x^2 = 4ay$ and $y = 8a^3/(x^2 + 4a^2)$, we get their points of intersection as $(\pm 2a, a)$.

Also both the curves are symmetrical about y-axis.

Now the required area $OPAQO = 2 \times$ area OPA (by symmetry)

$$= 2 \times [\text{area } OAPM - \text{area } OPM]$$

$$= 2\left[\int_0^{2a} y\,dx, \text{ for } y = 8a^3/(x^2 + 4a^2) - \int_0^{2a} y\,dx, \text{ for } x^2 = 4ay\right]$$

$$= 2\int_0^{2a} \frac{8a^3}{x^2 + 4a^2}\,dx - 2\int_0^{2a} \frac{x^2}{4a}\,dx$$

$$= 16\,a^3.\frac{1}{2a}\left[\tan^{-1}\frac{x}{2a}\right]_0^{2a} - \frac{1}{2a}\left[\frac{x^3}{3}\right]_0^{2a} = 2\pi a^2 - \frac{4a^2}{3} = [2\pi - \frac{4}{3}]\,a^2.$$

Problem 9: *Find by double integration the area bounded by the curves $y(x^2 + 2) = 3x$ and $4y = x^2$.*

Solution: Eliminating y from the given equations, we get

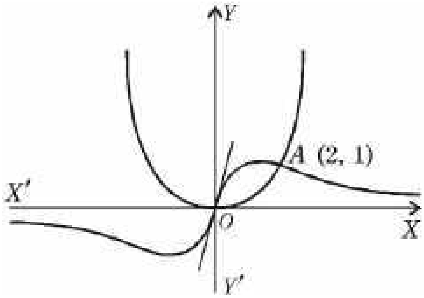
$$3x/(x^2 + 2) = x^2/4 \quad \text{or} \quad 12x = x^2(x^2 + 2)$$

or $\quad x(12 - x^3 - 2x) = 0,$

giving $x = 0$ and $x = 2$.

Thus the two curves intersect at the points where $x = 0$ and $x = 2$.

Both the curves have been shown in the figure.



The required area

$$= \int_{x=0}^{2} \int_{y=x^2/4}^{3x/(x^2+2)} dx\, dy,$$

the first integration to be performed w.r.t. y

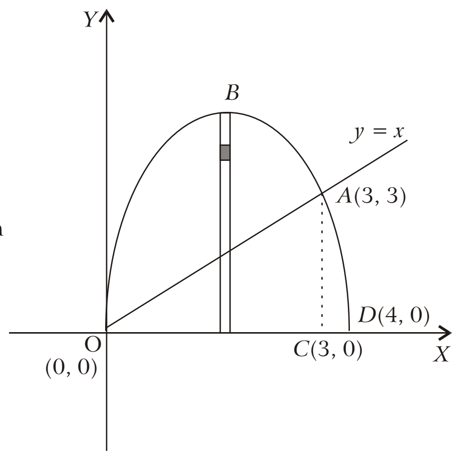
$$= \int_0^2 \left[y \right]_{x^2/4}^{3x/(x^2+2)} dx$$

$$= \int_0^2 \left[\frac{3x}{x^2+2} - \frac{x^2}{4} \right] dx$$

$$= \frac{3}{2} \int_0^2 \frac{2 . x\, dx}{x^2+2} - \frac{1}{4} \int_0^2 x^2\, dx = \frac{3}{2} [\log (x^2 + 2)]_0^2 - \frac{1}{4} \left[\frac{x^3}{3} \right]_0^2$$

$$= \frac{3}{2} [\log 6 - \log 2] - \frac{1}{12} . [8] = \frac{3}{2} \log (\frac{6}{2}) - \frac{2}{3} = \frac{3}{2} \log 3 - \frac{2}{3} .$$

Problem 10: *Find by double integration the area lying between the parabola $y = 4x - x^2$ and the straight line $y = x$.*



Solution: OA is the line

$$y = x \qquad \ldots(1)$$

and $OBAD$ is the parabola

$$y = 4x - x^2 \qquad \ldots(2)$$

On solving (1) and (2), we get $x = 0, 3$ and $y = 0, 3$.

∴ The line (1) meets the parabola (2) in $O(0, 0)$ and $A(3, 3)$.

We are to find the area $OBAO$

∴ Required area $OBAO$

= area $OCABO$ – area of ΔOCA

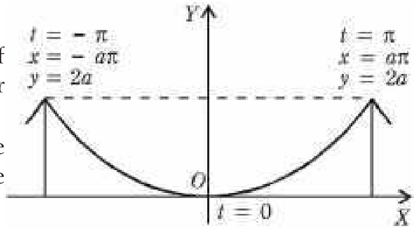
$$= \int_{x=0}^{3} \int_{y=0}^{4x-x^2} dx\, dy - \frac{1}{2} \cdot OC \cdot CA$$

$$= \int_0^3 (y)_0^{4x-x^2}\, dx - \frac{1}{2} \cdot 3 \cdot 3$$

$$= \int_0^3 (4x - x^2)\, dx - \frac{9}{2} = \left(2x^2 - \frac{x^3}{3} \right)_0^3 - \frac{9}{2} = (18-9) - \frac{9}{2} = 9 - \frac{9}{2} = \frac{9}{2} .$$

Comprehensive Problems 3

Problem 1: *Find the area included between the curve $x = a\,(t + \sin t)$, $y = a\,(1 - \cos t)$ and its base.* **(Agra 2005)**



Solution: The given cycloid has been shown in the figure. [To trace one complete arch of this cycloid proceed as in Example 11, after article 4].

Since the curve is symmetrical about the y-axis and its base is the line $y = 2a$, therefore the required area

$$= 2 \int_{y=0}^{2a} x\, dy$$

$$= 2 \int_{t=0}^{\pi} x \cdot \frac{dy}{dt} \cdot dt = 2 \int_0^{\pi} a\,(t + \sin t)\, a \sin t\, dt$$

$$= 2a^2 \int_0^{\pi} (t \sin t + \sin^2 t)\, dt = 2a^2 \int_0^{\pi} t \sin t\, dt + 2a^2 \int_0^{\pi} \sin^2 t\, dt$$

$$= 2a^2 \left[\{t\,.(-\cos t)\}_0^{\pi} - \int_0^{\pi} 1\,.(-\cos t)\, dt \right] + 4a^2 \int_0^{\pi/2} \sin^2 t\, dt$$

$$= 2a^2 \left[(\pi - 0) + \int_0^{\pi} \cos t\, dt \right] + 4a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 2a^2\,(\pi + 0) + \pi a^2,$$

$$\left[\because \int_0^{\pi} \cos t\, dt = 0 \right]$$

$$= 3\,\pi\, a^2.$$

Problem 2: *Find the area of a loop of the curve*

$$x = a \sin 2t,\ y = a \sin t \text{ or } a^2 x^2 = 4 y^2\,(a^2 - y^2).$$

Solution: To trace the given curve, we first find its cartesian equation by eliminating t. We have

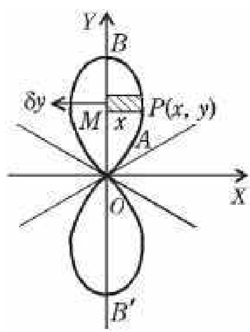
$$x = a \sin 2t = 2a \sin t \cos t.$$

$\therefore$
$$x^2 = 4a^2 \sin^2 t \cos^2 t = 4a^2 \sin^2 t\,(1 - \sin^2 t)$$

$$= 4a^2\,(y^2/a^2)\,\{1 - (y^2/a^2)\},$$

$$[\because y = a \sin t]$$

or
$$a^2 x^2 = 4 y^2\,(a^2 - y^2)$$



is the cartesian equation of the given curve.

Now we trace the curve from its cartesian equation. The curve is as shown in the figure.

At O, $x = 0$, $y = 0$ and so $t = 0$.

Again at B, $x = 0$, $y = a$ and so $t = \frac{1}{2}\pi$.

The required area of a loop of the curve

$$= 2 \times \text{area } OAB \qquad \text{[By symmetry]}$$

$$= 2 \int_{y=0}^{a} x\, dy = 2 \int_{t=0}^{\pi/2} x \cdot \frac{dy}{dt} \cdot dt = 2 \int_0^{\pi/2} a \sin 2t\,.\,a \cos t\, dt$$

$$= 2a^2 \int_0^{\pi/2} \sin 2t \cos t \, dt = 4a^2 \int_0^{\pi/2} \sin t \cos^2 t \, dt$$

$$= 4a^2 \frac{1}{3 . 1} \cdot 1, \quad \text{by Walli's formula}$$

$$= 4a^2/3.$$

Problem 3: *Show that the area bounded by the cissoid $x = a \sin^2 t$, $y = (a \sin^3 t)/\cos t$ and its asymptote is* $3\pi a^2/4$. **(Purvanchal 2006, 14; Avadh 09; Rohilkhand 11)**

Solution: Eliminating t from the given parametric equations, we get

$$y^2 = a^2 \frac{\sin^6 t}{\cos^2 t} = \frac{a^2 (x^3/a^3)}{1 - \sin^2 t} = \frac{x^3/a}{1-(x/a)} = \frac{x^3}{(a-x)}.$$

Therefore the Cartesian equation of the given curve is $y^2 (a - x) = x^3$. To trace this curve see Problem 6(i) of Comprehensive Problems 1.

The curve is symmetrical about x-axis, passes through the origin, the tangent there being $y = 0$. Also there are no real points of the curve if $x < 0$ or if $x > a$. The line $x = a$ is an asymptote of the curve. When $x = 0$, $t = 0$ and when $x = a$, $t = \pi/2$.

$\therefore$ The required area

$$= 2 \int_{x=0}^{a} y \, dx = 2 \int_{t=0}^{\pi/2} y \cdot \frac{dx}{dt} \cdot dt = 2 \int_0^{\pi/2} \frac{a \sin^3 t}{\cos t} \cdot 2a \sin t \cos t \, dt$$

$$= 4a^2 \int_0^{\pi/2} \sin^4 t \, dt = 4a^2 \frac{\Gamma \frac{5}{2} \Gamma \frac{1}{2}}{2 \Gamma 3} = 4a^2 \frac{\frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} . \sqrt{\pi}}{2.2.1} = \frac{3\pi a^2}{4}.$$

Problem 4: *Find the area of the loop of the curve*

$$x = a(1 - t^2), \ y = at(1 - t^2), \ \textit{where} \ -1 \le t \le 1.$$

Solution: Eliminating t, we have

$$y^2 = a^2 t^2 (1 - t^2)^2 = x^2 t^2 = x^2 \{1 - (x/a)\} = x^2 (a - x)/a.$$

Therefore $ay^2 = x^2 (a - x)$ is the cartesian equation of the given curve. To trace this curve see Problem 4(i) of Comprehensive Problems 1.

The required area of the loop

$$= 2 \int_{x=0}^{a} y \, dx = 2 \int_{t=1}^{0} y \cdot \frac{dx}{dt} \cdot dt,$$

$$= 2 \int_{t=1}^{0} y \cdot \frac{dx}{dt} \cdot dt, \qquad [\because \text{when } x = 0, t = 1 \text{ and when } x = a, t = 0]$$

$$= 2 \int_1^0 at(1 - t^2).(-2at) \, dt = 4a^2 \int_0^1 (t^2 - t^4) \, dt$$

$$= 4a^2 \left[\frac{1}{3} t^3 - \frac{1}{5} t^5\right]_0^1 = 4a^2 \left[\frac{1}{3} - \frac{1}{5}\right] = \frac{8}{15} a^2.$$

Comprehensive Problems 4

Problem 1: *Find the area between the following curves and the given radii vectors :*
(i) The spiral $r\,\theta^{1/2} = a\,; \theta = \alpha\,, \theta = \beta$.
(ii) The parabola $l/r = 1 + \cos\theta\,; \theta = 0\,, \theta = \alpha$.

Solution: **(i)** The required area

$$= \frac{1}{2}\int_{\theta=\alpha}^{\beta} r^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} \frac{a^2}{\theta}\, d\theta\,, \qquad [\because \; r\,\theta^{1/2} = a]$$

$$= \frac{a^2}{2}\int_{\alpha}^{\beta} \frac{1}{\theta}\, d\theta = \frac{a^2}{2}\Big[\log\theta\Big]_{\alpha}^{\beta} = \frac{a^2}{2}\Big[\log\beta - \log\alpha\Big]$$

$$= \frac{1}{2}a^2 \log(\beta/\alpha)\,.$$

(ii) The required area

$$= \frac{1}{2}\int_{0}^{\alpha} r^2\, d\theta = \frac{1}{2}\int_{0}^{\alpha} \frac{l^2}{(1+\cos\theta)^2}\, d\theta\,,$$

putting for r from the given equation of the curve

$$= \frac{1}{2}l^2\int_{0}^{\alpha} \frac{d\theta}{(2\cos^2\frac{1}{2}\theta)^2} = \frac{1}{2}l^2\int_{0}^{\alpha} \frac{d\theta}{4\cos^4\frac{1}{2}\theta}$$

$$= \frac{1}{8}l^2\int_{0}^{\alpha} \sec^4\frac{\theta}{2}\, d\theta\,.$$

Now put $\frac{\theta}{2} = t$ so that $\frac{1}{2}d\theta = dt$. Also when $\theta = 0\,, t = 0$ and when $\theta = \alpha\,, t = \frac{1}{2}\alpha$.

$\therefore$ Required area $= \frac{1}{4}l^2\int_{0}^{\alpha/2} \sec^4 t\, dt$

$$= \frac{1}{4}l^2\int_{0}^{\alpha/2} \sec^2 t\,.\sec^2 t\, dt = \frac{1}{4}l^2\int_{0}^{\alpha/2} (1+\tan^2 t)\sec^2 t\, dt$$

$$= \frac{1}{4}l^2\int_{0}^{\alpha/2} \{\sec^2 t + (\tan^2 t)\sec^2 t\}\, dt$$

$$= \frac{1}{4}l^2\Big[\tan t + \frac{1}{3}\tan^3 t\Big]_{0}^{\alpha/2}, \qquad \Big[\because \int (\tan t)^2 \sec^2 t\, dt = \frac{1}{3}(\tan t)^3\Big]$$

$$= \frac{1}{4}l^2\,[\tan\frac{1}{2}\alpha + \frac{1}{3}\tan^3\frac{1}{2}\alpha]\,.$$

Problem 2: *Find the area of the loop of the curve* $r = a\,\theta\cos\theta$ *between* $\theta = 0$ *and* $\theta = \pi/2$.
(Kumaun 2003; Kanpur 09)

Solution: The required area

$$= \frac{1}{2}\int_{0}^{\pi/2} r^2\, d\theta = \frac{1}{2}\int_{0}^{\pi/2} a^2\,\theta^2\cos^2\theta\, d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/2} \theta^2 \cos^2 \theta \, d\theta = \frac{a^2}{4} \int_0^{\pi/2} \theta^2 (1 + \cos 2\theta) \, d\theta$$

$$= \frac{a^2}{4} \int_0^{\pi/2} \theta^2 d\theta + \frac{a^2}{4} \int_0^{\pi/2} \theta^2 \cos 2\theta \, d\theta$$

$$= \frac{a^2}{4} \left[\frac{\theta^3}{3} \right]_0^{\pi/2} + \frac{a^2}{4} \left[\theta^2 \frac{\sin 2\theta}{2} \right]_0^{\pi/2} - \frac{a^2}{4} \int_0^{\pi/2} 2\theta \cdot \frac{\sin 2\theta}{2} \, d\theta$$

$$= \frac{a^2}{12} \cdot \frac{\pi^3}{8} + \frac{a^2}{8} \cdot 0 - \frac{a^2}{4} \left[\frac{\theta (- \cos 2\theta)}{2} \right]_0^{\pi/2} + \frac{a^2}{4} \int_0^{\pi/2} 1 \cdot \left[\frac{- \cos 2\theta}{2} \right] d\theta$$

$$= \frac{\pi^3 a^2}{96} + \frac{a^2}{8} \left[\frac{\pi}{2} (-1) \right] - \frac{a^2}{8} \left[\frac{\sin 2\theta}{2} \right]_0^{\pi/2} = \frac{\pi^3 a^2}{96} - \frac{\pi a^2}{16}$$

$$= \frac{\pi a^2}{96} (\pi^2 - 6) .$$

Problem 3(i): *Find the area of one loop of $r = a \cos 4\theta$.* **(Rohilkhand 2007)**

Solution: The given curve is $r = a \cos 4\theta$.

It is symmetrical about the initial line.

One loop is obtained by two consecutive values of θ for which r is zero. We have $r = 0$ when $\cos 4\theta = 0$ *i.e.*, $4\theta = -\frac{1}{2}\pi, \frac{1}{2}\pi$ *i.e.*, $\theta = -\pi/8, \pi/8$. Thus two consecutive values of θ for which r is zero are $-\pi/8$ and $\pi/8$. Therefore one loop of the curve lies between $\theta = -\pi/8$ and $\pi/8$ and this loop is symmetrical about the initial line $\theta = 0$.

Hence the area of a loop

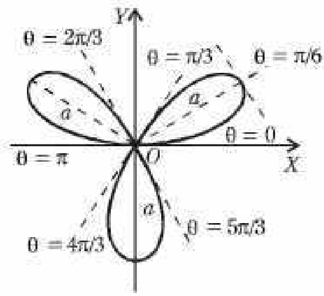
$$= 2 \int_0^{\pi/8} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi/8} a^2 \cos^2 4\theta \, d\theta, \qquad [\because r = a \cos 4\theta]$$

$$= \frac{1}{4} a^2 \int_0^{\pi/2} \cos^2 t \, dt, \text{ putting } 4\theta = t \text{ so that } 4 \, d\theta = dt$$

$$= \frac{1}{4} a^2 \cdot \frac{1}{2} \cdot \frac{1}{2} \pi, \text{ by Walli's formula} = \frac{1}{16} \pi a^2.$$

Problem 3(ii): *Find the area of a loop of the curve $r = a \sin 3\theta$.*

Solution: The given curve is not symmetrical about the initial line. We have $r = 0$ when $\sin 3\theta = 0$ *i.e.*, $3\theta = 0, \pi$ *i.e.*, $\theta = 0, \frac{1}{3}\pi$. Thus two consecutive values of θ for which r is zero are 0 and $\frac{1}{3}\pi$. Therefore one loop of the curve lies between $\theta = 0$ and $\frac{1}{3}\pi$. In all there are three loops as shown in the figure. For the first loop θ varies from $\theta = 0$ to $\theta = \frac{1}{3}\pi$.



Hence the area of a loop

$$= \frac{1}{2}\int_0^{\pi/3} r^2\, d\theta = \frac{1}{2}\int_0^{\pi/3} a^2 \sin^2 3\theta\, d\theta, \qquad [\because r = a \sin 3\theta]$$

$$= \frac{a^2}{4}\int_0^{\pi/3} (1 - \cos 6\theta)\, d\theta = \frac{1}{4}a^2\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \frac{a^2}{4}\left[\frac{\pi}{3}\right] = \frac{\pi a^2}{12}.$$

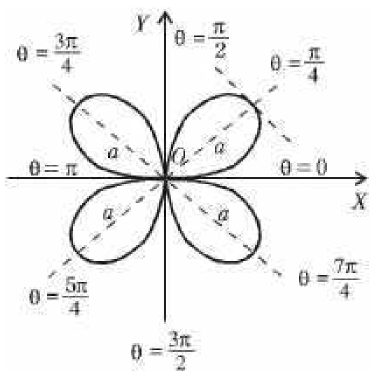
Note: Also whole area of the curve $r = a \sin 3\theta$

$$= 3 \times (\text{area of one loop}) = 3 \times (1/12)\, \pi a^2 = \frac{1}{4}\pi a^2.$$

Important: The above curve is a particular case of the curves of the type $r = a \sin n\theta$ which have n loops when n is odd and $2n$ loops when n is even.

Problem 4(i): *Find the whole area of the curve $r = a \sin 2\theta$.* **(Bundelkhand 2009)**

Solution: Here the given curve is $r = a \sin 2\theta$. Comparing with $r = a \sin n\theta$ we observe that $n = 2$ (*i.e.*, even), therefore the curve has four loops. This curve is not symmetrical about the initial line. Putting $r = 0$, we get $\sin 2\theta = 0$ *i.e.*, $2\theta = 0, \pi$ *i.e.*, $\theta = 0, \frac{1}{2}\pi$. Thus two consecutive values of θ for which r is zero are 0 and $\frac{1}{2}\pi$. Therefore for one loop of the curve θ varies from 0 to $\frac{1}{2}\pi$.



Now the whole area of the curve

$$= 4 \times \text{area of one loop}$$

$$= 4\int_0^{\pi/2} \frac{1}{2} r^2\, d\theta = 2\int_0^{\pi/2} a^2 \sin^2 2\theta\, d\theta = a^2 \int_0^{\pi/2} 2\sin^2 2\theta\, d\theta$$

$$= a^2 \int_0^{\pi/2} (1 - \cos 4\theta)\, d\theta = a^2\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} = \frac{1}{2}\pi a^2.$$

Problem 4(ii): *Find the whole area of the curve $r = a \cos 2\theta$.*

Solution: Comparing the given equation with $r = a \cos n\theta$ we observe that $n = 2$ (*i.e.*, even), therefore the number of loops

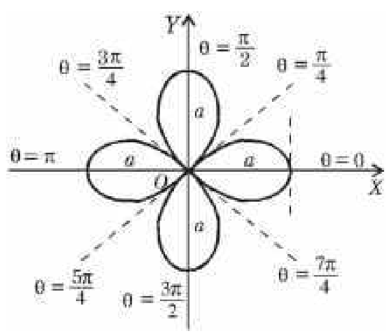
$$= 2n = 2 \times 2 = 4.$$

This curve is symmetrical about the initial line.

Putting $r = 0$, we get $\cos 2\theta = 0$

or $\quad 2\theta = \pm \frac{1}{2}\pi \quad$ or $\quad \theta = \pm \frac{1}{4}\pi$

i.e., for the first loop θ varies from $-\pi/4$



to $\pi/4$ and this loop is symmetrical about the initial line $\theta = 0$.

$\therefore$ Whole area of the curve = 4 × area of one loop

$$= 4 \times 2 \int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta = 4 \int_0^{\pi/4} a^2 \cos^2 2\theta \, d\theta .$$

Now put $2\theta = t$ so that $2\, d\theta = dt$. When $\theta = 0$, $t = 0$ and when $\theta = \frac{1}{4}\pi$, $t = \frac{1}{2}\pi$.

$\therefore$ The required area $= 4a^2 \int_0^{\pi/2} \cos^2 t \, . \frac{1}{2} dt = 2a^2 \int_0^{\pi/2} \cos^2 t \, dt$

$$= 2a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{1}{2} \pi a^2.$$

Problem 5(i): *Find the whole area of the curve* $r^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$.

Solution: The given curve is symmetrical about the initial line (the equation of the curve remains unchanged when θ is changed into $-\theta$), and the curve is symmetrical about the line $\theta = \pi/2$ (*i.e.*, y-axis) as the equation remains unchanged when θ is changed into $(\pi - \theta)$. Also there is symmetry about the pole as the equation of the curve remains unchanged when r is changed into $-r$.

In this curve r cannot be zero and r is real and finite for all values of θ . The figure of this curve is roughly like that of an ellipse.

$\therefore$ Whole area of the curve = 4 × area lying in the first quadrant

$$= 4 \times \frac{1}{2} \int_{\theta=0}^{\pi/2} r^2 \, d\theta = 2 \int_0^{\pi/2} (a^2 \cos^2 \theta + b^2 \sin^2 \theta) \, d\theta$$

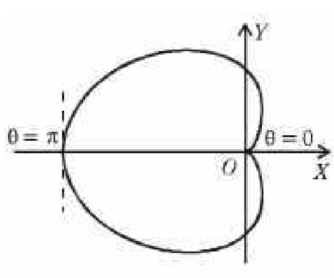
$$= 2a^2 \int_0^{\pi/2} \cos^2 \theta \, d\theta + 2b^2 \int_0^{\pi/2} \sin^2 \theta \, d\theta$$

$$= 2a^2 \cdot \frac{1}{2} \cdot \frac{1}{2} \pi + 2b^2 \cdot \frac{1}{2} \cdot \frac{1}{2} \pi = \frac{1}{2} \pi (a^2 + b^2) .$$

Problem 5(ii): *Find the area of the cardioid* $r = a(1 - \cos\theta)$.

(Agra 2001; Kumaun 02, 14)

Solution: The given curve is symmetrical about the initial line. We have $r = 0$, when $\cos\theta = 1$ *i.e.*, $\theta = 0$. Therefore the line $\theta = 0$ is tangent at the pole to the curve. Also r is maximum when $\cos\theta = -1$ *i.e.*, $\theta = \pi$ and then $r = 2a$. When θ increases from 0 to π , r increases from 0 to $2a$. Thus the curve is as shown in the figure.



The required area

$$= 2 \int_0^{\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi} a^2 (1 - \cos\theta)^2 \, d\theta$$

$$= a^2 \int_0^{\pi} (2 \sin^2 \tfrac{1}{2}\theta)^2 \, d\theta$$

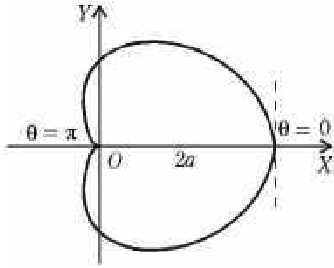
$$= 4a^2 \int_0^{\pi} \sin^4 \tfrac{1}{2}\theta \, d\theta = 8a^2 \int_0^{\pi/2} \sin^4 \phi \, d\phi ,$$

[Putting $\frac{1}{2}\theta = \phi$ so that $\frac{1}{2} d\theta = d\phi$ and adjusting the limits]

$$= 8a^2 \cdot \frac{3\,.\,1}{4\,.\,2} \cdot \frac{\pi}{2} = \frac{3\pi a^2}{2}.$$

Problem 6(i): *Show that the area of the limacon $r = a + b\cos\theta$, $(b < a)$ is equal to*

$$\pi\left(a^2 + \frac{1}{2}b^2\right).$$



Solution: The given curve is symmetrical about the initial line.

We have $r = 0$, when $\cos\theta = -a/b$ *i.e.*, $\theta = \cos^{-1}(-a/b)$ which is not real because $a > b$. Thus in this curve r cannot be zero. Also r is maximum when $\cos\theta = 1$ *i.e.*, $\theta = 0$ and then $r = a + b$. Again r is minimum when $\cos\theta = -1$ *i.e.*, $\theta = \pi$ and then $r = a - b$ which is positive. Some values of r and θ are as follows :

θ	0	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	$\frac{2}{3}\pi$	π
r	$a + b$	$a + \frac{1}{2}b$	a	$a - \frac{1}{2}b$	$a - b$

Hence the curve is as shown in the figure.

$\therefore$ The required area = 2 × area above the initial line

$$= 2\int_0^\pi \frac{1}{2} r^2\, d\theta = \int_0^\pi (a + b\cos\theta)^2\, d\theta,$$

putting for r from the given equation of the curve

$$= \int_0^\pi (a^2 + 2ab\cos\theta + b^2\cos^2\theta)\, d\theta$$

$$= a^2\Big[\theta\Big]_0^\pi + 2ab\Big[\sin\theta\Big]_0^\pi + \frac{b^2}{2}\int_0^\pi (1 + \cos 2\theta)\, d\theta$$

$$= a^2\pi + 0 + \frac{b^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^\pi = a^2\pi + \frac{b^2}{2}\pi = \pi\left(a^2 + \frac{1}{2}b^2\right).$$

Note : If we put $b = a$ or $-a$, we get the area of the cardioid $r = a(1 + \cos\theta)$

or $r = a(1 - \cos\theta)$.

Problem 6(ii): *Prove that the sum of the areas of the two loops of the limacon $r = a + b\cos\theta$, $(b > a)$ is equal to $\pi(2a^2 + b^2)/2$.*

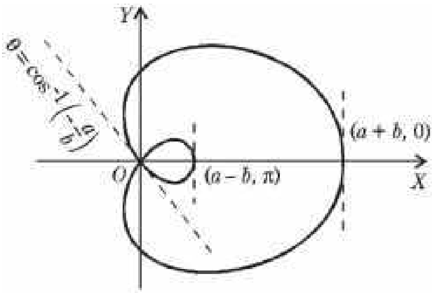
Solution: The given curve is symmetrical about the initial line. We have $r = 0$, when $\cos\theta = -a/b$ *i.e.*, $\theta = \cos^{-1}(-a/b) = \alpha$, (say). Since $\cos\alpha$ is –ive, therefore $\frac{1}{2}\pi < \alpha < \pi$.

Now r is maximum when $\cos\theta = 1$ *i.e.*, $\theta = 0$ and then $r = a + b$. Also r is minimum when $\cos\theta = -1$ *i.e.*, $\theta = \pi$ and then $r = a - b$ which is negative because $b < a$. Some values of r and θ are as follows :

θ	0	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	α	$\alpha < \theta < \pi$	π
r	$a + b$	$a + \frac{1}{2}b$	a	0	–ive	$a - b$

Thus the curve is as shown in the figure.

For the upper half of the larger loop θ varies from 0 to α *i.e.*, $\cos^{-1}(-ab)$ and for the lower half of the smaller loop θ varies from α to π.



$\therefore$ Required sum of the areas of the two loops

$$= 2\left[\int_0^{\alpha} \frac{1}{2} r^2 \, d\theta + \int_{\alpha}^{\pi} \frac{1}{2} r^2 \, d\theta\right]$$

$$= 2\int_0^{\pi} \frac{1}{2} r^2 \, d\theta,$$

by a property of definite integrals

$$= \int_0^{\pi} r^2 \, d\theta = \int_0^{\pi} (a + b\cos\theta)^2 \, d\theta$$

$$= \int_0^{\pi} (a^2 + 2ab\cos\theta + b^2\cos^2\theta) \, d\theta$$

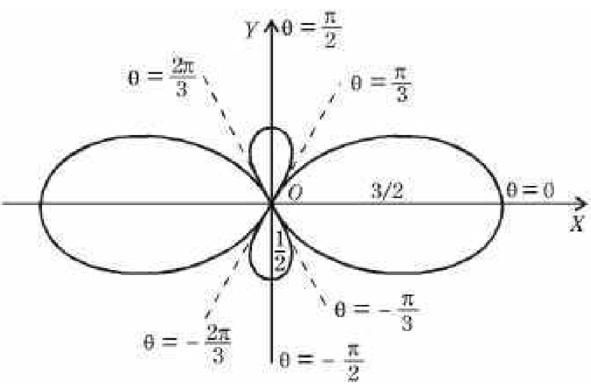
$$= \int_0^{\pi} a^2 \, d\theta + 2ab\int_0^{\pi} \cos\theta \, d\theta + b^2\int_0^{\pi} \cos^2\theta \, d\theta$$

$$= a^2\Big[\,\theta\,\Big]_0^{\pi} + 0 + 2b^2\int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$= a^2\pi + 2b^2 \cdot \frac{1}{2} \cdot \frac{1}{2}\pi = a^2\pi + \frac{1}{2}b^2\pi = \frac{1}{2}\pi(2a^2 + b^2).$$

Problem 7: *Calculate the ratio of the area of the larger to the area of the smaller loop of the curve* $r = \frac{1}{2} + \cos 2\theta$.

Solution: The given curve is symmetrical about the initial line. In the given equation of the curve $r = \frac{1}{2} + \cos 2\theta$ putting $r = 0$, we get $\cos 2\theta = -\frac{1}{2}$ *i.e.*, $2\theta = \pm 2\pi/3$ or $\pm 4\pi/3$ *i.e.*, $\theta = \pm \pi/3$ or $\pm 2\pi/3$.



The greatest radius vector of the loop lying between $\theta = -\frac{1}{3}\pi$ and $\theta = \frac{1}{3}\pi$ is given by $\theta = 0$ and it is equal to $\frac{3}{2}\cdot$ The greatest radius vector of the loop lying between $\theta = \frac{1}{3}\pi$ and $\theta = \frac{2}{3}\pi$ is given by $\theta = \frac{1}{2}\pi$ and its numerical value is $\frac{1}{2}\cdot$

Thus we observe that the larger loop lies between $\theta = -\pi/3$ and $\theta = \pi/3$ and it is symmetrical about the initial line $\theta = 0$.

Also the smaller loop lies between $\theta = \pi/3$ and $\theta = 2\pi/3$.

Hence area of the larger loop

$$= 2\int_0^{\pi/3} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi/3} (\frac{1}{2} + \cos 2\theta)^2 \, d\theta,$$

putting for r from the given equation of the curve

$$= \int_0^{\pi/3} (\frac{1}{4} + \cos 2\theta + \cos^2 2\theta)\, d\theta$$

$$= \int_0^{\pi/3} [\frac{1}{4} + \cos 2\theta + \frac{1}{2}(1 + \cos 4\theta)]\, d\theta$$

$$= \left[\frac{3}{4}\theta + \frac{1}{2}\sin 2\theta + \frac{1}{2}\frac{\sin 4\theta}{4}\right]_0^{\pi/3}$$

$$= \left[\frac{\pi}{4} + \frac{\sqrt{3}}{4} - \frac{1}{2}\frac{\sqrt{3}}{8}\right] = \frac{1}{4}\left[\pi + \frac{3\sqrt{3}}{4}\right] = \frac{1}{16}(4\pi + 3\sqrt{3}).$$

Area of the smaller loop (lying between $\theta = \pi/3$ and $\theta = 2\pi/3$)

$$= \int_{\pi/3}^{2\pi/3} \frac{1}{2} r^2 \, d\theta = \frac{1}{2}\int_{\pi/3}^{2\pi/3} (\frac{1}{2} + \cos 2\theta)^2 \, d\theta$$

$$= \frac{1}{2}\left[\frac{3}{4}\theta + \frac{1}{2}\sin 2\theta + \frac{1}{2}\cdot\frac{\sin 4\theta}{4}\right]_{\pi/3}^{2\pi/3}, \text{ as in the 1st case}$$

$$= \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) + \frac{1}{8}\left\{\frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right)\right\}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{8}\right] = \frac{1}{2}\left[\frac{\pi}{4} - \frac{3\sqrt{3}}{8}\right] = \frac{(2\pi - 3\sqrt{3})}{16}.$$

$$\therefore \text{ Required ratio} = \frac{\text{Area of the larger loop}}{\text{Area of the smaller loop}} = \frac{\frac{1}{16}(4\pi + 3\sqrt{3})}{\frac{1}{16}(2\pi - 3\sqrt{3})} = \frac{4\pi + 3\sqrt{3}}{2\pi - 3\sqrt{3}}.$$

Problem 8: *Show that the area of a loop of $r = a \cos n\theta$ is $\pi a^2/4n$, n being integral. Also prove that the whole area is $\pi a^2/4$ or $\pi a^2/2$ according as n is odd or even.*

Solution: The number of loops in $r = a \cos n\theta$ will be n or $2n$ according as n is odd or even.

The given curve is symmetrical about the initial line.

Also putting $r = 0$, we have $\cos n\theta = 0$ *i.e.*, $n\theta = -\frac{1}{2}\pi, \frac{1}{2}\pi$ *i.e.*, $\theta = -\pi/2n, \pi/2n$. Thus two consecutive values of θ for which r is zero are $-\pi/2n$ and $\pi/2n$.

$\therefore$ One loop lies between $\theta = -\pi/2n$ and $\theta = \pi/2n$ and it is symmetrical about the initial line $\theta = 0$.

$\therefore$ Area of one loop $= 2 . \int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta$, (By symmetry)

$$= \int_0^{\pi/2} r^2 \, d\theta = \int_0^{\pi/2n} a^2 \cos^2 n\theta \, d\theta .$$

Now put $n\theta = t$ so that $n \, d\theta = dt$. Also when $\theta = 0$, $t = 0$ and when $\theta = \pi/2n$, $t = \pi/2$.

$\therefore$ The area of one loop $= \frac{a^2}{n} \int_0^{\pi/2} \cos^2 t \, dt = \frac{a^2}{n} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi a^2}{4n}$.

Now if n is odd, the total number of loops will be n and the whole area

$$= n \times \text{ area of one loop}$$
$$= n (\pi a^2/4n) = \frac{1}{4} \pi a^2.$$

If n is even, the total number of loops will be $2n$ and then the whole area

$$= 2n \times \text{ area of one loop} = 2n \times (\pi a^2/4n) = \frac{1}{2} \pi a^2.$$

Problem 9: *Trace the curve $r = \sqrt{3} \cos 3\theta + \sin 3\theta$, and find the area of a loop.*

(Garhwal 2003)

Solution: The given equation of the curve is

$$r = \sqrt{3} \cos 3\theta + \sin 3\theta = 2 \{ (\sqrt{3}/2) \cos 3\theta + \frac{1}{2} \sin 3\theta \},$$ **(Note)**

$$= 2 \{ \sin \pi/3 \cos 3\theta + \cos \pi/3 \sin 3\theta \},$$

$[\because \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\cos \frac{\pi}{3} = \frac{1}{2}]$

$$= 2 \sin (3\theta + \pi/3) = 2 \sin 3 (\theta + \pi/9).$$

Now turning the initial line through an angle $-\pi/9$ the given equation of the curve becomes

$$r = 2 \sin 3 (\theta - \pi/9 + \pi/9) = 2 \sin 3\theta .$$

Now we shall trace the curve $r = 2 \sin 3\theta$.

This curve $r = 2 \sin 3\theta$ will have 3 loops and is not symmetrical about the initial line. Also putting $r = 0$, we get

$$\sin 3\theta = 0 \text{ i.e., } 3\theta = 0, \pi \text{ i.e., } \theta = 0, \pi/3 .$$

Therefore the lines $\theta = 0$ and $\theta = \pi/3$ are tangents to the curve at the pole and one loop of this curve lies between these two lines. For this loop r is greatest when $\sin 3\theta = 1$ *i.e.*, $3\theta = \pi/2$ *i.e.*, $\theta = \pi/6$. Thus this loop bends at $\theta = \pi/6$ and there r is equal to 2.

Here one loop of the curve lies in the region $0 < \theta < \pi/3$, one loop lies in the region $\pi/3 < \theta < 2\pi/3$ and one loop lies in the region $2\pi/3 < \theta < \pi$. If θ increases beyond π to 2π the same branches of the curve are repeated and we do not get any new branch. The shape of the curve is similar to that shown in Problem 3(ii). Here $a = 2$.

Now to get the location of the given curve turn the initial line back to its original position *i.e.*, in the figure of Problem 3(ii), turn the initial line through an angle $\pi/9$.
The required area of a loop

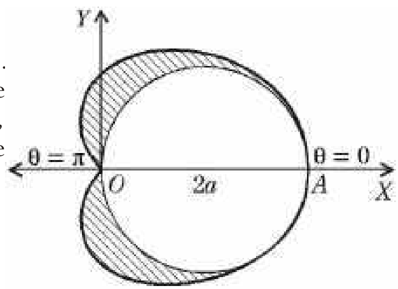
$$= 2 \times \int_0^{\pi/6} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi/6} 4 \sin^2 3\theta \, d\theta$$

$$= 2 \int_0^{\pi/6} (1 - \cos 6\theta) \, d\theta = 2 \left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/6} = \frac{\pi}{3}.$$

Comprehensive Problems 5

Problem 1: *Find the area outside the circle $r = 2a \cos\theta$ and inside the cardioid $r = a(1 + \cos\theta)$.*

Solution: The two curves intersect where $2a \cos\theta = a(1 + \cos\theta)$ *i.e.*, $\cos\theta = 1$ *i.e.*, $\theta = 0$. Besides this the two curves also intersect at the pole $r = 0$. Since for all values of θ, $2a \cos\theta$ *i.e.*, $a\cos\theta + a\cos\theta \le a(1 + \cos\theta)$, therefore the circle lies entirely within the cardioid.



Hence the required area

= Area of the cardioid – Area of the circle. ...(1)

Now area of the cardioid

$$= 2 \int_0^{\pi} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{\pi} a^2 (1 + \cos\theta)^2 \, d\theta = a^2 \int_0^{\pi} 4 \cos^4 \frac{1}{2}\theta \, d\theta$$

$$= 8a^2 \int_0^{\pi/2} \cos^4 \phi \, d\phi, \quad \left(\text{putting } \frac{1}{2}\theta = \phi \text{ so that } \frac{1}{2} d\theta = d\phi,\right.$$

$$\left.\text{also when } \theta = 0,\ \phi = 0 \text{ and when } \theta = \pi,\ \phi = \frac{1}{2}\pi\right)$$

$$= 8a^2 \frac{\Gamma\frac{5}{2}\Gamma\frac{1}{2}}{2\,\Gamma 3} = 8a^2 \cdot \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\sqrt{\pi}}{2.2.1} = \frac{3\pi a^2}{2}.$$

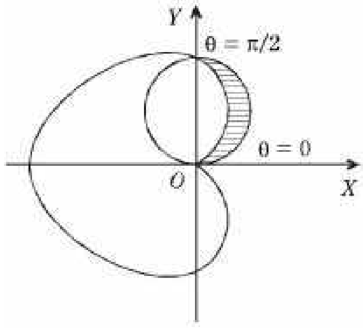
And area of the circle $= \pi a^2$, because the radius of the circle is a.

$\therefore$ From (1), the required area $= \frac{3}{2}\pi a^2 - \pi a^2 = \frac{1}{2}\pi a^2$.

Problem 2: *Find the total area inside $r = \sin\theta$ and outside $r = 1 - \cos\theta$.*

Solution: Eliminating r between the given equations, we have

$$\sin\theta = 1 - \cos\theta$$



or $\sin^2 \theta = (1 - \cos \theta)^2$

$= 1 + \cos^2 \theta - 2 \cos \theta$

or $1 - \cos^2 \theta = 1 + \cos^2 \theta - 2 \cos \theta$

or $2 \cos \theta (\cos \theta - 1) = 0$.

$\therefore$ $\cos \theta = 0$ or $\cos \theta = 1$ *i.e.*, $\theta = 0$ or $\pi/2$. Thus the two curves intersect at the points where $\theta = 0$ and $\theta = \pi/2$.

Draw the two curves in the same figure. The first curve is a circle passing through the pole and the diameter through the pole along the line $\theta = \pi/2$. The second curve is a cardioid.

$\therefore$ Required area

$$= \frac{1}{2} \int_0^{\pi/2} r^2 \, d\theta, \text{ for the curve } r = \sin \theta$$

$$- \frac{1}{2} \int_0^{\pi/2} r^2 \, d\theta, \text{ for the curve } r = 1 - \cos \theta$$

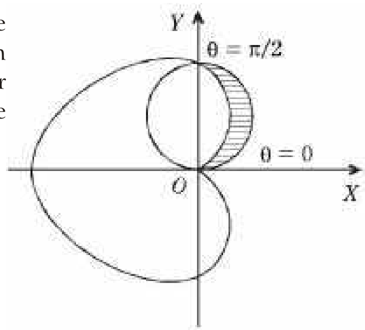
$$= \frac{1}{2} \int_0^{\pi/2} \sin^2 \theta \, d\theta - \frac{1}{2} \int_0^{\pi/2} (1 - \cos \theta)^2 \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left\{ \sin^2 \theta - (1 - \cos \theta)^2 \right\} d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} 2 (\cos \theta - \cos^2 \theta) \, d\theta = \Big[\sin \theta \Big]_0^{\pi/2} - \frac{1}{2} \cdot \frac{\pi}{2} = \left(1 - \frac{\pi}{4}\right).$$

Problem 3: *Find by double integration the area lying inside the circle* $r = a \sin \theta$ *and outside the cardioid* $r = a(1 - \cos \theta)$.

Solution: The given circle is $r = a \sin\theta$ and the cardioid is $r = a(1 - \cos\theta)$. Note that the given circle passes through the pole and the diameter through the pole makes an angle $\pi/2$ with the initial line.



Eliminating r between the two equations, we have

$a \sin\theta = a(1 - \cos\theta)$

or $$1 = \frac{\sin\theta}{1 - \cos\theta} = \frac{2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta}{2 \cos^2 \frac{1}{2}\theta} = \frac{\tan \theta}{2}$$

ro $\frac{1}{2}\theta = \frac{1}{4}\pi$ *i.e.*, $\theta = \pi/2$.

Thus the two curves meet at the point where $\theta = \pi/2$. Also for both the curves $r = 0$ when $\theta = 0$ and so the two curves also meet at the pole O where $\theta = 0$. To cover the required area the limits of integration for r are $a(1-\cos\theta)$ to $a\sin\theta$ and for θ are 0 to $\pi/2$. Therefore the required area

$$= \int_0^{\pi/2} \int_{a(1-\cos\theta)}^{a\sin\theta} r\, d\theta\, dr = \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} [a^2 \sin^2\theta - a^2(1-\cos\theta)^2]\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} [\sin^2\theta - 1 + 2\cos\theta - \cos^2\theta]\, d\theta$$

$$= \frac{a^2}{2}\left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{\pi}{2} + 2\cdot 1 - \frac{1}{2}\cdot\frac{\pi}{2}\right] = \frac{a^2}{2}\left[2 - \frac{\pi}{2}\right] = a^2\left[1 - \frac{\pi}{4}\right].$$

Problem 4: *Find the area common to the circle $r = a$ and the cardioid $r = a(1 + \cos\theta)$.*

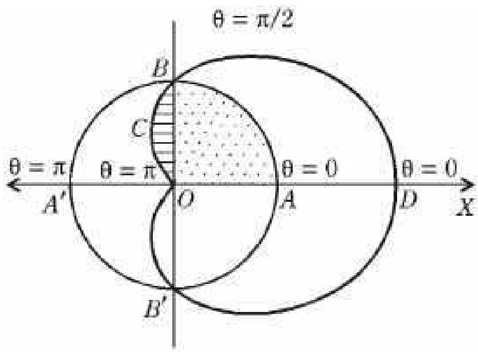
(Meerut 2007)

Solution: Eliminating r between the given equations, we get

$$a(1+\cos\theta) = a \quad \text{or} \quad \cos\theta = 0 \quad \text{or} \quad \theta = \pm\,\pi/2.$$

Thus the two curves cut each other at the point where $\theta = \pm\,\pi/2$.

Both the curves are symmetrical about the initial line and have been shown in the same figure.



Hence the required area

$= 2 \times$ area $ABCOA$, (By symmetry)

$= 2 \times$ (Area $OABO$ + Area $OBCO$) ...(1)

Now area $OABO$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2} r^2 d\theta, \text{ for } r = a$$

$$= \frac{1}{2}a^2\int_0^{\pi/2} d\theta = \frac{1}{2}a^2\int_0^{\pi/2} d\theta = \frac{1}{2}a^2[\theta]_0^{\pi/2} = \frac{1}{2}a^2\cdot\frac{\pi}{2} = \frac{\pi a^2}{4}.$$

And area $OBCO$

$$= \frac{1}{2}\int_{\pi/2}^{\pi} r^2\, d\theta, \text{ for } r = a(1+\cos\theta)$$

$$= \frac{1}{2}\int_{\pi/2}^{\pi} a^2(1+\cos\theta)^2\, d\theta = \frac{1}{2}a^2\int_{\pi/2}^{\pi}(1 + 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \frac{1}{2}a^2\int_{\pi/2}^{\pi}\{1 + 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\}\, d\theta$$

$$= \frac{1}{2}a^2\int_{\pi/2}^{\pi}(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\left[\frac{3\theta}{2} + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_{\pi/2}^{\pi} = \frac{a^2}{2}\left(\frac{3\pi}{2} - \frac{3\pi}{4} - 2\right) = \frac{a^2}{8}(3\pi - 8).$$

Hence from (1), the required area

$$= 2\left[\frac{\pi a^2}{4} + \frac{a^2}{8}(3\pi - 8)\right] = a^2\left(\frac{5\pi}{4} - 2\right).$$

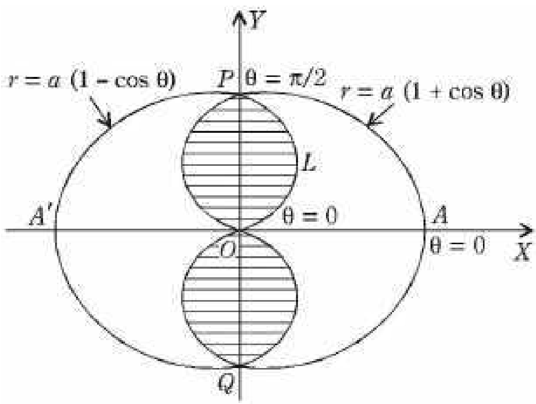
Problem 5: *Find the area of the portion included between the cardioids*

$$r = a\,(1 + \cos\theta) \text{ and } r = a\,(1 - \cos\theta).$$

Solution: Eliminating r between the given equations, we get

$$a\,(1 + \cos\theta) = a\,(1 - \cos\theta) \quad \text{or} \quad 2\cos\theta = 0 \text{ i.e., } \theta = \pm\,\pi/2,$$

showing that the two curves meet at the points $P\,(\theta = \pi/2)$ and $Q\,(\theta = -\,\pi/2)$.



Also by symmetry it is clear that the required area

$= 4 \times$ area $OLPO$,

where OLP is an arc of the cardioid $r = a\,(1 - \cos\theta)$

$$= 4\int_0^{\pi/2} \frac{1}{2} r^2 d\theta = 4\int_0^{\pi/2} \frac{1}{2} a^2\,(1 - \cos\theta)^2\, d\theta$$

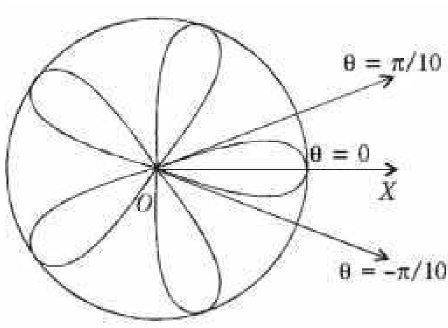
$$= 2a^2 \int_0^{\pi/2} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= 2a^2 \int_0^{\pi/2} \left[1 - 2\cos\theta + \frac{1}{2}(1 + \cos 2\theta)\right] d\theta$$

$$= 2a^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} = 2a^2\left(\frac{3}{4}\pi - 2\right).$$

Problem 6: *Show that the area contained between the circle $r = a$ and the curve $r = a\cos 5\theta$ is equal to three-fourth of the area of the circle.* **(Meerut 2013)**

Solution: The curve $r = a\cos 5\theta$ has five loops ($\because$ here $n = 5$ is odd). Also putting $r = 0$, we get



$\cos 5\theta = 0$ or $5\theta = \pm \frac{1}{2}\pi$ or $\theta = \pm \frac{\pi}{10}$.

Therefore θ varies from $-\pi/10$ to $\pi/10$ for the first loop. Also the curve is symmetrical about the initial line. Further giving values to θ from 0 to 2π in $r = a\cos 5\theta$, we observe that the maximum value of r is a and hence the curve $r = a\cos 5\theta$ lies completely inside the circle $r = a$ as shown in the figure.

Now area of the five loops of the curve $r = a\cos 5\theta$

$$= 5 \text{ times area of the one loop}$$

$$= 5\int_{-\pi/10}^{\pi/10} \frac{1}{2} r^2\, d\theta = \frac{5}{2}\int_{-\pi/10}^{\pi/10} a^2 \cos^2 5\theta\, d\theta$$

$$= \frac{5}{2}\cdot 2\int_0^{\pi/10} a^2 \cos^2 5\theta\, d\theta = a^2 \int_0^{\pi/2} \cos^2 \phi\, d\phi,$$

[putting $5\theta = \phi$ so that $5\, d\theta = d\phi$; also when $\theta = 0$, $\phi = 0$ and when $\theta = \pi/10$, $\phi = \pi/2$]

$$= a^2 \cdot \frac{1}{2}\cdot\frac{1}{2}\pi = \frac{1}{4}\pi a^2.$$

Also area of the circle $= \pi a^2$.

$\therefore$ Area contained between the two curves

$$= \text{area of the circle} - \text{area of the curve } r = a\cos 5\theta$$

$$= \pi a^2 - \frac{1}{4}\pi a^2 = (3/4).\pi a^2 = (3/4) \text{ of the area of the circle.}$$

Problem 7: *Find the area between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote.*

(Purvanchal 2010)

Solution: The given curve is symmetrical about the initial line. When $\theta = 0$, $r = 2a$.

The given equation of the curve can be written as

$$r = a\{(1/\cos\theta) + \cos\theta\} = a(1 + \cos^2\theta)/\cos\theta$$

or $$\frac{1}{r} = \frac{\cos\theta}{a(1 + \cos^2\theta)} = f(\theta), \text{ say.}$$

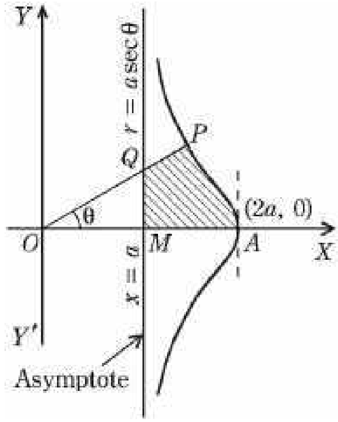
Now $f(\theta) = 0$ gives $\cos\theta = 0$ *i.e.*, $\theta = \pi/2$.

Also $$f'(\theta) = \frac{1}{a}\frac{-\sin\theta(1 + \cos^2\theta) - \cos\theta(-2\sin\theta\cos\theta)}{(1 + \cos^2\theta)^2}. \qquad ...(1)$$

Putting $\theta = \pi/2$ in (1), we get $f'(\pi/2) = -1/a$.

$\therefore$ Asymptote of the curve is

$$r\sin(\theta - \tfrac{1}{2}\pi) = \frac{1}{(-1/a)} \quad \text{or} \quad r\cos\theta = a \;\; i.e., \;\; r = a\sec\theta.$$



The cartesian equation of the asymptote is $x = a$. To find the area between the curve and its asymptote.

Let OQP be a radius vector cutting the curve at P and the asymptote $r = a \sec \theta$ at Q. Let $\angle XOP = \theta$. Then the shaded area $MAPQM$

$$= \text{area } OAPO - \text{area } OMQO.$$

Now area $OAPO$

$$= \int_0^\theta \frac{1}{2} r^2 \, d\theta, \quad \text{for the given curve}$$

$$= \frac{1}{2} \int_0^\theta a^2 (\sec \theta + \cos \theta)^2 \, d\theta$$

$$= \frac{1}{2} \int_0^\theta a^2 \, (\sec^2 \theta + \cos^2 \theta + 2) \, d\theta.$$

Also area $OMQO$

$$= \int_0^\theta \frac{1}{2} r^2 \, d\theta, \text{for the st. line } r = a \sec \theta$$

$$= \frac{1}{2} \int_0^\theta a^2 \sec^2 \theta \, d\theta.$$

$\therefore$ The shaded area $MAPQM$

$$= \frac{1}{2} \int_0^\theta a^2 \, (\sec^2 \theta + \cos^2 \theta + 2) \, d\theta - \frac{1}{2} \int_0^\theta a^2 \sec^2 \theta \, d\theta$$

$$= \frac{1}{2} \int_0^\theta a^2 \, (\cos^2 \theta + 2) \, d\theta = \frac{1}{2} a^2 \int_0^\theta \{\frac{1}{2} (1 + \cos 2\theta) + 2\} \, d\theta$$

$$= \frac{1}{2} a^2 \int_0^\theta \left(\frac{5}{2} + \frac{1}{2} \cos 2\theta\right) d\theta = \frac{1}{2} a^2 \left[\frac{5}{2} \theta + \frac{1}{2} \frac{\sin 2\theta}{2}\right]_0^\theta$$

$$= \frac{1}{2} a^2 \left(\frac{5}{2} \theta + \frac{1}{4} \sin 2\theta\right) = \frac{1}{8} a^2 \, (10 \, \theta + \sin 2\theta).$$

Now we move the point P further off along the curve. Its vectorial angle goes on increasing and ultimately when its distance from O tends to infinity its vectorial angle tends to $\pi/2$.

Hence the area between the curve and its asymptote lying above the x-axis

$$= \text{limit of the shaded area when } \theta \to \pi/2$$

$$= \lim_{\theta \to \pi/2} \frac{a^2}{8} [10 \, \theta + \sin 2\theta] = \frac{a^2}{8} \cdot 10 \cdot \left(\frac{1}{2} \pi\right) = \frac{5}{8} \pi \, a^2.$$

$\therefore$ By symmetry, the total area between the curve and the asymptote

$$= 2 \, .(5\pi \, a^2/8) = 5\pi \, a^2/4.$$

Aliter: The above area can also be easily found by changing the equation of the curve to cartesian form. The equation of the curve can be written as

$$r \cos \theta = a \, (1 + \cos^2 \theta) \quad \text{or} \quad r \cos \theta - a = a \cos^2 \theta$$

or $\quad r^2 (r\cos\theta - a) = a r^2 \cos^2\theta$, multiplying both sides by r^2.

Now putting $r\cos\theta = x$ and $r^2 = x^2 + y^2$, we get

$$(x^2 + y^2)(x - a) = ax^2 \quad \text{or} \quad y^2 (x - a) = ax^2 - x^2 (x - a)$$

or $\quad y^2 (x - a) = x^2 (2a - x)$,

which is the cartesian equation of the given curve.

Now trace the curve with the help of this cartesian equation. The curve is symmetrical about x-axis. It meets the x-axis at the point $(2a, 0)$ and the line $x = a$ is an asymptote of the curve. Here origin is a conjugate point because we get imaginary tangents at the origin. The curve does not exist in the regions $x > 2a$ and $x < a$. It exists only in the region $a < x \le 2a$.

Hence the required area

$$= 2\int_a^{2a} y\,dx = 2\int_a^{2a} x\sqrt{\left(\frac{2a - x}{x - a}\right)}\,dx,$$

(putting for y from the given equation of the curve).

Now put $x = a + a\sin^2\theta$ so that $dx = 2a\sin\theta\cos\theta\,d\theta$.

Also when $x = a$, $\sin\theta = 0$ or $\theta = 0$ and when $x = 2a$, $\sin\theta = 1$ or $\theta = \pi/2$.

$\therefore$ The required area

$$= 2\int_0^{\pi/2} (a + a\sin^2\theta)\frac{\cos\theta}{\sin\theta}\cdot 2a\sin\theta\cos\theta\,d\theta$$

$$= 4a^2\int_0^{\pi/2} (1 + \sin^2\theta)\cos^2\theta\,d\theta$$

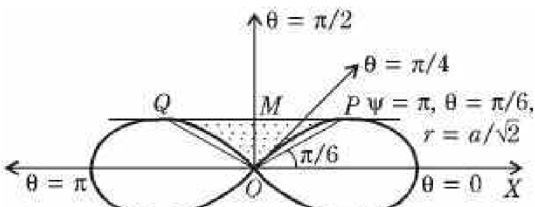
$$= 4a^2\int_0^{\pi/2} (\cos^2\theta + \cos^2\theta\sin^2\theta)\,d\theta$$

$$= 4a^2\left[\frac{1}{2}\cdot\frac{\pi}{2} + \frac{1.1}{4.2}\cdot\frac{\pi}{2}\right], \qquad \text{[By Walli's formula]}$$

$$= 4a^2\cdot\frac{1}{2}\cdot\frac{\pi}{2}\left[1 + \frac{1}{4}\right] = 4a^2\cdot\frac{1}{4}\pi\cdot\frac{5}{4} = \frac{5}{4}\pi a^2.$$

Problem 8: *O is the pole of the lemniscate $r^2 = a^2\cos 2\theta$ and PQ is a common tangent to its two loops. Find the area bounded by the line PQ and the arcs OP and OQ of the curve.*

Solution: The given curve is symmetrical about the initial line and also about the pole. The curve consists of two loops as shown in the figure.



From the figure it is clear that the common tangent to the two loops of the curve *i.e.*, the line PQ is parallel to the axis of x. Thus the tangent PQ at the point P makes an angle π with the x-axis *i.e.*, $\psi = \pi$ at P.

Now differentiating the given equation $r^2 = a^2 \cos 2\theta$, we get

$$2r \frac{dr}{d\theta} = a^2(-2\sin 2\theta), \quad \text{or} \quad \frac{dr}{d\theta} = -\frac{a^2 \sin 2\theta}{r}.$$

$$\therefore \quad \tan\phi = r\frac{d\theta}{dr} = -\frac{r \times r}{a^2 \sin 2\theta} = -\frac{a^2 \cos 2\theta}{a^2 \sin 2\theta} = -\cot 2\theta = \tan\left(\frac{1}{2}\pi + 2\theta\right).$$

$$\therefore \quad \phi = \frac{1}{2}\pi + 2\theta.$$

But for any point of the curve,

$$\psi = \theta + \phi = \theta + \left(\frac{1}{2}\pi + 2\theta\right) = \frac{1}{2}\pi + 3\theta.$$

Since at P, $\psi = \pi$, therefore at P, $\pi = \frac{1}{2}\pi + 3\theta$ or $\theta = \pi/6$.

Thus the vectorial angle θ of the point P is $\pi/6$ and the radius vector OP is given by $OP^2 = a^2 \cos\{2.(\pi/6)\} = a^2/2$.

$$\therefore \quad OP = a/\sqrt{2}.$$

Also putting $r = 0$ in the equation $r^2 = a^2 \cos 2\theta$, we get

$$\cos 2\theta = 0 \quad \text{or} \quad 2\theta = \pm \pi/2 \quad \text{or} \quad \theta = \pm \pi/4.$$

Thus $\theta = \pm \pi/4$ are the tangents at the pole to the curve.

Now the required area (*i.e.*, the dotted area)

$$= 2\,[\text{ area of the } \Delta\, OPM - \text{area of the segment } OPSO \text{ of the curve } r^2 = a^2\cos 2\theta\,]$$

$$= 2\left[\frac{1}{2} OM\,.OP - \int_{\pi/6}^{\pi/4} \frac{1}{2} r^2\, d\theta, \text{ for the curve } r^2 = a^2 \cos 2\theta\right]$$

$$= 2\left[\frac{1}{2}\left(OP \sin\frac{1}{6}\pi\right).\left(OP\cos\frac{1}{6}\pi\right) - \int_{\pi/6}^{\pi/4} \frac{1}{2} a^2 \cos 2\theta\, d\theta\right]$$

$$= 2\left[\frac{1}{2} OP^2 \sin\frac{1}{6}\pi \cos\frac{1}{6}\pi - \frac{1}{2}a^2\left(\frac{\sin 2\theta}{2}\right)_{\pi/6}^{\pi/4}\right]$$

$$= 2\left[\frac{1}{2}\cdot\frac{a^2}{2}\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} - \frac{1}{4}a^2\left(1 - \frac{\sqrt{3}}{2}\right)\right]$$

$$= 2a^2\left[\frac{\sqrt{3}}{16} - \frac{1}{4} + \frac{\sqrt{3}}{8}\right]$$

$$= \frac{2a^2}{16}[\sqrt{3} - 4 + 2\sqrt{3}]$$

$$= \frac{a^2}{8}(3\sqrt{3} - 4).$$

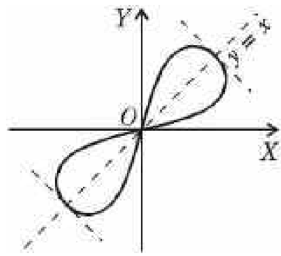
Comprehensive Problems 6

Problem 1: *Find the area of a loop of the curve* $x^4 + y^4 = 4a^2xy$.

Solution: Changing the equation of the curve $x^4 + y^4 = 4a^2xy$ into polar form by putting $x = r\cos\theta$, $y = r\sin\theta$, we have

$$r^4(\cos^4\theta + \sin^4\theta) = 4a^2r^2\cos\theta\sin\theta$$

or $$r^2 = 4a^2\sin\theta\cos\theta/(\cos^4\theta + \sin^4\theta). \quad ...(1)$$



From (1), $r = 0$, when $\sin\theta = 0$ or $\cos\theta = 0$ *i.e.*, $\theta = 0$ or $\pi/2$.

$\therefore$ A loop lies between $\theta = 0$ and $\theta = \pi/2$.

Hence the required area of a loop

$$= \frac{1}{2}\int_0^{\pi/2} r^2\, d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{4a^2\sin\theta\cos\theta}{\cos^4\theta + \sin^4\theta}\, d\theta, \text{ from (1)}$$

$$= 2a^2\int_0^{\pi/2} \frac{\tan\theta\sec^2\theta}{1 + \tan^4\theta}\, d\theta, \text{ dividing the Nr. and Dr. by } \cos^4\theta.$$

Now put $\tan^2\theta = t$ so that $2\tan\theta\sec^2\theta\, d\theta = dt$.

Also when $\theta = 0$, $t = 0$ and when $\theta \to \pi/2$, $t \to \infty$.

$\therefore$ The required area

$$= a^2\int_0^{\infty} \frac{dt}{1 + t^2} = a^2[\tan^{-1} t]_0^{\infty} = a^2[\tan^{-1}\infty - \tan^{-1}0] = a^2(\pi/2).$$

Problem 2: *Find the area of a loop of the curve* $(x^2 + y^2)^2 = 4axy^2$.

Solution: Changing to polars by putting $x = r\cos\theta$, $y = r\sin\theta$, the equation of the curve becomes

$$(r^2)^2 = 4a(r\cos\theta)(r\sin\theta)^2 \text{ or } r = 4a\cos\theta\sin^2\theta. \quad ...(1)$$

From (1), $r = 0$ when $\sin^2\theta = 0$ or $\cos\theta = 0$ *i.e.*, $\theta = 0$ or $\pi/2$.

$\therefore$ A loop lies between $\theta = 0$ and $\theta = \pi/2$.

Hence the required area of a loop

$$= \frac{1}{2}\int_0^{\pi/2} r^2\, d\theta = \frac{1}{2}\times 16a^2\int_0^{\pi/2}\cos^2\theta\sin^4\theta\, d\theta, \quad \text{[From (1)]}$$

$$= 8a^2\cdot\frac{1.3.1}{6.4.2}\cdot\frac{\pi}{2} = \frac{\pi a^2}{4}.$$

Problem 3: *Prove that the area of a loop of the curve* $x^6 + y^6 = a^2y^2x^2$ *is* $\pi a^2/12$.

Solution: Put $x = r\cos\theta$ and $y = r\sin\theta$ to change the equation of the curve into polar form. Then the curve becomes

$$r^6(\cos^6\theta + \sin^6\theta) = a^2r^4 . \cos^2\theta\sin^2\theta$$

or $$r^2 = a^2 \sin^2\theta \cos^2\theta/(\cos^6\theta + \sin^6\theta). \quad ...(1)$$

From (1), $r = 0$ when $\sin^2\theta = 0$ or $\cos^2\theta = 0$ *i.e.*, $\theta = 0$ or $\pi/2$.

$\therefore$ A loop lies between $\theta = 0$ and $\theta = \pi/2$.

Hence the required area of the loop

$$= \frac{1}{2}\int_0^{\pi/2} r^2 \, d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{a^2 \sin^2\theta \cos^2\theta \, d\theta}{\cos^6\theta + \sin^6\theta} \quad \text{[From (1)]}$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \frac{\tan^2\theta \sec^2\theta}{1 + \tan^2\theta} d\theta, \text{ dividing the Nr. and Dr. by } \cos^6\theta.$$

Now put $\tan^3\theta = t$ so that $3\tan^2\theta \sec^2\theta \, d\theta = dt$.

Also when $\theta = 0$, $t = 0$ and when $\theta \to \pi/2$, $t \to \infty$.

$\therefore$ The required area

$$= \frac{a^2}{6}\int_0^{\infty} \frac{dt}{1 + t^2} = \frac{a^2}{6}[\tan^{-1} t]_0^{\infty}$$

$$= \frac{1}{6} a^2 [\tan^{-1}\infty - \tan^{-1} 0]$$

$$= \frac{1}{6} a^2 \cdot \frac{\pi}{2} = \frac{1}{12}\pi a^2.$$

Problem 4: *Find the area of a loop of the curve* $x^4 + 3x^2y^2 + 2y^4 = a^2 xy$.

Solution: Put $x = r\cos\theta$ and $y = r\sin\theta$ to change the equation of the curve into polar form. Then the curve becomes

$$r^4(\cos^4\theta + 3\cos^2\theta \sin^2\theta + 2\sin^4\theta) = a^2 r^2 \cos\theta \sin\theta$$

or $$r^2 = \frac{a^2 \sin\theta \cos\theta}{\cos^4\theta + 3\cos^2\theta \sin^2\theta + 2\sin^4\theta}. \quad ...(1)$$

From (1), $r = 0$ when $\sin\theta = 0$ or $\cos\theta = 0$ *i.e*, $\theta = 0$ or $\pi/2$.

$\therefore$ A loop lies between $\theta = 0$ and $\theta = \pi/2$.

Hence the required area of the loop

$$= \frac{1}{2}\int_0^{\pi/2} r^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{a^2 \sin\theta \cos\theta \, d\theta}{\cos^4\theta + 3\cos^2\theta \sin^2\theta + 2\sin^4\theta}, \quad \text{[From (1)]}$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \frac{\tan\theta \sec^2\theta \, d\theta}{1 + 3\tan^2\theta + 2\tan^4\theta},$$

dividing the Nr. and Dr. by $\cos^4\theta$.

Now put $\tan^2\theta = t$ so that $2\tan\theta \sec^2\theta \, d\theta = dt$ and the new limits are $t = 0$ to $t = \infty$.

$\therefore$ The required area

$$= \frac{a^2}{4}\int_0^{\infty} \frac{dt}{1 + 3t + 2t^2} = \frac{a^2}{4}\int_0^{\infty} \frac{dt}{(1 + t)(1 + 2t)}$$

$$= \frac{a^2}{4} \int_0^{\infty} \left(\frac{2}{1+2t} - \frac{1}{1+t} \right) dt,$$ [By partial fractions]

$$= \frac{a^2}{4} \left[\log(1+2t) - \log(1+t) \right]_0^{\infty} = \frac{a^2}{4} \left[\log \left(\frac{1+2t}{1+t} \right) \right]_0^{\infty}$$

$$= \frac{a^2}{4} \lim_{t \to \infty} \log \left\{ \frac{(1/t)+2}{(1/t)+1} \right\} - \frac{a^2}{4} \left[\log \left(\frac{1+2t}{1+t} \right) \right]_{t=0}$$ **(Note)**

$$= \frac{1}{4} a^2 [\log(2/1)] - \frac{1}{4} a^2 [\log(1/1)] = \frac{1}{4} a^2 \log 2.$$

Problem 5: *Prove that the area of a loop of the curve $x^5 + y^5 = 5a\, x^2 y^2$ is five times the area of one loop of the curve $r^2 = a^2 \cos 2\theta$.* **(Purvanchal 2014)**

Solution: Changing the equation of the curve $x^5 + y^5 = 5a\, x^2 y^2$ into polar form by putting $x = r\cos\theta$ and $y = r\sin\theta$, we get

$$r^5 (\cos^5\theta + \sin^5\theta) = 5ar^4 \cos^2\theta \sin^2\theta$$

or $$r = a\sin^2\theta\cos^2\theta/(\cos^5\theta + \sin^5\theta).$$ ...(1)

From (1), $r = 0$ when $\sin^2\theta = 0$ or $\cos^2\theta = 0$ *i.e.*, when $\theta = 0$ or $\theta = \pi/2$.

$\therefore$ The loop of the curve (1) lies between $\theta = 0$ and $\theta = \pi/2$.

Hence the required area of the loop of $x^5 + y^5 = 5a\, x^2 y^2$

$$= \frac{1}{2} \int_0^{\pi/2} r^2 \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \frac{25a^2 \sin^4\theta \cos^4\theta}{(\sin^5\theta + \cos^5\theta)^2} d\theta,$$ [From (1)]

$$= \frac{25a^2}{2} \int_0^{\pi/2} \frac{\tan^4\theta \sec^2\theta \, d\theta}{(\tan^5\theta + 1)^2},$$ dividing the Nr. and Dr. by $\cos^{10}\theta$.

Now put $\tan^5\theta + 1 = t$ so that $5\tan^4\theta \sec^2\theta \, d\theta = dt$. Also when $\theta = 0$, $t = 1 + \tan 0 = 1$ and when $\theta \to \pi/2$, $t \to \infty$.

$\therefore$ The area of the loop

$$= \frac{25a^2}{2} \int_1^{\infty} \frac{1}{t^2} \frac{dt}{5} = \frac{5a^2}{2} \int_1^{\infty} \frac{1}{t^2} dt$$

$$= \frac{5a^2}{2} \left[\frac{-1}{t} \right]_1^{\infty} = \frac{5a^2}{2}.$$

Also from Example 13, the area of one loop of the curve $r^2 = a^2 \cos 2\theta$ is $\frac{1}{2} a^2$.

(Deduce it here.)

Thus the area of a loop of $x^5 + y^5 = 5ax^2 y^2$ is

$$= \frac{5}{2} a^2 = 5 \cdot (\frac{1}{2} a^2) = 5.$$ [area of a loop of $r^2 = a^2 \cos 2\theta$]

Hints to Objective Type Questions

Multiple Choice Questions

1. The area bounded by the axis of x, the curve $y = \sin^2 x$ and the ordinates $x = 0, x = \frac{\pi}{2}$ is

$$= \int_0^{\pi/2} y\,dx = \int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}\cdot$$

2. The curve $3ay^2 = x(x-a)^2$ meets the x-axis where $y = 0$ *i.e.*, where $x(x-a)^2 = 0$ *i.e.*, where $x = 0$ or where $x = a$.

When $0 < x < a$, y^2 is positive *i.e.*, y is real. So, the loop of the curve lies between $x = 0$, $x = a$.

3. The given curve is $r^2 = a^2 \cos 2\theta$, which is symmetrical about the initial line.

We have $r = 0$ when $\cos 2\theta = 0$ *i.e.*, when $2\theta = -\frac{\pi}{2}$ or $2\theta = \frac{\pi}{2}$ *i.e.*, when $\theta = -\frac{\pi}{4}$ or $\theta = \frac{\pi}{4}\cdot$ Thus, two consecutive values of θ for which r is zero are $-\pi/4$ and $\pi/4$. So, one loop of the curve lies between the lines $\theta = -\pi/4$ and $\theta = \pi/4$ and it is symmetrical about the initial line.

$\therefore$ Area of one loop of the curve $r^2 = a^2 \cos 2\theta$

$$= 2\,.\int_0^{\pi/4} \frac{1}{2} r^2\,d\theta = \int_0^{\pi/4} a^2 \cos 2\theta\,d\theta$$

$$= a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4} = \frac{a^2}{2}\left[\sin\frac{\pi}{2} - \sin 0\right] = \frac{a^2}{2}.$$

4. See Problem 3 (ii) of Comprehensive Problems 1.

5. Required area $= \frac{1}{2}\int_0^{2\pi} r^2\,d\theta = \frac{1}{2}\int_0^{2\pi} a^2\,d\theta = \frac{1}{2}a^2[\theta]_0^{2\pi} = \frac{2\pi a^2}{2} = \pi a^2.$

6. See Problem 8 of Comprehensive Problems 4.
7. See article 1.
8. See Example 2.
9. See Example 4.
10. See Example 15.

Fill in the Blanks

1. See article 1.
2. See article 2.

3. The required area

$$= \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} a^2 e^{2m\theta}\, d\theta$$

$$= \frac{1}{2} a^2 \left[\frac{e^{2m\theta}}{2m}\right]_{\alpha}^{\beta} = \frac{a^2}{4m}(e^{2m\beta} - e^{2m\alpha}).$$

4. The number of loops in the curve $r = a \sin n\theta$ is n if n is odd and is $2\,n$ if n is even. Here, $n = 3$ which is odd. So, the curve $r = a \sin 3\theta$ has three loops.

5. The required area

$$= \int_0^a y\, dx = \int_0^a c \cosh\left(\frac{x}{c}\right) dx = c\left[c \sinh \frac{x}{c}\right]_0^a = c^2\left[\sinh\frac{a}{c} - \sinh 0\right] = c^2 \sinh\frac{a}{c}.$$

True or False

1. See article 5.
2. The number of loops in $r = a \cos n\theta$ is $2n$ or n according as n is even or odd.
3. See Example 12.
4. See Problem 3 (ii) of Comprehensive Problems 1.

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Chapter-8

Rectification

(Lengths of Arcs and Intrinsic Equations of Plane Curves)

Comprehensive Problems 1

Problem 1(i): *Find the arc length of the curve* $y = \frac{1}{2} x^2 - \frac{1}{4} \log x$ *from* $x = 1$ *to* $x = 2$.

(Meerut 2012B)

Solution: The given curve is $y = \frac{1}{2} x^2 - \frac{1}{4} \log x$. ...(1)

Differentiating (1) w.r.t. x, we get $\frac{dy}{dx} = x - \frac{1}{4x} = \frac{4x^2 - 1}{4x}$.

$\therefore$ Required length of the curve

$$= \int_1^2 \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx = \int_1^2 \sqrt{\left\{1 + \frac{(4x^2 - 1)^2}{16x^2}\right\}}\, dx$$

$$= \int_1^2 \frac{4x^2 + 1}{4x}\, dx = \int_1^2 \left(x + \frac{1}{4x}\right) dx = \left[\frac{x^2}{2} + \frac{\log x}{4}\right]_1^2 = \frac{3}{2} + \frac{1}{4} \log 2 .$$

Problem 1(ii): *Find the length of the curve* $y = \log\,[(e^x - 1)/(e^x + 1)]$ *from* $x = 1$ *to* $x = 2$.

(Agra 2001, 02, 06; Meerut 04B; Avadh 08; Kanpur 11; Kashi 13; Rohilkhand 13)

Solution: The given curve is $y = \log\,[(e^x - 1)/(e^x + 1)]$

or $\quad y = \log\,(e^x - 1) - \log\,(e^x + 1)$. ...(1)

Differentiating (1) w.r.t. x, we get $\frac{dy}{dx} = \frac{e^x}{e^x - 1} - \frac{e^x}{e^x + 1} = \frac{2e^x}{(e^{2x} - 1)}$.

Now $\quad \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left\{\frac{2e^x}{e^{2x} - 1}\right\}^2 = 1 + \frac{4e^{2x}}{(e^{2x} - 1)^2}$

$$= \frac{(e^{2x} - 1)^2 + 4e^{2x}}{(e^{2x} - 1)^2} = \frac{(e^{2x} + 1)^2}{(e^{2x} - 1)^2}.$$

Measuring the arc length s in the direction of x increasing, we have

$$\frac{ds}{dx} = \frac{e^{2x} + 1}{e^{2x} - 1} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \text{ dividing the Nr. and the Dr. by } e^x$$

or $$ds = \frac{e^x + e^{-x}}{e^x - e^{-x}} dx.$$

$\therefore$ The required length s_1 is given by

$$s_1 = \int_1^2 \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \Big[\log (e^x - e^{-x})\Big]_1^2,$$

numerator being the diff. coeff. of denominator

$$= \log (e^2 - e^{-2}) - \log (e - e^{-1})$$

$$= \log [\{e^2 - (1/e^2)\}/\{e - (1/e)\}] = \log (e + 1/e).$$

Problem 2(i): *Show that in the catenary $y = c \cosh (x/c)$, the length of arc from the vertex to any point is given by $s = c \sinh (x/c)$.*

Solution: The given catenary is $y = c \cosh (x/c)$. ...(1)

The point $A\,(0, c)$ is the vertex of the catenary and let $P\,(x, y)$ be any point on it. We have to find the length of arc AP for which x varies from $x = 0$ to $x = x$.

Differentiating (1) w.r.t. x, we have

$$\frac{dy}{dx} = c \cdot \frac{1}{c} \sinh \frac{x}{c} = \sinh \frac{x}{c} \cdot$$

If s denotes the arc length of the catenary measured in the direction of x increasing, then

$$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left(1 + \sinh^2 \frac{x}{c}\right)} = \cosh \frac{x}{c}$$

or $$ds = \cosh (x/c)\, dx.$$

Integrating, we have

$$\int_0^s ds = \int_0^x \cosh \frac{x}{c}\, dx = \left[c \sinh \frac{x}{c}\right]_0^x = c \sinh \frac{x}{c} \text{ or } s = c \sinh (x/c),$$

which is the required arc length.

Problem 2(ii): *If s be the length of the arc of the catenary $y = c \cosh (x/c)$ from the vertex $(0, c)$ to the point (x, y), show that $s^2 = y^2 - c^2$.* **(Lucknow 2005)**

Solution: The given curve is $y = c \cosh (x/c)$. ...(1)

Proceeding as in Problem 2(i), the length s of arc extending from the vertex $(0, c)$ to any point (x, y) is given by $s = c \sinh (x/c)$. ...(2)

Squaring and subtracting (2) from (1), we get

$$y^2 - s^2 = c^2 \cosh^2 (x/c) - c^2 \sinh^2 (x/c) = c^2$$

or $$y^2 - c^2 = s^2.$$ This was to be proved.

Problem 3(i): *Find the length of an arc of the parabola $y^2 = 4ax$ measured from the vertex.*

Solution: Let the given parabola be $y^2 = 4ax$. ...(1)

Differentiating (1) w.r.t. x, we get $2y\,(dy/dx) = 4a$.

$\therefore \qquad dy/dx = 2a/y \quad$ or $\quad dx/dy = y/2a.$

If s denotes the arc length of the parabola $y^2 = 4ax$ measured from the vertex O to any point $P\ (x, y)$ lying on the upper half of the parabola, then s increases as y increases.

$\therefore$ The required length of arc $s = \int_0^y \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}\, dy$ **(Note)**

$$= \int_0^y \sqrt{\left(1 + \frac{y^2}{4a^2}\right)}\, dy = \frac{1}{2a}\int_0^y \sqrt{(y^2 + 4a^2)}\, dy$$

$$= \frac{1}{2a}\left[\frac{y}{2}\sqrt{(y^2 + 4a^2)} + \frac{4a^2}{2}\log\{y + \sqrt{(y^2 + 4a^2)}\}\right]_0^y$$

$$= \frac{1}{4a}[y\sqrt{(y^2 + 4a^2)} + 4a^2 \log\{y + \sqrt{(y^2 + 4a^2)}\} - 4a^2 \log 2a]$$

$$= \frac{1}{4a}\left[y\sqrt{(y^2 + 4a^2)} + 4a^2 \log\left\{\frac{y + \sqrt{(y^2 + 4a^2)}}{2a}\right\}\right].$$

Problem 3(ii): *Find the length of the arc of the parabola $y^2 = 4ax$ cut off by its latus rectum.*

(Kumaun 2001)

Solution: Here we have to find the arc length $L'OL$, which is double of the length found in Example 2.

Problem 4(i): *Find the length of the arc of the parabola $x^2 = 4ay$ from the vertex to an extremity of the latus rectum.* **(Kanpur 2008; Purvanchal 09)**

Solution: The given parabola is $x^2 = 4ay$, ...(1)

whose vertex is the point (0, 0) and whose axis is along the y-axis.

Let s_1 denote the arc length of the parabola measured from the vertex $O(0, 0)$ to an extremity of the latus rectum $(2a, a)$.

Differentiating (1) w.r.t. x, we get

$$2x = 4a\,(dy/dx) \quad \text{or} \quad (dy/dx) = x/2a.$$

$\therefore$ The required arc length $s_1 = \int_0^{2a} \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx$ **(Note)**

$$= \int_0^{2a} \sqrt{\left\{1 + \frac{x^2}{4a^2}\right\}}\, dx = \frac{1}{2a}\int_0^{2a} \sqrt{(x^2 + 4a^2)}\, dx$$

$$= \frac{1}{2a}\left[\frac{x}{2}\sqrt{(x^2 + 4a^2)} + \frac{4a^2}{2}\log\{x + \sqrt{(x^2 + 4a^2)}\}\right]_0^{2a}$$

$= a[\sqrt{2} + \log(1 + \sqrt{2})].$ [Proceeding as in Example 2]

Problem 4(ii): *Find the length of the arc of the parabola* $x^2 = 8y$ *from the vertex to an extremity of the latus rectum.*

Solution: Proceed exactly as in part (i). Here $a = 2$.

Problem 5(i): *Find the length of the arc of the parabola* $y^2 = 4ax$ *cut off by the line* $y = 3x$.

Solution: We have $y^2 = 4ax$ ...(1)

and $y = 3x$. ...(2)

Substituting for x in (1) from (2), we get

$$y^2 = 4a.(y/3) \quad \text{or} \quad 3y^2 - 4ay = 0 \text{ giving } y = 0, 4a/3.$$

$\therefore$ From (2), the corresponding values of x are 0 and $4a/9$. Hence the points of intersection of the parabola and the given line are $(0,0)$ and $(4a/9, 4a/3)$.

Also differentiating (1) w.r.t. x, we get $2y\,(dy/dx) = 4a$.

$\therefore$ $dy/dx = 2a/y$ or $dx/dy = y/2a$.

If s denotes the arc length of the parabola measured from the point $(0,0)$ to the point $(4a/9, 4a/3)$, then

$$s = \int_0^{4a/3} \sqrt{\left[1 + \left(\frac{dx}{dy}\right)^2\right]}\,dy = \int_0^{4a/3} \sqrt{\left[1 + \frac{y^2}{4a^2}\right]}\,dy$$

$$= \frac{1}{2a}\int_0^{4a/3} \sqrt{(y^2 + 4a^2)}\,dy$$

$$= \frac{1}{2a}\left[\frac{y}{2}\sqrt{(y^2 + 4a^2)} + \frac{4a^2}{2}\log\{y + \sqrt{(y^2 + 4a^2)}\}\right]_0^{4a/3}$$

$$= \frac{1}{2a}\left[\frac{1}{2}\cdot\frac{4}{3}a\sqrt{\left(\frac{16}{9}a^2 + 4a^2\right)} + 2a^2\log\left\{\frac{4}{3}a + \sqrt{\left(\frac{16}{9}a^2 + 4a^2\right)}\right\} - 2a^2\log(2a)\right]$$

$$= \frac{1}{2a}\left[\frac{2}{3}a\cdot\frac{2}{3}a\sqrt{(13)} + 2a^2\log\left\{\frac{4}{3}a + \frac{2}{3}a\sqrt{(13)}\right\} - 2a^2\log(2a)\right]$$

$$= a\left[\frac{2}{9}\sqrt{(13)} + \log\left\{\frac{2}{3} + \frac{1}{3}\sqrt{(13)}\right\}\right]$$

$$= a\left[\frac{2}{9}\sqrt{(13)} + \log\left\{\frac{2 + \sqrt{(13)}}{3}\right\}\right].$$

Problem 5(ii): *Show that the length of the arc of the parabola* $y^2 = 4ax$ *which is intercepted between the points of intersection of the parabola and the straight line* $3y = 8x$ *is* $a\left(\log 2 + \frac{15}{16}\right)$.

(Gorakhpur 2006; Purvanchal 06)

Solution: Solving $y^2 = 4ax$ and $3y = 8x$ for y, we get

$$y^2 = 4a.(3y/8) \quad \text{or} \quad 2y^2 - 3ay = 0 \quad \text{or} \quad y = 0, 3a/2.$$

Thus the parabola and the straight line intersect at the points where $y = 0$ and $3a/2$. We need not find the x-coordinates of these points.

Also differentiating $y^2 = 4ax$, we get

$$2y\,(dy/dx) = 4a \quad \text{or} \quad dy/dx = 2a/y \quad \text{or} \quad dx/dy = y/2a.$$

∴ If s denotes the arc length of the parabola measured in the direction of y increasing, then

$$\frac{ds}{dy} = \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}} = \frac{1}{2a}\sqrt{(4a^2 + y^2)} \quad \text{or} \quad ds = \frac{1}{2a}\sqrt{(4a^2 + y^2)}\,dy.$$

Let s_1 denote the required arc length from $y = 0$ to $y = 3a/2$. Then

$$\int_0^{s_1} ds = \int_0^{3a/2} \frac{1}{2a}\sqrt{(4a^2 + y^2)}\,dy$$

or
$$s_1 = \frac{1}{2a}\left[\frac{y}{2}\sqrt{(4a^2 + y^2)} + \frac{4a^2}{2}\log\{y + \sqrt{(4a^2 + y^2)}\}\right]_0^{3a/2}$$

$$= \frac{1}{4a}\left[\frac{3a}{2}\sqrt{\left(\frac{25a^2}{4}\right)} + 4a^2\log\left\{\frac{3a}{2} + \sqrt{\left(\frac{25a^2}{4}\right)}\right\} - 4a^2\log 2a\right]$$

$$= a\left[\frac{15}{16} + \log 2\right].$$

Problem 6(i): *Find the perimeter of the curve* $x^2 + y^2 = a^2$.

(Avadh 2010; Rohilkhand 13B)

Solution: The equation of the curve is $x^2 + y^2 = a^2$. ...(1)

Here the curve is the standard equation of the circle with centre $(0, 0)$ and radius a. Also it is symmetrical about both the axes. So the required perimeter will be four times the arc length lying in the first quadrant *i.e.*, between $x = 0$ to $x = a$.

Differentiating (1), w.r.t. x, we get $2x + 2y\,(dy/dx) = 0$

or $(dy/dx) = -(x/y) = -x/\sqrt{(a^2 - x^2)}$. [From (1)]

∴ The required perimeter

$= 4 \times$ {length of the arc in the first quadrant from $(0, a)$ to $(a, 0)$}

$$= 4\int_{x=0}^{a}\sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\,dx = 4\int_0^a \sqrt{\left[1 + \frac{x^2}{a^2 - x^2}\right]}\,dx, \text{ putting for } \frac{dy}{dx}$$

$$= 4a\int_0^a \frac{dx}{\sqrt{(a^2 - x^2)}} = 4a\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= 4a\,[\sin^{-1}(1) - \sin^{-1}(0)] = 4a\left[\frac{1}{2}\pi - 0\right] = 2a\pi.$$

Problem 6(ii): *Find the length of the arc of the semi-cubical parabola* $ay^2 = x^3$ *from the vertex to the point* (a, a).

(Bundelkhand 2010; Kanpur 15)

Solution: The given curve is $ay^2 = x^3$. ...(1)

It is symmetrical about the x-axis. We have to find the length of the arc from $x = 0$ to $x = a$ in the first quadrant.

Differentiating (1) w.r.t. x, we get $\quad 2ay\,(dy/dx)=3x^2$.

$$\therefore \qquad \frac{dy}{dx}=\frac{3x^2}{2ay}=\frac{3x^2}{2a\,(x^3/a)^{1/2}}, \text{ substituting for } y \text{ from (1)}$$

$$=\frac{3}{2}\,(x^{1/2}/a^{1/2}).$$

$$\therefore \quad \text{Required length} = \int_{x=0}^{a}\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}\,dx=\int_0^a\sqrt{\left\{1+\frac{9x}{4a}\right\}}\,dx$$

$$=\frac{1}{2\sqrt{a}}\int_0^a (9x+4a)^{1/2}\,dx=\frac{1}{2\sqrt{a}}\left[\frac{2}{3}\cdot(9x+4a)^{3/2}\cdot\frac{1}{9}\right]_0^a$$

$$=\frac{1}{27\sqrt{a}}\,[(13a)^{3/2}-(4a)^{3/2}]=(a/27)\,.\,[13\sqrt{(13)}-8].$$

Problem 7(i): *Show that the length of the arc of the curve* $x^2=a^2\,(1-e^{y/a})$ *measured from the origin to the point* $(x,\,y)$ *is* $a\log\{(a+x)/(a-x)\}-x$. **(Rohilkhand 2010B)**

Solution: The given equation of the curve is $\quad x^2=a^2\,(1-e^{y/a})$

$$\text{or} \qquad e^{y/a}=1-\frac{x^2}{a^2} \quad \text{or} \quad \frac{y}{a}=\log\left(1-\frac{x^2}{a^2}\right)$$

$$\text{or} \qquad y=a\log\left(\frac{a^2-x^2}{a^2}\right) \quad \text{or} \quad y=a\log(a^2-x^2)-a\log a^2 \qquad \text{...(1)}$$

Differentiating (1) w.r.t. x, we get $\quad \dfrac{dy}{dx}=a\cdot\dfrac{-2x}{a^2-x^2}\cdot$

$$\therefore \quad \text{Required arc length} = \int_a^x\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}dx=\int_0^x\left[1+\frac{4a^2x^2}{(a^2-x^2)^2}\right]^{1/2}dx$$

$$=\int_0^x\left(\frac{a^2+x^2}{a^2-x^2}\right)dx=\int_0^x\left(-1+\frac{2a^2}{a^2-x^2}\right)dx \qquad \textbf{(Note)}$$

$$=\left[-x+2a^2\cdot\frac{1}{2a}\log\frac{a+x}{a-x}\right]_0^x=a\log\frac{a+x}{a-x}-x.$$

Problem 7(ii): *Prove that the length of the loop of the curve* $3ay^2=x\,(x-a)^2$ *is* $4a/\sqrt{3}$.
(Meerut 2005B, 08, 09B)

Solution: The given curve is symmetrical about the x-axis.
At $y=0$, we have $x=0$ and $x=a$. So for the loop, x varies from 0 to a.
The equation of the given curve is

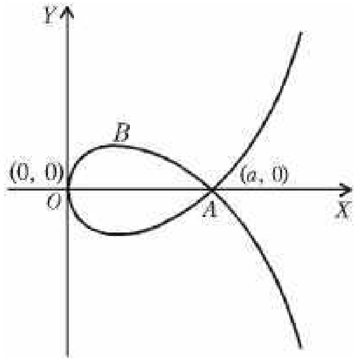
$$3ay^2=x\,(x-a)^2.$$

Taking logarithm, we have

$$\log 3a+2\log y=\log x+2\log(x-a).$$

Now differentiating w.r.t. x, we get

$$\frac{2}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{2}{x-a} = \frac{3x-a}{x(x-a)}.$$



Thus $$\left(\frac{dy}{dx}\right)^2 = \frac{(3x-a)^2}{x^2(x-a)^2}\cdot\frac{y^2}{4}$$

$$= \frac{(3x-a)^2\, x(x-a)^2}{x^2(x-a)^2 \,.12a}$$

$$= \frac{(3x-a)^2}{12ax}.$$

$\therefore$ $$\frac{ds}{dx} = \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}$$

$$= \sqrt{\left\{1+\frac{(3x-a)^2}{12ax}\right\}} = \frac{3x+a}{\sqrt{(12ax)}}.$$

$\therefore$ The required length of the whole loop

$$= 2\int_0^a \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\, dx \qquad \text{[By symmetry]}$$

$$= 2\int_0^a \frac{3x+a}{\sqrt{(12ax)}}\,dx = \frac{1}{\sqrt{(3a)}}\int_0^a (3x^{1/2} + ax^{-1/2})\,dx$$

$$= \frac{1}{\sqrt{(3a)}}\left[3\cdot\frac{2}{3}x^{3/2} + 2ax^{1/2}\right]_0^a = \frac{1}{\sqrt{(3a)}}[4a^{3/2}] = \frac{4a}{\sqrt{3}}.$$

Problem 8(i): *Find the perimeter of the loop of the curve* $9ay^2 = (x-2a)(x-5a)^2$.

Solution: The given equation of the curve is $9ay^2 = (x-2a)(x-5a)^2$. ...(1)

Shifting the origin to the point $(2a, 0)$, the equation (1) becomes $9ay^2 = x(x-3a)^2$. Now this is similar to the curve of Problem 7(ii). (Here we have $3a$ in place of a). So to find the required length, proceed exactly as in Problem 7(ii). The required length is

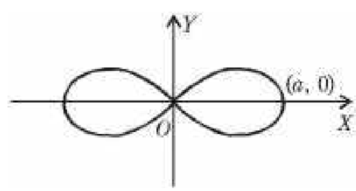
$$= 2\int_0^{3a} \frac{x+a}{\sqrt{(4ax)}}\,dx = 4a\sqrt{3}.$$

Problem 8(ii): *Show that the whole length of the curve* $x^2(a^2-x^2) = 8a^2y^2$ *is* $\pi a\sqrt{2}$.

(Bundelkhand 2006; Purvanchal 11)

Solution: The given curve is

$$x^2(a^2-x^2) = 8a^2y^2. \qquad ...(1)$$



The curve (1) is symmetrical about both the axes and it passes through the origin. Putting $y = 0$ in the given equation of the curve, we get

$$x^2(a^2-x^2) = 0$$

i.e., $$x = 0, x = \pm a.$$

So the curve passes through the points $(0, 0)$, $(a, 0)$ and $(-a, 0)$. Therefore for one loop x varies from 0 to a.

$\therefore$ The required whole length of the curve

$$= 4 \times \text{length of the half loop (lying above } x\text{-axis)}$$

$$= 4 \int_0^a \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx. \qquad \ldots(2)$$

Now differentiating (1) w.r.t. x, we get

$$16a^2 y \frac{dy}{dx} = 2a^2 x - 4x^3 \quad \text{or} \quad \frac{dy}{dx} = \frac{(a^2 - 2x^2)}{8a^2 y} x.$$

$$\therefore \quad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2 (a^2 - 2x^2)^2}{64a^4 y^2}$$

$$= 1 + \frac{x^2 (a^2 - 2x^2)^2}{8a^2 x^2 (a^2 - x^2)}, \quad [\text{Substituting for } 8a^2 y^2 \text{ from (1)}]$$

$$= 1 + \frac{(a^2 - 2x^2)^2}{8a^2 (a^2 - x^2)} = \frac{(3a^2 - 2x^2)^2}{8a^2 (a^2 - x^2)}.$$

$\therefore$ From (2), the required whole length of the curve

$$= 4 \int_0^a \sqrt{\left\{\frac{(3a^2 - 2x^2)^2}{8a^2 (a^2 - x^2)}\right\}}\, dx = \frac{2}{a\sqrt{2}} \int_0^a \frac{(3a^2 - 2x^2)}{\sqrt{(a^2 - x^2)}}\, dx$$

$$= \frac{\sqrt{2}}{a} \int_0^a \left[\frac{2(a^2 - x^2)}{\sqrt{(a^2 - x^2)}} + \frac{a^2}{\sqrt{(a^2 - x^2)}}\right] dx \qquad \textbf{(Note)}$$

$$= \frac{\sqrt{2}}{a} \int_0^a 2\sqrt{(a^2 - x^2)}\, dx + \frac{\sqrt{2}}{a} \int_0^a \frac{a^2}{\sqrt{(a^2 - x^2)}}\, dx$$

$$= \frac{2\sqrt{2}}{a} \left[\frac{x\sqrt{(a^2 - x^2)}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}\right]_0^a + \sqrt{2} \cdot a \left[\sin^{-1} \frac{x}{a}\right]_0^a$$

$$= \frac{2\sqrt{2}}{a} \left[0 + \frac{a^2}{2} \cdot \frac{\pi}{2}\right] = \sqrt{2} \cdot a \cdot \frac{\pi}{2} = \pi a \sqrt{2}.$$

Comprehensive Problems 2

Problem 1(i): *Find the whole length of the curve (astroid)* $x = a \cos^3 t,\ y = a \sin^3 t$.

(Rohilkhand 2011)

Solution: The given parametric equations of the astroid are

$$x = a \cos^3 t,\ y = a \sin^3 t. \qquad \ldots(1)$$

The shape of the curve is as shown in the figure of Example 4.

We have $dx/dt = -3a \cos^2 t \sin t$, $dy/dt = 3a \sin^2 t \cos t$.

The astroid is symmetrical about both the axes. For the arc of the astroid lying in the first quadrant, we have $t = 0$ at the point $(a, 0)$ and $t = \pi/2$ at the point $(0, a)$.

Now $$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9a^2 \cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t$$

$$= 9a^2 \sin^2 t \cos^2 t (\sin^2 t + \cos^2 t) = (3a \sin t \cos t)^2. \quad ...(2)$$

If s denotes the arc length of the astroid measured from the point $t = 0$ to any point P towards the point $t = \pi/2$, then s increases as t increases. Therefore ds/dt will be taken with positive sign. So taking square root of both sides of (2), we have

$$ds/dt = 3a \sin t \cos t \quad \text{or} \quad ds = 3a \sin t \cos t \, dt.$$

Let s_1 denote the arc length of the astroid lying in the first quadrant. Then

$$\int_0^{s_1} ds = \int_0^{\pi/2} 3a \sin t \cos t \, dt \quad \text{or} \quad s_1 = 3a \left[\frac{\sin^2 t}{2}\right]_0^{\pi/2} = \frac{3a}{2}.$$

Whole length of the curve

$$= 4 \times \text{length of the curve lying in the Ist quadrant} = 4 \, . \, (3a/2) = 6a.$$

Problem 1(ii): *Find the whole length of the curve (Hypocycloid)*

$$x = a \cos^3 t, \quad y = b \sin^3 t.$$

Solution: The given curve is similar to that of part (i) *i.e.*, the curve is symmetrical about both the axes and in the first quadrant t varies from 0 to $\frac{1}{2}\pi$.

Here $dx/dt = -3a \cos^2 t \sin t$, $dy/dt = 3b \sin^2 t \cos t$.

$\therefore$ The required whole length of the curve

$$= 4 \times \text{length of the curve in the first quadrant}$$

$$= 4 \int_0^{\pi/2} \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} \, dt$$

$$= 4 \int_0^{\pi/2} \sqrt{\{9a^2 \cos^4 t \sin^2 t + 9b^2 \sin^4 t \cos^2 t\}} \, dt$$

$$= 4 \int_0^{\pi/2} 3 \sin t \cos t \sqrt{(a^2 \cos^2 t + b^2 \sin^2 t)} \, dt.$$

Now put $a^2 \cos^2 t + b^2 \sin^2 t = z^2$,

so that $(-2a^2 \sin t \cos t + 2b^2 \sin t \cos t) \, dt = 2z \, dz$.

$\therefore$ $\sin t \cos t \, dt = \{z/(b^2 - a^2)\} \, dz$.

Also $z = a$ when $t = 0$ and $z = b$ when $t = \pi/2$.

Hence the required length

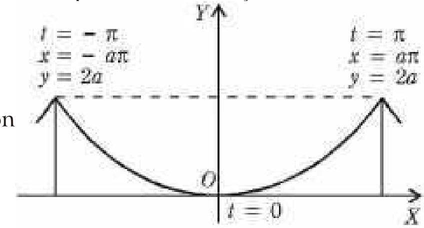
$$= 12 \int_a^b z \cdot \frac{z \, dz}{b^2 - a^2} = \frac{12}{b^2 - a^2} \int_a^b z^2 \, dz$$

$$= \frac{12}{b^2 - a^2} \left[\frac{z^3}{3}\right]_a^b = 4 \, \frac{b^3 - a^3}{b^2 - a^2} = 4 \cdot \frac{(b^2 + ab + a^2)}{b + a}.$$

Problem 2: *Rectify the curve or find the length of an arch of the curve*

$$x = a\,(t + \sin t\,),\ y = a\,(1 - \cos t\,). \qquad \textbf{(Rohilkhand 2000, 09B)}$$

Solution: Differentiating the given parametric equations of the cycloid w.r.t. t, we have



$$dx/dt = a\,(1 + \cos t\,)$$

and $$dy/dt = a \sin t\,.$$

If we measure the arc length s in the direction of t increasing, we have

$$\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}$$

$$= \sqrt{\{\,a^2\,(1 + \cos t\,)^2 + a^2 \sin^2 t\,\}}$$

$$= a\,\{\,1 + \cos^2 t + 2 \cos t + \sin^2 t\,\}^{1/2}$$

$$= a\,\sqrt{2}\,(1 + \cos t\,)^{1/2} = a\,\sqrt{2}\,(2 \cos^2 \tfrac{1}{2} t\,)^{1/2} = 2a \cos \tfrac{1}{2} t\,.$$

$\therefore$ $$ds = 2a \cos \frac{1}{2} t\, dt\,.$$

For an arch of the given cycloid lying between two successive cusps t varies from $-\pi$ to π. Also this arch is symmetrical about the y-axis and we have $t = 0$ at the vertex O.

$\therefore$ The required whole length of the arch

$$= 2 \times \text{ length of the arc from } t = 0 \text{ to } t = p$$

$$= 2 \int_0^{\pi} 2a \cos \frac{1}{2} t\, dt = 4a \int_0^{\pi} \cos \frac{1}{2} t\, dt = 4a \left[2 \sin \frac{1}{2} t\right]_0^{\pi}$$

$$= 4a\,[2 - 0] = 8a\,.$$

Problem 3: *Prove that the length of an arc of the cycloid* $x = a\,(t + \sin t\,),\ y = a\,(1 - \cos t\,)$ *from the vertex to the point* (x, y) *is* $\sqrt{(8ay)}$. **(Bundelkhand 2007; Meerut 12)**

Solution: Let s denote the arc length of the cycloid measured from the vertex to any point P (*i.e.*, from $t = 0$ to $t = t$).

Then proceeding as in Problem 2, the required arc length

$$s = \int_0^t \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}\, dt = 2a \int_0^t \cos \frac{1}{2} t\, dt$$

$$= 2a \left[2 \sin \frac{1}{2} t\right]_0^t = 4a \sin \frac{1}{2} t = \sqrt{(8a^2\,.\,2 \sin^2 \tfrac{1}{2} t\,)}$$

$$= \sqrt{\{\,8a\,.\,a\,(1 - \cos t)\,\}} = \sqrt{(8ay)}. \qquad [\because\ y = a\,(1 - \cos t\,)]$$

Problem 4: *Find the length of the arc of the curve* $x = e^t \sin t,\ y = e^t \cos t,$ *from* $t = 0$ *to* $t = \frac{1}{2}\pi$.

(Kumaun 2000, 08; Kanpur 09)

Solution: The given equations of the curve are $x = e^t \sin t,\ y = e^t \cos t$.
Differentiating w.r.t. t, we have

$$dx/dt = e^t \cos t + e^t \sin t = e^t\ (\cos t + \sin t)$$

and $$dy/dt = e^t\ (-\sin t) + e^t \cos t = e^t (\cos t - \sin t).$$

$\therefore$ The required arc length $= \int_0^{\pi/2} \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2\right\}}\, dt$

$$= \int_0^{\pi/2} \sqrt{\{e^{2t}(\cos t + \sin t)^2 + e^{2t}(\cos t - \sin t)^2\}}\, dt$$

$$= \int_0^{\pi/2} e^t \sqrt{\{2(\cos^2 t + \sin^2 t)\}}\, dt = \sqrt{2} \int_0^{\pi/2} e^t dt = \sqrt{2}\, [e^t]_0^{\pi/2}$$

$$= \sqrt{2}\,[e^{\pi/2} - 1].$$

Problem 5: *Show that in the epi-cycloid for which*

$$x = (a+b)\cos\theta - b\cos\{(a+b)/b\}\theta,\ y = (a+b)\sin\theta - b\sin\{(a+b)/b\}\theta,$$

the length of the arc measured from the point $\theta = \pi b/a$ *is* $\{4b\,(a+b)/a\}\cos\{(a/2b)\theta\}$.

Solution: Differentiating the given equations of the epicycloid w.r.t. θ, we have

$$dx/d\theta = (a+b)(-\sin\theta) - b\,[-\sin\{(a+b)/b\}\theta]\,.\,[(a+b)/b]$$
$$= -(a+b)\,[\sin\theta - \sin\{(a+b)/b\}\theta]$$

and $$dy/d\theta = (a+b)(\cos\theta) - b\,[\cos\{(a+b)/b\}\theta]\,.\,\{(a+b)/b\}$$
$$= (a+b)\,[\cos\theta - \cos\{(a+b)/b\}\theta].$$

We have $$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= (a+b)^2\left[\left\{\sin\theta - \sin\frac{a+b}{b}\theta\right\}^2 + \left\{\cos\theta - \cos\frac{a+b}{b}\theta\right\}^2\right]$$

$$= (a+b)^2\left[\sin^2\theta + \sin^2\frac{a+b}{b}\theta - 2\sin\theta\sin\frac{a+b}{b}\theta\right.$$
$$\left. + \cos^2\theta + \cos^2\frac{a+b}{b}\theta - 2\cos\theta\cos\frac{a+b}{b}\theta\right]$$

$$= (a+b)^2\left[1 + 1 - 2\left(\sin\theta\sin\frac{a+b}{b}\theta + \cos\theta\cos\frac{a+b}{b}\theta\right)\right]$$

$$= 2(a+b)^2\left[1 - \cos\frac{a}{b}\theta\right] = 4(a+b)^2\sin^2\frac{a}{2b}\theta. \quad \ldots(1)$$

If s denotes the arc length of the epicycloid measured from the point $\theta = \pi b/a$ to the point $\theta = \theta$ in the direction of θ decreasing, then s increases as θ decreases. Therefore $ds/d\theta$ will be negative. So taking square root of both sides of (1) and keeping the negative sign, we have

$$\frac{ds}{d\theta} = -2(a+b)\sin\frac{a}{2b}\theta \quad \textbf{(Note)}$$

or $$ds = -2(a+b)\sin(a/2b)\theta\, d\theta.$$

The required length s is now given by

$$s = -\int_{b\pi/a}^{\theta} 2\,(a+b)\sin(a/2b)\,\theta\, d\theta = -2\,(a+b)\cdot\frac{2b}{a}\left[-\cos\frac{a}{2b}\,\theta\right]_{b\pi/a}^{\theta}$$

$$= \frac{4b\,(a+b)}{a}\left[\cos\frac{a\theta}{2b} - \cos\frac{\pi}{2}\right] = \frac{4b\,(a+b)}{a}\cos\frac{a}{2b}\,\theta.$$

Problem 6: *In the ellipse $x = a\cos\phi,\ y = b\sin\phi$, show that*

$$ds = a\sqrt{(1 - e^2\cos^2\phi)}\,d\phi,$$

and hence show that the whole length of the ellipse is

$$2\,\pi a\left[1 - \left(\frac{1}{2}\right)^2\cdot\frac{e^2}{1} - \left(\frac{1.3}{2.4}\right)^2\cdot\frac{e^4}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2\cdot\frac{e^6}{5} - \ldots\right],$$

where e is the eccentricity of the ellipse. **(Meerut 2005)**

Solution: The given equations of the ellipse are $x = a\cos\phi,\ y = b\sin\phi$.

We have $dx/d\phi = -a\sin\phi,\ dy/d\phi = b\cos\phi$.

If we measure the length s in the direction of ϕ increasing,

$$\frac{ds}{d\phi} = \sqrt{\left\{\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2\right\}} = \sqrt{(a^2\sin^2\phi + b^2\cos^2\phi)}$$

$$= \sqrt{\{a^2\sin^2\phi + a^2(1-e^2)\cos^2\phi\}},$$

[$\because$ for ellipse, $b^2 = a^2(1-e^2)$]

$$= a\sqrt{(1 - e^2\cos^2\phi)}.$$

$$\therefore \quad ds = a\sqrt{(1 - e^2\cos^2\phi)}\,d\phi. \quad \ldots(1)$$

Also the ellipse is symmetrical about both the axes and in the first quadrant ϕ varies from 0 to $\frac{1}{2}\pi$. Therefore whole length of the ellipse = 4 × length of the ellipse lying in the first quadrant

$$= 4\int_0^{\pi/2} a\sqrt{(1 - e^2\cos^2\phi)}\,d\phi, \quad \text{[From (1)]}$$

$$= 4a\int_0^{\pi/2}(1 - e^2\cos^2\phi)^{1/2}\,d\phi$$

$$= 4a\int_0^{\pi/2}\left[1 - \frac{1}{2}e^2\cos^2\phi - \frac{1}{2.4}e^4\cos^4\phi - \frac{1.3}{2.4.6}e^6\cos^6\phi - \ldots\right]d\phi,$$

[On expanding by binomial theorem]

$$= 4a\left[\int_0^{\pi/2} 1\,.\,d\phi - \frac{e^2}{2}\int_0^{\pi/2}\cos^2\phi\,d\phi - \frac{e^4}{2.4}\int_0^{\pi/2}\cos^4\phi\,d\phi - \frac{1.3}{2.4.6}e^6\int_0^{\pi/2}\cos^6\phi\,d\phi - \ldots\right]$$

$$= 4a\left[\frac{\pi}{2} - \frac{e^2}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} - \frac{e^4}{2.4}\,\frac{3.1}{4.2}\cdot\frac{\pi}{2} - \frac{1.3}{2.4.6}\,e^6\cdot\frac{5.3.1}{6.4.2}\cdot\frac{\pi}{2} - \ldots\right]$$

$$= 2a\pi\left[1 - \frac{1}{2^2}\,e^2 - \frac{1^2.3}{2^2.4^2}\,e^4 - \frac{1^2.3^2.5}{2^2.4^2.6^2}\,e^6 - \ldots\right]$$

$$= 2\pi a\left[1 - \left(\frac{1}{2}\right)^2\cdot\frac{e^2}{1} - \left(\frac{1.3}{2.4}\right)^2\cdot\frac{e^4}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2\cdot\frac{e^6}{5} - \ldots\right].$$

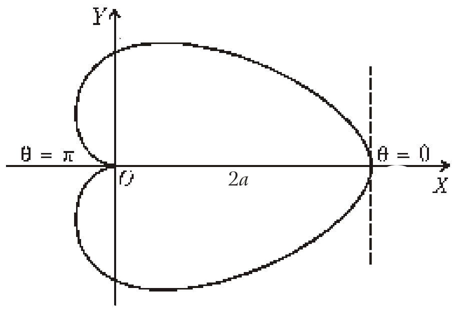
Comprehensive Problems 3

Problem 1: *Find the entire length of the cardioid $r = a\,(1 + \cos\theta)$.*

(Purvanchal 2007; Rohilkhand 09, 11B)

Solution: The given curve is

$r = a\,(1 + \cos\theta)$. ...(1)



It is symmetrical about the initial line and for the portion of the curve lying above the initial line θ varies from $\theta = 0$ to $\theta = \pi$.

Now differentiating (1) w.r.t. θ, we have $dr/d\theta = -\,a\sin\theta$.

If s denotes the arc length of the cardioid measured from the vertex (*i.e.*, the point $\theta = 0$) to any point $P\,(r, \theta)$ in the direction of θ increasing, we have

$$\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} = \sqrt{\{a^2\,(1 + \cos\theta)^2 + (-\,a\sin\theta)^2\}}$$

$$= \sqrt{\{a^2\,(2\cos^2\tfrac{1}{2}\theta)^2 + a^2.\,(2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta)^2\}}$$

$$= 2a\sqrt{\{\cos^4\tfrac{1}{2}\theta + \sin^2\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta\}}$$

$$= 2a\sqrt{\{\cos^2\tfrac{1}{2}\theta\,(\cos^2\tfrac{1}{2}\theta + \sin^2\tfrac{1}{2}\theta)\}}$$

$$= 2a\sqrt{(\cos^2\tfrac{1}{2}\theta)} = 2a\cos\tfrac{1}{2}\theta.$$

$\therefore$ $$ds = 2a\cos\tfrac{1}{2}\theta\,d\theta. \quad ...(2)$$

Let s_1 denote the arc length of the upper half of the cardioid (*i.e.*, from $\theta = 0$ to $\theta = \pi$).

Then $$\int_0^{s_1} ds = \int_0^{\pi} 2a\cos\tfrac{1}{2}\theta\,d\theta = 2a\left[2\sin\tfrac{1}{2}\theta\right]_0^{\pi}$$

or $$s_1 = 4a\left[\sin \frac{1}{2}\pi - \sin 0\right] = 4a(1-0) = 4a.$$

∴ By symmetry, the whole length of the cardioid

= 2 × the arc length of the upper half of the cardioid = $2 \cdot 4a = 8a$.

Problem 2: *Find the perimeter of the curve* $r = a(1 + \cos\theta)$ *and show that arc of the upper half is bisected by* $\theta = \pi/3$. **(Gorakhpur 2005; Purvanchal 07)**

Solution: As proved in Problem 1 length s_1 of the upper half of the cardioid

$r = a(1 + \cos\theta)$ is $4a$. **(Prove it here)**

∴ The perimeter of the cardioid $= 2 \cdot 4a = 8a$.

Again, if s_2 denotes the arc length of the cardioid from the point $\theta = 0$ to the point $\theta = \pi/3$, then

$$s_2 = \int_0^{\pi/3} 2a \cos \frac{1}{2}\theta \, d\theta, \qquad \text{[Refer the result (2) of Problem 1]}$$

$$= 2a\left[2 \sin \frac{1}{2}\theta\right]_0^{\pi/3} = 4a\left[\sin \frac{\pi}{6} - \sin 0\right] = 4a \cdot \frac{1}{2}$$

$$= \frac{1}{2}(s_1) = \text{half the arc length of the upper half of the cardioid.}$$

Problem 3: *Prove that the line* $4r\cos\theta = 3a$ *divides the cardioid* $r = a(1 + \cos\theta)$ *into two parts such that lengths of the arc on either side of the line are equal.*

Solution: The given equations of the cardioid and the line are

$$r = a(1 + \cos\theta) \quad \text{and} \quad 4r\cos\theta = 3a.$$

Eliminating r, we have $a(1 + \cos\theta) = 3a/(4\cos\theta)$

or $$4\cos\theta + 4\cos^2\theta = 3 \quad \text{or} \quad 4\cos^2\theta + 4\cos\theta - 3 = 0$$

or $$(2\cos\theta - 1)(2\cos\theta + 3) = 0 \;\; i.e., \cos\theta = \frac{1}{2} \quad \text{or} \quad \cos\theta = -\frac{3}{2}.$$

But the values of $\cos\theta$ cannot be numerically greater than 1. Therefore $\cos\theta = -\frac{3}{2}$ is inadmissible.

So we have $\cos\theta = \frac{1}{2}$ *i.e.*, $\theta = \pi/3$.

Hence the vectorial angle of the point of intersection of the cardioid with the given line is $\pi/3$. The cartesian equation of the given line is $x = 3a/4$ *i.e.*, the line is perpendicular to the x-axis. So now we have to prove that the arc of the upper half of the cardioid is bisected by $\theta = \pi/3$. This is the same as Problem 2 and so prove it here.

Problem 4: *Show that the arc of the upper half of the curve*

$r = a(1 - \cos\theta)$ *is bisected by* $\theta = 2\pi/3$. **(Kumaun 2003)**

Solution: As proved in Example 8, the arc length of the upper half of the cardioid $r = a(1 - \cos\theta)$ is $4a$. (Prove it here)

Also, the arc length of the cardioid from the point $\theta = 0$ to the point $\theta = 2\pi/3$

$$= 2a \int_0^{2\pi/3} \sin \frac{1}{2} \theta \, d\theta, \qquad \text{[Refer the result (3) of Example 8]}$$

$$= 2a \left[-2 \cos \frac{1}{2} \theta \right]_0^{2\pi/3} = -4a \left[\cos \frac{\pi}{3} - \cos 0 \right] = -4a \left[\frac{1}{2} - 1 \right]$$

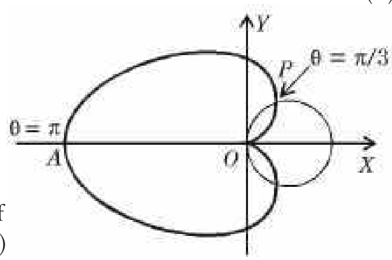
$$= 2a = \frac{1}{2} (4a) = \text{half the arc length of the upper half of the cardioid.}$$

Hence the arc of the upper half of the curve $r = a(1 - \cos \theta)$ is bisected by $\theta = 2\pi/3$.

Problem 5: *Find the length of the cardioid $r = a(1 - \cos \theta)$ lying outside the circle $r = a \cos \theta$.*

Solution: The given cardioid is $r = a(1 - \cos \theta)$. ...(1)

It meets the circle $r = a \cos \theta$. ...(2)



Eliminating r between (1) and (2), we have

$a(1 - \cos \theta) = a \cos \theta$

or $1 - \cos \theta = \cos \theta$

or $2 \cos \theta = 1$

or $\cos \theta = \frac{1}{2}$ *i.e.,* $\theta = \frac{1}{3} \pi$.

Hence the vectorial angle of the point of intersection P of the cardioid $r = a(1 - \cos \theta)$ with the circle $r = a \cos \theta$ is $\pi/3$.

So for the portion of the cardioid $r = a(1 - \cos \theta)$ lying outside the circle $r = a \cos \theta$ (above the initial line) θ varies from $\theta = \pi/3$ to $\theta = \pi$.

Also differentiating (1), we get $dr/d\theta = -a \sin \theta$.

By symmetry, the required length of the cardioid

$$= 2 \times \text{the arc length from } \theta = \pi/3 \text{ to } \theta = \pi \text{ of the upper half of the cardioid}$$

$$= 2 \int_{\theta = \pi/3}^{\pi} \sqrt{\left\{ r^2 + \left(\frac{dr}{d\theta} \right)^2 \right\}} \, d\theta$$

$$= 2 \int_{\pi/3}^{\pi} \sqrt{\{ a^2 (1 - \cos \theta)^2 + (-a \sin \theta)^2 \}} \, d\theta$$

$$= 2a \int_{\pi/3}^{\pi} \sqrt{\{ (2 \sin^2 \tfrac{1}{2} \theta)^2 + 4 \sin^2 \tfrac{1}{2} \theta \cos^2 \tfrac{1}{2} \theta \}} \, d\theta$$

$$= 4a \int_{\pi/3}^{\pi} \sqrt{\{ \sin^2 \tfrac{1}{2} \theta (\sin^2 \tfrac{1}{2} \theta + \cos^2 \tfrac{1}{2} \theta) \}} \, d\theta$$

$$= 4a \int_{\pi/3}^{\pi} \sin \frac{1}{2} \theta \, d\theta = 4a \left[-2 \cos \frac{1}{2} \theta \right]_{\pi/3}^{\pi}$$

$$= -8a \left[\cos \frac{1}{2} \pi - \cos \frac{1}{6} \pi \right] = -8a \left[0 - \frac{1}{2} \sqrt{3} \right] = 4a \sqrt{3}.$$

Problem 6: *Find the length of the arc of the equiangular spiral $r = ae^{\theta \cot \alpha}$, taking $s = 0$ when, $\theta = 0$.* **(Kanpur 2007)**

Solution: The given equiangular curve is $r = ae^{\theta \cot \alpha}$. ...(1)

Differentiating (1) w.r.t. θ, we get

$$dr/d\theta = ae^{\theta \cot \alpha} . \cot \alpha = r \cot \alpha, \text{ from (1).}$$

If s denotes the arc length of the equiangular spiral measured from $\theta = 0$ to any point $P(r, \theta)$ in the direction of θ increasing, we have

$$\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} = \sqrt{(r^2 + r^2 \cot^2 \alpha)}$$

$$= r \operatorname{cosec} \alpha = ae^{\theta \cot \alpha} \cdot \operatorname{cosec} \alpha$$

or $$ds = ae^{\theta \cot \alpha} \cdot \operatorname{cosec} \alpha \, d\theta$$

∴ Integrating, $\int_0^s ds = a \operatorname{cosec} \alpha \int_0^\theta e^{\theta \cot \alpha} \, d\theta$

or $$s = a \operatorname{cosec} \alpha \cdot \frac{1}{\cot \alpha} \cdot [e^{\theta \cot \alpha}]_0^\theta$$

$$= a \sec \alpha \, [e^{\theta \cot \alpha} - e^0] = a \sec \alpha \, [e^{\theta \cot \alpha} - 1].$$

Problem 7: *Find the length of any arc of the cissoid* $r = a (\sin^2 \theta / \cos \theta)$.

Solution: The given curve is $r = a (\sin^2 \theta / \cos \theta)$

or $$r = a \tan \theta \sin \theta. \qquad ...(1)$$

Differentiating (1) w.r.t. θ, we have

$$dr/d\theta = a [\tan \theta \cos \theta + \sec^2 \theta \sin \theta]$$

$$= a [\sin \theta + \sec^2 \theta \sin \theta] = a \sin \theta [1 + \sec^2 \theta].$$

We have $$(ds/d\theta)^2 = r^2 + (dr/d\theta)^2$$

$$= a^2 \tan^2 \theta \sin^2 \theta + a^2 \sin^2 \theta (1 + \sec^2 \theta)^2$$

$$= a^2 \sin^2 \theta [\tan^2 \theta + (1 + \sec^2 \theta)^2]$$

$$= a^2 \sin^2 \theta [\tan^2 \theta + 1 + \sec^4 \theta + 2 \sec^2 \theta]$$

$$= a^2 \sin^2 \theta [\sec^2 \theta + \sec^4 \theta + 2 \sec^2 \theta]$$

$$= a^2 \sin^2 \theta . \sec^2 \theta [3 + \sec^2 \theta]$$

$$= a^2 \tan^2 \theta [3 + \sec^2 \theta]. \qquad ...(2)$$

If s denotes the arc length of the cissoid measured from the point $\theta = \theta_1$ in the direction of θ increasing, then

$$ds = a \tan \theta \sqrt{(3 + \sec^2 \theta)} \, d\theta,$$

on taking square root of (2) and keeping the +ive sign.

Let s_1 denote the required arc length from $\theta = \theta_1$ to $\theta = \theta_2$. Then

$$\int_0^{s_1} ds = \int_{\theta = \theta_1}^{\theta_2} a \tan \theta \sqrt{(3 + \sec^2 \theta)} \, d\theta$$

$$= a \int_{\theta_1}^{\theta_2} \frac{\sin \theta}{\cos \theta} \sqrt{\left(3 + \frac{1}{\cos^2 \theta}\right)} \, d\theta$$

or $$s_1 = a \int_{\theta_1}^{\theta_2} \frac{\sin\theta}{\cos^2\theta} \sqrt{(1 + 3\cos^2\theta)}\, d\theta \qquad \ldots(3)$$

Let us first evaluate the indefinite integral

$$I = \int \frac{\sin\theta}{\cos^2\theta} \sqrt{\{1 + 3\cos^2\theta\}}\, d\theta.$$

Put $\cos\theta = t$ so that $-\sin\theta\, d\theta = dt$.

Then $$I = -\int \frac{\sqrt{(1+3t^2)}}{t^2}\, dt = -\int \frac{1+3t^2}{t^2\sqrt{(1+3t^2)}}\, dt$$

$$= -\int \frac{dt}{t^2\sqrt{(1+3t^2)}} - \int \frac{3\, dt}{\sqrt{(1+3t^2)}}.$$

To evaluate the first integral, put $t = 1/z$ so that $dt = -(1/z^2)\, dz$.

Then $$-\int \frac{dt}{t^2\sqrt{(1+3t^2)}} = \int \frac{dz}{\sqrt{(1+3/z^2)}} = \frac{1}{2}\int \frac{2z\ dz}{\sqrt{(z^2+3)}} = \sqrt{(z^2+3)},$$

by power formula

$$= \sqrt{\left(\frac{1}{t^2} + 3\right)} = \sqrt{\left(\frac{1}{\cos^2\theta} + 3\right)} = \sqrt{(\sec^2\theta + 3)}.$$

Also the second integral

$$= -\int \frac{3\, dt}{\sqrt{(1+3t^2)}} = -\sqrt{3}\int \frac{dt}{\sqrt{\left(\frac{1}{3} + t^2\right)}}$$

$$= -\sqrt{3}\log\{t + \sqrt{(t^2 + \tfrac{1}{3})}\} = -\sqrt{3}\log\{\cos\theta + \sqrt{(\cos^2\theta + \tfrac{1}{3})}\}.$$

$\therefore$ $$I = \sqrt{(\sec^2\theta + 3)} - \sqrt{3}\log\{\cos\theta + \sqrt{(\cos^2\theta + \tfrac{1}{3})}\}.$$

Hence from (3), we get the required arc length

$$s_1 = \left[a\sqrt{(\sec^2\theta + 3)} - \sqrt{3}.a\log\{\cos\theta + \sqrt{(\cos^2\theta + \tfrac{1}{3})}\}\right]_{\theta_1}^{\theta_2}$$

$$= f(\theta_2) - f(\theta_1),$$

where $$f(\theta) = a\sqrt{(\sec^2\theta + 3)} - a\sqrt{3}\log\{\cos\theta + \sqrt{(\cos^2\theta + \tfrac{1}{3})}\}.$$

Problem 8: *Show that the whole length of the limacon $r = a + b\cos\theta$, $(a > b)$ is equal to that of an ellipse whose semi-axes are equal in length to the maximum and minimum radii vectors of the limacon.*

Solution: As shown in Example 10, the whole length of the limacon

$$= 2\int_0^{\pi} \sqrt{(a^2 + b^2 + 2ab\cos\theta)}\, d\theta, \qquad \text{(prove it here).} \qquad \ldots(1)$$

Also the maximum and minimum radii vectors of the limacon are given by $\cos\theta = 1$ and $\cos\theta = -1$ and they are respectively $a + b$ and $a - b$.

Now the parametric equations of the ellipse with semi-major axis as $(a+b)$ and semi-minor axis as $(a-b)$ may be taken as

$$x=(a+b)\cos\phi,\ y=(a-b)\sin\phi. \qquad ...(2)$$

Differentiating (2) w.r.t. ϕ, we have

$$dx/d\phi=-(a+b)\sin\phi,\ dy/d\phi=(a-b)\cos\phi.$$

Now the ellipse (2) is symmetrical in all the four quadrants and for the portion of the ellipse lying in the first quadrant ϕ varies from $\phi=0$ to $\phi=\frac{1}{2}\pi$.

By symmetry, the perimeter (whole length) of the ellipse $=4\times$ the arc length of the ellipse lying in the first quadrant

$$=4\int_0^{\pi/2}\sqrt{\left\{\left(\frac{dx}{d\phi}\right)^2+\left(\frac{dy}{d\phi}\right)^2\right\}}\,d\phi$$

$$=4\int_0^{\pi/2}\sqrt{[\{-(a+b)\sin\phi\}^2+\{(a-b)\cos\phi\}^2]}\,d\phi$$

$$=4\int_0^{\pi/2}\sqrt{\{a^2\sin^2\phi+b^2\sin^2\phi+2ab\sin^2\phi+a^2\cos^2\phi+b^2\cos^2\phi-2ab\cos^2\phi\}}\,d\phi$$

$$=4\int_0^{\pi/2}\sqrt{\{a^2+b^2-2ab(\cos^2\phi-\sin^2\phi)\}}\,d\phi$$

$$=4\int_0^{\pi/2}\sqrt{\{a^2+b^2-2ab\cos 2\phi\}}\,d\phi.$$

Now put $2\phi=t$ so that $2d\phi=dt$. Also when $\phi=0$, $t=0$ and when $\phi=\pi/2$, $t=\pi$.
Then the whole length of the ellipse

$$=4\int_0^{\pi}\sqrt{\{a^2+b^2-2ab\cos t\}}\,\frac{1}{2}\,dt$$

$$=2\int_0^{\pi}\sqrt{\{a^2+b^2-2ab\cos t\}}\,dt$$

$$=2\int_0^{\pi}\sqrt{\{a^2+b^2-2ab\cos(\pi-t)\}}\,dt,$$

$$\left[\because \int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right]$$

$$=2\int_0^{\pi}\sqrt{(a^2+b^2+2ab\cos t)}\,dt$$

$$=2\int_0^{\pi}\sqrt{(a^2+b^2+2ab\cos\theta)}\,d\theta, \qquad \left[\because \int_0^a f(x)\,dx=\int_0^a f(t)\,dt\right]$$

= the whole length of the limacon. [From (1)]

Comprehensive Problems 4

Problem 1: *Prove that the intrinsic equation of the parabola $x^2=4ay$ is*

$$s=a\tan\psi\sec\psi+a\log(\tan\psi+\sec\psi).$$

Solution: Proceeding exactly as in Example 13, we get

$$\tan\psi = dy/dx = x/2a \,. \qquad \text{...(1)}$$

Also
$$s = \int_0^x \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx = \frac{1}{2a}\int_0^x \sqrt{(4a^2 + x^2)}\, dx$$

$$= \frac{1}{4a}\left[x\sqrt{(4a^2 + x^2)} + 4a^2 \log \frac{x + \sqrt{(x^2 + 4a^2)}}{2a}\right], \qquad \text{...(2)}$$

proceeding as in Example 13.

Eliminating x from (1) and (2), we get

$$s = a\,[\tan\psi \sec\psi + \log(\tan\psi + \sec\psi)],$$

which is the required intrinsic equation.

Problem 2: *Find the intrinsic equation of the parabola $y^2 = 4ax$. Hence deduce the length of the arc measured from the vertex to an extremity of the latus rectum.* **(Garhwal 2003)**

Solution: We have already obtained the intrinsic equation of the parabola $y^2 = 4ax$ in Example 13, as

$$s = a\,[\operatorname{cosec}\psi \cot\psi + \log(\operatorname{cosec}\psi + \cot\psi)], \qquad \text{...(1)}$$

where ψ is the angle between the x-axis and the tangent at the point whose arcual distance from the vertex is s. **(Prove it here)**

Now in the intrinsic equation (1) of the parabola the arc length s has been measured from the vert ex. We want to find the length of the arc from the vertex to an extremity of the latus rectum. Let this length be s_1.

At an extremity of the latus rectum, $y = 2a$. Also $\tan\psi = y/2a$. So at an extremity of the latus rectum, $\tan\psi = 2a/2a = 1$ *i.e.*, $\psi = \pi/4$. So putting $\psi = \pi/4$ in (1), we get

$$s_1 = a\left[\operatorname{cosec}\frac{1}{4}\pi \cot\frac{1}{4}\pi + \log\left(\operatorname{cosec}\frac{1}{4}\pi + \cot\frac{1}{4}\pi\right)\right]$$

$$= a\,[\sqrt{2} + \log(1 + \sqrt{2})].$$

Problem 3: *Show that the intrinsic equation of the semi-cubical parabola*

$3ay^2 = 2x^3$ is $9s = 4a(\sec^3\psi - 1)$.

(Meerut 2005, 09B; Rohilkhand 08B)

Solution: The given semicubical parabola is $3ay^2 = 2x^3$. ...(1)

Differentiating (1) w.r.t. x, we get $6ay\,(dy/dx) = 6x^2$

or
$$\frac{dy}{dx} = \frac{x^2}{ay} = \frac{x^2}{a\sqrt{(2x^3/3a)}} = \sqrt{\left(\frac{3x}{2a}\right)}.$$

$\therefore$
$$\tan\psi = \frac{dy}{dx} = \sqrt{\left(\frac{3x}{2a}\right)}. \qquad \text{...(2)}$$

If s denotes the arc length of the given curve measured from the point $(0, 0)$ to any point $P(x, y)$ in the direction of x increasing, then

$$s = \int_0^x \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx = \int_0^x \sqrt{\left(1 + \frac{3x}{2a}\right)}\, dx$$

$$= \int_0^x \left(1+\frac{3x}{2a}\right)^{1/2} dx = \left[\frac{\{1+(3x/2a)\}^{3/2}}{(3/2).(3/2a)}\right]_0^x = \frac{4a}{9}\left[\left(1+\frac{3x}{2a}\right)^{3/2} - 1\right]$$

...(3)

Eliminating x between (2) and (3), we get

$$s = \frac{4a}{9}[(1+\tan^2\psi)^{3/2} - 1] = \frac{4a}{9}(\sec^3\psi - 1),$$

which is the required intrinsic equation of the curve.

Problem 4: *Find the intrinsic equation of the catenary* $y = c \cosh (x/c)$.

(Garhwal 2001, 03; Rohilkhand 07; Kumaun 08; Kanpur 14)

Hence show that $c\rho = c^2 + s^2$, *where* ρ *is the radius of curvature.*

Solution: The given curve is $y = c \cosh (x/c)$. ...(1)

Differentiating (1) w.r.t. x, we get

$$dy/dx = c \sinh (x/c).(1/c) = \sinh (x/c).$$

$\therefore$ $\tan\psi = dy/dx = \sinh (x/c)$. ...(2)

If s denotes the arc length of the catenary measured from the vertex $(0, c)$ to any point (x, y) in the direction of x increasing, then

$$s = \int_0^x \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\, dx = \int_0^x \sqrt{\left\{1+\sinh^2\frac{x}{c}\right\}}\, dx$$

$$= \int_0^x \cosh\frac{x}{c}\, dx = c \sinh\frac{x}{c}. \quad ...(3)$$

Eliminating x between (2) and (3), we get

$s = c \tan\psi$, which is the required intrinsic equation of the catenary.

Also $\rho = \dfrac{ds}{d\psi} = c \sec^2\psi = c(1+\tan^2\psi) = c\left(1+\dfrac{s^2}{c^2}\right)$ or $c\rho = c^2 + s^2$.

Problem 5: *Prove that the intrinsic equation of the curve*

$x = a(1+\sin t), y = a(1+\cos t)$ *is* $s + a\psi = 0$.

Solution: The given curve is $x = a(1+\sin t), y = a(1+\cos t)$. ...(1)

$\therefore$ $dx/dt = a\cos t$ and $dy/dt = -a\sin t$.

$\therefore$ $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-a\sin t}{a\cos t} = -\tan t$.

Hence $\tan\psi = dy/dx = -\tan t = \tan(-t)$

so that $\psi = -t$. ...(2)

Measuring the arc length s from the point $t = 0$, we have

$$s = \int_0^t \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2\right\}}\, dt$$

$$= \int_0^t \sqrt{(a^2 \cos^2 t + a^2 \sin^2 t)}\, dt = a \int_0^t dt = at. \quad ...(3)$$

Eliminating t from (2) and (3), the intrinsic equation is

$$s = a(-\psi) \quad \text{or} \quad s + a\psi = 0.$$

Problem 6: *Find the intrinsic equation of the cardioid* $r = a(1 - \cos\theta)$.

(Avadh 2005, 12; Meerut 07B; Rohilkhand 12; Kanpur 15)

Solution: The given curve is $r = a(1 - \cos\theta)$. ...(1)

Differentiating (1) w.r.t. θ, we have $dr/d\theta = a \sin\theta$.

$$\therefore \quad \tan\phi = r\frac{d\theta}{dr} = \frac{r}{dr/d\theta} = \frac{r}{a\sin\theta} = \frac{a(1-\cos\theta)}{a\sin\theta}$$

$$= \frac{2\sin^2 \frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \tan\frac{1}{2}\theta.$$

Therefore $\phi = \frac{1}{2}\theta$, so that $\psi = \theta + \phi = \theta + \frac{1}{2}\theta = \frac{3}{2}\theta$, giving $\theta = \frac{2}{3}\psi$. ...(2)

If s denotes the arc length of the cardioid measured from the cusp O (*i.e.*, the point $\theta = 0$) to any point $P(r, \theta)$ in the direction of θ increasing, we have

$$s = \int_0^\theta \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta = a\int_0^\theta \sqrt{\{(1-\cos\theta)^2 + \sin^2\theta\}}\, d\theta$$

$$= 2a\int_0^\theta \sin\frac{1}{2}\theta\, d\theta = 4a\left[-\cos\frac{1}{2}\theta\right]_0^\theta$$

$$= 4a\left(1 - \cos\frac{1}{2}\theta\right) = 8a\sin^2\frac{1}{4}\theta. \quad ...(3)$$

Eliminating θ between (2) and (3), we get

$$s = 8a\sin^2\left\{\frac{1}{4}\cdot\frac{2}{3}\psi\right\} \quad \text{or} \quad s = 8a\sin^2\frac{1}{6}\psi,$$

which is the required intrinsic equation.

Problem 7: *Find the intrinsic equation of* $r = a\, e^{\theta \cot \alpha}$, *where s is measured from the point* $(a, 0)$.

Solution: The given curve is $r = a\, e^{\theta\cot\alpha}$. ...(1)

Differentiating (1), w.r.t. θ, we have

$$(dr/d\theta) = a\cot\alpha \,.\, e^{\theta\cot\alpha} = r\cot\alpha.$$

We have $$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{dr/d\theta} = \frac{r}{r\cot\alpha} = \tan\alpha$$

or $$\phi = \alpha \text{ so that } \psi = \theta + \phi = \theta + \alpha \quad \text{or} \quad \theta = \psi - \alpha. \quad ...(2)$$

If we measure the arc length s from the point $\theta = 0$ to any point $P(r, \theta)$ in the direction of θ increasing, we have

$$s = \int_0^\theta \sqrt{\{(dr/d\theta)^2 + r^2\}}\, d\theta = \int_0^\theta \sqrt{\{r^2 \cot^2 \alpha + r^2\}}\, d\theta$$

$$= \int_0^\theta r\sqrt{(1+\cot^2 \alpha)}\, d\theta = \operatorname{cosec} \alpha \int_0^\theta r\, d\theta$$

$$= \operatorname{cosec} \alpha \int_0^\theta a\, e^{\theta \cot \alpha}\, d\theta, \qquad [\because r = a\, e^{\theta \cot \alpha}]$$

$$= a \operatorname{cosec} \alpha \left[\frac{e^{\theta \cot \alpha}}{\cot \alpha}\right]_0^\theta = a \sec \alpha\, [e^{\theta \cot \alpha} - 1]. \qquad \text{...(3)}$$

Eliminating θ between (2) and (3), we get $s = a \sec \alpha\, [e^{(\psi - \alpha)\cot \alpha} - 1]$, which is the required intrinsic equation.

Problem 8: *Find the intrinsic equation of the spiral $r = a\theta$, the arc being measured from the pole.*

Solution: The given curve is $r = a\theta$. ...(1)

Differentiating (1) w.r.t. θ, we have $dr/d\theta = a$.

Therefore $$\tan \phi = r\frac{d\theta}{dr} = \frac{r}{dr/d\theta} = \frac{a\theta}{a} = \theta.$$

$\therefore$ $\phi = \tan^{-1}\theta$ so that $\psi = \theta + \phi = \theta + \tan^{-1}\theta$. ...(2)

If s denotes the arc length of the spiral measured from the pole $(0, 0)$ to any point $P\,(r, \theta)$, then

$$s = \int_0^\theta \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta = \int_0^\theta \sqrt{(a^2\theta^2 + a^2)}\, d\theta$$

$$= a\int_0^\theta \sqrt{(\theta^2+1)}\, d\theta = a\left[\frac{\theta}{2}\sqrt{(\theta^2+1)} + \frac{1}{2}\log\{\theta + \sqrt{(\theta^2+1)}\}\right]_0^\theta$$

$$= \frac{1}{2}a\,[\theta\sqrt{(1+\theta^2)} + \log\{\theta + \sqrt{(1+\theta^2)}\}]. \qquad \text{...(3)}$$

The required intrinsic equation is obtained by eliminating θ between (2) and (3).

Problem 9: *Find the intrinsic equation of the curve $p^2 = r^2 - a^2$.*

Solution: The given curve is $p^2 = r^2 - a^2$ ...(1)

Differentiating (1) w.r.t. r, we have

$$2p\,(dp/dr) = 2r \quad \text{or} \quad r\,(dr/dp) = p.$$

$\therefore$ $$\rho = \frac{ds}{d\psi} = r\frac{dr}{dp} = p = \sqrt{(r^2 - a^2)}. \qquad \text{...(2)}$$

[From (1)]

Also from the equation of the curve we have $p = 0$ for $r = a$.

If we measure the arc length s (from $r = a$) in the direction of r increasing, we have

$$s = \int_a^r \frac{r\, dr}{\sqrt{(r^2 - p^2)}} = \int_a^r \frac{r\, dr}{a}, \qquad [\because r^2 - p^2 = a^2]$$

$$= \frac{1}{a}\left[\frac{r^2}{2}\right]_a^r = \frac{1}{2a}[r^2 - a^2]$$

or $\qquad 2as = r^2 - a^2 \quad \text{or} \quad \sqrt{(2as)} = \sqrt{(r^2 - a^2)}$ ...(3)

Eliminating r between (2) and (3), we have

$$\frac{ds}{d\psi} = \sqrt{(2as)} \qquad \text{or} \qquad \frac{ds}{\sqrt{s}} = \sqrt{(2a)}\, d\psi.$$

If $s = 0$ when $\psi = 0$, then integrating, we have $\int_a^s \frac{ds}{\sqrt{s}} = \sqrt{(2a)} \int_0^\psi d\psi.$

$\therefore \qquad 2\sqrt{s} = \sqrt{(2a)}\,\psi \qquad \text{or} \qquad s = \frac{1}{2} a\psi^2,$

which is the required intrinsic equation.

Problem 10: *In the four-cusped astroid* $x^{2/3} + y^{2/3} = a^{2/3}$, *show that*

(*i*) $s = \frac{3}{4} a \cos 2\psi$, *s being measured from the vertex*;

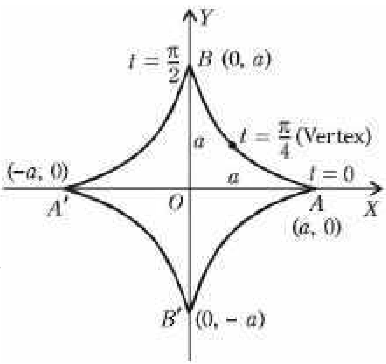
(*ii*) $s = \frac{3}{2} a \sin^2 \psi$, *s being measured from the cusp on x-axis*;

(Purvanchal 2014)

(*iii*) *whole length of the curve is* $6a$.

Solution: The parametric equations of the given curve are

$$x = a\cos^3 t,\ y = a\sin^3 t. \qquad ...(1)$$



We have

$$dx/dt = -3a\cos^2 t \sin t,$$

and $\qquad dy/dt = 3a\sin^2 t \cos t.$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t \sin t} = -\tan t.$$

So we have

$$\tan\psi = dy/dx = -\tan t = \tan(-t).$$

$\therefore \qquad \psi = -t.$ ...(2)

Now $\qquad \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (9a^2\cos^4 t\sin^2 t) + (9a^2\sin^4 t\cos^2 t)$

$$= 9a^2\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t)$$

$$= 9a^2\sin^2 t\cos^2 t. \qquad ...(3)$$

(i) If s denotes the arc length of the given curve measured from the vertex (*i.e.*, the middle point of the arc in the Ist quadrant) to any point P lying towards the cusp on x-axis, then s increases as t decreases. Therefore ds/dt will be negative, so from (3), we have

$$ds/dt = -3a\sin t\cos t$$

or $\qquad ds = -3a\sin t\cos t\, dt.$ ...(4)

Now at the vertex of the given curve, we have $t = \pi/4$.

$\therefore$ From (4), the arcual distance s measured from the vertex is given by

$$s = -3a \int_{\pi/4}^{t} \sin t \cos t \, dt = -\frac{3a}{2} \int_{\pi/4}^{t} \sin 2t \, dt$$

$$= -\frac{3a}{2} \left[-\frac{\cos 2t}{2} \right]_{\pi/4}^{t} = \frac{3}{4} a \cos 2t. \qquad ...(5)$$

Eliminating t between (2) and (5), the required intrinsic equation of the curve is

$$s = \frac{3}{4} a \cos \{ 2 (-\psi) \} = \frac{3}{4} a \cos 2\psi.$$

$[\because \cos(-\theta) = \cos\theta]$

(ii) If s denotes the arc length of the given curve measured from the cusp on x-axis to any point P lying towards the second cusp on y-axis, then s increases as t increases. Therefore ds/dt will be positive. Hence from (3), we have

$$ds/dt = 3a \sin t \cos t \quad \text{or} \quad ds = 3a \sin t \cos t \, dt.$$

Also at the cusp on x-axis, we have $t = 0$.

$$\therefore \qquad s = \int_0^t 3a \sin t \cos t \, dt = 3a \left[\frac{\sin^2 t}{2} \right]_0^t = \frac{3}{2} a \sin^2 t. \qquad ...(6)$$

Eliminating t between (2) and (6), the required intrinsic equation of the curve is

$$s = \frac{3}{2} a \sin^2 (-\psi) \quad \text{or} \quad s = \frac{3}{2} a \sin^2 \psi.$$

(iii) The whole length of the curve is already obtained as $6a$ in Example 4.

Problem 11: *Find the cartesian equation of the curve whose intrinsic equation is $s = c \tan\psi$ when it is given that at $\psi = 0, x = 0$ and $y = c$.*

Solution: The given intrinsic equation of the curve is $s = c \tan\psi$. ...(1)

Differentiating (1) w.r.t. 'x', we have $ds/dx = c \sec^2 \psi \cdot (d\psi/dx)$. ...(2)

$$\text{Also} \qquad \frac{ds}{dx} = \sqrt{\left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\}} = \sqrt{(1 + \tan^2 \psi)} = \sec\psi. \qquad ...(3)$$

Equating the values of ds/dx from (2) and (3), we get

$$c \sec^2 \psi \cdot (d\psi/dx) = \sec\psi \quad \text{or} \quad dx = c \sec\psi \, d\psi. \qquad ...(4)$$

Integrating both sides of (1), we get

$x + A = c \log(\sec\psi + \tan\psi)$, where A is constant of integration.

But as given $\psi = 0$ when $x = 0$ so that $A = 0$.

$$\text{Therefore} \qquad x = c \log(\sec\psi + \tan\psi) \quad \text{or} \quad e^{x/c} = \sec\psi + \tan\psi. \qquad ...(5)$$

$$\text{Now} \qquad e^{-x/c} = \frac{1}{e^{x/c}} = \frac{1}{\sec\psi + \tan\psi} = \frac{\sec\psi - \tan\psi}{\sec^2\psi - \tan^2\psi} = \sec\psi - \tan\psi. \qquad ...(6)$$

Subtracting (6) from (5), we get

$$e^{x/c} - e^{-x/c} = 2 \tan\psi \quad \text{or} \quad \tan\psi = \frac{e^{x/c} - e^{-x/c}}{2}$$

or $\dfrac{dy}{dx} = \sinh \dfrac{x}{c}$ **(Note)**

or $dy = \sinh (x/c)\, dx$.

Integrating both sides, we get $y + B = c \cosh (x/c)$. ...(7)

But (as given) when $x = 0$, $y = c$ so that $B = 0$.

Therefore putting $B = 0$ in (7), we get $y = c \cosh (x/c)$,

which is the required cartesian equation of the given curve.

Hints to Objective Type Questions

Multiple Choice Questions

1. See article 3.

2. The given curve is $r = a\,(1 + \cos \theta)$.

$\therefore \quad \dfrac{dr}{d\theta} = -\, a \sin \theta.$

Now, $\left(\dfrac{ds}{d\theta}\right)^2 = r^2 + \left(\dfrac{dr}{d\theta}\right)^2 = a^2\,(1 + \cos\theta)^2 + a^2 \sin\theta$

$$= a^2 \left[\left(2\cos^2 \frac{\theta}{2}\right) + \left(2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}\right)^2\right]$$

$$= 4a^2 \cos^2 \frac{\theta}{2}\left[\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2}\right] = 4a^2 \cos^2 \frac{\theta}{2}.$$

$\therefore \quad \dfrac{ds}{d\theta} = 2\, a \cos \dfrac{\theta}{2}.$

3. We have, $\left(\dfrac{ds}{dt}\right)^2 = \left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2$

$$= (-\,3a \cos^2 t \sin t)^2 + (3a \sin^2 t \cos t)^2$$

$$= 9a^2 \cos^2 t \sin^2 t\,(\cos^2 t + \sin^2 t)$$

$$= 9a^2 \cos^2 t \sin^2 t = (3a \sin t \cos t)^2.$$

4. See Problem 1 of Comprehensive Problems 3.

5. See article 1.

6. See Example 1.

7. See Example 4.

8. See Example 9.

9. See Problem 4 of Comprehensive Problems 2.

10. See Example 14.

11. See Example 16.

Fill in the Blank(s)

1. See article 1.

2. See article 2.

3. See Problem 1(i) of Comprehensive Problems 1.

4. The given curve is $r = a\, e^{\theta \cot \alpha}$.

$\therefore \qquad \dfrac{dr}{d\theta} = a\, e^{\theta \cot \alpha} \,.\, \cot \alpha = r \cot \alpha$.

Now, $\qquad \left(\dfrac{ds}{d\theta}\right)^2 = r^2 + \left(\dfrac{dr}{d\theta}\right)^2 = r^2 + r^2 ct^2 \alpha = r^2 (1 + \cot^2 \alpha) = r^2 \operatorname{cosec}^2 \alpha.$

$\therefore \qquad \dfrac{ds}{d\theta} = r \operatorname{cosec} \alpha$ or $ds = r \operatorname{cosec} \alpha \, d\theta.$

5. We know that $\left(\dfrac{ds}{dr}\right)^2 = 1 + \left(r \dfrac{d\theta}{dr}\right)^2$. $\quad \therefore \dfrac{ds}{dr} = \sqrt{1 + \left(r \dfrac{d\theta}{dr}\right)^2}$.

6. See Example 5.

True or False

1. See article 2.

2. The relation between s and ψ for any curve is called its intrinsic equation.

3. See article 4.

4. The arc length is given by $\displaystyle\int_{\theta_1}^{\theta_2} \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\, d\theta.$

5. See Example 4.

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Chapter-9

Volumes and Surfaces of Solids of Revolution

Comprehensive Problems 1

Problem 1(i): *Find the volume of a hemisphere.*

Solution: Proceed exactly as in Example 1. The hemisphere is generated by the revolution of a quadrant of the circle $x^2 + y^2 = a^2$ about x-axis. The limits for the volume will be from 0 to a. The required volume of hemisphere is $\frac{2}{3}\pi a^3$.

Problem 1(ii): *Find the volume of a spherical cap of height h cut off from a sphere of radius a.*

(Kanpur 2010)

Solution: The limits for the volume of the spherical cap of height h will be from $a - h$ to a. Proceeding as in Example 1, we get the required volume

$$= \int_{a-h}^{a} \pi y^2 \, dx = \pi \int_{a-h}^{a} (a^2 - x^2)\, dx = \pi \left[a^2 x - \frac{x^3}{3} \right]_{a-h}^{a}$$

$$= \pi \left[a^3 - \frac{1}{3} a^3 - a^2 (a - h) + \frac{1}{3} (a - h)^3 \right]$$

$$= \pi \left[a^3 - \frac{1}{3} a^3 - a^3 + a^2 h + \frac{1}{3} a^3 - a^2 h + ah^2 - \frac{1}{3} h^3 \right]$$

$$= \pi \left[ah^2 - \frac{1}{3} h^3 \right] = \pi h^2 \left[a - \frac{1}{3} h \right].$$

Problem 2(i): *A segment is cut off from a sphere of radius a by a plane at a distance* $\frac{1}{2}a$ *from the centre. Show that the volume of the segment is 5/32 of the volume of the sphere.*

Solution: Draw the figure as in Example 1. Let BC be the line $x = \frac{1}{2} a$.

The segment of the sphere is generated by revolving the area $ABCA$ of the circle about the x-axis. Hence the limits for the volume of the segment will be from $x = \frac{1}{2} a$ to $x = a$.

$\therefore$ The volume of the segment of the sphere

$$= \int_{a/2}^{a} \pi y^2 \, dx = \int_{a/2}^{a} \pi (a^2 - x^2)\, dx, \qquad [\because x^2 + y^2 = a^2]$$

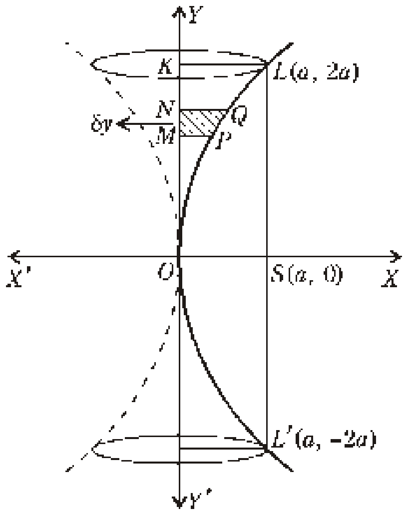
$$= \pi \left[a^2 x - \frac{x^3}{3} \right]_{a/2}^{a} = \pi \left[a^3 - \frac{1}{3} a^3 - \left(\frac{1}{2} a^3 - \frac{1}{24} a^3 \right) \right] = \pi \left[\frac{5}{24} a^3 \right] = \frac{5}{32} \left[\frac{4}{3} \pi a^3 \right].$$

Also volume of the sphere $= \frac{4}{3}\pi a^3$. [See Example 1]

$$\therefore \quad \frac{\text{Volume of the segment}}{\text{Volume of the sphere}} = \frac{\frac{5}{32}\cdot[\frac{4}{3}\pi a^3]}{\frac{4}{3}\pi a^3} = \frac{5}{32}$$

i.e., Volume of the segment $= \frac{5}{32}$ of the volume of the sphere.

Problem 2(ii): *The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the volume of the reel thus generated.*



Solution: The given parabola is $y^2 = 4ax$.

It is symmetrical about x-axis. The tangent at the vertex is $x = 0$ *i.e.*, y-axis. LL' is the latus rectum.

A reel is formed by revolving about y-axis the area enclosed between the arc $L'OL$ of the parabola and the axis of y. The volume of the reel generated by the revolution of the arc cut off by the latus rectum LL' about y-axis $= 2 \times$ volume generated by revolving the area OLK about y-axis.

Consider an elementary strip $PMNQ$ parallel to the axis of x, where P is the point (x, y) and Q is the point $(x + \delta x, y + \delta y)$ on the parabola $y^2 = 4ax$.

Then $PM = x$

and $NM = ON - OM = (y + \delta y) - y = \delta y$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about y-axis

$$= \pi (PM)^2 \cdot (NM) = \pi x^2 \delta y.$$

Also as the length of the semi-latus rectum SL is $2a$, therefore y varies from 0 to $2a$.

$\therefore$ The required volume

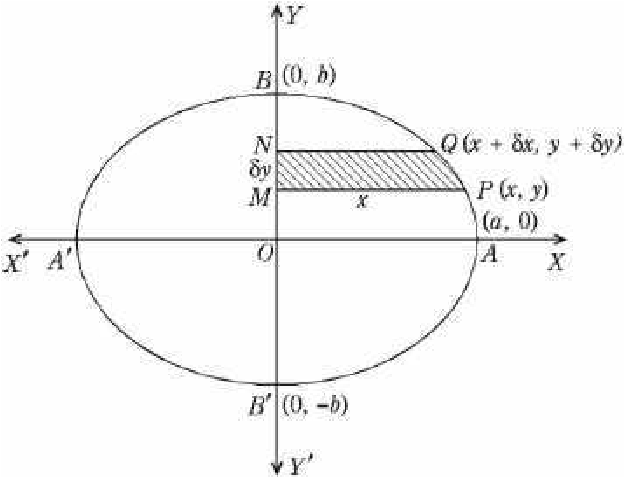
$$= 2\int_0^{2a} \pi x^2 dy = 2\int_0^{2a} \pi \left[\frac{y^2}{4a}\right]^2 dy \qquad [\because y^2 = 4ax]$$

$$= \frac{\pi}{8a^2}\int_0^{2a} y^4\, dy = \frac{\pi}{8a^2}\left[\frac{y^5}{5}\right]_0^{2a} = \frac{\pi}{40a^2}\cdot 32a^5 = \frac{4}{5}\pi a^3.$$

Problem 3: *Prove that the volume of the solid generated by the revolution of an ellipse round its minor axis is a mean proportional between those generated by the revolution of the ellipse and of the auxiliary circle about the major axis.* **(Rohilkhand 2010)**

Solution: Let the ellipse be $x^2/a^2 + y^2/b^2 = 1$. ...(1)

Also the equation of its auxiliary circle is $x^2 + y^2 = a^2$. ...(2)



Take an elementary strip *PMNQ* perpendicular to the axis of *x*. We have $PM = y$ and $MN = \delta x$. Now volume of the elementary disc formed by revolving the strip *PMNQ* about the *x*-axis $= \pi . (PM)^2 \cdot MN = \pi y^2 \, \delta x$.

Also the ellipse is symmetrical about the *y*-axis and for the portion of the curve lying in the first quadrant *x* varies from 0 to *a*. Therefore the required volume of the solid formed

$$= 2\pi \int_0^a y^2 \, dx = 2\pi \int_0^a \frac{b^2}{a^2} (a^2 - x^2) \, dx \qquad \text{[From (1)]}$$

$$= 2\pi \frac{b^2}{a^2} \left[a^2 x - \frac{1}{3} x^3 \right]_0^a = 2\pi \frac{b^2}{a^2} (a^3 - \frac{1}{3} a^3) = \frac{4}{3} \pi a b^2 = V_1, \text{ say.}$$

Also the volume of the sphere formed by the revolution of (2) about *x*-axis is

$$\frac{4}{3} \pi a^3 = V_2 \text{ , say.} \qquad \text{[See Example 1; prove it here]}$$

Now we have to find the volume of the solid formed by the revolution of the ellipse about the minor axis *i.e.*, the axis of *y*. Consider an elementary strip *PMNQ* perpendicular to *y*-axis. Then the volume of the elementary disc formed by revolving the strip *PMNQ* about *y*-axis $= \pi \, x^2 \, \delta y$.

The ellipse is symmetrical about *x*-axis and on the arc *AB* of the ellipse *y* varies from 0 to *b*.

∴ The volume of the solid generated by the revolution of the ellipse about *y*-axis

$$= 2 \int_0^b \pi \, x^2 \, dy \qquad \textbf{(Note)}$$

$$= 2\pi \frac{a^2}{b^2} \int_0^b (b^2 - y^2) \, dy \qquad \left[\because \text{ From(1), } x^2 = \frac{a^2}{b^2} (b^2 - y^2) \right]$$

$$= \frac{2\pi a^2}{b^2}\left[b^2 y - \frac{y^3}{3}\right]_0^b = \frac{2\pi a^2}{b^2}\left[b^3 - \frac{b^3}{3}\right] = \frac{4\pi a^2 b}{3} = V_3 \text{ , say.}$$

Now mean proportional between V_1 and V_2

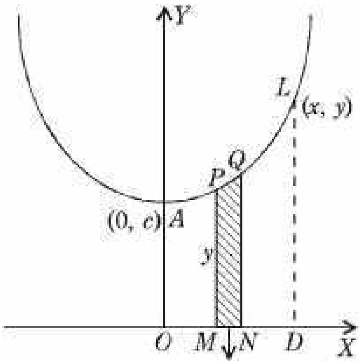
$$= \sqrt{(V_1 V_2)} = \sqrt{\left(\frac{4}{3}\pi a b^2 \cdot \frac{4}{3}\pi a^3\right)} = \frac{4}{3}\pi a^2 b = V_2$$

= volume generated when ellipse is revolved about minor axis.

Problem 4(i): *Find the volume of the solid generated by the revolution of an arc of the catenary $y = c \cosh(x/c)$ about the x-axis.* **(Meerut 2009B; Purvanchal 11)**

Solution: The given equation of catenary is

$$y = c \cosh(x/c).$$



Let AL be an arc of this catenary where L is the point (x, y).

Take an elementary strip $PMNQ$ perpendicular to the axis of x, so that $PM = y$ and $MN = dx$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the axis of x

$$= \pi . PM^2 . MN = \pi y^2 \, dx.$$

$\therefore$ the required volume

$$= \int_0^x \pi \, y^2 \, dx$$

$$= \pi \int_0^x c^2 \cosh^2 \frac{x}{c} \, dx, \qquad [\because \; y = c \cosh(x/c)]$$

$$= \frac{\pi c^2}{2}\int_0^x \left(1 + \cosh \frac{2x}{c}\right) dx = \frac{\pi c^2}{2}\left[x + \frac{c}{2}\sinh \frac{2x}{c}\right]_0^x$$

$$= \frac{\pi c^2}{2}\left[x + \frac{c}{2}\sinh \frac{2x}{c}\right].$$

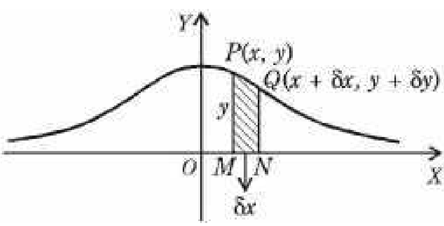
The solid of revolution formed by revolving a catenary about its directrix is called a **catenoid**.

Problem 4(ii): *Find the volume of the solid generated by the revolution of the curve $y = a^3/(a^2 + x^2)$ about its asymptote.* **(Garhwal 2001; Agra 02; Meerut 09)**

Solution: The given curve is

$$y = a^3/(a^2 + x^2)$$

or $$x^2 y = a^2 (a - y). \quad ...(1)$$



Equating to zero, the coefficient of the highest power of x, the asymptote parallel to x-axis is $y = 0$ *i.e.*, x-axis. The shape of the curve is as shown in the figure.

Take an elementary strip $PMNQ$ where $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ are two neighbouring points on the curve. We have $PM = y$ and $MN = \delta x$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the axis of x is

$$= \pi PM^2 . MN = \pi y^2 \delta x.$$

The curve is symmetrical about y-axis and for the portion of the curve in the positive quadrant x varies from 0 to ∞.

$\therefore$ The required volume

$$= 2\int_0^{\infty} \pi y^2 \, dx = 2\pi \int_0^{\infty} \frac{a^6}{(x^2+a^2)^2} dx, \qquad \text{[From (1)]}$$

$$= 2\pi a^6 \int_0^{\pi/2} \frac{a \sec^2\theta \, d\theta}{a^4 (1+\tan^2\theta)^2},$$

putting $x = a\tan\theta$ so that $dx = a\sec^2\theta \, d\theta$

$$= 2\pi a^3 \int_0^{\pi/2} \frac{\sec^2\theta \, d\theta}{\sec^4\theta} = 2\pi a^3 \int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$= \pi a^3 \cdot \frac{1}{2} \cdot \frac{1}{2} \pi = \frac{1}{2}\pi^2 a^3.$$

Problem 5: *If the hyperbola $x^2/a^2 - y^2/b^2 = 1$ revolves about the x-axis, show that the volume included between the surface thus generated, the cone generated by the asymptotes and two planes perpendicular to the axis of x, at a distance h apart, is equal to that of a circular cylinder of height h and radius b.*

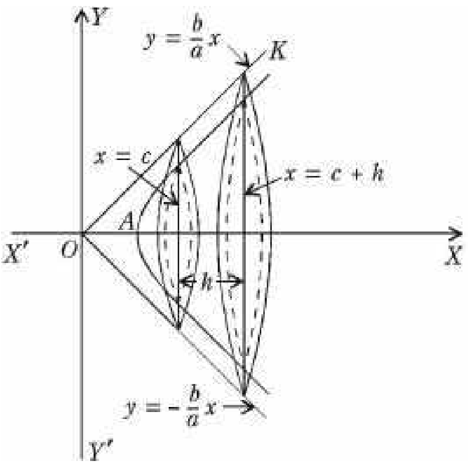
Solution: The given hyperbola is

$$x^2/a^2 - y^2/b^2 = 1, \qquad \ldots(1)$$

and the equation of its asymptotes is

$$x^2/a^2 - y^2/b^2 = 0$$

or $\quad y = \pm (b/a)\, x$.



Let the two given planes be at distances c and $(c+h)$ from the origin O. Then the volume of the portion of the cone generated by the asymptotes between $x = c$ and $x = c + h$ is

$$= \int_c^{c+h} \pi y^2 \, dx,$$

where $y = \frac{b}{a} x$

(Note)

$$= \pi \int_c^{c+h} \frac{b^2}{a^2} x^2 \, dx = \frac{\pi b^2}{a^2} \left[\frac{x^3}{3} \right]_c^{c+h} = \frac{\pi b^2}{3a^2} [(c+h)^3 - c^3]$$

$$= \frac{\pi b^2}{3a^2} [h^3 + 3ch^2 + 3c^2 h] = V_1, \text{ say.}$$

Now the volume of the portion of the solid generated by the hyperbola between the two given planes (*i.e.*, between $x = c$ and $x = c + h$) is

$$= \int_c^{c+h} \pi\, y^2\, dx, \quad \text{where } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \textbf{(Note)}$$

$$= \frac{\pi b^2}{a^2} \int_c^{c+h} (x^2 - a^2)\, dx = \frac{\pi b^2}{a^2} \left[\frac{x^3}{3} - a^2 x \right]_c^{c+h}$$

$$= \frac{\pi b^2}{a^2} \left[\frac{1}{3} \{ (c+h)^3 - c^3 \} - a^2 \{ (c+h) - c \} \right]$$

$$= \frac{\pi b^2}{3a^2} [h^3 + 3ch^2 + 3c^2 h - 3a^2 h] = V_2, \text{ say.}$$

$\therefore$ The required volume $= V_1 - V_2$

$$= \frac{\pi b^2}{3a^2} [h^3 + 3ch^2 + 3c^2 h] - \frac{\pi b^2}{3a^2} [h^3 + 3ch^2 + 3c^2 h - 3a^2 h]$$

$$= \frac{\pi b^2}{3a^2} \cdot 3a^2 h = \pi b^2 h$$

= volume of the cylinder of radius b and height h.

Problem 6(i): *Find the volume formed by the revolution of the loop of the curve* $y^2 (a + x) = x^2 (a - x)$ *about the axis of* x. **(Kanpur 2008)**

Solution: Proceed exactly as in Example 2. The required volume

$$= 2\pi a^3 \left(\log 2 - \frac{2}{3}\right).$$

Problem 6(ii): *Find the volume of the solid generated by the revolution of the loop of the curve* $y^2 = x^2 (a - x)$ *about the axis of* x. **(Kanpur 2011)**

Solution: The given equation of the curve is $y^2 = x^2 (a - x)$ ...(1)

Putting $y = 0$ in (1) we get $x = 0$, $x = a$ *i.e.*, the loop is formed between $(0, 0)$ and $(a, 0)$.

$\therefore$ The required volume $= \int_0^a \pi y^2\, dx$ [Proceeding as in Example 2]

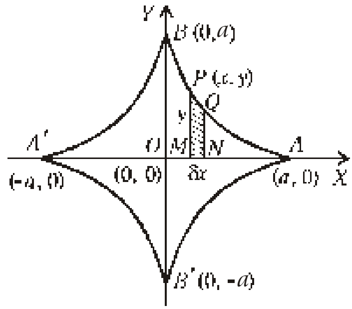
$$= \pi \int_0^a x^2 (a - x)\, dx \qquad \text{[From (1)]}$$

$$= \pi \int_0^a (x^2 a - x^3)\, dx = \pi \left[a \frac{x^3}{3} - \frac{x^4}{4} \right]_0^a = \pi a^4 \left[\frac{1}{3} - \frac{1}{4} \right] = \frac{1}{12} \pi a^4.$$

Problem 7: *Show that the volume of the solid generated by the revolution of the upper half of the loop of the curve* $y^2 = x^2 (2 - x)$ *about x-axis is* $\frac{4}{3} \pi$. **(Meerut 2005)**

Solution: Put $a = 2$ and proceed exactly as in Problem 6(ii).

Problem 8: *The area of the curve* $x^{2/3} + y^{2/3} = a^{2/3}$ *lying in the first quadrant revolves about x-axis. Find the volume of the solid generated.* **(Agra 2014)**



Solution: The given curve is

$$x^{2/3} + y^{2/3} = a^{2/3} \qquad \ldots(1)$$

The curve is symmetrical about both the axes. The coordinates of B are $(0, a)$ and those of A are $(a, 0)$.
Take an elementary strip $PMNQ$ where P is the point (x, y) and Q is the point $(x + \delta x, y + \delta y)$ on the curve. We have $PM = y$ and $MN = \delta x$.
Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the axis of x is

$$= \pi y^2 \, \delta x.$$

$\therefore$ The required volume

$$= \int_0^a xy^2 \, dx, \text{ by symmetry}$$

$$= \pi \int_0^a (a^{2/3} - x^{2/3})^3 \, dx,$$

$$[\because \text{ from (1)}, \ y^{2/3} = (a^{2/3} - x^{2/3}) \text{ so that } y^2 = (a^{2/3} - x^{2/3})^3]$$

$$= \pi \int_0^{\pi/2} a^2 \cos^6\theta \cdot 3a\sin^2\theta\cos\theta \, d\theta,$$

$$\text{putting } x = a\sin^3\theta \text{ so that } dx = 3a\sin^2\theta\cos\theta \, d\theta$$

$$= 3\pi a^3 \int_0^{\pi/2} \sin^2\theta \cos^7\theta \, d\theta = 3\pi a^3 \cdot \frac{1.6.4.2}{9.7.5.3.1} = \frac{16\pi a^3}{105}.$$

Problem 9: *Find the volume of the solid obtained by revolving the loop of the curve* $a^2 y^2 = x^2 (2a - x)(x - a)$ *about x-axis.*

Solution: The given curve is $a^2 y^2 = x^2 (2a - x)(x - a)$. ...(1)

The curve (1) is symmetrical about x-axis. It passes through the origin but the origin is a conjugate point. The curve cuts the x-axis at the points $(a, 0)$ and $(2a, 0)$ and so the loop of the curve is formed between $(a, 0)$ and $(2a, 0)$.

$$\therefore \text{ The required volume} = \int_{x=a}^{2a} \pi \, y^2 dx = \pi \int_a^{2a} \frac{x^2 (2a - x)(x - a)}{a^2} \, dx, \quad [\text{ From (1)}]$$

$$= \frac{\pi}{a^2} \int_a^{2a} (-x^4 + 3ax^3 - 2a^2 x^2) \, dx = \frac{\pi}{a^2} \left[-\frac{x^5}{5} + \frac{3ax^4}{4} - \frac{2a^2 x^3}{3} \right]_a^{2a}$$

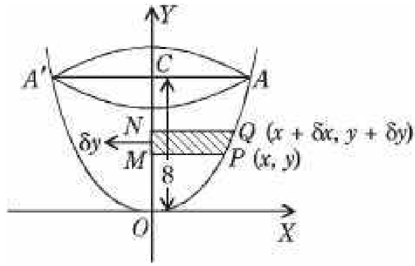
$$= \frac{\pi}{a^2} \left[\left(-\frac{32a^5}{5} + 12a^5 - \frac{16a^5}{3} \right) - \left(\frac{a^5}{5} + \frac{3a^5}{4} - \frac{2a^5}{3} \right) \right]$$

$$= \pi a^3 \left[-\frac{32}{5} + 12 - \frac{16}{3} - \frac{1}{5} - \frac{3}{4} + \frac{2}{3} \right] = \frac{23}{60} \pi a^3.$$

Problem 10: *A basin is formed by the revolution of the curve* $x^3 = 64y, (y > 0)$ *about the axis of y. If the depth of the basin is 8 inches, how many cubic inches of water it will hold ?*

Solution: The given curve is

$$x^3 = 64y. \quad \ldots(1)$$



The curve (1) passes through the origin and the tangent there is the line $y = 0$. When $y > 0$, we have $x > 0$ and so no portion of the curve lies in the second quadrant.

Take an elementary strip $PMNQ$ where $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ are two neighbouring points on the curve and PM and QN are perpendiculars from the points P and Q respectively, on the y-axis.

We have $PM = x$ and $MN = \delta y$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about y-axis is

$$= \pi \,.\, PM^2 .\, MN = \pi x^2 \, \delta y.$$

Clearly to form the required basin y varies from 0 to 8.

$\therefore$ The required volume (*i.e.*, the capacity in cubic inches)

$$= \int_{y=0}^{8} \pi \, x^2 \, dy = \int_0^8 \pi \, (64\, y)^{2/3}, \quad \text{[From (1)]}$$

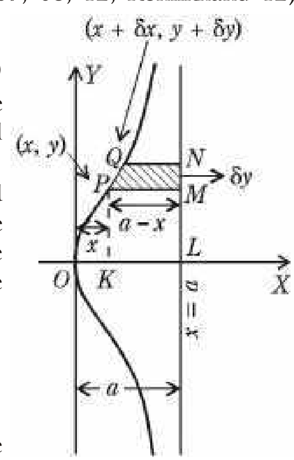
$$= 16\pi \int_0^8 y^{2/3} \, dy = 16\pi \,.\left[\frac{3}{5}\, y^{5/3}\right]_0^8 = \frac{48\pi}{5} \cdot 32 = \frac{1536\pi}{5} \text{ cubic inches.}$$

Problem 11: *Show that the volume of the solid generated by the revolution of the curve* $(a - x)\, y^2 = a^2 x$, *about its asymptote is* $\frac{1}{2}\pi^2 a^3$.

(Meerut 2004B, 06B; Kumaun 07, 08, 12; Rohilkhand 12)

Solution: The given curve is

$$(a - x)\, y^2 = a^2\, x. \quad \ldots(1)$$



Its shape is as shown in the figure. Equating to zero, the coefficient of highest power of y, the asymptote parallel to the axis of y is $a - x = 0$ *i.e.*, $x = a$.

Take an elementary strip $PMNQ$, where $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ are two neighbouring points on the curve and PM, QN are perpendiculars to the asymptote from the points P and Q respectively. We have $PM = OL - OK = a - x$ and $MN = \delta y$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the line $x = a$ is

$$= \pi \,.\, PM^2 .\, MN = \pi\,(a - x)^2 \, \delta y.$$

The given curve is symmetrical about x-axis and for the portion of the curve above x-axis y varies from 0 to ∞.

$\therefore$ The required volume $= 2 \int_{y=0}^{\infty} \pi \,(a - x)^2 \, dy$

$$= 2\pi \int_0^{\infty} \left(a - \frac{ay^2}{y^2 + a^2} \right)^2 dy, \qquad \left[\because \text{ from (1)}, x = \frac{ay^2}{y^2 + a^2} \right]$$

$$= 2\pi\, a^6 \int_0^{\infty} \frac{dy}{(y^2 + a^2)^2}.$$

Now put $y = a \tan\theta$ so that $dy = a \sec^2 \theta\, d\theta$.

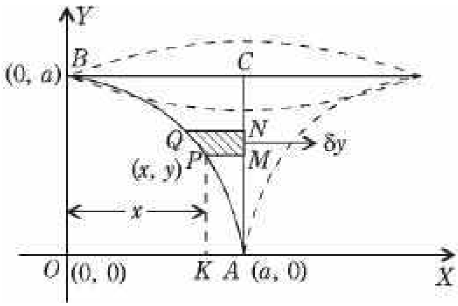
When $y = 0, \theta = 0$ and when $y \to \infty, \theta \to \pi/2$. Therefore the required volume

$$= 2\pi\, a^6 \int_0^{\pi/2} \frac{a \sec^2 \theta\, d\theta}{a^4 \sec^4 \theta} = 2\pi\, a^3 \int_0^{\pi/2} \cos^2 \theta\, d\theta$$

$$= 2\pi\, a^3 \cdot \frac{1}{2} \cdot \frac{1}{2}\, \pi = \frac{1}{2}\, \pi^2\, a^3.$$

Problem 12: *The figure bounded by a quadrant of a circle of radius a and tangents at its extremities revolves about one of the tangents. Prove that the volume of the solid generated is* $(\frac{5}{3} - \frac{1}{2}\pi)\, \pi a^3$.

Solution: Let the arc AB be the quadrant of the circle $x^2 + y^2 = a^2$. The area bounded by the arc AB and the tangents AC and BC (*i.e.*, the area $ABCA$) is revolved about the tangent AC, (say). Take an elementary strip $PMNQ$ where $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ are two neighbouring points on the arc AB and PM, QN are perpendiculars from P and Q on the tangent AC.



We have $\qquad PM = OA - OK = a - x$

and $\qquad MN = AN - AM = y + \delta y - y = \delta y.$

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the tangent AC (*i.e.*, the line $x = a$) is

$$= \pi \,.\, PM^2 \,.\, MN = \pi\,(a - x)^2\, \delta y\,.$$

Also for the arc AB, y varies from 0 to a.

$\therefore$ The required volume

$$= \int_{y=0}^{a} \pi\,(a - x)^2\, dy = \pi \int_0^a (a^2 + x^2 - 2ax)\, dy$$

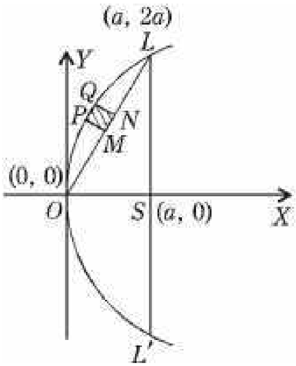
$$= \pi \int_0^a \{ a^2 + (a^2 - y^2) - 2a\sqrt{(a^2 - y^2)} \}\, dy, \qquad [\because x^2 = a^2 - y^2]$$

$$= \pi \left[2a^2 y - \frac{y^3}{3} - 2a \left\{ \frac{1}{2}\, y\sqrt{(a^2 - y^2)} + \frac{1}{2}\, a^2 \sin^{-1}(y/a) \right\} \right]_0^a$$

$$= \pi\, [2a^3 - \frac{1}{3}\, a^3 - a^3 \sin^{-1}(1)] = \pi a^3\, [\frac{5}{3} - \frac{1}{2}\, \pi].$$

Problem 13: *The area cut off from the parabola $y^2 = 4ax$ by the chord joining the vertex to an end of the latus rectum is rotated through four right angles about the chord. Find the volume of the solid generated.* **(Rohilkhand 2009; Bundelkhand 09)**

Solution: The given parabola is $y^2 = 4ax$.



Let LL' be the latus rectum. The area bounded by the arc OL and the chord OL is revolved about the chord OL.

Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be any two neighbouring points on the arc OL and PM, QN be the perpendiculars from P and Q respectively on the axis of revolution OL.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the chord OL is

$$= \pi . PM^2 . MN = \pi . PM^2 . d(OM).$$

(Note)

Also equation of the chord OL is

$$y - 0 = \frac{2a - 0}{a - 0}(x - 0) \text{ i.e., } 2x - y = 0 \qquad \text{...(1)}$$

$\therefore$ PM = the length of the perpendicular from (x, y) to (1)

$$= \frac{2x - y}{\sqrt{(2^2 + 1^2)}} = \frac{2x - y}{\sqrt{5}},$$

and $$OM = \sqrt{(OP^2 - MP^2)} = \sqrt{\left\{(x^2 + y^2) - \frac{(2x - y)^2}{5}\right\}} = \frac{x + 2y}{\sqrt{5}}.$$

Now the required volume

$$= \int_{x=0}^{a} \pi (PM)^2 \, d(OM),$$

[$\because$ for the arc OL, x varies from 0 to a]

$$= \int_{x=0}^{a} \pi \left(\frac{2x - y}{\sqrt{5}}\right)^2 d\left(\frac{x + 2y}{\sqrt{5}}\right)$$

$$= \int_{x=0}^{a} \pi \left(\frac{2x - 2\sqrt{(ax)}}{\sqrt{5}}\right)^2 \frac{d}{dx}\left(\frac{x + 2 . 2\sqrt{(ax)}}{\sqrt{5}}\right) dx, \quad [\because \ y = 2\sqrt{(ax)}]$$

$$= \frac{\pi}{5\sqrt{5}} \int_0^a \{2x - 2\sqrt{(ax)}\}^2 \left(1 + 4\sqrt{a} \cdot \frac{1}{2\sqrt{x}}\right) dx, \qquad \textbf{(Note)}$$

$$= \frac{4\pi}{5\sqrt{5}} \int_0^a [x^2 - 2\sqrt{a}\, x^{3/2} + ax]\left[1 + 2\sqrt{\left(\frac{a}{x}\right)}\right] dx$$

$$= \frac{4\pi}{5\sqrt{5}} \int_0^a [x^2 - 3ax + 2a^{3/2}\sqrt{x}]\, dx = \frac{4\pi}{5\sqrt{5}}\left[\frac{x^3}{3} - \frac{3ax^2}{2} + \frac{2a^{3/2}\, x^{3/2}}{3/2}\right]_0^a$$

$$= \frac{4\pi}{5\sqrt{5}}\left[\frac{a^3}{3} - \frac{3a^3}{2} + \frac{4a^3}{3}\right] = \frac{2\pi a^3}{15\sqrt{5}} = \frac{2\sqrt{5}}{75}\pi a^3.$$

Comprehensive Problems 2

Problem 1: *Find the volume of the solid generated by the revolution of the cycloid*

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta), 0 \le \theta \le \pi,$$

(*i*) *about the x-axis.* (*ii*) *about the base.*

Solution: The equations of the cycloid are $x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$.

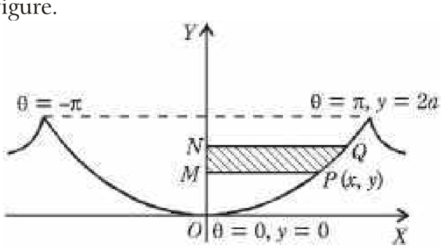
The cycloid is symmetrical about the y-axis. For half of the curve θ varies from 0 to π. Now proceed exactly as in Example 5.

Problem 2: *Show that the volume of the solid generated by the revolution of the cycloid*

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta), 0 \le \theta \le \pi,$$

about the y-axis is $\pi a^3 \left(\frac{3}{2}\pi^2 - \frac{8}{3}\right)$.

Solution: The curve is as shown in the figure.



The required volume

$$= \int_{y=0}^{2a} \pi x^2\, dy = \pi \int_{\theta=0}^{\pi} x^2 \frac{dy}{d\theta}\, d\theta$$

$$= \pi \int_{\theta=0}^{\pi} a^2 (\theta + \sin\theta)^2\, a \sin\theta\, d\theta$$

$$= \pi a^3 \int_{\theta=0}^{\pi} (\theta^2 + 2\theta\sin\theta + \sin^2\theta)\sin\theta\, d\theta$$

$$= \pi a^3 \int_0^{\pi} (\theta^2 \sin\theta + 2\theta\sin^2\theta + \sin^3\theta)\, d\theta$$

$$= \pi a^3 (I_1 + 2I_2 + I_3), \text{ say}$$

where $$I_1 = \int_0^{\pi} \theta^2 \sin\theta\, d\theta = \left[-\theta^2\cos\theta\right]_0^{\pi} + \int_0^{\pi} 2\theta\cos\theta\, d\theta$$

$$= \pi^2 + \left[2\theta \sin\theta\right]_0^{\pi} - 2\int_0^{\pi} \sin\theta\, d\theta = \pi^2 + 2\left[\cos\theta\right]_0^{\pi}$$

$$= \pi^2 + 2(-1-1) = \pi^2 - 4,$$

$$I_2 = \int_0^{\pi} \theta \sin^2\theta\, d\theta = \int_0^{\pi} (\pi - \theta)\sin^2(\pi - \theta)\, d\theta,$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi} \pi \sin^2\theta\, d\theta - \int_0^{\pi} \theta\sin^2\theta\, d\theta = \pi\int_0^{\pi} \sin^2\theta\, d\theta - I_2$$

so that $$2I_2 = \pi\int_0^{\pi} \sin^2\theta\, d\theta = 2\pi\int_0^{\pi/2} \sin^2\theta\, d\theta$$

or $$I_2 = \pi\int_0^{\pi/2} \sin^2\theta\, d\theta = \pi\cdot\frac{1}{2}\cdot\frac{1}{2}\pi = \frac{1}{4}\pi^2,$$

and $$I_3 = \int_0^{\pi} \sin^3\theta\, d\theta = 2\int_0^{\pi/2} \sin^3\theta\, d\theta = 2\cdot\frac{2}{3.1} = \frac{4}{3}.$$

$\therefore$ The required volume $= \pi a^3 (I_1 + 2I_2 + I_3)$

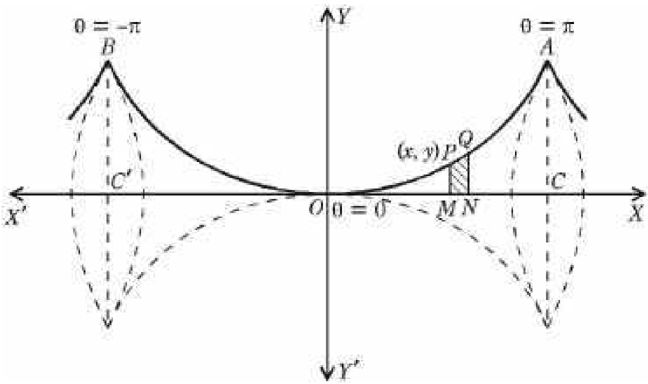
$$= \pi a^3 [(\pi^2 - 4) + 2 \cdot \frac{1}{4} \pi^2 + \frac{4}{3}] = \pi a^3 [\frac{3}{2} \pi^2 - \frac{8}{3}].$$

Problem 3: *Prove that the volume of the reel formed by the revolution of the cycloid*

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$$

about the tangent at the vertex is $\pi^2 a^3$. **(Agra 2003; Kumaun 13)**

Solution: The given cycloid is symmetrical about the y-axis and the tangent at the vertex is x-axis. The reel is formed by the revolution about x-axis of the area enclosed between the cycloid and the x-axis. For the arc OA of the curve θ varies from 0 to π.



Take an elementary strip $PMNQ$ where P is the point (x, y) and Q is the point $(x + \delta x, y + \delta y)$. We have $PM = y$ and $MN = \delta x$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the tangent at the vertex (*i.e.*, about x-axis) is $= \pi PM^2 . MN = \pi y^2 \delta x$.

$\therefore$ The required volume

$$= 2 \int \pi y^2 \, dx \text{, between the limits of integration from } O \text{ to } A$$

$$= 2 \int_0^{\pi} \pi y^2 \frac{dx}{d\theta} d\theta$$

$$= 2\pi \int_0^{\pi} a^2 (1 - \cos\theta)^2 . a (1 + \cos\theta) \, d\theta, \text{ putting for } y \text{ and } \frac{dx}{d\theta}$$

$$= 2\pi a^3 \int_0^{\pi} \left(2 \sin^2 \frac{\theta}{2}\right)^2 \cdot \left(2 \cos^2 \frac{\theta}{2}\right) d\theta$$

$$= 2\pi a^3 \int_0^{\pi/2} 4 \sin^4 t . 2 \cos^2 t . 2 \, dt \text{, putting } \frac{\theta}{2} = t \text{ so that } d\theta = 2 \, dt$$

$$= 32 \pi a^3 \int_0^{\pi/2} \sin^4 t \cos^2 t \, dt$$

$$= 32 \pi a^3 \cdot \frac{3.1.1}{6.4.2} \cdot \frac{\pi}{2} = \pi^2 a^3.$$

Problem 4: *Prove that the volume of the solid generated by the revolution about the x-axis of the loop of the curve* $x = t^2,\ y = t - \frac{1}{3}t^3$ *is* $\frac{3}{4}\pi$.

Solution: The given parametric equations of the curve are

$$x = t^2,\ y = t - \frac{1}{3}t^3. \qquad ...(1)$$

Eliminating t, we have

$$y^2 = t^2\left(1 - \frac{1}{3}t^2\right)^2 = x\left(1 - \frac{1}{3}x\right)^2.$$

The curve is thus symmetrical about the x-axis. The curve cuts the x-axis at the points $(0, 0)$ and $(3, 0)$. Therefore the loop of the curve lies between these points. Putting $y = 0$ in (1), we get

$$t\left(1 - \frac{1}{3}t^2\right) = 0 \text{ giving } t = 0, \pm\sqrt{3}.$$

Therefore for the upper half of the loop t varies from 0 to $\sqrt{3}$.

$\therefore$ The required volume $= \int_0^{\sqrt{3}} \pi\, y^2 \cdot \frac{dx}{dt}\, dt = \int_0^{\sqrt{3}} \pi\left(t - \frac{1}{3}t^3\right)^2 2t\, dt$, from (1)

$$= 2\pi\int_0^{\sqrt{3}} t\left(t^2 + \frac{1}{9}t^6 - \frac{2}{3}t^4\right) dt = 2\pi\int_0^{\sqrt{3}}\left(t^3 + \frac{1}{9}t^7 - \frac{2}{3}t^5\right) dt$$

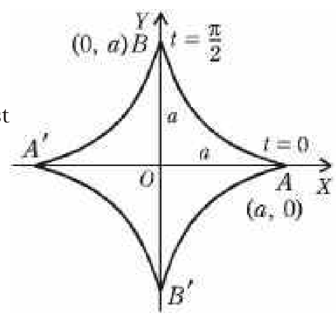
$$= 2\pi\left[\frac{t^4}{4} + \frac{1}{9}\cdot\frac{t^8}{8} - \frac{2}{3}\cdot\frac{t^6}{6}\right]_0^{\sqrt{3}} = 2\pi\left[\frac{9}{4} + \frac{1}{9}\cdot\frac{81}{8} - \frac{2}{3}\cdot\frac{27}{6}\right]$$

$$= 2\pi\left[\frac{9}{4} + \frac{9}{8} - 3\right] = \frac{3\pi}{4}.$$

Problem 5: *Find the volume of the spindle shaped solid generated by revolving the astroid* $x^{2/3} + y^{2/3} = a^{2/3}$ *about the x-axis.*

Solution: The parametric equations of the given curve $x^{2/3} + y^{2/3} = a^{2/3}$ are

$$x = a\cos^3 t,\ y = a\sin^3 t. \qquad ...(1)$$



The curve is symmetrical about both the axes.

At the point B, $x = 0$ and so $t = \frac{1}{2}\pi$.

Again at the point A, $x = a$ and so $t = 0$.

Therefore for the portion of the curve lying in the first quadrant t varies from $\frac{1}{2}\pi$ to 0.

$\therefore$ The required volume = 2 × volume generated by revolving the area lying in the Ist quadrant

$$= 2\int_{x=0}^{a} \pi\, y^2\, dx = 2\int_{\pi/2}^{0} \pi\, y^2 \cdot \frac{dx}{dt}\, dt$$

$$= 2\pi\int_{\pi/2}^{0} a^2\sin^6 t\,.\left(-3a\cos^2 t\sin t\, dt\right),$$

[From (1)]

$$= 6\pi a^3 \int_0^{\pi/2} \sin^7 t \,.\, \cos^2 t\, dt = 6\pi a^3 \cdot \frac{6\,.\,4\,.\,2\,.\,1}{9\,.\,7\,.\,5\,.\,3\,.\,1} = \frac{32}{105}\,\pi a^3.$$

Problem 6: *Find the volume of the solid generated by the revolution of the cissoid* $x = 2a \sin^2 t,\ y = 2a \sin^3 t / \cos t$ *about its asymptote.* **(Kanpur 2006; Bundelkhand 14)**

Solution: The given parametric equations of the cissoid are

$$x = 2a \sin^2 t,\ y = 2a \sin^3 t\ / \cos t. \qquad \textbf{[Remember]}$$

Let us eliminate t between these equations.

We have $\quad \sin^2 t = x/2a. \quad$ ...(1)

Now
$$y^2 = \left[2a \frac{\sin^3 t}{\cos t}\right]^2 = 4a^2 \frac{\sin^6 t}{\cos^2 t} = 4a^2 \frac{(\sin^2 t)^3}{1 - \sin^2 t}$$
$$= \frac{\{4a^2 (x/2a)^3\}}{\{1 - (x/2a)\}}, \qquad \text{[From (1)]}$$
$$= \frac{x^3}{2a - x}.$$

Thus $y^2 (2a - x) = x^3$ is the cartesian equation of the given cissoid and for the shape of the curve see Example 3.

Proceeding as in Example 3, the required volume

$$= 2\pi \int_{y=0}^{\infty} (2a - x)^2\, dy = 2\pi \int_{t=0}^{\pi/2} (2a - x)^2 \frac{dy}{dt}\, dt$$

$$\left[\because\ t = 0, \text{ when } y = 0 \text{ and } t \to \frac{1}{2}\pi \text{ when } y \to \infty\right]$$

$$= 2\pi \int_0^{\pi/2} (2a - 2a \sin^2 t)^2 \,.\, 2a \frac{3 \sin^2 t \cos^2 t + \sin^4 t}{\cos^2 t}\, dt$$

$$= 16\pi a^3 \int_0^{\pi/2} \cos^2 t\, (3 \sin^2 t \cos^2 t + \sin^4 t)\, dt$$

$$= 16\pi a^3 \left[\int_0^{\pi/2} 3 \sin^2 t \cos^4 t\, dt + \int_0^{\pi/2} \sin^4 t \cos^2 t\, dt\right]$$

$$= 16\pi a^3 \left[3 \cdot \frac{1\,.\,3\,.\,1}{6\,.\,4\,.\,2} \cdot \frac{1}{2}\pi + \frac{3\,.\,1\,.\,1}{6\,.\,4\,.\,2} \cdot \frac{1}{2}\pi\right]$$

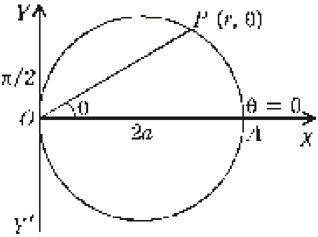
$$= 16\pi a^3 \cdot \frac{\pi}{32} \cdot (3 + 1) = 2\pi^2 a^3.$$

Comprehensive Problems 3

Volumes Of Solids Of Revolution (Polar Equations).

Problem 1: *Find the volume of the solid generated by the revolution of* $r = 2a \cos\theta$ *about the initial line.*

Solution: The given curve $r = 2a \cos \theta$ is a circle passing through the pole. It is symmetrical about the initial line (*i.e.*, *x*-axis). We have $\theta = 0$ at the point A and $\theta = \pi/2$ at the point O where $r = 0$. Thus for the upper half of the circle θ varies from 0 to $\frac{1}{2}\pi$.



∴ The required volume

$$= \frac{2}{3}\int_0^{\pi/2} \pi r^3 \sin\theta \, d\theta$$

[**Note:** We have used the formula given in article 2 (i)]

$$= \frac{2}{3}\pi \int_0^{\pi/2} (2a\cos\theta)^3 \sin\theta \, d\theta \qquad [\because \; r = 2a\cos\theta]$$

$$= \frac{16\pi a^3}{3}\int_0^{\pi/2} \cos^3\theta \sin\theta \, d\theta = -\frac{16\pi a^3}{3}\int_0^{\pi/2} \cos^3\theta \, . \, (-\sin\theta) \, d\theta$$

$$= -\frac{16\pi a^3}{3} \cdot \left[\frac{\cos^4\theta}{4}\right]_0^{\pi/2}$$

$$= -\frac{4}{3}\pi a^3 [0-1] = \frac{4}{3}\pi a^3 .$$

Problem 2: *The arc of the cardioid* $r = a(1+\cos\theta)$, *specified by* $-\pi/2 \le \theta \le \pi/2$, *is rotated about the line* $\theta = 0$, *prove that the volume generated is* $\frac{5}{2}\pi a^3$.

Solution: See figure of Example 7. Here the portion $B'AB$ of the cardioid is rotated about the initial line (*i.e.*, *x*-axis). Obviously the volume generated is the same as the volume generated by the revolution of the portion AB about *x*-axis. For the portion AB, θ varies from 0 to $\frac{1}{2}\pi$.

∴ The required volume

$$= \int_0^{\pi/2} \frac{2}{3}\pi r^3 \sin\theta \, d\theta$$

$$= \frac{2\pi a^3}{3}\int_0^{\pi/2} (1+\cos\theta)^3 \sin\theta \, d\theta$$

$$= -\frac{2\pi a^3}{3}\left[\frac{(1+\cos\theta)^4}{4}\right]_0^{\pi/2}$$

$$= -\frac{1}{6}\pi a^3 [1-16] = \frac{15}{6}\pi a^3 = \frac{5}{2}\pi a^3 .$$

Problem 3: *Find the volume of the solid generated by the revolution of the cardioid* $r = a(1-\cos\theta)$ *about the initial line.* **(Rohilkhand 2010)**

Solution: Proceed exactly as in Problem 2. Required volume $= \frac{8}{3}\pi a^3$.

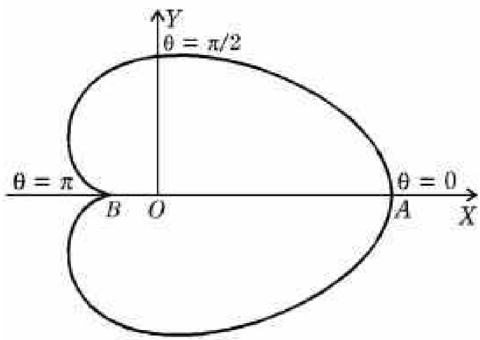
Problem 4: *Show that the volume of the solid formed by the revolution of the curve* $r = a + b\cos\theta \ (a > b)$ *about the initial line is* $\dfrac{4}{3}\pi a (a^2 + b^2)$.

(Meerut 2008)

Solution: The given equation of the curve is

$$r = a + b\cos\theta \quad (a > b). \qquad \text{...(1)}$$

It is symmetrical about the initial line and for the upper half of the curve θ varies from 0 to π.



$\therefore$ The required volume formed by revolving the whole curve about the initial line

$$= \int_0^{\pi} \frac{2}{3}\pi r^3 \sin\theta \, d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi} (a + b\cos\theta)^3 \sin\theta \, d\theta, \qquad \text{[From (1)]}$$

$$= -\frac{2\pi}{3b}\int_0^{\pi} (a + b\cos\theta)^3 (-b\sin\theta)\, d\theta$$

$$= -\frac{2\pi}{3b}\left[\frac{(a + b\cos\theta)^4}{4}\right]_0^{\pi}$$

$$= -\frac{2\pi}{3b}\left[\frac{(a-b)^4}{4} - \frac{(a+b)^4}{4}\right] = \frac{\pi}{6b}\left[(a+b)^4 - (a-b)^4\right]$$

$$= \frac{\pi}{6b}\left[(a+b)^2 + (a-b)^2\right]\left[(a+b)^2 - (a-b)^2\right]$$

$$= \frac{\pi}{6b} 2(a^2 + b^2) \cdot 4ab = \frac{4\pi a}{3}(a^2 + b^2).$$

Note: If $b = a$, then the given curve becomes $r = a(1 + \cos\theta)$ *i.e.*, a cardioid and hence the volume of the solid generated by the revolution of the cardioid $r = a(1 + \cos\theta)$ about the initial line $= \dfrac{4}{3}\pi a (a^2 + a^2) = \dfrac{8}{3}\pi a^3$.

Problem 5: *Find the volume of the solid generated by revolving one loop of the lemniscate* $r^2 = a^2 \cos 2\theta$ *about the line* $\theta = \dfrac{1}{2}\pi$.

(Garhwal 2002; Meerut 06; Kumaun 07, 11)

Solution: The given curve is

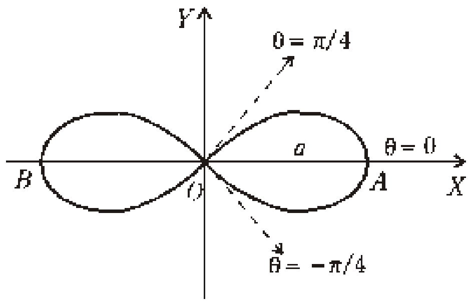
$$r^2 = a^2 \cos 2\theta \qquad \ldots(1)$$

It is symmetrical about the initial line.

We have $r = 0$ when $\cos 2\theta = 0$

i.e. $\quad 2\theta = \pm \frac{1}{2}\pi \quad$ or $\quad \theta = \pm \frac{1}{4}\pi.$

Thus for one loop θ varies from $-\pi/4$ to $\pi/4$. And for the upper half of one loop θ varies from 0 to $\frac{1}{4}\pi$.



Hence the required volume of the solid generated by revolving one loop about the line $\theta = \frac{1}{2}\pi$ (*i.e.*, y-axis)

$$= 2\int_0^{\pi/4} \frac{2}{3}\pi r^3 \cos\theta \, d\theta$$

[**Note.** We have used the formula given in article 4 (ii)]

$$= \frac{4\pi}{3}\int_0^{\pi/4} a^3 (\cos 2\theta)^{3/2} \cos\theta \, d\theta, \qquad \text{[From (1)]}$$

$$= \frac{4\pi a^3}{3}\int_0^{\pi/4} (1 - 2\sin^2\theta)^{3/2} \cos\theta \, d\theta.$$

Now put $\sqrt{2}\sin\theta = \sin\phi$ so that $\sqrt{2}\cos\theta \, d\theta = \cos\phi \, d\phi$. **[Note the substitution]**

Also when $\theta = 0$, $\phi = 0$ and when $\theta = \pi/4$, $\phi = \pi/2$.

Then the required volume

$$= \frac{4\pi a^3}{3}\int_0^{\pi/2} (1 - \sin^2\phi)^{3/2} \cdot \frac{1}{\sqrt{2}} \cos\phi \, d\phi$$

$$= \frac{4\pi a^3}{3\sqrt{2}}\int_0^{\pi/2} \cos^4\phi \, d\phi = \frac{4\pi a^3}{3\sqrt{2}} \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = \frac{\pi^2 a^3}{4\sqrt{2}}.$$

Comprehensive Problems 4

Problem 1: *Find the surface of a sphere of radius a.* **(Agra 2000; Kanpur 06)**

Solution: Suppose the sphere is generated by the revolution of a semi-circle of radius a about its bounding diameter (say x-axis).

Let the equation of the circle be $x^2 + y^2 = a^2$, $\qquad \ldots(1)$

the centre being the origin.

Then as in Example 10, we get $ds/dx = a/y$.

Also for the semi-circle, x varies from $-a$ to a.

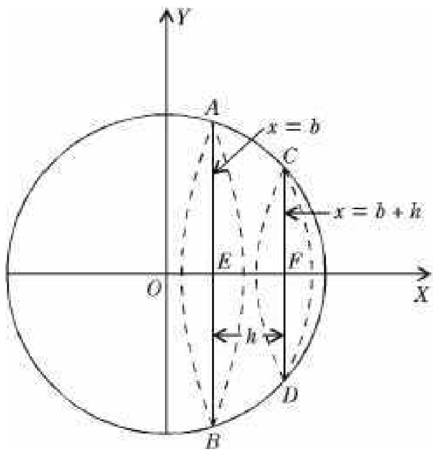
$\therefore$ The required surface

$$= 2\pi \int_{x=-a}^{a} y \, ds = 2\pi \int_{-a}^{a} y \frac{ds}{dx} dx = 2\pi \int_{-a}^{a} y \cdot \frac{a}{y} dx$$

$$= 2\pi \int_{-a}^{a} a\, dx = 2\pi a \left[x \right]_{-a}^{a} = 2\pi a\,(a + a) = 4\pi a^2.$$

Problem 2: *Show that the surface of the spherical zone contained between two parallel planes is $2\pi ah$ where a is the radius of the sphere and h the distance between the planes.*

(Kanpur 2009)



Solution: Let the sphere be generated by the revolution about the x-axis of the circle

$$x^2 + y^2 = a^2. \qquad \ldots(1)$$

Let the two parallel planes bounding the spherical zone be formed by the revolution of the lines $x = b$ and $x = b + h$. Then the required surface is generated by the revolution of the arc AC about x-axis. Proceeding as in Example 10, we get

$$\frac{ds}{dx} = \frac{a}{y} \cdot$$

$\therefore$ The required surface

$$= \int_{b}^{b+h} 2\pi y \frac{ds}{dx}\, dx$$

$$= 2\pi \int_{b}^{b+h} y \cdot \frac{a}{y}\, dx$$

$$= 2\pi a \int_{b}^{b+h} dx = 2\pi a\,[x]_{b}^{b+h} = 2\pi a\,(b + h - h) = 2\pi ab\,.$$

Problem 3: *Find the area of the surface formed by the revolution of the parabola $y^2 = 4ax$ about the x-axis by the arc from the vertex to one end of the latus rectum.*

(Rohilkhand 2011)

Solution: The given parabola is $y^2 = 4ax$. ...(1)

Differentiating (1) w.r.t. x, we get

$$2y\,(dy/dx) = 4a \quad \text{or} \quad dy/dx = 2a/y\,.$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left(1 + \frac{4a^2}{y^2}\right)} = \sqrt{\left(\frac{y^2 + 4a^2}{y^2}\right)}$$

$$= \frac{\sqrt{(4ax + 4a^2)}}{y}, \qquad \text{[From (1)]}$$

$$= \frac{2\sqrt{a}\sqrt{(x + a)}}{y}. \qquad \ldots(2)$$

For the given arc from the vertex $(0, 0)$ to one end of the latus rectum, say the end $(a, 2a)$, x varies from 0 to a.

∴ The required surface

$$= \int_{x=0}^{a} 2\pi\, y \frac{ds}{dx}\, dx = 2\pi \int_0^a y \cdot \frac{2\sqrt{a}\sqrt{(x+a)}}{y}\, dx, \qquad \text{[From (2)]}$$

$$= 4\pi\sqrt{a} \int_0^a (x+a)^{1/2}\, dx = 4\pi\sqrt{a} \cdot \left[\frac{2}{3}(x+a)^{3/2}\right]_0^a$$

$$= \frac{8\pi\sqrt{a}}{3}\left[(2a)^{3/2} - a^{3/2}\right] = \frac{8\pi a^2}{3}\left[2\sqrt{2} - 1\right].$$

Problem 4: *Find the surface generated by the revolution of an arc of the catenary $y = c \cosh (x / c)$ about the x-axis, between the planes $x = a$ and $x = b$.* **(Agra 2003)**

Solution: Proceeding exactly as in Example 11, the required surface

$$= 2\pi \int_{x=a}^{b} y \frac{ds}{dx}\, dx = \pi c\left[x + \frac{c}{2}\sinh\left(\frac{2x}{c}\right)\right]_a^b$$

$$= \pi c\left[(b-a) + \frac{c}{2}\sinh\frac{2b}{c} - \frac{c}{2}\sinh\frac{2a}{c}\right].$$

Problem 5: *For a catenary $y = a \cosh (x/a)$, prove that $aS = 2V = \pi a\,(ax + sy)$, where s is the length of the arc from the vertex , S and V are respectively the area of the curved surface and volume of the solid generated by the revolution of the arc about x-axis.*

Solution: The given equation of catenary is $y = a \cosh (x/a)$. ...(1)

The vertex of the catenary (1) is the point $(0, a)$.

Differentiating (1) w.r.t. x, we get $dy/dx = \sinh (x/a)$.

$$\therefore \qquad \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left\{1 + \sinh^2\frac{x}{a}\right\}} = \cosh\frac{x}{a}$$

or $\qquad ds = \cosh (x/a)\, dx$.

If s is the length of the arc from the vertex $(x = 0)$ to any point (x, y) on the catenary, then we have

$$s = \int_0^x \cosh\left(\frac{x}{a}\right) dx = \left[a \sinh\frac{x}{a}\right]_0^a = a \sinh\frac{x}{a}. \qquad \text{...(2)}$$

Now S = area of the curved surface of the solid generated by the revolution of the arc about x-axis

$$= \int_0^x 2\pi\, y \frac{ds}{dx}\, dx = \pi\, a\left[x + \frac{a}{2}\sin h\frac{2x}{a}\right] \qquad \text{...(3)}$$

[As proved in Ex. 11 after article 6. Prove it here.]

Also V = the volume generated by the revolution of the arc about x-axis

$$= \int_0^x \pi\, y^2\, dx = \pi\int_0^x a^2\cosh^2\frac{x}{a}\, dx = \frac{\pi a^2}{2}\int_0^x 2\cosh^2\frac{x}{a}\, dx$$

$$= \frac{\pi a^2}{2}\int_0^x \left[1 + \cosh\frac{2x}{a}\right] dx = \frac{\pi a^2}{2}\left[x + \frac{a}{2}\sinh\frac{2x}{a}\right]_0^x$$

$$= \frac{\pi a^2}{2}\left[x + \frac{a}{2}\sinh\frac{2x}{a}\right]. \qquad ...(4)$$

From (3), $aS = \pi a^2 \, [x + \frac{1}{2} a \sinh (2x/a)]$. ...(5)

From (4), $2V = \pi a^2 \, [x + \frac{1}{2} a \sinh (2x/a)] = aS$. ...(6)

From (5) and (6), we have $aS = 2V$.

Also from (2), $\pi a \, (ax + sy) = \pi a \, [ax + a^2 \sinh (x/a) \cosh (x/a)]$, $[\because \; y = a \cosh (x/a)]$

$$= \pi a\left[ax + \frac{a^2}{2}\cdot 2\sinh\frac{x}{a}\cosh\frac{x}{a}\right]$$

$$= \pi a^2 \, [\, x + (a/2) \sinh (2x/a)] = aS. \qquad \text{[From (5)]}$$

Hence $aS = 2V = \pi a \, (ax + sy)$.

Problem 6: *Find the surface of the solid generated by the revolution of the ellipse* $x^2 + 4y^2 = 16$ *about its major axis.* **(Meerut 2005, 06)**

Solution: The given ellipse is $x^2 + 4y^2 = 16$. ...(1)

The equation (1) of the ellipse can be written as $\dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$.

Comparing it with the standard form of the equation of the ellipse $x^2/a^2 + y^2/b^2 = 1$, we see that $a = 4$, $b = 2$ and the major axis is along the x-axis. So we have to revolve the curve (1) about x-axis.

Differentiating (1) w.r.t. x, we get $2x + 8y\dfrac{dy}{dx} = 0$ or $\dfrac{dy}{dx} = -\dfrac{x}{4y}$.

$$\therefore \quad \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left\{1 + \frac{x^2}{16y^2}\right\}} = \sqrt{\left(\frac{16y^2 + x^2}{16y^2}\right)}$$

$$= \sqrt{\{(64 - 4x^2) + x^2\}}/4y = \sqrt{(64 - 3x^2)}/4y. \qquad ...(2)$$

Now the ellipse (1) is symmetrical about both axes and for the arc of the ellipse lying in the first quadrant x varies from 0 to 4.

$\therefore$ The required surface

$= 2 \times$ surface generated by the revolution of the arc in the first quadrant

$$= 2\int_{x=0}^{4} 2\pi \, y \frac{ds}{dx}\, dx = 4\pi \int_0^4 y \cdot \frac{\sqrt{(64 - 3x^2)}}{4y}\, dx, \qquad \text{[From (2)]}$$

$$= \pi \int_0^4 \sqrt{(64 - 3x^2)}\, dx = \pi\sqrt{3}\int_0^4 \sqrt{\left\{\left(\frac{8}{\sqrt{3}}\right)^2 - x^2\right\}}\, dx$$

$$= \pi\sqrt{3}\left[\frac{x}{2}\sqrt{\left\{\left(\frac{8}{\sqrt{3}}\right)^2 - x^2\right\}} + \frac{1}{2}\cdot\frac{64}{3}\sin^{-1}\left(\frac{x\sqrt{3}}{8}\right)\right]_0^4$$

$$= \pi\sqrt{3}\left[2\sqrt{\left(\frac{64}{3}-16\right)}+\frac{32}{3}\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right] = \pi\sqrt{3}\left[\frac{8}{\sqrt{3}}+\frac{32}{3}\cdot\frac{\pi}{3}\right],$$

$$\left[\because\ \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}\right]$$

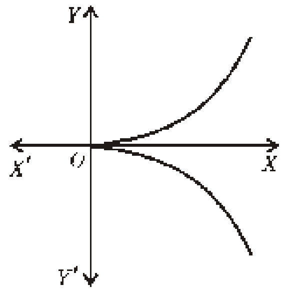
$$= 8\pi\left[1+\frac{4\pi}{3\sqrt{3}}\right].$$

Problems on revolution about *y*-axis.

Problem 7: *Find the surface of the solid formed by the revolution, about the axis of y, of the part of the curve $ay^2 = x^3$ from $x = 0$ to $x = 4a$ which is above the x-axis.*

Solution: The given curve is

$$ay^2 = x^3. \qquad \ldots(1)$$



Differentiating (1) w.r.t. x, we get

$$2ay\frac{dy}{dx} = 3x^2 \quad \text{or} \quad \frac{dy}{dx} = \frac{3x^2}{2ay}.$$

$$\therefore \qquad \frac{ds}{dx} = \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left\{1+\frac{9x^4}{4a^2y^2}\right\}}$$

$$= \sqrt{\left\{1+\frac{9x^4}{4a\,.\,x^3}\right\}}, \qquad \text{[From (1)]}$$

$$= \sqrt{\left(1+\frac{9x}{4a}\right)} = \frac{1}{2\sqrt{a}}\sqrt{(4a+9x)}.$$

$\therefore$ The required surface

$$= \int_{x=0}^{4a} 2\pi x\, ds$$

$$= \int_0^{4a} 2\pi x \frac{ds}{dx}\, dx \qquad \textbf{(Note)}$$

$$= \int_0^{4a} 2\pi x\cdot\frac{1}{2\sqrt{a}}\sqrt{(4a+9x)}\, dx$$

$$= \frac{\pi}{\sqrt{a}}\int_0^{a} x\sqrt{(4a+9x)}\, dx.$$

Put $4a + 9x = t^2$ so that $9\,dx = 2t\,dt$. When $x = 0, t = \sqrt{(4a)}$ and when $x = 4a, t = \sqrt{(40a)}$.

$\therefore$ The required surface

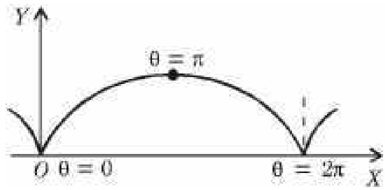
$$= \frac{\pi}{\sqrt{a}}\int_{\sqrt{(4a)}}^{\sqrt{(40a)}} \frac{t^2-4a}{9}\cdot t\cdot\frac{2t\,dt}{9} = \frac{2\pi}{81\sqrt{a}}\int_{\sqrt{(4a)}}^{\sqrt{(40a)}}(t^4-4at^2)\,dt$$

$$= \frac{2\pi}{81\sqrt{a}}\left[\frac{1}{5}t^5-\frac{4}{3}at^3\right]_{\sqrt{(4a)}}^{\sqrt{(40a)}} = \frac{128}{1215}\pi a^2\left[125\sqrt{(10)}+1\right].$$

Comprehensive Problems 5

Problem 1: *Find the surface area of the solid generated by revolving the cycloid* $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ *about the x-axis.*

Solution: The given parametric equations of the cycloid are



$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta). \quad ...(1)$$

$$\therefore \quad \frac{dx}{d\theta} = a(1 - \cos\theta) \quad \text{and} \quad \frac{dy}{d\theta} = a\sin\theta.$$

Hence, $$\frac{ds}{d\theta} = \sqrt{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]}$$

$$= \sqrt{[a^2(1 - \cos\theta)^2 + a^2\sin^2\theta]} = a\sqrt{[1 - 2\cos\theta + \cos^2\theta + \sin^2\theta]}$$

$$= a\sqrt{[2(1 - \cos\theta)]} = a\sqrt{[2 . 2\sin^2(\theta/2)]} = 2a\sin(\theta/2). \quad ...(2)$$

We have $y = 0$ when $1 - \cos\theta = 0$ *i.e.*, $\cos\theta = 1$ giving $\theta = 0$ and 2π. When $\theta = 0$, $x = 0$ and when $\theta = 2\pi$, $x = 2a\pi$. Also y is maximum when $\cos\theta = -1$ *i.e.*, $\theta = \pi$ and then $y = 2a$ and $x = a\pi$. Thus for one arch of the given curve θ varies from 0 to 2π and this arch is symmetrical about the line $x = a\pi$ which meets the curve at the point $\theta = \pi$.

$$\therefore \quad \text{The required surface} = 2\int_0^{\pi} 2\pi\, y\, \frac{ds}{d\theta}\, d\theta$$

$$= 4\pi\int_0^{\pi} a(1 - \cos\theta) . 2a\sin(\theta/2)\, d\theta, \text{ from (1) and (2)}$$

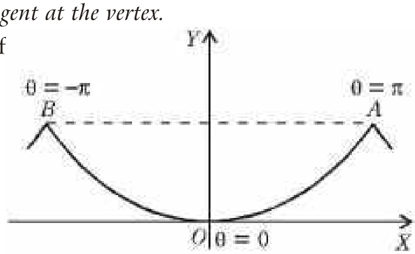
$$= 8\pi a^2\int_0^{\pi} 2\sin^2\frac{\theta}{2}\cdot\sin\frac{\theta}{2}\, d\theta = 16\pi a^2\int_0^{\pi}\sin^3\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2\int_0^{\pi/2}\sin^3 t . 2\, dt, \text{ putting } \frac{\theta}{2} = t \text{ so that } d\theta = 2\, dt$$

$$= 32\pi a^2\cdot\frac{2}{3 . 1} = \frac{64\pi a^2}{3}.$$

Problem 2: *Find the area of the surface generated by revolving an arch of the cycloid* $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ *about the tangent at the vertex.*

Solution: The given parametric equations of the cycloid are



$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta). \quad ...(1)$$

$$\therefore \quad \frac{dx}{d\theta} = a(1 + \cos\theta) \quad \text{and} \quad \frac{dy}{d\theta} = a\sin\theta.$$

Hence $$\frac{ds}{d\theta} = \sqrt{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]}$$

$$= \sqrt{[a^2(1 + \cos\theta)^2 + a^2\sin^2\theta]}$$

$$= a\sqrt{[1 + 2\cos\theta + \cos^2\theta + \sin^2\theta]} = a\,.\,\sqrt{[2(1+\cos\theta)]}$$

$$= a\,.\,\sqrt{\{2\,.\,2\cos^2(\theta/2)\}} = 2a\cos(\theta/2)\,. \qquad \ldots(2)$$

Also for one arch of the given curve, θ varies from $-\pi$ to π and this arch is symmetrical about the y-axis which meets the curve at the point $\theta = 0$.

$\therefore$ The required surface $= 2\int_0^{\pi} 2\pi\, y\, \dfrac{ds}{d\theta}\, d\theta$

$$= 4\pi \int_0^{\pi} a(1-\cos\theta)\,.\,2a\cos(\theta/2)\,d\theta, \text{ from (1) and (2)}$$

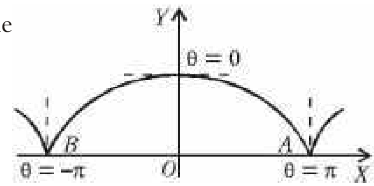
$$= 8\pi a^2 \int_0^{\pi} 2\sin^2\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\,d\theta = 16\pi a^2 \int_0^{\pi} \sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta$$

$$= 16\pi a^2 \int_0^{\pi/2} \sin^2 t\cos t\,.\,2\,dt, \text{ putting } \frac{\theta}{2} = t \text{ so that } d\theta = 2\,dt$$

$$= 32\pi a^2 \int_0^{\pi/2} \sin^2 t\cos t\,dt = 32\pi a^2\cdot\frac{1}{3\,.\,1} = \frac{32\pi a^2}{3}\,.$$

Problem 3: *The portion between the consecutive cusps of the cycloid* $x = a(\theta + \sin\theta)$, $y = a(1+\cos\theta)$ *is revolved about the x-axis. Prove that the area of the surface so formed is to the area of the cycloid as* 64 : 9.

Solution: The given parametric equations of the cycloid are



$$x = a(\theta + \sin\theta),\ y = a(1+\cos\theta)\,. \qquad \ldots(1)$$

$$\therefore \qquad \frac{dx}{d\theta} = a(1+\cos\theta),\ \frac{dy}{d\theta} = -a\sin\theta.$$

Hence $\dfrac{ds}{d\theta} = \sqrt{\left\{\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2\right\}} = \sqrt{[a^2(1+\cos\theta)^2 + a^2\sin^2\theta]}$

$$= a\sqrt{[1 + 2\cos\theta + \cos^2\theta + \sin^2\theta]}$$

$$= a\sqrt{[2(1+\cos\theta)]} = a\sqrt{[2\,.\,2\cos^2(\theta/2)]} = 2a\cos(\theta/2)\,. \qquad \ldots(2)$$

For one arch of the given curve (*i.e.*, for the portion between two successive cusps) θ varies from $-\pi$ to π. Also this arch is symmetrical about the y-axis which meets the arch at the point where $\theta = 0$. The base of the given cycloid is the axis of x.

$\therefore$ The surface S generated by the revolution of the cycloid about the x-axis

$$= 2\int_0^{\pi} 2\pi\, y\,\frac{ds}{d\theta}\,d\theta = 4\pi\int_0^{\pi} a(1+\cos\theta)\,.\,2a\cos(\theta/2)\,d\theta,$$

[From (1) and (2)]

$$= 8\pi a^2 \int_0^{\pi} 2\cos^2\frac{\theta}{2}\cos\frac{\theta}{2}\,d\theta = 16\pi a^2\int_0^{\pi}\cos^3\frac{\theta}{2}\,d\theta$$

$$= 16\pi a^2 \int_0^{\pi/2} (\cos^3 t)\,.\,2\,dt, \text{ putting } \frac{\theta}{2} = t \text{ so that } d\theta = 2\,dt$$

$$= 32\pi a^2\cdot\frac{2}{3\,.\,1} = \frac{64\pi a^2}{3}\,. \qquad \ldots(3)$$

Also the area A of the given cycloid

$$= 2\int_0^{\pi} y\,\frac{dx}{d\theta}\,d\theta = 2\int_0^{\pi} a\,(1+\cos\theta)\,.\,a\,(1+\cos\theta)\,d\theta$$

$$= 2a^2\int_0^{\pi}(1+\cos\theta)^2\,d\theta = 2a^2\int_0^{\pi}\left(2\cos^2\frac{\theta}{2}\right)^2 d\theta$$

$$= 8a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\,d\theta = 8a^2\int_0^{\pi/2}(\cos^4 t)\,.\,2\,dt\text{, putting } t=\frac{\theta}{2}$$

$$= 16a^2\cdot\frac{3\,.\,1}{4\,.\,2}\cdot\frac{\pi}{2} = 3\pi a^2. \qquad \ldots(4)$$

From (3) and (4), we get the required ratio $= \dfrac{S}{A} = \dfrac{\frac{64}{3}\pi a^2}{3\pi a^2} = \dfrac{64}{9}\cdot$

Problem 4: *Prove that the surface area of the solid generated by the revolution, about the x-axis of the loop of the curve* $x = t^2,\ y = t - \frac{1}{3}t^3$ *is* 3π.

Solution: The given equations of the curve are $x = t^2,\ y = t - \frac{1}{3}t^3$. ...(1)

$\therefore \qquad dx/dt = 2t \quad \text{and} \quad dy/dt = 1 - t^2.$

Hence
$$\frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} = \sqrt{[(2t)^2 + (1-t^2)^2]}$$

$$= \sqrt{[4t^2 + 1 - 2t^2 + t^4]} = \sqrt{(1+t^2)^2} = (1+t^2). \qquad \ldots(2)$$

Putting $y = 0$ in (1), we get $t - \frac{1}{3}t^3 = 0$ which gives $t = 0$ or $t = \pm\sqrt{3}$. For the upper half of the loop y is positive and so for the upper half of the loop t varies from 0 to $\sqrt{3}$.

$\therefore$ The required surface $= \displaystyle\int_0^{\sqrt{3}} 2\pi\,y\,\frac{ds}{dt}\,dt$

$$= 2\pi\int_0^{\sqrt{3}}\left(t - \frac{1}{3}t^3\right)(1+t^2)\,dt = 2\pi\int_0^{\sqrt{3}}\left(t + \frac{2}{3}t^3 - \frac{1}{3}t^5\right)dt$$

$$= 2\pi\left[\frac{t^2}{2} + \frac{2}{3}\cdot\frac{t^4}{4} - \frac{1}{3}\,\frac{t^6}{6}\right]_0^{\sqrt{3}} = 2\pi\left[\frac{t^2}{2} + \frac{t^4}{6} - \frac{t^6}{18}\right]_0^{\sqrt{3}}$$

$$= 2\pi\left[\frac{3}{2} + \frac{9}{6} - \frac{27}{18}\right] = 3\pi.$$

Problem 5: *Prove that the surface of the oblate spheroid formed by the revolution of the ellipse of the semi-major axis a and eccentricity e is*

$$2\pi a^2\left[1 + \frac{1-e^2}{2e}\,log\left(\frac{1+e}{1-e}\right)\right].$$

Solution: [**Note : Oblate spheroid** *is generated by the revolution of the ellipse about its minor axis*.]

Let the parametric equations of the ellipse be

$$x = a\cos t,\ y = b\sin t, \text{ where } b^2 = a^2(1-e^2). \qquad ...(1)$$

$\therefore$ $$dx/dt = -a\sin t \ \text{ and } \ dy/dt = b\cos t.$$

$\therefore$ $$\frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} = \sqrt{\{a^2\sin^2 t + b^2\cos^2 t\}}$$

$$= \sqrt{\{a^2\sin^2 t + a^2(1-e^2)\cos^2 t\}} = a\sqrt{(1-e^2\cos^2 t)} \qquad ...(2)$$

The ellipse is symmetrical about both the axes and for the arc of the ellipse lying in the first quadrant t varies from 0 to $\pi/2$.

We have to revolve the ellipse about its minor axis which is the y-axis.

$\therefore$ The required surface $= 2\int_0^{\pi/2} 2\pi x \frac{ds}{dt}\,dt$ **(Note)**

$$= 4\pi\int_0^{\pi/2} a\cos t \,.\, a\sqrt{(1-e^2\cos^2 t)}\,dt$$

$$= 4\pi a^2\int_0^{\pi/2} \sqrt{(1-e^2+e^2\sin^2 t)}\cos t\,dt$$

$$= \frac{4\pi a^2}{e}\int_0^{e} \sqrt{[(1-e^2)+z^2]}\,dz,$$

putting $e\sin t = z$ so that $e\cos t\,dt = dz$

$$= \frac{4\pi a^2}{e}\left[\frac{z}{2}\sqrt{\{(1-e^2)+z^2\}} + \frac{1}{2}(1-e^2)\log\{z+\sqrt{(1-e^2+z^2)}\}\right]_0^e$$

$$= \frac{2\pi a^2}{e}[e + (1-e^2)\log(e+1) - (1-e^2)\log\sqrt{(1-e^2)}]$$

$$= \frac{2\pi a^2}{e}\left[e + (1-e^2)\log\frac{1+e}{\sqrt{(1-e^2)}}\right]$$

$$= 2\pi a^2\left[1 + \frac{1-e^2}{e}\log\sqrt{\left(\frac{1+e}{1-e}\right)}\right] = 2\pi a^2\left[1 + \frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right].$$

Comprehensive Problems 6

Problems on surfaces of revolution (Polar equations):

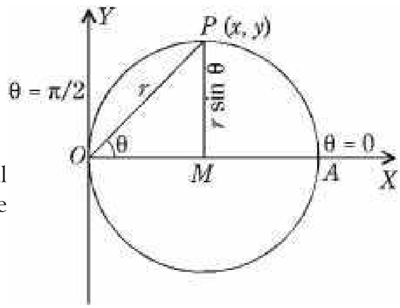
Problem 1: *Find the area of the surface of revolution formed by revolving the curve $r = 2a\cos\theta$ about the initial line.*

Solution: The given curve is

$$r = 2a\cos\theta, \qquad ...(1)$$

which is clearly a circle of radius a passing through the pole and having diameter through the pole as initial line. At O, $r = 0$ and so (1) gives $\theta = \pi/2$ at O.

Differentiating (1) w.r.t θ, we have $dr/d\theta = -2a\sin\theta$.



$$\therefore \quad \frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}$$

$$= \sqrt{\{4a^2 \cos^2\theta + 4a^2 \sin^2\theta\}}$$

$$= 2a\sqrt{(\cos^2\theta + \sin^2\theta)} = 2a.$$

The given curve is revolved about the initial line (*i.e.*, the x-axis) and for the upper half of the curve, θ varies from 0 to $\pi/2$.

$\therefore$ The required surface $= \int_0^{\pi/2} 2\pi\, y \frac{ds}{d\theta}\, d\theta$,

where $y = r\sin\theta$

$$= 2\pi \int_0^{\pi/2} r\sin\theta \,.\, 2a\, d\theta = 4a\pi \int_0^{\pi/2} 2a\cos\theta\sin\theta\, d\theta, \qquad \text{[From (1)]}$$

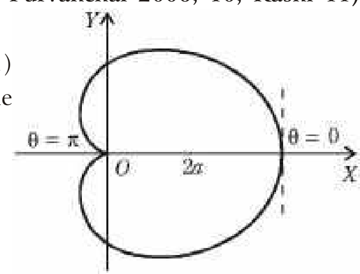
$$= 8\pi a^2 \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = 8\pi a^2 \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}$$

$$= 8\pi a^2 \left(\frac{1}{2} - 0\right) = 4\pi a^2.$$

Problem 2: *Find the surface of the solid formed by the revolution of the cardioid $r = a(1 + \cos\theta)$ about the initial line.* **(Agra 2003; Purvanchal 2006, 10; Kashi 11)**

Solution: The given curve is

$$r = a(1 + \cos\theta). \qquad \text{...(1)}$$



It is symmetrical about the initial line and for the upper half of the curve, θ varies from 0 to π.

Differentiating (1) w.r.t. θ, we get

$$dr/d\theta = a(-\sin\theta) = -a\sin\theta.$$

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}$$

$$= \sqrt{[a^2(1 + \cos\theta)^2 + a^2\sin^2\theta]}$$

$$= a\sqrt{[2(1 + \cos\theta)]} = 2a\cos\frac{1}{2}\theta. \qquad \text{...(2)}$$

$\therefore$ The required surface

$$= \int_0^{\pi} 2\pi\, y \frac{ds}{d\theta}\, d\theta, \text{ where } y = r\sin\theta$$

$$= \int_0^{\pi} 2\pi \,.\, r\sin\theta \,.\, 2a\cos\frac{1}{2}\theta\, d\theta$$

$$= 2\pi \int_0^{\pi} a(1 + \cos\theta)\sin\theta\cos\frac{1}{2}\theta\, d\theta \qquad \text{[From (1)]}$$

$$= 4\pi a^2 \int_0^{\pi} 2\cos^2\frac{\theta}{2} \cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \cdot \cos\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \cos^4 \frac{\theta}{2} \sin \frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2 \int_0^{\pi/2} (\cos^4 t \sin t)\,.\,2\,dt, \text{ putting } \frac{\theta}{2} = t \text{ so that } d\theta = 2\,dt$$

$$= 32\pi a^2 \int_0^{\pi/2} \cos^4 t \sin t\, dt$$

$$= 32\pi a^2 \cdot \frac{3\,.\,1\,.\,1}{5\,.\,3\,.\,1} = \frac{32\pi a^2}{5}.$$

Problem 3: *The arc of the cardioid $r = a(1 + \cos\theta)$ included between $-\frac{1}{2}\pi \leq \theta \leq \frac{1}{2}\pi$ is rotated about the line $\theta = \frac{1}{2}\pi$. Find the area of the surface generated.* **(Purvanchal 2010)**

Solution: The given cardioid is $r = a(1 + \cos\theta)$, ...(1)

which is symmetrical about the initial line.

As proved in problem 2, $ds/d\theta = 2a \cos(\theta/2)$. (Prove it here)

The curve is rotated about the line $\theta = \pi/2$ *i.e.*, the y-axis and for the upper half of the cardioid lying in the first quadrant, θ varies from 0 to $\frac{1}{2}\pi$.

$\therefore$ The required surface $= \int_0^{\pi/2} 2\pi x \frac{ds}{d\theta}\, d\theta$, **(Note)**

$$= 4\pi \int_0^{\pi/2} r\cos\theta\,.\,2a\cos\frac{\theta}{2}\, d\theta, \quad [\because x = r\cos\theta]$$

$$= 8\pi a \int_0^{\pi/2} a(1 + \cos\theta)\,.\,\cos\theta\,.\,\cos\frac{\theta}{2}\, d\theta, \quad \text{[From (1)]}$$

$$= 8\pi a^2 \int_0^{\pi/2} 2\cos^2\frac{\theta}{2} \cdot \left(1 - 2\sin^2\frac{\theta}{2}\right) \cos\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2 \int_0^{\pi/2} \left(1 - \sin^2\frac{\theta}{2}\right)\left(1 - 2\sin^2\frac{\theta}{2}\right) \cos\frac{\theta}{2}\, d\theta \quad \textbf{(Note)}$$

$$= 16\pi a^2 \int_0^{\pi/2} \left(1 - 3\sin^2\frac{\theta}{2} + 2\sin^4\frac{\theta}{2}\right) \cos\frac{\theta}{2}\, d\theta.$$

Put $\sin(\theta/2) = t$ so that $\frac{1}{2}\cos\left(\frac{\theta}{2}\right) d\theta = dt$.

Also when $\theta = 0$, $t = 0$ and when $\theta = \pi/2$, $t = 1/\sqrt{2}$.

$\therefore$ The required surface $= 16\pi a^2 \int_0^{1/\sqrt{2}} (1 - 3t^2 + 2t^4)\,.\,2\,dt$

$$= 32\pi a^2 \left[t - t^3 + 2 \cdot \frac{t^5}{5}\right]_0^{1/\sqrt{2}} = 32\pi a^2 \left[\frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}} + \frac{1}{10\sqrt{2}}\right]$$

$$= \frac{48\sqrt{2}}{5}\pi a^2.$$

Problems on surfaces of revolution formed by rotation about any axis:

Problem 4: *A quadrant of a circle of radius a revolves about its chord. Show that the surface of the spindle generated is* $2\pi a^2 \sqrt{2}\,(1 - \frac{1}{4}\pi)$.

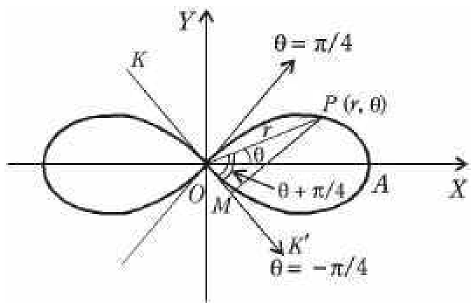
Solution: Let us take the quadrant of the circle in such a way that it is placed symmetrically about the x-axis. The quadrant of the circle subtends an angle $\pi/2$ at the centre.

Now draw the figure as in Example 17 by taking $2\alpha = \pi/2$ and then proceed exactly in the same way as in Example 17 by taking $\alpha = \pi/4$. Thus the required result is obtained by putting $\alpha = \pi/4$ in the result of Example 17.

Problem 5: *The lemniscate* $r^2 = a^2 \cos 2\theta$ *revolves about a tangent at the pole. Show that the surface of the solid generated is* $4\pi a^2$. **(Meerut 2005B; Kumaun 12)**

Solution: The given curve is

$$r^2 = a^2 \cos 2\theta. \quad \ldots(1)$$



Proceeding as in Example 16, we get

$$ds/d\theta = a^2/r.$$

Putting $r = 0$ in (1), we get $\cos 2\theta = 0$ giving $2\theta = \pm \frac{1}{2}\pi$ *i.e.*, $\theta = \pm \frac{1}{4}\pi$. Therefore one loop of the curve lies between $\theta = -\frac{1}{4}\pi$ and $\theta = \frac{1}{4}\pi$. The curve consists of two loops and both the lines $\theta = -\frac{1}{4}\pi$ and $\theta = \frac{1}{4}\pi$ are tangents at the pole. Let the curve be revolved about the line KOK' which is a tangent at the pole.

Take any point $P(r, \theta)$ on the curve and draw PM perpendicular to the axis of rotation KOK'. Then $\angle POM = \frac{1}{4}\pi + \theta$ and $PM = OP \sin(\frac{1}{4}\pi + \theta) = r \sin(\frac{1}{4}\pi + \theta)$.

Also for one loop θ varies from $-\frac{1}{4}\pi$ to $\frac{1}{4}\pi$.

$\therefore$ The required surface = 2 × surface generated by one loop

$$= 2 \times \int_{-\pi/4}^{\pi/4} 2\pi (PM) \frac{ds}{d\theta}\, d\theta = 4\pi \int_{-\pi/4}^{\pi/4} r \sin(\tfrac{1}{4}\pi + \theta) \cdot \frac{a^2}{r}\, d\theta,$$

$$\left[\because \frac{ds}{d\theta} = \frac{a^2}{r}, PM = r \sin(\tfrac{1}{4}\pi + \theta)\right]$$

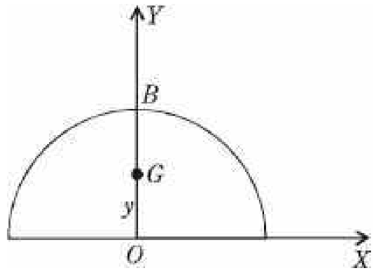
$$= 4\pi a^2 \int_{-\pi/4}^{\pi/4} \sin(\tfrac{1}{4}\pi + \theta)\, d\theta = 4\pi a^2 \left[-\cos(\tfrac{1}{4}\pi + \theta)\right]_{-\pi/4}^{\pi/4}$$

$$= 4\pi a^2 [0 + 1] = 4\pi a^2.$$

Comprehensive Problems 7

Problem 1: *Find the position of the centroid of a semi-circular area.*

Solution: Clearly the C.G. of the semi-circular area will lie somewhere on the radius which is perpendicular to the bounding diameter. Let the distance of the centroid from the centre O be $\bar{y}$. Also a sphere will be generated by the rotation of the semi-circular area about the bounding diameter.



$\therefore$ By Pappus theorem, volume of the solid of revolution

= area of semi-circle

× Circumference of the circle generated by the centroid of this area.

Hence $\frac{4}{3}\pi a^3 = \frac{1}{2}\pi a^2 . 2\pi\bar{y}$, where a is the radius of the semi-circle

or $$\bar{y} = \frac{1}{2\pi} \cdot \frac{\frac{4}{3}\pi a^3}{\frac{1}{2}\pi a^2} = \frac{4a}{3\pi}.$$

Problem 2: *Find the volume generated by the revolution of an ellipse having semi-axes a and b about a tangent at the vertex.*

Solution: Area of the given ellipse = πab. ...(1)

The C.G. of the ellipse will describe a circle of radius a when revolved about the tangent at A or a circle of radius b when revolved about the tangent at B. [See fig. of Example 19]

Hence the length of the arc described by the C.G. will be

$$2\pi . 2a \text{ i.e., } 4\pi a \quad \textbf{or} \quad 2\pi . 2b \text{ i.e., } 4\pi b.$$

$\therefore$ By Pappus theorem, the required volume

(*i*) when revolved about the tangent at the vertex $A = \pi ab . 2\pi a = 2\pi^2 a^2 b$,

(*ii*) when revolved about the tangent at the vertex $B = \pi ab . 2\pi b = 2\pi^2 ab^2$.

Problem 3: *Find by using Pappus theorem the volume of the ring generated by the revolution of an ellipse of eccentricity* $1/\sqrt{2}$ *about a straight line parallel to the minor axis and situated at a distance from the centre equal to three times the major axis.*

Solution: Let a be the semi-major axis of the ellipse. Then its semi-minor axis

$$b = a\sqrt{(1-e^2)} = a\sqrt{(1-\frac{1}{2})} = a/\sqrt{2}. \qquad [\because e = 1/\sqrt{2}]$$

$\therefore$ Area of the ellipse = $\pi ab = \pi a . (a/\sqrt{2}) = \pi a^2/\sqrt{2}$.

Distance of the C.G. of the ellipse from the axis of revolution is = $3 . 2a = 6a$, (given).

As the ellipse revolves about the given line its C.G. will describe a circle of radius $6a$ whose perimeter will be $= 2\pi \cdot 6a = 12\pi a$.

Now by Pappus theorem, the required volume

$$= \text{area of the ellipse} \times \text{length of the arc described by its C.G.}$$
$$= (\pi a^2/\sqrt{2}) \cdot 12\pi a = 12\pi^2 a^3/\sqrt{2}$$
$$= 6\sqrt{2}\pi^2 a^3, \text{ where } a \text{ is the semi-major axis.}$$

Problem 4: *Find the volume of the ring generated by the revolution of the cardioid $r = a(1 + \cos\theta)$ about the line $r\cos\theta + a = 0$, given that the centroid of the cardioid is at a distance $5a/6$ from the origin.*

Solution: The given curve is

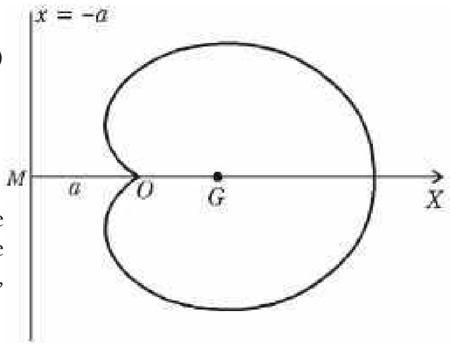
$$r = a(1 + \cos\theta). \qquad \ldots(1)$$

And the given line of rotation is

$$r\cos\theta + a = 0$$

or $\quad x + a = 0, \quad [\because x = r\cos\theta]$

or $\quad x = -a$.



By symmetry the centre of gravity G of the cardioid lies on the initial line OX. If G be the centroid of the area of the cardioid, then $OG = 5a/6$ (given).

Also $\quad GM$ = the length of the perpendicular from G on the line of rotation

$$= GO + OM = (5a/6) + a = 11a/6.$$

Also the area A of the cardioid

$$= 2 \times \int_0^\pi \frac{1}{2} r^2 \, d\theta = \int_0^\pi a^2 (1 + \cos\theta)^2 \, d\theta, \qquad \text{[From (1)]}$$

$$= a^2 \int_0^\pi \left(2\cos^2 \frac{1}{2}\theta\right)^2 d\theta$$

$$= 8a^2 \int_0^{\pi/2} \cos^4 \phi \, d\phi, \text{ putting } \theta/2 = \phi \text{ so that } d\theta = 2\, d\phi$$

$$= 8a^2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{1}{2}\pi = \frac{3}{2}\pi a^2.$$

$\therefore$ By Pappus theorem, the required volume

$$= (2\pi \cdot GM) \times A = 2\pi(11a/6) \times \frac{3}{2}\pi a^2 = \frac{11}{2}\pi^2 a^3.$$

Problem 5: *A semi-circular bend of lead has a mean radius of 8 inches; the initial diameter of the pipe is 4 inches and the thickness of the lead is $\frac{1}{2}$ inch. Applying the theorem of Pappus and Guldin find the volume of the lead and its weight, given that 1 cubic inch of lead weighs 0.4 lb.*

Solution: Internal diameter of pipe = 4 inches.

$$\text{Thickness of metal} = \frac{1}{2} \text{ inch}$$

$\therefore$ External diameter of the pipe $= 4 + 1 = 5$ inches.

$\therefore$ Area of lead $= \frac{1}{4}\pi(5^2 - 4^2) = \frac{9}{4}\pi.$

The centroid of this area is at a distance of 8 inches from the axis of rotation. Therefore the length of path traced out by its centroid in describing a semi-circle $= 8\pi$ inches.

$\therefore$ Volume of the lead $= 8\pi \times \frac{9}{4}\pi = 18\pi^2$ cu. inch.

$\therefore$ Weight of the pipe = volume × density $= 18\pi^2 \times 0.4$ lb. $= 71.1$ lb.

Problem 6: *State the theorems of Pappus and Guldin.* **(Meerut 2008)**

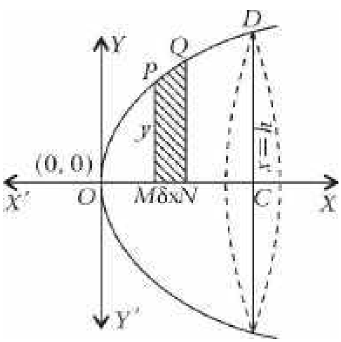
Solution: See article 10.

Hints to Objective Type Questions

Multiple Choice Questions

1. See article 4.

2. The given parabola is $y^2 = 4ax$.



Take an elementary strip $PMNQ$, where P is the point (x, y) and Q is the point $(x + \delta x, y + \delta y)$. Then

$$PM = y$$

and $$MN = ON - OM = (x + \delta x) - x = \delta x.$$

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the x-axis

$$= \pi . PM^2 . MN = \pi y^2 \delta x.$$

The paraboloid is formed by the revolution of the area ODC about x-axis.

Also for the area ODC, x varies from $x = 0$ to $x = h$.

$\therefore$ The required volume

$$= \int_0^h \pi y^2 \, dx = \int_0^h \pi (4ax) \, dx = 4\pi a \left[\frac{x^2}{2}\right]_0^h = 2\pi a h^2.$$

3. See article 6, part (b).

4. See Example 16.

5. See Example 7.

Fill in the Blank(s)

1. See article 2, part (a).

2. See article 4.

3. See article 8.

4. See article 7.

True or False

1. See article 4.

2. See article 6, part (a).
Also see remark after article 6.

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