Krishna's

TEXT BOOK on

INTEGRAL CALCULUS

(For B.A. and B.Sc. IInd Semester students of Kumaun University)

Kumaun University Semester Syllabus *w.e.f.* 2016-17

By

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Dedicated

to

Lord Krishna

Authors & Publishers

Preface

This book on **Integral Calculus** has been specially written according to the latest semester **Syllabus** to meet the requirements of **B.A. and B.Sc. Semester-II Students** of all colleges affiliated to Kumaun University.

The subject matter has been discussed in such a simple way that the students will find no difficulty to understand it. The proofs of various theorems and examples have been given with minute details. Each chapter of this book contains complete theory and a fairly large number of solved examples. Sufficient problems have also been selected from various university examination papers. At the end of each chapter an exercise containing objective questions has been given.

We have tried our best to keep the book free from misprints. The authors shall be grateful to the readers who point out errors and omissions which, inspite of all care, might have been there.

The authors, in general, hope that the present book will be warmly received by the students and teachers. We shall **indeed** be very thankful to our colleagues for their recommending this book to their students.

The authors wish to express their thanks to **Mr. S.K. Rastogi**, *M.D.,* **Mr. Sugam Rastogi**, *Executive Director,* **Mrs. Kanupriya Rastogi**, *Director* and **entire team** of **KRISHNA Prakashan Media (P) Ltd., Meerut** for bringing out this book in the present nice form.

The authors will feel amply rewarded if the book serves the purpose for which it is meant. Suggestions for the improvement of the book are always welcome.

—Authors

Syllabus

Integral Calculus

B.A./B.Sc. II Semester

Kumaun University

Second Semester – Second Paper

B.A./B.Sc. Paper-II MM 60

Paper II: Integral Calculus

Definite Integrals: Integral as a limit of sum, Properties of Definite integrals, Fundamental theorem of integral calculus, Summation of series by integration, Infinite integrals, Differentiation and integration under the integral sign.

Functions Defined by Infinite Integrals: Beta function, Properties and various forms, Gamma function, Recurrence formula and other relations, Relation between Beta and Gamma function, Evaluation of integrals using Beta and Gamma functions.

Multiple Integrals: Double integrals, Repeated integrals, Evaluation of Double integrals, Double integral in polar coordinates, Change of variables and Introduction to Jacobians, Change of order of integration in Double integrals, Triple integrals, Evaluation of Triple integrals, Drichlet's theorem and its Liovelle's extension.

Geometrical Applications of Definite Integrals: Area bounded by curves (quadrature), Rectification (length of curves), Volumes and Surfaces of Solids of revolution.

Brief Contents

Krishna's

INTEGRAL CALCULUS

Chapters

Chapter 1

Definite Integrals

1 Definite Integral

Sometimes in geometrical and other applications of integral calculus it becomes necessary to find the difference in the values of an integral of a function $f(x)$ for two given values of the variable x, say a and b. This difference is called the *definite integral* of $f(x)$ from a to b or between the *limits* a and b.

This definite integral is denoted by

$$\int_a^b f(x)\,dx$$

and is read as "*the integral of $f(x)$ with respect to x between the limits a and b*".

It is often written thus:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a),$$

where $F(x)$ is an integral of $f(x)$, $F(b)$ is the value of $F(x)$ at $x = b$, and $F(a)$ is the value of $F(x)$ at $x = a$.

The number a is called the *lower limit* and the number b, the *upper limit* of integration. The interval (a, b) is called the *range of integration.*

Fundamental Theorem of Integral Calculus: Let $f \in \mathbf{R}[a, b]$ and let ϕ be a differentiable function on $[a, b]$ such that $\phi'(x) = f(x)$ for all $x \in [a, b]$. Then

$$\int_a^b f(x)\, dx = \phi(b) - \phi(a).$$

2 Fundamental Properties of Definite Integrals

Property 1: *We have* $\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$, *i.e., the value of a definite integral does not change with the change of variable of integration (also called '**argument**') provided the limits of integration remain the same.*

Proof: Let $\int f(x)\, dx = F(x)$; then $\int f(t)\, dt = F(t)$.

Now $$\int_a^b f(x)\, dx = [F(x)]_a^b = F(b) - F(a), \qquad \ldots(1)$$

and $$\int_a^b f(t)\, dt = [F(t)]_a^b = F(b) - F(a), \qquad \ldots(2)$$

From (1) and (2), we see that $\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$.

Property 2: *We have* $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$, *i.e., interchanging the limits of a definite integral does not change the absolute value but changes only the sign of the integral.*

Proof : Let $\int f(x)\, dx = F(x)$. Then

$$\int_a^b f(x)\, dx = [F(x)]_a^b = F(b) - F(a) \qquad \ldots(1)$$

Also $$-\int_b^a f(x)\, dx = -[F(x)]_b^a = -[F(a) - F(b)] = F(b) - F(a). \qquad \ldots(2)$$

From (1) and (2), we see that $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$.

Property 3: *We have* $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$.

Proof: Let $\int f(x)\, dx = F(x)$.

Then the R.H.S.

$$= [F(x)]_a^c + [F(x)]_c^b = \{F(c) - F(a)\} + \{F(b) - F(c)\}$$

$$= F(b) - F(a) = \int_a^b f(x)\, dx = \text{L.H.S.}$$

Note 1: This property also holds true even if the point c is exterior to the interval (a, b).

Note 2: In place of one additional point c, we can take several points. Thus

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + \ldots$$
$$+ \int_{c_{r-1}}^{c_r} f(x)\,dx + \ldots + \int_{c_n}^{b} f(x)\,dx.$$

Property 4: *We have* $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$

Proof: Let $\displaystyle I = \int_0^a f(x)\,dx.$

Put $x = a - t$, so that $dx = -\,dt$.

When $x = 0, t = a$ and when $x = a, t = 0$.

$$\therefore \quad I = \int_a^0 f(a-t)\,(-\,dt) = \int_0^a f(a-t)\,dt, \qquad \text{[by property 2]}$$

$$= \int_0^a f(a-x)\,dx. \qquad \text{[by property 1]}$$

Property 5: $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$ *or* $\displaystyle\int_0^a f(x)\,dx$, *according as* $f(x)$ *is an odd or an even function of* x.

Proof: Odd and even functions. A function $f(x)$ is said to be

(i) *an odd function* of x if $f(-x) = -f(x)$,

(ii) *an even function* of x if $f(-x) = f(x)$.

Now $$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx, \text{ by property 3.} \qquad \ldots(1)$$

Let $\displaystyle u = \int_{-a}^{0} f(x)\,dx$. In the integral u, put $x = -\,t$ so that $dx = -\,dt$.

Also $t = a$, when $x = -\,a$ and $t = 0$ when $x = 0$.

$$\therefore \quad u = \int_a^0 f(-t)\,(-\,dt) = \int_0^a f(-t)\,dt, \qquad \text{[by property 2]}$$

$$= \int_0^a f(-x)\,dx, \qquad \text{[by property 1]}$$

$$= -\int_0^a f(x)\,dx, \text{ if } f(x) \text{ is an odd function of } x,$$

or $$= \int_0^a f(x)\,dx, \text{ if } f(x) \text{ is an even function of } x.$$

$\therefore$ from (1), we get

$$\int_{-a}^{a} f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 0\,,$$

if $f\ (x)$ is an odd function of x

and
$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 2\int_0^a f(x)\,dx,$$

if $f\ (x)$ is an even function of x.

Property 6: $\displaystyle\int_0^{2a} \boldsymbol{f(x)\,dx = 2\int_0^a f(x)\,dx}$, *if* $\boldsymbol{f\,(2a - x) = f(x)}$

and $\displaystyle\int_0^{2a} \boldsymbol{f(x)\,x = 0}$, *if* $\boldsymbol{f\,(2a - x) = -f(x).}$

Proof : We have $\displaystyle\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$

$$= \int_0^a f(x)\,dx - \int_a^0 f(2a - y)\,dy,$$

[putting $x = 2a - y$ in the second integral and changing the limits]

$$= \int_0^a f(x)\,dx + \int_0^a f(2a - y)\,dy,$$

interchanging the limits in the second integral

$$= \int_0^a f(x)\,dx + \int_0^a f(2a - x)\,dx,$$

changing the argument from y to x in the second integral

$$= 2\int_0^a f(x)\,dx, \text{ if } f\ (2a - x) = f\ (x)$$

or $\quad = 0, \text{ if } f(2a - x) = -\,f(x).$

Corollary: $\displaystyle\int_0^{2a} \boldsymbol{f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a - x)\,dx.}$

Remember:

(i) $\displaystyle\int_{-\pi/2}^{\pi/2} f(\sin x)\,dx = 2\int_0^{\pi/2} f(\sin x)\,dx \text{ or } = 0$

as if, $f\ (\sin x)$ is an *even* or an *odd* function respectively.

(ii) $\displaystyle\int_0^{\pi} f(\sin x)\,dx = 2\int_0^{\pi/2} f(\sin x)\,dx,$ [by property 6, because $\sin(\pi - x) = \sin x$]

(iii) $\displaystyle\int_{-\pi/2}^{\pi/2} f(\cos x)\,dx = 2\int_0^{\pi/2} f(\cos x)\,dx,$ [by property 5]

(iv) $\int_0^{\pi} f(\cos x)\, dx = 2\int_0^{\pi/2} f(\cos x)\, dx$ or $= 0$,

as if, f (cos x) is an *even* or an *odd* function respectively.

(v) $\int_0^{\pi/2} f(\sin x)\, dx = \int_0^{\pi/2} f\left\{\sin\left(\frac{1}{2}\pi - x\right)\right\} dx,$ [by property 4]

$$= \int_0^{\pi/2} f(\cos x)\, dx.$$

(vi) $\int_0^{\pi} \sin^m x \cos^n x\, dx = 2\int_0^{\pi/2} \sin^m x \cos^n x\, dx$ or $= 0$,

according as n is an even or an odd integer, (by property 6).

Illustrative Examples

Example 1: *Evaluate* $\int_0^{\pi} \cos^{2n} x\, dx$.

Solution: We have $\int_0^{\pi} \cos^{2n} x\, dx = 2\int_0^{\pi/2} \cos^{2n} x\, dx,$

$$\left[\because \int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx \text{ if } f(2a - x) = f(x).\right.$$

Here taking $f(x) = \cos^{2n} x$, we see that

$$\left. f(\pi - x) = \cos^{2n}(\pi - x) = (-\cos x)^{2n} = \cos^{2n} x = f(x)\right]$$

$$= 2 \cdot \frac{(2n-1)(2n-3)\ldots\ldots 3.1}{2n(2n-2)(2n-4)\ldots\ldots 4.2} \cdot \frac{\pi}{2}, \text{ by Walli's formula}$$

$$= \frac{(2n-1)(2n-3)\ldots 3.1}{2^n . n!} \cdot \pi.$$

Example 2: *Evaluate* $\int_0^{\pi} \theta \sin^3 \theta\, d\theta$.

Solution: Let $I = \int_0^{\pi} \theta . \sin^3 \theta\, d\theta.$ ...(1)

Then $I = \int_0^{\pi} (\pi - \theta) \sin^3 (\pi - \theta)\, d\theta,$

$$\left[\because \int_0^{a} f(x)\, dx = \int_0^{a} f(a - x)\, dx, \text{ refer prop. } 4\right]$$

$$= \int_0^{\pi} (\pi - \theta) \sin^3 \theta\, d\theta. \quad \ldots(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} [\theta \sin^3 \theta + (\pi - \theta) \sin^3 \theta]\, d\theta = \int_0^{\pi} (\theta + \pi - \theta) \sin^3 \theta\, d\theta$$

$$= \int_0^{\pi} \pi \sin^3 \theta\, d\theta = \pi \int_0^{\pi} \sin^3 \theta\, d\theta$$

$$= 2\pi \int_0^{\pi/2} \sin^3 \theta\, d\theta, \text{ by a property of definite integrals; refer prop. 6}$$

$$= 2\pi \cdot \frac{2}{3.1} .1, \quad \text{by Walli's formula}$$

$$= 4\pi/3.$$

$\therefore \qquad I = \frac{2}{3} \pi .$

Example 3: *Prove without performing integration that*

$$\int_{-a}^{2a} \frac{x\, dx}{x^2 + p^2} = \int_{a}^{2a} \frac{x\, dx}{x^2 + p^2} .$$

Solution: We have

$$\int_{-a}^{2a} \frac{x\, dx}{x^2 + p^2} = \int_{-a}^{a} \frac{x\, dx}{x^2 + p^2} + \int_{a}^{2a} \frac{x\, dx}{x^2 + p^2} . \qquad \text{...(1)}$$

But if $f(x) = \dfrac{x}{x^2 + p^2}$, then $f(-x) = \dfrac{-x}{x^2 + p^2} = -f(x)$.

Therefore $f(x)$ is an odd function of x.

$\therefore \qquad \int_{-a}^{a} \dfrac{x\, dx}{x^2 + p^2} = 0.$

So from (1), we get $\displaystyle\int_{-a}^{2a} \frac{x\, dx}{x^2 + p^2} = \int_{a}^{2a} \frac{x\, dx}{x^2 + p^2} .$

Example 4: *Evaluate* $\displaystyle\int_0^{\pi} \frac{x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x} .$

(Kumaun 2012)

Solution: Let $\displaystyle I = \int_0^{\pi} \frac{x\, dx}{a^2 \cos^2 x + b^2 \sin^2 x} . \qquad \text{...(1)}$

Then $\displaystyle I = \int_0^{\pi} \frac{(\pi - x)\, dx}{a^2 \cos^2 (\pi - x) + b^2 \sin^2 (\pi - x)} ,$

$$\left[\because \int_0^{a} f(x)\, dx = \int_0^{a} f(a - x)\, dx\right]$$

$$= \int_0^{\pi} \frac{(\pi - x)\, dx}{a^2 \cos^2 x + b^2 \sin^2 x} . \qquad \text{...(2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{x + (\pi - x)}{a^2 \cos^2 x + b^2 \sin^2 x} dx = \pi \int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$

$$= 2\pi \int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x},$$

by a property of definite integrals, refer prop. 6.

$\therefore$ $$I = \pi \int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \pi \int_0^{\pi/2} \frac{\sec^2 x \, dx}{a^2 + b^2 \tan^2 x},$$

dividing the numerator and the denominator by $\cos^2 x$.

Now put $b \tan x = t$. Then $b \sec^2 x \, dx = dt$.

Also when $x = 0$, $t = 0$ and when $x \to \pi/2, t \to \infty$.

$\therefore$ $$I = \frac{\pi}{b} \int_0^{\infty} \frac{dt}{a^2 + t^2} = \frac{\pi}{b} \cdot \frac{1}{a} \left[\tan^{-1} \frac{t}{a} \right]_0^{\infty}$$

$$= \frac{\pi}{ab} [\tan^{-1} \infty - \tan^{-1} 0] = \frac{\pi}{ab} \left[\frac{\pi}{2} - 0 \right] = \frac{\pi^2}{2ab}.$$

Example 5: *Evaluate* $\int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx.$

Solution: Let $I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx.$

Then $$I = \int_0^{\pi/2} \frac{\cos\left(\frac{1}{2}\pi - x\right) - \sin\left(\frac{1}{2}\pi - x\right)}{1 + \sin\left(\frac{1}{2}\pi - x\right)\cos\left(\frac{1}{2}\pi - x\right)} dx,$$ [Refer prop. 4]

$$= \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \cos x \sin x} dx = -\int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx = -I.$$

$\therefore$ $2I = 0$ or $I = 0$.

Example 6: *Evaluate* $\int_0^{\pi} \frac{x \, dx}{1 + \sin x}.$

Solution: Let $I = \int_0^{\pi} \frac{x \, dx}{1 + \sin x} = \int_0^{\pi} \frac{(\pi - x) \, dx}{1 + \sin (\pi - x)},$ [Refer prop. 4]

$$= \int_0^{\pi} \frac{(\pi - x)}{1 + \sin x} dx = \int_0^{\pi} \frac{\pi}{1 + \sin x} dx - \int_0^{\pi} \frac{x}{1 + \sin x} dx$$

$$= \pi \int_0^{\pi} \frac{1}{1 + \sin x} dx - I.$$

$\therefore$ $$2I = \pi \int_0^{\pi} \frac{dx}{1 + \sin x} = 2\pi \int_0^{\pi/2} \frac{dx}{1 + \sin x},$$ [Refer prop. 6] or $$I = \pi \int_0^{\pi/2} \frac{dx}{1+\sin x} = \pi \int_0^{\pi/2} \frac{dx}{1+\sin\left(\frac{1}{2}\pi - x\right)},$$ [Refer prop. 4]

$$= \pi \int_0^{\pi/2} \frac{dx}{1+\cos x} = \pi \int_0^{\pi/2} \frac{dx}{2\cos^2 \frac{1}{2}x} = \pi \int_0^{\pi/2} \frac{1}{2} \sec^2 \frac{1}{2} x \, dx$$

$$= \pi \left[\tan \frac{1}{2} x\right]_0^{\pi/2} = \pi \left[\tan \frac{1}{4}\pi - \tan 0\right] = \pi (1-0) = \pi.$$

Example 7: *Show that* $\int_0^{\pi/2} \log \sin x \, dx = -\frac{1}{2}\pi \log 2$ *or* $\frac{1}{2}\pi \log \frac{1}{2}$.

Solution: Let $$I = \int_0^{\pi/2} \log \sin x \, dx. \quad ...(1)$$

Then $$I = \int_0^{\pi/2} \log \sin\left(\frac{1}{2}\pi - x\right) dx,$$ $\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$

$$= \int_0^{\pi/2} \log \cos x \, dx. \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \log \sin x \, dx + \int_0^{\pi/2} \log \cos x \, dx$$

$$= \int_0^{\pi/2} \log(\sin x \cos x)\, dx$$ **(Note)**

$$= \int_0^{\pi/2} \log \left\{\frac{\sin 2x}{2}\right\} dx = \int_0^{\pi/2} (\log \sin 2x - \log 2)\, dx$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx - \int_0^{\pi/2} \log 2 \, dx$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx - (\log 2)\,[x]_0^{\pi/2}$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx - \frac{\pi}{2} \log 2.$$

Now put $2x = t$, so that $2\,dx = dt$. Also $t = 0$ when $x = 0$ and $t = \pi$ when $x = \frac{1}{2}\pi$.

$\therefore$ $$2I = \frac{1}{2}\int_0^{\pi} \log \sin t \, dt - \frac{\pi}{2} \log 2$$

$$= \frac{1}{2}\int_0^{\pi} \log \sin x \, dx - \frac{\pi}{2} \log 2,$$ [Refer prop. 1]

$$= \frac{1}{2} . 2 \int_0^{\pi/2} \log \sin x \, dx - \frac{\pi}{2} \log 2,$$ [Refer prop. 6]

$$= I - \frac{1}{2}\pi \log 2.$$

Therefore $\quad 2I - I = -\frac{1}{2}\pi \log 2$

or $\quad I = -\frac{1}{2}\pi \log 2 = \frac{1}{2}\pi \log (2)^{-1} = \frac{1}{2}\pi \log \frac{1}{2}.$

Example 8: *Show that* $\int_0^{\pi/2} x \cot x \, dx = \frac{1}{2}\pi \log 2.$

Solution: Let $I = \int_0^{\pi/2} x \cot x \, dx$. Integrating by parts taking cot x as the second function, we get

$$I = [x \log \sin x]_0^{\pi/2} - \int_0^{\pi/2} 1 . \log \sin x \, dx$$

$$= \left[\frac{\pi}{2}\log 1 - \lim_{x \to 0} x \log \sin x\right] - \int_0^{\pi/2} \log \sin x \, dx$$

$$= 0 - \lim_{x \to 0} x \log \sin x - \int_0^{\pi/2} \log \sin x \, dx.$$

Now $\quad \lim_{x \to 0} x \log \sin x = \lim_{x \to 0} \frac{\log \sin x}{1/x} \qquad \left[\text{form } \frac{\infty}{\infty}\right]$

$$= \lim_{x \to 0} \frac{(1/\sin x)\cos x}{-1/x^2} = \lim_{x \to 0} \frac{-x^2 \cos x}{\sin x} \qquad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x \to 0} \frac{-2x \cos x + x^2 \sin x}{\cos x} = \frac{0}{1} = 0.$$

$\therefore \quad I = 0 - \int_0^{\pi/2} \log \sin x \, dx = -\int_0^{\pi/2} \log \sin x \, dx.$

Now let $\quad u = \int_0^{\pi/2} \log \sin x \, dx.$

Then proceeding as in Example 7, we have $u = -\frac{1}{2}\pi \log 2.$

$\therefore \quad I = -u = \frac{1}{2}\pi \log 2.$

Example 9: *Show that* $\int_0^{\pi/4} \log (1 + \tan \theta) \, d\theta = \frac{\pi}{8} \log 2.$

Solution: Let $I = \int_0^{\pi/4} \log (1 + \tan \theta) \, d\theta.$

Then $\quad I = \int_0^{\pi/4} \log \left\{1 + \tan\left(\frac{1}{4}\pi - \theta\right)\right\} d\theta, \quad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$

$$= \int_0^{\pi/4} \log \left[1 + \frac{(1 - \tan \theta)}{(1 + \tan \theta)}\right] d\theta = \int_0^{\pi/4} \log \left\{\frac{2}{1 + \tan \theta}\right\} d\theta$$

$$= \int_0^{\pi/4} \log 2 \, . \, d\theta - \int_0^{\pi/4} \log (1 + \tan \theta) \, d\theta$$

$$= \log 2 \, . \, [\theta]_0^{\pi/4} - I.$$

$\therefore$ $$2I = \frac{1}{4} \pi \log 2 \quad \text{or} \quad I = \frac{1}{8} \pi \log 2 \, .$$

Example 10: *Show that* $\int_0^{\pi/2} \frac{\sin x \, dx}{\sin x + \cos x} = \frac{\pi}{4}$.

(Lucknow 2014)

Solution: Let $I = \int_0^{\pi/2} \frac{\sin x \, dx}{\sin x + \cos x}$. ...(1)

Then $$I = \int_0^{\pi/2} \frac{\sin \left(\frac{1}{2} \pi - x \right)}{\sin \left(\frac{1}{2} \pi - x \right) + \cos \left(\frac{1}{2} \pi - x \right)}$$ [Refer prop. 4]

$$= \int_0^{\pi/2} \frac{\cos x \, dx}{\cos x + \sin x} \, . \qquad \text{...(2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sin x \, dx}{\sin x + \cos x} + \int_0^{\pi/2} \frac{\cos x \, dx}{\sin x + \cos x}$$

$$= \int_0^{\pi/2} \left[\frac{\sin x}{\sin x + \cos x} + \frac{\cos x}{\sin x + \cos x} \right] dx$$

$$= \int_0^{\pi/2} 1 \, . \, dx = [x]_0^{\pi/2} = \frac{\pi}{2} \, .$$

$\therefore$ $$I = \frac{1}{4} \pi \, .$$

Comprehensive Exercise 1

Evaluate the following integrals :

1. (i) $\int_0^{\pi} \cos^6 x \, dx.$ (ii) $\int_0^{\pi} \sin^3 x \, dx.$

2. (i) $\int_{-1}^{1} \frac{x^2 \sin^{-1} x}{\sqrt{(1 - x^2)}} dx.$ (ii) $\int_{-a}^{a} x \sqrt{(a^2 - x^2)} \, dx.$

(iii) $\int_{-1}^{1} \frac{x \sin^{-1} x}{\sqrt{(1 - x^2)}} dx.$

3. (i) $\int_0^{\pi} \frac{dx}{a + b \cos x}$. (ii) $\int_0^{2\pi} \frac{dx}{a + b \cos x + c \sin x}$.

4. (i) Show that $\int_0^{\pi} \frac{x\,dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} = \frac{\pi^2 (a^2 + b^2)}{4a^3 b^3}$. **(Kumaun 2007, 09)**

(ii) Show that $\int_0^{\pi} \frac{x\,dx}{a^2 - \cos^2 x} = \frac{\pi^2}{2a \sqrt{(a^2 - 1)}}, (a > 1)$.

(iii) Evaluate $\int_0^{\pi} \frac{x\,dx}{1 + \cos^2 x}$.

5. (i) Evaluate $\int_0^{\pi/2} (\sin x - \cos x) \log (\sin x + \cos x)\,dx$.

(ii) Evaluate $\int_0^{\pi/2} \sin 2x \log \tan x\,dx$.

6. (i) Evaluate $\int_0^{\pi} \frac{x \sin x}{(1 + \cos^2 x)}\,dx$. **(Kumaun 2007)**

(ii) Evaluate $\int_0^{\pi} x \sin^6 x \cos^4 x\,dx$.

7. (i) Prove that $\int_0^{\pi} \frac{x \sin x}{1 + \sin x}\,dx = \pi \left(\frac{\pi}{2} - 1 \right)$. **(Kumaun 2011)**

(ii) Show that $\int_0^{\pi} \frac{x \tan x}{\sec x + \cos x}\,dx = \frac{1}{4} \pi^2$.

(iii) Show that $\int_0^{\pi} \frac{x \tan x\,dx}{\sec x + \tan x} = \pi \left(\frac{1}{2} \pi - 1 \right)$.

8. (i) Evaluate $\int_0^{\pi} \sin^3 \theta\,(1 + 2 \cos \theta)\,(1 + \cos \theta)^2\,d\theta$.

(ii) Evaluate $\int_0^{\pi} \sin^5 x\,(1 - \cos x)^3\,dx$.

9. (i) Show that $\int_0^{\pi/2} \log (\tan x)\,dx = 0$. **(Kumaun 2010)**

(ii) Prove that $\int_0^{1} \log \sin \left(\frac{\pi}{2} y \right) dy = \log \frac{1}{2}$.

10. (i) Evaluate $\int_0^{\pi} x \log \sin x\,dx$.

(ii) Evaluate $\int_0^{\pi/2} \log \cos x\,dx$.

(iii) Evaluate $\int_0^{\pi/2} \log \sin 2x\,dx$.

(iv) Evaluate $\int_0^{\infty} \frac{\tan^{-1} x\,dx}{x (1 + x^2)}$.

11. (i) Show that $\int_0^{\pi/2} \left(\frac{\theta}{\sin \theta} \right)^2 d\theta = \pi \log 2$.

(ii) Show that $\int_0^{\infty} (\cot^{-1} x)^2 \, dx = \pi \log 2.$

(iii) Show that $\int_0^1 \frac{\sin^{-1} x}{x} \, dx = \frac{1}{2} \pi \log 2.$

(iv) Show that $\int_0^{\pi} \log (1 + \cos x) \, dx = \pi \log \frac{1}{2} \cdot$

12. (i) Show that $\int_0^{\infty} \log \left(x + \frac{1}{x} \right) \frac{dx}{1 + x^2} = \pi \log 2.$

(ii) Show that $\int_0^{\infty} \frac{\log (1 + x^2) \, dx}{(1 + x^2)} = \pi \log 2.$

(iii) Show that $\int_0^1 \frac{\log (1 + x)}{1 + x^2} \, dx = \frac{1}{8} \pi \log 2.$ **(Kumaun 2008)**

13. (i) Evaluate $\int_0^{\pi/2} \frac{dx}{1 + \tan x} \cdot$

(ii) Evaluate $\int_0^{\pi/2} \frac{dx}{1 + \cot x} \cdot$

14. (i) Show that $\int_0^{\infty} \frac{x \, dx}{(1 + x)(1 + x^2)} = \frac{\pi}{4} \cdot$

(ii) Show that $\int_0^{a} \frac{dx}{x + \sqrt{(a^2 - x^2)}} = \frac{\pi}{4} \cdot$

15. (i) Show that $\int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}} \, dx = \frac{\pi}{4} \cdot$ **(Lucknow 2007)**

(ii) Show that $\int_0^{\pi/2} \frac{\tan x}{\tan x + \cot x} \, dx = \frac{\pi}{4} \cdot$

(iii) Show that $\int_0^{\pi/2} \frac{dx}{1 + \sqrt{(\tan x)}} = \frac{\pi}{4} \cdot$

(iv) Show that $\int_0^{\pi/2} \frac{\sqrt{(\tan x)} \, dx}{1 + \sqrt{(\tan x)}} = \frac{\pi}{4} \cdot$

16. (i) Prove that $\int_0^{\pi/2} \frac{\sqrt{(\tan x)}}{\sqrt{(\tan x)} + \sqrt{(\cot x)}} \, dx = \frac{\pi}{4} \cdot$

(ii) Show that $\int_0^{\pi/2} \frac{\sin^2 x \, dx}{(\sin x + \cos x)} = \frac{1}{\sqrt{2}} \log (\sqrt{2} + 1).$

17. (i) Evaluate $\int_0^{\pi/2} \frac{\cos^2 x}{(\sin x + \cos x)} \, dx.$

(ii) Evaluate $\int_0^{a} \frac{a \, dx}{\{x + \sqrt{(a^2 - x^2)}\}^2} \cdot$

(iii) Evaluate $\int_0^{\pi/2} \frac{x \, dx}{\sin x + \cos x} \cdot$

18. (i) Show that $\int_0^{\pi/2} \phi(\sin 2x) \sin x \, dx = \int_0^{\pi/2} \phi(\sin 2x) \cos x \, dx$

$$= \sqrt{2} \int_0^{\pi/4} \phi(\cos 2x) \cos x \, dx.$$

(Kumaun 2007)

(ii) Show that $\int_0^{\pi} \frac{x^2 \sin 2x \sin\left(\frac{1}{2}\pi \cos x\right)}{2x - \pi} dx = \frac{8}{\pi}$.

Answers 1

1. (i) $\frac{5\pi}{16}$ (ii) $\frac{4}{3}$ **2.** (i) 0

(ii) 0 (iii) 2 **3.** (i) $\frac{\pi}{\sqrt{(a^2 - b^2)}}$

(ii) $\frac{2\pi}{\sqrt{(a^2 - b^2 - c^2)}}$ **4.** (iii) $\frac{\pi^2 \sqrt{2}}{4}$ **5.** (i) 0

(ii) 0 **6.** (i) $\frac{1}{4}\pi^2$ (ii) $\frac{3\pi^2}{512}$

8. (i) $\frac{8}{3}$ (ii) $\frac{32}{21}$ **10.** (i) $\frac{1}{2}\pi^2 \log \frac{1}{2}$

(ii) $\frac{1}{2}\pi \log \frac{1}{2}$ (iii) $\frac{1}{2}\pi \log \frac{1}{2}$ (iv) $\frac{1}{2}\pi \log 2$

13. (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{4}$ **17.** (i) $\frac{1}{\sqrt{2}} \log(\sqrt{2} + 1)$

(ii) $\frac{1}{\sqrt{2}} \log(\sqrt{2} + 1)$ (iii) $\frac{\pi}{2\sqrt{2}} \log(\sqrt{2} + 1)$

3 The Definite Integrals as the Limit of a Sum

So far integration has been defined as the inverse process of differentiation. But it is also possible to regard a definite integral as the limit of the sum of certain number of terms, when the number of terms tends to infinity and each term tends to zero.

Definition: *Let $f(x)$ be a single valued continuous function defined in the interval (a, b) where $b > a$ and let the interval (a, b) be divided into n equal parts each of length h, so that $nh = b - a$; then we define*

$$\int_a^b f(x)\, dx = \lim h\,[f(a) + f(a + h) + f(a + 2h) + \ldots + f\{a + (n - 1)h\}],$$

when $n \to \infty$, $h \to 0$ and $nh \to b - a$.

Thus $\int_a^b f(x)\,dx = \lim_{h\to 0} h \sum_{r=0}^{n-1} f(a+rh)$, where $n \to \infty$ as $h \to 0$ and nh remains equal to $b-a$. We call $\int_a^b f(x)\,dx$ as the definite integral of $f(x)$ w.r.t. x between the limits a and b.

Illustrative Examples

Example 11: *Evaluate* $\int_a^b x^2\,dx$ *directly from the definition of the integral as the limit of a sum.*

Solution: From the definition of a definite integral as the limit of a sum, we know that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} h[f(a) + f(a+h) + f(a+2h) + \ldots + f\{a+(n-1)h\}].$$

where $h \to 0$ as $n \to \infty$ and $nh \to b-a$.

Here $f(x) = x^2$; therefore $f(a), f(a+h), f(a+2h)$, etc. will be $a^2, (a+h)^2, (a+2h)^2, \ldots$, respectively.

$$\therefore \quad \int_a^b x^2\,dx = \lim_{n\to\infty} h[a^2 + (a+h)^2 + (a+2h)^2 + \ldots + \{a+(n-1)h\}^2],$$

where $h \to 0$ as $n \to \infty$ and $nh \to b-a$

$$= \lim_{n\to\infty} h[na^2 + 2ah\{1+2+3+\ldots+(n-1)\}$$
$$+ h^2\{1^2 + 2^2 + 3^2 + \ldots + (n-1)^2\}].$$

But we know that

$$\Sigma n = \frac{n(n+1)}{2}$$

and $$\Sigma n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Taking $n = (n-1)$ in the above results, we get

$$\int_a^b x^2\,dx = \lim_{n\to\infty} h\left[na^2 + 2ah \cdot \frac{(n-1)n}{2} + \frac{h^2}{6}(n-1)n(2n-1)\right]$$

$$= \lim_{n\to\infty}\left[(nh)a^2 + a(nh)(n-1)h + \frac{1}{6}(nh)(n-1)h(2n-1)h\right]$$

$$= \lim_{n\to\infty}\left[(nh)a^2 + a(nh)^2\left(1-\frac{1}{n}\right) + \frac{1}{6}\cdot 2(nh)^3\left(1-\frac{1}{n}\right)\left(1-\frac{1}{2n}\right)\right].$$

Now as $n \to \infty$, $h \to 0$ and $nh \to b-a$.

$$\therefore \quad \int_a^b x^2\,dx = (b-a)\,a^2 + a\,(b-a)^2 + \frac{1}{3}(b-a)^3$$

$$= \frac{1}{3}(b-a)\,\{3a^2 + 3\,(b-a)\,a + b^2 - 2ab + a^2\}$$

$$= \frac{1}{3}(b-a)\,(a^2 + ab + b^2)$$

$$= \frac{1}{3}(b^3 - a^3).$$

Example 12: *Show that* $\int_a^b x^m\,dx = \dfrac{b^{m+1} - a^{m+1}}{(m+1)}$.

Solution: Here $f(x) = x^m$; therefore $f(a) = a^m$, $f(a+h) = (a+h)^m$, etc.

$$\therefore \quad \int_a^b x^m\,dx = \lim_{h\to 0} h[a^m + (a+h)^m + \ldots + \{a + (n-1)\,h\}^m],$$

where $b - a = nh$.

Now $$\lim_{h\to 0} \frac{(t+h)^{m+1} - t^{m+1}}{h} = \frac{d}{dt}\,t^{m+1} = (m+1)\,t^m.$$

$$\therefore \quad \lim_{h\to 0} \frac{(t+h)^{m+1} - t^{m+1}}{h \cdot t^m} = (m+1), \text{ i.e., constant} \qquad \ldots(1)$$

Putting $t = a, (a+h), (a+2h)$, etc., in (1), we get

$$\lim_{h\to 0} \frac{(a+h)^{m+1} - a^{m+1}}{h \cdot a^m} = \lim_{h\to 0} \frac{(a+2h)^{m+1} - (a+h)^{m+1}}{h\,(a+h)^m} = \ldots$$

$$\ldots = \lim_{h\to 0} \frac{(a+nh)^{m+1} - \{a + (n-1)\,h\}^{m+1}}{h\,\{a + (n-1)\,h\}^m}$$

$$= (m+1) \text{ i.e., a constant.} \qquad \ldots(2)$$

Also we know that if $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \ldots\ldots$, then each of these ratios is equal to

$$\frac{a + c + e + \ldots.}{b + d + f + \ldots} \qquad \ldots(3)$$

Now we apply the property (3) to various limits given in (2). Thus forming a new numerator and denominator by adding the numerators and denominators of the various ratios in (2), we get

$$\lim_{h\to 0} \frac{(a+nh)^{m+1} - a^{m+1}}{h\,[a^m + (a+h)^m + \ldots + \{a + (n-1)\,h\}^m]} = (m+1)$$

or $$\lim_{h\to 0} \frac{[a + (b-a)]^{m+1} - a^{m+1}}{h\,[a^m + (a+h)^m + \ldots + \{a + (n-1)\,h\}^m]} = (m+1). \qquad [\because nh = b - a]$$

or $$\lim_{h\to 0} h\,[a^m + (a+h)^m + \ldots + \{a+(n-1)\,h\}^m] = \frac{b^{m+1} - a^{m+1}}{m+1}.$$

$\therefore$ $$\int_a^b x^m\,dx = \frac{b^{m+1} - a^{m+1}}{(m+1)}.$$

Example 13: *From the definition of a definite integral as the limit of a sum, evaluate* $\int_a^b e^x\,dx.$

Solution: Here $f(x) = e^x$; therefore $f(a) = e^a$, $f(a+h) = e^{a+h}$, etc.

$\therefore$ $$\int_a^b e^x\,dx = \lim_{h\to 0} h\,\{e^a + e^{a+h} + e^{a+2h} + \ldots + e^{a+(n-1)h}\}.$$

where $nh = b - a$ and $n \to \infty$ as $h \to 0$

$$= \lim_{h\to 0} he^a\{1 + e^h + e^{2h} + \ldots + e^{(n-1)h}\}$$

$$= \lim_{h\to 0} he^a \left\{\frac{(e^h)^n - 1}{e^h - 1}\right\}, \text{ summing the G.P.}$$

$$= \lim_{h\to 0} he^a \left[\frac{e^{nh} - 1}{e^h - 1}\right]$$

$$= \lim_{h\to 0} he^a \left[\frac{e^{b-a} - 1}{e^h - 1}\right], \qquad [\because nh = (b-a)]$$

$$= e^a\,(e^{b-a} - 1), \qquad \left[\lim_{h\to 0} \frac{h}{e^h - 1} = \lim_{h\to 0} \frac{1}{e^h} = 1\right]$$

$$= e^b - e^a.$$

Example 14: *Evaluate by summation* $\int_a^b \sin x\,dx.$

Solution: Here $f(x) = \sin x$; therefore $f(a) = \sin a$, $f(a+h) = \sin(a+h)$, etc.

$\therefore$ $$\int_a^b \sin x\,dx = \lim_{h\to 0} h\,[\sin a + \sin(a+h) + \ldots + \sin\{a + (n-1)\,h\}],$$

where $nh = b - a$ and $n \to \infty$ as $h \to 0$

$$= \lim_{h\to 0} h\left[\frac{\sin\left(\frac{1}{2}nh\right)}{\sin\frac{1}{2}h}\cdot\sin\left\{a + \frac{1}{2}(n-1)\,h\right\}\right], \text{ from Trigonometry}$$

$$= \lim_{h\to 0} 2\cdot\frac{\frac{1}{2}h}{\sin\frac{1}{2}h}\cdot\sin\left(\frac{b-a}{2}\right)\cdot\sin\left(a + \frac{b-a-h}{2}\right), \qquad [\because nh = b - a]$$

$$= 2 \cdot 1 \cdot \sin\left(\frac{b-a}{2}\right) \sin\left(a + \frac{b-a}{2}\right), \qquad \left[\because \lim_{\theta \to 0} \frac{\theta}{\sin \theta} = 1\right]$$

$$= 2 \sin \frac{b-a}{2} \sin \frac{a+b}{2} = \cos a - \cos b.$$

Comprehensive Exercise 2

1. Find by summation the value of $\int_a^b x \, dx$.
2. Evaluate by summation $\int_1^2 x \, dx$.
3. Evaluate by summation $\int_0^2 x^3 \, dx$.
4. Using the definition of integral as the limit of a sum, show that
$\int_a^b \cos x \, dx = \sin b - \sin a$.
5. Evaluate by summation $\int_0^{\pi/2} \sin x \, dx$.
6. Evaluate by summation $\int_0^{\pi/2} \cos x \, dx$.
7. Evaluate by summation $\int_a^b \frac{1}{x^2} \, dx$.

Answers 2

1. $\frac{1}{2}(b^2 - a^2)$
2. $\frac{3}{2}$
3. 4
5. 1
6. 1
7. $\frac{1}{a} - \frac{1}{b}$

4 Summation of Series with the Help of Definite Integrals

The definition of a definite integral as the limit of a sum (article 3) helps us to evaluate the limit of the sums of some special types of series. We know that

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} h[f(a) + f(a+h) + \ldots + f\{a + (n-1)h\}]$$

$$= \lim_{n\to\infty} h \sum_{r=0}^{n-1} f(a+rh), \quad \text{where } nh = b-a.$$

Putting $a = 0$ and $b = 1$, so that $h = (1/n)$, we get

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right).$$

Thus the limit of the sum of a series can be expressed in the form of a definite integral provided the series has the following properties :

(a) Each term of the series should have $(1/n)$ as a common factor which tends to zero as $n \to \infty$.

(b) The general term of the series should be the product of $1/n$ and a function $f(r/n)$ of r/n, so that the various terms of the series can be obtained from it by giving different values to r, say $r = 0, 1, 2, \ldots, n-1$.

(c) There should be n terms in the series, but if however the number of terms differs by a finite number from n, then the required limit does not change because each term tends to zero. Thus

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=p}^{n+q} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx,$$

if p and q are independent of n.

Working Rule :

(i) Write down the general term [say r th term or $(r-1)$th term etc., as convenient] of the series. Take out $(1/n)$ as a factor from the general term and thus write the series in the form $\dfrac{1}{n}\sum_{r=0}^{n-1} f\left(\dfrac{r}{n}\right)$. We may have some other limits of r in the summation; for example, r may vary from 1 to n or from 0 to $2n$, etc. .

(ii) Now to evaluate $\lim_{n\to\infty} \dfrac{1}{n}\sum_{r=0}^{n-1} f\left(\dfrac{r}{n}\right)$, replace r/n by x, $1/n$ by dx and $\lim_{n\to\infty} \Sigma$ by the sign of integration *i.e.*, by $\int$.

(iii) To find the limits of integration of x first note carefully the limits of r in the summation $\Sigma\, f(r/n)$. Divide these limits by n to get the values of r/n. Take limits of these values of r/n as $n \to \infty$ and get the limits of integration of x.

Illustrative Examples

Example 15: *Show that the limit of the sum* $\dfrac{1}{n} + \dfrac{1}{n+1} + \dfrac{1}{n+2} + \ldots + \dfrac{1}{3n}$, *when n is indefinitely increased is log* 3 . **(Kumaun 2013)**

Solution: Here the general term of the series is $\dfrac{1}{n+r}$ and r varies from 0 to $2n$.

Now we have to find $\lim\limits_{n\to\infty}\sum\limits_{r=0}^{2n}\dfrac{1}{n+r}$.

We have $$\lim_{n\to\infty}\sum_{r=0}^{2n}\frac{1}{n+r}=\lim_{n\to\infty}\sum_{r=0}^{2n}\frac{1}{n\{1+(n/r)\}},$$

expressing the general term in the form $(1/n)\,f(r/n)$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{2n}\frac{1}{1+(r/n)},$$ taking $\dfrac{1}{n}$ outside the sign of summation.

Now $\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{r=0}^{2n}\dfrac{1}{1+(r/n)}$ is of the form

$$\lim_{n\to\infty}\frac{1}{n}\Sigma f\left(\frac{r}{n}\right),\quad \text{where } f\left(\frac{r}{n}\right)=\frac{1}{1+(r/n)}.$$

The limits of r in this summation are 0 to $2n$. When $r=0$, $\dfrac{r}{n}=\dfrac{0}{n}=0$ and when $r=2n$, $\dfrac{r}{n}=\dfrac{2n}{n}=2$. As $n\to\infty$, these values of $\dfrac{r}{n}$ tend to 0 and 2 respectively, giving us the limits of integration.

Now replacing r/n by x, $1/n$ by dx, $\lim\limits_{n\to\infty}\Sigma$ by the sign of integration $\int$, taking the limits of integration of x from 0 to 2, we get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{2n}\frac{1}{1+(r/n)}=\int_0^2\frac{1}{1+x}\,dx=[\log(1+x)]_0^2$$

$$=\log 3-\log 1=\log 3-0=\log 3.$$

Example 16: *Evaluate* $\lim\limits_{n\to\infty}\left[\dfrac{n}{n^2}+\dfrac{n}{n^2+1^2}+\dfrac{n}{n^2+2^2}+\ldots+\dfrac{n}{n^2+(n+1)^2}\right]$.

Solution: Here, the rth term $=\dfrac{n}{n^2+(r-1)^2}\cdot$ As the r th term contains $(r-1)$, we consider the $(r+1)$th term.

The $(r+1)$th term $=\dfrac{n}{n^2+r^2}=\dfrac{n}{n^2\{1+(r/n)^2\}}=\dfrac{1}{n}\cdot\left\{\dfrac{1}{1+(r/n)^2}\right\}$,

and r varies from 0 to $n+1$.

$\therefore$ the given limit $=\lim\limits_{n\to\infty}\sum\limits_{r=0}^{n+1}\dfrac{1}{n}\left[\dfrac{1}{1+(r/n)^2}\right].$

Also the lower limit of integration

$$= \lim_{n\to\infty} \left(\frac{0}{n}\right) = \lim_{n\to\infty} 0 = 0. \qquad [\because r = 0 \text{ for the 1st term}]$$

and the upper limit $= \lim_{n\to\infty} \left(\frac{n+1}{n}\right) = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1.$

[$\because r = (n+1)$ for the last term]

$\therefore$ the required limit $= \int_0^1 \frac{1}{1+x^2}\, dx = [\tan^{-1} x]_0^1 = \tan^{-1} 1 = \frac{\pi}{4}.$

Example 17: *Prove that* $\lim_{n\to\infty} \left[\frac{1^2}{1^3+n^3} + \frac{2^2}{2^3+n^3} + \ldots + \frac{n^2}{n^3+n^3}\right] = \frac{1}{2} \log 2.$

Solution: Here the r th term $= \frac{r^2}{r^3+n^3} = \frac{1}{n^3}\left\{\frac{r^2}{(r/n)^3+1}\right\} = \frac{1}{n} \cdot \left\{\frac{(r/n)^2}{(r/n)^3+1}\right\},$

and r varies from 1 to n.

$\therefore$ the given limit $= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \left\{\frac{(r/n)^2}{(r/n)^3+1}\right\} = \int_0^1 \frac{x^2\, dx}{x^3+1}$

$$= \left[\frac{1}{3} \log (x^3+1)\right]_0^1 = \frac{1}{3} \log 2 - \frac{1}{3} \log 1 = \frac{1}{3} \log 2.$$

Example 18: *Evaluate* $\lim_{n\to\infty} \frac{1}{n}\left[\sin^{2k} \frac{\pi}{2n} + \sin^{2k} \frac{2\pi}{2n} + \sin^{2k} \frac{3\pi}{2n} + \ldots + \sin^{2k} \frac{\pi}{2}\right].$

Solution: Here the r th term $= \frac{1}{n} \cdot \sin^{2k} \frac{r\pi}{2n}$, and r varies from 1 to n.

$\therefore$ the given limit $= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \sin^{2k} \frac{r\pi}{2n}$

$$= \int_0^1 \sin^{2k}\left(\frac{\pi}{2} \cdot x\right) dx = \frac{2}{\pi} \int_0^{\pi/2} \sin^{2k} t\, dt, \text{ putting } \frac{\pi x}{2} = t$$

so that $\frac{1}{2}\pi\, dx = dt$ and the limits for t are 0 to $\pi/2$

$$= \frac{2}{\pi} \cdot \frac{(2k-1)}{2k} \cdot \frac{(2k-3)}{(2k-2)} \ldots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, \text{ by Walli's formula}$$

$$= \frac{(2k-1)(2k-3)\ldots 3.1}{2k.(2k-2)\ldots 4.2}.$$

Example 19: *Find the limit, as* $n \to \infty$, *of the product*

$$\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)^{1/2}\left(1+\frac{3}{n}\right)^{1/3} \ldots \left(1+\frac{n}{n}\right)^{1/n}.$$

(Lucknow 2013)

Solution: Let $P = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)^{1/2}\left(1+\frac{3}{n}\right)^{1/3}\dots\left(1+\frac{n}{n}\right)^{1/n}$.

Then $$\log P = \lim_{n\to\infty}\left[\log\left(1+\frac{1}{n}\right)+\frac{1}{2}\log\left(1+\frac{2}{n}\right)\right.$$
$$\left.+\frac{1}{3}\log\left(1+\frac{3}{n}\right)+\dots.+\frac{1}{n}\log\left(1+\frac{n}{n}\right)\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r}\log\left(1+\frac{r}{n}\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\cdot\frac{1}{(r/n)}\log\left(1+\frac{r}{n}\right)$$ **(Note)**

$$= \int_0^1 \frac{1}{x}\log(1+x)\,dx = \int_0^1 \frac{1}{x}\left[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots\right]dx$$

$$= \int_0^1\left(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\dots\right)dx = \left[x-\frac{x^2}{4}+\frac{x^3}{9}-\frac{x^4}{16}+\dots\right]_0^1$$

or $$\log P = 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots\infty = \frac{\pi^2}{12},$$ from trigonometry.

$\therefore$ $$P = e^{\pi^2/12}.$$

Example 20: *Evaluate* $\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\dots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$.

(Lucknow 2008; Kumaun 07, 13)

Solution: Let $P = \lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\dots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$.

$\therefore$ $$\log P = \lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)\right.$$
$$\left.+\log\left(1+\frac{3^2}{n^2}\right)+\dots+\log\left(1+\frac{n^2}{n^2}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r^2}{n^2}\right)$$

$$= \int_0^1 \log(1+x^2)\,dx = \int_0^1 \log(1+x^2)\,.1\,dx$$

$$= [x\log(1+x^2)]_0^1 - \int_0^1 \frac{2x\,.\,x\,dx}{1+x^2},$$

integrating by parts taking 1 as the 2nd function

$$= \log 2 - 2 \int_0^1 \frac{(1 + x^2) - 1}{1 + x^2} dx = \log 2 - 2 \int_0^1 \left[1 - \frac{1}{1 + x^2}\right] dx$$

$$= \log 2 - 2 [x - \tan^{-1} x]_0^1 = \log 2 - 2 [1 - \frac{1}{4} \pi].$$

Thus $\log P = \log 2 + \frac{1}{2} (\pi - 4)$ or $\log (P/2) = \frac{1}{2} (\pi - 4)$

or $P = 2e^{(\pi - 4)/2}$.

Example 21: *Find the limit of* $\left\{\frac{n!}{n^n}\right\}^{1/n}$ *when n tends to infinity.*

(Kumaun 2010)

Solution: $P = \lim_{n \to \infty} \left\{\frac{n!}{n^n}\right\}^{1/n} = \lim_{n \to \infty} \left\{\frac{1 \cdot 2 \cdot 3 \cdot 4 \ldots n}{n \cdot n \cdot n \cdot n \ldots n}\right\}^{1/n} = \lim_{n \to \infty} \left\{\frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \ldots \frac{n}{n}\right\}^{1/n}.$

$\therefore$ $$\log P = \lim_{n \to \infty} \frac{1}{n} \left[\log \left(\frac{1}{n}\right) + \log \left(\frac{2}{n}\right) + \log \left(\frac{3}{n}\right) + \ldots + \log \left(\frac{n}{n}\right)\right]$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \log \left(\frac{r}{n}\right) = \int_0^1 \log x \, dx = \int_0^1 (\log x) \cdot 1 \, dx$$

$$= [(\log x) \cdot x]_0^1 - \int_0^1 \frac{1}{x} \cdot x \, dx, \text{ integrating by parts}$$

$$= 0 - [x]_0^1 = -1.$$

$\therefore$ $P = e^{-1} = 1/e.$

Comprehensive Exercise 3

Evaluate the following :

1. $\lim_{n \to \infty} \left[\frac{1}{n + 1} + \frac{1}{n + 2} + \ldots + \frac{1}{2n}\right].$

2. $\lim_{n \to \infty} \left[\frac{1}{n + m} + \frac{1}{n + 2m} + \ldots + \frac{1}{n + nm}\right].$

3. $\lim_{n \to \infty} \left[\frac{n}{(n + 1)^2} + \frac{n}{(n + 2)^2} + \ldots + \frac{n}{(n + n)^2}\right].$

4. $\lim_{n \to \infty} [\{\sqrt{(n + 1)} + \sqrt{(n + 2)} + \ldots + \sqrt{(2n)}\} / n \sqrt{n}].$

5. $\lim_{n \to \infty} n \left[\frac{1}{(n + 1)(n + 2)} + \frac{1}{(n + 2)(n + 4)} + \frac{1}{(n + 3)(n + 6)} + \ldots + \frac{1}{6n^2}\right].$

6. $\lim_{n \to \infty} \left[\frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \ldots + \frac{1}{2n}\right].$

7. $\lim_{n\to\infty}\left[\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\ldots+\frac{1}{n}\right]$.

8. $\lim_{n\to\infty}\left[\frac{1}{n^3}(1+4+9+16+\ldots+n^2)\right]$.

9. $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\ldots+\frac{1}{8n}\right]$.

10. $\lim_{n\to\infty}\left[\frac{n^{1/2}}{n^{3/2}}+\frac{n^{1/2}}{(n+3)^{3/2}}+\frac{n^{1/2}}{(n+6)^{3/2}}+\ldots+\frac{n^{1/2}}{\{n+3(n-1)\}^{3/2}}\right]$. **(Lucknow 2006)**

11. $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{\sqrt{(n^2-1^2)}}+\frac{1}{\sqrt{(n^2-2^2)}}+\ldots+\frac{1}{\sqrt{\{n^2-(n-1)^2\}}}\right]$.

12. $\lim_{x\to\infty}\left[\frac{1}{n^2}\sec^2\frac{1}{n^2}+\frac{2}{n^2}\sec^2\frac{4}{n^2}+\frac{3}{n^2}\sec^2\frac{9}{n^2}+\ldots+\frac{1}{n}\sec^2 1\right]$. **(Lucknow 2010)**

13. $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^3}{r^4+m^4}$.

14. $\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{n}\cdot\sqrt{\left(\frac{n+r}{n-r}\right)}$. **(Lucknow 2014)**

15. $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{n}}{\sqrt{r}\cdot(3\sqrt{r}+4\sqrt{n})^2}$.

16. $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n^2}{(n^2+r^2)^{3/2}}$.

17. $\lim_{n\to\infty}\left[\frac{1}{\sqrt{(2n-1^2)}}+\frac{1}{\sqrt{(4n-2^2)}}+\ldots+\frac{1}{n}\right]$.

18. $\lim_{n\to\infty}\left[\frac{n}{(n+1)\sqrt{(2n+1)}}+\frac{n}{(n+2)\sqrt{\{2(2n+2)\}}}+\ldots+\frac{n}{2n\sqrt{(n.3n)}}\right]$.

19. $\lim_{n\to\infty}\left[\frac{(n-m)^{1/3}}{n}+\frac{(2^2n-m)^{1/3}}{2n}+\frac{(3^2n-m)^{1/3}}{3n}+\ldots+\frac{(n^3-m)^{1/3}}{n^2}\right]$.

20. $\lim_{n\to\infty}\left[\frac{1}{na}+\frac{1}{na+1}+\frac{1}{na+2}+\ldots+\frac{1}{nb}\right]$. **(Kumaun 2011)**

21. $\lim_{n\to\infty}\frac{1+2^{10}+3^{10}+\ldots+n^{10}}{n^{11}}$.

22. Prove that $\lim_{n\to\infty}\frac{1^m+2^m+3^m+\ldots+n^m}{n^{m+1}}=\frac{1}{m+1},(m>1)$.

23. Evaluate $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right]^{1/n}$. **(Lucknow 2009)**

24. Evaluate $\lim_{n\to\infty}\left[\dfrac{(n+1)(n+2)(n+3)\dots(n+n)}{n^n}\right]^{1/n}$.

25. Evaluate $\lim_{n\to\infty}\left[\left(1+\dfrac{1}{n^4}\right)\left(1+\dfrac{2^4}{n^4}\right)^{1/2}\left(1+\dfrac{3^4}{n^4}\right)^{1/3}\dots\left(1+\dfrac{n^4}{n^4}\right)^{1/n}\right]$.

26. Evaluate $\lim_{n\to\infty}\left[\sin\dfrac{\pi}{2n}\sin\dfrac{2\pi}{2n}\sin\dfrac{3\pi}{2n}\dots\sin\dfrac{n\pi}{2n}\right]^{1/n}$.

27. Evaluate $\lim_{n\to\infty}\left[\tan\dfrac{\pi}{2n}\tan\dfrac{2\pi}{2n}\tan\dfrac{3\pi}{2n}\dots\tan\dfrac{n\pi}{2n}\right]^{1/n}$. **(Kumaun 2012)**

28. Evaluate the limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{2/n^2}\left(1+\frac{2^2}{n^2}\right)^{4/n^2}\left(1+\frac{3^2}{n^2}\right)^{6/n^2}\dots\left(1+\frac{n^2}{n^2}\right)^{2n/n^2}.$$

(Kumaun 2009)

Answers 3

1. $\log 2$ **2.** $(1/m)\log(1+m)$ **3.** $\dfrac{1}{2}$

4. $\dfrac{2}{3}[2\sqrt{2}-1]$ **5.** $\log\dfrac{3}{2}$ **6.** $\dfrac{\pi}{4}$

7. $\dfrac{1}{2}\log 2+\dfrac{\pi}{4}$ **8.** $\dfrac{1}{3}$ **9.** $\dfrac{3}{8}$ **10.** $\dfrac{1}{3}$

11. $\dfrac{1}{2}\pi$ **12.** $\dfrac{1}{2}\tan 1$ **13.** $\dfrac{1}{4}\log 2$ **14.** $\dfrac{\pi}{2}+1$ **15.** $\dfrac{1}{14}$

16. $\dfrac{1}{\sqrt{2}}$ **17.** $\dfrac{\pi}{2}$ **18.** $\dfrac{\pi}{3}$ **19.** $\dfrac{3}{2}$ **20.** $\log(b/a)$

21. $\dfrac{1}{11}$ **23.** $4/e$ **24.** $4/e$ **25.** $e^{\pi^2/48}$ **26.** $\dfrac{1}{2}$

27. 1 **28.** $4/e$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The value of the integral $\int_{-\pi/2}^{\pi/2}\sin^2 x\,dx$ is

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{4}$ (c) $\dfrac{\pi}{8}$ (d) $\dfrac{1}{2}$

2. The value of the limit, $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^3}{r^4+n^4}$ is

(a) $\log 2$ (b) $\frac{1}{4}\log 2$ (c) $\frac{1}{2}\log 2$ (d) $\frac{1}{8}\log 2$

3. The value of the integral $\int_{-\pi/2}^{\pi/2} \sin^3 x\, dx$ shall be

(a) 1 (b) 0 (c) −1 (d) π

(Kumaun 2010, 12)

4. The integral $\int_0^{\pi} f(\sin x)\, dx$ is equal to

(a) $2\int_0^{\pi/2} f(\sin x)\, dx$ (b) $\int_0^{\pi} f(\cos x)\, dx$

(c) $\int_0^{\pi/2} f(\cos x)\, dx$ (d) $\int_0^{\pi/2} f(\sin x)\, dx$

(Kumaun 2010, 13)

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. If $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)\, dx = \ldots\ldots$.

2. If $f(2a-x) = -f(x)$, then $\int_0^{2a} f(x)\, dx = \ldots\ldots$.

3. If $f(-x) = f(x)$, then $\int_{-a}^{a} f(x)\, dx = 2\ldots\ldots$.

4. If $f(2a-x) = f(x)$, then $\int_0^{2a} f(x)\, dx = 2\ldots\ldots$.

5. $\int_{-\pi/2}^{\pi/2} \sin^3 x \cos^2 x\, dx = \ldots\ldots$.

6. $\int_{-1}^{1} \frac{x^2 \sin^{-1} x}{\sqrt{(1-x^2)}}\, dx = \ldots\ldots$.

7. $\int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x}\, dx = \ldots\ldots$.

8. $\int_0^{\pi/2} \frac{dx}{1+\sqrt{(\tan x)}} = \ldots\ldots$.

9. $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx = \ldots\ldots$.

10. $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1+(r/n)} = \ldots\ldots$.

True or False

Write 'T' for true and 'F' for false statement.

1. $\int_0^a f(x)\,dx = \int_0^a f(a+x)\,dx.$

2. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$

3. If m is a positive integer, then

$$\int_0^{\pi} \sin^m x \cos^{2m+1} x\,dx = 0.$$

4. $\int_0^{\pi/2} \frac{x\,dx}{\sin x + \cos x} = \frac{\pi}{2}\int_0^{\pi/2} \frac{dx}{\sin x + \cos x}.$

5. $\int_{-\pi/2}^{\pi/2} \cos^3 x\,dx = 0.$

6. $\lim_{n\to\infty} \sum_{r=0}^{2n} \frac{1}{n+r} = \int_0^1 \frac{1}{1+x}\,dx.$

7. $\lim_{n\to\infty} \sum_{r=0}^{n+1} \frac{1}{n}\left[\frac{1}{1+(r/n)^2}\right] = \frac{\pi}{4}.$

8. $\lim_{n\to\infty} \left[\frac{1}{n^3}(1+4+9+16+\ldots+n^2)\right] = \frac{1}{2}.$

Answers

Multiple Choice Questions

1. (a) 2. (b) 3. (b) 4. (a)

Fill in the Blank(s)

1. 0 2. 0 3. $\int_0^a f(x)\,dx$ 4. $\int_0^a f(x)\,dx$

5. 0 6. 0 7. 0 8. $\frac{\pi}{4}$

9. $\frac{\pi}{4}$ 10. log 2

True or False

1. *F* 2. *T* 3. *T* 4. *F* 5. *F*

6. *F* 7. *T* 8. *F*

□

Chapter 2

Improper Integrals (Infinite Integrals)

1 Some Definitions

1. Infinite Interval: The interval whose length (range) is infinite is said to be an *infinite interval*. Thus the intervals (a, ∞), $(-\infty, b)$ and $(-\infty, \infty)$ are infinite intervals.

2. Bounded Functions: A function $f(x)$ is said to be *bounded* over the interval I if there exist two real numbers a and b $(b > a)$ such that

$$a \leq f(x) \leq b \text{ for all } x \in I.$$

A function $f(x)$ is said to be unbounded at a point, if it becomes infinite at that point. Thus the function

$$f(x) = x / \{(x-1)(x-2)\}$$

is unbounded at each of the points $x = 1$ and $x = 2$.

3. Monotonic functions: A real valued function f defined on an interval I is said to be **monotonically** increasing if

$$x > y \Rightarrow f(x) > f(y) \ \forall \ x, y \in I$$

and **monotonically decreasing** if

$$x > y \Rightarrow f(x) < f(y) \ \forall \ x, y \in I.$$

A function f defined on an interval I is said to be a monotonic function if it is either monotonically decreasing or monotonically increasing on I.
For example the function f defined by $f(x) = \sin x$ is monotonically increasing in the interval $0 \leq x \leq \frac{1}{2}\pi$ and monotonically decreasing in the interval $\frac{1}{2}\pi \leq x \leq \pi$.

4. **Proper Integral:** The definite integral $\int_a^b f(x)\,dx$ is said to be a *proper integral* if the range of integration is finite and the integrand $f(x)$ is bounded. The integral $\int_0^{\pi/2} \sin x\,dx$ is a proper integral. Also $\int_0^1 \frac{\sin x}{x}\,dx$ is an example of a proper integral because $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

5. **Improper Integrals:** The definite integral $\int_a^b f(x)\,dx$ is said to be an *improper integral* if (*i*) the interval (a, b) is not finite (*i.e.*, is infinite) and the function $f(x)$ is bounded over this interval; or (*ii*) the interval (a, b) is finite and $f(x)$ is not bounded over this interval; or (*iii*) neither the interval (a, b) is finite nor $f(x)$ is bounded over it.

6. **Improper integrals of the first kind or infinite integrals:** A definite integral $\int_a^b f(x)\,dx$ in which the range of integration is infinite (*i.e.*, either $b = \infty$ or $a = -\infty$ or both) and the integrand $f(x)$ is bounded, is called an improper integral of the first kind or an infinite integral. Thus $\int_0^\infty \frac{dx}{1+x^2}$ is an improper integral of the first kind since the upper limit of integration is infinite and the integrand $1/(1+x^2)$ is bounded. Similarly $\int_{-\infty}^0 e^x\,dx$ is an example of an improper integral of the first kind because here the lower limit of integration is infinite. Also $\int_{-\infty}^\infty \frac{dx}{1+x^2}$ is an improper integral of the first kind.

In case the **interval (a, b) is infinite and the integrand $f(x)$ is bounded**, we define

(*i*) $\int_a^\infty f(x)\,dx = \lim_{x \to \infty} \int_a^x f(x)\,dx,$

provided that the limit exists finitely *i.e.*, the limit is equal to a definite real number.

(*ii*) $\int_{-\infty}^b f(x)\,dx = \lim_{x \to \infty} \int_{-x}^b f(x)\,dx,$

provided that the limit exists finitely.

(*iii*) $\int_{-\infty}^\infty f(x)\,dx = \lim_{x_1 \to \infty} \int_{-x_1}^c f(x)\,dx + \lim_{x_2 \to \infty} \int_c^{x_2} f(x)\,dx$

provided that both these limits exist finitely.

7. **Improper integrals of the second kind:** A definite integral $\int_a^b f(x)\,dx$ in which the range of integration is finite but the integrand $f(x)$ is unbounded at one or more points of the interval $a \leq x \leq b$, is called an improper integral of the second kind.

Thus $$\int_0^4 \frac{dx}{(x-2)(x-3)}$$

and $\int_0^1 \frac{1}{x^2}\, dx$ are improper integrals of the second kind.

In the case of the definite integral

$$\int_a^b f(x)\, dx,$$

if the range of integration (a, b) is finite and the integrand **$f(x)$ is unbounded at one or more points of the given interval**, we define the value of the integral as follows :

(i) If $f(x)$ is unbounded at $x = b$ only *i.e.*, if $f(x) \to \infty$ as $x \to b$ only, then we define

$$\int_a^b f(x)\, dx = \lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x)\, dx,$$

provided that the limit exists finitely. Here ε is a small positive number.

(ii) If $f(x) \to \infty$ as $x \to a$ only, then we define

$$\int_a^b f(x)\, dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b} f(x)\, dx,$$

provided that the limit exists finitely.

(iii) If $f(x) \to \infty$ as $x \to c$ only, where $a < c < b$, then we define

$$\int_a^b f(x)\, dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)\, dx + \lim_{\varepsilon' \to 0} \int_{c+\varepsilon'}^{b} f(x)\, dx,$$

provided that both these limits exist finitely.

(iv) If $f(x)$ is unbounded at both the points a and b of the interval (a, b) and is bounded at each other point of this interval, we write

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx,$$

where $a < c < b$ and the value of the integral exists only if each of the integrals on the right hand side exists.

2 Convergence of Improper Integrals

When the limit of an improper integral as defined above, is a definite finite number, we say that the given integral is **convergent** and the value of the integral is equal to the value of that limit. When the limit is ∞ or $-\infty$, the integral is said to be **divergent** *i.e.*, the value of the integral does not exist.

In case the limit is neither a definite number nor ∞ or $-\infty$, the integral is said to be **oscillatory** and in this case also the value of the integral does not exist *i.e.*, the integral is not convergent. We can define the convergence of the infinite integral $\int_a^\infty f(x)\, dx$ as follows :

Definition: *The integral* $\int_a^{\infty} f(x)\, dx$ *is said to converge to the value* I*, if for any arbitrarily chosen positive number* ε*, however small but not zero, there exists a corresponding positive number* N *such that*

$$\left| \int_a^b f(x)\, dx - I \right| < \varepsilon \text{ for all values of } b \geq N.$$

Similarly we can define the convergence of an integral, when the lower limit is infinite, or when the integrand becomes infinite at the upper or lower limit.

Illustrative Examples

Example 1: *Discuss the convergence of the following integrals by evaluating them*

(*i*) $\int_1^{\infty} \frac{dx}{\sqrt{x}}$, (*ii*) $\int_1^{\infty} \frac{dx}{x^{3/2}}$.

Solution: (i) We have

$$\int_1^{\infty} \frac{dx}{\sqrt{x}} = \lim_{x \to \infty} \int_1^x \frac{dx}{\sqrt{x}}, \text{ (By def.)}$$

$$= \lim_{x \to \infty} \int_1^x x^{-1/2}\, dx = \lim_{x \to \infty} \left[\frac{x^{1/2}}{1/2} \right]_1^x$$

$$= \lim_{x \to \infty} [2\sqrt{x} - 2] = \infty.$$

Thus the limit does not exist finitely and therefore the given integral is divergent (*i.e.*, the integral does not exist).

(ii) We have

$$\int_1^{\infty} \frac{dx}{x^{3/2}} = \lim_{x \to \infty} \int_1^x \frac{dx}{x^{3/2}}, \quad \text{(By def.)}$$

$$= \lim_{x \to \infty} \int_1^x x^{-3/2}\, dx = \lim_{x \to \infty} \left[\frac{x^{-1/2}}{-1/2} \right]_1^x = \lim_{x \to \infty} \left[-\frac{2}{\sqrt{x}} \right]_1^x$$

$$= \lim_{x \to \infty} \left[-\frac{2}{\sqrt{x}} + 2 \right] = 2.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent and its value is 2.

Example 2: *Test the convergence of* $\int_0^{\infty} e^{-mx}\, dx, \quad (m > 0)$.

Solution: We have $\int_0^{\infty} e^{-mx}\, dx = \lim_{x \to \infty} \int_0^{\infty} e^{-mx}\, dx, \quad \text{(by def.)}$

$$= \lim_{x \to \infty} \left[\frac{e^{-mx}}{-m} \right]_0^x = \lim_{x \to \infty} \left\{ -\frac{1}{m} (e^{-mx} - 1) \right\}$$

$$= -\frac{1}{m} [0 - 1] = \frac{1}{m}.$$

Thus the limit exists and is unique and finite, therefore the given integral is convergent.

Example 3: *Test the convergence of* $\int_0^{\infty} \frac{4a\,dx}{x^2 + 4a^2}$.

Solution: We have $\int_0^{\infty} \frac{4a\,dx}{x^2+4a^2} = \lim_{x \to \infty} \int_0^x \frac{4a\,dx}{x^2+(2a)^2}$, (By def.)

$$= \lim_{x \to \infty} \left[4a \cdot \frac{1}{2a} \tan^{-1} \frac{x}{2a}\right]_0^x = 2 \lim_{x \to \infty} \left[\tan^{-1} \frac{x}{2a}\right]_0^x$$

$$= 2 \,.\, \lim_{x \to \infty} \left[\tan^{-1} \frac{x}{2a} - 0\right] = 2 \,.\, [\tan^{-1} \infty] = 2 \,.\, \frac{\pi}{2} = \pi.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent.

Example 4: *Test the convergence of* (*i*) $\int_{-\infty}^{0} e^{x}\,dx$; (*ii*) $\int_{-\infty}^{0} e^{-x}\,dx$.

Solution: (i) We have

$$\int_{-\infty}^{0} e^x\,dx = \lim_{x \to \infty} \int_{-x}^{0} e^x\,dx, \quad \text{(By def.)}$$

$$= \lim_{x \to \infty} [e^x]_{-x}^{0} = \lim_{x \to \infty} [1 - e^{-x}] = [1 - 0] = 1.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent.

(ii) We have $\int_{-\infty}^{0} e^{-x}\,dx = \lim_{x \to \infty} \int_{-x}^{0} e^{-x}\,dx$, (By def.)

$$= \lim_{x \to \infty} \left[\frac{e^{-x}}{-1}\right]_{-x}^{0} = -\lim_{x \to \infty} [e^0 - e^x] = \infty.$$

Thus the limit does not exist finitely and therefore the given integral is divergent(*i.e.*, the integral does not exist).

Example 5: *Test the convergence of* $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

(Kanpur 2008; Gorakhpur 11)

Solution: We have

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2}$$

$$= \lim_{x \to \infty} \int_{-x}^{0} \frac{dx}{1+x^2} + \lim_{x \to \infty} \int_0^x \frac{dx}{1+x^2}$$

$$= \lim_{x \to \infty} [\tan^{-1} x]_{-x}^{0} + \lim_{x \to \infty} [\tan^{-1} x]_0^x$$

$$= \lim_{x \to \infty} [0 - \tan^{-1} (-x)] + \lim_{x \to \infty} [\tan^{-1} x - 0]$$

$$= -(-\pi/2) + \pi/2 = \pi.$$

Thus the limit exists and is unique and finite; therefore the given integral is convergent.

Example 6: *Evaluate* $\int_0^1 \frac{dx}{\sqrt{x}}$.

(Gorakhpur 2010)

Solution: In the given integral, the integrand $1/\sqrt{x}$ becomes infinite at the lower limit $x = 0$. Therefore we have

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0} \int_{0+\varepsilon}^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0} [2\sqrt{x}]_{\varepsilon}^1$$

$$= \lim_{\varepsilon \to 0} [2 - 2\sqrt{\varepsilon}] = 2.$$

Hence the given integral is convergent and its value is 2.

Example 7: *Evaluate* $\int_0^1 \frac{dx}{\sqrt{(1-x)}}$.

Solution: Here the integrand *i.e.*, $1/\sqrt{(1-x)}$ becomes unbounded *i.e.*, infinite at the upper limit (*i.e.*, $x = 1$).

$$\therefore \quad \int_0^1 \frac{dx}{\sqrt{(1-x)}} = \lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} \frac{dx}{\sqrt{(1-x)}}$$

$$= \lim_{\varepsilon \to 0} [-2\sqrt{(1-x)}]_0^{1-\varepsilon} = \lim_{\varepsilon \to 0} [-2\sqrt{\varepsilon} + 2] = 2,$$

which is a definite real number. Hence the given integral is convergent and its value is 2.

Example 8: *Evaluate* $\int_{-1}^1 \frac{dx}{x^2}$.

Solution: Here the integrand becomes infinite at $x = 0$ and $-1 < 0 < 1$.

$$\therefore \quad \int_{-1}^1 \frac{dx}{x^2} = \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} \frac{dx}{x^2} + \lim_{\varepsilon' \to 0} \int_{\varepsilon'}^1 \frac{dx}{x^2}$$

$$= \lim_{\varepsilon \to 0} \left[-\frac{1}{x}\right]_{-1}^{-\varepsilon} + \lim_{\varepsilon' \to 0} \left[-\frac{1}{x}\right]_{\varepsilon'}^1$$

$$= \lim_{\varepsilon \to 0} \left[\frac{1}{\varepsilon} - 1\right] + \lim_{\varepsilon' \to 0} \left[-1 + \frac{1}{\varepsilon'}\right].$$

Since both the limits do not exist finitely, therefore the integral does not exist and is divergent.

Comprehensive Exercise 1

Evaluate the following integrals and discuss their convergence :

1. $\int_1^\infty \frac{dx}{x}$.
2. $\int_3^\infty \frac{dx}{(x-2)^2}$.
3. $\int_0^\infty e^{2x}\, dx$.
4. $\int_0^\infty \frac{dx}{(1+x)^{2/3}}$.
5. $\int_{-\infty}^0 \sinh x\, dx$.
6. $\int_{-\infty}^0 \cosh x\, dx$.

7. $\int_0^\infty \cos x \, dx$.

8. $\int_{-\infty}^{\infty} e^{-x} \, dx$.

9. $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2}$.

10. $\int_0^1 \frac{dx}{x^3}$.

11. $\int_0^1 \frac{dx}{1 - x}$.

12. $\int_{-1}^{1} \frac{dx}{x^{2/3}}$.

(Gorakhpur 2011)

Answers 1

1. ∞, divergent
2. 1, convergent
3. ∞, divergent
4. ∞, divergent
5. $-\infty$, divergent
6. ∞, divergent
7. Oscillates and so not convergent
8. ∞, divergent
9. π, convergent
10. ∞, divergent
11. ∞, divergent
12. 6, convergent

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The integral $\int_1^\infty \frac{dx}{\sqrt{x}}$ is

(a) convergent
(b) divergent
(c) uniformly convergent
(d) none of these

2. The integral $\int_{-\infty}^{0} e^x \, dx$ is

(a) convergent
(b) divergent
(c) uniformly convergent
(d) none of these

3. Value of the integral $\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$ is

(a) $\pi / 2$
(b) $-\pi / 2$
(c) π
(d) $-\pi$

4. Value of the integral $\int_0^1 \frac{dx}{\sqrt{(1 - x)}}$ is

(a) 2
(b) -2
(c) 1
(d) -1

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. The definite integral $\int_a^b f(x)\,dx$ is said to be a ...... if the range of integration (a, b) is finite and the integrand $f(x)$ is bounded over (a, b).
2. The definite integral $\int_a^b f(x)\,dx$ is said to be an improper integral if the interval (a, b) is finite and $f(x)$ is not ...... over this interval.
3. The definite integral $\int_a^b f(x)\,dx$ is said to be an ...... if the interval (a, b) is not finite and $f(x)$ is bounded over (a, b).

True or False

Write 'T' for true and 'F' for false statement.

1. A definite integral $\int_a^b f(x)\,dx$ in which the range of integration (a, b) is finite but the integrand $f(x)$ is unbounded at one or more points of the interval $a \le x \le b$, is called an improper integral of the second kind.
2. The integral $\int_0^\infty \frac{dx}{1+x^2}$ is an improper integral of the second kind.
3. The integral $\int_0^4 \frac{dx}{(x-2)(x-3)}$ is an improper integral of the first kind.

Answers

Multiple Choice Questions

1. (b) 2. (a) 3. (c) 4. (a)

Fill in the Blank(s)

1. proper integral 2. bounded
3. improper integral of the first kind

True or False

1. T 2. F 3. T

□

Chapter

3

Differentiation and Integration Under the Sign of Integration

1 Method of Differentiation Under the Sign of Integration

By this method the value of a definite integral can be determined by differentiating the integrand with respect to a quantity of which the limits of integration are independent. This method can also be applied to indefinite integrals. It is a very important method of integration. By this method the values of many definite integrals can be determined which are otherwise not easily integrable. This method is applied in two ways. First, new integrals can be deduced by differentiating certain known integrals. Secondly, the value of a given integral can be found by first differentiating the integrand, then evaluating the new integral thus obtained and finally integrating the result with respect to the same quantity with respect to which the integrand was first differentiated.

The following examples will illustrate the method.

Illustrative Examples

Example 1: *Evaluate* $\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx.$

Solution: Let $u=\int_0^{\infty}\dfrac{\tan^{-1}(ax)}{x(1+x^2)}\,dx.$ ...(1)

Here the integrand $\dfrac{\tan^{-1}(ax)}{x(1+x^2)}$ is a function of two variables x and a. Obviously u is a function of a. So differentiating both sides of (1) with respect to a, we get

$$\frac{du}{da}=\frac{d}{da}\left[\int_0^{\infty}\frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx\right]=\int_0^{\infty}\left[\frac{\partial}{\partial a}\left\{\frac{\tan^{-1}(ax)}{x(1+x^2)}\right\}\right]dx$$

$$=\int_0^{\infty}\frac{1}{x(1+x^2)}\cdot\frac{1}{1+a^2x^2}\cdot x\,dx=\int_0^{\infty}\frac{dx}{(1+x^2)(1+a^2x^2)}$$

$$=\int_0^{\infty}\left[\frac{1}{(1-a^2)(1+x^2)}-\frac{a^2}{(1-a^2)(1+a^2x^2)}\right]dx,$$

resolving the integrand into partial fractions

$$=\frac{1}{(1-a^2)}[\tan^{-1}x]_0^{\infty}-\frac{a^2}{1-a^2}\cdot\frac{1}{a}[\tan^{-1}ax]_0^{\infty}$$

$$=\frac{1}{(1-a^2)}\cdot\frac{\pi}{2}-\frac{a}{1-a^2}\left[\frac{\pi}{2}\right]$$ [Assuming that a is positive]

$$=\frac{\pi}{2}\cdot\frac{1}{1-a^2}(1-a)=\frac{\pi}{2(1+a)}\cdot$$

Thus $$\frac{du}{da}=\frac{\pi}{2(1+a)}\cdot$$

Integrating both sides with respect to a, we get

$$u=\frac{\pi}{2}\log(1+a)+C.$$ ...(2)

When $a=0$, we have from (1) $u=0$.

$\therefore$ from (2), $0=0+C$ or $C=0$.

Putting $C=0$ in (2), we get $u=\frac{1}{2}\pi\log(1+a)$.

Hence $$\int_0^{\infty}\frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx=\frac{\pi}{2}\log(1+a),\text{ if } a>0.$$

Case when a is negative :

If a is negative, we have

$$\frac{du}{da}=\frac{1}{(1-a^2)}[\tan^{-1}x]_0^{\infty}-\frac{a^2}{1-a^2}\cdot\frac{1}{a}[\tan^{-1}ax]_0^{\infty}$$

$$=\frac{1}{(1-a^2)}\left[\frac{\pi}{2}-a\left(-\frac{\pi}{2}\right)\right]$$ $\left[\because \tan^{-1}(-\infty)=-\frac{\pi}{2}\right]$

$$=\frac{\pi}{2(1-a)}\cdot$$

Integrating both sides w.r.t. a, we get

$$u = -\frac{\pi}{2} \log (1-a) + C_1 . \qquad \ldots(3)$$

Again, when $a = 0, u = 0$. Putting these values in (3), we have $C_1 = 0$.

$\therefore \qquad u = -\frac{\pi}{2} \log (1-a).$

Hence $\quad \int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^2)} dx = -\frac{\pi}{2} \log (1-a)$, if $a < 0$.

Example 2: *Show that*

$$\int_0^{\pi/2} \log (\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta)\, d\theta = \pi \log \left(\frac{\alpha+\beta}{2}\right).$$

(Garhwal 2000)

Solution: Let $u = \int_0^{\pi/2} \log (\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta)\, d\theta. \qquad \ldots(1)$

Then $\quad \frac{du}{d\alpha} = \int_0^{\pi/2} \frac{2\alpha \cos^2 \theta}{\alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta} d\theta = \int_0^{\pi/2} \frac{2\alpha \cos^2 \theta}{(\alpha^2 - \beta^2) \cos^2 \theta + \beta^2} d\theta$

$$= \frac{2\alpha}{(\alpha^2-\beta^2)} \int_0^{\pi/2} \frac{(\alpha^2-\beta^2)\cos^2\theta + \beta^2 - \beta^2}{(\alpha^2-\beta^2)\cos^2\theta + \beta^2} d\theta$$

$$= \frac{2\alpha}{(\alpha^2-\beta^2)} \int_0^{\pi/2} \left[1 - \frac{\beta^2}{(\alpha^2-\beta^2)\cos^2\theta + \beta^2}\right] d\theta$$

$$= \frac{2\alpha}{(\alpha^2-\beta^2)} \int_0^{\pi/2} \left[1 - \frac{\beta^2 \sec^2\theta}{(\alpha^2-\beta^2) + \beta^2 \sec^2\theta}\right] d\theta$$

$$= \frac{2\alpha}{(\alpha^2-\beta^2)} \int_0^{\pi/2} \left[1 - \frac{\beta^2 \sec^2\theta}{\alpha^2 + \beta^2 \tan^2\theta}\right] d\theta$$

$$= \frac{2\alpha}{\alpha^2-\beta^2} \left[\theta - \beta \cdot \frac{1}{\alpha} \tan^{-1}\left(\frac{\beta \tan\theta}{\alpha}\right)\right]_0^{\pi/2}$$

[Putting $\beta \tan\theta = t$ so that $\beta \sec^2\theta\, d\theta = dt$]

$$= \frac{2\alpha}{\alpha^2-\beta^2}\left[\frac{\pi}{2} - \frac{\beta}{\alpha}\cdot\frac{\pi}{2}\right] = \frac{\pi}{\alpha+\beta}.$$

Thus $\quad \frac{du}{d\alpha} = \frac{\pi}{\alpha+\beta}.$

Integrating both sides with respect to α, we get

$$u = \pi \log (\alpha + \beta) + C. \qquad \ldots(2)$$

From (1), when $\alpha = \beta$, we have

$$u = \int_0^{\pi/2} \log \{\alpha^2 (\cos^2\theta + \sin^2\theta)\}\, d\theta = \int_0^{\pi/2} \log \alpha^2\, d\theta$$

$$= \frac{\pi}{2} \log \alpha^2 = \pi \log \alpha.$$

So putting $\beta = \alpha$ in (2), we get

$$\pi \log \alpha = \pi \log (2\alpha) + C \quad \text{or} \quad C = \pi \log \frac{1}{2} \cdot$$

Hence putting $C = \pi \log \frac{1}{2}$ in (2), we get

$$u = \pi \log (\alpha + \beta) + \pi \log \frac{1}{2} = \pi \log \left(\frac{\alpha + \beta}{2} \right) \cdot$$

Example 3: *Show that* $\int_0^{\pi/2} \frac{dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} = \frac{\pi (a^2 + b^2)}{4 a^3 b^3} \cdot$

Solution: Let us first evaluate

$$u = \int_0^{\pi/2} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} \cdot$$

We have
$$u = \int_0^{\pi/2} \frac{\sec^2 x \, dx}{a^2 \tan^2 x + b^2} = \frac{1}{a} \cdot \frac{1}{b} \left[\tan^{-1} \left(\frac{a \tan x}{b} \right) \right]_0^{\pi/2},$$

putting $a \tan x = t$ so that $a \sec^2 x \, dx = dt$

$$= \frac{1}{ab} \cdot \frac{\pi}{2} = \frac{\pi}{2ab} \cdot$$

$\therefore$
$$\int_0^{\pi/2} \frac{dx}{a^2 \sin^2 x + b^2 \cos^2 x} = \frac{\pi}{2ab} \cdot \qquad \ldots(1)$$

Differentiating both sides of (1) w.r.t. 'a', we get

$$\int_0^{\pi/2} \frac{-2a \sin^2 x}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} dx = -\frac{\pi}{2a^2 b}$$

or
$$\int_0^{\pi/2} \frac{\sin^2 x \, dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} = \frac{\pi}{4a^3 b} \cdot \qquad \ldots(2)$$

Similarly differentiating both sides of (1) w.r.t. 'b', we get

$$\int_0^{\pi/2} \frac{\cos^2 x \, dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} = \frac{\pi}{4ab^3} \cdot \qquad \ldots(3)$$

Adding (2) and (3), we get

$$\int_0^{\pi/2} \frac{dx}{(a^2 \sin^2 x + b^2 \cos^2 x)^2} = \frac{\pi}{4a^3 b} + \frac{\pi}{4ab^3} = \frac{\pi (a^2 + b^2)}{4a^3 b^3} \cdot$$

Example 4: *Prove that* $\int_0^{\infty} \frac{e^{-a x} \sin bx}{x} dx = \tan^{-1} \frac{b}{a} \cdot$

Hence also deduce the value of $\int_0^{\infty} \frac{\sin bx}{x} dx.$

Solution: Let $u = \int_0^\infty \frac{e^{-ax} \sin bx}{x} dx.$ ...(1)

Then $$\frac{du}{db} = \int_0^\infty \frac{e^{-ax} x \cos bx}{x} dx = \int_0^\infty e^{-ax} \cos bx \, dx$$

$$= \frac{1}{a^2 + b^2} \left[e^{-ax} (-a \cos bx + b \sin bx) \right]_0^\infty$$

$$\left[\because \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) \right]$$

$$= \frac{1}{a^2 + b^2} \cdot \lim_{x \to \infty} [e^{-ax} (-a \cos bx + b \sin bx)] - \frac{1}{(a^2 + b^2)} [-a]$$

$$= \frac{a}{a^2 + b^2}, \text{ if } a > 0. \qquad \left[\because \text{if } a > 0, \text{then } \lim_{x \to \infty} e^{-ax} = 0 \right]$$

Thus $$\frac{du}{db} = \frac{a}{a^2 + b^2}.$$

Integrating both sides w.r.t. 'b', we get

$$u = a \cdot \frac{1}{a} \tan^{-1} \frac{b}{a} + C = \tan^{-1} \frac{b}{a} + C. \qquad ...(2)$$

From (1), when $b = 0$, we have $u = 0$.

$\therefore$ from (2), we have

$$0 = 0 + C \quad \text{or} \quad C = 0.$$

Putting $C = 0$ in (2), we get

$$u = \tan^{-1} (b / a).$$

Thus $$\int_0^\infty \frac{e^{-ax} \sin bx}{x} dx = \tan^{-1} \left(\frac{b}{a} \right). \qquad ...(3)$$

Now putting $a = 0$ on both sides of (3), we get

$$\int_0^\infty \frac{\sin bx}{x} dx = \begin{cases} \pi / 2, & \text{if } b > 0 \\ -\pi / 2, & \text{if } b < 0. \end{cases}$$

$$[\because \tan^{-1} \infty = \pi / 2 \text{ and } \tan^{-1} (-\infty) = -\pi / 2]$$

Example 5: *Show that* $\int_0^\infty \left(\frac{\sin ax}{x} \right)^2 dx = \frac{\pi a}{2}, a > 0.$

Solution: We have

$$\int_0^\infty \left(\frac{\sin ax}{x} \right)^2 dx = \int_0^\infty \frac{1}{x^2} \sin^2 ax \, dx$$

$$= \left[\left(-\frac{1}{x} \right) \sin^2 ax \right]_0^\infty - \int_0^\infty \left(-\frac{1}{x} \right) 2a \sin ax \cos ax \, dx,$$

integrating by parts taking $1 / x^2$ as the second function

$$= 0 + a \int_0^{\infty} \frac{\sin 2ax}{x}\, dx \quad \left[\because \lim_{x \to \infty} \frac{1}{x} \sin^2 ax = 0 \text{ and } \lim_{x \to 0} \frac{\sin^2 ax}{x} = 0 \right]$$

$$= a \cdot \frac{\pi}{2} \quad \left[\because \text{ from Ex. 4, we have } \int_0^{\infty} \frac{\sin bx}{x}\, dx = \frac{\pi}{2}, \text{ if } b > 0 \right]$$

Example 6: *Show that* $\displaystyle\int_0^{\infty} \frac{dx}{(x^2 + a)^{n+1}} = \frac{\pi}{2} \cdot \frac{1.3. \le (2n-1)}{2^n\, n!\, a^{(2n+1)/2}}.$

Solution: We have $\displaystyle\int_0^{\infty} \frac{dx}{x^2 + a} = \frac{1}{\sqrt{a}} \left[\tan^{-1} \frac{x}{\sqrt{a}} \right]_0^{\infty} = \frac{1}{\sqrt{a}} \cdot \frac{\pi}{2} = \frac{\pi}{2} a^{-1/2}.$

Differentiating both sides n times w.r.t. 'a', we get

$$\int_0^{\infty} \frac{(-1)^n\, n!}{(x^2 + a)^{n+1}}\, dx = \frac{\pi}{2} \frac{(-1)^n\, 1.3.5. \le (2n-1)}{2^n\, a^{(2n+1)/2}}$$

or

$$\int_0^{\infty} \frac{dx}{(x^2 + a)^{n+1}} = \frac{\pi}{2} \frac{1.3.5. \le (2n-1)}{2^n\, n!\, a^{(2n+1)/2}}.$$

Example 7: *Prove that* $\displaystyle\int_0^{\infty} \frac{\tan^{-1} \alpha x \tan^{-1} \beta x}{x^2}\, dx = \frac{1}{2} \pi \log \left\{ \frac{(\alpha + \beta)^{\alpha + \beta}}{\alpha^{\alpha}\, \beta^{\beta}} \right\}.$

Solution: Let $\displaystyle u = \int_0^{\infty} \frac{\tan^{-1} \alpha x \tan^{-1} \beta x}{x^2}\, dx.$ ...(1)

Then $$\frac{\partial u}{\partial \alpha} = \int_0^{\infty} \frac{x}{1 + \alpha^2 x^2} \frac{\tan^{-1} \beta x}{x^2}\, dx = \int_0^{\infty} \frac{\tan^{-1} \beta x}{x (1 + \alpha^2 x^2)} \quad \text{...(2)}$$

$\therefore$ $$\frac{\partial^2 u}{\partial \alpha\, \partial \beta} = \int_0^{\infty} \frac{dx}{(1 + \alpha^2 x^2)(1 + \beta^2 x^2)}$$

$$= \int_0^{\infty} \frac{1}{\alpha^2 - \beta^2} \left[\frac{\alpha^2}{1 + \alpha^2 x^2} - \frac{\beta^2}{1 + \beta^2 x^2} \right] dx,$$

resolving into partial fractions

$$= \frac{1}{\alpha^2 - \beta^2} \left[\alpha^2 \cdot \frac{1}{\alpha} \tan^{-1} \alpha x - \beta^2 \cdot \frac{1}{\beta} \tan^{-1} \beta x \right]_0^{\infty}$$

$$= \frac{1}{\alpha^2 - \beta^2} \left[\alpha \cdot \frac{\pi}{2} - \beta \cdot \frac{\pi}{2} \right] = \frac{\pi}{2} \frac{\alpha - \beta}{\alpha^2 - \beta^2} = \frac{\pi}{2 (\alpha + \beta)}.$$

Integrating both sides w.r.t. 'β', we get

$$\frac{\partial u}{\partial \alpha} = \frac{\pi}{2} \log (\alpha + \beta) + C \quad \text{...(3)}$$

But when $\beta = 0$, we have from (2), $\dfrac{\partial u}{\partial \alpha} = 0.$

$\therefore$ from (3), $0 = \dfrac{1}{2} \pi \log \alpha + C$ or $C = -\dfrac{1}{2} \pi \log \alpha.$

Putting this value of C in (3), we get

$$\frac{\partial u}{\partial \alpha} = \frac{\pi}{2} \log (\alpha + \beta) - \frac{\pi}{2} \log \alpha.$$

Now integrating both sides w.r.t. 'α', we get

$$u = \frac{\pi}{2} \left[\left\{ \alpha \log (\alpha + \beta) - \int \frac{\alpha}{\alpha + \beta} \, d\alpha \right\} - \left\{ \alpha \log \alpha - \int \alpha \cdot \frac{1}{\alpha} \, d\alpha \right\} \right] + C_1$$

$$= \frac{\pi}{2} \left[\alpha \log (\alpha + \beta) - \int \left(1 - \frac{\beta}{\alpha + \beta} \right) d\alpha - \alpha \log \alpha + \alpha \right] + C_1$$

$$= \frac{\pi}{2} [\alpha \log (\alpha + \beta) - \alpha + \beta \log (\alpha + \beta) - \alpha \log \alpha + \alpha] + C_1$$

$$= \frac{\pi}{2} [(\alpha + \beta) \log (\alpha + \beta) - \alpha \log \alpha] + C_1 \qquad \ldots (4)$$

From (1), when $\alpha = 0$, we have $u = 0$.

$\therefore$ from (4), $\quad 0 = \frac{\pi}{2} \beta \log \beta + C_1 \qquad \left[\because \lim_{\alpha \to 0} \alpha \log \alpha = 0 \right]$

or $\quad C_1 = -\frac{\pi}{2} \beta \log \beta.$

Putting this value of C_1 in (4), we get

$$u = \frac{\pi}{2} [(\alpha + \beta) \log (\alpha + \beta) - \alpha \log \alpha] - \frac{\pi}{2} \beta \log \beta$$

$$= \frac{\pi}{2} [(\alpha + \beta) \log (\alpha + \beta) - (\alpha \log \alpha + \beta \log \beta)]$$

$$= \frac{\pi}{2} \cdot \log \left\{ \frac{(\alpha + \beta)^{\alpha + \beta}}{\alpha^{\alpha} \beta^{\beta}} \right\}.$$

Example 8: *Evaluate* $\int_0^\infty \frac{\cos mx}{1 + x^2} \, dx, m > 0$ *and deduce the value of* $\int_0^\infty \frac{x \sin mx}{1 + x^2} \, dx.$

Solution: Let $\quad u = \int_0^\infty \frac{\cos mx}{1 + x^2} \, dx. \qquad \ldots (1)$

Then

$$\frac{du}{dm} = -\int_0^\infty \frac{x \sin mx}{1 + x^2} \, dx = -\int_0^\infty \frac{x^2 \sin mx}{x (1 + x^2)} \, dx$$

$$= -\int_0^\infty \frac{\{(1 + x^2) - 1\} \sin mx}{x (1 + x^2)} \, dx$$

$$= -\int_0^\infty \frac{\sin mx}{x} \, dx + \int_0^\infty \frac{\sin mx}{x (1 + x^2)} \, dx$$

$$= -\frac{\pi}{2} + \int_0^\infty \frac{\sin mx}{x (1 + x^2)} \, dx. \qquad \ldots (2)$$

Differentiating both sides of (2) w.r.t. m, we get

$$\frac{d^2 u}{dm^2} = \int_0^\infty \frac{x \cos mx}{x (1 + x^2)} \, dx = \int_0^\infty \frac{\cos mx}{1 + x^2} \, dx = u$$

or $\quad (D^2 - 1)\, u = 0$, where $D \equiv \dfrac{d}{dm}\cdot$

The general solution of this differential equation is

$$u = A\, e^{m} + B\, e^{-m}.$$

$$\therefore \quad \frac{du}{dm} = A\, e^{m} - B\, e^{-m}.$$

Now when $m = 0$, from (1) we have

$$u = \int_0^\infty \frac{dx}{1+x^2} = [\tan^{-1} x\,]_0^\infty = \frac{\pi}{2}$$

and from (2), $\dfrac{du}{dm} = -\dfrac{\pi}{2}\cdot$

So from (3) and (4), we get

$$\frac{\pi}{2} = A + B \text{ and } -\frac{\pi}{2} = A - B.$$

Solving these equations for A and B, we get

$$A = 0 \text{ and } B = \pi / 2.$$

Putting these values of A and B in (3), we get

$$u = \frac{\pi}{2}\, e^{-m}.$$

Hence $\quad \displaystyle\int_0^\infty \frac{\cos mx}{1+x^2}\, dx = \frac{\pi}{2}\, e^{-m}.$

Differentiating both sides w.r.t. m, we get

$$-\int_0^\infty \frac{x \sin mx}{1+x^2}\, dx = -\frac{\pi}{2}\, e^{-m} \quad \text{or} \quad \int_0^\infty \frac{x \sin mx}{1+x^2}\, dx = \frac{\pi}{2}\, e^{-m}.$$

2 Method of Integration Under the Sign of Integration

This method is similar to that of differentiation. We illustrate the method by means of the following examples.

Remember: The order of integration can be changed if the limits of integration are independent of the variables.

Illustrative Examples

Example 9: *From the integral* $\displaystyle\int_0^1 x^n\, dx = \frac{1}{n+1}$, *prove that* $\displaystyle\int_0^1 \frac{x^a - x^b}{\log x}\, dx = \log\left(\frac{a+1}{b+1}\right)\cdot$

Solution: We have

$$\int_0^1 x^n\, dx = \frac{1}{n+1}\cdot \qquad \ldots(1)$$

Integrating both sides of (1) with respect to n between the limits $n = b$ to $n = a$, we get

$$\int_{n=b}^{a} \left[\int_0^1 x^n \, dx\right] dn = \int_b^a \frac{1}{n+1} \, dn$$

or $$\int_{x=0}^{1} \left[\int_{n=b}^{a} x^n \, dn\right] dx = [\log (n+1)]_{n=b}^{a},$$ changing the order of integration because the limits of integration of both x and n are constants

or $$\int_{x=0}^{1} \left[\frac{x^n}{\log x}\right]_{n=b}^{a} dx = \log (a+1) - \log (b+1)$$

or $$\int_0^1 \frac{x^a - x^b}{\log x} \, dx = \log \left(\frac{a+1}{b+1}\right).$$

Example 10: *Prove that* $\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x} \cos px \, dx = \frac{1}{2} \log \left(\frac{\beta^2 + p^2}{\alpha^2 + p^2}\right).$

Solution: We know that

$$\int_0^\infty e^{-ax} \cos px \, dx = \frac{a}{a^2 + p^2}. \qquad \ldots(1)$$

Integrating both sides of (1) w.r.t. 'a', between the limits $a = \alpha$ to $a = \beta$, we get

$$\int_{x=0}^{\infty} \left[\int_{a=\alpha}^{\beta} e^{-ax} \, da\right] \cos px \, dx = \frac{1}{2} \int_{a=\alpha}^{\beta} \frac{2a}{a^2 + p^2} \, da$$

or $$\int_{x=0}^{\infty} \left[\frac{e^{-ax}}{-x}\right]_{a=\alpha}^{\beta} \cos px \, dx = \frac{1}{2} [\log (a^2 + p^2)]_{a=\alpha}^{\beta}$$

or $$\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x} \cos px \, dx = \frac{1}{2} \log \left(\frac{\beta^2 + p^2}{\alpha^2 + p^2}\right).$$

Example11: *Prove that* $\int_0^\infty \frac{\sin bx}{x} \, dx = \frac{\pi}{2}$, *if* $b > 0$.

Solution: We have $\int_0^\infty e^{-zx} \, dz = \left[\frac{e^{-zx}}{-x}\right]_{z=0}^{\infty} = \frac{1}{x}. \qquad \ldots(1)$

Now multiplying both sides of (1) by $\sin bx$ and then integrating w.r.t. 'x' between the limits $x = 0$ to $x = \infty$, we get

$$\int_0^\infty \frac{\sin bx}{x} \, dx = \int_{z=0}^{\infty} \left[\int_{x=0}^{\infty} e^{-zx} \sin bx \, dx\right] dz$$

$$= \int_{z=0}^{\infty} \frac{b}{z^2 + b^2} \, dz$$

$$= \left[\tan^{-1} \frac{z}{b}\right]_{z=0}^{\infty} = \frac{\pi}{2}, \text{ if } b > 0.$$

Example 12: *Evaluate the integrals*

(i) $\int_0^\infty e^{-x^2}\, dx$ (ii) $\int_0^\infty \exp\left\{-\left(x^2+\frac{\alpha^2}{x^2}\right)\beta^2\right\} dx$

(iii) $\int_0^\infty e^{-a^2x^2} \cos 2bx\, dx.$

Solution: (i) Let $I=\int_0^\infty e^{-x^2}\, dx.$

Putting $x=\alpha\, y$, we have

$$I=\int_0^\infty \alpha . e^{-\alpha^2 y^2}\, dy, \text{ if } \alpha>0. \qquad \ldots(1)$$

Multiplying both sides of (1) by $e^{-\alpha^2}$, we have

$$I\, e^{-\alpha^2}=\int_0^\infty \alpha . \exp\{-\alpha^2(1+y^2)\}\, dy. \qquad \ldots(2)$$

Now integrating both sides of (2) w.r.t. 'α' between the limits $\alpha=0$ to $\alpha=\infty$, we have

$$I\int_0^\infty e^{-\alpha^2}\, d\alpha=\int_{y=0}^\infty \left[\int_{\alpha=0}^\infty \alpha . \exp\{-\alpha^2(1+y^2)\}\, d\alpha\right] dy$$

or

$$I^2=\int_{y=0}^\infty \left[\frac{e^{-\alpha^2(1+y^2)}}{-2(1+y^2)}\right]_{\alpha=0}^\infty dy$$

$$=\frac{1}{2}\int_0^\infty \frac{dy}{1+y^2}=\frac{1}{2}[\tan^{-1} y]_0^\infty=\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4}.$$

$\therefore$ $$I=\frac{\sqrt{\pi}}{2}.$$

Thus $$\int_0^\infty e^{-x^2}\, dx=\frac{\sqrt{\pi}}{2}.$$ **(Remember)**

Remark 1: $\int_{-\infty}^\infty e^{-x^2}\, dx=2\int_0^\infty e^{-x^2}\, dx=2\cdot\frac{\sqrt{\pi}}{2}=\sqrt{\pi}.$

Remark 2: From (1), we have

$$\int_0^\infty e^{-\alpha^2 y^2}\, dy=\frac{I}{\alpha}=\frac{1}{\alpha}\cdot\frac{\sqrt{\pi}}{2}=\frac{\sqrt{\pi}}{2\alpha}, \text{ if } \alpha>0.$$

(ii) Let $$I=\int_0^\infty \exp\left\{-\beta^2\left(x^2+\frac{\alpha^2}{x^2}\right)\right\} dx. \qquad \ldots(1)$$

Then $$\frac{dI}{d\alpha}=-2\int_0^\infty \exp\left\{-\beta^2\left(x^2+\frac{\alpha^2}{x^2}\right)\right\}\cdot\frac{\alpha\beta^2}{x^2}\, dx.$$

Putting $\frac{\alpha}{x}=z$ so that $-\frac{\alpha}{x^2}\, dx=dz$ and adjusting the limits, we get

$$\frac{dI}{d\alpha}=-2\beta^2\int_0^\infty \exp\left\{-\left(z^2+\frac{\alpha^2}{z^2}\right)\beta^2\right\} dz=-2I\beta^2$$

$\therefore \quad \dfrac{dI}{I} = -2\beta^2 \, d\alpha.$

Integrating both sides, we get

$$\int \frac{dI}{I} = -2\beta^2 \int d\alpha + \log C$$

or $\quad \log I = -2\alpha\beta^2 + \log C$

or $\quad I = C\, e^{-2\alpha\beta^2}. \qquad \ldots(2)$

To determine C, let $\alpha = 0$; then from (1), we have

$$I = \int_0^\infty e^{-\beta^2 x^2}\, dx = \frac{\sqrt{\pi}}{2\beta}.$$

$\therefore$ from (2), we have $\dfrac{\sqrt{\pi}}{2\beta} = C.$

Putting this value of C in (2), we get

$$I = \int_0^\infty \exp\left\{-\beta^2\left(x^2 + \frac{\alpha^2}{x^2}\right)\right\} dx = \frac{\sqrt{\pi}}{2\beta}\, e^{-2\alpha\beta^2}.$$

Remark: If we put $\beta = 1$, we have

$$\int_0^\infty \exp\left\{-\left(x^2 + \frac{\alpha^2}{x^2}\right)\right\} dx = \frac{\sqrt{\pi}}{2}\, e^{-2\alpha}.$$

(iii) Let $\quad I = \displaystyle\int_0^\infty e^{-a^2x^2} \cos 2bx\, dx. \qquad \ldots(1)$

Then $\quad \dfrac{dI}{db} = \displaystyle\int_0^\infty e^{-a^2x^2}(-2x)\sin 2bx\, dx$

$$= \left[\frac{e^{-a^2x^2}}{a^2}\sin 2bx\right]_0^\infty - \int_0^\infty \frac{e^{-a^2x^2}}{a^2}\cdot 2b\cos 2bx\, dx,$$ integrating by parts taking $e^{-a^2x^2}(-2x)$ as the second function

$$= 0 - \frac{2b}{a^2}\int_0^\infty e^{-a^2x^2}\cos 2bx\, dx = -\frac{2b}{a^2} I.$$

$\therefore \quad \dfrac{dI}{I} = -\dfrac{2b}{a^2}\, db.$

Integrating both sides, we get

$$\int \frac{dI}{I} = -\frac{2}{a^2}\int b\, db + \log C$$

or $\quad \log I = -\dfrac{b^2}{a^2} + \log C \quad$ or $\quad I = C\, e^{-b^2/a^2}. \qquad \ldots(2)$

When $b = 0$, we have from (1)

$$I = \int_0^\infty e^{-a^2x^2}\, dx = \frac{\sqrt{\pi}}{2a}.$$

$\therefore$ from (2), we have $\dfrac{\sqrt{\pi}}{2a} = C.$

Putting this value of C in (2), we get

$$I = \int_0^\infty e^{-a^2x^2} \cos 2bx \, dx = \frac{\sqrt{\pi}}{2a} e^{-b^2/a^2}.$$

Example 13: *Evaluate the integral* $\int_0^\infty \dfrac{\cos mx}{a^2 + x^2} dx$ *and deduce the values of the integrals*

$$\int_0^\infty \frac{x \sin mx}{a^2 + x^2} dx \text{ and } \int_0^\infty \frac{\sin mx}{x(a^2 + x^2)} dx.$$

Solution: Let $I = \int_0^\infty \dfrac{\cos mx}{a^2 + x^2} dx.$

We have $$\int_0^\infty 2z \exp\{-(a^2 + x^2) z^2\} dz = \left[-\frac{e^{-(a^2+x^2)z^2}}{a^2 + x^2}\right]_{z=0}^\infty = \frac{1}{a^2 + x^2}.$$

Now multiplying both sides by $\cos mx$ and integrating from 0 to ∞ w.r.t. 'x', we have

$$I = \int_0^\infty \frac{\cos mx}{a^2 + x^2} dx = \int_0^\infty \int_0^\infty [\cos mx . 2z \exp\{-(a^2 + x^2) z^2\}] \, dz \, dx$$

$$= \int_0^\infty 2ze^{-a^2z^2} \left[\int_0^\infty e^{-x^2z^2} \cos mx \, dx\right] dz$$

$$= \int_0^\infty 2ze^{-a^2z^2} \frac{\sqrt{\pi}}{2z} \exp\left(-\frac{m^2}{4z^2}\right) dz \qquad \text{[See Ex. 4. part (iii)]}$$

$$= \sqrt{\pi} \int_0^\infty \exp\left(-a^2z^2 - \frac{m^2}{4z^2}\right) dz$$

$$= \sqrt{\pi} \int_0^\infty \exp\left\{-a^2\left(z^2 + \frac{m^2}{4a^2z^2}\right)\right\} dz$$

$$= \sqrt{\pi} \cdot \frac{\sqrt{\pi}}{2a} \exp\left(-2 \cdot \frac{m}{2a} \cdot a^2\right). \qquad \text{[See Ex. 4. part (ii)]}$$

Hence $$\int_0^\infty \frac{\cos mx}{a^2 + x^2} dx = \frac{\pi}{2a} e^{-ma}. \qquad \ldots(1)$$

Differentiating both sides of (1) w.r.t. 'm', we have

$$\int_0^\infty \frac{-x \sin mx}{a^2 + x^2} dx = -a \cdot \frac{\pi}{2a} e^{-ma}$$

or $$\int_0^\infty \frac{x \sin mx}{a^2 + x^2} dx = \frac{\pi}{2} e^{-ma}. \qquad \ldots(2)$$

Again integrating both sides of (1) w.r.t. 'm' between the limits 0 and m, we have

$$\int_0^\infty \frac{\sin mx}{x(a^2+x^2)}\,dx = \frac{\pi}{2a}\left[\frac{e^{-ma}}{-a}\right]_{m=0}^{m}$$

$$= \frac{\pi}{2a^2}[1-e^{-ma}]. \qquad \ldots(3)$$

Deductions: If in the above results we put $a = 1$, we have

$$\int_0^\infty \frac{\cos mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}\,;\ \int_0^\infty \frac{x\sin mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}\,;$$

and
$$\int_0^\infty \frac{\sin mx}{x(1+x^2)}\,dx = \frac{\pi}{2}(1-e^{-m}).$$

Example 14: *Show that* $\int_{-\infty}^\infty \frac{\sin rx}{(x-b)^2+a^2}\,dx = \frac{\pi}{a}e^{-ar}\sin br$*, where* $r > 0$.

Solution: Put $x - b = y$ so that $dx = dy$.

When $x = -\infty$, $y = -\infty$ and when $x = \infty$, $y = \infty$.

$$\therefore \quad \int_{-\infty}^\infty \frac{\sin rx}{(x-b)^2+a^2}\,dx = \int_{-\infty}^\infty \frac{\sin r(b+y)}{y^2+a^2}\,dy$$

$$= \int_{-\infty}^\infty \frac{\sin rb\cos ry + \cos rb\sin ry}{y^2+a^2}\,dy$$

$$= 2\sin rb\int_0^\infty \frac{\cos ry}{y^2+a^2}\,dy,$$ the second integral vanishes because the integrand $\frac{\sin ry}{y^2+a^2}$ is an odd function of y

$$= 2\sin rb\cdot\frac{\pi}{2a}e^{-ra} \qquad \left[\because \int_0^\infty \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi}{2a}e^{-ma}, m > 0. \text{ See Ex. 5.}\right]$$

$$= \frac{\pi}{a}e^{-ar}\sin br.$$

Comprehensive Exercise 1

1. Evaluate $\int_0^{\pi/2} \frac{\log(1+\cos\alpha\cos x)}{\cos x}\,dx$.

2. Show that $\int_0^{\pi/2} \log\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right)\frac{d\theta}{\sin\theta} = \pi\sin^{-1}\frac{b}{a}$, $a > b$.

3. Evaluate $\int_0^{\pi} \dfrac{dx}{a + b\cos x}$ (when $a > 0$, $|b| < a$) and deduce that

$$\int_0^{\pi} \frac{dx}{(a + b\cos x)^2} = \frac{\pi a}{(a^2 - b^2)^{3/2}}.$$

4. Evaluate $\int_0^{\infty} \dfrac{\log(1 + a^2x^2)}{1 + b^2x^2}\, dx$.

5. Evaluate $\int_0^{\infty} \dfrac{\sin ax \cos bx}{x}\, dx$.

6. Prove that $\int_0^{\infty} \dfrac{1 - \cos mx}{x}\, e^{-x}\, dx = \dfrac{1}{2}\log(1 + m^2)$.

7. Prove that $\int_0^{1} \dfrac{x^n - 1}{\log x}\, dx = \log(n + 1)$.

8. Show that if $-1 < a < 1$ and $-\pi/2 < \sin^{-1} a < \pi/2$, then

$$\int_0^{\pi} \frac{\log(1 + a\cos x)}{\cos x}\, dx = \pi \sin^{-1} a.$$

9. Show that if $a > 0$, then

$$\int_0^{\pi/2} \frac{\log(1 + a\sin^2 x)}{\sin^2 x}\, dx = \pi[\sqrt{(1 + a)} - 1].$$

10. From the integrals $\int_0^{\infty} e^{-ax} \cos bx\, dx$ and $\int_0^{\infty} e^{-ax} \sin bx\, dx$, deduce by the method of differentiation, the values of the integrals

$$\int_0^{\infty} e^{-ax} x^n \cos bx\, dx \quad \text{and} \quad \int_0^{\infty} e^{-ax} x^n \sin bx\, dx.$$

11. Prove that $\int_0^{\infty} \dfrac{e^{-\alpha x} - e^{-\beta x}}{x} \sin px\, dx = \tan^{-1}\left(\dfrac{\beta}{p}\right) - \tan^{-1}\left(\dfrac{\alpha}{p}\right)$.

12. Prove that $\int_0^{\infty} \dfrac{\cos \alpha x - \cos \beta x}{x}\, dx = \log\left(\dfrac{\beta}{\alpha}\right)$.

13. Show that $\int_0^{\infty} \dfrac{\cos mx}{(a^2 + x^2)^2}\, dx = \dfrac{\pi}{2^2}\left(\dfrac{m}{a^2} + \dfrac{1}{a^3}\right) e^{-ma}$.

14. Prove that $\int_0^{\pi/2} \cos(m \tan\theta)\, d\theta = \dfrac{\pi}{2} e^{-m}$.

15. Show that $\int_{-\infty}^{\infty} \dfrac{\cos rx}{(x - b)^2 + a^2}\, dx = \dfrac{\pi}{a} e^{-ar} \cos br$, where $r > 0$.

16. Show that $\int_0^{\infty} e^{-ax^2}\, dx = \dfrac{1}{2}\sqrt{\left(\dfrac{\pi}{a}\right)}$, $(a > 0)$ and deduce that

$$\int_0^{\infty} x^{2n} e^{-ax^2}\, dx = \frac{\sqrt{\pi}}{2} \cdot \frac{1.3.5.\leq(2n - 1)}{2^n a^{n+(1/2)}}.$$

Answers

1. $\frac{1}{2}\left(\frac{\pi^2}{4}-\alpha^2\right)$.

3. $\frac{\pi}{\sqrt{(a^2-b^2)}}$.

4. $\frac{\pi}{b}\log\left(\frac{a+b}{b}\right)$.

5. $$\int_0^\infty \frac{\sin ax \cos bx}{x}\,dx = \begin{cases} \pi/2, & \text{when } a>0 \\ b, & \text{when } a<b \\ \pi/4, & \text{when } a=b. \end{cases}$$

10. $\frac{n!\cos\{(n+1)\tan^{-1}(b/a)\}}{(a^2+b^2)^{(n+1)/2}}$; $\frac{n!\sin\{(n+1)\tan^{-1}(b/a)\}}{(a^2+b^2)^{(n+1)/2}}$.

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If $u=\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx$ and $a>0$, then

(a) $\frac{du}{db}=\frac{b}{a^2+b^2}$

(b) $\frac{du}{db}=\frac{1}{a^2+b^2}$

(c) $\frac{du}{db}=\frac{a}{a^2+b^2}$

(d) $\frac{du}{db}=\frac{ab}{a^2+b^2}$

2. The value of the integral $\int_0^\infty \frac{\sin bx}{x}\,dx$, if $b>0$, is

(a) π

(b) $\frac{\pi}{2}$

(c) $\frac{\pi}{4}$

(d) $\frac{3\pi}{2}$

3. The value of the integral $\int_0^1 \frac{x^n-1}{\log x}\,dx$ is

(a) $\log n$

(b) $\log(n+1)$

(c) $\frac{1}{n}$

(d) $\frac{1}{\log(n+1)}$

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. If $u = \int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx$, then $\frac{du}{da} = \int_0^{\infty} \frac{dx}{\ldots\ldots}$.

2. If $\int_0^{\pi} \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{(a^2 - b^2)}}$, then $\int_0^{\pi} \frac{dx}{(a + b\cos x)^2} = \ldots\ldots$.

3. If $\int_0^{\infty} \frac{\cos mx}{1 + x^2}\,dx = \frac{\pi}{2} e^{-m}$, then $\int_0^{\infty} \frac{x \sin mx}{1 + x^2}\,dx = \ldots\ldots$.

4. If $\int_0^{\infty} \frac{\cos mx}{a^2 + x^2}\,dx = \frac{\pi}{2a} e^{-ma}$, then $\int_0^{\infty} \frac{\sin mx}{x(a^2 + x^2)}\,dx = \ldots\ldots$.

True or False

Write 'T' for true and 'F' for false statement.

1. $\int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{4}$.

2. If $u = \int_0^{\infty} \frac{1 - \cos mx}{x} e^{-x}\,dx$, then $\frac{du}{dm} = \frac{1}{1 + m^2}$.

Answers

Multiple Choice Questions

1. (c). **2.** (b). **3.** (b).

Fill in the Blank(s)

1. $(1 + x^2)(1 + a^2x^2)$.

2. $\frac{a\pi}{(a^2 - b^2)^{3/2}}$.

3. $\frac{\pi}{2} e^{-m}$.

4. $\frac{\pi}{2a^2}(1 - e^{-ma})$.

True or False

1. *F*. **2.** *F*.

□

Chapter 4

Beta and Gamma Functions

1 Beta Function

Euler's Integrals : Beta and Gamma Functions:

Definition: *The definite integral*

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \quad \textit{for} \quad m>0, n>0$$

is called the **Beta function** *and is denoted by* $\mathbf{B}(m,n)$ [*read as "Beta m, n"*] .

Thus $\mathbf{B}(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx,$

where m, n are any positive numbers, integral or fractional. Beta function is also called the **Eulerian integral of the first kind.**

2 Some Simple Properties of Beta Function

(i) Symmetry of Beta function *i.e.*, $B(m,n)=B(n,m)$. **(Lucknow 2011)**

We have $B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx,$ by the def. of the Beta function

$$= \int_0^1 (1-x)^{m-1} \{1-(1-x)\}^{n-1}\, dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$= \int_0^1 (1-x)^{m-1} x^{n-1}\, dx = \int_0^1 x^{n-1} (1-x)^{m-1}\, dx$$

$= \mathbf{B}(n, m)$, by the def. of Beta function.

Hence $\mathbf{B}(m, n) = \mathbf{B}(n, m)$.

(ii) If *m* or *n* is a positive integer, *B* (*m*, *n*) can be evaluated in an explicit form.

Case I. When *n* is a positive integer. If $n = 1$, the result is obvious because

$$\mathbf{B}(m, 1) = \int_0^1 x^{m-1} (1-x)^{1-1}\, dx = \int_0^1 x^{m-1}\, dx = \left[\frac{x^m}{m}\right]_0^1 = \frac{1}{m}.$$

So let us take $n > 1$. We have

$$\mathbf{B}(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$$

$$= \int_0^1 (1-x)^{n-1} x^{m-1}\, dx$$

$$= \left[(1-x)^{n-1} \cdot \frac{x^m}{m}\right]_0^1 - \int_0^1 (n-1)(1-x)^{n-2}(-1) \cdot \frac{x^m}{m}\, dx,$$

integrating by parts taking x^{m-1} as the second function

$$= 0 + \frac{n-1}{m} \cdot \int_0^1 x^m (1-x)^{n-2}\, dx \qquad [\because n > 1]$$

$$= \frac{n-1}{m} \cdot \int_0^1 x^{(m+1)-1} (1-x)^{(n-1)-1}\, dx$$

$$= \frac{n-1}{m} \mathbf{B}(m+1, n-1).$$

By the repeated application of this process, we get

$$\mathbf{B}(m, n) = \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} \cdots\cdots \frac{1}{m+n-2} \mathbf{B}(m+n-1, 1)$$

$$= \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} \cdots\cdots \frac{1}{m+n-2} \int_0^1 x^{m+n-2} (1-x)^0\, dx$$

$$= \frac{n-1}{m} \cdot \frac{n-2}{m+1} \cdot \frac{n-3}{m+2} \cdots\cdots \frac{1}{m+n-2} \int_0^1 x^{m+n-2}\, dx$$

$$= \frac{(n-1)!}{m(m+1)(m+2)\cdots\cdots(m+n-2)} \cdot \left[\frac{x^{m+n-1}}{m+n-1}\right]_0^1.$$

$$\therefore \qquad \mathbf{B}(m, n) = \frac{1}{m(m+1)(m+2)\cdots\cdots(m+n-2)(m+n-1)}.$$

Case II. When *m* is a positive integer. Since the Beta function is symmetrical in m and n *i.e.*, $\mathbf{B}(m, n) = \mathbf{B}(n, m)$, therefore by case I, we conclude that

$$\mathbf{B}(m, n) = \frac{(m-1)!}{n(n+1)(n+2)\ldots\ldots(n+m-2)(n+m-1)}.$$

(iii) If both m and n are positive integers, then

$$\mathbf{B}(m, n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

From (ii), we have

$$\mathbf{B}(m, n) = \frac{(n-1)!}{m(m+1)(m+2)\ldots\ldots(m+n-2)(m+n-1)}$$

$$= \frac{(n-1)!\,(m-1)!}{(m+n-1)(m+n-2)\ldots\ldots(m+1)\,m\,(m-1)!},$$

writing the denominator in the reversed order and multiplying the Nr and Dr by $(m-1)!$

$$= \frac{(m-1)!\,(n-1)!}{(m+n-1)!}.$$

Illustrative Examples

Example 1: *Express the following integrals in terms of Beta function* :

(i) $\int_0^1 x^m (1-x^2)^n \, dx, m > -1, n > -1$; **(Lucknow 2010)** *(ii)* $\int_0^1 \frac{x^2}{\sqrt{(1-x^5)}} dx$

(iii) $\int_0^1 x^{m-1} (1-x^2)^{n-1} \, dx$.

(Garhwal 2003)

Solution: (i) We have

$$\int_0^1 x^m (1-x^2)^n \, dx = \int_0^1 x^{m-1} (1-x^2)^n \,.\, x \, dx \qquad \textbf{[Note]}$$

$$= \int_0^1 y^{(m-1)/2} (1-y)^n \cdot \frac{dy}{2},$$

putting $x^2 = y$ so that $2x \, dx = dy$

$$= \frac{1}{2} \int_0^1 y^{(m-1)/2} (1-y)^n \, dy$$

$$= \frac{1}{2} \int_0^1 y^{[(m+1)/2]-1} (1-y)^{(n+1)-1} \, dy$$

$$= \frac{1}{2} \mathbf{B}\left(\frac{1}{2}(m+1), n+1\right).$$

(ii) We have

$$\int_0^1 \frac{x^2}{\sqrt{(1-x^5)}} dx = \int_0^1 x^2 (1-x^5)^{-1/2} \, dx$$

$$= \int_0^1 x^2 \cdot \frac{1}{x^4} (1 - x^5)^{-1/2} \,.\, x^4 \, dx = \int_0^1 x^{-2} (1 - x^5)^{-1/2} \, x^4 \, dx$$

$$= \int_0^1 y^{-2/5} (1 - y)^{-1/2} \cdot \frac{1}{5} dy, \quad \text{putting } x^5 = y \text{ so that } 5x^4 \, dx = dy$$

$$= \frac{1}{5} \int_0^1 y^{-2/5} (1 - y)^{-1/2} \, dy = \frac{1}{5} \int_0^1 y^{(3/5) - 1} (1 - y)^{(1/2) - 1} \, dy$$

$$= \frac{1}{5} \mathbf{B} \left(\frac{3}{5}, \frac{1}{2} \right) .$$

(iii) Proceed as in part (i).

Example 2: *Prove that*

$$\int_0^a (a - x)^{m-1} \,.\, x^{n-1} \, dx = a^{m+n-1} \, \mathbf{B} (m, n) = \frac{a^{m+n-1} \, \Gamma m \, \Gamma n}{\Gamma (m + n)} .$$

Solution: We have

$$\int_0^a (a - x)^{m-1} x^{n-1} \, dx$$

$$= \int_0^1 (a - ay)^{m-1} (ay)^{n-1} a \, dy, \quad \text{putting } x = ay$$

$$= \int_0^1 a^{(m-1) + (n-1) + 1} (1 - y)^{m-1} y^{n-1} \, dy$$

$$= a^{m+n-1} \int_0^1 y^{n-1} (1 - y)^{m-1} \, dy$$

$$= a^{m+n-1} \, \mathbf{B} (n, m) = a^{m+n-1} \, \mathbf{B} (m, n) \qquad [\because \mathbf{B} (m, n) = \mathbf{B} (n, m)]$$

$$= \frac{a^{m+n-1} \, \Gamma m \, \Gamma n}{\Gamma (m + n)} . \qquad \left[\because \mathbf{B} (m, n) = \frac{\Gamma m \, \Gamma n}{\Gamma (m + n)} \right]$$

Example 3: *Show that if m, n are positive, then*

$$\int_a^b (x - a)^{m-1} (b - x)^{n-1} \, dx = (b - a)^{m+n-1} \,.\, \mathbf{B} (m, n)$$

$$= (b - a)^{m+n-1} \cdot \frac{\Gamma m \, \Gamma n}{\Gamma (m + n)} .$$

(Agra 2003)

Solution: The given integral is

$$\int_a^b (x - a)^{m-1} (b - x)^{n-1} \, dx .$$

Put $x = a + (b - a) \, y$ so that $dx = (b - a) \, dy$.

Also when $x = a$, $y = 0$ and when $x = b$, $y = 1$.

$$\therefore \quad \int_a^b (x - a)^{m-1} (b - x)^{n-1} \, dx$$

$$= \int_0^1 [(b - a) \, y]^{m-1} [b - a - (b - a) \, y]^{n-1} \,.\, (b - a) \, dy$$

$$= \int_0^1 (b - a)^{m-1} \,.\, y^{m-1} \,.\, (b - a)^{n-1} \,.\, (1 - y)^{n-1} \,.\, (b - a) \, dy$$

$$= (b-a)^{m+n-1} \int_0^1 y^{m-1} (1-y)^{n-1}\, dy = (b-a)^{m+n-1}\, \mathbf{B}(m,n)$$

$$= (b-a)^{m+n-1} \cdot \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)} \cdot \qquad \left[\because \mathbf{B}(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}\right]$$

Example 4: *Show that* $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\, dx = \frac{1}{(a+b)^m \,.\, a^n}\, \mathbf{B}(m,n)$.

(Garhwal 2009)

Solution: The given integral

$$I = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\, dx$$

$$= \int_0^1 \left(\frac{x}{a+bx}\right)^{m-1} \cdot \left(\frac{1-x}{a+bx}\right)^{n-1} \cdot \frac{1}{(a+bx)^2}\, dx\,.$$

[Note]

Put $\displaystyle\frac{x}{a+bx} = \frac{y}{a+b}$ so that $\displaystyle\frac{(a+bx)\,.\,1 - x\,.\,b}{(a+bx)^2}\, dx = \frac{dy}{a+b}$

i.e., $\displaystyle\frac{1}{(a+bx)^2}\, dx = \frac{dy}{a(a+b)}\cdot$

Further $\displaystyle\frac{1-x}{a+bx} = \frac{1}{a}\,\frac{a-ax}{a+bx} = \frac{1}{a}\left[\frac{a+bx-ax-bx}{a+bx}\right] = \frac{1}{a}\left[1 - \frac{x(a+b)}{a+bx}\right] = \frac{1-y}{a}\cdot$

Also when $x = 0$, $y = 0$ and when $x = 1$, $y = 1$.

$$\therefore \qquad I = \int_0^1 \left(\frac{y}{a+b}\right)^{m-1} \left(\frac{1-y}{a}\right)^{n-1} \cdot \frac{dy}{a(a+b)}$$

$$= \frac{1}{(a+b)^m \,.\, a^n} \int_0^1 y^{m-1}(1-y)^{n-1}\, dy = \frac{\mathbf{B}(m,n)}{(a+b)^m \,.\, a^n}\cdot$$

Example 5: *Prove that* $\displaystyle\frac{\mathbf{B}(m+1,n)}{\mathbf{B}(m,n)} = \frac{m}{m+n}\cdot$

Solution: We have $\mathbf{B}(m+1,n) = \mathbf{B}(n, m+1)$

[By the symmetry of Beta function]

$$= \int_0^1 x^{n-1}(1-x)^{(m+1)-1}\, dx = \int_0^1 (1-x)^m\, x^{n-1}\, dx$$

[Note]

$$= \left[(1-x)^m \cdot \frac{x^n}{n}\right]_0^1 - \int_0^1 m(1-x)^{m-1}(-1)\cdot\frac{x^n}{n}\, dx\,,$$

(integrating by parts)

$$= 0 + \frac{m}{n}\int_0^1 x^{n-1}.\, x(1-x)^{m-1}\, dx$$

$$= \frac{m}{n}\int_0^1 x^{n-1}[1-(1-x)](1-x)^{m-1}\, dx$$

$$= \frac{m}{n}\left[\int_0^1 x^{n-1}(1-x)^{m-1}\,dx - \int_0^1 x^{n-1}(1-x)^m\,dx\right]$$

$$= \frac{m}{n}[\mathbf{B}(n, m) - \mathbf{B}(n, m+1)] = \frac{m}{n}\mathbf{B}(m, n) - \frac{m}{n}\mathbf{B}(m+1, n)$$

or $$\left(1+\frac{m}{n}\right)\mathbf{B}(m+1, n) = \frac{m}{n}\mathbf{B}(m, n)$$ [By transposition]

or $$(n+m)\,\mathbf{B}(m+1, n) = m\,\mathbf{B}(m, n) \quad \text{or} \quad \frac{\mathbf{B}(m+1, n)}{\mathbf{B}(m, n)} = \frac{m}{m+n}.$$

3 Another Form of Beta Function

$$\mathbf{B}(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx,\ m>0,\ n>0.$$

(Lucknow 2007)

Proof: By the definition of Beta function, we have

$$\mathbf{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

Put $x = \frac{1}{1+y}$ so that $dx = -\frac{1}{(1+y)^2}\,dy$.

Also when $x \to 0$, $y \to \infty$ and when $x = 1$, $y = 0$.

$\therefore$ $$\mathbf{B}(m, n) = \int_\infty^0 \frac{1}{(1+y)^{m-1}} \cdot \left[1 - \frac{1}{1+y}\right]^{n-1} \cdot \left[-\frac{1}{(1+y)^2}\right] dy$$

$$= \int_0^\infty \frac{1}{(1+y)^{m-1+2}} \cdot \frac{y^{n-1}}{(1+y)^{n-1}}\,dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx. \quad \ldots(1)$$

[By a property of definite integrals]

Again since Beta function is symmetrical in m and n, we have

$$\mathbf{B}(m, n) = \mathbf{B}(n, m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx, \text{ by (1).}$$

Thus $$\mathbf{B}(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx,\ m>0,\ n>0.$$

Illustrative Examples

Example 6: *Prove that* $\int_0^\infty \frac{x^{m-1} - x^{n-1}}{(1+x)^{m+n}}\,dx = 0,\ m>0,\ n>0.$

(Lucknow 2009)

Solution: The given integral is $= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx - \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$

$$= \mathbf{B}(m, n) - \mathbf{B}(n, m)$$

$$= \mathbf{B}(m, n) - \mathbf{B}(m, n) = 0 .$$

Example 7: *Express* $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} dx$ *in terms of Beta function where* $m > 0, n > 0$, $a > 0, b > 0$. **(Kumaun 2013)**

Solution: In the given integral put $bx = ay$ *i.e.*, $x = (a / b)\, y$ so that $dx = (a / b)\, dy$. When $x = 0$, $y = 0$ and when $x \to \infty$, $y \to \infty$.

$$\therefore \quad \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} dx = \int_0^\infty \left(\frac{a}{b}\, y\right)^{m-1} \cdot \frac{1}{(a+ay)^{m+n}} \cdot \frac{a}{b}\, dy$$

$$= \int_0^\infty \frac{a^{m-1}\, y^{m-1}}{b^{m-1} . a^{m+n} (1+y)^{m+n}} \frac{a}{b}\, dy = \frac{1}{a^n b^m} \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

$$= \frac{1}{a^n b^m} \mathbf{B}(m, n) . \qquad \text{[By article 3]}$$

4 Gamma Function

Definition: *The definite integral*

$$\int_0^\infty e^{-x}\, x^{n-1}\, dx , \text{ for } n > 0$$

is called the ***Gamma Function*** *and is denoted by* $\Gamma(n)$ [*read as "Gamma n"*].

(Gorakhpur 2006)

Thus $\quad \mathbf{\Gamma(n) = \int_0^\infty e^{-\infty} x^{n-1} dx, \text{ for } n > 0.}$

Gamma function is also called **Eulerian integral of the second kind**.

5 Elementary Properties of Gamma Function

(i) $\Gamma(n+1) = n\,\Gamma(n)$, *where* $n > 0$ **(Lucknow 2007, 08, 10)**

and *(ii)* $\Gamma(n) = (n-1)!$, *where n is a positive integer.* **(Gorakhpur 2006)**

Proof: By the definition of gamma function, we have

$$\Gamma(n+1) = \int_0^\infty e^{-x}\, x^{(n+1)-1}\, dx = \int_0^\infty x^n e^{-x}\, dx$$

$$= [-e^{-x}\, x^n]_0^\infty + \int_0^\infty e^{-x} . nx^{n-1}\, dx , \qquad ...(1)$$

integrating by parts taking e^{-x} as the second function.

Now $$\lim_{x \to \infty} \frac{x^n}{e^x} = \lim_{x \to \infty} \frac{x^n}{1 + x + (x^2 / 2!) + \ldots + (x^n / n!) + \ldots}$$

$$= \lim_{x \to \infty} \frac{1}{\frac{1}{x^n} + \frac{1}{x^{n-1}} + \ldots + \frac{1}{n!} + \frac{x}{(n+1)!} + \ldots\ldots} = \frac{1}{\infty} = 0 .$$

$\therefore$ from (1), we get $\Gamma(n+1) = 0 + n \int_0^\infty e^{-x} x^{n-1} dx,$ $[\because n > 0]$

$= n\,\Gamma(n),$ which proves the result (i).

(ii) We have $\Gamma(n) = \Gamma[(n-1)+1] = (n-1)\,\Gamma(n-1)$. $[\because \Gamma(n+1) = n\,\Gamma(n)]$

Similarly $\Gamma(n-1) = (n-2)\,\Gamma(n-2), \ldots$ etc.

Hence if n is a +ive integer, then proceeding as above, we get

$$\Gamma(n) = (n-1)(n-2)\ldots 2.1\,\Gamma(1).$$

But $$\Gamma(1) = \int_0^\infty e^{-x} x^{1-1} dx = \int_0^\infty e^{-x} x^0 dx = \int_0^\infty e^{-x}.1\, dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^\infty = -\left[\lim_{x \to \infty} \frac{1}{e^x} - e^0\right] = -[0-1] = 1.$$

Hence $\Gamma(n) = (n-1)(n-2)\ldots 2.1.1 = (n-1)!$ if n is a + ive integer.

Remember: $\Gamma(n) = (n-1)\,\Gamma(n-1)$, where $n > 1$ and $\Gamma(1) = 1$.

Also it may be remarked that $\Gamma(0) = \infty$ and $\Gamma(-n) = \infty$ where n is a positive integer.

6 Some Transformations of Gamma Function

We have $$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx. \qquad \ldots(1)$$

(i) Put $x = ay$ so that $dx = a\,dy$; when $x = 0$, $y = 0$ and when $x \to \infty$, $y \to \infty$.

$\therefore$ $$\Gamma(n) = \int_0^\infty e^{-ay} a^n y^{n-1} dy.$$

Hence $$\int_0^\infty e^{-ay} y^{n-1} dy = \frac{\Gamma(n)}{a^n}.$$ **(Remember)**

(ii) In (1) if we put $x = \log(1/y)$ or $y = e^{-x}$ so that $dy = -e^{-x} dx$,

then $$\Gamma(n) = -\int_1^0 \left(\log \frac{1}{y}\right)^{n-1} dy = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy.$$

(Kanpur 2006, Garhwal 12)

(iii) In (1) if we put $x^n = y$ so that $nx^{n-1} dx = dy$, we get

$$\Gamma(n) = \frac{1}{n} \int_0^\infty e^{-(y)^{1/n}} dy$$

or $$\int_0^\infty e^{-(y)^{1/n}} dy = n\,\Gamma(n) = \Gamma(n+1).$$

7 Relation between Beta and Gamma Functions

$$\mathbf{B}\,(\boldsymbol{m}, \boldsymbol{n}) = \frac{\Gamma\,(\boldsymbol{m})\,\Gamma\,(\boldsymbol{n})}{\Gamma\,(\boldsymbol{m}+\boldsymbol{n})}, \textit{ where } \boldsymbol{m} > \mathbf{0},\ \boldsymbol{n} > \mathbf{0}.$$

(Agra 2001; Lucknow 07; Kanpur 09; Garhwal 10; Kumaun 15)

Proof: We have

$$\frac{\Gamma\,(m)}{z^m} = \int_0^\infty e^{-zx}\, x^{m-1}\, dx.$$ [See article 6, part (ii)]

$\therefore$ $$\Gamma\,(m) = z^m \int_0^\infty e^{-zx} . x^{m-1}\, dx = \int_0^\infty z^m e^{-zx} x^{m-1}\, dx.$$

Multiplying both sides by $e^{-z} z^{n-1}$, we get

$$\Gamma\,(m)\, e^{-z} z^{n-1} = \int_0^\infty e^{-z(1+x)} z^{m+n-1} x^{m-1}\, dx. \qquad ...(1)$$

Now integrating both sides of (1) with respect to z from 0 to ∞, we get

$$\Gamma\,(m) \int_0^\infty e^{-z} z^{n-1}\, dz = \int_0^\infty \left[\int_0^\infty e^{-z(1+x)} z^{m+n-1} x^{m-1}\, dx \right] dz$$

or $$\Gamma\,(m)\,\Gamma\,(n) = \int_0^\infty \left[\int_0^\infty e^{-z(1+x)} z^{m+n-1}\, dz \right] x^{m-1}\, dx$$

$$= \int_0^\infty \frac{\Gamma\,(m+n)}{(1+x)^{m+n}}\, x^{m-1}\, dx$$ [By article 6, part (ii)]

$$= \Gamma\,(m+n) \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

$$= \Gamma\,(m+n) \,.\, B\,(m,n), \text{ by article 3}.$$

$\therefore$ $$\mathbf{B}\,(m,n) = \frac{\Gamma\,(m)\,\Gamma\,(n)}{\Gamma\,(m+n)}.$$

Thus remember that $\displaystyle\int_0^1 x^{m-1} (1-x)^{n-1}\, dx = \frac{\Gamma\,(m)\,\Gamma\,(n)}{\Gamma\,(m+n)}.$

(Kanpur 2009)

Corollary: $\boldsymbol{\Gamma\,(n)\,\Gamma\,(1-n) = \dfrac{\pi}{\sin n\pi}}$, **where $\boldsymbol{0 < n < 1}$.**

(Garhwal 2006)

Proof: We know that $\mathbf{B}\,(m,n) = \displaystyle\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$, [See article 3]

and $$\mathbf{B}\,(m,n) = \frac{\Gamma\,(m)\,\Gamma\,(n)}{\Gamma\,(m+n)}, \text{ where } m > 0 \text{ and } n > 0.$$

$\therefore$ $$\frac{\Gamma\,(m)\,\Gamma\,(n)}{\Gamma\,(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx.$$

Putting $m+n=1$ or $m=1-n$ in the above relation, we get

$$\frac{\Gamma\,(1-n)\,\Gamma\,(n)}{\Gamma\,(1)} = \int_0^\infty \frac{x^{n-1}}{1+x}\, dx, \text{ where } 0 < n < 1.$$

[Note that $m > 0 \Rightarrow 1-n > 0 \Rightarrow n < 1$.]

But $\Gamma(1) = 1$. Also

$$\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\pi}{\sin n\pi}. \qquad \textbf{(Remember)}$$

$\therefore$
$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}, \text{ where } 0 < n < 1.$$

8 The Value of $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(Agra 2000)

Proof: We know that $B(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ ...(1)

If we take $m = \frac{1}{2}, n = \frac{1}{2}$, then from (1), we have

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{\Gamma(1)} = \left[\Gamma\left(\frac{1}{2}\right)\right]^2 \qquad [\because\ \Gamma(1) = 1]$$

Thus
$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = B\left(\frac{1}{2}, \frac{1}{2}\right) = \int_0^1 x^{1/2-1}(1-x)^{1/2-1}\,dx,$$

by the definition of Beta function

$$= \int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx.$$

Now put $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$.

Also when $x = 0, \theta = 0$ and when $x = 1, \theta = \frac{1}{2}\pi$.

$\therefore$
$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \int_0^{\pi/2} \frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta}\cdot 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2\,[\theta]_0^{\pi/2}$$

$$= 2\left[\frac{1}{2}\pi - 0\right] = \pi.$$

Taking square root of both the sides, we get

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \qquad \textbf{(Remember)}$$

Important Deduction: *To prove that* $\int_0^\infty e^{-x^2}\,dx = \dfrac{\sqrt{\pi}}{2}$.

Proof: Let $I = \int_0^\infty e^{-x^2}\,dx$.

Put $x^2 = z$ so that $2x\,dx = dz$

or $\qquad dx = \dfrac{1}{2x}\, dz = \dfrac{1}{2\sqrt{z}}\, dz = \dfrac{1}{2} z^{-1/2}\, dz$.

Also when $x = 0, z = 0$ and when $x \to \infty, z \to \infty$.

$$\therefore \qquad I = \int_0^\infty e^{-z} \frac{1}{2} z^{-1/2}\, dz$$

$$= \frac{1}{2} \int_0^\infty e^{-z} z^{1/2-1}\, dz = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Hence $\qquad \displaystyle\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{\mathbf{2}}$. **(Remember)**

9 $\displaystyle\int_0^{\pi/2} \cos^m \theta \sin^n \theta\, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$, $m > -1, n > -1$

(Garhwal 2005, 07)

Proof: Put $\sin^2 \theta = x$ so that $2 \sin\theta \cos\theta\, d\theta = dx$

or $\qquad 2 \sin\theta . \sqrt{(1 - \sin^2 \theta)}\, d\theta = dx \quad$ or $\quad 2 x^{1/2} . \sqrt{(1 - x)}\, d\theta = dx$.

$$\therefore \qquad d\theta = \frac{dx}{2 x^{1/2} (1 - x)^{1/2}}.$$

Also when $\theta = 0$, $x = 0$ and when $\theta = \frac{1}{2}\pi$, $x = 1$.

$$\therefore \qquad \int_0^{\pi/2} \cos^m \theta \sin^n \theta\, d\theta = \int_0^{\pi/2} (1 - \sin^2 \theta)^{m/2} . \sin^n \theta\, d\theta$$

$$= \int_0^1 (1 - x)^{m/2} . x^{n/2} . \frac{dx}{2 x^{1/2} (1 - x)^{1/2}}$$

$$= \frac{1}{2} \int_0^1 x^{(n-1)/2} (1 - x)^{(m-1)/2}\, dx$$

$$= \frac{1}{2} \int_0^1 x^{\{(n+1)/2\} - 1} (1 - x)^{\{(m+1)/2\} - 1}\, dx$$

$$= \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right), \text{ provided } m > -1 \text{ and } n > -1$$

$$= \frac{1}{2} \frac{\Gamma \frac{1}{2}(m+1) \Gamma \frac{1}{2}(n+1)}{\Gamma \frac{1}{2}(m + 1 + n + 1)} \qquad \left[\because \mathbf{B}(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m + n)}\right]$$

$$= \frac{\Gamma \frac{1}{2}(m+1) \Gamma \frac{1}{2}(n+1)}{2 \Gamma \frac{1}{2}(m + n + 2)}.$$

10 Some Important Transformations of Beta Function

Beta function can be transformed into many other forms. A few of them are given below.

(i) We know that $\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \mathbf{B}(m,n)$.

Now $$\int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}}\,dy + \int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy.$$

Making the substitution $y = 1/x$ in the last integral, we get

$$\int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy = \int_0^1 \frac{x^{n-1}\,dx}{(1+x)^{m+n}}.$$

$$\therefore \quad \mathbf{B}(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

$$= \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}}\,dy + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx.$$

Hence $$\boldsymbol{\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = \mathbf{B}(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}}.$$

(ii) We know that $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \mathbf{B}(m,n)$.

If we put $x = \frac{ay}{b}$, so that $dx = \frac{a}{b}\,dy$, we get

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = a^m b^n \int_0^\infty \frac{y^{m-1}}{(ay+b)^{m+n}}\,dy.$$

$$\therefore \quad \int_0^\infty \frac{y^{m-1}}{(ay+b)^{m+n}}\,dy = \frac{1}{a^m b^n}\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \frac{1}{a^m b^n}\,\mathbf{B}(m,n).$$

Hence $$\boldsymbol{\int_0^\infty \frac{y^{m-1}}{(ay+b)^{m+n}}\,dy = \frac{\Gamma(m)\,\Gamma(n)}{a^m b^n\,\Gamma(m+n)}}.$$

Again putting $y = \tan^2\theta$ *i.e.*, $dy = 2\tan\theta\sec^2\theta\,d\theta$ in the integral just obtained, we get

$$\boldsymbol{\int_0^{\pi/2} \frac{\sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \frac{\Gamma(m)\,.\Gamma(n)}{2\,a^m b^n\,\Gamma(m+n)}}.$$

(iii) We know that $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \mathbf{B}(m, n)$.

Putting $x = \sin^2\theta$, so that $dx = 2\sin\theta\cos\theta\,d\theta$, we have

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta.$$

$$\therefore \quad \int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta = \frac{\mathbf{B}(m, n)}{2} = \frac{\Gamma(m)\Gamma(n)}{2\,\Gamma(m+n)}.$$

This result may also be written in the form

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\cdot\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)},$$

by putting $2m-1 = p$ and $2n-1 = q$.

(iv) We know that $\int_0^1 y^{m-1}(1-y)^{n-1}\,dy = \mathbf{B}(m, n)$.

Putting $y = \dfrac{x-b}{a-b}$, so that $dy = \dfrac{dx}{a-b}$, we have

$$\int_0^1 y^{m-1}.(1-y)^{n-1}\,dy = \int_b^a \left(\frac{x-b}{a-b}\right)^{m-1}\left(\frac{a-x}{a-b}\right)^{n-1}\cdot\frac{dx}{a-b}$$

$$= \frac{1}{(a-b)^{m+n-1}}\int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx.$$

$$\therefore \quad \int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx = (a-b)^{m+n-1}\int_0^1 y^{m-1}(1-y)^{n-1}\,dy$$

or
$$\int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx = (a-b)^{m+n-1}\,\mathbf{B}(m, n)$$

$$= (a-b)^{m+n-1}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

11 Duplication Formula

$$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m), \text{ where } m > 0.$$

(Agra 2001, 03; Kanpur 09; Garhwal 11, 13)

Proof: We know that

$$\mathbf{B}(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \text{ where } m > 0, n > 0.$$

If we take $n = m$, then

$$\mathbf{B}\,(m, m) = \frac{[\Gamma\,(m)]^2}{\Gamma\,(2\,m)}\,. \qquad ...(1)$$

Again by the definition of Beta function, we have

$$\mathbf{B}\,(m, m) = \int_0^1 x^{m-1}\,(1-x)^{m-1}\,dx\,.$$

Let us put $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\,d\theta$.

Also when $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \frac{1}{2}\pi$.

Then $$\mathbf{B}\,(m, m) = \int_0^{\pi/2} \sin^{2\,(m-1)}\theta\,.\cos^{2\,(m-1)}\theta . 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\,.\cos^{2m-1}\theta\,d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1}\,d\theta$$

$$= 2\int_0^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^{2m-1} d\theta = \frac{1}{2^{2m-2}}\int_0^{\pi/2} \sin^{2m-1} 2\theta\,d\theta$$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi} \sin^{2m-1}\phi\cdot\frac{d\phi}{2},$$

putting $2\theta = \phi$ so that $d\theta = \frac{1}{2}d\phi$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1}\phi\,d\phi$$

$$= \frac{1}{2^{2m-1}}\cdot 2\int_0^{\pi/2} \sin^{2m-1}\phi\,d\phi \qquad \textbf{(Note)}$$

$$= \frac{1}{2^{2m-2}}\int_0^{\pi/2} \sin^{2m-1}\phi\,.\cos^0\phi\,d\phi \qquad \textbf{(Note)}$$

$$= \frac{1}{2^{2m-2}}\,\frac{\Gamma\frac{1}{2}(2m-1+1)\,\Gamma\frac{1}{2}(0+1)}{2\,\Gamma\frac{1}{2}(2m-1+0+2)} = \frac{1}{2^{2m-1}}\cdot\frac{\Gamma\,(m)\,\Gamma\,(\frac{1}{2})}{\Gamma\,(m+\frac{1}{2})}$$

$$= \frac{1}{2^{2m-1}}\cdot\frac{\Gamma\,(m)\,\sqrt{\pi}}{\Gamma\,(m+\frac{1}{2})}\,. \qquad ...(2)$$

$[\because\ \Gamma\,(\frac{1}{2}) = \sqrt{\pi}]$

Now equating the two values of $B\,(m, m)$ obtained in (1) and (2), we get

$$\frac{[\Gamma\,(m)]^2}{\Gamma\,(2m)} = \frac{1}{2^{2m-1}}\cdot\frac{\Gamma\,(m)\,.\sqrt{\pi}}{\Gamma\,(m+\frac{1}{2})}$$

or $$\Gamma\,(m)\,\Gamma\,(m+\frac{1}{2}) = \frac{\sqrt{\pi}}{2^{2\,m-1}}\,\Gamma\,(2\,m)\,. \qquad \textbf{(Remember)}$$

12 $\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)=\frac{(2\pi)^{(n-1)/2}}{n^{1/2}}$,

where n is a positive integer.

Proof: Let $A=\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)$. ...(1)

Writing the above expression in the reverse order, we have

$$A=\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\ldots\Gamma\left(1-\frac{n-2}{n}\right)\Gamma\left(1-\frac{n-1}{n}\right). \quad ...(2)$$

Multiplying (1) and (2), we get

$$A^2=\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(1-\frac{2}{n}\right).....\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(1-\frac{n-1}{n}\right)$$

$$=\frac{\pi^{n-1}}{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{n-1}{n}\pi}. \quad \text{[See corollary of article 7]} \quad ...(3)$$

To calculate this expression, we factorize $1-x^{2n}$.

Now the roots of the equation $x^{2n}-1=0$ are given by

$$x=(1)^{1/2n}=(\cos 2r\pi+i\sin 2r\pi)^{1/2n}$$

$$=\cos\frac{r\pi}{n}+i\sin\frac{r\pi}{n}, \quad \text{where } r=0,1,2,\ldots,2n-1.$$

Hence, we have

$$1-x^{2n}=(1-x)(1+x)\left(x-\cos\frac{\pi}{n}-i\sin\frac{\pi}{n}\right)\left(x-\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}\right)\ldots$$

$$\ldots\left(x-\cos\frac{n-1}{n}\pi-i\sin\frac{n-1}{n}\pi\right)\left(x-\cos\frac{n-1}{n}\pi+i\sin\frac{n-1}{n}\pi\right)$$

$$=(1-x^2)\left(1-2x\cos\frac{\pi}{n}+x^2\right)\left(1-2x\cos\frac{2\pi}{n}+x^2\right)\ldots$$

$$\ldots\left(1-2x\cos\frac{n-1}{n}\pi+x^2\right).$$

$$\therefore \quad \frac{1-x^{2n}}{1-x^2}=\left(1-2x\cos\frac{\pi}{n}+x^2\right)\left(1-2x\cos\frac{2\pi}{n}+x^2\right)\ldots\left(1-2x\cos\frac{n-1}{n}\pi+x^2\right).$$

Putting $x=1$ and $x=-1$ respectively, we have in the limit,

$$n=\left(2-2\cos\frac{\pi}{n}\right)\left(2-2\cos\frac{2\pi}{n}\right)\ldots\left(2-2\cos\frac{n-1}{n}\pi\right)$$

and

$$n=\left(2+2\cos\frac{\pi}{n}\right)\left(2+2\cos\frac{2\pi}{n}\right)\ldots\left(2+2\cos\frac{n-1}{n}\pi\right).$$

Multiplying these, we get

$$n^2 = 2^{2n-2} \sin^2 \frac{\pi}{n} \cdot \sin^2 \frac{2\pi}{n} \ldots \sin^2 \frac{n-1}{n} \pi$$

or $$n = 2^{n-1} . \sin \frac{\pi}{n} \cdot \sin \frac{2\pi}{n} \ldots \sin \frac{n-1}{n} \pi .$$

Hence, from (3), we get

$$A^2 = \frac{\pi^{n-1}}{n / 2^{n-1}} = \frac{(2\pi)^{n-1}}{n} \quad \text{or} \quad A = \frac{(2\pi)^{(n-1)/2}}{n^{1/2}} .$$

Remark: The value of $\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \ldots \sin \frac{n-1}{n} \pi$ can also be found by using the trigonometrical identity

$$\frac{\sin n\theta}{\sin \theta} = 2^{n-1} \sin \left(\theta + \frac{\pi}{n} \right) \sin \left(\theta + \frac{2\pi}{n} \right) \sin \left(\theta + \frac{3\theta}{n} \right) \ldots \sin \left(\theta + \frac{n-1}{n} \pi \right) .$$

From the above identity, we have

$$\frac{\sin n\theta}{n\theta} \cdot \frac{\theta}{\sin \theta} \cdot n = 2^{n-1} \sin \left(\theta + \frac{\pi}{n} \right) \sin \left(\theta + \frac{2\pi}{n} \right) \ldots \sin \left(\theta + \frac{n-1}{n} \pi \right) .$$

Taking limit as $\theta \to 0$, we get

$$n = 2^{n-1} \sin \frac{\pi}{n} \sin \frac{2\pi}{n} \sin \frac{3\pi}{n} \ldots \sin \frac{n-1}{n} \pi .$$

13 **(i)** $\int_0^\infty e^{-ax} \cos bx . x^{m-1} \, dx = \frac{\Gamma m}{k^m} \cos m\alpha$

(ii) $\int_0^\infty e^{-ax} \sin bx . x^{m-1} dx = \frac{\Gamma m}{k^m} \sin m\alpha,$

where $k = \sqrt{(a^2 + b^2)}$ **and** $\alpha = \tan^{-1} \left(\frac{a}{b} \right)$

Proof: We have

$$\int_0^\infty e^{-ax} e^{ibx} x^{m-1} dx = \int_0^\infty e^{-(a-ib)x} x^{m-1} dx = \frac{\Gamma (m)}{(a-ib)^m}$$

[See article 6, part (i)]

$$= (a - ib)^{-m} \Gamma (m) . \qquad \ldots(1)$$

Let us first separate $(a - ib)^{-m}$ into real and imaginary parts.

Put $a = k \cos \alpha$ and $b = k \sin \alpha$ so that

$$\alpha = \tan^{-1} (b / a) \text{ and } k = \sqrt{(a^2 + b^2)} .$$

Then $$(a - ib)^{-m} = [k (\cos \alpha - i \sin \alpha)]^{-m}$$

$$= k^{-m} (\cos \alpha - i \sin \alpha)^{-m}$$

$$= k^{-m} (\cos m\alpha + i \sin m\alpha), \text{ by De-Moivre's theorem.}$$

Now from (1), we have

$$\int_0^\infty e^{-ax} e^{ibx} x^{m-1} dx = k^{-m} (\cos m\alpha + i \sin m\alpha) \Gamma(m)$$

or
$$\int_0^\infty e^{-ax} (\cos bx + i \sin bx) x^{m-1} dx = \frac{\Gamma(m)}{k^m} (\cos m\alpha + i \sin m\alpha),$$

$$[\because e^{i\theta} = \cos\theta + i \sin\theta, \text{ by Euler's theorem}]$$

or
$$\int_0^\infty e^{-ax} \cos bx \,.\, x^{m-1} dx + i \int_0^\infty e^{-ax} \sin bx \,.\, x^{m-1} dx$$

$$= \frac{\Gamma(m)}{k^m} \cos m\alpha + i \frac{\Gamma(m)}{k^m} \sin m\alpha. \qquad ...(2)$$

Equating real and imaginary parts in (2), we get

$$\int_0^\infty e^{-ax} \cos bx \,.\, x^{m-1} dx = \frac{\Gamma(m)}{k^m} \cos m\alpha,$$

and
$$\int_0^\infty e^{-ax} \sin bx \,.\, x^{m-1} dx = \frac{\Gamma(m)}{k^m} \sin m\alpha,$$

where $k = \sqrt{(a^2 + b^2)}$ and $\alpha = \tan^{-1}(b/a)$.

Deductions: **(i)** If we put $a = 0$, then $\alpha = \pi/2$ and $k = b$.

Hence
$$\int_0^\infty x^{m-1} \cos bx \, dx = \frac{\Gamma(m)}{b^m} \cos \frac{m\pi}{2}$$

and
$$\int_0^\infty x^{m-1} \sin bx \, dx = \frac{\Gamma(m)}{b^m} \sin \frac{m\pi}{2}.$$

(ii) If we put $m = 1$, then

$$\int_0^\infty e^{-ax} \cos bx \, dx = \frac{\Gamma(1)}{k} \cos\alpha = \frac{k \cos\alpha}{k^2} = \frac{a}{a^2 + b^2}$$

and
$$\int_0^\infty e^{-ax} \sin bx \, dx = \frac{\Gamma(1)}{k} \sin\alpha = \frac{k \sin\alpha}{k^2} = \frac{b}{a^2 + b^2}.$$

Illustrative Examples

Example 8: *Evaluate the following integrals:*

(i) $\displaystyle\int_0^\infty \frac{x^8 (1 - x^6)}{(1 + x)^{24}} dx,$

(Garhwal 2000)

(ii) $\displaystyle\int_0^\infty \frac{x^4 (1 + x^5)}{(1 + x)^{15}} dx,$

(iii) $\displaystyle\int_0^\infty \frac{x \, dx}{1 + x^6}.$

Solution: **(i)** We have

$$\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = \int_0^\infty \frac{x^8\,dx}{(1+x)^{24}} - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\,dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

$$= \mathbf{B}(9,15) - \mathbf{B}(15,9), \qquad \text{[By article 3]}$$

$$= \mathbf{B}(9,15) - \mathbf{B}(9,15),$$

by symmetry of Beta function

$$= 0.$$

(ii) We have $\displaystyle\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx = \int_0^\infty \frac{x^4\,dx}{(1+x)^{15}} + \int_0^\infty \frac{x^9\,dx}{(1+x)^{15}}$

$$= \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}}\,dx + \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+5}}\,dx$$

$$= \mathbf{B}(5,10) + \mathbf{B}(10,5) = \mathbf{B}(5,10) + \mathbf{B}(5,10)$$

$$= 2\,\mathbf{B}(5,10) = 2\,\frac{\Gamma 5\,\Gamma 10}{\Gamma 15}$$

$$= 2\cdot\frac{4\cdot 3\cdot 2\cdot 1}{14\cdot 13\cdot 12\cdot 11\cdot 10} = \frac{1}{5005}.$$

(iii) Let $\displaystyle I = \int_0^\infty \frac{x\,dx}{1+x^6}.$

Put $x^6 = y$ or $x = y^{1/6}$, so that $dx = \frac{1}{6}y^{-5/6}\,dy$.

$$\therefore \quad I = \frac{1}{6}\int_0^\infty \frac{y^{1/6}\cdot y^{-5/6}}{1+y}\,dy = \frac{1}{6}\int_0^\infty \frac{y^{-2/3}}{1+y}\,dy$$

$$= \frac{1}{6}\int_0^\infty \frac{y^{(1/3)-1}}{(1+y)^{(1/3)+(2/3)}}\,dy = \frac{1}{6}B\left(\frac{1}{3},\frac{2}{3}\right), \qquad \text{[By article 3]}$$

$$= \frac{1}{6}\,\frac{\Gamma\frac{1}{3}\,\Gamma\frac{2}{3}}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)} = \frac{1}{6}\,\frac{\Gamma\frac{1}{3}\,\Gamma\left(1-\frac{1}{3}\right)}{\Gamma 1}$$

$$= \frac{1}{6}\cdot\frac{\pi}{\sin\frac{1}{3}\pi} \qquad \left[\because\ \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{1}{6}\cdot\frac{\pi}{(\sqrt{3}/2)} = \frac{1}{6}\cdot\frac{2\pi}{\sqrt{3}} = \frac{\pi}{3\sqrt{3}}.$$

Example 9: *Show that* $\displaystyle\int_0^1 \frac{dx}{(1-x^n)^{1/2}} = \frac{\sqrt{\pi}\,\Gamma(1/n)}{n\,\Gamma(1/n+1/2)}.$

(Kumaun 2012)

Solution: Let $x^n = \sin^2 \theta$ *i.e.*, $x = \sin^{2/n} \theta$ so that $dx = \frac{2}{n} \sin^{(2/n)-1} \theta \cos \theta \, d\theta$.

Then
$$\int_0^1 \frac{dx}{\sqrt{(1-x^n)}} = \frac{2}{n} \int_0^{\pi/2} \frac{\sin^{(2/n)-1} \theta \cos \theta \, d\theta}{\cos \theta}$$
$$= \frac{2}{n} \int_0^{\pi/2} \sin^{(2/n)-1} \theta \cos^0 \theta \, d\theta$$
$$= \frac{2}{n} \cdot \frac{\Gamma(1/n)\,\Gamma(\frac{1}{2})}{2\,\Gamma(1/n+1/2)} = \frac{\sqrt{\pi}}{n} \cdot \frac{\Gamma(1/n)}{\Gamma(1/n+1/2)}.$$

Example 10: *Evaluate* $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$.

(Kumaun 2002, 10)

Solution: We have
$$\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx = \int_0^1 \frac{x^{m-1}\, dx}{(1+x)^{m+n}} + \int_0^1 \frac{x^{n-1}\, dx}{(1+x)^{m+n}}. \qquad \ldots(1)$$

Now in the second integral on the R.H.S. of (1), we put $x = 1/y$ so that $dx = -(1/y^2)\, dy$; also when $x \to 0$, $y \to \infty$ and when $x = 1$, $y = 1$.

$\therefore$
$$\int_0^1 \frac{x^{n-1}\, dx}{(1+x)^{m+n}} = \int_\infty^1 \frac{(1/y)^{n-1}}{(1+1/y)^{m+n}} \left(-\frac{1}{y^2} dy\right)$$
$$= -\int_\infty^1 \frac{y^{m+n}\, dy}{(1+y)^{m+n} \cdot y^{n-1} \cdot y^2} = \int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

(Note)

$$= \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx. \qquad \left[\because \int_a^b f(x)\, dx = \int_a^b f(y)\, dy\right]$$

Now from (1), we have
$$\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$
$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx,$$

by a property of definite integrals

$$= \mathbf{B}(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}. \qquad \text{[Refer articles 3 and 7]}$$

Example 11: *Show that* $\int_0^{\pi/2} (\tan x)^n\, dx = \frac{\pi}{2} \sec \frac{n\pi}{2}$, *where* $-1 < n < 1$.

Solution: We have

$$\int_0^{\pi/2} \tan^n x\,dx = \int_0^{\pi/2} \frac{\sin^n x}{\cos^n x}\,dx = \int_0^{\pi/2} \sin^n x \cos^{-n} x\,dx$$

$$= \frac{\Gamma \frac{1}{2}(n+1) . \Gamma \frac{1}{2}(-n+1)}{2\,\Gamma \frac{1}{2}(n-n+2)},$$

where $-n+1>0$ *i.e.*, $n<1$ and $n+1>0$ *i.e.*, $n>-1$

$$= \frac{1}{2} \Gamma \frac{1}{2}(n+1)\,\Gamma \frac{1}{2}(1-n)$$

$$= \frac{1}{2} \Gamma \frac{1}{2}(n+1)\,\Gamma\,[1-\frac{1}{2}(n+1)] = \frac{1}{2} \frac{\pi}{\sin \frac{1}{2}(n+1)\,\pi},$$

$$\left[\because\ \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}, \text{by cor. to article 7}\right]$$

$$= \frac{\pi}{2} \cdot \frac{1}{\sin(\frac{1}{2}\pi + \frac{1}{2}n\pi)} = \frac{\pi}{2} \cdot \frac{1}{\cos(\frac{1}{2}n\pi)}$$

$$= \frac{\pi}{2} \sec \frac{n\pi}{2}, \quad \text{where } -1<n<1.$$

Example 12: *Prove that*

(i) $\int_0^\infty x^{2n-1} e^{-ax^2}\,dx = \frac{\Gamma(n)}{2a^n}$;

(ii) $\int_0^\infty x^m e^{-ax^n}\,dx = \frac{\Gamma[(m+1)/n]}{na^{(m+1)/n}}$;

(iii) $\int_0^1 \frac{dx}{\sqrt{(-\log x)}} = \sqrt{\pi}$.

Solution: **(i)** Let $I = \int_0^\infty x^{2n-1} e^{-ax^2}\,dx = \int_0^\infty x^{2n-2} e^{-ax^2}\,x\,dx$.

Put $ax^2 = z$ so that $2ax\,dx = dz$. When $x=0$, $z=0$ and when $x \to \infty$, $z \to \infty$.

$$\therefore \quad I = \int_0^\infty \left(\frac{z}{a}\right)^{n-1} e^{-z} \frac{1}{2a}\,dz = \frac{1}{2a^n}\int_0^\infty e^{-z} z^{n-1}\,dz$$

$$= \frac{1}{2a^n}\Gamma(n), \text{ by definition of Gamma function.}$$

(ii) Let $I = \int_0^\infty x^m e^{-ax^n}\,dx = \int_0^\infty \frac{x^m}{x^{n-1}} e^{-ax^n}\,x^{n-1}\,dx$ **[Note]**

$$= \int_0^\infty x^{m-n+1} e^{-ax^n}\,x^{n-1}\,dx.$$

Put $ax^n = t$ so that $nax^{n-1}\,dx = dt$. Also when $x=0$, $t=0$ and when $x \to \infty$, $t \to \infty$.

$$\therefore \quad I = \int_0^\infty \left(\frac{t}{a}\right)^{(m-n+1)/n} e^{-t} \cdot \frac{1}{na}\,dt, \qquad \left[\because\ ax^n = t \Rightarrow x = \left(\frac{t}{a}\right)^{1/n}\right]$$

$$= \frac{1}{na \,.\, a^{(m-n+1)/n}} \int_0^\infty t^{\{(m+1)/n\}-1} e^{-t} \, dt$$

$$= \frac{1}{na^{(m+1)/n}} \Gamma\{(m+1)/n\}, \text{ by the definition of Gamma function.}$$

(iii) Let $I = \int_0^1 \frac{dx}{\sqrt{(-\log x)}} = \int_0^1 \frac{dx}{\sqrt{\{\log (1/x)\}}} = \int_0^1 \left(\log \frac{1}{x}\right)^{-1/2} dx$.

Put $\log (1/x) = y$ *i.e.*, $1/x = e^y$ *i.e.*, $x = e^{-y}$ so that $dx = -e^{-y} \, dy$.

Also when $x \to \infty$, $y \to \infty$ and when $x = 1$, $y = 0$.

$\therefore$ $I = -\int_\infty^0 y^{-1/2} e^{-y} \, dy = \int_0^\infty e^{-y} y^{1/2-1} \, dy$

$= \Gamma(\frac{1}{2})$, by the def. of Gamma function

$= \sqrt{\pi}$.

Example 13: *Evaluate the integral*

$\int_a^b (x-a)^p (b-x)^q \, dx$, *where p and q are positive integers.*

Solution: Let $I = \int_a^b (x-a)^p (b-x)^q \, dx$.

Put $x = a\cos^2\theta + b\sin^2\theta$ so that

$dx = -2a\cos\theta\sin\theta \, d\theta + 2b\sin\theta\cos\theta \, d\theta$

i.e., $dx = 2(b-a)\cos\theta\sin\theta \, d\theta$.

Also $x - a = a\cos^2\theta + b\sin^2\theta - a = b\sin^2\theta - a(1-\cos^2\theta)$

$= b\sin^2\theta - a\sin^2\theta = (b-a)\sin^2\theta$

and $b - x = b - a\cos^2\theta - b\sin^2\theta = b(1-\sin^2\theta) - a\cos^2\theta$

$= (b-a)\cos^2\theta$.

To find the limits for θ, when $x = a$, we have

$a = a\cos^2\theta + b\sin^2\theta$

i.e., $(b-a)\sin^2\theta = 0$ *i.e.*, $\sin^2\theta = 0$ as $a \neq b$ *i.e.*, $\theta = 0$

and when $x = b$, we have

$b = a\cos^2\theta + b\sin^2\theta$

i.e., $(a-b)\cos^2\theta = 0$ *i.e.*, $\cos^2\theta = 0$ as $a \neq b$ *i.e.*, $\theta = \pi/2$.

Thus the new limits for θ are 0 to $\pi/2$. Hence the given integral

$$I = \int_0^{\pi/2} (b-a)^p \sin^{2p}\theta \,.\, (b-a)^q \cos^{2q}\theta \,.\, 2(b-a)\cos\theta\sin\theta \, d\theta$$

$$= 2(b-a)^{p+q+1} \int_0^{\pi/2} \sin^{2p+1}\theta \cos^{2q+1}\theta \, d\theta$$

$$= 2\,(b-a)^{p+q+1}\,\frac{\Gamma\{\frac{1}{2}(2p+1+1)\}\,\Gamma\{\frac{1}{2}(2q+1+1)\}}{2\,\Gamma\,(2p+1+2q+1+2)},$$

provided $2p+1>-1$ and $2q+1>-1$ *i.e.*, $p>-1$ and $q>-1$ which is so because p and q are given to be + ive integers

$$= (b-a)^{p+q+1}\,\frac{\Gamma\,(p+1)\,\Gamma\,(q+1)}{\Gamma\,(p+q+1+1)}$$

$$= (b-a)^{p+q+1}\,\frac{p\,!\,q\,!}{(p+q+1)\,!},$$

because $\Gamma\,(n+1)=n\,!$ if n is a positive integer.

Example 14: *Find the value of* $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\ldots\Gamma\left(\frac{8}{9}\right).$

Solution: We know that

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)=\frac{(2\pi)^{(n-1)/2}}{n^{1/2}},$$

where n is a positive integer.

Putting $n=9$ in the above relation, we get

$$\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\ldots\Gamma\left(\frac{8}{9}\right)=\frac{(2\,\pi)^{(9-1)/2}}{9^{1/2}}=\frac{(2\,\pi)^4}{3}=\frac{16}{3}\,\pi^4.$$

Example 15: *Show that*

(i) $2^n\,\Gamma\,(n+\frac{1}{2})=1.3.5.\ldots(2n-1)\,\sqrt{\pi}$, *where n is a + ive integer,*

(Lucknow 2009)

(ii) $\Gamma\left(\frac{3}{2}-x\right)\Gamma\left(\frac{3}{2}+x\right)=\left(\frac{1}{4}-x^2\right)\pi\sec\pi x$, *provided* $-1<2x<1$.

(Kumaun 2011)

Solution: **(i)** We have

$$\Gamma\,(n+\tfrac{1}{2})=(n-\tfrac{1}{2})\,\Gamma\,(n-\tfrac{1}{2})=\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$$

$$=\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\ldots\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)$$

$$=\frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdot\frac{2n-5}{2}\ldots\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$$

$$=\frac{1}{2^n}\,(2n-1)\,(2n-3)\,(2n-5)\ldots3.1.\sqrt{\pi}\,.$$

$\therefore$ $$2^n\,\Gamma\,(n+\tfrac{1}{2})=1.3.5.\ldots(2n-1)\,\sqrt{\pi}\,.$$

(ii) We have

$$\Gamma\left(\frac{3}{2}-x\right)\Gamma\left(\frac{3}{2}+x\right) = (\frac{1}{2}-x)\,\Gamma\,(\frac{1}{2}-x)\,.\,(\frac{1}{2}+x)\,\Gamma\,(\frac{1}{2}+x)$$

$$=\left(\frac{1}{4}-x^2\right)\Gamma\left(\frac{1-2x}{2}\right)\Gamma\left(\frac{1+2x}{2}\right)$$

$$=\left(\frac{1}{4}-x^2\right)\Gamma\left(\frac{1-2x}{2}\right)\Gamma\left(1-\frac{1-2x}{2}\right)$$

$$=\left(\frac{1}{4}-x^2\right)\frac{\pi}{\sin\left(\frac{1-2x}{2}\pi\right)}$$

$$=\left(\frac{1}{4}-x^2\right)\cdot\frac{\pi}{\sin\left(\frac{1}{2}\pi-x\pi\right)}$$

$$=\left(\frac{1}{4}-x^2\right)\cdot\frac{\pi}{\cos x\pi}=\left(\frac{1}{4}-x^2\right)\cdot\pi\sec\pi x\,.$$

Example 16: *With certain restrictions on the values of a, b, m and n, prove that*

$$\int_0^\infty\int_0^\infty e^{-(ax^2+by^2)}\,x^{2m-1}\,y^{2n-1}\,dx\,dy=\frac{\Gamma(m)\,\Gamma(n)}{4a^m b^n}\,.$$

Solution: Let us denote the given integral by I. Then

$$I=\int_0^\infty e^{-ax^2}\,x^{2m-1}\,dx\times\int_0^\infty e^{-by^2}\,y^{2n-1}\,dy=I_1\times I_2\,.$$

To evaluate I_1, put $ax^2=t$ so that $2ax\,dx=dt$.

$\therefore$
$$I_1=\int_0^\infty e^{-t}\,(t/a)^{(2m-1)/2}\cdot\frac{dt}{2\sqrt{(at)}}$$

$$=\frac{1}{2a^m}\int_0^\infty e^{-t}\,t^{m-1}\,dt$$

$$=\frac{\Gamma(m)}{2a^m},$$ provided a and m are +ive.

Similarly, $I_2=\dfrac{\Gamma(n)}{2b^n}$, provided b and n are +ive.

Hence
$$I=\frac{\Gamma m\,\Gamma n}{4a^m b^n}\cdot$$

Example 17: *Show that the sum of the series*

$$\frac{1}{n+1}+m\,\frac{1}{n+2}+\frac{m(m+1)}{2!}\cdot\frac{1}{n+3}+\frac{m(m+1)(m+2)}{3!}\cdot\frac{1}{n+4}+\ldots$$

is $\dfrac{\Gamma(n+1)\,\Gamma(1-m)}{\Gamma(n-m+2)}$, *where* $-1<n<1$.

Solution: We have

$$\frac{\Gamma(n+1)\,\Gamma(1-m)}{\Gamma(n-m+2)} = \mathbf{B}\,(n+1, 1-m)$$

$$= \int_0^1 x^n (1-x)^{-m}\, dx$$

$$= \int_0^1 x^n \left[1 + mx + \frac{m(m+1)}{2!} x^2 + \frac{m(m+1)(m+2)}{3!} x^3 + \ldots\right] dx$$

$$= \int_0^1 \left[x^n + mx^{n+1} + \frac{m(m+1)}{2!} x^{n+2} + \frac{m(m+1)(m+2)}{3!} x^{n+3} + \ldots\right] dx$$

$$= \left[\frac{x^{n+1}}{n+1} + m\frac{x^{n+2}}{n+2} + \frac{m(m+1)}{2!}\frac{x^{n+3}}{n+3} + \frac{m(m+1)(m+2)}{3!}\frac{x^{n+4}}{n+4} + \ldots\right]_0^1$$

$$= \frac{1}{n+1} + m.\frac{1}{n+2} + \frac{m(m+1)}{2!}\cdot\frac{1}{n+3} + \frac{m(m+1)(m+2)}{3!}\frac{1}{n+4} + \ldots$$

Example 18: *Prove that* $\int_0^\infty \frac{\sin bz}{z}\, dz = \frac{\pi}{2}\cdot$

Solution: We have

$$I = \int_0^\infty \int_0^\infty e^{-xz} \sin bz\, dx\, dz$$

$$= \int_0^\infty \left[\frac{e^{-xz}}{-z}\right]_0^\infty \sin bz\, dz \text{, on first integrating w.r.t. } x$$

$$= \int_0^\infty \frac{\sin bz}{z}\, dz\,. \qquad \ldots(1)$$

Again on first integrating w.r.t. z, we have

$$I = \int_0^\infty \int_0^\infty e^{-xz} \sin bz\, dx\, dz = \int_0^\infty \left[\int_0^\infty e^{-xz} \sin bz\, dz\right] dx$$

$$= \int_0^\infty \frac{b}{b^2+x^2}\, dx\,,$$ [See article 13, Deduction (ii)]

$$= \left[\tan^{-1}\frac{x}{b}\right]_0^\infty = \frac{\pi}{2}\cdot \qquad \ldots(2)$$

Hence equating the two values (1) and (2) of I, we have

$$\int_0^\infty \frac{\sin bz}{z}\, dz = \frac{\pi}{2}\cdot$$

Example 19: *Show that*

$$\int_0^\infty \cos(bz^{1/n})\, dz = \frac{1}{b^n}\Gamma(n+1) \,.\, \cos\frac{n\pi}{2}.$$

Solution: Put $z^{1/n} = x$ *i.e.*, $z = x^n$, so that $dz = nx^{n-1}\, dx$.

$$\therefore \quad \int_0^\infty \cos(bz^{1/n})\, dz = \int_0^\infty \cos(bx)\,.\, nx^{n-1}\, dx$$

$$= n\int_0^\infty x^{n-1}\cos(bx)\, dx$$

$$= \text{real part of } n\int_0^\infty e^{-ibx}\, x^{n-1}\, dx$$

$$= \text{real part of } n\,\frac{\Gamma(n)}{(ib)^n}$$

$$= \text{real part of } \frac{n\,\Gamma(n)}{b^n}\cdot\left(\cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi\right)^{-n}$$

$$= \text{real part of } \frac{\Gamma(n+1)}{b^n}\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)$$

$$= \frac{1}{b^n}\cdot\Gamma(n+1)\,.\cos\left(\frac{n\pi}{2}\right).$$

Example 20: *Show that* $\displaystyle\int_0^\infty \frac{x^c}{c^x}\, dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}, c > 1.$

(Gorakhpur 2005; Kanpur 07; Lucknow 10)

Solution: We have

$$I = \int_0^\infty \frac{x^c}{c^x}\, dx = \int_0^\infty \frac{x^c}{e^{\log c^x}}\, dx$$

$$= \int_0^\infty \frac{x^c}{e^{x\log c}}\, dx = \int_0^\infty e^{-x\log c}\, x^c\, dx.$$

Put $x\log c = y$ so that $(\log c)\, dx = dy$.

When $x = 0$, we have $y = 0$ and when $x \to \infty$, $y \to \infty$.

Also $c > 1 \Rightarrow \log c > 0$.

$$\therefore \quad I = \int_0^\infty e^{-y}\left(\frac{y}{\log c}\right)^c \frac{dy}{\log c}$$

$$= \frac{1}{(\log c)^{c+1}}\int_0^\infty e^{-y}\, y^{(c+1)-1}\, dy$$

$$= \frac{1}{(\log c)^{c+1}}\Gamma(c+1),$$

provided $c + 1 > 0$ which is so because $c > 1$.

Comprehensive Exercise 1

1. Prove that

(i) $\int_0^a \frac{dx}{(a^n - x^n)^{1/n}} = \frac{1}{n} \cdot \frac{\pi}{\sin(\pi / n)}$.

(Kanpur 2010)

(ii) $\int_0^2 (8 - x^3)^{-1/3}\, dx = \frac{2\pi}{3\sqrt{3}}$.

(Kumaun 2008)

2. Show that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}\, dx = \frac{\mathbf{B}(m, n)}{a^n (1+a)^m} = \frac{\Gamma(m)\Gamma(n)}{a^n (1+a)^m \Gamma(m+n)}$.

[Hint. Put $\frac{x(1+a)}{a+x} = y$].

3. Show that $\int_0^{\pi/2} \sin^p \theta \cos^q \theta\, d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right), p > -1, q > -1.$

Deduce that $\int_0^2 x^4 (8 - x^3)^{-1/3}\, dx = \frac{16}{3} B\left(\frac{5}{3}, \frac{2}{3}\right)$.

4. Prove that $B(m, n) = B(m+1, n) + B(m, n+1)$ for $m > 0, n > 0$.

(Kanpur 2005; Gorakhpur 05; Bundelkhand 11)

5. Prove that

(i) $\int_0^\infty e^{-ax} x^{n-1}\, dx = \frac{\Gamma(n)}{a^n}$. (ii) $\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx = \Gamma(n)$.

(Kumaun 2007)

6. Show that, if $m > -1$, then

$$\int_0^\infty x^m e^{-n^2 x^2}\, dx = \frac{1}{2\, n^{m+1}} \Gamma\left(\frac{m+1}{2}\right).$$

(Kumaun 2009)

7. Prove that $\int_0^1 x^m (1 - x^n)^p\, dx = \frac{1}{n} B\left(\frac{m+n}{n}, p+1\right)$ **(Lucknow 2010)**

8. Prove that $\int_0^1 (1 - x^n)^{1/n}\, dx = \frac{1}{n} \frac{[\Gamma(1/n)]^2}{2\,\Gamma(2/n)}$.

9. Show that $\Gamma(0 \cdot 1)\, \Gamma(0 \cdot 2)\, \Gamma(0 \cdot 3) \ldots \Gamma(0 \cdot 9) = \frac{(2\pi)^{9/2}}{\sqrt{(10)}}$.

10. Show that $\int_0^{\pi/2} \sqrt{(\sin\theta)}\, d\theta \times \int_0^{\pi/2} \frac{d\theta}{\sqrt{(\sin\theta)}} = \pi.$ **(Lucknow 2009)**

11. Show that $\int_0^1 \frac{x^2\, dx}{(1 - x^4)^{1/2}} \times \int_0^1 \frac{dx}{(1 + x^4)^{1/2}} = \frac{\pi}{4\sqrt{2}}$.

(Lucknow 2006, 11)

12. Show that $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = 2\int_0^{\pi/2} \sqrt{(\tan\theta)}\, d\theta = 4\int_0^\infty \frac{x^2\, dx}{1 + x^4} = \pi\sqrt{2}$.

(Lucknow 2008, 11)

13. Show that the perimeter of a loop of the curve $r^n = a^n \cos n\theta$ is

$$\frac{a}{n} \cdot 2^{(1/n)-1} \cdot \frac{[\Gamma(1/2n)]^2}{\Gamma(1/n)}.$$

14. Prove that $\int_0^1 \frac{dx}{\sqrt{(1-x^4)}} = \frac{\sqrt{2}}{8\sqrt{\pi}} \left[\Gamma\left(\frac{1}{4}\right)\right]^2.$

15. Show that $\int_0^{\pi/2} \sin^p \theta \, d\theta = \frac{1}{2} B\left(\frac{1}{2}, \frac{p+1}{2}\right).$

16. Show that

(i) $\int_0^\infty xe^{-\alpha x} \cos \beta x \, dx = \frac{(\alpha^2 - \beta^2)}{(\alpha^2 + \beta^2)^2}$

(ii) $\int_0^\infty xe^{-\alpha x} \sin \beta x \, dx = \frac{2\alpha\beta}{(\alpha^2 + \beta^2)^2}.$

17. Prove that $\int_{-\infty}^\infty \cos\left(\frac{1}{2}\pi x^2\right) dx = 1.$

18. Show that $\mathbf{B}(m, m) \,.\, \mathbf{B}\left(m + \frac{1}{2}, m + \frac{1}{2}\right) = \frac{\pi m^{-1}}{2^{4m-1}}.$

(Rohilkhand 2005)

19. Prove that $\int_0^\pi \frac{\sin^{n-1} x \, dx}{(a + b\cos x)^n} = \frac{2^{n-1}}{(a^2 - b^2)^{n/2}} \cdot \mathbf{B}\left(\frac{n}{2}, \frac{n}{2}\right), \; a > b.$

(Kumaun 2008)

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. For $m > 0, n > 0,$

(a) $B(m, n) = \frac{\Gamma(m)}{\Gamma(n)}$

(b) $B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

(c) $B(m, n) = \frac{\Gamma(n)}{\Gamma(m)}$

(d) $B(m, n) = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}$

2. The value of the integral $\int_0^\infty e^{-x} x^{-1/2} \, dx$ is

(a) $\frac{\sqrt{\pi}}{2}$

(b) $\frac{\pi}{2}$

(c) $\sqrt{\pi}$

(d) π

(Kumaun 2008, 09, 11, 13)

3. For $m>0, n>0$,

(a) $B(m,n)=\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$ (b) $B(m,n)=\int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

(c) $B(m,n)=\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$ (d) $B(m,n)=\int_0^\infty \frac{x^{m}}{(1+x)^{m+n}}\,dx$

4. If $a>0$ and $n>0$, then the value of the integral

$$\int_0^\infty e^{-ax}\,x^{n-1}\,dx \text{ is}$$

(a) $a^n\,\Gamma(n)$ (b) $a^{-n}\,\Gamma(n)$

(c) $\frac{\Gamma(n)}{2\,a^n}$ (d) $\frac{\Gamma(n)}{n^2}$

5. The value of $\int_0^1 x^4(1-x)^3\,dx$ is

(a) $\frac{1}{280}$ (b) $\frac{1}{180}$

(c) $\frac{1}{380}$ (d) $\frac{1}{80}$ **(Garhwal 2001)**

6. The value of the integral $\int_0^{\pi/2} \sin^4 x\cos^2 x\,dx$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{8}$

(c) $\frac{\pi}{16}$ (d) $\frac{\pi}{32}$ **(Garhwal 2001)**

7. The value of $\int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}}$ is

(a) $\Gamma(m)+\Gamma(n)$ (b) $\frac{\Gamma(m)}{\Gamma(n)}$

(c) $\Gamma(m).\Gamma(n)$ (d) $\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ **(Garhwal 2002)**

8. If $m,n>0$, then the value of $\int_0^1 x^{n-1}\left(\log\frac{1}{x}\right)^{m-1}dx$ is equal to

(a) $\frac{\Gamma(m)}{n^m}$ (b) $\frac{\Gamma(n)}{n^m}$

(c) $\frac{\Gamma(n)}{m^n}$ (d) $\frac{\Gamma(m)}{m^m}$ **(Garhwal 2003)**

9. The value of $\Gamma(\frac{1}{4}) \cdot \Gamma(\frac{3}{4})$ is ....

(a) π

(b) $\frac{\pi}{\sqrt{2}}$

(c) $\pi\sqrt{2}$

(d) $\sqrt{\frac{\pi}{2}}$

(Garhwal 2004, 11, 14)

10. The value of integral $\int_0^1 \frac{dx}{\sqrt{-\log x}}$ is

(a) $\sqrt{\pi}$

(b) π

(c) $\sqrt{\frac{\pi}{2}}$

(d) none of these

(Garhwal 2004)

11. If $m > 0, n > 0$, then $B(m, n)$ is defined as

(a) $\int_0^1 x^m (1-x)^n \, dx$

(b) $\int_0^1 x^{m-1} (1-x)^n \, dx$

(c) $\int_0^1 x^{m-1} (1-x)^{n-1} \, dx$

(d) $\int_0^1 x^{m-1} (1-x)^{n+1} \, dx$

(Garhwal 2006; Kumaun 15)

12. $\int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta = \dfrac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$ when

(a) $m > 0, \ n > 0$

(b) $m > -1, \ n > -1$

(c) $m > -\frac{1}{2}, \ n > -\frac{1}{2}$

(d) $0 < m < 1, 0 < n < 1$

(Garhwal 2006)

13. The value of $\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx$ is

(a) 1

(b) zero

(c) $(n-1)!$

(d) Γn

(Garhwal 2008)

14. The value of $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)\ldots\Gamma\left(\frac{8}{9}\right)$ is

(a) $\frac{1}{6}\pi^3$

(b) $16\pi^4$

(c) $\frac{1}{3}\pi^4$

(d) $\frac{16}{3}\pi^4$

(Garhwal 2009)

15. Integral $\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$ is equal to

(a) $B(m+1, n+1)$

(b) $B(m, n+1)$

(c) $B(m, n)$

(d) none of these

(Garhwal 2012)

16. Value of integral $\int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta$ is

(a) $\dfrac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$ (b) $\dfrac{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

(c) $\dfrac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+1}{2}\right)}$ (d) $\dfrac{\Gamma\left(\frac{m-1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{2\Gamma\left(\frac{m-n-2}{2}\right)}$

(Garhwal 2013)

17. The integral $\int_0^{\infty} x^{n-1} e^{-x} \, dx$ is known as

(a) Beta function (b) Gamma function

(c) Beta and Gamma function (d) none of these

(Kumaun 2007)

18. The value of $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)$ shall be

(a) $\dfrac{\sqrt{\pi}}{2}$ (b) $\dfrac{\pi}{\sqrt{2}}$

(c) $\dfrac{\sqrt{3}\,\pi}{2}$ (d) $\dfrac{2\pi}{\sqrt{3}}$

(Kumaun 2010)

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. The definite integral $\int_0^1 x^{m-1} (1-x)^{n-1} \, dx$, for $m > 0, n > 0$ is called the ……

2. The definite integral $\int_0^{\infty} e^{-x} x^{n-1} \, dx$, for $n > 0$ is called the …… .

(Garhwal 2001)

3. $B(m, n) = \dfrac{\Gamma(m)\Gamma(n)}{\ldots\ldots}$.

4. $\dfrac{B(m+1, n)}{B(m, n)} = \dfrac{m}{\ldots\ldots}$.

5. For $m > 0, n > 0, \int_0^{\infty} \dfrac{x^{m-1} - x^{n-1}}{(1+x)^{m+n}} \, dx = \ldots\ldots$.

6. For $n > 0, \Gamma(n+1) = \ldots\ldots \Gamma(n)$.

7. If n is a positive integer, then $\Gamma(n) = \ldots\ldots$.

8. If $0 < n < 1$, then $\Gamma(n)\,\Gamma(1-n) = \ldots\ldots$

9. $\Gamma\left(\frac{1}{2}\right) = \ldots\ldots$ (Kumaun 2014)

10. If $m > -1, n > -1$, then $\int_0^{\pi/2} \cos^m \theta \sin^n \theta \, d\theta = \dfrac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\ldots\ldots}$

11. $\int_0^\infty e^{-x^2}\, dx = \ldots\ldots$

12. For $m > 0, \Gamma(m)\,\Gamma\left(m + \frac{1}{2}\right) = \dfrac{\sqrt{\pi}}{\ldots\ldots}\,\Gamma(2m)$.

13. For $a > 0, n > 0, \int_0^\infty e^{-ax} x^{n-1}\, dx = \dfrac{\Gamma(n)}{\ldots\ldots}$.

14. The value of $\Gamma\left(\frac{1}{3}\right) \cdot \Gamma\left(\frac{2}{3}\right)$ is …… . (Agra 2002)

True or False

Write 'T' for true and 'F' for false statement.

1. $\int_0^\infty e^{-x} x^{1/2}\, dx = \Gamma\left(\frac{1}{2}\right)$.

2. $\int_0^\infty \dfrac{x\, dx}{1 + x^6} = \dfrac{\pi}{3\sqrt{3}}$.

3. $\int_0^\infty \dfrac{x^8 (1 - x^6)}{(1 + x)^{24}}\, dx = 1$.

4. For $m > 0, n > 0, B(m, n) = \int_0^\infty \dfrac{x^{n-1}}{(1 + x)^{m+n}}\, dx$.

5. For $m > 0, n > 0, \int_0^\infty \dfrac{x^{m-1}}{(1 + x)^{m+n}}\, dx = \int_0^\infty \dfrac{x^{n-1}}{(1 + x)^{m+n}}\, dx$.

6. $\Gamma(6) = 120$.

7. For $m > 0, n > 0, B(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m + n)}$.

8. $\int_0^\infty e^{-x^2}\, dx = \dfrac{\sqrt{\pi}}{4}$.

9. $B(m + 1, n) + B(m, n + 1) = B(m + 1, n + 1)$. (Agra 2003)

Answers

Multiple Choice Questions

1.	(b)	2.	(c)	3.	(a)	4.	(b)	5.	(a)
6.	(d)	7.	(d)	8.	(a)	9.	(c)	10.	(a)
11.	(a)	12.	(b)	13.	(d)	14.	(d)	15.	(c)
16.	(a)	17.	(b)	18.	(d)				

Fill in the Blank(s)

1.	Beta function	2.	Gamma function	3.	$\Gamma(m+n)$
4.	$m+n$	5.	0	6.	n
7.	$(n-1)!$	8.	$\dfrac{\pi}{\sin n\pi}$	9.	$\sqrt{\pi}$
10.	$2\,\Gamma\left(\dfrac{m+n+2}{2}\right)$	11.	$\dfrac{\sqrt{\pi}}{2}$	12.	2^{2m-1}
13.	a^n.	14.	$\dfrac{2\pi}{\sqrt{3}}$		

True or False

1.	F	2.	T	3.	F	4.	T	5.	T
6.	T	7.	F	8.	F	9.	F		

□

Chapter

5

Dirichlet's and Liouville's Integrals

1 Dirichlet's Theorem for three Variables

Theorem: *If l, m, n are all positive, then the triple integral*

$$\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)},$$

where the integral is extended to all positive values of the variables x, y and z subject to the condition $x + y + z \le 1$. **(Lucknow 2007)**

Proof: Let us first consider the double integral

$$I_2 = \iint x^{l-1} y^{m-1}\, dx\, dy,$$

where the integral is extended to all positive values of the variables x and y subject to the condition $x + y \le 1$.

Obviously the region of integration of I_2, in the 2-dimensional Euclidean space, is bounded by the straight lines $x = 0$, $y = 0$ and $x + y = 1$. The limits of integration for this region can be expressed as $0 \le x \le 1, 0 \le y \le 1 - x$.

$\therefore \qquad I_2 = \int_{x=0}^{1} \int_{y=0}^{1-x} x^{l-1} y^{m-1} \, dx \, dy$

$$= \int_0^1 x^{l-1} \left[\frac{y^m}{m} \right]_0^{1-x} dx = \int_0^1 \frac{1}{m} x^{l-1} (1-x)^m \, dx$$

$$= \frac{1}{m} \int_0^1 x^{l-1} (1-x)^{m+1-1} \, dx$$

$$= \frac{1}{m} \mathbf{B}(l, m+1), \quad \text{by the def. of Beta function}$$

$$= \frac{1}{m} \frac{\Gamma(l) \Gamma(m+1)}{\Gamma(l+m+1)} = \frac{1}{m} \frac{\Gamma(l) \cdot m \Gamma(m)}{\Gamma(l+m+1)}, \qquad [\because \Gamma(n+1) = n\Gamma(n)]$$

$$= \frac{\Gamma(l) \Gamma(m)}{\Gamma(l+m+1)}. \qquad ...(1)$$

(Remember)

This is Dirichlet's theorem for two variables.

Now consider the double integral $U_2 = \iint x^{l-1} y^{m-1} \, dx \, dy$,

where the integral is extended to all positive values of the variables x and y subject to the condition $x + y \le h$.

We have $\qquad x + y \le h \Rightarrow \frac{x}{h} + \frac{y}{h} \le 1$.

So putting $x/h = u$ and $y/h = v$ so that $dx = h\,du$ and $dy = h\,dv$, the integral U_2 becomes

$$U_2 = \iint (hu)^{l-1} (hv)^{m-1} h^2 \, du \, dv$$

$$= h^{l+m} \iint u^{l-1} v^{m-1} \, du \, dv, \text{where } u + v \le 1$$

$$= h^{l+m} \frac{\Gamma(L) \Gamma(m)}{\Gamma(l+m+1)}, \text{ by (1).} \qquad ...(2)$$

Now we consider the triple integral

$$I_3 = \iiint x^{l-1} y^{m-1} z^{n-1} \, dx \, dy \, dz,$$

subject to the condition $x + y + z \le 1$ *i.e.*, $y + z \le 1 - x$ and $0 \le x \le 1$.

We have

$$I_3 = \int_{x=0}^{1} \left[\iint y^{m-1} z^{n-1} \, dy \, dz \right] x^{l-1} \, dx, \text{where } y + z \le 1 - x$$

$$= \int_0^1 (1-x)^{m+n} \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n+1)} x^{l-1} \, dx, \qquad \text{by using (2)}$$

$$= \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n+1)} \int_0^1 x^{l-1} (1-x)^{m+n+1-1} \, dx$$

$$= \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n+1)} \mathbf{B}(l, m+n+1)$$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n+1)} \cdot \frac{\Gamma(l)\Gamma(m+n+1)}{\Gamma(l+m+n+1)}$$

$$= \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}, \text{ which proves the required result.}$$

Remark: The triple integral

$$\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = h^{l+m+n} \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)},$$

where the integral is extended to all positive values of the variables x, y and z subject to the condition $x+y+z \le h$.

Alternative proof of Dirichlet's theorem for three variables:

Let $$I_3 = \iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz,$$

where the integral is extended to all positive values of the variables x, y and z subject to the condition $x+y+z \le 1$.

Obviously the region of integration, in the 3-dimensional Euclidean space, is the volume bounded by the coordinate planes $x=0$, $y=0$, $z=0$ and the plane $x+y+z=1$. After a little geometric consideration, we observe that the limits of integration for this region can be expressed as

$$0 \le x \le 1, 0 \le y \le 1-x, 0 \le z \le 1-x-y.$$

Hence the triple integral I_3 may be written as

$$I_3 = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^{1-x} x^{l-1} y^{m-1} \left[\frac{z^n}{n}\right]_0^{1-x-y} dx\, dy$$

$$= \frac{1}{n}\int_0^1 \int_0^{1-x} x^{l-1} y^{m-1} (1-x-y)^n\, dx\, dy$$

$$= \frac{1}{n}\int_0^1 x^{l-1} \left[\int_0^{1-x} y^{m-1} \{(1-x)-y\}^n\, dy\right] dx.$$

To integrate w.r.t. y, put $y=(1-x)t$ so that $dy=(1-x)\,dt$; also when $y=0$, $t=0$ and when $y=1-x$, $t=1$.

$\therefore$ the required integral

$$I_3 = \frac{1}{n}\int_0^1 x^{l-1} \left[\int_0^1 (1-x)^{m-1} t^{m-1} \{(1-x)^n (1-t)^n\} (1-x)\, dt\right] dx$$

$$= \frac{1}{n}\int_0^1 \int_0^1 x^{l-1} (1-x)^{m+n} t^{m-1} (1-t)^n\, dx\, dt$$

$$= \frac{1}{n}\int_0^1 x^{l-1} (1-x)^{m+n}\, dx \times \int_0^1 t^{m-1} .(1-t)^n\, dt$$

$$= \frac{1}{n} B(l, m+n+1)\, B(m, n+1),$$

(by the definition of Beta function)

$$= \frac{1}{n} \cdot \frac{\Gamma(l)\,\Gamma(m+n+1)}{\Gamma(l+m+n+1)} \cdot \frac{\Gamma(m)\,\Gamma(n+1)}{\Gamma(m+n+1)} = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}.$$

$[\because\ \Gamma(n+1) = n\,\Gamma(n)]$

Note: Dirichlet's theorem holds good even if the condition is taken as $x+y+z<1$ in place of $x+y+z \le 1$.

Corollary: *Evaluate without using Dirichlet's Theorem* $\iiint x^p\, y^q\, z^r\, dx\, dy\, dz$, *where x, y, z are always positive and* $x+y+z \le 1$.

2 Dirichlet's Theorem for *n* Variables

The theorem states that

$$\iint \dots \int x_1^{l_1-1}\, x_2^{l_2-1} \dots x_n^{l_n-1}\, dx_1\, dx_2 \dots dx_n = \frac{\Gamma(l_1)\,\Gamma(l_2)\dots\Gamma(l_n)}{\Gamma(1+l_1+l_2+\dots+l_n)},$$

where the integral is extended to all positive values of the variables $x_1, x_2, \dots, x_n$ *subject to the condition* $x_1+x_2+\dots+x_n \le 1$.

Proof: We shall prove the theorem by mathematical induction. To start the induction we shall first show that the theorem is true for two variables *i.e.*, for $n=2$.

So let us consider the integral $I_2 = \iint x_1^{l_1-1}\, x_2^{l_2-1}\, dx_1\, dx_2$

subject to the condition $x_1+x_2 \le 1$.

Now proceeding as in article 1, show that $I_2 = \dfrac{\Gamma(l_1)\,\Gamma(l_2)}{\Gamma(1+l_1+l_2)}$. ...(1)

The result (1) shows that the theorem is true for two variables *i.e.*, for $n=2$.

Now assume as our induction hypothesis that the theorem is true for *n* variables *i.e.*, assume that

$$I_n = \iint \dots \int x_1^{l_1-1}\, x_2^{l_2-1} \dots x_n^{l_n-1}\, dx_1\, dx_2 \dots dx_n$$

$$= \frac{\Gamma(l_1)\,\Gamma(l_2)\dots\Gamma(l_n)}{\Gamma(1+l_1+l_2+\dots+l_n)}, \quad \text{...(2)}$$

subject to the condition $x_1+x_2+\dots+x_n \le 1$.

If the condition be $x_1+x_2+\dots+x_n \le h$, then putting

$\dfrac{x_1}{h} = u_1, \dfrac{x_2}{h} = u_2, \dots, \dfrac{x_n}{h} = u_n$, so that

$dx_1 = h\,du_1,\ dx_2 = h\,du_2, \dots, dx_n = h\,du_n$, we have

$$\iint \dots \int x_1^{l_1-1}\, x_2^{l_2-1} \dots x_n^{l_n-1}\, dx_1\, dx_2 \dots dx_n$$

$$= h^{l_1+l_2+\dots+l_n} \iint \dots \int u_1^{l_1-1}\, u_2^{l_2-1} \dots u_n^{l_n-1}\, du_1\, du_2 \dots du_n$$

subject to the condition $u_1+u_2+\dots+u_n \le 1$

$$= h^{l_1 + l_2 + \ldots + l_n} \frac{\Gamma(l_1)\Gamma(l_2)\ldots\Gamma(l_n)}{\Gamma(1 + l_1 + l_2 + \ldots + l_n)}, \qquad \ldots(3)$$

using the assumed result (2).

Now for $n + 1$ variables the condition is

$$x_1 + x_2 + \ldots + x_n + x_{n+1} \leq 1$$

i.e., $x_2 + x_3 + \ldots + x_n + x_{n+1} \leq 1 - x_1$, and $0 \leq x_1 \leq 1$.

We then have

$$\iint \ldots \int x_1^{l_1 - 1} x_2^{l_2 - 1} \ldots x_n^{l_n - 1} x_{n+1}^{l_{n+1} - 1} dx_1\, dx_2 \ldots dx_n\, dx_{n+1},$$

where $x_1 + x_2 + \ldots + x_{n+1} \leq 1$

$$= \int_{x_1 = 0}^{1} x_1^{l_1 - 1} \left[\iint \ldots \int x_2^{l_2 - 1} \ldots x_{n+1}^{l_{n+1} - 1} dx_2 \ldots dx_{n+1} \right] dx_1$$

$$= \int_{x_1 = 0}^{1} x_1^{l_1 - 1} \cdot \frac{\Gamma(l_2)\Gamma(l_3)\ldots\Gamma(l_{n+1})}{\Gamma(1 + l_2 + l_3 + \ldots + l_n + l_{n+1})} \cdot (1 - x_1)^{l_2 + l_3 + \ldots + l_{n+1}} dx_1,$$

[using (3)]

$$= \frac{\Gamma(l_2)\Gamma(l_3)\ldots\Gamma(l_{n+1})}{\Gamma(1 + l_2 + \ldots + l_n + l_{n+1})} \cdot \int_0^1 x_1^{l_1 - 1} (1 - x_1)^{(1 + l_2 + l_3 + \ldots + l_{n+1}) - 1} dx_1$$

$$= \frac{\Gamma(l_2)\Gamma(l_3)\ldots\Gamma(l_{n+1})}{\Gamma(1 + l_2 + \ldots + l_n + l_{n+1})} \cdot \frac{\Gamma(l_1)\Gamma(1 + l_2 + \ldots + l_{n+1})}{\Gamma(1 + l_1 + l_2 + \ldots + l_n + l_{n+1})}$$

$$= \frac{\Gamma(l_1)\Gamma(l_2)\ldots\Gamma(l_{n+1})}{\Gamma(1 + l_1 + l_2 + \ldots + l_{n+1})}. \qquad \ldots(4)$$

The result (4) shows that the theorem holds for $(n + 1)$ variables if it holds for n variables. But we have seen that the theorem is true for two variables. Hence by mathematical induction the theorem is true for all values of n.

Illustrative Examples

Example 1: *Evaluate* $\iint x^{2l - 1} y^{2m - 1}\, dx\, dy$ *for all positive values of* x *and* y *such that* $x^2 + y^2 \leq c^2$.

(Lucknow 2007)

Solution: Let us denote the given integral by I. Then we have to find the value of I extended to all positive values of x and y subject to the condition

$$\left(\frac{x}{c}\right)^2 + \left(\frac{y}{c}\right)^2 \leq 1.$$

Put $(x / c)^2 = u$ *i.e.*, $x = cu^{1/2}$, so that $dx = \frac{1}{2} cu^{-1/2}\, du$,

and $(y / c)^2 = v$ *i.e.*, $y = cv^{1/2}$, so that $dy = \frac{1}{2} cv^{-1/2}\, dv$.

Then the required integral

$$I = \iint (cu^{1/2})^{2l-1} (cv^{1/2})^{2m-1} \cdot \frac{1}{2} cu^{-1/2} \cdot \frac{1}{2} cv^{-1/2} \, du \, dv$$

$$= \frac{1}{4} c^{2l+2m} \iint u^{l-1} v^{m-1} \, du \, dv,$$ where u, v take all +ive values subject to the condition $u + v \le 1$

$$= \frac{1}{4} c^{2l+2m} \cdot \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m+1)},$$ by Dirichlet's theorem.

Example 2: *Find the value of* $\iint \dots \int dx_1 \, dx_2 \dots dx_n$ *extended to all positive values of the variables, subject to the condition* $x_1^2 + x_2^2 + \dots + x_n^2 < R^2$.

Solution: Let us denote the given integral by I. Then we have to find the value of I extended to all positive values of $x_1, x_2, \dots, x_n$ subject to the condition

$$\frac{x_1^2}{R^2} + \frac{x_2^2}{R^2} + \dots + \frac{x_n^2}{R^2} < 1.$$

Put $(x_1 / R)^2 = u_1$ *i.e.*, $x_1 = R u_1^{1/2}$, so that $dx_1 = \frac{1}{2} R u_1^{-1/2} \, du_1$,

$(x_2 / R)^2 = u_2$ *i.e.*, $x_2 = R u_2^{1/2}$, so that $dx_2 = \frac{1}{2} R u_2^{-1/2} \, du_2$,

and so on.

Then the required integral

$$I = \iint \dots \int \left(\frac{1}{2}\right)^n R^n u_1^{-1/2} u_2^{-1/2} \dots u_n^{-1/2} \, du_1 \, du_2 \dots du_n$$

$$= \left(\frac{R}{2}\right)^n \iint \dots \int u_1^{(1/2)-1} u_2^{(1/2)-1} \dots u_n^{(1/2)-1} \, du_1 \, du_2 \dots du_n,$$ subject to the condition $u_1 + u_2 + \dots + u_n < 1$

$$= \left(\frac{R}{2}\right)^n \frac{\{\Gamma(\frac{1}{2})\}^n}{\Gamma(1 + n \cdot \frac{1}{2})},$$ by Dirichlet's theorem

$$= \left(\frac{R}{2}\right)^n \cdot \frac{\pi^{n/2}}{\Gamma(1 + \frac{1}{2} n)}. \qquad [\because \Gamma(\tfrac{1}{2}) = \sqrt{\pi}]$$

Example 3: *Find the volume of the solid surrounded by the surface*

$$(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1.$$ **(Kumaun 2011)**

Solution: Since the equation $(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1$ does not change by putting $-x$ for x, $-y$ for y and $-z$ for z, therefore the surface represented by this equation is symmetrical in all the eight octants.

So the volume of the solid surrounded by this surface $= 8 \times$ the volume of the portion of this solid lying in the positive octant.

Now the volume of a small element situated at any point $(x, y, z) = dx\,dy\,dz$.

$\therefore$ the volume of the solid in the positive octant

$$= \iiint dx\,dy\,dz\,,$$

where the integral is extended to all positive values of the variables x, y, z subject to the condition $(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} \le 1$.

Now put $(x/a)^{2/3} = u,\ (y/b)^{2/3} = v,\ (z/c)^{2/3} = w$

i.e., $x = au^{3/2},\ y = bv^{3/2},\ z = cw^{3/2}$

so that $dx = \frac{3}{2} au^{1/2}\,du,\ dy = \frac{3}{2} bv^{1/2}\,dv,\ dz = \frac{3}{2} cw^{1/2}\,dw$.

$\therefore$ the volume in the positive octant

$$= \iiint \frac{27}{8} abc\, u^{(3/2)-1} v^{(3/2)-1} w^{(3/2)-1}\, du\,dv\,dw\,,$$

where $u + v + w \le 1$

$$= \frac{27}{8} abc \frac{[\Gamma(3/2)]^3}{\Gamma\left(\frac{9}{2}+1\right)} = \frac{27}{8} abc \cdot \frac{\left(\frac{1}{2}\cdot\sqrt{\pi}\right)^3}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$$

$$= \frac{27}{8} abc \cdot \frac{\pi}{8} \cdot \frac{32}{27.35} = \frac{\pi\, abc}{8} \cdot \frac{4}{35}.$$

Hence the required volume

$$= 8 \cdot \frac{\pi\, abc}{8} \cdot \frac{4}{35} = \frac{4\pi\, abc}{35}.$$

3 Liouville's Extension of Dirichlet's Theorem

Theorem: *If the variables x, y, z are all positive such that*

$$h_1 \le x + y + z \le h_2,$$

then the triple integral

$$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1} dx\,dy\,dz$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \int_{h_1}^{h_2} f(u)\, u^{l+m+n-1} du.$$

Proof: Let $I = \iiint x^{l-1} y^{m-1} z^{n-1}\, dx\,dy\,dz$, integrated over some region.

Subject to the condition $x + y + z \le u$, we have by Dirichlet's theorem

$$I = u^{l+m+n} \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}. \quad ...(1)$$

If the condition be $x + y + z \le u + \delta u$, then

$$I = (u+\delta u)^{l+m+n} \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}. \quad ...(2)$$

Therefore the value of the integral I extended to all such positive values of the variables as make the sum of the variables lie between u and $u + \delta u$ is

$$= \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}[(u+\delta u)^{l+m+n} - u^{l+m+n}],$$

[subtracting (2) from (1)]

$$= \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)} u^{l+m+n}\left[\left(1+\frac{\delta u}{u}\right)^{l+m+n} - 1\right]$$

$$= \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)} u^{l+m+n}\left[1+(l+m+n)\frac{\delta u}{u}+\ldots - 1\right]$$

$$= \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)} (l+m+n)\, u^{l+m+n-1}\,\delta u,$$

to the first order of approximation

$$= \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)} u^{l+m+n-1}\,\delta u.$$

Now consider the integral

$$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz,$$

subject to the condition $h_1 \le x+y+z \le h_2$.

If $x+y+z$ lies between u and $u+\delta u$, the value of $f(x+y+z)$ can only differ from $f(u)$ by a small quantity of the same order as δu. Hence neglecting square of δu, the part of the integral

$$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$$

which arises from supposing the sum of the variables to lie between u and $u+\delta u$ is ultimately equal to $\dfrac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)} f(u) \,.\, u^{l+m+n-1}\,\delta u$.

Therefore the whole integral

$$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz,$$

where $h_1 \le x+y+z \le h_2$, is equal to

$$\frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)} \int_{h_1}^{h_2} f(u)\,.\,u^{l+m+n-1}\, du.$$

Remark: The above theorem holds good even if we take the condition as $h_1 < x+y+z < h_2$ in place of $h_1 \le x+y+z \le h_2$.

Illustrative Examples

Example 4: *Find the value of* $\iiint \log(x+y+z)\,dx\,dy\,dz$, *the integral extending over all positive values of* x, y, z *subject to the condition* $x+y+z<1$.

(Lucknow 2009; Kumaun 10)

Solution: Here the integral is to be extended for all positive values of x, y and z such that $0<x+y+z<1$.

$\therefore$ the required integral

$$= \iiint \log(x+y+z)\,dx\,dy\,dz, \text{ where } 0<x+y+z<1$$

$$= \iiint \log(x+y+z)\,x^{1-1}\,y^{1-1}\,z^{1-1}\,dx\,dy\,dz \qquad \textbf{(Note)}$$

$$= \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1)} \int_0^1 (\log u)\,u^{1+1+1-1}\,du,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{1}{\Gamma(3)} \int_0^1 u^2 \log u\,du, \qquad [\because\ \Gamma(1)=1]$$

$$= \frac{1}{2.1}\left[\left((\log u)\cdot\frac{u^3}{3}\right)_0^1 - \int_0^1 \frac{1}{u}\cdot\frac{u^3}{3}\,du\right],$$

integrating by parts taking u^2 as the second function

$$= \frac{1}{2}\left[0 - \frac{1}{3}\lim_{u\to 0} u^3 \log u - \frac{1}{3}\int_0^1 u^2\,du\right]$$

$$= -\frac{1}{6}\left[\frac{u^3}{3}\right]_0^1, \qquad \left[\because\ \lim_{u\to 0} u^3 \log u = 0\right]$$

$$= -\frac{1}{18}.$$

Note: $\lim_{u\to 0} u^3 \log u = \lim_{u\to 0} \frac{\log u}{1/u^3} = \lim_{u\to 0} \frac{1/u}{-3/u^4} = \lim_{u\to 0} -\frac{1}{3}u^3 = 0.$

Example 5: *Prove that*

$$\iiint \frac{dx\,dy\,dz}{\sqrt{(a^2-x^2-y^2-z^2)}} = \frac{\pi^2 a^2}{8},$$

the integral being extended for all positive values of the variables for which the expression is real.

(Lucknow 2008, 11)

Solution: The given expression is real when $x^2+y^2+z^2<a^2$.

Therefore the required integral is to be extended to all positive values of x, y and z such that

$$0<x^2+y^2+z^2<a^2 \quad i.e., \quad 0<x^2/a^2+y^2/a^2+z^2/a^2<1.$$

Put $(x^2 / a^2) = u_1$, $(y^2 / a^2) = u_2$ and $(z^2 / a^2) = u_3$

i.e., $x = a u_1^{1/2}$, $y = a u_2^{1/2}$ and $z = a u_3^{1/2}$

so that $dx = \frac{1}{2} a u_1^{-1/2} du_1$, $dy = \frac{1}{2} a u_2^{-1/2} du_2$ and $dz = \frac{1}{2} a u_3^{-1/2} du_3$.

With these substitutions the given condition reduces to

$$0 < u_1 + u_2 + u_3 < 1$$

and the required integral becomes

$$= \iiint \frac{(\frac{1}{2})^3 . a^3 \, u_1^{-1/2} \, u_2^{-1/2} \, u_3^{-1/2} \, du_1 \, du_2 \, du_3}{a \sqrt{\{1 - (u_1 + u_2 + u_3)\}}}$$

$$= \frac{a^2}{8} \iiint \frac{u_1^{1/2 - 1} \, u_2^{1/2 - 1} \, u_3^{1/2 - 1} \, du_1 \, du_2 \, du_3}{\sqrt{\{1 - (u_1 + u_2 + u_3)\}}}$$

$$= \frac{a^2}{8} \cdot \frac{[\Gamma (1/2)]^3}{\Gamma (\frac{3}{2})} \cdot \int_0^1 u^{3/2 - 1} \cdot \frac{1}{\sqrt{(1 - u)}} \, du,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{a^2}{8} \cdot \frac{[\sqrt{\pi}]^3}{\frac{1}{2} \cdot \sqrt{\pi}} \cdot \int_0^{\pi/2} \frac{\sin \theta \, .2 \sin \theta \cos \theta \, d\theta}{\sqrt{(1 - \sin^2 \theta)}}, \text{ putting } u = \sin^2 \theta \text{ etc.}$$

$$= \frac{\pi a^2}{2} \int_0^{\pi/2} \sin^2 \theta \, d\theta = \frac{\pi a^2}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi^2 a^2}{8}.$$

Example 6: *Prove that when x and y are positive and* $x + y < h$,

$$\iint f'(x + y) \, x^{l - 1} \, y^{-l} \, dx \, dy = \frac{\pi}{\sin l\pi} [f(h) - f(0)].$$

(Kumaun 2008)

Solution: The given integral

$$I = \iint f'(x + y) \, x^{l - 1} \, y^{(1 - l) - 1} \, dx \, dy, \text{ where } 0 < x + y < h$$

$$= \frac{\Gamma (l) \Gamma (1 - l)}{\Gamma (l + 1 - l)} \int_0^h f'(u) \, u^{l + (1 - l) - 1} \, du,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{\Gamma (l) \Gamma (1 - l)}{\Gamma (1)} \int_0^h f'(u) \, du$$

$$= \frac{\pi}{\sin \pi l} [f(u)]_0^h = \frac{\pi}{\sin \pi l} [f(h) - f(0)].$$

Example 7: *Evaluate* $\iiint x^{\alpha} \, y^{\beta} \, z^{\gamma} \, (1 - x - y - z)^{\lambda} \, dx \, dy \, dz$ *over the interior of the tetrahedron formed by the coordinate planes and the plane* $x + y + z = 1$.

Solution: Here the region of integration is bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$. So the variables x, y, z take all positive values subject to the condition

$$0 < x + y + z < 1.$$

Hence the given integral

$$= \iiint x^{(\alpha+1)-1}\, y^{(\beta+1)-1}\, z^{(\gamma+1)-1}\, [1-(x+y+z)]^{\lambda}\, dx\, dy\, dz$$

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)} \cdot \int_0^1 u^{\alpha+1+\beta+1+\gamma+1-1} (1-u)^{\lambda}\, du,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)} \cdot \int_0^1 u^{(\alpha+\beta+\gamma+3)-1} (1-u)^{(\lambda+1)-1}\, du$$

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)}\, B(\alpha+\beta+\gamma+3, \lambda+1)$$

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)} \cdot \frac{\Gamma(\alpha+\beta+\gamma+3)\,\Gamma(\lambda+1)}{\Gamma(\alpha+\beta+\gamma+\lambda+4)}$$

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)\,\Gamma(\lambda+1)}{\Gamma(\alpha+\beta+\gamma+\lambda+4)}.$$

Example 8: *Evaluate*

$$\iiint \sqrt{(a^2b^2c^2 - b^2c^2x^2 - c^2a^2y^2 - a^2b^2z^2)}\, dx\, dy\, dz$$

taken throughout the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

(Lucknow 2009; Kumaun 12)

Solution: The given ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is symmetrical in all the eight octants. Let us first evaluate the given integral over the region of the ellipsoid which lies in the positive octant *i.e.*, where x, y, z are all positive.

Put $x^2/a^2 = u$, $y^2/b^2 = v$, $z^2/c^2 = w$.

Then $x = au^{1/2}$, $dx = \frac{1}{2} au^{-1/2}\, du$ etc.

Now the given integral extended over the positive octant of the given ellipsoid is

$$I = abc \iiint \sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)}\, dx\, dy\, dz,$$

where $0 < x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1$

$$= abc \iiint \sqrt{(1-u-v-w)} \cdot \frac{1}{8} abc\, u^{-1/2} v^{-1/2} w^{-1/2}\, du\, dv\, dw,$$

where $0 < u + v + w \le 1$

$$= \frac{a^2b^2c^2}{8} \iiint u^{(1/2)-1} v^{(1/2)-1} w^{(1/2)-1} \sqrt{\{1-(u+v+w)\}}\, du\, dv\, dw$$

$$= \frac{a^2b^2c^2}{8} \cdot \frac{[\Gamma(\frac{1}{2})]^3}{\Gamma(3/2)} \int_0^1 \sqrt{(1-t)}\,.\, t^{1/2+1/2+1/2-1}\, dt,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{a^2 b^2 c^2}{8} \cdot \frac{(\sqrt{\pi})^3}{\frac{1}{2} \cdot \sqrt{\pi}} \int_0^1 (1-t)^{(3/2)-1} t^{(3/2)-1} \, dt$$

$$= \frac{a^2 b^2 c^2}{8} \cdot 2\pi \cdot \frac{\Gamma(3/2)\,\Gamma(3/2)}{\Gamma(3)} = \frac{\pi^2 a^2 b^2 c^2}{32}.$$

Hence if the integration is extended throughout the ellipsoid, the given integral

$$= 8I = 8 \cdot \frac{\pi^2 a^2 b^2 c^2}{32} = \frac{\pi^2 a^2 b^2 c^2}{4}.$$

Example 9: *Evaluate* $\iint \sqrt{\left\{\frac{1 - x^2/a^2 - y^2/b^2}{1 + x^2/a^2 + y^2/b^2}\right\}} \, dx\, dy$,

where $x^2/a^2 + y^2/b^2 \le 1$.

Solution: The ellipse $x^2/a^2 + y^2/b^2 = 1$ is symmetrical in all the four quadrants. Let us first evaluate the given integral over the region of the ellipse $x^2/a^2 + y^2/b^2 = 1$ which lies in the first quadrant *i.e.*, where x and y are both positive.

Put $\quad x^2/a^2 = u, \; y^2/b^2 = v.$

Then $\quad x = a u^{1/2}, \; dx = \frac{1}{2} a u^{-1/2} \, du,$

$$y = b v^{1/2}, \; dy = \frac{1}{2} b v^{-1/2} \, dv.$$

∴ the given integral extended over the region of the ellipse $x^2/a^2 + y^2/b^2 = 1$ which lies in the first quadrant is given by

$$I = \iint \sqrt{\left(\frac{1-u-v}{1+u+v}\right)} \cdot \frac{1}{2} ab u^{-1/2} v^{-1/2} \, du\, dv, \quad \text{where} \quad 0 < u + v \le 1$$

$$= \frac{ab}{4} \iint \sqrt{\left\{\frac{1-(u+v)}{1+(u+v)}\right\}} \, u^{(1/2)-1} v^{(1/2)-1} \, du\, dv$$

$$= \frac{ab}{4} \cdot \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(1)} \int_0^1 \sqrt{\left(\frac{1-t}{1+t}\right)} \cdot t^{1/2 + 1/2 - 1} \, dt$$

$$= \frac{\pi\, ab}{4} \cdot \int_0^1 \frac{1-t}{\sqrt{(1-t^2)}} \, dt = \frac{\pi\, ab}{4} \int_0^{\pi/2} \frac{1 - \sin\theta}{\cos\theta} \cos\theta \, d\theta,$$

putting $t = \sin\theta$ so that $dt = \cos\theta \, d\theta$

$$= \frac{\pi\, ab}{4} \int_0^{\pi/2} (1 - \sin\theta) \, d\theta = \frac{\pi\, ab}{4} [\theta + \cos\theta]_0^{\pi/2}$$

$$= \frac{\pi\, ab}{4} \left[\left(\frac{\pi}{2} + 0\right) - (0 + 1)\right] = \frac{\pi\, ab}{4} \left(\frac{\pi}{2} - 1\right).$$

Hence the given integral extended over the whole region of the ellipse

$$x^2/a^2 + y^2/b^2 = 1 = 4.I = \pi\, ab \left(\frac{1}{2}\pi - 1\right).$$

Example 10: *Prove that* $I = \iiiint dx\, dy\, dz\, dw$, *for all positive values of the variables for which* $x^2 + y^2 + z^2 + w^2$ *is not less than* a^2 *and not greater than* b^2 *is* $\pi^2 (b^4 - a^4) / 32$.

Solution: We have to evaluate I subject to the condition

$$a^2 < x^2 + y^2 + z^2 + w^2 < b^2.$$

Putting $x^2 = u_1$ *i.e.*, $x = u_1^{1/2}, dx = \frac{1}{2} u_1^{-1/2}\, du_1$ etc., we get

$$I = \iiiint \frac{1}{2} u_1^{-1/2} \cdot \frac{1}{2} u_2^{-1/2} \cdot \frac{1}{2} u_3^{-1/2} \cdot \frac{1}{2} u_4^{-1/2}\, du_1\, du_2\, du_3\, du_4$$

subject to the condition $a^2 < u_1 + u_2 + u_3 + u_4 < b^2$

or $$I = \frac{1}{16} \iiiint u_1^{(1/2)-1} u_2^{(1/2)-1} u_3^{(1/2)-1} u_4^{(1/2)-1}\, du_1\, du_2\, du_3\, du_4$$

$$= \frac{1}{16} \cdot \frac{[\Gamma(\frac{1}{2})]^4}{\Gamma(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2})} \int_{a^2}^{b^2} t^{1/2+1/2+1/2+1/2-1}\, dt,$$

by Liouville's theorem

$$= \frac{(\sqrt{\pi})^4}{16\,\Gamma(2)} \int_{a^2}^{b^2} t\, dt = \frac{\pi^2}{16} \cdot \left[\frac{t^2}{2}\right]_{a^2}^{b^2} = \frac{\pi^2}{32} (b^4 - a^4).$$

Comprehensive Exercise 1

1. Show that the integral $\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$ integrated over the region in the first octant below the surface $(x/a)^p + (y/b)^q + (z/c)^r = 1$ is

$$\frac{a^l b^m c^n}{pqr} \cdot \frac{\Gamma(l/p)\,\Gamma(m/q)\,\Gamma(n/r)}{\Gamma(l/p + m/q + n/r + 1)}.$$

2. Show that if l, m, n are all positive,

$$\iiint x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz = \frac{a^l b^m c^n}{8} \cdot \frac{\Gamma(l/2)\,\Gamma(m/2)\,\Gamma(n/2)}{\Gamma(l/2 + m/2 + n/2 + 1)},$$

where the triple integral is taken throughout the part of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, which lies in the positive octant.

3. Prove that the area in the positive quadrant between the curve $x^n + y^n = a^n$ and the coordinate axes is

$$\frac{a^2 [\Gamma(1/n)]^2}{2n\,\Gamma(2/n)}.$$

4. (i) Find the volume in the positive octant of the ellipsoid
$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$$
(ii) Find the volume of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$
(Kumaun 2015)

(iii) Evaluate $\iiint dx\, dy\, dz$, where $x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1.$
(Kumaun 2007)

5. Evaluate $\iiint xyz\, dx\, dy\, dz$ for all positive values of the variables throughout the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$

6. Find the volume of the tetrahedron bounded by the plane $x/a + y/b + z/c = 1$ and the coordinate axes.

7. The plane $x/a + y/b + z/c = 1$ meets the coordinate axes in the points A, B, C. Use Dirichlet's integral to evaluate the mass of the tetrahedron $OABC$, the density at any point (x, y, z) being $k\, xyz$.

8. Evaluate the integral $\iiint x^2\, y\, z\, dx\, dy\, dz$ over the volume enclosed by the region $x, y, z, \geq 0$ and $x + y + z \leq 1.$

9. Evaluate the double integral $\iint_D x^{1/2}\, y^{1/2}\, (1 - x - y)^{2/3}\, dx\, dy$

over the domain D bounded by the lines $x = 0, y = 0, x + y = 1.$

10. Evaluate $\iint_T x^{1/2} y^{1/2}\, (1 - x - y)^{3/2}\, dx\, dy$, where T is the region bounded by $x \geq 0, y \geq 0,\ x + y \leq 1.$

11. Find the value of $\iint x^{l-1}\, y^{-l}\, e^{x+y}\, dx\, dy$,

extended to all positive values of x and y subject to $x + y < h$.

12. Evaluate $\iiint e^{x+y+z}\, dx\, dy\, dz$ taken over the positive octant such that $x + y + z \leq 1.$

13. Evaluate $\iiint x^{-1/2}\, y^{-1/2}\, z^{-1/2}\, (1 - x - y - z)^{1/2}\, dx\, dy\, dz$ extended to all positive values of the variables subject to the condition $x + y + z < 1.$

14. Prove that $\iiint \dfrac{dx\, dy\, dz}{\sqrt{(1 - x^2 - y^2 - z^2)}} = \dfrac{\pi^2}{8}$, the integral being extended to all positive values of the variables for which the expression is real. **(Lucknow 2006)**

15. Show that $\displaystyle\iint \ldots \int \frac{dx_1\, dx_2 \ldots dx_n}{\sqrt{(1 - x_1^2 - x_2^2 - \ldots - x_n^2)}} = \frac{\pi^{(n+1)/2}}{2^n\, \Gamma\left(\frac{n+1}{2}\right)}$

the integral being extended to all positive values of the variables for which the expression is real.

16. If S is a unit sphere with its centre at the origin, then prove that
$$\iiint_S \frac{dx\, dy\, dz}{\sqrt{(1 - x^2 - y^2 - z^2)}} = \pi^2.$$

17. Evaluate $\iiint_R (x+y+z+1)^2\, dx\, dy\, dz$, where R is the region defined by

$x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z \leq 1.$

18. Show that $\iint \left(\dfrac{1-x^2-y^2}{1+x^2+y^2}\right)^{1/2} dx\, dy = \dfrac{\pi}{8}(\pi - 2)$

over the positive quadrant of the circle $x^2 + y^2 = 1$.

19. Find the value of $\iiint xyz \sin(x+y+z)\, dx\, dy\, dz$,

the integral being extended to all positive values of the variables subject to the condition $x + y + z \leq \pi/2$.

20. Evaluate $\iiint \sqrt{\left(\dfrac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2}\right)}\, dx\, dy\, dz$ integral being taken over all positive values of x, y, z such that $x^2 + y^2 + z^2 \leq 1$.

21. Prove that $\iint_D e^{-x^2-y^2}\, dx\, dy = \dfrac{\pi}{4}(1 - e^{-R^2})$, where D is the region defined by $x \geq 0,\ y \geq 0,\ x^2 + y^2 \leq R^2$. **(Lucknow 2009)**

22. (i) Evaluate $\iint_R \sqrt{(x^2+y^2)}\, dx\, dy$

where R is the region in the xy-plane bounded by $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

(ii) Evaluate $\iint_R \sqrt{(x^2+y^2)}\, dx\, dy$ where R is the region $x^2 + y^2 \leq a^2$.

23. Evaluate the integral $\iiint_R \sqrt{(1-x^2-y^2-z^2)}\, dx\, dy\, dz$

where R is the region interior to the sphere $x^2 + y^2 + z^2 = 1$.

24. Find the mass of the region bounded by the ellipsoid

$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$

if the density varies as the square of the distance from its centre.

25. Prove that $\iiint \dfrac{dx\, dy\, dz}{(x+y+z+1)^3} = \dfrac{1}{2}\left[\log 2 - \dfrac{5}{8}\right]$

throughout the volume bounded by the coordinate planes and the plane $x + y + z = 1$.

26. Evaluate the integral

$\iiint_R (ax^2 + by^2 + cz^2)\, dx\, dy\, dz$

where R is the region given by $x^2 + y^2 + z^2 \leq d^2$.

27. Evaluate the following integrals :

(i) $\int_0^a \int_0^{\sqrt{(a^2-y^2)}} (a^2 - x^2 - y^2)\, dx\, dy$

(ii) $\int_0^2 \int_0^{\sqrt{(4-x^2)}} (x^2 + y^2)\, dx\, dy.$

Answers 1

4. (i) $\pi\, abc / 6$ (ii) $\frac{4}{3}\pi\, abc$ (iii) $\frac{4}{3}\pi\, abc$

5. $a^2b^2c^2 / 48$ **6.** $abc / 6$ **7.** $k\, a^2b^2c^2 / 720$

8. $1/2520$ **9.** $27\pi / 1760$ **10.** $2\pi / 315$

11. $\frac{\pi}{\sin l\pi}(e^h - 1)$ **12.** $(e-2)/2$ **13.** $\pi^2 / 4$

17. $31/60$ **19.** $(\pi^4 - 48\pi^2 + 384)/384$

20. $\frac{1}{8}\pi\left[B\left(\frac{3}{4},\frac{1}{2}\right) - B\left(\frac{5}{4},\frac{1}{2}\right)\right]$ **22.** (i) $38\pi/3$ (ii) $2\pi a^3/3$

23. $\pi^2/4$

24. $8\pi\, abck\,(a^2 + b^2 + c^2)/30$, where k is constant.

26. $\frac{4}{15} d^5\, \pi\,(a+b+c)$ **27.** (i) $\pi a^4/8$ (ii) 2π

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If $x, y, z \geq 0$ and $h_1 \leq x + y + z \leq h_2$, the value of

$$\iiint F(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz \text{ is equivalent to}$$

(a) $\dfrac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)}\displaystyle\int_{h_1}^{h_2} F(u)\, u^{l+m+n-1}\, du$

(b) $\dfrac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}\displaystyle\int_{h_1}^{h_2} F(u)\, u^{l+m+n-1}\, du$

(c) $\dfrac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)}\displaystyle\int_{h_1}^{h_2} F(u)\, u^{l+m+n}\, du$

(d) $\dfrac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n+1)}\displaystyle\int_{h_1}^{h_2} F(u)\, u^{l+m+n}\, du$

2. Dirichlet's theorem can be generalized for n variables, where

(a) $n \le 4$ (b) $n \le 100$

(c) n is any positive integer (d) none of these **(Kumaun 2008)**

3. Dirichlet's theorem

$$\iint \ldots \int x_1^{m_1-1} x_2^{m_2-1} \ldots x_n^{m_n-1}\, dx_1\, dx_2 \ldots dx_n = \frac{\Gamma(m_1)\Gamma(m_2)\ldots\Gamma(m_n)}{\Gamma(1+m_1+m_2+\ldots+m_n)},$$

hold to the condition

(a) $x_1 + x_2 + \ldots + x_n = 1$ (b) $x_1 + x_2 + \ldots + x_n \ge 1$

(c) $x_1 + x_2 + \ldots + x_n \le 1$ (d) none of these **(Kumaun 2007)**

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. If l, m, n are all positive, then the triple integral

$$\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\ldots\ldots},$$

where the integral is extended to all positive values of the variables x, y and z subject to the condition $x + y + z \le 1$.

2. If the variables x, y, z are all positive such that $h_1 \le x + y + z \le h_2$,

then the triple integral

$$\iiint F(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$$

$$= \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\ldots\ldots} \int_{h_1}^{h_2} F(t)\, t^{l+m+n-1}\, dt.$$

True or False

Write 'T' for true and 'F' for false statement.

1. $$\iiint \frac{dx\, dy\, dz}{(x+y+z+1)^3} = \frac{1}{3}\int_0^1 \frac{u^2}{(u+1)^3}\, du,$$

where the region of integration is the volume bounded by the coordinate planes and the plane $x + y + z = 1$.

2. $$\iiint (x+y+z+1)^2\, dx\, dy\, dz = \frac{1}{2}\int_0^1 u^2 (u+1)^2\, du,$$

where the region of integration is the volume bounded by the coordinate planes and the plane

3. If l and m are both positive, then the double integral

$$\iint x^{l-1} y^{m-1}\, dx\, dy = \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)},$$

where the integral is extended to all positive values of the variables x and y subject to the condition $x + y \le 1$.

Answers

Multiple Choice Questions

1. (a) 2. (c) 3. (c)

Fill in the Blank(s)

1. $\Gamma\,(l + m + n + 1)$ 2. $\Gamma\,(l + m + n)$.

True or False

1. F 2. T 3. F

□

Chapter

6

Double and Triple Integrals

(Multiple Integrals, Change of Order of Integration)

1 Double Integrals

The concept of double integral is an extension of the concept of a definite integral to the case of two arguments (*i.e.* a two dimensional space). Let a function $f(x, y)$ of the independent variables x and y be continuous inside some domain (region) A and on its boundary. Divide the domain A into n subdomains $A_1, A_2, \dots, A_n$ of areas $\delta A_1, \delta A_2, \dots, \delta A_n$. Let (x_r, y_r) be any point inside the rth elementary area δA_r. From the sum

$$S_n = f(x_1, y_1)\,\delta A_1 + f(x_2, y_2)\,\delta A_2 + \dots + f(x_r, y_r)\,\delta A_r + \dots + f(x_n, y_n)\,\delta A_n$$

$$= \sum_{r=1}^{n} f(x_r, y_r)\,\delta A_r . \qquad \dots(1)$$

Now take the limit of the sum (1) as $n \to \infty$ in such a way that the largest of the areas δA_r approaches to zero. This limit, if it exists, is called the **double integral** of the function $f(x, y)$ over the domain A. It is denoted by $\iint_A f(x, y)\,dA$ and is read as "the double integral of $f(x, y)$ over A".

Suppose the domain (region) A is divided into rectangular partitions by a network of lines parallel to the coordinate axes. Let dx be the length of a sub-rectangle and dy be its width so that $dx\, dy$ is an element of area in Cartesian coordinates. The integral $\iint f(x, y)\, dA$ is written as $\iint_A f(x, y)\, dx\, dy$ and is called the *double integral* of $f(x, y)$ over the region A.

2 Evaluation of Double Integrals

If the region A be given by the inequalities $a \le x \le b$, $c \le y \le d$, then the double integral

$$\iint_A f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dx\, dy$$

$$= \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx, \qquad \ldots(1)$$

or

$$\iint_A f(x, y)\, dx\, dy = \int_c^d \int_a^b f(x, y)\, dy\, dx$$

$$= \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy \qquad \ldots(2)$$

i.e., in this case the order of integration is immaterial, provided the limits of integration are changed accordingly.

Note: In formula (1) the definite integral $\int_c^d f(x, y)\, dy$ is calculated first. During this integration x is regarded as a constant. While in the formula (2) the definite integral $\int_a^b f(x, y)\, dx$ is calculated first and during this integration y is regarded as a constant.

3 Evaluation of Double Integrals by Repeated Integrals

The double integrals over domains that have special shapes can be reduced to a pair of ordinary integrals. If the region A is bounded by the curves

$y = f_1(x)$, $y = f_2(x)$, $x = a$ and $x = b$, then

$$\iint_A f(x, y)\, dx\, dy = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y)\, dx\, dy$$

$$= \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x, y)\, dy \right] dx, \qquad \ldots(1)$$

where the integration with respect to y is performed first treating x as a constant.

Similarly, if the region A is bounded by the curves

$x = f_1(y)$, $x = f_2(y)$, $y = c$, $y = d$, we have

$$\iint_A f(x, y)\, dx\, dy = \int_c^d \int_{f_1(y)}^{f_2(y)} f(x, y)\, dy\, dx$$

$$= \int_c^d \left[\int_{f_1(y)}^{f_2(y)} f(x, y)\, dx \right] dy, \qquad \ldots(2)$$

where the integration with respect to x is performed first treating y as a constant. The integrals in the right hand sides of (1) and (2) are called **repeated integrals**.

Remember: While evaluating double integrals, first integrate w.r.t. the variable having variable limits (treating the other variable as constant) and then integrate w.r.t. the variable with constant limits.

Remark: In the double integral $\int_a^b \int_c^d f(x, y)\, dx\, dy$, it is generally understood that the limits of integration c to d are those of y and the limits of integration a to b are those of x. However this is not a standard convention. Some authors regard these limits in the reverse order *i.e.* they regard the limits c to d as those of x and the limits a to b as those of y. So it is better to write this double integral as $\int_{x=a}^{b} \int_{y=c}^{d} f(x, y)\, dx\, dy$ so that there is no confusion about the limits. However in the double integral $\int_a^b \int_{f_1(x)}^{f_2(x)} F(x, y)\, dx\, dy$, there is no confusion about the limits. Obviously, the variable limits are those of y because they are in terms of x and so the constant limits must be those of x. Here the first integration must be performed with respect to y regarding x as constant.

4 Properties of Double Integral

I. If the region A is partitioned into two parts, say A_1 and A_2, then

$$\iint_A f(x, y)\, dx\, dy = \iint_{A_1} f(x, y)\, dx\, dy + \iint_{A_2} f(x, y)\, dx\, dy.$$

Similarly for a sub-division of A into three or more parts.

II. The double integral of the algebraic sum of a fixed number of functions is equal to the algebraic sum of the double integrals taken for each term. Thus

$$\iint_A [f_1(x, y) + f_2(x, y) + f_3(x, y) + \ldots]\, dx\, dy$$

$$= \iint_A f_1(x, y)\, dx\, dy + \iint_A f_2(x, y)\, dx\, dy$$

$$+ \iint_A f_3(x, y)\, dx\, dy + \ldots$$

III. A constant factor may be taken outside the integral sign. Thus

$$\iint_A m\, f(x, y)\, dx\, dy = m \iint_A f(x, y)\, dx\, dy,$$

where m is a constant.

Illustrative Examples

Example 1: *Evaluate the following double integrals :*

(i) $\int_0^a \int_0^b (x^2 + y^2)\, dx\, dy$ **(Kanpur 2006; Lucknow 10; Purvanchal 14)**

(ii) $\int_1^2 \int_0^x \frac{dx\, dy}{x^2 + y^2}$.

(Kumaun 2015)

Solution: **(i)** We have $\int_0^a \int_0^b (x^2 + y^2)\, dx\, dy = \int_0^a \left[x^2 y + \frac{y^3}{3}\right]_{y=0}^b dx$,

(integrating w.r.t. y treating x as constant)

$$= \int_0^a \left[bx^2 + \frac{b^3}{3}\right] dx = \left[b\frac{x^3}{3} + \frac{b^3}{3}x\right]_0^a = \frac{ba^3}{3} + \frac{b^3 a}{3} = \frac{1}{3} ab\,(a^2 + b^2).$$

(ii) We have $\int_1^2 \int_0^x \frac{dx\, dy}{x^2 + y^2} = \int_1^2 \left[\int_0^x \frac{dy}{x^2 + y^2}\right] dx$

$$= \int_1^2 \left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_{y=0}^x dx$$

(integrating w.r.t. y treating x as constant)

$$= \int_1^2 \left[\frac{1}{x}(\tan^{-1} 1 - \tan^{-1} 0)\right] dx = \frac{\pi}{4}\int_1^2 \frac{dx}{x} = \frac{\pi}{4}[\log x]_1^2$$

$$= \frac{1}{4}\pi[\log 2 - \log 1] = \frac{1}{4}\pi \log 2.$$

Example 2: *Evaluate :*

(i) $\int_0^3 \int_1^2 xy\,(1 + x + y)\, dx\, dy$

(ii) $\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dx\, dy}{1 + x^2 + y^2}$.

(Gorakhpur 2005; Kanpur 12; Avadh 14; Kumaun 14)

Solution: **(i)** We have $\int_0^3 \int_1^2 xy\,(1 + x + y)\, dx\, dy$

$$= \int_0^3 \left[x \cdot \frac{y^2}{2} + x^2 \cdot \frac{y^2}{2} + x \cdot \frac{y^3}{3}\right]_{y=1}^2 dx,$$

(integrating w.r.t. y treating x as constant)

$$= \int_0^3 \left[\frac{x}{2}(4-1) + \frac{x^2}{2}(4-1) + \frac{x}{3}(8-1)\right] dx$$

$$= \int_0^3 \left[\left(\frac{3}{2} + \frac{7}{3}\right)x + \frac{3}{2}x^2\right] dx = \left[\frac{23}{6}\cdot\frac{x^2}{2} + \frac{3}{2}\cdot\frac{x^3}{3}\right]_0^3$$

$$= \frac{23}{6} \cdot \frac{9}{2} + \frac{27}{2} = \frac{123}{4} = 30\frac{3}{4} \cdot$$

(ii) We have $\int_0^1 \int_0^{\sqrt{(1+x^2)}} \frac{dx\,dy}{1+x^2+y^2}$

$$= \int_0^1 \frac{1}{\sqrt{(1+x^2)}} \left[\tan^{-1} \frac{y}{\sqrt{(1+x^2)}}\right]_{y=0}^{\sqrt{(1+x^2)}} dx,$$

(integrating w.r.t. y treating x as constant)

$$= \int_0^1 \frac{1}{\sqrt{(1+x^2)}} [\tan^{-1} 1 - \tan^{-1} 0]\, dx = \frac{\pi}{4} \int_0^1 \frac{dx}{\sqrt{(1+x^2)}}$$

$$= \frac{\pi}{4} [\log \{x + \sqrt{(1+x^2)}\}]_0^1 = \frac{\pi}{4} \log (1+\sqrt{2}).$$

Example 3: *Evaluate* $\int_0^a \int_0^{\sqrt{(a^2-y^2)}} \sqrt{(a^2-x^2-y^2)}\, dy\, dx$.

Solution: Here the variable limits are those of x and so the first integration must be performed w.r.t. x taking y as constant.

$\therefore$

$$\int_0^a \int_0^{\sqrt{(a^2-y^2)}} \sqrt{(a^2-x^2-y^2)}\, dy\, dx$$

$$= \int_0^a \left[\int_0^{\sqrt{(a^2-y^2)}} \sqrt{\{(a^2-y^2)-x^2\}}\, dx\right] dy$$

$$= \int_0^a \left[\frac{x\sqrt{(a^2-y^2-x^2)}}{2} + \frac{(a^2-y^2)}{2} \sin^{-1} \frac{x}{\sqrt{(a^2-y^2)}}\right]_{x=0}^{\sqrt{(a^2-y^2)}} dy,$$

(integrating w.r.t. x treating y as constant)

$$= \int_0^a \left[0 + \frac{a^2-y^2}{2} \cdot \frac{\pi}{2}\right] dy = \frac{\pi}{4}\left[a^2 y - \frac{y^3}{3}\right]_0^a = \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right] = \frac{1}{6}\pi a^3.$$

Example 4: *Show that* $\int_1^2 \int_0^{y/2} y\, dy\, dx = \int_1^2 \int_0^{x/2} x\, dx\, dy$.

Solution: We have

$$\int_1^2 \int_0^{y/2} y\, dy\, dx = \int_1^2 \left[y \int_0^{y/2} dx\right] dy = \int_1^2 y\,[x]_0^{y/2}\, dy,$$

(integrating w.r.t. x treating y as a constant)

$$= \int_1^2 y\left[\frac{y}{2} - 0\right] dy = \frac{1}{2}\int_1^2 y^2\, dy = \frac{1}{2}\left[\frac{y^3}{3}\right]_1^2 = \frac{1}{6}[8-1] = \frac{7}{6} \qquad \ldots(1)$$

Again $\int_1^2 \int_0^{x/2} x\, dx\, dy = \int_1^2 x\left[\int_0^{x/2} dy\right] dx = \int_1^2 x\,[y]_0^{x/2}\, dx,$

(integrating w.r.t. y treating x as a constant)

$$= \int_1^2 x\left[\frac{x}{2} - 0\right] dx = \frac{1}{2}\int_1^2 x^2\, dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_1^2 = \frac{1}{6}(8-1) = \frac{7}{6} \qquad \text{...(2)}$$

From (1) and (2), we see that

$$\int_1^2 \int_0^{y/2} y\, dy\, dx = \int_1^2 \int_0^{x/2} x\, dx\, dy\,.$$

Examples on the region of integration (*Double Integration*)

Example 5: *Evaluate* $\iint x^2 y^2\, dx\, dy$ *over the region* $x^2 + y^2 \le 1$.

Solution: Let R denote the region $x^2 + y^2 \le 1$. Then R is the region in the xy-plane bounded by the circle $x^2 + y^2 = 1$. The limits of integration for this region can be expressed either as

$$-1 \le x \le 1, -\sqrt{(1-x^2)} \le y \le \sqrt{(1-x^2)}$$

or as

$$-\sqrt{(1-y^2)} \le x \le \sqrt{(1-y^2)}, -1 \le y \le 1.$$

Because from the equation of the circle $x^2 + y^2 = 1$, we have $x^2 = 1 - y^2$ so that $x = \pm\sqrt{(1-y^2)}$. Thus for a fixed value of y, x varies from $-\sqrt{(1-y^2)}$ to $\sqrt{(1-y^2)}$ in the area bounded by the circle $x^2 + y^2 = 1$. Also y varies from -1 to 1 to cover the whole area of the circle $x^2 + y^2 = 1$. Therefore if the first integration is to be performed w.r.t. x regarding y as constant, then

$$\iint_R x^2 y^2\, dx\, dy = \int_{y=-1}^{1} \int_{x=-\sqrt{(1-y^2)}}^{\sqrt{(1-y^2)}} x^2 y^2 dx\, dy$$

$$= \int_{y=-1}^{1} y^2 \left[\int_{x=-\sqrt{(1-y^2)}}^{\sqrt{(1-y^2)}} x^2 dx\right] dy$$

$$= \int_{-1}^{1} y^2 \left[2\int_{x=0}^{\sqrt{(1-y^2)}} x^2\, dx\right] dy$$

$$= \int_{-1}^{1} 2y^2 \left[\frac{x^3}{3}\right]_0^{\sqrt{(1-y^2)}} dy$$

$$= \int_{-1}^{1} \frac{2}{3} y^2 (1-y^2)^{3/2}\, dy$$

$$= 2 \cdot \frac{2}{3} \int_0^1 y^2 (1-y^2)^{3/2}\, dy.$$

Put $y = \sin\theta$ so that $dy = \cos\theta\, d\theta$;

when $y = 0, \theta = 0$ and when $y = 1, \theta = \pi/2$.

$$\therefore \quad \iint_R x^2 y^2\, dx\, dy = \frac{4}{3}\int_0^{\pi/2} \sin^2\theta\, (1-\sin^2\theta)^{3/2} . \cos\theta\, d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2} \sin^2\theta \cos^4\theta\, d\theta = \frac{4}{3} \cdot \frac{1.3.1}{6.4.2} \cdot \frac{\pi}{2} = \frac{\pi}{24}.$$

Example 6: *Find by double integration the area of the region bounded by the circle* $x^2 + y^2 = a^2$.

(Agra 2007; Kanpur 09)

Solution: The area of a small element situated at any point (x, y) is $dx\, dy$. To find the area bounded by the circle $x^2 + y^2 = a^2$, the region of integration R can be expressed as $-a \le y \le a, -\sqrt{(a^2 - y^2)} \le x \le \sqrt{(a^2 - y^2)}$,

where the first integration is to be performed w.r.t. x regarding y as constant.

$\therefore$ the required area

$$= \iint_R dx\, dy = \int_{y=-a}^{a} \int_{x=-\sqrt{(a^2-y^2)}}^{\sqrt{(a^2-y^2)}} 1 . dx\, dy$$

$$= \int_{-a}^{a} \left[2 \int_0^{\sqrt{(a^2-y^2)}} 1 . dx \right] dy = 2 \int_{-a}^{a} [x]_0^{\sqrt{(a^2-y^2)}} dy$$

$$= 2 \int_{-a}^{a} \sqrt{(a^2 - y^2)}\, dy = 2.2 \int_0^a \sqrt{(a^2 - y^2)}\, dy \qquad \textbf{(Note)}$$

$$= 4 \left[\frac{y \sqrt{(a^2 - y^2)}}{2} + \frac{a^2}{2} \sin^{-1} \frac{y}{a} \right]_0^a = 4 \left[0 + \frac{a^2}{2} \sin^{-1} 1 \right]$$

$$= 4 . \frac{1}{2} a^2 . \frac{1}{2} \pi = \pi a^2 .$$

Example 7: *Evaluate* $\iint (x + y)^2\, dx\, dy$ *over the area bounded by the ellipse* $x^2 / a^2 + y^2 / b^2 = 1$. *Hence find the mass of an elliptic plate whose density per unit area is given by* $\rho = k(x + y)^2$.

Solution: The region of integration can be considered as bounded by

$$y = -b \sqrt{(1 - x^2 / a^2)}, y = b \sqrt{(1 - x^2 / a^2)}, x = -a \text{ and } x = a .$$

$$\therefore \qquad \iint (x + y)^2\, dx\, dy = \int_{-a}^{a} \int_{-b\sqrt{(1 - x^2 a^2)}}^{b\sqrt{(1 - x^2/a^2)}} (x^2 + y^2 + 2xy)\, dx\, dy ,$$

the first integration to be performed w.r.t. y regarding x as a constant

$$= \int_{-a}^{a} 2 \int_0^{b\sqrt{(1 - x^2/a^2)}} (x^2 + y^2)\, dx\, dy ,$$

[$\because$ $2xy$ being an odd function of y, its integration under the given limits of y is 0]

$$= 2 \int_{-a}^{a} \left[x^2 y + \frac{y^3}{3} \right]_0^{b\sqrt{(1 - x^2/a^2)}} dx$$

$$= 2 \int_{-a}^{a} \left\{ x^2 b \sqrt{\left(1 - \frac{x^2}{a^2}\right)} + \frac{b^3}{3} \left(1 - \frac{x^2}{a^2}\right)^{3/2} \right\} dx$$

$$= 4 \int_0^a \left\{ x^2 \, b \sqrt{\left(1 - \frac{x^2}{a^2}\right)} + \frac{b^3}{3} \left(1 - \frac{x^2}{a^2}\right)^{3/2} \right\} dx$$

$$= 4b \int_0^{\pi/2} \left\{ a^2 \sin^2 \theta \cos \theta + \frac{b^2}{3} \cos^3 \theta \right\} a \cos \theta \, d\theta ,$$

putting $x = a \sin \theta$ so that $dx = a \cos \theta \, d\theta$

$$= 4ab \int_0^{\pi/2} \left\{ a^2 \sin^2 \theta \cos^2 \theta + \frac{b^2}{3} \cos^4 \theta \right\} d\theta$$

$$= 4ab \left[a^2 \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta \, d\theta + \frac{b^2}{3} \int_0^{\pi/2} \cos^4 \theta \, d\theta \right]$$

$$= 4ab \left[a^2 \cdot \frac{1.1}{4.2} \cdot \frac{\pi}{2} + \frac{b^2}{3} \cdot \frac{3.1}{4.2} \cdot \frac{\pi}{2} \right] \quad \text{[By Walli's formula]}$$

$$= 4ab \left[\frac{1}{16} \pi a^2 + \frac{1}{16} \pi \, b^2 \right] = \frac{1}{4} \pi \, ab \, (a^2 + b^2) .$$

The mass of an elliptic plate whose density is given by $\rho = k \, (x + y)^2$

$= \iint_A k \, (x + y)^2 \, dx \, dy$, where the integration is to be performed over the area A of the ellipse

$$= k \cdot \frac{1}{4} \cdot \pi \, ab \, (a^2 + b^2) .$$

Example 8: *Evaluate* $\iint (x^2 + y^2) \, dx \, dy$ *over the region in the positive quadrant for which* $x + y \le 1$. **(Rohilkhand 2012; Avadh 14)**

Solution: The region of integration R is the area bounded by the coordinate axes and the straight line $x + y = 1$. Therefore the region R is bounded by $y = 0$, $y = 1 - x$ and $x = 0$, $x = 1$.

Therefore

$$\iint_R (x^2 + y^2) \, dx \, dy = \int_{x=0}^{1} \int_{y=0}^{1-x} (x^2 + y^2) \, dx \, dy ,$$

the first integration to be performed w.r.t. y regarding x as constant

$$= \int_0^1 \left[x^2 \, y + \frac{y^3}{3} \right]_0^{1-x} dx = \int_0^1 \left[x^2 \, (1 - x) + \frac{(1 - x)^3}{3} \right] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1 - x)^4}{3 \times 4} \right]_0^1 = \left[\frac{1}{3} - \frac{1}{4} + \frac{1}{12} \right] = \frac{1}{6} .$$

Example 9: *Evaluate* $\iint xy \, (x + y) \, dx \, dy$ *over the area between* $y = x^2$ *and* $y = x$. **(Gorakhpur 2005, 06)**

Solution: Draw the given curves $y = x^2$ and $y = x$ in the same figure. The two curves intersect at the points whose abscissae are given by $x^2 = x$ or $x \, (x - 1) = 0$ *i.e.*, $x = 0$ or 1.

When $0 < x < 1$, we have $x > x^2$. So the area of integration can be considered as lying between the curves $y = x^2, y = x, x = 0$ and $x = 1$.

Therefore the required integral

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} xy\,(x + y)\,dx\,dy = \int_0^1 \left[\int_{x^2}^{x} (x^2 y + xy^2)\,dy \right] dx$$

$$= \int_0^1 \left[\frac{x^2 y^2}{2} + \frac{xy^3}{3} \right]_{x^2}^{x} dx = \int_0^1 \left[\left(\frac{x^4}{2} + \frac{x^4}{3} \right) - \left(\frac{x^6}{2} + \frac{x^7}{3} \right) \right] dx$$

$$= \int_0^1 \left[\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3} \right] dx = \left[\frac{x^5}{6} - \frac{x^7}{14} - \frac{x^8}{24} \right]_0^1$$

$$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = \frac{28 - 12 - 7}{168} = \frac{9}{168} = \frac{3}{56}.$$

Example 10: *Prove by the method of double integration that the area lying between the parabolas* $y^2 = 4ax$ *and* $x^2 = 4ay$ *is* $\frac{16}{3} a^2$.

Solution: Draw the two parabolas in the same figure. The two parabolas intersect at the points whose abscissae are given by $(x^2 / 4a)^2 = 4ax$ *i.e.,* $x\,(x^3 - 64a^3) = 0$ *i.e.,* $x = 0$ and $x^3 = 64a^3$. Thus the two parabolas intersect at the points where $x = 0$ and $x = 4a$.

Now the area of a small element situated at any point $(x, y) = dx\,dy$.

$\therefore$ the required area

$$= \int_{x=0}^{4a} \int_{y=x^2/4a}^{\sqrt{(4ax)}} dx\,dy = \int_0^{4a} [y]_{x^2/4a}^{\sqrt{(4ax)}} dx$$

$$= \int_0^{4a} \left[2\sqrt{a}.\,x^{1/2} - \frac{1}{4a}.\,x^2 \right] dx = \left[2\sqrt{a}\,.\,x^{3/2}\,.\frac{2}{3} - \frac{1}{4a}\cdot\frac{x^3}{3} \right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}.\,(4a)^{3/2} - \frac{1}{12a}.\,64a^3 = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2.$$

Comprehensive Exercise 1

Evaluate the following double integrals :

1. (i) $\int_0^2 \int_0^{\sqrt{(4+x^2)}} \frac{dx\,dy}{4 + x^2 + y^2}$. **(Rohilkhand 2005)**

(ii) $\int_1^a \int_1^b \frac{dx\,dy}{xy}$.

(iii) $\int_0^{\pi/2} \int_{\pi/2}^{\pi} \cos(x + y)\,dy\,dx$. **(Kanpur 2007, 11)**

(iv) $\int_0^1 \int_0^{x^2} e^{y/x}\, dx\, dy.$ (v) $\int_1^2 \int_0^{3y} y\, dy\, dx.$

(vi) $\int_0^2 \int_0^{\sqrt{(2x - x^2)}} x\, dx\, dy.$

(Lucknow 2006; Kanpur 08)

2. (i) $\int_0^1 \int_0^1 \dfrac{dx\, dy}{\sqrt{\{(1 - x^2)(1 - y^2)\}}}.$

(ii) $\int_0^1 \int_0^{\sqrt{(1 - y^2)}} 4y\, dy\, dx.$

(Lucknow 2008)

(iii) $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dx\, dy.$ (iv) $\int_2^3 \int_0^{y-1} \dfrac{dy\, dx}{y}.$

(v) $\int_0^a \int_0^{\sqrt{(a^2 - y^2)}} (a^2 - x^2 - y^2)\, dy\, dx.$

(vi) $\int_0^a \int_0^{\sqrt{(a^2 - x^2)}} (x + y)\, dx\, dy.$

3. Show that (i) $\int_1^2 \int_3^4 (xy + e^y)\, dx\, dy = \int_3^4 \int_1^2 (xy + e^y)\, dy\, dx.$

(Kumaun 2015)

(ii) $\int_0^1 dx \int_0^1 \dfrac{x - y}{(x + y)^3}\, dy \neq \int_0^1 dy \int_0^1 \dfrac{x - y}{(x + y)^3}\, dx.$

Find the values of the two integrals.

4. (i) Evaluate the double integral $\int_0^a \int_0^{\sqrt{(a^2 - x^2)}} x^2 y\, dx\, dy.$

Mention the region of integration involved in this double integral.

(ii) Evaluate $\iint x^2 y^3\, dx\, dy$ over the circle $x^2 + y^2 = a^2.$ **(Rohilkhand 2013B)**

5. Evaluate $\iint (x + y + a)\, dx\, dy$ over the circular area $x^2 + y^2 \leq a^2.$

6. Evaluate $\iint x^2 y^2\, dxdy$ over the region bounded by $x = 0, y = 0$ and $x^2 + y^2 = 1.$

(Avadh 2012)

7. Evaluate $\iint xy\, dx\, dy$ over the region in the positive quadrant for which $x + y \leq 1.$

8. Evaluate $\iint e^{2x+3y}\, dx\, dy$ over the triangle bounded by $x = 0,\ y = 0$ and $x + y = 1.$

9. Evaluate $\iint \dfrac{xy}{\sqrt{(1 - y^2)}}\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = 1.$

10. Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,$ by double integration.

11. Compute the value of $\iint_R y\, dx\, dy,$ where R is the region in the first quadrant bounded by the ellipse $x^2 / a^2 + y^2 / b^2 = 1.$

12. Find the mass of a plate in the form of a quadrant of an ellipse $x^2 / a^2 + y^2 / b^2 = 1$ whose density per unit area is given by $\rho = k\, x\, y$.

13. Show by double integration that the area between the parabolas $y^2 = 4\,ax$ and $x^2 = 4by$ is $(16/3)\,ab$.

14. Find by double integration the area lying between the parabola $y = 4\,x - x^2$ and the line $y = x$.

15. Evaluate $\iint y\, dx\, dy$ over the area between the parabolas $y^2 = 4\,x$ and $x^2 = 4\,y$.

16. Find by double integration the area of the region enclosed by the circle $x^2 + y^2 = a^2$ and the line $x + y = a$ (in the first quadrant).

Answers 1

1. (i) $\frac{1}{4}\pi \log(1 + \sqrt{2})$ (ii) $(\log b)(\log a)$ (iii) -2
   (iv) $\frac{1}{2}$ (v) 7 (vi) $\frac{1}{2}\pi$

2. (i) $\frac{1}{4}\pi^2$ (ii) $\frac{4}{3}$ (iii) $3/35$
   (iv) $1 - \log(3/2)$ (v) $\pi a^4 / 8$ (vi) $2\,a^3 / 3$

3. (ii) $\frac{1}{2}$ and $-\frac{1}{2}$

4. (i) $a^5 / 15$. The area of the circle $x^2 + y^2 = a^2$ in the positive quadrant.
   (ii) 0

5. πa^3

6. $\pi / 96$

7. $\frac{1}{24}$

8. $\frac{1}{6}(e - 1)^2\,(2\,e + 1)$

9. $\frac{1}{6}$

10. $\pi\, ab$

11. $ab^2 / 3$

12. $k\, a^2\, b^2 / 8$

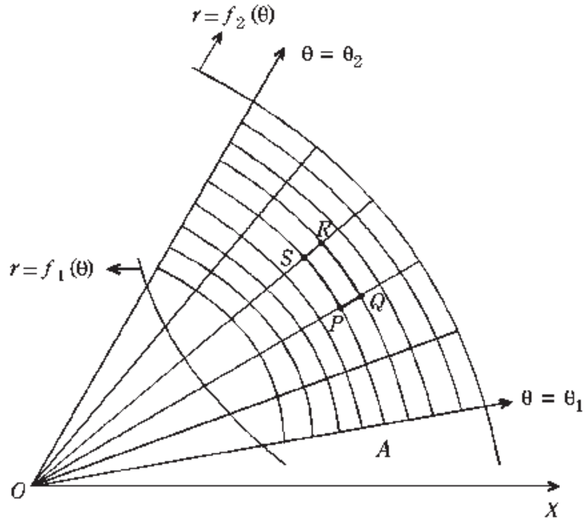
14. $\frac{9}{2}$

15. 48 / 5

16. $\frac{1}{4}(\pi - 2)\, a^2$

5 To Express a Double Integral in Terms of Polar Coordinates

Let a function $f(r, \theta)$ of the polar coordinates (r, θ) be continuous inside some region A and on its boundary. Let the region A be bounded by the curves $r = f_1(\theta)$, $r = f_2(\theta)$ and the lines $\theta = \theta_1$, $\theta = \theta_2$.



Divide the area A into elements by a series of concentric circular arcs with centre at origin and successive radii differing by equal amounts and a series of straight lines drawn through the origin at equal intervals of angles. Let δr be the distance between two consecutive circles and $\delta\theta$ be the angle between two consecutive lines. There is thus a network of elementary areas (say n in number) of which a typical one is $PQRS$. If P is the point (r, θ), the area of the element $PQRS$ situated at the point P is $\frac{1}{2}(r + \delta r)^2 \delta\theta - \frac{1}{2} r^2 \delta\theta = r\, \delta\theta\, \delta r$, by neglecting the term $\frac{1}{2}(\delta r)^2 \delta\theta$ being an infinitesimal of higher order.

Now by the definition of the double integral of $f(r, \theta)$ over the region A, we have

$$\iint_A f(r, \theta)\, dA = \lim_{\substack{n \to \infty \\ \delta r \to 0,\ \delta\theta \to 0}} \sum_{k=1}^{n} f(r_k, \theta_k)\, r_k\, \delta\theta\, \delta r,$$

where $r_k\, \delta\theta \delta r$ is the area of the element situated at the point (r_k, θ_k).

Using the area of integration, this double integral is generally written as

$$\int_{\theta_1}^{\theta_2} \int_{f_1(\theta)}^{f_2(\theta)} f(r, \theta)\, d\theta\, dr \quad \text{or} \quad \int_{\theta_1}^{\theta_2} d\theta \int_{f_1(\theta)}^{f_2(\theta)} f(r, \theta)\, dr.$$

The first integration is performed with respect to r, keeping θ as a constant. After substituting the limits for r, the second integration with respect to θ is performed.

Remark: The area of the typical element $PQRS$ situated at the point $P(r, \theta)$ can also be found as below :

We have $OP = r$, $OQ = r + \delta r$ so that $PQ = \delta r$. Also PS is the arc of a circle of radius r subtending an angle $\delta\theta$ at the centre of the circle and so arc $PS = r\, \delta\theta$. Therefore the area of the element $PQRS$ is $\delta r \cdot r\, \delta\theta$ *i.e.*, $r\, \delta\theta\, \delta r$.

Illustrative Examples

Example 11: *Evaluate* $\int_0^{\pi} \int_0^{a(1+\cos\theta)} r^2 \cos\theta \, d\theta \, dr.$

Solution: We have

$$\int_0^{\pi} \int_0^{a(1+\cos\theta)} r^2 \cos\theta \, d\theta \, dr = \int_0^{\pi} \cos\theta \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{3}\int_0^{\pi} \cos\theta \cdot a^3 (1+\cos\theta)^3 \, d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi} \cos\theta\,(1+3\cos\theta+3\cos^2\theta+\cos^3\theta)\, d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi} [\cos\theta+3\cos^2\theta+3\cos^3\theta+\cos^4\theta]\, d\theta$$

$$= 2\cdot\frac{a^3}{3}\int_0^{\pi/2} [3\cos^2\theta+\cos^4\theta]\, d\theta \qquad \left[\because \int_0^{\pi} \cos^n\theta\, d\theta = 0 \text{ or } 2\int_0^{\pi/2} \cos^n\theta\, d\theta \text{ according as } n \text{ is odd or even}\right]$$

$$= \frac{2a^3}{3}\left[3\cdot\frac{1}{2}\cdot\frac{\pi}{2}+\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}\right] = \frac{2a^3}{3}\cdot\frac{3\pi}{4}\left[1+\frac{1}{4}\right] = \frac{2a^3}{3}\cdot\frac{3\pi}{4}\cdot\frac{5}{4} = \frac{5\pi a^3}{8}.$$

Example 12: *Evaluate* $\iint \frac{r\, d\theta\, dr}{\sqrt{(a^2+r^2)}}$ *over one loop of the lemniscate* $r^2 = a^2 \cos 2\theta$.

Solution: In the equation of the lemniscate $r^2 = a^2\cos 2\theta$, putting $r = 0$, we get $\cos 2\theta = 0$ *i.e.*, $2\theta = \pm\pi/2$ *i.e.*, $\theta = \pm\pi/4$. Therefore for one loop of the given lemniscate θ varies from $-\pi/4$ to $\pi/4$ and r varies from 0 to $a\sqrt{(\cos 2\theta)}$.

Therefore the required integral

$$= \int_{\theta=-\pi/4}^{\pi/4} \int_{r=0}^{a\sqrt{(\cos 2\theta)}} \frac{r\, d\theta\, dr}{\sqrt{(a^2+r^2)}}$$

$$= \int_{-\pi/4}^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} \frac{1}{2}(a^2+r^2)^{-1/2}(2r)\, d\theta\, dr$$

$$= \int_{-\pi/4}^{\pi/4} [(a^2+r^2)^{1/2}]_0^{a\sqrt{(\cos 2\theta)}}\, d\theta$$

$$= \int_{-\pi/4}^{\pi/4} [a(1+\cos 2\theta)^{1/2} - a]\, d\theta$$

$$= 2a\int_0^{\pi/4} [(2\cos^2\theta)^{1/2} - 1]\, d\theta = 2a\int_0^{\pi/4} (\sqrt{2}\cos\theta - 1)\, d\theta$$

$$= 2a[\sqrt{2}\sin\theta - \theta]_0^{\pi/4} = 2a\left[\sqrt{2}\cdot\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right] = 2a\left[1-\frac{\pi}{4}\right] = \frac{a}{2}(4-\pi).$$

Example 13: *Find by double integration the area lying inside the circle $r = a \sin \theta$ and outside the cardioid $r = a (1 - \cos \theta)$.*

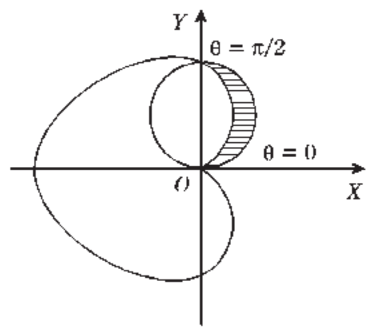
Solution: The given circle is $r = a \sin \theta$ and the cardioid is $r = a (1 - \cos \theta)$. Note that the given circle passes through the pole and the diameter through the pole makes an angle $\pi / 2$ with the initial line.

Eliminating r between the two equations, we have

$$a \sin \theta = a (1 - \cos \theta)$$

or
$$1 = \frac{\sin \theta}{1 - \cos \theta} = \frac{2 \sin \frac{1}{2} \theta \cos \frac{1}{2} \theta}{2 \cos^2 \frac{1}{2} \theta} = \tan \frac{\theta}{2}$$

or
$$\frac{1}{2} \theta = \frac{1}{4} \pi \text{ i.e., } \theta = \pi / 2.$$



Thus the two curves meet at the point where $\theta = \pi / 2$. Also for both the curves $r = 0$ when $\theta = 0$ and so the two curves also meet at the pole O where $\theta = 0$. To cover the required area the limits of integration for r are $a (1 - \cos \theta)$ to $a \sin \theta$ and for θ are 0 to $\pi / 2$. Therefore the required area

$$= \int_0^{\pi/2} \int_{a (1 - \cos \theta)}^{a \sin \theta} r \, d\theta \, dr$$

$$= \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_{a (1 - \cos \theta)}^{a \sin \theta} d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} [a^2 \sin^2 \theta - a^2 (1 - \cos \theta)^2] \, d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi/2} [\sin^2 \theta - 1 + 2 \cos \theta - \cos^2 \theta] \, d\theta$$

$$= \frac{a^2}{2} \left[\frac{1}{2} \cdot \frac{\pi}{2} - \frac{\pi}{2} + 2.1 - \frac{1}{2} \cdot \frac{\pi}{2} \right] = \frac{a^2}{2} \left[2 - \frac{\pi}{2} \right] = \frac{a^2}{4} (4 - \pi).$$

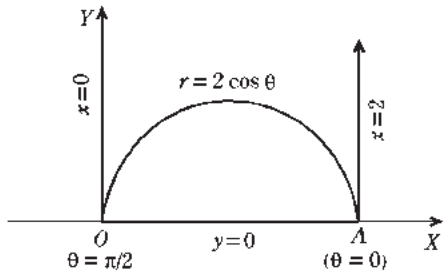
Example 14: *Transform the integral* $\int_0^2 \int_0^{\sqrt{(2x - x^2)}} \frac{x \, dx \, dy}{\sqrt{(x^2 + y^2)}}$ *by changing to polar coordinates and hence evaluate it.* **(Kumaun 2008)**

Solution: From the limits of integration it is obvious that the region of integration is bounded by $y = 0$, $y = \sqrt{(2x - x^2)}$ and $x = 0$, $x = 2$ *i.e.*, the region of integration is the area of the circle $x^2 + y^2 - 2x = 0$ between the lines $x = 0$, $x = 2$ and lying above the axis of x *i.e.*, the line $y = 0$.

Putting $x = r \cos \theta$, $y = r \sin \theta$ the corresponding polar equation of the circle is

$$r^2 (\cos^2 \theta + \sin^2 \theta) - 2 r \cos \theta = 0, \quad \text{or} \quad r = 2 \cos \theta.$$

From the figure it is obvious that r varies from 0 to $2 \cos \theta$ and θ varies from 0 to $\pi / 2$. Note that at the point A of the circle, $\theta = 0$ and at the point O, $r = 0$ and so from $r = 2 \cos \theta$, we get $\theta = \pi / 2$ at O.



The polar equivalent of elementary area $dx\, dy$ is $r\, d\theta\, dr$.

$$\therefore \qquad \iint_A f(x, y)\, dx\, dy = \iint_A f(r \cos \theta, r \sin \theta)\, r\, d\theta\, dr,$$

where A is the region of integration.

Hence transforming to polar coordinates, the given double integral

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{2 \cos \theta} \frac{r \cos \theta}{r}\, r\, d\theta\, dr = \int_0^{\pi 2} \cos \theta \left[\frac{r^2}{2} \right]_0^{2 \cos \theta} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2} \cos \theta . 4 \cos^2 \theta\, d\theta = 2 \int_0^{\pi/2} \cos^3 \theta\, d\theta = 2 . \frac{2}{3} = \frac{4}{3}.$$

Comprehensive Exercise 2

1. (i) Evaluate $\int_0^{\pi} \int_0^{a \sin \theta} r\, d\theta\, dr.$ **(Kashi 2013)**

(ii) Evaluate $\int_0^{\pi/2} \int_0^{a \cos \theta} r \sin \theta\, d\theta\, dr.$

(iii) Evaluate $\int_0^{\pi} \int_0^{a (1 + \cos \theta)} r^3 \sin \theta \cos \theta\, d\theta\, dr.$ **(Agra 2003; Kumaun 09)**

2. Evaluate $\iint r^2\, d\theta\, dr$ over the area of the circle $r = a \cos \theta$. **(Kanpur 2010)**

3. Integrate $r \sin \theta$ over the area of the cardioid $r = a (1 + \cos \theta)$, lying above the initial line. **(Kanpur 2010)**

4. Find the mass of a loop of the lemniscate $r^2 = a^2 \sin 2 \theta$ if density $\rho = kr^2$.

5. Find by double integration the area lying inside the cardioid $r = a (1 + \cos \theta)$ and outside the circle $r = a$.

6. Find by double integration the area lying inside the cardioid $r = 1 + \cos \theta$ and outside the parabola $r (1 + \cos \theta) = 1$.

Transform the following double integrals to polar coordinates and hence evaluate them :

7. (i) $\int_{y=0}^{a} \int_{x=0}^{\sqrt{(a^2 - y^2)}} (a^2 - x^2 - y^2)\, dx\, dy.$

(ii) $\int_0^1 \int_x^{\sqrt{(2x - x^2)}} (x^2 + y^2)\, dx\, dy.$

(iii) $\int_0^a \int_0^{\sqrt{(a^2 - x^2)}} y^2 \sqrt{(x^2 + y^2)}\, dx\, dy.$

Answers 2

1. (i) $\frac{1}{4}\pi a^2$ (ii) $\frac{a^2}{6}$ (iii) $\frac{16}{15}a^4$

2. $\frac{4a^3}{9}$ 3. $\frac{4a^3}{3}$ 4. $\frac{\pi k a^4}{16}$

5. $\frac{1}{4}a^2(\pi+8)$ 6. $\frac{9\pi+16}{12}$

7. (i) $\int_0^{\pi/2}\int_0^a (a^2-r^2)\, r\, d\theta\, dr\,;\ \frac{\pi a^4}{8}$

(ii) $\int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} r^3\, d\theta\, dr\,;\left(\frac{3\pi}{8}\right)-1$

(iii) $\int_0^{\pi/2}\int_0^a r^4\sin^2\theta\, d\theta\, dr\,;\ \frac{\pi a^5}{20}$

6 Triple Integrals

Let the function $f(x, y, z)$ of the point $P(x, y, z)$ be continuous for all points within a finite region V and on its boundary. Divide the region V into n parts; let $\delta V_1, \delta V_2, \ldots, \delta V_n$ be their volumes. Take a point in each part and from the sum

$$S_n = f(x_1, y_1, z_1)\,\delta V_1 + f(x_2, y_2, z_2)\,\delta V_2 + \ldots + f(x_n, y_n, z_n)\,\delta V_n$$

$$= \sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta V_r\,. \qquad \ldots(1)$$

Then the limit to which the sum (1) tends when n tends to infinity and the dimensions of each sub-division tend to zero, is called the **triple integral** of the function $f(x, y, z)$ over the region V. This is denoted by

$$\iiint_V f(x, y, z)\, dV \quad \text{or} \quad \iiint_V f(x, y, z)\, dx\, dy\, dz\,.$$

7 Evaluation of Triple Integrals

(a) If the region V be specified by the inequalities

$$a \le x \le b,\ c \le y \le d,\ e \le z \le f,$$

then the triple integral

$$\iiint_V f(x, y, z)\, dx\, dy\, dz = \int_a^b\int_c^d\int_e^f f(x, y, z)\, dx\, dy\, dz$$

$$= \int_a^b dx \int_c^d dy \int_e^f f(x, y, z)\, dz\,.$$

Here the order of integration is immaterial and the integration with respect to any of x, y and z can be performed first.

(b) If the limits of z are given as functions of x and y, the limits of y as functions of x while x takes the constant values say from $x = a$ to $x = b$, then

$$\iiint_V f(x, y, z)\, dx\, dy\, dz = \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x,\, y)}^{z_2(x,\, y)} f(x, y, z)\, dz .$$

The integration with respect to z is performed first regarding x and y as constants, then the integration w.r.t. y is performed regarding x as a constant and in the last we perform the integration w.r.t. x.

Illustrative Examples

Example 15: *Evaluate* $\int_{y=0}^{3} \int_{x=0}^{2} \int_{z=0}^{1} (x + y + z)\, dz\, dx\, dy$.

Solution: The given integral

$$= \int_{y=0}^{3} \int_{x=0}^{2} \left\{ \int_0^1 (x + y + z)\, dz \right\} dx\, dy$$

$$= \int_{y=0}^{3} \int_{x=0}^{2} \left\{ xz + yz + \frac{z^2}{2} \right\}_0^1 dx\, dy = \int_0^3 \left\{ \int_0^2 \left(x + y + \frac{1}{2} \right) dx \right\} dy$$

$$= \int_0^3 \left\{ \frac{x^2}{2} + xy + \frac{x}{2} \right\}_0^2 dy = \int_0^3 (3 + 2y)\, dy = \left[3y + \frac{2y^2}{2} \right]_0^3 = 18 .$$

Example 16: *Evaluate the following integrals.*

(i) $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} xyz\, dx\, dy\, dz$;

(ii) $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2 + y^2 + z^2)\, dx\, dy\, dz$.

Solution: **(i)** We have

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} x\, y\, z\, dx\, dy\, dz = \int_0^1 \int_0^{1-x} xy \left[\frac{z^2}{2} \right]_0^{1-x-y} dx\, dy ,$$

integrating w.r.t. z regarding x and y as constants

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} xy\, \{(1 - x) - y\}^2\, dx\, dy$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} x\, [\, y\, (1 - x)^2 - 2\, (1 - x)\, y^2 + y^3\,]\, dx\, dy$$

$$= \frac{1}{2} \int_0^1 x \left[\frac{(1 - x)^2\, y^2}{2} - \frac{2\, (1 - x)\, y^3}{3} + \frac{y^4}{4} \right]_0^{1-x} dx ,$$

integrating w.r.t. y regarding x as constant

$$= \frac{1}{24} \int_0^1 x \left[6 (1-x)^4 - 8 (1-x)^4 + 3 (1-x)^4 \right] dx$$

$$= \frac{1}{24} \int_0^1 x (1-x)^4 \, dx = \frac{1}{24} \int_0^{\pi/2} \sin^2 \theta \cos^8 \theta . 2 \sin \theta \cos \theta \, d\theta ,$$

putting $x = \sin^2 \theta$ so that $dx = 2 \sin \theta \cos \theta \, d\theta$

$$= \frac{1}{12} \int_0^{\pi/2} \sin^3 \theta \cos^9 \theta \, d\theta = \frac{1}{12} \cdot \frac{2.8.6.4.2}{12.10.8.6.4.2} = \frac{1}{720} .$$

(ii) Here the integrand $x^2 + y^2 + z^2$ is a symmetrical expression in x, y and z and therefore the limits of integration can be assigned at pleasure. We have the given integral

$$= \int_{z=-c}^{c} \int_{y=-b}^{b} \int_{x=-a}^{a} (x^2 + y^2 + z^2) \, dx \, dy \, dz$$

$$= 2 \int_{z=-c}^{c} \int_{y=-b}^{b} \int_{x=0}^{a} (x^2 + y^2 + z^2) \, dx \, dy \, dz ,$$

because $x^2 + y^2 + z^2$ is an even function of x

$$= 2 \int_{z=-c}^{c} \int_{y=-b}^{b} \left[\frac{x^3}{3} + (y^2 + z^2) x \right]_0^a dy \, dz ,$$

integrating w.r.t. x regarding y and z as constants

$$= 2 \int_{z=-c}^{c} \int_{y=-b}^{b} \left[\frac{a^3}{3} + ay^2 + az^2 \right] dy \, dz$$

$$= 4 \int_{z=-c}^{c} \int_0^b \left[\frac{a^3}{3} + az^2 + ay^2 \right] dy \, dz ,$$

because $\frac{a^3}{3} + az^2 + ay^2$ is an even function of y

$$= 4 \int_{z=-c}^{c} \left[\frac{a^3}{3} y + az^2 y + \frac{ay^3}{3} \right] dz ,$$

integrating w.r.t. y regarding z as constant

$$= 4 \int_{z=-c}^{c} \left[\frac{a^3 b}{3} + abz^2 + \frac{ab^3}{3} \right] dz = 8 \int_0^c \left[\frac{a^3 b}{3} + abz^2 + \frac{ab^3}{3} \right] dz$$

$$= 8 \left[\frac{a^3 b}{3} z + ab \frac{z^3}{3} + \frac{ab^3}{3} z \right]_0^c$$

$$= \frac{8}{3} (a^3 bc + abc^3 + ab^3 c) = \frac{8}{3} abc \, (a^2 + b^2 + c^2) .$$

Example 17: *Evaluate* $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{(4z - x^2)}} dz \, dx \, dy$.

Solution: The given triple integral is

$$= \int_0^4 \int_0^{2\sqrt{z}} \left[\int_0^{\sqrt{(4z - x^2)}} dy \right] dz\, dx = \int_0^4 \int_0^{2\sqrt{z}} [y]_0^{\sqrt{(4z - x^2)}}\, dz\, dx$$

$$= \int_0^4 [\sqrt{(4z - x^2)}\, dx]_0^{2\sqrt{z}}\, dz$$

$$= \int_0^4 \left[\frac{x}{2} \sqrt{(4z - x^2)} + \frac{4z}{2} \sin^{-1} \frac{x}{2\sqrt{z}} \right]_0^{2\sqrt{z}} dz$$

$$= \int_0^4 \left[0 + \frac{4z}{2} \sin^{-1} \frac{2\sqrt{z}}{2\sqrt{z}} \right] dz = \int_0^4 2z \cdot \frac{\pi}{2}\, dz = \int_0^4 \pi z\, dz$$

$$= \pi \left[\frac{z^2}{2} \right]_0^4 = \frac{\pi}{2} [16] = 8\pi .$$

Example 18: *Find the volume of the tetrahedron bounded by the coordinate planes and the plane* $x + y + z = 1$. **(Rohilkhand 2013B**

Solution: Here the region of integration V to cover the volume of the tetrahedron can be expressed as $0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1 - x - y$.

Therefore the required volume of the tetrahedron

$$= \iiint_V dx\, dy\, dz = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx\, dy\, dz \qquad \textbf{(Note)}$$

$$= \int_0^1 \int_0^{1-x} [z]_0^{1-x-y}\, dx\, dy = \int_0^1 \int_0^{1-x} (1 - x - y)\, dx\, dy$$

$$= \int_0^1 \left[(1 - x) y - \frac{y^2}{2} \right]_0^{1-x} dx = \int_0^1 \left[(1 - x)^2 - \frac{(1 - x)^2}{2} \right] dx$$

$$= \int_0^1 \frac{1}{2} (1 - x)^2\, dx = \frac{1}{2} \left[\frac{(1 - x)^3}{3 \cdot (-1)} \right]_0^1 = -\frac{1}{6} [0 - 1] = \frac{1}{6} \cdot$$

Example 19: *Evaluate* $\iiint (x + y + z)\, dx\, dy\, dz$ *over the tetrahedron* $x = 0, y = 0, z = 0$ *and* $x + y + z = 1$.

Solution: The region of integration V for the given tetrahedron can be expressed as

$$0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1 - x - y .$$

Hence the required triple integral $= \iiint_V (x + y + z)\, dx\, dy\, dz$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x + y + z)\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^{1-x} \left[(x + y) z + \frac{z^2}{2} \right]_0^{1-x-y} dx\, dy$$

$$= \int_0^1 \int_0^{1-x} \left[(x + y)(1 - x - y) + \frac{(1 - x - y)^2}{2} \right] dx\, dy$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\left(x+y+\frac{1-x-y}{2}\right) dx\, dy$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2}(1-x-y)(1+x+y)\, dx\, dy$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} [1-(x+y)^2]\, dx\, dy = \frac{1}{2}\int_0^1 \left[y - \frac{(x+y)^3}{3}\right]_0^{1-x} dx$$

(Note)

$$= \frac{1}{2}\int_0^1 \left(1 - x - \frac{1}{3} + \frac{x^3}{3}\right) dx = \frac{1}{2}\int_0^1 \left(\frac{2}{3} - x + \frac{x^3}{3}\right) dx$$

$$= \frac{1}{2}\left[\frac{2}{3}x - \frac{x^2}{2} + \frac{x^4}{3 \times 4}\right]_0^1 = \frac{1}{2}\left[\frac{2}{3} - \frac{1}{2} + \frac{1}{12}\right] = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8}.$$

Example 20: *Evaluate* $\iiint z^2\, dx\, dy\, dz$ *over the sphere* $x^2 + y^2 + z^2 = 1$.

Solution: Here the region of integration can be expressed as

$$-1 \le x \le 1, -\sqrt{(1-x^2)} \le y \le \sqrt{(1-x^2)}, -\sqrt{(1-x^2-y^2)} \le z \le \sqrt{(1-x^2-y^2)}.$$

$\therefore$ the required triple integral

$$= \int_{-1}^1 \int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} \int_{-\sqrt{(1-x^2-y^2)}}^{\sqrt{(1-x^2-y^2)}} z^2\, dx\, dy\, dz$$

$$= \int_{-1}^1 \int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} \left[\frac{z^3}{3}\right]_{-\sqrt{(1-x^2-y^2)}}^{\sqrt{(1-x^2-y^2)}} dx\, dy$$

$$= \frac{1}{3}\int_{-1}^1 \left[\int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} 2(1-x^2-y^2)^{3/2}\, dy\right] dx$$

$$= \frac{2}{3}\int_{-1}^1 \left[\int_{-\pi/2}^{\pi/2} [(1-x^2)\cos^2\theta]^{3/2}.\sqrt{(1-x^2)}.\cos\theta\, d\theta\right] dx$$

[putting $y = \sqrt{(1-x^2)}\sin\theta$ so that $dy = \sqrt{(1-x^2)}\cos\theta\, d\theta$;
also when $y = 0$, $\theta = 0$ and when $y = \sqrt{(1-x^2)}$, $\theta = \pi/2$]

$$= \frac{2}{3}\int_{-1}^1 \left[2.\int_0^{\pi/2} (1-x^2)^2 \cos^4\theta\, d\theta\right] dx$$

$$= \frac{4}{3}\int_{-1}^1 (1-x^2)^2 \cdot \frac{3.1}{4.2}\cdot\frac{\pi}{2}\, dx = \frac{\pi}{4}\int_{-1}^1 (1-x^2)^2\, dx$$

$$= \frac{\pi}{4}.2\int_0^1 (1-2x^2+x^4)\, dx = \frac{\pi}{2}\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_0^1$$

$$= \frac{\pi}{2}\left[1 - \frac{2}{3} + \frac{1}{5}\right] = \frac{\pi}{2}\cdot\frac{8}{15} = \frac{4\pi}{15}.$$

Comprehensive Exercise 3

Evaluate the following integrals :

1. (i) $\int_{x=0}^{1} \int_{y=0}^{2} \int_{z=1}^{2} x^2 \ yz \ dz \ dy \ dx.$

(ii) $\int_0^1 \int_0^1 \int_0^1 e^{\ x+y+z} \ dx \ dy \ dz.$

(iii) $\int_{-1}^{1} \int_0^{z} \int_{x-z}^{x+z} (x + y + z) \ dy \ dx \ dz.$

(iv) $\int_0^{\log 2} \int_0^{x} \int_0^{x + \log y} e^{x+y+z} \ dx \ dy \ dz.$

2. (i) $\int_0^1 \int_{y^2}^{1} \int_0^{1-x} x \ dy \ dx \ dz.$

(ii) $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \dfrac{dx \ dy \ dz}{(1 + x + y + z)^3}.$ **(Kanpur 2008; Avadh 13)**

(iii) $\int_1^3 \int_{1/x}^{1} \int_0^{\sqrt{(xy)}} xyz \ dx \ dy \ dz.$

(iv) $\int_0^{\pi/2} d\theta \int_0^{a \sin \theta} dr \int_0^{(a^2 - r^2)/a} r \ dz.$

3. (i) $\int_0^{a} \int_0^{x} \int_0^{x+y} e^{\ x+y+z} \ dx \ dy \ dz.$

(ii) $\int_0^{a} \int_0^{a-x} \int_0^{a-x-y} x^2 \ dx \ dy \ dz.$

4. Evaluate the triple integral of the function $f(x, y, z) = x^2$ over the region V enclosed by the planes $x = 0, y = 0, z = 0$ and $x + y + z = a.$
(Avadh 2012; Rohilkhand 12)

5. Find the volume of the tetrahedron bounded by the plane $x / a + y / b + z / c = 1$ and the coordinate planes.

6. (i) Evaluate $\iiint \dfrac{dx \ dy \ dz}{(x + y + z + 1)^3}$ over the region $x \geq 0, y \geq 0, z \geq 0,$ $x + y + z \leq 1.$ **(Avadh 2013)**

(ii) Evaluate $\iiint xyz \ dx \ dy \ dz$ over the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$
(Kanpur 2011)

(iii) Evaluate $\iiint (z^5 + z) \ dx \ dy \ dz$ over the sphere $x^2 + y^2 + z^2 = 1.$

(iv) Evaluate $\iiint_R u^2 \ v^2 \ w \ du \ dv \ dw,$ where R is the region $u^2 + v^2 \leq 1,$ $0 \leq w \leq 1.$

Answers 3

1. (i) 1 (ii) $(e-1)^3$ (iii) 0
(iv) $\frac{8}{3}\log 2-\frac{19}{9}$

2. (i) $\frac{4}{35}$ (ii) $\frac{1}{2}\left(\log 2-\frac{5}{8}\right)$ (iii) $\frac{1}{6}\left(\frac{26}{3}-\log 3\right)$
(iv) $\frac{5a^3\pi}{64}$

3. (i) $\frac{1}{8}(e^{4a}-6e^{2a}+8e^{a}-3)$ (ii) $\frac{a^5}{60}$

4. $\frac{a^5}{60}$ 5. $\frac{abc}{6}$

6. (i) $\frac{1}{2}\left(\log 2-\frac{5}{8}\right)$ (ii) 0 (iii) 0 (iv) $\frac{\pi}{48}$

8 Change of Order of Integration

If in a double integral the limits of integration of both x and y are constant, we can generally integrate $\int\int f(x, y)\,dx\,dy$ in either order. But if the limits of y are functions of x, we must first integrate w.r.t. y regarding x as constant and then integrate w.r.t. x. In this case the order of integration can be changed only if we find the new limits of x as functions of y and the new constant limits of y. This is usually best obtained from geometrical considerations as will be clear from the examples that follow.

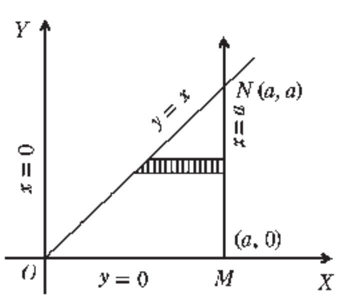
Illustrative Examples

Example 21: *Change the order of integration in the double integral*

$$\int_0^a \int_0^x f(x, y)\,dx\,dy.$$

(Lucknow 2006, 08; Kashi 13)

Solution: In the given integral the limits of integration are given by the straight lines $y=0$, $y=x$, $x=0$ and $x=a$. Draw these lines bounding the region of integration in the same figure. We observe that the region of integration is the area ONM.



In the given integral, the limits of integration of y being variable, we are required to integrate first w.r.t. y regarding x as constant and then w.r.t. x.

To reverse the order of integration, we have to integrate first w.r.t. x regarding y as constant and then w.r.t. y. This is done by dividing the area ONM into strips parallel to the x-axis. Let us take strips parallel to the x-axis starting from the line ON (*i.e.*, $y = x$) and terminating on the line MN (*i.e.*, $x = a$). Thus for this region ONM, x varies from y to a and y varies from 0 to a.

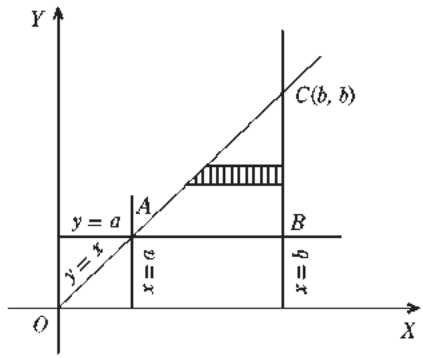
Hence by changing the order of integration, we have

$$\int_0^a \int_0^x f(x, y)\, dx\, dy = \int_0^a \int_y^a f(x, y)\, dy\, dx.$$

Example 22: *Prove that* $\int_a^b dx \int_a^x f(x, y)\, dy = \int_a^b dy \int_y^b f(x, y)\, dx$.

(Lucknow 2007)

Solution: Let $I = \int_a^b dx \int_a^x f(x, y)\, dy$.



We are required to change the order of integration in the integral I. In the integral I the limits of integration of y are given by the straight lines $y = a$ and $y = x$. Also the limits of integration of x are given by the straight lines $x = a$ and $x = b$. Draw the straight lines $y = a$, $y = x$, $x = a$ and $x = b$, bounding the region of integration, in the same figure. We observe that the region of integration is the area of the triangle ABC.

In the integral I we are required to integrate first w.r.t. y and then w.r.t. x. To reverse the order of integration we have to integrate first w.r.t. x and then w.r.t. y. This is done by dividing the area ABC into strips parallel to the x-axis. Let us take strips parallel to the x-axis starting from the line AC (*i.e.*, $y = x$) and terminating on the line BC (*i.e.*, $x = b$). Thus for the region ABC, x varies from y to b and y varies from a to b.
Hence by changing the order of integration, we have

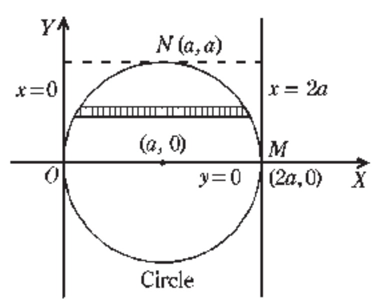
$$\int_a^b dx \int_a^x f(x, y)\, dy = \int_a^b dy \int_y^b f(x, y)\, dx.$$

Example 23: *Change the order of integration in* $\int_0^{2a} \int_0^{\sqrt{(2ax - x^2)}} f(x, y)\, dx\, dy$.

(Meerut 2013B)

Solution: In the given integral the limits of integration of y are given by $y = 0$ (*i.e.*, the x-axis) and $y = \sqrt{(2ax - x^2)}$ *i.e.*, $y^2 = 2ax - x^2$

i.e., $(x - a)^2 + y^2 = a^2$ which is a circle with centre $(a, 0)$ and radius a. Again the limits of integration of x are given by the straight lines $x = 0$ (*i.e.*, the y-axis) and $x = 2a$.



Draw the curves $(x-a)^2+y^2=a^2$, $y=0$, $x=0$ and $x=2a$, bounding the region of integration, in the same figure. From figure we observe that the area of integration is $OMNO$.

In the given integral we are required to integrate first w.r.t. y regarding x as a constant and then w.r.t. x.

To reverse the order of integration, divide the area $OMNO$ into strips parallel to the x-axis. These strips will have their extremities on the portions ON and NM of the circle. Solving the equation of circle $(x-a)^2+y^2=a^2$ for x, we get

$$(x-a)^2=a^2-y^2 \text{ i.e., } x-a=\pm\sqrt{(a^2-y^2)} \text{ i.e., } x=a\pm\sqrt{(a^2-y^2)}.$$

So for the region $OMNO$, x varies from $a-\sqrt{(a^2-y^2)}$ to $a+\sqrt{(a^2-y^2)}$ and y varies from 0 to a.

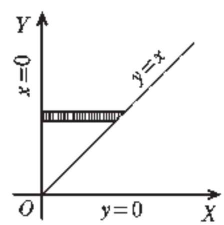
Therefore, changing the order of integration, the given double integral transforms to $\int_0^a \int_{a-\sqrt{(a^2-y^2)}}^{a+\sqrt{(a^2-y^2)}} f(x,y)\,dy\,dx$.

Example 24: *Change the order of integration in the double integral*

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dx\,dy$$

and hence find its value. **(Agra 2002; Kumaun 01; Avadh 07; Kashi 14; Purvanchal 14)**

Solution: In the given integral the limits of integration are given by the lines $y=x$, $y=\infty$, $x=0$ and $x=\infty$. Therefore the region of integration is bounded by $x=0$, $y=x$ and, an infinite boundary. In the given integral the limits of integration of y are variable while those of x are constant. Thus we have to first integrate with respect to y regarding x as constant and then we integrate w.r.t. x. This is done by first integrating w.r.t. y along a strip drawn parallel to the y-axis and then integrating w.r.t. x along all such strips so drawn as to cover the whole region of integration.



If we want to reverse the order of integration, we have to first integrate w.r.t. x regarding y as constant and then we integrate w.r.t. y. This is done by dividing this area into strips parallel to the x-axis. So we take strips parallel to the x-axis starting from the line $x=0$ and terminating on the line $y=x$. Now the limits for x are 0 to y and the limits for y are 0 to ∞.

Hence by changing the order of integration, we have

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dx\,dy=\int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dy\,dx=\int_0^\infty \frac{e^{-y}}{y}[x]_0^y\,dy$$

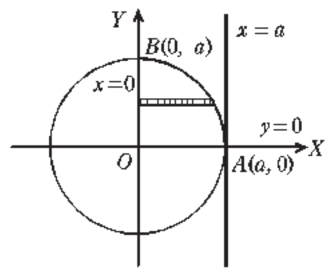
$$=\int_0^\infty \frac{e^{-y}}{y}\cdot y\,dy=\int_0^\infty e^{-y}\,dy=\left[\frac{e^{-y}}{-1}\right]_0^\infty=1.$$

Example 25: *Change the order of integration in the integral* $\int_0^a \int_0^{\sqrt{(a^2 - x^2)}} f(x, y)\, dx\, dy$.

Solution: In the given integral the limits of integration of y are given by the straight line $y = 0$ (*i.e.*, the x-axis) and the curve

$$y = \sqrt{(a^2 - x^2)} \text{ i.e., } y^2 = a^2 - x^2 \text{ i.e., } x^2 + y^2 = a^2$$

which is a circle with centre at the origin and radius a. Again the limits of integration of x are given by the lines $x = 0$ and $x = a$.



We draw the curves $y = 0$, $x^2 + y^2 = a^2$, $x = 0$ and $x = a$, giving the limits of integration, in the same very figure and we observe that the region of integration is the area OAB of the quadrant of the circle $x^2 + y^2 = a^2$.

To change the order of integration in the given integral, we have to first integrate w.r.t. x regarding y as a constant and then we integrate w.r.t. y. This is done by covering the area OAB by strips drawn parallel to the x-axis. These strips start from the line OB (*i.e.*, $x = 0$) and terminate on the arc AB of the circle $x^2 + y^2 = a^2$. So on these strips x varies from 0 to $\sqrt{(a^2 - y^2)}$. Also to cover the area OAB, y varies from 0 to a. Hence by changing the order of integration, we have the given integral

$$= \int_0^a \int_0^{\sqrt{(a^2 - y^2)}} f(x, y)\, dy\, dx.$$

Example 26: *Change the order of integration in*

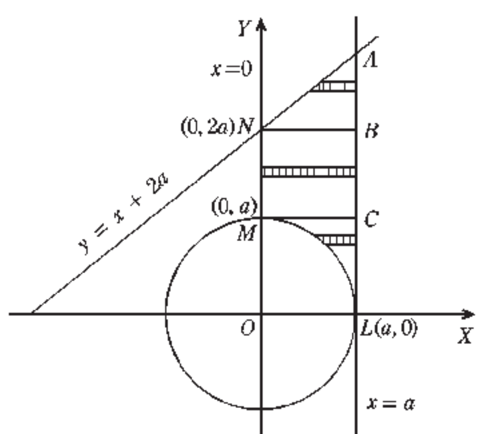
$$\int_0^a \int_{\sqrt{(a^2 - x^2)}}^{x + 2a} f(x, y)\, dx\, dy.$$

(Kumaun 2009, 15)

Solution: Here the area of integration is bounded by the curves

$$y = \sqrt{(a^2 - x^2)} \quad \text{i.e.,} \quad x^2 + y^2 = a^2$$

which is a circle with centre $(0, 0)$ and radius a, $y = x + 2a$ which is a straight line passing through $(0, 2a)$, $x = 0$ *i.e.*, the y-axis and the line $x = a$ which is a line parallel to the y-axis at a distance a from the origin.



We draw the curves $x^2 + y^2 = a^2$, $y = x + 2a$, $x = 0$ and $x = a$, giving the limits of integration, in the same figure. We observe that the region of integration is the area $MLANM$.

To reverse the order of integration, cover this area of integration $ML\ ANM$ by strips parallel to the x-axis. Draw the lines MC and NB parallel to the x-axis so that the region of integration $ML\ ANM$ is divided into three portions MLC, $NMCB$ and NAB.

For the region MLC, x varies from the arc ML of the circle $x^2 + y^2 = a^2$ to the line $x = a$ *i.e.*, x varies from $\sqrt{(a^2 - y^2)}$ to a and y varies from 0 to a.

For the region $NMCB$, x varies from 0 to a and y varies from a to $2a$.

For the region NBA, x varies from $y - 2a$ to a and y varies from $2a$ to $3a$.

Therefore, changing the order of integration, the given integral transforms to

$$\int_0^a \int_{\sqrt{(a^2 - y^2)}}^a f(x, y)\, dy\, dx + \int_a^{2a} \int_0^a f(x, y)\, dy\, dx + \int_{2a}^{3a} \int_{y-2a}^a f(x, y)\, dy\, dx.$$

Comprehensive Exercise 4

Change the order of integration in the following integrals.

1. $\int_0^1 \int_x^{x(2-x)} f(x, y)\, dx\, dy.$

2. $\int_0^3 \int_1^{\sqrt{(4-y)}} (x + y)\, dy\, dx.$

3. $\int_0^{a \cos \alpha} \int_{x \tan \alpha}^{\sqrt{(a^2 - x^2)}} f(x, y)\, dx\, dy.$ **(Kanpur 2005; Avadh 11; Kumaun 02, 10)**

4. $\int_0^a \int_{mx}^{lx} f(x, y)\, dx\, dy.$ **(Lucknow 2010)**

5. $\int_0^{2a} \int_{x^2/4a}^{3a - x} f(x, y)\, dx\, dy.$

6. $\int_0^a \int_0^{b/(b+x)} f(x, y)\, dx\, dy.$

7. $\int_0^a \int_x^{a^2/x} f(x, y)\, dx\, dy.$ **(Lucknow 2009; Kanpur 10; Kumaun 12)**

8. $\int_c^a \int_{(b/a)\sqrt{(a^2 - x^2)}}^b f(x, y)\, dx\, dy,$ where $c < a$.

9. $\int_0^{a/2} \int_{x^2/a}^{x - (x^2/a)} f(x, y)\, dx\, dy.$

10. $\int_0^{2a} \int_{\sqrt{(2ax - x^2)}}^{\sqrt{(2ax)}} f(x, y)\, dx\, dy.$ **(Kumaun 2013)**

11. $\int_0^{ab/(a^2 + b^2)} \int_0^{(a/b)\sqrt{(b^2 - y^2)}} f(x, y)\, dy\, dx.$

12. $\int_0^{\pi/2} \int_0^{2a\cos\theta} f(r, \theta)\, d\theta\, dr.$

(Kanpur 2009; Kumaun 11)

13. Change the order of integration in the double integral

$$\int_0^a \int_0^x \frac{\phi'(y)\, dx\, dy}{\sqrt{\{(a-x)(x-y)\}}} \quad \text{and hence find its value.}$$

[**Hint:** Put $x = a\cos^2\theta + y\sin^2\theta$]

Answers 4

1. $\int_0^1 \int_{1-\sqrt{(1-y)}}^{y} f(x, y)\, dy\, dx$

2. $\int_1^2 \int_0^{4-x^2} (x+y)\, dx\, dy$

3. $\int_0^{a\sin\alpha} \int_0^{y\cot\alpha} f(x, y)\, dy\, dx + \int_{a\sin\alpha}^{a} \int_0^{\sqrt{(a^2-y^2)}} f(x, y)\, dy\, dx$

4. $\int_0^{am} \int_{y/l}^{y/m} f(x, y)\, dy\, dx + \int_{am}^{al} \int_{y/l}^{a} f(x, y)\, dy\, dx$

5. $\int_0^a \int_0^{\sqrt{(4ay)}} f(x, y)\, dy\, dx + \int_a^{3a} \int_a^{3a-y} f(x, y)\, dy\, dx$

6. $\int_0^{b/(a+b)} \int_0^a f(x, y)\, dy\, dx + \int_{b/(a+b)}^{1} \int_0^{b(1-y)/y} f(x, y)\, dy\, dx$

7. $\int_0^a \int_0^y f(x, y)\, dy\, dx + \int_a^{\infty} \int_0^{a^2/y} f(x, y)\, dy\, dx$

8. $\int_0^{b\sqrt{\{1-(c^2/a^2)\}}} \int_{a\sqrt{\{1-(y^2/b^2)\}}}^{a} f(x, y)\, dy\, dx$

$$+ \int_{b\sqrt{\{1-(c^2/a^2)\}}}^{a} \int_c^a f(x, y)\, dy\, dx$$

9. $\int_0^{a/4} \int_{\frac{1}{2}[a-\sqrt{(a^2-4ay)}]}^{\sqrt{(ay)}} f(x, y)\, dy\, dx$

10. $\int_0^a \int_{y^2/2a}^{a-\sqrt{(a^2-y^2)}} f(x, y)\, dy\, dx + \int_0^a \int_{a+\sqrt{(a^2-y^2)}}^{2a} f(x, y)\, dy\, dx$

$$+ \int_0^{2a} \int_{y^2/2a}^{2a} f(x, y)\, dy\, dx$$

11. $\int_0^{ab/\sqrt{(a^2+b^2)}} \int_0^{ab/\sqrt{(a^2+b^2)}} f(x, y)\, dx\, dy$

$$+ \int_{ab/\sqrt{(a^2+b^2)}}^{a} \int_0^{(b/a)\sqrt{(a^2-x^2)}} f(x, y)\, dx\, dy$$

12. $\int_0^{2a} \int_0^{\cos^{-1}(r/2a)} f(r, \theta)\, dr\, d\theta$

13. $\int_0^a \int_y^a \frac{\phi'(y)\, dy\, dx}{\sqrt{\{(a-x)(x-y)\}}} = \pi\, [\phi(a) - \phi(0)]$

9 Change of Variables in a Double Integral

Sometimes, the evaluation of a double integral becomes more convenient by a suitable change of variables from one system to another system.

Let the variables in the double integral $\iint_A f(x, y)\, dx\, dy$ be changed from x, y to u, v where $x = \phi(u, v)$ and $y = \psi(u, v)$.

Then on substituting for x and y, the double integral is transformed to $\iint_{A'} F(u, v)\, J\, du\, dv$, where $J(u, v)$ is the Jacobian of x, y w.r.t. u, v *i.e.*,

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix},$$

and A' is the region in the uv-plane corresponding to the region A in the xy-plane. Thus remember that $dx\, dy = J\, du\, dv$.

Special case: Change to polar coordinates from the cartesian co-ordinates.

To change the variables from cartesian to polar coordinates we put $x = r\cos\theta$, $y = r\sin\theta$. In this case

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} == \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r,$$

and therefore $dx\, dy = J\, d\theta\, dr = r\, d\theta\, dr$.

This change is specially useful when the region of integration is a circle or a part of a circle.

Illustrative Examples

Example 27: *Transform* $\iint f(x, y)\, dx\, dy$ *by the substitution* $x + y = u$, $y = uv$.

Solution: We have $x + y = u$ and $y = uv$. ...(1)

From these, we have

$$x = u - y = u - uv \quad \text{and} \quad y = uv. \quad ...(2)$$

$\therefore \qquad \dfrac{\partial x}{\partial u} = 1 - v, \dfrac{\partial x}{\partial v} = -u, \dfrac{\partial y}{\partial u} = v$ and $\dfrac{\partial y}{\partial v} = u$.

$$\therefore \qquad J = \frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 - v & -u \\ v & u \end{vmatrix} = u.$$

$\therefore \qquad dx\, dy = J\, du\, dv = u\, du\, dv$.

Hence the given integral transforms to

$$\iint F(u, v)\, u\, du\, dv.$$

Example 28: *Transform* $\iint f(x, y)\, dx\, dy$ *to polar coordinates.*

Solution: We have $x = r\cos\theta$, $y = r\sin\theta$.

$$\text{Now} \qquad J = \frac{\partial (x, y)}{\partial (r, \theta)} == \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

$\therefore \qquad dx\, dy = J\, d\theta\, dr = r\, d\theta\, dr$.

Hence the given integral transforms to $\iint F(r, \theta)\, r\, d\theta\, dr$.

Example 29: *Evaluate* $\iint \sqrt{(a^2 - x^2 - y^2)}\, dx\, dy$ *over the semi-circle* $x^2 + y^2 = ax$ *in the positive quadrant.*

Solution: Here the region of integration is a semi-circle. Therefore, for the sake of convenience, changing to polar coordinates by putting $x = r\cos\theta$ and $y = r\sin\theta$ in $x^2 + y^2 = ax$, we have

$$r^2\cos^2\theta + r^2\sin^2\theta = a\, r\cos\theta \quad \text{or} \quad r^2(\sin^2\theta + \cos^2\theta) = ar\cos\theta$$

or $\qquad r = a\cos\theta$.

The equation $r = a\cos\theta$ represents a circle passing through the pole and diameter through the pole along the initial line.

For the given region r varies from 0 to $a\cos\theta$ and θ varies from 0 to $\pi/2$.

$$\therefore \qquad \iint \sqrt{(a^2 - x^2 - y^2)}\, dx\, dy = \int_0^{\pi/2} \int_0^{a\cos\theta} \sqrt{(a^2 - r^2)}\,.\, r\, d\theta\, dr,$$

$$[\because x^2 + y^2 = r^2 \text{ and } dx\, dy = r\, d\theta\, dr]$$

$$= \int_0^{\pi/2} \left[\int_0^{a\cos\theta} -\frac{1}{2}(a^2 - r^2)^{1/2}\,.\,(-2r)\, dr\right] d\theta \qquad \textbf{(Note)}$$

$$= \int_0^{\pi/2} \left[-\frac{1}{2}\cdot\frac{2}{3}(a^2 - r^2)^{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2} (a^3\sin^3\theta - a^3)\, d\theta = -\frac{a^3}{3}\left[\frac{2}{3.1} - \frac{\pi}{2}\right] = \frac{1}{3}a^3\left(\frac{1}{2}\pi - \frac{2}{3}\right).$$

Comprehensive Exercise 5

1. Transform $\int_0^a \int_0^{a-x} f(x, y)\, dx\, dy$, by the substitution $x + y = u,\ y = uv$.
2. By using the transformation $x + y = u,\ y = uv$, show that
$$\int_0^1 \int_0^{1-x} e^{y/(x+y)}\, dx\, dy = \frac{1}{2}(e - 1).$$
3. By using the transformation $x + y = u,\ y = uv$, prove that
$$\iint \{xy(1 - x - y)\}^{1/2}\, dx\, dy$$
taken over the area of the triangle bounded by the lines
$x = 0,\ y = 0,\ x + y = 1$ is $2\pi/105$.
4. Evaluate $\iint (x^2 + y^2)^{7/2}\, dx\, dy$ over the circle $x^2 + y^2 = 1$.
5. Evaluate $\iint xy(x^2 + y^2)^{3/2}\, dx\, dy$ over the positive quadrant of the circle $x^2 + y^2 = 1$.
6. Evaluate $\iint e^{-(x^2+y^2)}\, dx\, dy$ over the circle $x^2 + y^2 = a^2$.

Answers 5

1. $\int_0^a \int_0^1 F(u, v)\, u\, du\, dv$ 4. $2\pi/9$ 5. $1/14$ 6. $\pi(1 - e^{-a^2})$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The value of the double integral $\int_{\theta=0}^{2\pi} \int_{r=0}^{a} r\, d\theta\, dr$ is

(a) πa^2 (b) $\frac{\pi a^2}{2}$ (c) πa (d) $2\pi a^2$

2. The value of the triple integral $\int_0^1 \int_0^1 \int_0^1 xyz\, dx\, dy\, dz$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{8}$ (c) $\frac{1}{4}$ (d) 1

(Kumaun 2009, 11)

3. The value of the double integral $\int_0^a \int_0^{\sqrt{(a^2 - y^2)}} dy\, dx$ is

(a) πa^2 (b) $2\pi a^2$ (c) $\frac{\pi a^2}{2}$ (d) $\frac{\pi a^2}{4}$

(Kumaun 2013)

4. The value of the triple integral
$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \cos^2 x \cos^2 y \cos^2 z\, dx\, dy\, dz$ is

(a) $\frac{\pi^2}{16}$ (b) $\frac{\pi}{64}$ (c) $\frac{\pi^3}{8}$ (d) $\frac{\pi^3}{64}$

5. The value of the triple integral $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z}\, dx\, dy\, dz$ is

(a) e^3 (b) $\frac{e^3}{4}$ (c) $(e-1)^3$ (d) $(e+1)^3$

(Kumaun 2012)

6. The value of the triple integral
$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\pi/2} \cos x \cos y \cos z\, dx\, dy\, dz$ is

(a) 1 (b) $\frac{\pi}{2}$ (c) π (d) $\frac{3\pi}{2}$

(Rohilkhand 2005)

7. The value of $\int_0^{\pi/2} \int_0^{\sin\theta} r\, d\theta\, dr$ is equal to

(a) $\int_0^{\pi/2} \sin\theta\, d\theta$ (b) $\int_0^{\sin\theta} \frac{\pi}{2} r\, dr$

(c) $\int_0^{\pi/2} \frac{\sin^2\theta}{2} d\theta$ (d) none of these

(Garhwal 2003)

8. Value of $\int_0^2 \int_0^2 (x^2 + y^2)\, dx\, dy$ is

(a) $\frac{3}{13}$ (b) $\frac{32}{3}$ (c) $\frac{34}{4}$ (d) 1

(Kumaun 2014)

9. Value of $\int_0^a \int_0^b \frac{dx\, dy}{xy}$ is

(a) ab (b) $(\log b) \cdot (\log a)$

(c) 0 (d) a / b

(Kumaun 2015)

10. Value of $\int_0^{\pi/2} \int_0^a \cos\theta\, r \sin\theta\, d\theta\, dr$ is

(a) a^2 (b) $6 / a^2$ (c) $a^2 / 6$ (d) 0

(Kumaun 2015)

11. Value of $\int_0^{\pi} \int_0^x \sin y\, dy\, dx$ is

(a) $\sqrt{\pi}$ (b) 0 (c) 1 (d) π

(Kumaun 2015)

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. The value of the double integral $\int_0^3 \int_1^2 dx\, dy$ is …… . **(Agra 2002)**

2. The value of the double integral $\int_0^1 \int_0^1 xy\, dx\, dy$ is …… .

3. The value of the double integral $\int_0^1 \int_0^x xy\, dx\, dy$ is …… .

4. The value of the double integral $\int_0^{\pi/2} \int_0^{2a\cos\theta} r\, d\theta\, dr$ is …… .

5. The value of the triple integral $\int_0^2 \int_0^2 \int_0^2 xyz\, dx\, dy\, dz$ is …… .

6. The value of the triple integral $\int_1^2 \int_1^2 \int_1^3 dx\, dy\, dz$ is …… .

7. The value of the double integral $\int_{-a}^{a} \int_0^{\sqrt{(a^2-x^2)}} dx\, dy$ is …… .

True or False

Write 'T' for true and 'F' for false statement.

1. The value of the double integral $\int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{a} r\, d\theta\, dr$ is $\dfrac{\pi a^2}{2}$.

2. The value of the double integral $\int_{-a}^{a} \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} dx\, dy$ is πa^2.

3. The value of the double integral $\int_{-a}^{a} \int_0^{\sqrt{(a^2-x^2)}} x\, dx\, dy$ is 0.

Answers

Multiple Choice Questions

1. (a)	2. (b)	3. (d)	4. (d)	5. (c)
6. (a)	7. (c)	8. (b)	9. (b)	10. (c)
11. (d)				

Fill in the Blank(s)

1. 3	2. $\frac{1}{4}$	3. $\frac{1}{8}$	4. $\frac{\pi a^2}{2}$
5. 8	6. 2	7. $\frac{\pi a^2}{2}$	

True or False

1. *T*	2. *T*	3. *T*

□

Chapter 7

Areas of Curves

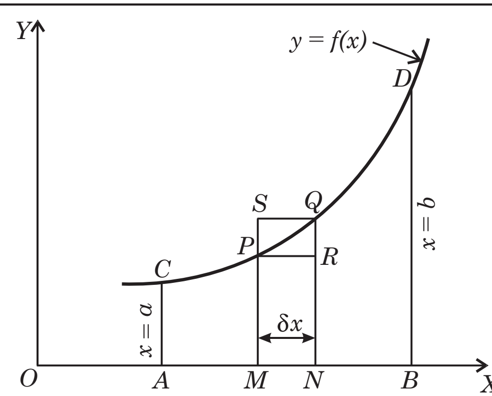
1 Quadrature

The process of finding the area of any bounded portion of a curve is called **quadrature.**

2 Areas of Curves Given by Cartesian Equations

If $f(x)$ is a continuous and single valued function of x, then the area bounded by the curve $y = f(x)$, the axis of x and the ordinates $x = a$ and $x = b$ is

$$\int_a^b y\,dx, \quad \text{or} \quad \int_a^b f(x)\,dx.$$



Proof: Let CD be the arc of the curve $y = f(x)$ and AC and BD be the two ordinates $x = a$ and $x = b$.

Consider $P(x, y)$ and $Q(x + \delta x, y + \delta y)$, the two neighbouring points on the curve. Draw PM and QN perpendiculars to the axis of x, then

$$PM = y, QN = y + \delta y \text{ and } MN = \delta x.$$

Draw PR and QS perpendiculars to NQ and MP produced respectively. The area $AMPC$ depends upon the position of P on the curve. Let A denote the area $AMPC$ and $A + \delta A$ be the area $ANQC$. Then the area

$$MNQP = \text{area } ANQC - \text{area } AMPC$$

$$= A + \delta A - A = \delta A.$$

But clearly this area δA (*i.e.*, the area $MNQP$) lies in magnitude between the areas of the rectangles $MNRP$ and $MNQS$.

Thus, we have

$$\text{area of the rectangle } MNQS > \delta A > \text{area of the rectangle } MNRP$$

i.e., $(y + \delta y)\, \delta x > \delta A > y\, \delta x$

or $y + \delta y > \dfrac{\delta A}{\delta x} > y.$

Now as $Q \to P, \delta x \to 0$ and $\delta y \to 0$. Therefore we have

$$\frac{dA}{dx} = y = f(x)$$

or $dA = y\, dx.$

Integrating both sides between the limits $x = a$ and $x = b$, we have

$$\int_{x=a}^{x=b} dA = \int_a^b y\, dx$$

or $[A]_{x=a}^{x=b} = \int_a^b y\, dx$

or (Area A when $x = b$) – (Area A when $x = a$) $= \int_a^b y\, dx$

or Area $ABDC - 0 = \int_a^b y\, dx$

or **Area $ABDC = \int_a^b y\, dx = \int_a^b f(x)\, dx.$**

Similarly, it can be shown that the area bounded by the curve $x = f(y)$, the axis of y and the abscissae $y = a$ and $y = b$ is

$$\int_a^b x\, dy, \quad or \quad \int_a^b f(y)\, dy.$$

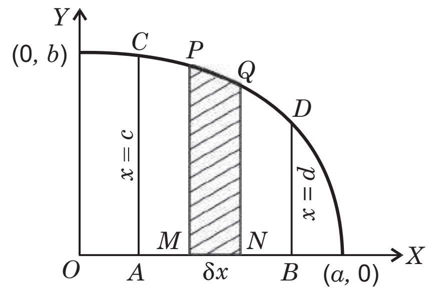
Note 1: In choosing the limits of integration, the lower limit of integration should be taken as the smaller value of the independent variable while the greater value gives us the upper limit of integration.

Note 2: If the curve is symmetrical about x-axis or y-axis or both, then we shall find the area of one symmetrical part and multiply it by the number of symmetrical parts to get the whole area.

Illustrative Examples

Example 1: *Find the area bounded by the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, *the ordinates* $x=c$, $x=d$ *and the x-axis.* **(Meerut 2000)**

Solution: Equation of the ellipse is



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

or $$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

giving $$y=\frac{b}{a}\sqrt{(a^2-x^2)}. \qquad ...(1)$$

$\therefore$ the required area = the area $ABDC$

$$=\int_c^d y\,dx$$

$$=\int_c^d \frac{b}{a}\sqrt{(a^2-x^2)}\,dx, \text{ from (1)}$$

$$=\frac{b}{a}\left[\frac{1}{2}x\sqrt{(a^2-x^2)}+\frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)\right]_c^d$$

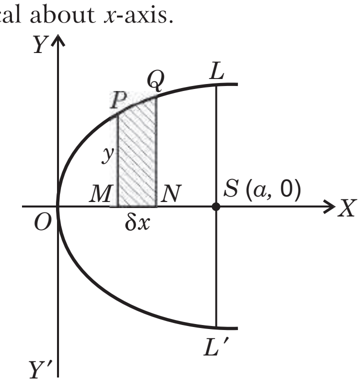
$$=\frac{b}{2a}\left[d\sqrt{(a^2-d^2)}-c\sqrt{(a^2-c^2)}+a^2\left(\sin^{-1}\frac{d}{a}-\sin^{-1}\frac{c}{a}\right)\right].$$

Example 2: *Find the area bounded by the parabola* $y^2=4ax$ *and its latus rectum.*

(Garhwal 2003; Agra 05; Avadh 05; Bundelkhand 08)

Solution: Latus rectum is a line through the focus $S(a,0)$ and perpendicular to x-axis *i.e.*, its equation is $x=a$. Also the curve is symmetrical about x-axis.

$\therefore$ the required area LOL'



$$=2\times\text{area } OSL=2\cdot\int_0^a y\,dx$$

$$=2\int_0^a \sqrt{(4ax)}\,dx,$$

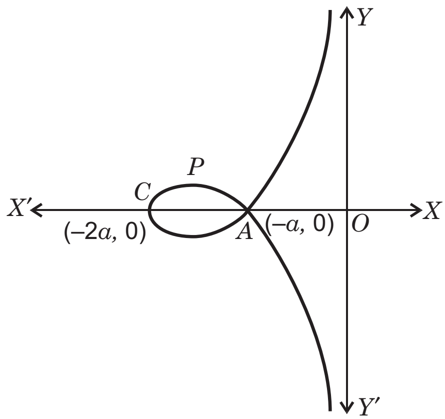
$$[\because y^2=4ax, \text{ i.e., } y=\sqrt{(4ax)}]$$

$$=2\sqrt{(4a)}\left[\frac{2}{3}x^{3/2}\right]_0^a=\frac{8}{3}\sqrt{(a)}\cdot a^{3/2}=\frac{8}{3}a^2.$$

Example 3: *Find the area of a loop of the curve* $xy^2+(x+a)^2(x+2a)=0$.

Solution: The curve is symmetrical about x-axis. Putting $y=0$, we get $x=-a$ and $x=-2a$.

The loop is formed between $x = -2a$ and $x = -a$.



To find the area of the loop, we first shift the origin to the point $(-a, 0)$. The equation of the curve then becomes

$$(x-a)\,y^2 + \{(x-a)+a\}^2\,(x-a+2a) = 0$$

or $$y^2\,(x-a) + x^2\,(x+a) = 0$$

or $$y^2 = \frac{x^2\,(a+x)}{a-x}. \qquad ...(1)$$

Note that the shifting of the origin only changes the equation of the curve and has no effect on its shape. Now the origin being at the point A, the new limits for the loop are $x = -a$ to $x = 0$.

$\therefore$ required area of the loop

$$= 2 \times \text{area } CPA = 2\int_{-a}^{0} y\,dx, \qquad \text{[the value of } y \text{ to be put from (1)]}$$

$$= 2\int_{-a}^{0}\left\{-x\sqrt{\left(\frac{a+x}{a-x}\right)}\right\}dx,$$ [Note that in the equation (1), for the portion CPA, $y = -x\sqrt{\{(a+x)/(a-x)\}}$]

$$= 2\int_{-a}^{0}\frac{-x\,(a+x)}{\sqrt{(a^2-x^2)}}\,dx,$$

multiplying the numerator and the denominator by $\sqrt{(a+x)}$

$$= 2\int_{\pi/2}^{0}\frac{-(-a\sin\theta)\,(a-a\sin\theta)}{a\cos\theta}\cdot(-a\cos\theta)\,d\theta,$$

putting $x = -a\sin\theta$ and $dx = -a\cos\theta\,d\theta$

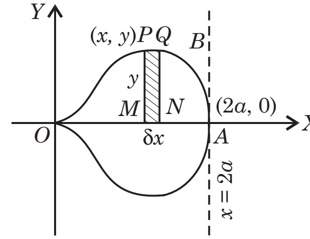
$$= -2a^2\int_{\pi/2}^{0}(\sin\theta-\sin^2\theta)\,d\theta = 2a^2\int_{0}^{\pi/2}(\sin\theta-\sin^2\theta)\,d\theta$$

$$= 2a^2\,[1-\frac{1}{2}\cdot\frac{1}{2}\pi], \text{ by Walli's formula} = 2a^2\,(1-\frac{1}{4}\pi).$$

Example 4: *Find the whole area of the curve* $a^2y^2 = x^3\,(2a-x)$.

(Meerut 2006B; Bundelkhand 12; Avadh 13; Kumaun 13; Rohilkhand 14)

Solution: The given curve is $a^2 y^2 = x^3 (2a - x)$. ...(1)



It is symmetrical about x-axis and it cuts the x-axis at the points $(0,0)$ and $(2a,0)$. The curve does not exist for $x > 2a$ and $x < 0$. Thus the curve consists of a loop lying between $x = 0$ and $x = 2a$.

$\therefore$ the required area = 2 × area OBA

$$= 2\int_0^{2a} y\, dx$$

$$= 2\int_0^{2a} \frac{x^{3/2}\sqrt{(2a-x)}}{a}\, dx, \text{ from (1).}$$

Now put $x = 2a\sin^2\theta$ so that $dx = 4a\sin\theta\cos\theta\, d\theta$.

When $x = 0, \theta = 0$ and when $x = 2a, \theta = \frac{1}{2}\pi$.

$\therefore$ the required area

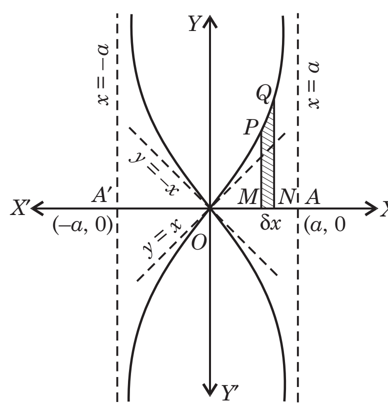
$$= \frac{2}{a}\int_0^{\pi/2} (2a)^{3/2}\sin^3\theta . \sqrt{(2a)} .\cos\theta . 4a\sin\theta\cos\theta\, d\theta$$

$$= 32a^2\int_0^{\pi/2} \sin^4\theta\cos^2\theta\, d\theta = 32a^2 . \frac{3.1.1}{6.4.2}\cdot\frac{\pi}{2}, \text{ by Walli's formula.}$$

$$= \pi a^2.$$

Example 5: *Find the whole area between the curve $x^2 y^2 = a^2 (y^2 - x^2)$ and its asymptotes.*

Solution: The given curve is symmetrical about both the axes and passes through the origin. The tangents at $(0,0)$ are given by $y^2 - x^2 = 0$ *i.e.*, $y = \pm x$ are the tangents at the origin.



Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2) the asymptotes parallel to y-axis are given by $x^2 - a^2 = 0$ *i.e.*, $x = \pm a$.

The asymptotes parallel to x-axis are given by $y^2 + a^2 = 0$ which gives two imaginary asymptotes.

$\therefore$ the required area = 4 × area lying in the first quadrant

$$= 4\int_0^a y\, dx = 4\int_0^a \sqrt{\left(\frac{a^2x^2}{a^2 - x^2}\right)} dx,$$

[$\because$ from the equation of the given curve, $y^2 = a^2 x^2/(a^2 - x^2)$]

$$= 4 \int_0^a \frac{ax\, dx}{\sqrt{(a^2 - x^2)}} = - 2a \int_0^a \frac{-2x\, dx}{\sqrt{(a^2 - x^2)}} = - 2a \left[\frac{(a^2 - x^2)^{1/2}}{1/2} \right]_0^a$$

$$= - 4a\,[0 - a] = 4a^2.$$

Comprehensive Exercise 1

1. Find the area bounded by the axis of x, and the following curves and the given ordinates :
   (i) $y = \log x\,; x = a\,, x = b, (b > a > 1)$.
   (ii) $xy = c^2\,; x = a\,, x = b\,, (a > b > 0)$. **(Kashi 2012)**
2. (i) Find the area bounded by the curve $y = x^3$, the y-axis and the lines $y = 1$ and $y = 8$.
   (ii) Show that the area cut off a parabola by any double ordinate is two thirds of the corresponding rectangle contained by that double ordinate and its distance from the vertex.
3. (i) Find the area of the quadrant of an ellipse $(x^2/a^2) + (y^2/b^2) = 1$.
   (Bundelkhand 2010; Kanpur 11)
   (ii) Find the whole area of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$.
   (Avadh 2010; Rohilkhand 10B; Kumaun 12)
4. (i) Trace the curve $ay^2 = x^2 (a - x)$ and show that the area of its loop is $8a^2/15$. **(Avadh 2008)**
   (ii) Find the area of the loop of the curve $3ay^2 = x(x - a)^2$.
   (iii) Find the area of the loop of the curve $y^2 = x(x - 1)^2$.
5. Find the area
   (i) of the loop of the curve $x(x^2 + y^2) = a(x^2 - y^2)$
   or $y^2 (a + x) = x^2 (a - x)$.
   (ii) of the portion bounded by the curve and its asymptotes. **(Meerut 2004)**
6. (i) Trace the curve $y^2 (2a - x) = x^3$ and find the entire area between the curve and its asymptotes. **(Avadh 2011)**
   (ii) Find the area between the curve $y^2 (4 - x) = x^2$ and its asymptote.
   (Avadh 2012; Kanpur 14; Bundelkhand 14)
   (iii) Find the whole area of the curve $a^2 x^2 = y^3 (2a - y)$.
7. (i) Find the area bounded by the curve $xy^2 = 4a^2 (2a - x)$ and its asymptote.
   (Rohilkhand 2009 B)
   (ii) Find the area enclosed by the curve $xy^2 = a^2 (a - x)$ and y-axis.

(iii) Trace the curve $a^2 y^2 = a^2 x^2 - x^4$ and find the whole area within it.

(Rohilkhand 2012; Avadh 12, Bundelkhand 14)

8. (i) Prove that the area of a loop of the curve $a^4 y^2 = x^4 (a^2 - x^2)$ is $\pi a^2/8$.

(ii) Show that the whole area of the curve $a^4 y^2 = x^5 (2a - x)$ is to that of the circle whose radius is a, as 5 to 4. **(Kanpur 2010)**

9. (i) Find the area between the curve $y^2 (a - x) = x^3$ (cissoid) and its asymptotes. Also find the ratio in which the ordinate $x = a/2$ divides the area.

(ii) Find the area of the loop of the curve $y^2 (a - x) = x^2 (a + x)$.

(Purvanchal 2011)

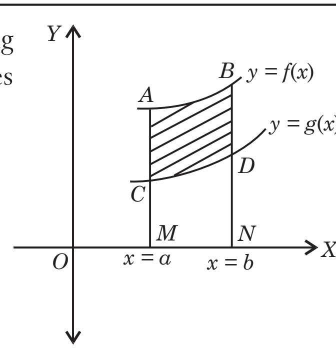
10. Trace the curve $y^2 (a + x) = (a - x)^3$. Find the area between the curve and its asymptote. **(Purvanchal 2007)**

Answers 1

1.	(i) $b \log (b/e) - a \log (a/e)$	(ii) $c^2 \log (a/b)$	
2.	(i) $(45)/4$.		
3.	(i) $\pi ab/4$.	(ii) πab	
4.	(ii) $8a^2/(15\sqrt{3})$.	(iii) $8/(15)$	
5.	(i) $\frac{1}{2} a^2 (4 - \pi)$.	(ii) $\frac{1}{2} a^2 (4 + \pi)$	
6.	(i) $3\pi a^2$.	(ii) $(64)/3$.	(iii) πa^2
7.	(i) $4\pi a^2$	(ii) πa^2	(iii) $4a^2/3$
9.	(i) $3\pi a^2/4 ; (3\pi - 8) : (3\pi + 8)$	(ii) $\frac{1}{2} a^2 (4 - \pi)$	
10.	$3\pi a^2$.		

3 Area between Two Curves

It is clear from the adjacent figure that the area lying between the curves $y = f(x)$, $y = g(x)$ and the ordinates $x = a$, $x = b$ is



$$= \text{area } ABNM - \text{area } CDNM$$

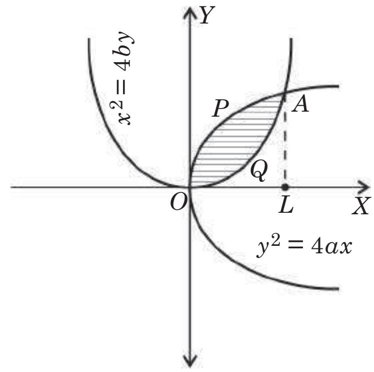
$$= \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

$$= \int_a^b \{ f(x)\, dx - g(x)\}\, dx.$$

Illustrative Examples

Example 6: *Find the area included between the curves $y^2 = 4ax$ and $x^2 = 4by$.*

(Bundelkhand 2011; Rohilkhand 10; Kumaun 09, 10))



Solution: Solving the equations of the two given curves, we have $y^4 = 16a^2(4by) = 64a^2 by$.

$\therefore \qquad y(y^3 - 64a^2 b) = 0,$

giving $\qquad y = 0, \; 4a^{2/3} b^{1/3}$.

When $\qquad y = 0, x = 0$ and when

$y = 4a^{2/3} b^{1/3}, x = 4a^{1/3} b^{2/3}$.

Hence, the points of intersection of the given curves are $O(0,0)$ and $A(4a^{1/3} b^{2/3}, 4b^{1/3} a^{2/3})$.

$\therefore$ the required area (*i.e.*, the shaded area)

$$= \text{area } OPAL - \text{area } OQAL$$

$$= \left[\int_0^{4a^{1/3}b^{2/3}} y\,dx, \text{ from the curve } y^2 = 4ax\right]$$

$$- \left[\int_0^{4a^{1/3}b^{2/3}} y\,dx, \quad \text{from the curve } y^2 = 4by\right]$$

(Note that for the required area x varies from 0 to $4a^{1/3} b^{2/3}$)

$$= \int_0^{4a^{1/3}b^{2/3}} \sqrt{(4ax)}\,dx - \int_0^{4a^{1/3}b^{2/3}} \left(\frac{x^2}{4b}\right) dx$$

$$= 2\sqrt{a}\left[\frac{2x^{3/2}}{3}\right]_0^{4a^{1/3}\,b^{2/3}} - \frac{1}{4b}\left[\frac{x^3}{3}\right]_0^{4a^{1/3}\,b^{2/3}}$$

$$= \frac{4\sqrt{a}}{3}[8\sqrt{(a)}\,.\,b] - \frac{1}{12b}(64\,ab^2) = \frac{32}{3}ab - \frac{16}{3}ab = \frac{16}{3}ab.$$

Example 7: *Find the area of the segment cut off from the parabola $y^2 = 2x$ by the straight line $y = 4x - 1$.*

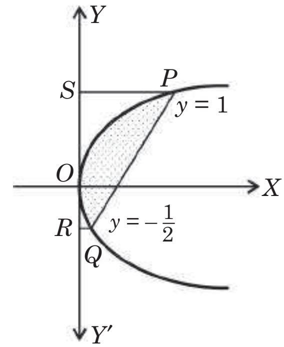
Solution: The given curves are $y^2 = 2x$, ...(1)

and $\qquad y = 4x - 1$. ...(2)

The two curves have been shown in the figure.

Solving (1) and (2) for y, we have

$$y^2 = 2 \cdot \frac{1}{4}(y+1) \quad \text{or} \quad 2y^2 - y - 1 = 0$$



or $\qquad (y-1)(2y+1) = 0$.

$\therefore \qquad y = -\frac{1}{2}, 1$.

Thus the curves (1) and (2) intersect at the points where

$$y = -\frac{1}{2} \quad \text{and} \quad y = 1.$$

Now the required area of the segment POQ (*i.e.*, the dotted area)

= the area bounded by the st. line $y = 4x - 1$ and the y-axis from $y = -\frac{1}{2}$ to $y = 1$

– the area bounded by the parabola $y^2 = 2x$ and the y-axis from $y = -\frac{1}{2}$ to $y = 1$

$$= \left[\int_{-1/2}^{1} x\, dy, \text{from (2)}\right] - \left[\int_{-1/2}^{1} x\, dy, \text{from (1)}\right]$$

$$= \int_{-1/2}^{1} \frac{1}{4}(y+1)\, dy - \int_{-1/2}^{1} \frac{1}{2} y^2\, dy$$

$$= \frac{1}{4}\left[\frac{1}{2}y^2 + y\right]_{-1/2}^{1} - \frac{1}{6}[y^3]_{-1/2}^{1}$$

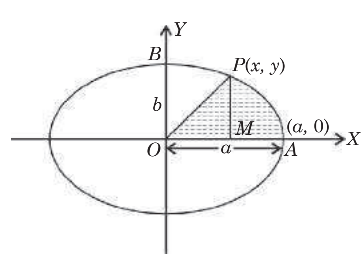
$$= \frac{1}{4}\left[\frac{3}{2} - \left(\frac{1}{8} - \frac{1}{2}\right)\right] - \frac{1}{6}\left(1 + \frac{1}{8}\right)$$

$$= \frac{1}{4}\left(\frac{3}{2} + \frac{3}{8}\right) - \frac{1}{6} \cdot \frac{9}{8} = \frac{1}{4} \cdot \frac{15}{8} - \frac{1}{6} \cdot \frac{9}{8}$$

$$= \frac{15}{32} - \frac{3}{16} = \frac{9}{32}.$$

Example 8: *If $P(x, y)$ be any point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ and S be the sectorial area bounded by the curve, the x-axis and the line joining the origin to P, show that $x = a \cos(2S/ab)$, $y = b \sin(2S/ab)$.*

Solution: The given ellipse is shown in the figure.



We have

S = the sectorial area OAP

(*i.e.*, the dotted area)

= the area of the Δ OMP + the area PMA

$= \frac{1}{2} OM \cdot MP + \int_x^a y\, dx$, for the ellipse

$$= \frac{1}{2} xy + \int_x^a \frac{b}{a} \sqrt{(a^2 - x^2)}\, dx$$

[$\because$ from the equation of the ellipse, $y = (b/a)\sqrt{(a^2 - x^2)}$]

$$= \frac{1}{2} x \cdot \frac{b}{a} \sqrt{(a^2 - x^2)} + \frac{b}{a}\left[\frac{x}{2}\sqrt{(a^2 - x^2)} + \frac{1}{2} a^2 \sin^{-1}\frac{x}{a}\right]_x^a$$

$$= \frac{bx}{2a}\sqrt{(a^2 - x^2)} + \frac{b}{a}\left[0 + \frac{1}{2} a^2 \cdot \frac{\pi}{2} - \frac{x}{2}\sqrt{(a^2 - x^2)} - \frac{1}{2} a^2 \sin^{-1}\frac{x}{a}\right]$$

$$= \frac{bx}{2a}\sqrt{(a^2 - x^2)} + \frac{b}{a} \cdot \frac{1}{2} a^2 \left(\frac{\pi}{2} - \sin^{-1}\frac{x}{a}\right) - \frac{bx}{2a}\sqrt{(a^2 - x^2)}$$

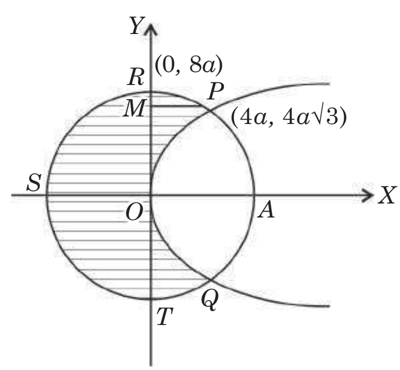
$$= \frac{ab}{2}\left(\frac{\pi}{2} - \sin^{-1}\frac{x}{a}\right) = \frac{ab}{2}\cos^{-1}\frac{x}{a}.$$

Thus $\quad S = \dfrac{ab}{2}\cos^{-1}\dfrac{x}{a}.$

$\therefore \quad \cos^{-1}\dfrac{x}{a} = \dfrac{2S}{ab}$ or $\dfrac{x}{a} = \cos\dfrac{2S}{ab}$ or $x = a\cos\dfrac{2S}{ab}.$

Also $\quad y = \dfrac{b}{a}\sqrt{(a^2 - x^2)} = \dfrac{b}{a}\sqrt{\{a^2 - a^2\cos^2(2S/ab)\}} = b\sin\dfrac{2S}{ab}.$

Example 9: *Show that the larger of the two areas into which the circle $x^2 + y^2 = 64a^2$ is divided by the parabola $y^2 = 12ax$ is $\frac{16}{3}a^2[8\pi - \sqrt{3}]$.*



Solution: $x^2 + y^2 = 64a^2$ is a circle with centre $(0,0)$ and radius $8a$ and $y^2 = 12ax$ is a parabola whose vertex is at $(0,0)$ and latus rectum $12a$. Both the curves are symmetrical about x-axis. Solving the two equations, the co-ordinates of the common point P are $(4a, 4a\sqrt{3})$. Draw PM perpendicular from P to the y-axis.

Now the area of the larger portion of the circle (*i.e.*, the shaded area) = the area $PRSTQOP$

= the area of the semi-circle RST + 2 area OPR

$$= \frac{1}{2} \cdot \pi (8a)^2 + 2\,[\text{area } OPM + \text{area } MPR]$$

$$= \frac{1}{2}\pi(8a)^2 + 2\int_0^{4a\sqrt{3}} x\,dy, \text{ for } y^2 = 12ax + 2\int_{4a\sqrt{3}}^{8a} x\,dy,$$

for $x^2 + y^2 = 64a^2$

$$= 32\pi a^2 + 2\int_0^{4a\sqrt{3}} \frac{y^2}{12a}\,dy + 2\int_{4a\sqrt{3}}^{8a} \sqrt{(64a^2 - y^2)}\,dy$$

$$= 32\,\pi a^2 + \frac{1}{6a}\left[\frac{y^3}{3}\right]_0^{4a\sqrt{3}} + 2\left[\frac{1}{2}\,y\,\sqrt{(64a^2 - y^2)} + \frac{64a^2}{2}\sin^{-1}\frac{y}{8a}\right]_{4a\sqrt{3}}^{8a}$$

$$= 32\,\pi a^2 + \frac{1}{6a}\left[\frac{64 \times 3\sqrt{3}a^3}{3}\right]$$

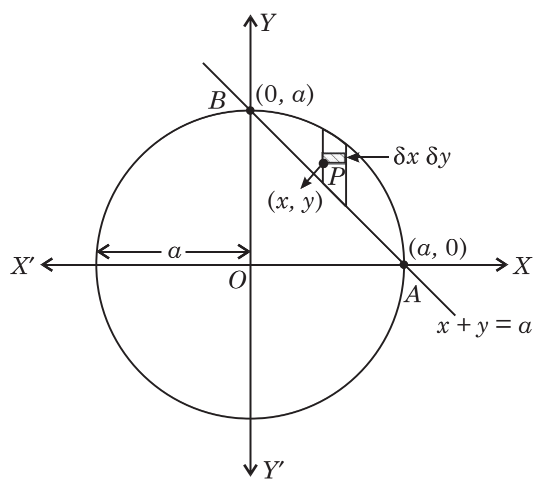
$$+ 2\left[\{0 - 8a^2\sqrt{3}\} + 32a^2\{\sin^{-1}1 - \sin^{-1}(\sqrt{3}/2)\}\right]$$

$$= 32\,\pi a^2 + \frac{32\sqrt{3}a^2}{3} - 16a^2\sqrt{3} + \frac{32}{3}a^2\pi$$

$$= \frac{128}{3}a^2\pi - \frac{16}{3}a^2\sqrt{3} = \frac{16}{3}a^2(8\pi - \sqrt{3}).$$

Example 10: *Find by double integration the area of the region enclosed by the curves* $x^2 + y^2 = a^2$, $x + y = a$ *(in the first quadrant).*

Solution: The given equations of the circle $x^2 + y^2 = a^2$



[centre $(0,0)$ and radius a] and of the straight line $x + y = a$ (with equal intercepts a on both the axes) can be easily traced as shown in the figure.

The required area is the area bounded by the arc AB and the line AB. To find it with the help of double integration take any point $P(x, y)$ in this portion and consider an elementary area $\delta x\,\delta y$ at P. The required area can now be covered by first moving y from the straight line $x + y = a$ to the arc of the circle $x^2 + y^2 = a^2$ and then moving x from 0 to a.

$\therefore$ the required area $= \displaystyle\int_{x=0}^{a}\int_{y=(a-x)}^{\sqrt{(a^2-x^2)}} dx\,dy$, the first integration to be performed w.r.t. y whose limits are variable

$$= \int_0^a [y]_{(a-x)}^{\sqrt{(a^2-x^2)}}\,dx = \int_0^a [\sqrt{(a^2 - x^2)} - (a - x)]\,dx$$

$$= \left[\left\{\frac{1}{2}x\sqrt{(a^2 - x^2)} + \frac{1}{2}a^2\sin^{-1}(x/a)\right\} - ax + \frac{1}{2}x^2\right]_0^a$$

$$= \frac{1}{2}a^2.\left(\frac{1}{2}\pi\right) - a^2 + \frac{1}{2}a^2 = \frac{1}{2}a^2\left(\frac{1}{2}\pi - 1\right)$$

$$= \frac{1}{4}a^2(\pi - 2).$$

Note: The required area can also be covered by first moving x from the straight line $x + y = a$ to the arc of the circle $x^2 + y^2 = a^2$ and then moving y from 0 to a.

Comprehensive Exercise 2

1. Find the common area between the curves $y^2 = 4ax$ and $x^2 = 4ay$.

(Meerut 2004B, 08; Agra 14)

2. (i) Find the area included between $y^2 = 4ax$ and $y = mx$.

(ii) Find the area of the segment cut off from the parabola $y^2 = 4x$ by the line $y = 8x - 1$.

3. (i) Find the area common to the two curves $y^2 = ax$, $x^2 + y^2 = 4ax$.

(Meerut 2005B, 06, 09B)

(ii) Find the area lying above x-axis and included between the circle $x^2 + y^2 = 2ax$ and the parabola $y^2 = ax$. **(Bundelkhand 2007)**

4. (i) Show that the area included between the parabolas $y^2 = 4a(x + a)$, $y^2 = 4b(b - x)$ is $\frac{8}{3}(a + b)\sqrt{(ab)}$. **(Rohilkhand 2013)**

(ii) Show that the area common to the ellipses $a^2 x^2 + b^2 y^2 = 1$, $b^2 x^2 + a^2 y^2 = 1$, where $0 < a < b$, is $4(ab)^{-1} \tan^{-1}(a/b)$.

5. If A is the vertex, O the centre and P any point (x, y) on the hyperbola $x^2/a^2 - y^2/b^2 = 1$, show that $x = a \cosh(2S/ab)$, $y = b \sinh(2S/ab)$, where S is the sectorial area OPA.

6. Prove that the area of a sector of the ellipse of semi-axes a and b between the major axis and a radius vector from the focus is $\frac{1}{2} ab(\theta - e \sin\theta)$, where θ is the eccentric angle of the point to which the radius vector is drawn.

7. Find the area common to the circle $x^2 + y^2 = 4$ and the ellipse $x^2 + 4y^2 = 9$.

(Purvanchal 2009)

8. Find the area included between the parabola $x^2 = 4ay$ and the curve $y = 8a^3/(x^2 + 4a^2)$. **(Rohilkhand 2008B)**

9. Find by double integration the area bounded by the curves $y(x^2 + 2) = 3x$ and $4y = x^2$.

10. Find by double integration the area lying between the parabola $y = 4x - x^2$ and the straight line $y = x$.

Answers 2

1. $16a^2/3$ **2.** (i) $8a^2/3m^3$ (ii) $9/(64)$

3. (i) $a^2\left(3\sqrt{3}+\frac{4}{3}\pi\right)$ (ii) $a^2\left[\frac{1}{4}\pi-\frac{2}{3}\right]$

7. $4\pi+9\sin^{-1}\left\{\frac{1}{3}\sqrt{(7/3)}\right\}-8\sin^{-1}\left\{\frac{1}{2}\cdot\sqrt{(7/3)}\right\}$

8. $\left[2\pi-\frac{4}{3}\right]a^2$ **9.** $(3/2)\log 3-(2/3)$

4 Areas of Curves given by Parametric Equations

To find the area of a curve given by parametric equations is explained by the following examples.

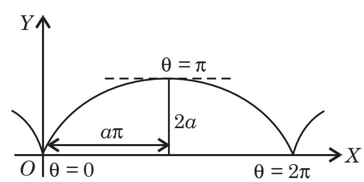
Example 11: *Find the area included between the cycloid $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ and its base.* **(Kumaun 2001, 11; Meerut 07B; Purvanchal 07; Kashi 13)**

Solution: The parametric equations of the given cycloid are $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$.

We have $dx/d\theta=a(1-\cos\theta)$, $dy/d\theta=a\sin\theta$.

$$\therefore \quad \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{a\sin\theta}{a(1-\cos\theta)}$$

$$=\frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta}=\cot\frac{1}{2}\theta.$$



In this curve $y=0$ when $a(1-\cos\theta)=0$

i.e., $\cos\theta=1$ *i.e.,* $\theta=0$.

When $\theta=0$, $x=a(0-\sin 0)=0$, $y=0$ and $dy/dx=\cot 0=\infty$. Thus the curve passes through the point $(0,0)$ and the axis of y is tangent at this point.

In this curve y is **maximum** when $\cos\theta=-1$ *i.e.,* $\theta=\pi$. When $\theta=\pi$, $x=a(\pi-\sin\pi)=a\pi$, $y=2a$,

$\frac{dy}{dx}=\cot\frac{1}{2}\pi=0$. Thus at the point $\theta=\pi$, whose cartesian co-ordinates are $(a\pi,2a)$, the tangent to the curve is parallel to x-axis. This curve does not exist in the region $y>2a$.

In this curve y cannot be –ive because $\cos\theta$ cannot be greater than 1. Thus one complete arch of the given cycloid is as shown in the figure.

Now this cycloid is symmetrical with respect to the line $x=a\pi$ (axis of the cycloid) and its base is the x-axis. Therefore the required area

$$= 2\int_{x=0}^{a\pi} y\,dx = 2\int_{\theta=0}^{\pi} y\,\frac{dx}{d\theta}\cdot d\theta = 2\int_0^{\pi} a\,(1-\cos\theta)\,.\,a\,(1-\cos\theta)\,d\theta$$

$$= 2a^2\int_0^{\pi}(1-\cos\theta)^2\,d\theta = 2a^2\int_0^{\pi}(2\sin^2\tfrac{1}{2}\theta)^2\,d\theta = 8a^2\int_0^{\pi}\sin^4\tfrac{1}{2}\theta\,d\theta$$

$$= 8a^2\int_0^{\pi/2}\sin^4\phi\,2\,d\phi\text{, putting }\tfrac{1}{2}\theta = \phi\text{ so that }\tfrac{1}{2}\,d\theta = d\phi$$

$$= 16a^2\int_0^{\pi/2}\sin^4\phi\,d\phi = 16a^2\cdot\frac{3.1}{4.2}\cdot\frac{\pi}{2}\text{, by Walli's formula}$$

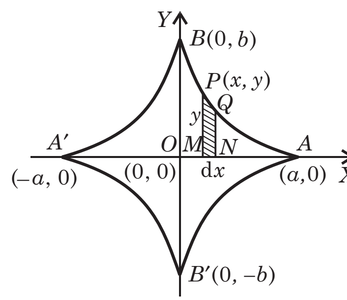
$$= 3\pi\,a^2.$$

Example 12: *Find the whole area of the curve (hypocycloid) given by the equations* $x = a\cos^3 t$, $y = b\sin^3 t$. **(Gorakhpur 2005; Rohilkhand 09; Kashi 11)**

Solution: Eliminating t from the given equations the cartesian equation of the curve is obtained as

$$(x/a)^{2/3} + (y/b)^{2/3} = 1$$

i.e., $$\{(x/a)^2\}^{1/3} + \{(y/b)^2\}^{1/3} = 1.$$



Since the powers of x and y are all even, the curve is symmetrical about both the axes. It does not pass through the origin. It cuts the axis of x at the points $(\pm a, 0)$ and the axis of y at the points $(0, \pm b)$. The tangent at the point $(a, 0)$ is x-axis. At the point B, $x = 0$ and $t = \frac{1}{2}\pi$. At the point A, $x = a$ and $t = 0$.

$\therefore$ the required area $= 4 \times$ area OAB

$$= 4\int_{x=0}^{a} y\,dx = 4\int_{t=\theta/2}^{0} y\cdot\frac{dx}{dt}\cdot dt$$

$$= 4\int_{\pi/2}^{0} b\sin^3 t\,.(-3a\cos^2 t\sin t)\,dt,\ \text{(putting for } y \text{ and } dx/dt)$$

$$= 12ab\int_0^{\pi/2}\sin^4 t\cos^2 t\,dt \qquad \textbf{(Note)}$$

$$= 12ab\cdot\frac{3.1.1}{6.4.2}\cdot\frac{\pi}{2} = \frac{3}{8}\pi\,ab.$$

Comprehensive Exercise 3

1. Find the area included between the curve $x = a\,(t + \sin t)$, $y = a\,(1 - \cos t)$ and its base. **(Agra 2005)**
2. Find the area of a loop of the curve $x = a\sin 2t$, $y = a\sin t$ or $a^2x^2 = 4y^2\,(a^2 - y^2)$.

3. Show that the area bounded by the cissoid $x = a\sin^2 t$, $y = (a\sin^3 t)/\cos t$ and its asymptote is $3\pi a^2/4$.

4. Find the area of the loop of the curve

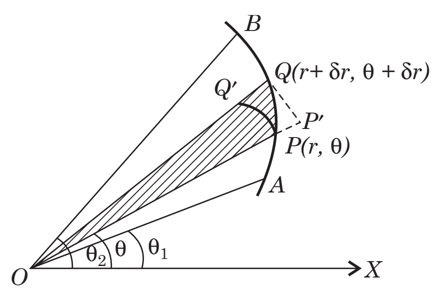
$$x = a(1-t^2),\ y = at(1-t^2),\ \text{where } -1 \le t \le 1.$$

Answers 3

1. $3\pi a^2$ **2.** $4a^2/3$ **3.** $3\pi a^2/4$ **4.** $8a^2/(15)$

5 Areas of Curves given by Polar Equations

If $r = f(\theta)$ be the equation of a curve in polar coordinates where $f(\theta)$ is a single valued continuous function of θ, then the area of the sector enclosed by the curve and the two radii vectors $\theta = \theta_1$ and $\theta = \theta_2$ $(\theta_1 < \theta_2)$, is equal to $\frac{1}{2}\int_{\theta=\theta_1}^{\theta_2} r^2\, d\theta.$



Proof: Let OAB be the area of the curve $r = f(\theta)$ between the radii vectors $\theta = \theta_1$ and $\theta = \theta_2$.

Let $P(r, \theta)$ be any point on the curve between A and B. Take a point $Q(r+\delta r, \theta+\delta\theta)$ on the curve very near to P and draw the radius vector OQ. Let the sectorial areas AOP and AOQ be denoted by A and $A + \delta A$ respectively.

Then the curvilinear area $OPQO$

$$= A + \delta A - A = \delta A.$$

Also we have $OP = r$; $OQ = r + \delta r$ and $\angle POQ = \delta\theta$.

The area of the circular sector POQ'

$$= \frac{1}{2}(\text{radius} \times \text{arc}) = \frac{1}{2} r \,.\, r\,\delta\theta = \frac{1}{2} r^2\,\delta\theta,$$

and the area of the circular sector $P'OQ$

$$= \frac{1}{2}(r+\delta r)\,.\,(r+\delta r)\,\delta\theta = \frac{1}{2}(r+\delta r)^2\,\delta\theta.$$

Now, area $POQ' <$ area $OPQ <$ area $P'OQ$,

i.e., $$\frac{1}{2} r^2\,\delta\theta < \delta A < \frac{1}{2}(r+\delta r)^2\,\delta\theta,\ \textit{i.e.,}\ \frac{1}{2} r^2 < \delta A/\delta\theta < \frac{1}{2}(r+\delta r)^2.$$

Proceeding to limits as $\delta\theta \to 0$, we get

$$\frac{dA}{d\theta} = \frac{1}{2} r^2 \quad \text{or} \quad dA = \frac{1}{2} r^2 \, d\theta.$$

$$\therefore \qquad [A]_{\theta_1}^{\theta_2} = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta.$$

Now the L.H.S. = the value of A for θ equal to θ_2 – the value of A for θ equal to θ_1

= (the area AOB) – 0 = area AOB.

Hence the required area $AOB = \frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 \, d\theta.$

Note: In some cases it is more convenient to find the required area by using double integration. In that case the area is given by $\int_{\theta=\theta_1}^{\theta_2} \int_{r=0}^{f(\theta)} r \, d\theta \, dr, (\theta_1 < \theta_2)$.

Remember: The number of loops in $r = a \cos n\theta$ or $r = a \sin n\theta$ is n or $2n$ according as n is odd or even.

Illustrative Examples

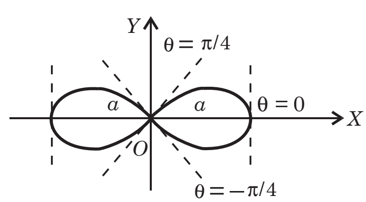
Example 13: *Find the area of the curve $r^2 = a^2 \cos 2\theta$.*

(Agra 2006, 07; Rohilkhand 07; Meerut 10B; Kumaun 11)

Solution: The given curve is symmetrical about the initial line $\theta = 0$ and about the pole. Putting $r = 0$ in the given equation of the curve, we get

$$\cos 2\theta = 0 \quad \text{or} \quad 2\theta = \pm \frac{1}{2} \pi \quad \text{or} \quad \theta = \pm \frac{1}{4} \pi.$$

Thus two consecutive values of θ for which r is zero are $-\frac{1}{4}\pi$ and $\frac{1}{4}\pi$. Therefore for one loop of the curve θ varies from $-\pi/4$ to $\pi/4$.



When $\frac{1}{2}\pi < 2\theta < \frac{3}{2}\pi$ *i.e.,* $\frac{1}{4}\pi < \theta < \frac{3}{4}\pi$, r^2 is negative *i.e.,* r is imaginary. Therefore this curve does not exist in the region $\frac{1}{4}\pi < \theta < \frac{3}{4}\pi$.

Hence this curve has only two loops as shown in the figure.

$\therefore$ whole area of the curve = 2 × area of one loop

$$= 2 \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 \, d\theta = \int_{-\pi/4}^{\pi/4} a^2 \cos 2\theta \, d\theta, \qquad [\because r^2 = a^2 \cos 2\theta]$$

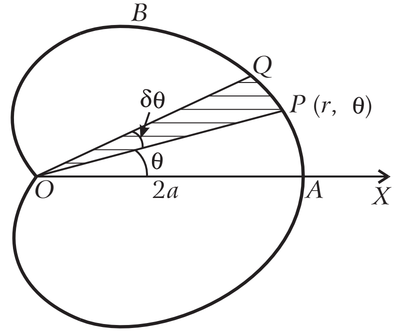
$$= 2a^2 \int_0^{\pi/4} \cos 2\theta \, d\theta, \qquad \text{[by a property of definite integrals]}$$

$$= 2a^2 \left[\frac{\sin 2\theta}{2} \right]_0^{\pi/4} = \frac{2a^2}{2} = a^2.$$

Example 14: *Find the area of the cardioid* $r = a(1 + \cos\theta)$.

(Agra 2002; Garhwal 02; Meerut 03, 04B, 10B; Kashi 12)

Solution: The given curve is symmetrical about the initial line since its equation remains unaltered when θ is changed into $-\theta$.



We have $r = 0$, when $\cos\theta = -1$ *i.e.*, $\theta = \pi$. Therefore the line $\theta = \pi$ is tangent at the pole to the curve. Also r is maximum when $\cos\theta = 1$ *i.e.*, $\theta = 0$ and then $r = 2a$.

When θ increases from 0 to π, r decreases from $2a$ to 0. Thus the curve is as shown in the figure.

Now the required area $= 2 \times$ area of the upper half of the curve

$$= 2\int_0^{\pi} \frac{1}{2} r^2 \, d\theta = 2\int_0^{\pi} \frac{1}{2} a^2 (1 + \cos\theta)^2 \, d\theta, \qquad [\because r = a(1 + \cos\theta)]$$

$$= a^2 \int_0^{\pi} (2\cos^2 \tfrac{1}{2}\theta)^2 \, d\theta = 4a^2 \int_0^{\pi} \cos^4 \tfrac{1}{2}\theta \, d\theta.$$

Now put $\frac{1}{2}\theta = \phi$ so that $\frac{1}{2} d\theta = d\phi$.

Also when $\theta = 0$, $\phi = 0$ and when $\theta = \pi$, $\phi = \pi/2$.

$\therefore$ the required area $= 8a^2 \int_0^{\pi/2} \cos^4 \phi \, d\phi = 8a^2 \cdot \frac{3.1}{4.2} \cdot \frac{\pi}{2}$, by Walli's formula

$$= 3\pi a^2 / 2.$$

Example 15: *Find the area of a loop of the curve* $r = a\cos 3\theta + b\sin 3\theta$. **(Meerut 2000)**

Solution: In the given equation of the curve put $a = k\cos\alpha$, $b = k\sin\alpha$ so that $k = \sqrt{(a^2 + b^2)}$ and $\alpha = \tan^{-1}(b/a)$.

Thus the given equation reduces to $r = k\cos 3\theta \cos\alpha + k\sin 3\theta \sin\alpha$

or $$r = k\cos(3\theta - \alpha) = k\cos 3(\theta - \tfrac{1}{3}\alpha). \qquad \textbf{(Note)}$$

Now rotating the initial line through an angle $\alpha/3$, the given equation of the curve becomes

$$r = k\cos 3(\theta + \tfrac{1}{3}\alpha - \tfrac{1}{3}\alpha) = k\cos 3\theta. \qquad \textbf{(Note)}$$

It should be noted that the rotation of the initial line changes only the equation of the curve and has no effect on its shape. Therefore the area of a loop of the given curve is the same as the area of a loop of the curve $r = k\cos 3\theta$.

The curve $r = k\cos 3\theta$ is symmetrical about the initial line.

Putting $r = 0$ in it, we have

$$\cos 3\theta = 0 \;\; i.e., 3\theta = \pm\pi/2 \;\; i.e., \theta = \pm\pi/6.$$

$\therefore$ one loop of this curve lies between $\theta = -\pi/6$ and $\theta = +\pi/6$ and it is symmetrical about the initial line.

$\therefore$ the required area $= 2 \cdot \int_0^{\pi/6} \frac{1}{2} r^2 \, d\theta,$ (By symmetry)

$$= \int_0^{\pi/6} k^2 \cos^2 3\theta \, d\theta .$$

Now put $3\theta = t$, so that $3\, d\theta = dt$. Also when $\theta = 0, t = 0$ and when $\theta = \pi/6, t = \pi/2$.

$\therefore$ the required area $= \frac{k^2}{3} \int_0^{\pi/2} \cos^2 t \, dt = \frac{k^2}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} \pi = \frac{k^2}{12} \pi$

$= (a^2 + b^2) \pi / 12.$ $[\because k^2 = a^2 + b^2]$

Comprehensive Exercise 4

1. Find the area between the following curves and the given radii vectors :
   (i) The spiral $r\,\theta^{1/2} = a;\ \theta = \alpha, \theta = \beta$.
   (ii) The parabola $l/r = 1 + \cos\theta;\ \theta = 0, \theta = \alpha$.
2. Find the area of the loop of the curve $r = a\,\theta \cos\theta$ between $\theta = 0$ and $\theta = \pi/2$.
   (Kanpur 2009)
3. (i) Find the area of one loop of $r = a \cos 4\theta$.
   (ii) Find the area of a loop of the curve $r = a \sin 3\theta$.
4. (i) Find the whole area of the curve $r = a \sin 2\theta$. **(Bundelkhand 2009)**
   (ii) Find the whole area of the curve $r = a \cos 2\theta$.
5. (i) Find the whole area of the curve $r^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$.
   (ii) Find the area of the cardioid $r = a(1 - \cos\theta)$. **(Kumaun 2014)**
6. (i) Show that the area of the limacon $r = a + b\cos\theta, (b < a)$ is equal to
   $$\pi\left(a^2 + \frac{1}{2}b^2\right).$$
   (ii) Prove that the sum of the areas of the two loops of the limacon $r = a + b\cos\theta$, $(b > a)$ is equal to $\pi(2a^2 + b^2)/2$.
7. Calculate the ratio of the area of the larger to the area of the smaller loop of the curve $r = \frac{1}{2} + \cos 2\theta$.
8. Show that the area of a loop of $r = a\cos n\theta$ is $\pi a^2/4n$, n being integral. Also prove that the whole area is $\pi a^2/4$ or $\pi a^2/2$ according as n is odd or even.
9. Trace the curve $r = \sqrt{3}\cos 3\theta + \sin 3\theta$, and find the area of a loop.

Answers 4

1. (i) $\frac{1}{2} a^2 \log (\beta / \alpha)$ (ii) $\frac{1}{4} l^2 \left[\tan \frac{1}{2} \alpha + \frac{1}{3} \tan^3 \frac{1}{2} \alpha \right]$

2. $\frac{\pi a^2}{96} (\pi^2 - 6)$

3. (i) $\pi a^2 / (16)$ (ii) $\pi a^2 / (12)$

4. (i) $\pi a^2 / 2$ (ii) $\pi a^2 / 2$

5. (i) $\frac{1}{2} \pi (a^2 + b^2)$ (ii) $3 \pi a^2 / 2$

7. $\dfrac{4\pi + 3\sqrt{3}}{2\pi - 3\sqrt{3}}$

9. $\pi / 3$

6 Area Bounded by Two Curves (Polar equations).

To find the area bounded by two curves given in polar form is explained by the following examples.

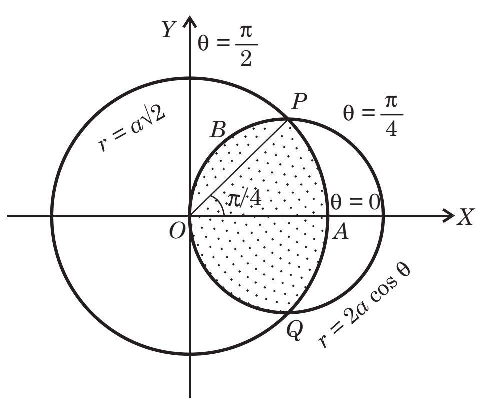
Illustrative Examples

Example 16: *Find the area common to the circles $r = a\sqrt{2}$ and $r = 2a \cos \theta$.*

(Agra 2000; Kumaun 08; Meerut 05, 11, 12; Rohilkhand 11B; Kanpur 14; Purvanchal 14)

Solution: The given equations of circles are $r = a\sqrt{2}$ and $r = 2a \cos \theta$. The first equation represents a circle with centre at pole and radius $a\sqrt{2}$. The second equation represents a circle passing through the pole and the diameter through the pole as the initial line. Both these circles are symmetrical about the initial line. Eliminating r between the two equations, we have at the points of intersection

$$a\sqrt{2} = 2a \cos \theta, \textit{ i.e.}, \cos \theta = 1 / \sqrt{2}, \textit{ i.e.}, \theta = \pm \pi / 4.$$



Thus at $P, \theta = \pi / 4$. For the circle $r = 2a \cos \theta$, at O, $r = 0$ and so $\cos \theta = 0$ *i.e.*, $\theta = \frac{1}{2} \pi$.

Now the required area = Area *OQAPBO*

$$= 2 \text{ (area } OAPBO), \quad \text{(by symmetry)}$$

$$= 2 [\text{Area } OAP + \text{Area } OPBO]$$

$$= 2 \left[\frac{1}{2} \int_0^{\pi/4} r^2 \, d\theta, \text{ for the circle } r = a\sqrt{2} \right.$$

$$\left. + \frac{1}{2} \int_{\pi/4}^{\pi/2} r^2 \, d\theta, \text{ for the circle } r = 2a \cos \theta \right]$$

$$= \int_0^{\pi/4} (a\sqrt{2})^2 \, d\theta + \int_{\pi/4}^{\pi/2} (2a \cos \theta)^2 \, d\theta$$

$$= 2a^2 [\theta]_0^{\pi/4} + 2a^2 \int_{\pi/4}^{\pi/2} (1 + \cos 2\theta) \, d\theta$$

$$= 2a^2 \left(\frac{\pi}{4} \right) + 2a^2 \left[\theta + \frac{\sin 2\theta}{2} \right]_{\pi/4}^{\pi/2}$$

$$= \frac{\pi a^2}{2} + 2a^2 \left[\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2} \right]$$

$$= \frac{1}{2} \pi a^2 + \frac{1}{2} \pi a^2 - a^2 = \pi a^2 - a^2 = a^2 (\pi - 1).$$

Example 17: *Find the ratio of the two parts into which the parabola $2a = r(1 + \cos \theta)$ divides the area of the cardioid $r = 2a(1 + \cos \theta)$.*

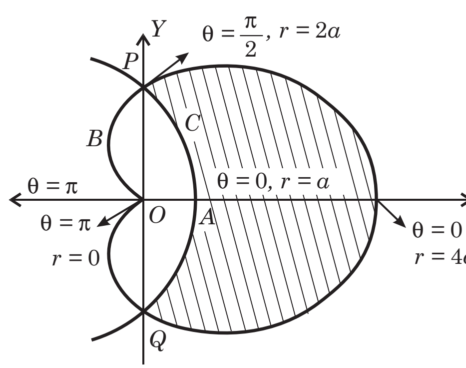
Solution: Eliminating r between the given equations of the curves, we get

$$2a(1 + \cos \theta) = 2a / (1 + \cos \theta) \quad \text{or} \quad (1 + \cos \theta)^2 = 1$$

or $\cos \theta (\cos \theta + 2) = 0$

or $\cos \theta = 0$, $[\because \cos \theta \neq -2]$

or $\theta = \pm \pi / 2$.



Thus at the point of intersection P of the two curves, $\theta = \pi / 2$.

Now area of the whole cardioid

$$= 2 \times \frac{1}{2} \int_0^{\pi} r^2 \, d\theta,$$

(by symmetry)

$$= \int_0^{\pi} 4a^2 (1 + \cos \theta)^2 \, d\theta$$

$$= 4a^2 \int_0^{\pi} \left(2\cos^2 \frac{1}{2}\theta\right)^2 d\theta = 16a^2 \int_0^{\pi} \cos^4 \frac{1}{2}\theta \, d\theta$$

$$= 16a^2 \int_0^{\pi/2} \cos^4 \phi . 2\, d\phi, \qquad \left(\text{putting } \frac{1}{2}\theta = \phi \text{ so that } \frac{1}{2}d\theta = d\phi;\right.$$

$$\left.\text{also when } \theta = 0, \phi = 0 \text{ and when } \theta = \pi, \phi = \frac{1}{2}\pi\right)$$

$$= 32a^2 \cdot \frac{3.1}{4.2} \cdot \frac{\pi}{2} = 6\pi a^2. \qquad ...(1)$$

Area $OACPO = \frac{1}{2}\int_0^{\pi/2} r^2 \, d\theta$, for the parabola $r = \frac{2a}{1+\cos\theta}$

$$= \frac{1}{2} \cdot 4a^2 \int_0^{\pi/2} \frac{d\theta}{(1+\cos\theta)^2} = \frac{a^2}{2}\int_0^{\pi/2} \sec^4 \frac{1}{2}\theta \, d\theta$$

$$= \frac{1}{2} a^2 \int_0^{\pi/2} \left(1+\tan^2 \frac{1}{2}\theta\right) \sec^2 \frac{1}{2}\theta \, d\theta$$

$$= a^2 \left[\tan\frac{1}{2}\theta + \frac{1}{3}\tan^3 \frac{1}{2}\theta\right]_0^{\pi/2} = a^2\left(1+\frac{1}{3}\right) = \frac{4a^2}{3}. \qquad ...(2)$$

Also area $OPBO = \frac{1}{2}\int_{\pi/2}^{\pi} r^2 \, d\theta$, for the cardioid $r = 2a(1+\cos\theta)$

$$= \frac{1}{2}\int_{\pi/2}^{\pi} 4a^2 (1+\cos\theta)^2 \, d\theta = 2a^2 \int_{\pi/2}^{\pi} [1 + 2\cos\theta + \cos^2\theta] \, d\theta$$

$$= 2a^2 \int_{\pi/2}^{\pi} \left(1 + 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= 2a^2 \left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_{\pi/2}^{\pi} = \frac{3}{2}\pi a^2 - 4a^2. \qquad ...(3)$$

Adding (2) and (3) and multiplying by 2, we get the whole area included between the two curves *i.e.*, the area of the smaller portion of the cardioid

$$= 2 \times \left[\frac{4}{3}a^2 + \left(\frac{3}{2}\pi a^2 - 4a^2\right)\right] = a^2\left[3\pi - \frac{16}{3}\right]$$

$$= \frac{1}{3}a^2 [9\pi - 16]. \qquad ...(4)$$

Also the shaded area (*i.e.*, the area of the larger portion of the cardioid)

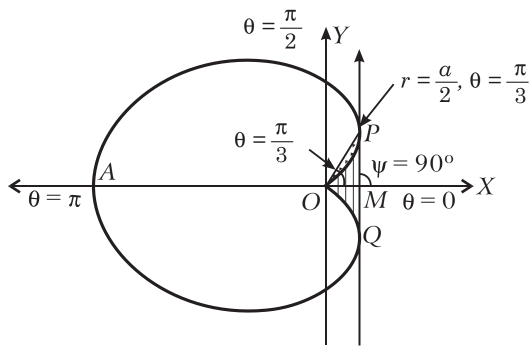
= (Area of the whole cardioid) – (unshaded area) *i.e.*, = (1) – (4)

$$= 6\pi a^2 - \frac{1}{3}a^2 (9\pi - 16) = \frac{1}{3}a^2 (9\pi + 16). \qquad ...(5)$$

$\therefore$ Ratio of the two parts $= \dfrac{\text{Larger area}}{\text{Smaller area}} = \dfrac{\frac{1}{3}a^2(9\pi+16)}{\frac{1}{3}a^2(9\pi-16)} = \dfrac{9\pi+16}{9\pi-16}.$

Example 18: *Find the area lying between the cardioid $r = a(1 - \cos\theta)$ and its double tangent.*

Solution: Let PQ be the double tangent of the cardioid. Clearly it is perpendicular to OX *i.e.*, it must be inclined at an angle of $90°$ to the initial line *i.e.*, $\psi = 90°$ at P.



Also we know that at any point of a curve,

$$\psi = \theta + \phi. \qquad ...(1)$$

Now

$$\tan\phi = r\,(d\theta/dr) = r/(dr/d\theta)$$

$$= a(1 - \cos\theta)/(a\sin\theta), \qquad [\because r = a(1-\cos\theta)]$$

$$= \frac{2\sin^2\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \tan\frac{1}{2}\theta.$$

$$\therefore \qquad \phi = \frac{1}{2}\theta.$$

Putting the value of ϕ in (1), we get

$$\psi = \theta + \frac{1}{2}\theta = \frac{3}{2}\theta.$$

Since at P, $\psi = \frac{1}{2}\pi$, therefore at P, $\frac{1}{2}\pi = \frac{3}{2}\theta$ or $\theta = \frac{\pi}{3}$.

$\therefore$ the vectorial angle of the point of contact P of the double tangent is $\pi/3$ *i.e.*, $60°$. Substituting this value of θ in the equation of the curve, we get the radius vector $OP = a(1 - \cos 60°) = a/2$.

Thus in the triangle OPM,

$$OP = \frac{1}{2}a, \angle POM = 60°, \angle PMO = 90°.$$

$$\therefore \qquad OM = \frac{1}{2}a\cos 60° = \frac{1}{2}a \cdot \frac{1}{2} = \frac{1}{4}a$$

and

$$PM = \frac{1}{2}a\sin 60° = \frac{1}{2}a(\sqrt{3}/2).$$

$\therefore$ area of the triangle $OPM = \frac{1}{2}OM \,.\, PM = \frac{1}{2}\left(\frac{1}{4}a\right)(\sqrt{3}a/4) = (1/32)\,a^2\sqrt{3}.$

Also the sectorial area OPO of the cardioid $r = a(1 - \cos\theta)$ *i.e.*, the dotted area

$$= \frac{1}{2}\int_0^{\pi/3} r^2\,d\theta = \frac{1}{2}\int_0^{\pi/3} a^2(1-\cos\theta)^2\,d\theta$$

$$= \frac{1}{2} a^2 \int_0^{\pi/3} (1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta)\, d\theta$$

$$= \frac{1}{2} a^2 \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/3}$$

$$= \frac{1}{2} a^2 \left(\frac{1}{2}\pi - \sqrt{3} + \frac{1}{8}\sqrt{3}\right) = \frac{1}{16} a^2 (4\pi - 7\sqrt{3}).$$

Hence the required area (*i.e.*, the area shaded by vertical lines)

$$= 2\,[\text{area of } \Delta\, OPM - \text{area of sector } OPO]$$

$$= 2\left[\frac{1}{32} a^2 \sqrt{3} - \frac{1}{16} a^2 (4\pi - 7\sqrt{3})\right] = \frac{1}{16} a^2 (15\sqrt{3} - 8\pi).$$

Example 19: *Find the area of a loop of the curve $r = a \sin 3\theta$ outside the circle $r = a/2$ and hence find the whole area of the curve outside the circle $r = a/2$.*

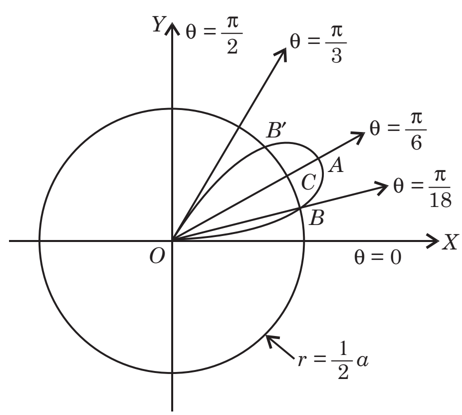
Solution: Eliminating r between the two given equations, we get $(a/2) = a \sin 3\theta$

i.e., $\sin 3\theta = \frac{1}{2}$

i.e., $3\theta = \pi/6$ or $5\pi/6$

i.e., $\theta = \pi/18$ or $5\pi/18$

i.e., $\theta = 10^\circ$ or 50°.



Thus the loop of the curve $r = a \sin 3\theta$ lying between $\theta = 0$ and $\theta = \pi/3$ intersects the circle $r = a/2$ at the points B and B' where $\theta = 10^\circ$ at B and $\theta = 50^\circ$ at B'. This loop is symmetrical about OA and $\theta = \pi/6$ at A.

Now the required area of a loop of the curve $r = a \sin 3\theta$ lying outside the circle $r = a/2$

= the area $BAB'CB$ (*i.e.*, the shaded area)

$= 2 \times$ area $BACB$, (by symmetry)

$= 2 \times$ [(area of the curve $r = a \sin 3\theta$ between the radii vectors OB and OA *i.e.*, $\theta = \pi/18$ and $\theta = \pi/6$) − (area of the circle $r = a/2$ between the radii vectors OB and OC *i.e.*, $\theta = \pi/18$ and $\theta = \pi/6$)]

$$= 2\left[\frac{1}{2}\int_{\pi/18}^{\pi/6} r^2\, d\theta, \text{ for the curve } r = a\sin 3\theta \right.$$

$$\left. - \frac{1}{2}\int_{\pi/18}^{\pi/6} r^2\, d\theta, \text{ for the circle } r = \frac{a}{2}\right]$$

$$= \int_{\pi/18}^{\pi/6} a^2 \sin^2 3\theta\, d\theta - \int_{\pi/18}^{\pi/6} \frac{a^2}{4}\, d\theta$$

$$= \frac{a^2}{2} \int_{\pi/18}^{\pi/6} (1 - \cos 6\theta)\, d\theta - \frac{a^2}{4} [\theta]_{\pi/18}^{\pi/6}$$

$$= \frac{a^2}{2} \left[\theta - \frac{\sin 6\theta}{6} \right]_{\pi/18}^{\pi/6} - \frac{a^2}{4} \left[\frac{\pi}{6} - \frac{\pi}{18} \right]$$

$$= \frac{a^2}{2} \left[\left\{ \frac{\pi}{6} - \frac{\sin \pi}{6} \right\} - \left\{ \frac{\pi}{18} - \frac{1}{6} \sin \frac{\pi}{3} \right\} \right] - \frac{a^2}{4} \cdot \frac{\pi}{9}$$

$$= \frac{a^2}{2} \left[\frac{\pi}{6} - \left\{ \frac{\pi}{18} - \frac{1}{6} \cdot \frac{\sqrt{3}}{2} \right\} \right] - \frac{a^2 \pi}{36}$$

$$= \frac{a^2}{2} \left[\frac{\pi}{9} + \frac{\sqrt{3}}{12} \right] - \frac{a^2 \pi}{36} = \frac{a^2}{72} [2\pi + 3\sqrt{3}] .$$

Again the curve $r = a \sin 3\theta$ has 3 equal loops. [$\because n = 3$ which is odd.]

$\therefore$ whole area of the curve $r = a \sin 3\theta$ outside the circle $r = a/2$

$= 3 \times$ area $BAB'CB$ *i.e.*, 3 times the shaded area

$$= 3 \times \frac{1}{72} a^2 [2\pi + 3\sqrt{3}] = \frac{1}{24} a^2 [2\pi + 3\sqrt{3}].$$

Comprehensive Exercise 5

1. Find the area outside the circle $r = 2a \cos \theta$ and inside the cardioid $r = a(1 + \cos \theta)$.
2. Find the total area inside $r = \sin \theta$ and outside $r = 1 - \cos \theta$.
3. Find by double integration the area lying inside the circle $r = a \sin \theta$ and outside the cardioid $r = a(1 - \cos \theta)$.
4. Find the area common to the circle $r = a$ and the cardioid $r = a(1 + \cos \theta)$.
   (Meerut 2007)
5. Find the area of the portion included between the cardioids $r = a(1 + \cos \theta)$ and $r = a(1 - \cos \theta)$.
6. Show that the area contained between the circle $r = a$ and the curve $r = a \cos 5\theta$ is equal to three-fourth of the area of the circle.
7. Find the area between the curve $r = a(\sec \theta + \cos \theta)$ and its asymptote.
   (Purvanchal 2010)
8. O is the pole of the lemniscate $r^2 = a^2 \cos 2\theta$ and PQ is a common tangent to its two loops. Find the area bounded by the line PQ and the arcs OP and OQ of the curve.

Answers 5

1. $\pi a^2 / 2$
2. $1 - (\pi / 4)$
3. $a^2 \{1 - (\pi / 4)\}$
4. $a^2 \left(\frac{5}{4}\pi - 2\right)$
5. $2a^2 \left(\frac{3}{4}\pi - 2\right)$
7. $5\pi a^2 / 4$
8. $\frac{1}{8} a^2 (3\sqrt{3} - 4)$

7 Cartesian Equations Changed to Polar Form

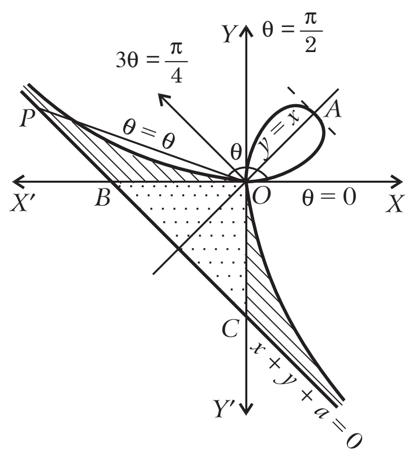
Sometimes it is convenient to find the required area if the given cartesian equation of the curve is changed to polar form by putting $x = r\cos\theta$ and $y = r\sin\theta$.

Example 20: *Find the area of a loop of the folium $x^3 + y^3 = 3axy$.* **(Meerut 2001)**

Solution: Changing the equation of the curve $x^3 + y^3 = 3axy$ into polar form by putting $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$(r\cos\theta)^3 + (r\sin\theta)^3 = 3a\,(r\cos\theta)\,.(r\sin\theta)$$

or $\quad r = 3a\cos\theta\sin\theta / (\cos^3\theta + \sin^3\theta)$. ...(1)



From (1), $r = 0$ when $\theta = 0$ and when $\theta = \pi / 2$.

$\therefore$ the loop lies between $\theta = 0$ and $\theta = \pi / 2$.

Hence the required area of the loop

$$= \frac{1}{2}\int_0^{\pi/2} r^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(\frac{3a\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}\right)^2 d\theta, \qquad \text{putting for } r \text{ from (1)}$$

$$= \frac{9a^2}{2}\int_0^{\pi/2} \frac{\cos^2\theta\sin^2\theta}{(\cos^3\theta + \sin^3\theta)^2}\, d\theta$$

$$= \frac{9a^2}{2}\int_0^{\pi/2} \frac{\tan^2\theta\sec^2\theta}{(1 + \tan^3\theta)^2}\, d\theta,$$

dividing the numerator and the denominator by $\cos^6\theta$.

Now put $1 + \tan^3\theta = t$ so that $3\tan^2\theta\sec^2\theta\, d\theta = dt$.

Also when $\theta = 0$, $t = 1$ and when $\theta \to \pi / 2, t \to \infty$.

$\therefore$ area of the loop

$$= \frac{9a^2}{2}\int_1^{\infty} \frac{1}{t^2}\cdot\frac{dt}{3} = \frac{3a^2}{2}\left[-\frac{1}{t}\right]_1^{\infty} = \frac{3a^2}{2}.$$

Comprehensive Exercise 6

1. Find the area of a loop of the curve $x^4 + y^4 = 4a^2 xy$.
2. Find the area of a loop of the curve $(x^2 + y^2)^2 = 4axy^2$.
3. Prove that the area of a loop of the curve $x^6 + y^6 = a^2 \; y^2 \; x^2$ is $\pi \, a^2 \,/\, 12$.
4. Find the area of a loop of the curve $x^4 + 3\,x^2 \; y^2 + 2\,y^4 = a^2 \; xy$.
5. Prove that the area of a loop of the curve $x^5 + y^5 = 5ax^2 y^2$ is five times the area of one loop of the curve $r^2 = a^2 \cos 2\theta$. **(Purvanchal 2014)**

Answers 6

1. $a^2 (\pi / 2)$ 2. $\pi a^2 / 4$ 3. $\frac{1}{4} a^2 \log 2$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The area bounded by the axis of x, and the curve $y = \sin^2 x$ and the given ordinates $x = 0, \; x = \frac{\pi}{2}$ is

   (a) $\frac{\pi}{4}$ (b) $\frac{\pi^2}{4}$ (c) $\frac{\pi}{2}$ (d) π

2. The loop of the curve $3\,a\;y^2 = x\,(x - a)^2$ will lie between

   (a) $x = 0, x = a$ (b) $x = -\,a, x = a$
   (c) $x = 0, x = -\,a$ (d) $y = 0, y = a$

3. The area of one loop of the curve $r^2 = a^2 \cos 2\,\theta$ is

   (a) a^2 (b) $\frac{a^2}{2}$ (c) $\frac{3a^2}{2}$ (d) $\frac{a^2}{4}$

4. The whole area of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is

(a) $\dfrac{\pi}{2}ab$ (b) $\pi\, ab$ (c) $\pi^2\, ab$ (d) $\dfrac{2\pi}{3}ab$

(Agra 2006; Bundelkhand 06, 08; Kumaun 09)

5. The area of the curve $r = a$ is

(a) πa^2 (b) $2\pi a$ (c) $2\pi a^2$ (d) $4\pi a^2$

(Rohilkhand 2006)

6. The whole area of the curve $r = a\cos n\theta$ is $\pi a^2/4$ if n is:

(a) odd (b) even

(c) odd and even both (d) none of these **(Kumaun 2008)**

7. Quadrature is the process of evaluating the

(a) length of plane curves (b) area under plane curves

(c) volume formed by revolution of curve about the axis

(d) none of these **(Kumaun 2010)**

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. The process of finding the area of any bounded portion of a curve is called …… .
2. If $f(x)$ is a continuous and single valued function of x, then the area bounded by the curve $y = f(x)$ the axis of x and the ordinates $x = a$ and $x = b$ is …… .
3. The area between the curve $r = a\, e^{m\theta}$ and the given radii vectors $\theta = \alpha, \theta = \beta$ is …… .
4. The curve $r = a \sin 3\theta$ has …… loops.
5. The area bounded by the axis of x, and the curve $y = c\cosh\left(\dfrac{x}{c}\right)$ and the ordinates $x = 0, x = a$ is …… .

True or False

Write 'T' for true and 'F' for false statement.

1. If $r = f(\theta)$ be the equation of a curve in polar co-ordinates where $f(\theta)$ is a single-valued continuous function of θ, then the area of the sector enclosed by the curve and the two radii vectors $\theta = \theta_1$ and $\theta = \theta_2$ $(\theta_1 < \theta_2)$, is equal to $\dfrac{1}{2}\displaystyle\int_{\theta=\theta_1}^{\theta=\theta_2} r^2\, d\theta$.
2. The number of loops in $r = a\cos n\theta$ is n or $2n$ according as n is even or odd.
3. The area of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ is $\dfrac{3}{8}\pi a^2$. **(Agra 2003)**
4. The area of ellipse $x^2/a^2 + y^2/b^2 = 1$ is $2a^2 b$. **(Agra 2002)**

Answers

Multiple Choice Questions

1. (a) 2. (a) 3. (b) 4. (b) 5. (a)
6. (a) 7. (b)

Fill in the Blank(s)

1. quadrature
2. $\int_a^b f(x)\,dx$
3. $\dfrac{a^2}{4m}(e^{2m\beta} - e^{2m\alpha})$
4. three
5. $c^2 \sinh \dfrac{a}{c}$

True or False

1. T 2. F 3. T 4. F

□

Chapter

8

Rectification

(Lengths of Arcs and Intrinsic Equations of Plane Curves)

1 Rectification

The process of finding the length of an arc of a curve between two given points is called ***rectification.***

2 Lengths of Curves (Meerut 2009B)

If s denotes the arc length of a curve measured from a *fixed point* to any point on it, then as proved in Differential Calculus, we have

$$\frac{ds}{dx} = \pm \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}},$$

where +ive or –ive sign is to be taken before the radical sign according as x increases or decreases as s increases. Hence if s increases as x increases, we have

$$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} \quad \text{or} \quad ds = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} \, dx .$$

Integrating, we have $s = \int_a^x \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx,$

where a is the abscissa of the fixed point from which s is measured.

Hence the arc length of the curve $y = f(x)$ included between two points for which $x = a$ and $x = b$ is equal to $\int_a^b \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx, \quad (b > a).$

Sometimes it is more convenient to take y as the independent variable. Then the length of the arc of the curve $x = f(y)$ between $y = a$ and $y = b$ is equal to

$$\int_a^b \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}\, dy, \quad (b > a).$$

Remark: Suppose we have to find the length of the arc of a curve (whose cartesian equation is given) lying between the points (x_1, y_1) and (x_2, y_2). We can use either of the two formulae

$$s = \int_{x_1}^{x_2} \sqrt{\{1 + (dy/dx)^2\}}\, dx \quad \text{and} \quad s = \int_{y_1}^{y_2} \sqrt{\{1 + (dx/dy)^2\}}\, dy.$$

If we feel any difficulty in integration while using one of these two formulae, we must try the other formula also.

Illustrative Examples

Example 1: *Show that the length of the curve $y = \log \sec x$ between the points where $x = 0$ and $x = \frac{1}{3}\pi$ is $\log(2 + \sqrt{3})$.*

(Kanpur 2005; Rohilkhand 14)

Solution: The given curve is $y = \log \sec x$. ...(1)

Differentiating (1) w.r.t. x, we get $\frac{dy}{dx} = \frac{1}{\sec x} \sec x \tan x = \tan x$.

Now $$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2 x = \sec^2 x. \quad ...(2)$$

If the arc length s of the given curve is measured from $x = 0$ in the direction of x increasing, we have $\frac{ds}{dx} = \sec x$ or $ds = \sec x\, dx$.

Therefore if s_1 denotes the arc length from $x = 0$ to $x = \frac{1}{3}\pi$, then

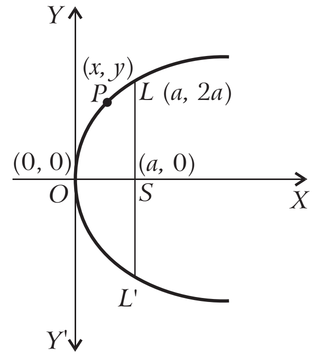
$$\int_0^{s_1} ds = \int_0^{\pi/3} \sec x\, dx = [\log(\sec x + \tan x)]_0^{\pi/3}$$

or $$s_1 = \left[\log\left(\sec\frac{1}{3}\pi + \tan\frac{1}{3}\pi\right) - \log 1\right] = \log(2 + \sqrt{3}).$$

Example 2: *Find the length of the arc of the parabola $y^2 = 4ax$ extending from the vertex to an extremity of the latus rectum.* **(Lucknow 2005; Meerut 09)**

Solution: The given equation of parabola is

$$y^2 = 4ax. \quad \text{...(1)}$$



The point $O(0,0)$ is the vertex of the parabola and the point $L(a, 2a)$ is an extremity of the latus rectum LSL'. We have to find the length of arc OL. Differentiating (1) w.r.t. x, we get $2y\,(dy/dx) = 4a$.

$\therefore \qquad dy/dx = 2a/y \quad \text{or} \quad dx/dy = y/2a.$

Now
$$\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{4a^2}$$

$$= \frac{1}{4a^2}(4a^2 + y^2). \quad \text{...(2)}$$

If 's' denotes the arc length of the parabola measured from the vertex O to any point $P(x, y)$ towards the point L, then s increases as y increases. Therefore ds/dy will be positive. So extracting the square root of (2) and keeping the positive sign, we have

$$\frac{ds}{dy} = \frac{1}{2a}\sqrt{(4a^2 + y^2)} \quad \text{or} \quad ds = \frac{1}{2a}\sqrt{(4a^2 + y^2)}\,dy.$$

Let s_1 denote the arc length OL. Then

$$\int_0^{s_1} ds = \int_0^{2a} \frac{1}{2a}\sqrt{(4a^2 + y^2)}\,dy$$

or
$$s_1 = \frac{1}{2a}\left[\frac{y}{2}\sqrt{(4a^2 + y^2)} + \frac{4a^2}{2}\log\{y + \sqrt{(4a^2 + y^2)}\}\right]_0^{2a}$$

$$= \frac{1}{2a}[a\sqrt{(4a^2 + 4a^2)} + 2a^2\log\{2a + \sqrt{(8a^2)}\} - 0 - 2a^2\log(2a)]$$

$$= \frac{1}{2a}[2\sqrt{2}a^2 + 2a^2\log\{(2a + 2\sqrt{2}a)/2a\}]$$

$$= \frac{2a^2}{2a}[\sqrt{2} + \log(1 + \sqrt{2})] = a[\sqrt{2} + \log(1 + \sqrt{2})].$$

Example 3: *Find the perimeter of the loop of the curve $3ay^2 = x^2(a - x)$.*

(Meerut 2000, 04, 06B, 07B, 11B; Purvanchal 10; Kashi 14)

Solution: The given curve is $3ay^2 = x^2(a - x)$. ...(1)

Here the curve is symmetrical about the x-axis. Putting $y = 0$, we get $x = 0, x = a$. So the loop lies between $x = 0$ and $x = a$. Differentiating (1) w.r.t. x, we get

$$6ay\frac{dy}{dx} = 2ax - 3x^2 \quad \text{or} \quad \frac{dy}{dx} = \frac{x(2a - 3x)}{6ay}.$$

$\therefore \qquad 1+\left(\frac{dy}{dx}\right)^2=1+\frac{x^2\,(2a-3x)^2}{36a^2\,y^2}=1+\frac{x^2\,(2a-3x)^2}{12a\,x^2\,(a-x)}$

[Substituting for $3ay^2$ from (1)]

$$=1+\frac{(2a-3x)^2}{12a\,(a-x)}=\frac{12a^2-12ax+(2a-3x)^2}{12a\,(a-x)}=\frac{(4a-3x)^2}{12a\,(a-x)}.$$

$\therefore$ the required length of the loop

= twice the length of the half loop lying above the x-axis

[By symmetry]

$$=2\int_0^a \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx=2\int_0^a \sqrt{\left\{\frac{(4a-3x)^2}{12a\,(a-x)}\right\}}\,dx$$

$$=\frac{1}{\sqrt{(3a)}}\int_0^a \frac{(4a-3x)}{\sqrt{(a-x)}}\,dx=\frac{1}{\sqrt{(3a)}}\int_0^a \frac{3\,(a-x)+a}{\sqrt{(a-x)}}\,dx$$

$$=\frac{1}{\sqrt{(3a)}}\int_0^a \left[\frac{3\,(a-x)}{\sqrt{(a-x)}}+\frac{a}{\sqrt{(a-x)}}\right]dx$$

$$=\frac{1}{\sqrt{(3a)}}\int_0^a [3\sqrt{(a-x)}+a\,(a-x)^{-1/2}]\,dx$$

$$=\frac{1}{\sqrt{(3a)}}\left[-3\cdot\frac{2}{3}(a-x)^{3/2}-a\,.\,2\,(a-x)^{1/2}\right]_0^a$$

$$=\frac{1}{\sqrt{(3a)}}[2a^{3/2}+2a^{3/2}]=\frac{4a}{\sqrt{3}}.$$

Example 4: *Find the length of the astroid* $x^{2/3}+y^{2/3}=a^{2/3}$.

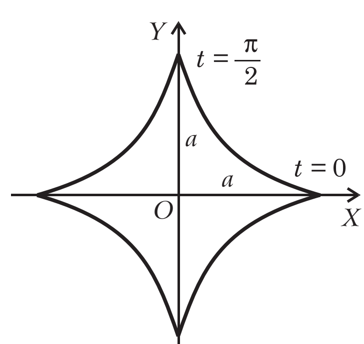
(Meerut 2002, 03, 13B; Kumaun 02, 10; Agra 05; Purvanchal 08; Kashi 12)

Solution: The given astroid is $x^{2/3}+y^{2/3}=a^{2/3}$. ...(1)

The curve is symmetrical in all the four quadrants. For the arc of the curve in the first quadrant x varies from 0 to a. Differentiating (1), w.r.t. x, we get

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\frac{dy}{dx}=0$$

so that $\qquad \frac{dy}{dx}=-\left(\frac{y}{x}\right)^{1/3}.$



$\therefore$ the required whole length of the curve

= 4 × length of the curve lying in the Ist quadrant

$$=4\int_0^a \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx=4\int_0^a \sqrt{\left(1+\frac{y^{2/3}}{x^{2/3}}\right)}\,dx$$

$$= 4\int_0^a \frac{\sqrt{(x^{2/3} + y^{2/3})}}{x^{1/3}}\, dx = 4\int_0^a \frac{\sqrt{(a^{2/3})}}{x^{1/3}}\, dx$$

$$= 4a^{1/3} \int_0^a x^{-1/3}\, dx = 4a^{1/3} \left[\frac{3}{2} x^{2/3}\right]_0^a = 6a.$$

Comprehensive Exercise 1

1. (i) Find the arc length of the curve $y = \frac{1}{2}x^2 - \frac{1}{4}\log x$ from $x = 1$ to $x = 2$.

(Meerut 2012B)

(ii) Find the length of the curve $y = \log \frac{e^x - 1}{e^x + 1}$ from $x = 1$ to $x = 2$.

(Meerut 2004B; Agra 06; Avadh 08; Kanpur 11; Rohilkhand 13; Kashi 13)

2. (i) Show that in the catenary $y = c \cosh (x / c)$, the length of arc from the vertex to any point is given by $s = c \sinh (x / c)$.

(ii) If s be the length of the arc of the catenary $y = c \cosh (x / c)$ from the vertex $(0, c)$ to the point (x, y), show that $s^2 = y^2 - c^2$.

3. (i) Find the length of an arc of the parabola $y^2 = 4ax$ measured from the vertex.

(ii) Find the length of the arc of the parabola $y^2 = 4ax$ cut off by its latus rectum.

4. (i) Find the length of the arc of the parabola $x^2 = 4ay$ from the vertex to an extremity of the latus rectum. **(Kanpur 2008; Purvanchal 09)**

(ii) Find the length of the arc of the parabola $x^2 = 8y$ from the vertex to an extremity of the latus rectum.

5. (i) Find the length of the arc of the parabola $y^2 = 4ax$ cut off by the line $y = 3x$.

(ii) Show that the length of the arc of the parabola $y^2 = 4ax$ which is intercepted between the points of intersection of the parabola and the straight line $3y = 8x$ is $a\left(\log 2 + \frac{15}{16}\right)$.

(Gorakhpur 2006; Purvanchal 06)

6. (i) Find the perimeter of the curve $x^2 + y^2 = a^2$.

(Avadh 2010; Rohilkhand 12B)

(ii) Find the length of the arc of the semi-cubical parabola $ay^2 = x^3$ from the vertex to the point (a, a). **(Bundelkhand 2010)**

7. (i) Show that the length of the arc of the curve $x^2 = a^2 (1 - e^{y/a})$ measured from the origin to the point (x, y) is $a \log \{(a + x) / (a - x)\} - x$.

(Rohilkhand 2010B)

(ii) Prove that the length of the loop of the curve $3ay^2 = x(x - a)^2$ is $4a/\sqrt{3}$.

(Meerut 2005B, 08, 09B)

8. (i) Find the perimeter of the loop of the curve $9ay^2 = (x-2a)(x-5a)^2$.

(ii) Show that the whole length of the curve $x^2(a^2-x^2) = 8a^2y^2$ is $\pi a\sqrt{2}$.

(Bundelkhand 2006; Purvanchal 11)

Answers 1

1. (i) $\frac{3}{2}+\frac{1}{4}\log 2$ (ii) $\log\left(e+\frac{1}{e}\right)$

3. (i) $\frac{1}{4a}\left[y\sqrt{(y^2+4a^2)}+4a^2\log\left\{\frac{y+\sqrt{(y^2+4a^2)}}{2a}\right\}\right]$

(ii) $2a[\sqrt{2}+\log(1+\sqrt{2})]$

4. (i) $a[\sqrt{2}+\log(1+\sqrt{2})]$ (ii) $2[\sqrt{2}+\log(1+\sqrt{2})]$

5. (i) $a\left[\frac{2\sqrt{(13)}}{9}+\log\left\{\frac{2+\sqrt{(13)}}{3}\right\}\right]$

6. (i) $2a\pi$ (ii) $\frac{1}{27}a[13\sqrt{(13)}-8]$

8. (i) $4a\sqrt{3}$

3 Equations of the Curve in Parametric Form (Meerut 2009B)

If the equations of the curve be given in the parametric form $x = f(t)$, $y = \phi(t)$, then s is obviously a function of t. In this case if we measure the arc length s in the direction of t increasing, we have

$$\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}} \quad \text{or} \quad ds = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}}\, dt.$$

On integrating between proper limits, the required length

$$s = \int_{t_1}^{t_2}\sqrt{\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}}\, dt.$$ **(Meerut 2003)**

Illustrative Examples

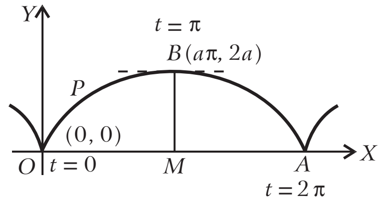
Example 5: *Show that 8a is the length of an arch of the cycloid whose equations are*

$$x = a(t-\sin t),\ y = a(1-\cos t).$$

(Agra 2002; Meerut 06; Rohilkhand 08; Kashi 11; Avadh 12; Purvanchal 14)

Solution: The given equations of the cycloid are $x = a(t - \sin t)$, $y = a(1 - \cos t)$.
We have $dx/dt = a(1 - \cos t)$, and $dy/dt = a \sin t$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a(1 - \cos t)} = \frac{2 \sin \frac{1}{2} t \cos \frac{1}{2} t}{2 \sin^2 \frac{1}{2} t} = \cot \frac{1}{2} t.$$



Now $y = 0$ when $\cos t = 1$ *i.e.*, $t = 0$. At $t = 0$, $x = 0$, $y = 0$ and $dy/dx = \infty$. Thus the curve passes through the point $(0, 0)$ and the tangent there is perpendicular to the x-axis.

Again y is maximum when $\cos t = -1$ *i.e.*, $t = \pi$. When $t = \pi$, $x = a\pi$, $y = 2a$, $dy/dx = 0$. Thus at the point $(a\pi, 2a)$ the tangent to the curve is parallel to the x-axis.

Also in this curve y cannot be negative. Thus an arch OBA of the given cycloid is as shown in the figure. It is symmetrical about the line BM which is the axis of the cycloid.

We have $$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \{a(1 - \cos t)\}^2 + (a \sin t)^2$$

$$= a^2 \{(2 \sin^2 \tfrac{1}{2} t)^2 + (2 \sin \tfrac{1}{2} t \cos \tfrac{1}{2} t)^2\}$$

$$= 4a^2 \sin^2 \tfrac{1}{2} t (\sin^2 \tfrac{1}{2} t + \cos^2 \tfrac{1}{2} t) = 4a^2 \sin^2 \tfrac{1}{2} t. \qquad \text{...(1)}$$

If s denotes the arc length of the cycloid measured from the cusp O to any point P towards the vertex B, then s increases as t increases. Therefore ds/dt will be taken with positive sign. So taking square root of both sides of (1), we have

$$ds/dt = 2a \sin \tfrac{1}{2} t \quad \text{or} \quad ds = 2a \sin \tfrac{1}{2} t \, dt.$$

At the cusp O, $t = 0$, and at the vertex B, $t = \pi$.

Now the length of the arch $OBA = 2 \times$ length of the arc OB

$$= 2 \int_0^\pi 2a \sin \tfrac{1}{2} t \, dt = 4a \left[-2 \cos \tfrac{1}{2} t\right]_0^\pi = -8a \left[\cos \tfrac{1}{2} t\right]_0^\pi$$

$$= -8a [0 - 1] = 8a.$$

Example 6: *Find the length of the loop of the curve* $x = t^2$, $y = t - \frac{1}{3} t^3$. **(Kanpur 2010)**

Solution: Eliminating the parameter t from $x = t^2$ and $y = t - \frac{1}{3} t^3$, we get $y^2 = x(1 - \frac{1}{3}x)^2$ as the cartesian equation of the curve and hence we observe that the curve is symmetrical about the x-axis. The loop of the curve extends from the point $(0,0)$ to the point $(3,0)$. Putting $y = 0$ in $y = t - \frac{1}{3}t^3$, we get $t = 0$ and $t = \sqrt{3}$. Therefore the arc of the upper half of the loop extends from $t = 0$ to $t = \sqrt{3}$.

Now the required length of the loop

$$= 2 \times \text{length of the half of the loop which lies above } x\text{-axis}$$

$$= 2\int_0^{\sqrt{3}} \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}\, dt$$

$$= 2\int_0^{\sqrt{3}} \sqrt{\{(2t)^2 + (1 - \tfrac{1}{3}\cdot 3t^2)^2\}}\, dt$$

$$= 2\int_0^{\sqrt{3}} \sqrt{(1 + 2t^2 + t^4)}\, dt = 2\int_0^{\sqrt{3}} (1 + t^2)\, dt$$

$$= 2\left[t + \frac{t^3}{3}\right]_0^{\sqrt{3}} = 2[\sqrt{3} + \sqrt{3}] = 4\sqrt{3}.$$

Example 7: *Show that the length of an arc of the curve*

$x \sin t + y \cos t = f'(t)$, $x \cos t - y \sin t = f''(t)$ is given by

$s = f(t) + f''(t)$, where c is the constant of integration. **(Agra 2003)**

Solution: The given equations of the curve are $x \sin t + y \cos t = f'(t)$ ...(1)

and $x \cos t - y \sin t = f''(t)$. ...(2)

Multiplying (1) by $\sin t$ and (2) by $\cos t$ and adding, we get

$$x(\sin^2 t + \cos^2 t) = \sin t \,.\, f'(t) + \cos t \,.\, f''(t)$$

or $x = \sin t\, f'(t) + \cos t\, f''(t)$. ...(3)

Again, multiplying (1) by $\cos t$ and (2) by $\sin t$ and subtracting, we get

$y = \cos t\, f'(t) - \sin t\, f''(t)$. ...(4)

Now differentiating (3) and (4) w.r.t. t, we get

$$dx/dt = \cos t\, f'(t) + \sin t\, f''(t) + \cos t\, f'''(t) - \sin t\, f''(t)$$

$$= [f'(t) + f'''(t)] \cos t$$

and $dy/dt = -[f'(t) + f'''(t)] \sin t$.

Now if s be the arc length in the direction of t increasing, then

$$\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}$$

$$= \sqrt{[\cos^2 t\{f'(t) + f'''(t)\}^2 + \sin^2 t\{f'(t) + f'''(t)\}^2]}$$

$$= [f'(t) + f'''(t)]\sqrt{(\cos^2 t + \sin^2 t)} = f'(t) + f'''(t).$$

Integrating both sides, we have $s = \int [f'(t) + f'''(t)]\, dt + c$

$= f(t) + f''(t) + c$, where c is the constant of integration.

Comprehensive Exercise 2

1. (i) Find the whole length of the curve (astroid) $x = a \cos^3 t,\ y = a \sin^3 t$.

(Rohilkhand 2011)

(ii) Find the whole length of the curve (Hypocycloid)

$x = a \cos^3 t,\ y = b \sin^3 t$.

2. Rectify the curve or find the length of an arch of the curve

$x = a(t + \sin t),\ y = a(1 - \cos t)$. **(Rohilkhand 2009B)**

3. Prove that the length of an arc of the cycloid $x = a(t + \sin t),\ y = a(1 - \cos t)$ from the vertex to the point (x, y) is $\sqrt{(8ay)}$. **(Bundelkhand 2007; Meerut 12)**

4. Find the length of the arc of the curve

$x = e^t \sin t,\ y = e^t \cos t$, from $t = 0$ to $t = \frac{1}{2}\pi$.

(Kumaun 2008; Kanpur 09)

5. Show that in the epi-cycloid for which

$x = (a + b)\cos\theta - b\cos\{(a + b)/b\}\theta,$

$y = (a + b)\sin\theta - b\sin\{(a + b)/b\}\theta,$

the length of the arc measured from the point $\theta = \pi b/a$ is

$\{4b(a + b)/a\}\cos\{(a/2b)\theta\}$.

6. In the ellipse $x = a\cos\phi,\ y = b\sin\phi$, show that $ds = a\sqrt{(1 - e^2\cos^2\phi)}\, d\phi$, and hence show that the whole length of the ellipse is

$$2\pi a\left[1 - \left(\frac{1}{2}\right)^2 \cdot \frac{e^2}{1} - \left(\frac{1.3}{2.4}\right)^2 \cdot \frac{e^4}{3} - \left(\frac{1.3.5}{2.4.6}\right)^2 \cdot \frac{e^6}{5} - \dots\right],$$

where e is the eccentricity of the ellipse. **(Meerut 2005)**

Answers 2

1. (i) $6a$ (ii) $4(b^2 + ab + a^2)/(b + a)$
2. $8a$
4. $\sqrt{2}\,[e^{\pi/2} - 1]$

4 Equation of the Curve in Polar Form

For the curve $r = f\ (\theta)$, if we measure the arc length s in the direction of θ increasing, we have

$$\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} \quad \text{or} \quad ds = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta.$$

On integrating between proper limits, the required length

$$s = \int_{\theta_1}^{\theta_2} \sqrt{\left\{r^2 + \left(\frac{dr}{dq}\right)^2\right\}}\, dq.$$

(Meerut 2003)

If the equation of the curve be $\theta = f\ (r)$, then the required length is given by

$$s = \int_{r_1}^{r_2} \sqrt{\left\{1 + \left(r\frac{dq}{dr}\right)^2\right\}}\, dr.$$

Illustrative Examples

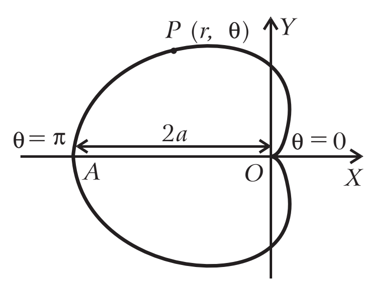
Example 8: *Find the perimeter of the cardioid $r = a\ (1 - \cos\theta)$.*

(Meerut 2007; Bundelkhand 11)

Solution: The given curve is $r = a\ (1 - \cos\theta)$. ...(1)

It is symmetrical about the initial line.

We have $r = 0$ when $\cos\theta = 1$ *i.e.*, $\theta = 0$. Also r is maximum when $\cos\theta = -1$ *i.e.*, $\theta = \pi$ and then $r = 2a$. As θ increases from 0 to π, r increases from 0 to $2a$. So the curve is as shown in the figure.



By symmetry, the perimeter of the cardioid

= 2 × the arc length of the upper half of the cardioid.

Now differentiating (1) w.r.t. θ, we have

$$dr / d\theta = a \sin\theta.$$

We have
$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2\ (1 - \cos\theta)^2 + a^2 \sin^2\theta$$

$$= a^2\ (2\sin^2 \tfrac{1}{2}\theta)^2 + a^2\ (2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta)^2$$

$$= 4a^2 \sin^2\tfrac{1}{2}\theta\,(\sin^2\tfrac{1}{2}\theta + \cos^2\tfrac{1}{2}\theta)$$

$$= 4a^2 \sin^2 \tfrac{1}{2}\theta. \qquad ...(2)$$

If s denotes the arc length of the cardioid measured from the cusp O (*i.e.*, the point $\theta = 0$) to any point $P(r, \theta)$ in the direction of θ increasing, then s increases as θ increases. Therefore $ds / d\theta$ will be positive.

Hence from (2), we have

$$ds / d\theta = 2a \sin \frac{1}{2} \theta, \quad \text{or} \quad ds = 2a \sin \frac{1}{2} \theta \, d\theta. \qquad ...(3)$$

At the cusp O, $\theta = 0$ and at the vertex A, $\theta = \pi$.

$\therefore$ the length of the arc $OPA = \int_0^{\pi} 2a \sin \frac{1}{2} \theta \, d\theta$

$$= 4a \left[-\cos \frac{\theta}{2} \right]_0^{\pi} = -4a \left[\cos \frac{\theta}{2} \right]_0^{\pi} = -4a(0 - 1) = 4a.$$

$\therefore$ the perimeter of the cardioid $= 2 \times 4a = 8a$.

Example 9: *Find the length of the arc of the equiangular spiral $r = a e^{\theta \cot \alpha}$, between the points for which radii vectors are r_1 and r_2.* **(Kanpur 2007; Kumaun 09)**

Solution: The given equiangular spiral is $r = ae^{\theta \cot \alpha}$. ...(1)

Differentiating (1) w.r.t. θ, we get $dr / d\theta = ae^{\theta \cot \alpha} . \cot \alpha = r \cot \alpha$, from (1),

$$\therefore \quad d\theta / dr = 1 / (r \cot \theta), \text{ i.e., } (r \, d\theta / dr) = \tan \theta \qquad ...(2)$$

If s denotes the arc length of the given curve measured in the direction of r increasing, we have

$$\frac{ds}{dr} = \sqrt{\left\{ 1 + r^2 \left(\frac{d\theta}{dr} \right)^2 \right\}} \qquad \textbf{(Note)}$$

$$= \sqrt{(1 + \tan^2 \alpha)} = \sqrt{(\sec^2 \alpha)} = \sec \alpha, \text{ from (2)}$$

or $\quad ds = \sec \alpha \, dr.$ **(Meerut 2001B)**

Let s_1 denote the required arc length, *i.e.*, from $r = r_1$ to $r = r_2$.

Then $\quad \int_0^{s_1} ds = \int_{r_1}^{r_2} \sec \alpha \, dr = (\sec \alpha) [r]_{r_1}^{r_2}$ or $s_1 = (\sec \alpha)(r_2 - r_1)$.

Example 10: *Prove that the perimeter of the limacon $r = a + b \cos \theta$, if b / a be small, is approximately $2 \pi a \left(1 + \frac{1}{4} b^2 / a^2 \right)$.*

Solution: The given curve is

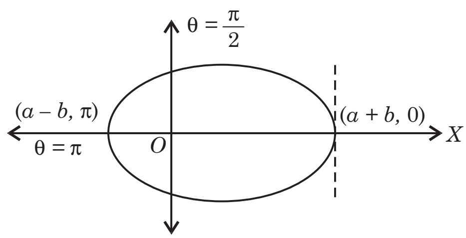
$$r = a + b \cos \theta, (a > b). \qquad ...(1)$$

Note that b / a is given to be small so we must have $b < a$. The curve (1) is symmetrical about the initial line and for the portion of the curve lying above the initial line θ varies from $\theta = 0$ to $\theta = \pi$.

By symmetry, the perimeter of the limacon

$= 2 \times$ the arc length of the upper half of the limacon.

Now differentiating (1) w.r.t. θ, we have $dr/d\theta = -b\sin\theta$.



We have $$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = (a + b\cos\theta)^2 + (-b\sin\theta)^2$$

$$= a^2 + b^2\cos^2\theta + 2ab\cos\theta + b^2\sin^2\theta$$

$$= a^2 + b^2 + 2ab\cos\theta.$$

If we measure the arc length s in the direction of θ increasing, we have

$$ds/d\theta = \sqrt{(a^2 + b^2 + 2ab\cos\theta)}$$

or $$ds = \sqrt{(a^2 + b^2 + 2ab\cos\theta)}\, d\theta.$$

The arc length of the upper half of the limacon

$$= \int_0^\pi \sqrt{(a^2 + b^2 + 2ab\cos\theta)}\, d\theta = a\int_0^\pi \left(1 + \frac{2b}{a}\cos\theta + \frac{b^2}{a^2}\right)^{1/2} d\theta$$

$$= a\int_0^\pi \left[1 + \frac{b}{a}\cos\theta + \frac{1}{2}\cdot\frac{b^2}{a^2} + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2\,!}\left(4\frac{b^2}{a^2}\cos^2\theta\right)\right] d\theta$$

[Expanding by binomial theorem and neglecting powers of b/a higher than two because b/a is small]

$$= a\int_0^\pi \left[1 + \frac{b}{a}\cos\theta + \frac{1}{2}\frac{b^2}{a^2}(1 - \cos^2\theta)\right] d\theta$$

$$= a\int_0^\pi \left[1 + \frac{b}{a}\cos\theta + \frac{1}{2}\frac{b^2}{a^2}\sin^2\theta\right] d\theta$$

$$= a\left[\left\{\theta + \frac{b}{a}\sin\theta\right\}_0^\pi + \frac{1}{2}\frac{b^2}{a^2}\, 2\int_0^{\pi/2} \sin^2\theta\, d\theta\right]$$

$$= a\left[\pi + \frac{1}{2}\frac{b^2}{a^2}\cdot 2\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] = a\pi\left[1 + \frac{b^2}{4a^2}\right].$$

$\therefore$ the perimeter of the limacon $= 2 \times a\pi\,[1 + (b^2/4a^2)] = 2a\pi\,[1 + (b^2/4a^2)]$.

Example 11: *If s be the length of the curve* $r = a \tanh \frac{1}{2}\theta$ *between the origin and* $\theta = 2\pi$, *and* Δ *be the area under the curve between the same two points, prove that* $\Delta = a(s - a\pi)$.

Solution: The given curve is $r = a \tanh \frac{1}{2}\theta$. ...(1)

Differentiating (1) w.r.t. θ, we get $dr/d\theta = a \cdot \frac{1}{2} \operatorname{sech}^2 \frac{1}{2}\theta$.

We have $$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2 \tanh^2 \frac{1}{2}\theta + \frac{a^2}{4} \operatorname{sech}^4 \frac{1}{2}\theta$$

$$= \frac{1}{4}a^2 \left[4 \tanh^2 \frac{1}{2}\theta + \operatorname{sech}^4 \frac{1}{2}\theta\right]$$

$$= \frac{1}{4}a^2 \left[4\left(1 - \operatorname{sech}^2 \frac{1}{2}\theta\right) + \operatorname{sech}^4 \frac{1}{2}\theta\right] = \frac{1}{4}a^2 \left[2 - \operatorname{sech}^2 \frac{1}{2}\theta\right]^2. \qquad ...(2)$$

If we measure the arc length s in the direction of θ increasing, we have

$$ds/d\theta = \frac{1}{2}a\left(2 - \operatorname{sech}^2 \frac{1}{2}\theta\right)$$

[Retaining +ive sign while taking the square root of (2)]

or $$ds = \frac{1}{2}a\left(2 - \operatorname{sech}^2 \frac{1}{2}\theta\right) d\theta.$$

Now at the origin $r = 0$ and putting $r = 0$ in (1), we get $\theta = 0$.

∴ the arc length of the given curve between the origin $(\theta = 0)$ and $\theta = 2\pi$ is given by

$$s = \frac{1}{2}a\int_0^{2\pi} \left(2 - \operatorname{sech}^2 \frac{1}{2}\theta\right) d\theta$$

$$= \frac{1}{2}a\int_0^{2\pi} 2\, d\theta - \frac{1}{2}a\int_0^{2\pi} \operatorname{sech}^2 \frac{1}{2}\theta\, d\theta$$

$$= \frac{1}{2}a \cdot 2\,[\theta]_0^{2\pi} - \frac{1}{2}a\left[2 \tan \frac{1}{2}\theta\right]_0^{2\pi}$$

$$= 2a\pi - a \tanh \pi. \qquad ...(3)$$

Also the area between the radii vectors $\theta = 0, \theta = 2\pi$ and the curve is

$$\Delta = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta = \frac{1}{2}a^2\int_0^{2\pi} \tanh^2 \frac{1}{2}\theta\, d\theta$$

$$= \frac{1}{2}a^2\int_0^{2\pi} \left(1 - \operatorname{sech}^2 \frac{1}{2}\theta\right) d\theta = \frac{1}{2}a^2\left[\theta - 2\tanh\frac{1}{2}\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}a^2\,[2\pi - 2\tanh\pi] = a^2\,[\pi - \tanh\pi]$$

$$= a\,[a\pi - a\tanh\pi] = a\,[(2a\pi - a\tanh\pi) - a\pi]$$

$$= a(s - a\pi).$$ [From (3)]

5 Equation of the Curve in Pedal Form

Let $p = f(r)$ be the equation of the curve and r_1 and r_2 be the values of r at two given points of the curve. Then by differential calculus we know that

$$\frac{ds}{dr} = \frac{r}{\sqrt{(r^2 - p^2)}}$$

or $$ds = \frac{r}{\sqrt{(r^2 - p^2)}}\,dr, \quad \text{where } s \text{ increases as } r \text{ increases.}$$

On integrating between proper limits, the required length

$$s = \int_{r_1}^{r_2} \frac{r}{\sqrt{(r^2 - p^2)}}\,dr.$$

The value of p should be put in terms of r from the equation of the curve.

Remark: If the curve is symmetrical about one or more lines, then find out the length of one symmetrical part and then multiply it by the number of symmetrical parts.

Illustrative Examples

Example 12: *Prove the formula* $s = \int \frac{r\,dr}{\sqrt{(r^2 - p^2)}}$.

Show that the arc of the curve $p^2(a^4 + r^4) = a^4 r^2$ *between the limits* $r = b$, $r = c$ *is equal in length to the arc of the hyperbola* $xy = a^2$ *between the limits* $x = b, x = c$.

Solution: From differential calculus, we know that

$$\tan\phi = r\frac{d\theta}{dr} \text{ and } \frac{ds}{dr} = \sqrt{\left[1 + \left(r\frac{d\theta}{dr}\right)^2\right]}.$$

$$\therefore \quad \frac{ds}{dr} = \sqrt{(1 + \tan^2\phi)} = \sqrt{(\sec^2\phi)} = \sec\phi$$

$$= \frac{1}{\cos\phi} = \frac{1}{\sqrt{(1 - \sin^2\phi)}} = \frac{1}{\sqrt{\{1 - (p^2/r^2)\}}} \qquad [\because p = r\sin\phi]$$

$$= \frac{r}{\sqrt{(r^2 - p^2)}}.$$

Thus $$ds = \frac{r}{\sqrt{(r^2 - p^2)}}\,dr.$$

Integrating between the given limits, we get $s = \int \frac{r}{\sqrt{(r^2 - p^2)}}\,dr.$ ...(1)

Now the given curve is $p^2(a^4 + r^4) = a^4 r^2$

or $$p^2 = a^4 r^2 / (a^4 + r^4).$$

We have $$r^2 - p^2 = r^2 - \frac{a^4 r^2}{(a^4 + r^4)} = \frac{r^6}{(a^4 + r^4)}. \quad ...(2)$$

Therefore from (1), the arc of the given curve between the limits $r = b, r = c$ is

$$= \int_b^c \frac{r\,dr}{\sqrt{(r^2 - p^2)}} = \int_b^c \frac{r\,dr}{\sqrt{\{r^6 / (a^4 + r^4)\}}} \quad \text{[From (2)]}$$

$$= \int_b^c \frac{r\sqrt{(a^4 + r^4)}}{r^3}\,dr = \int_b^c \frac{\sqrt{(a^4 + r^4)}}{r^2}\,dr. \quad ...(3)$$

Also, for the hyperbola $xy = a^2$ *i.e.*, $y = a^2 / x, dy / dx = - a^2 / x^2$.

$\therefore$ the arc length of the hyperbola $xy = a^2$ between the limits $x = b, x = c$

$$= \int_b^c \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\,dx = \int_b^c \sqrt{\left\{1 + \frac{a^4}{x^4}\right\}}\,dx$$

$$= \int_b^c \frac{\sqrt{(x^4 + a^4)}}{x^2}\,dx = \int_b^c \frac{\sqrt{(r^4 + a^4)}}{r^2}\,dr$$ [Changing the variable from x to r by a property of definite integrals]

$$= \int_b^c \frac{\sqrt{(a^4 + r^4)}}{r^2}\,dr. \quad ...(4)$$

From (3) and (4) we observe that the two lengths are equal.

Comprehensive Exercise 3

1. Find the entire length of the cardioid $r = a(1 + \cos\theta)$.
(Purvanchal 2007; Rohilkhand 09, 11B)
2. Find the perimeter of the curve $r = a(1 + \cos\theta)$ and show that arc of the upper half is bisected by $\theta = \pi / 3$. **(Gorakhpur 2005; Purvanchal 07)**
3. Prove that the line $4r\cos\theta = 3a$ divides the cardioid $r = a(1 + \cos\theta)$ into two parts such that lengths of the arc on either side of the line are equal.
4. Show that the arc of the upper half of the curve $r = a(1 - \cos\theta)$ is bisected by $\theta = 2\pi / 3$.
5. Find the length of the cardioid $r = a(1 - \cos\theta)$ lying outside the circle $r = a\cos\theta$.
6. Find the length of the arc of the equiangular spiral $r = a e^{\theta \cot\alpha}$, taking $s = 0$ when $\theta = 0$.
7. Find the length of any arc of the cissoid $r = a(\sin^2\theta / \cos\theta)$.

8. Show that the whole length of the limacon $r = a + b\cos\theta$, $(a > b)$ is equal to that of an ellipse whose semi-axes are equal in length to the maximum and minimum radii vectors of the limacon.

Answers 3

1. $8a$ 5. $4a\sqrt{3}$ 6. $a\sec\alpha\,[e^{\theta\cot\alpha} - 1]$

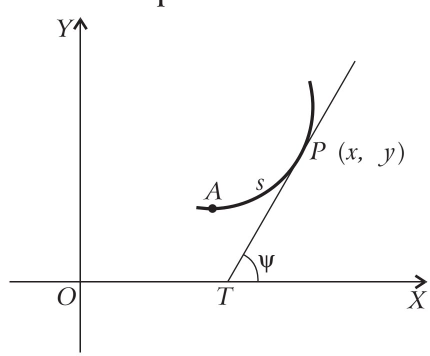
7. $f(\theta_2) - f(\theta_1)$, where $f(\theta) = a\sqrt{(\sec^2\theta + 3)} - a\sqrt{3}\log\left\{\cos\theta + \sqrt{\left(\cos^2\theta + \frac{1}{3}\right)}\right\}$

6 Intrinsic Equations

Definition: By the *intrinsic equation* of a curve we mean a relation between s and ψ, where s is the length of the arc AP of the curve measured from a fixed point A on it to a variable point P, and ψ is the angle which the tangent to the curve at P makes with a fixed straight line usually taken as the positive direction of the axis of x.

The co-ordinates s and ψ are known as **Intrinsic Co-ordinates**.

(a) To find the intrinsic equation from the cartesian equation:



Let the equation of the given curve be $y = f(x)$. Take A as the fixed point on the curve from which s is measured and take the axis of x as the fixed straight line with reference to which ψ is measured. Let $P(x, y)$ be any point on the curve and PT be the tangent at the point P to the curve.

Let arc $AP = s$ and $\angle PTX = \psi$.

Now, we have $\tan\psi = dy/dx = f'(x)$. ...(1)

Let a be the abscissa of the point A from which s is measured. Then

$$s = \int_a^x \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]}\,dx = \int_a^x \sqrt{[1 + \{f'(x)\}^2]}\,dx. \quad ...(2)$$

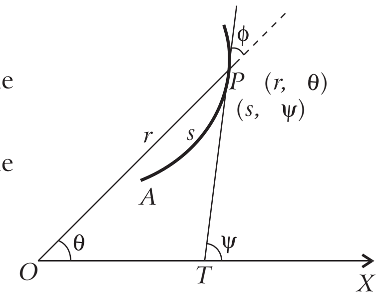
Eliminating x between (1) and (2), we obtain the required intrinsic equation.

Note: To find the intrinsic equation from the parametric equations we use $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ and then proceed as in case (a).

(b) Intrinsic equation from Polar equation:

Let the equation of the given curve be $r = f(\theta)$.

Take A as the fixed point on the curve from which s is measured.



Let P be any point (r, θ) on the curve.

Let arc $AP = s$ and $\angle PTX = \psi$, where OX is the initial line.

If ϕ is the angle between the radius vector and the tangent at P, then

$$\tan \phi = r\frac{d\theta}{dr} = \frac{r}{dr/d\theta}$$

$$= \frac{f(\theta)}{f'(\theta)}, \qquad ...(1)$$

and $$\psi = \theta + \phi. \qquad ...(2)$$

Let α be the vectorial angle of the point A. Then we have

$$s = \int_{\alpha}^{\theta} \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta = \int_{\alpha}^{\theta} \sqrt{[\{f(\theta)\}^2 + \{f'(\theta)\}^2]}\, d\theta \qquad ...(3)$$

Eliminating θ and ϕ between (1), (2) and (3), we get a relation between s and ψ, which is the intrinsic equation of the curve.

(c) Intrinsic equation from Pedal Equation:

Let the pedal equation of the curve be $p = f(r)$. ...(1)

Then $$s = \int_a^r \frac{r\, dr}{\sqrt{(r^2 - p^2)}}, \qquad ...(2)$$

the arc length s being measured from the point $r = a$.

Also the radius of curvature $\rho = \dfrac{ds}{d\psi} = r\dfrac{dr}{dp}$. ...(3)

Eliminating p and r between (1), (2) and (3), we obtain the required intrinsic equation.

Illustrative Examples

Example 13: *Show that the intrinsic equation of the parabola* $y^2 = 4ax$ *is*

$$s = a \cot \psi \operatorname{cosec} \psi + a \log(\cot \psi + \operatorname{cosec} \psi),$$

ψ being the angle between the x-axis and the tangent at the point whose arcual distance from the vertex is s.

Solution: The given parabola is $y^2 = 4ax$. ...(1)

Differentiating (1) w.r.t. x, we get $2y\,(dy/dx) = 4a$.

$\therefore$ $$\tan \psi = dy/dx = 4a/2y = 2a/y. \qquad ...(2)$$

If s denotes the arc length of the parabola measured from the vertex $(0, 0)$ in the direction of y increasing, then

$$\frac{ds}{dy}=\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}=\sqrt{\left\{1+\frac{y^2}{4a^2}\right\}} \qquad \left[\because \frac{dx}{dy}=\frac{y}{2a}\right]$$

$$=\sqrt{\left\{\frac{4a^2+y^2}{4a^2}\right\}}=\frac{1}{2a}\sqrt{(4a^2+y^2)}.$$

$\therefore$
$$ds=\frac{1}{2a}\sqrt{(4a^2+y^2)}\,dy.$$

Integrating,
$$\int_0^s ds=\frac{1}{2a}\int_0^y \sqrt{(4a^2+y^2)}\,dy$$

or
$$s=\frac{1}{2a}\left[\frac{1}{2}y\sqrt{(4a^2+y^2)}+\frac{1}{2}\cdot 4a^2\log\{y+\sqrt{(4a^2+y^2)}\}\right]_0^y$$

$$=\left(\frac{1}{2a}\right)\left[\frac{1}{2}y\sqrt{(4a^2+y^2)}+\frac{1}{2}\cdot 4a^2\log\{y+\sqrt{(4a^2+y^2)}\}-\frac{1}{2}\cdot 4a^2\log 2a\right]$$

$$=\frac{1}{4a}\left[y\sqrt{(4a^2+y^2)}+4a^2\log\frac{y+\sqrt{(4a^2+y^2)}}{2a}\right]. \qquad ...(3)$$

Now to obtain the intrinsic equation of the given parabola we eliminate y between (2) and (3). From (2), we have $y=2a\cot\psi$. Putting this value of y in (3), we get

$$s=\frac{1}{4a}\left[2a\cot\psi\sqrt{(4a^2+4a^2\cot^2\psi)}+4a^2\log\frac{2a\cot\psi+\sqrt{(4a^2+4a^2\cot^2\psi)}}{2a}\right]$$

$$=\frac{1}{4a}[(2a\cot\psi)\,.\,2a\sqrt{(1+\cot^2\psi)}+4a^2\log\{\cot\psi+\sqrt{(1+\cot^2\psi)}\}]$$

$$=a\cot\psi\,\text{cosec}\,\psi+a\log(\cot\psi+\text{cosec}\,\psi),$$

which is the required intrinsic equation.

Example 14: *Show that the intrinsic equation of the cycloid*

$$x=a(t+\sin t),\ y=a(1-\cos t)\quad \textit{is}\quad s=4a\sin\psi.$$

Hence or otherwise find the length of the complete cycloid.

(Meerut 2001, 06B, 07, 10; Agra 01; Kanpur 04; Avadh 04, 09, 10; Rohilkhand 07 B)

Solution: The given equations of the cycloid are

$$x=a(t+\sin t),\ y=a(1-\cos t). \qquad ...(1)$$

We have $dx/dt=a(1+\cos t)$, and $dy/dt=a\sin t$.

$\therefore$
$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{a\sin t}{a(1+\cos t)}=\frac{2\sin\frac{1}{2}t\cos\frac{1}{2}t}{2\cos^2\frac{1}{2}t}=\tan\frac{1}{2}t.$$

Hence
$$\tan\psi=dy/dx=\tan\frac{1}{2}t\quad\text{or}\quad \psi=\frac{1}{2}t. \qquad ...(2)$$

If s denotes the arc length of the cycloid measured from the vertex (*i.e.*, the point $t=0$) to any point P (*i.e.*, the point 't') in the direction of t increasing, then

$$s=\int_0^t \sqrt{\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}}\,dt=\int_0^t \sqrt{\{a^2(1+\cos t)^2+a^2\sin^2 t\}}\,dt$$

$$=\int_0^t \sqrt{\{2a^2(1+\cos t)\}}\,dt$$

$$=2a\int_0^t \cos\frac{1}{2}t\,dt=2a\left[2\sin\frac{1}{2}t\right]_0^t=4a\sin\frac{1}{2}t \qquad \text{...(3)}$$

Eliminating t from (2) and (3), we get $s=4a\sin\psi$, ...(4)
which is the required intrinsic equation of the cycloid.

Second Part: In the intrinsic equation (4) of the cycloid the arc length s has been measured from the vertex *i.e.*, the point $\psi=0$. At a cusp, we have $t=\pi$ and $\psi=\pi/2$. If s_1 denotes the length of the arc extending from the vertex to a cusp, then from (4), we have $s_1=4a\sin\frac{1}{2}\pi=4a$.

$\therefore$ the whole length of an arch of the cycloid $=2\times 4a=8a$.

Example 15: *Find the intrinsic equation of the cardioid $r=a(1+\cos\theta)$,* **(Garhwal 2003)** *and hence, or otherwise, prove that $s^2+9\rho^2=16a^2$, where ρ is the radius of curvature at any point, and s is the length of the arc intercepted between the vertex and the point.*

(Meerut 2005B)

Solution: The given curve is $r=a(1+\cos\theta)$. ...(1)
Differentiating (1) w.r.t. θ, we have $dr/d\theta=-a\sin\theta$.

$$\therefore \qquad \tan\phi=r\frac{d\theta}{dr}=\frac{r}{dr/d\theta}=\frac{a(1+\cos\theta)}{-a\sin\theta}=\frac{2\cos^2\frac{1}{2}\theta}{-2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}$$

$$=-\cot\frac{1}{2}\theta=\tan\left(\frac{1}{2}\pi+\frac{1}{2}\theta\right).$$

Therefore $\phi=\frac{1}{2}\pi+\frac{1}{2}\theta$,

so that $\qquad \psi=\theta+\phi=\theta+\frac{1}{2}\pi+\frac{1}{2}\theta=\frac{1}{2}\pi+\frac{3}{2}\theta$

or $\qquad \frac{1}{2}\theta=\frac{1}{3}\left(\psi-\frac{1}{2}\pi\right)$. ...(2)

If s denotes the arc length of the cardioid measured from the vertex (*i.e.*, $\theta=0$) to any point P (*i.e.*, $\theta=\theta$) in the direction of θ increasing, then

$$s=\int_0^\theta \sqrt{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}}\,d\theta=2a\int_0^\theta \sqrt{\{(1+\cos\theta)^2+\sin^2\theta\}}\,d\theta$$

$$=2a\int_0^\theta \sqrt{\{1+2\cos\theta+\cos^2\theta+\sin^2\theta\}}\,d\theta$$

$$=2a\int_0^\theta \sqrt{\{2(1+\cos\theta)\}}\,d\theta=2a\int_0^\theta \cos\frac{1}{2}\theta\,d\theta$$

$$=2a\left[2\sin\frac{1}{2}\theta\right]_0^\theta=4a\sin\frac{1}{2}\theta. \qquad \text{...(3)}$$

Eliminating θ between (2) and (3), we get $s = 4a \sin \left\{\frac{1}{3}\left(\psi - \frac{1}{2}\pi\right)\right\}$, ...(4)

which is the required intrinsic equation.

Also $$\rho = \frac{ds}{d\psi} = \frac{4a}{3} \cos \frac{1}{3}(\psi - \frac{1}{2}\pi),$$ from (4)

or $$3\rho = 4a \cos \frac{1}{3}(\psi - \frac{1}{2}\pi). \quad ...(5)$$

Squaring and adding (4) and (5), we get

$$s^2 + 9\rho^2 = (4a)^2 \{\sin^2 \frac{1}{3}(\psi - \frac{1}{2}\pi) + \cos^2 \frac{1}{3}(\psi - \frac{1}{2}\pi)\}$$

$$= 16a^2 .1 = 16a^2.$$

Example 16: *Find the intrinsic equation of the equiangular spiral $p = r \sin \alpha$.*

(Meerut 2000, 01, 04, 06, 09, 10B)

Solution: The given pedal equation of the curve is

$$p = r \sin \alpha. \quad ...(1)$$

Differentiating (1) w.r.t. r, we have

$$dp / dr = \sin \alpha.$$

∴ $$\rho = \frac{ds}{d\psi} = r\frac{dr}{dp} = \frac{r}{dp / dr} = \frac{r}{\sin \alpha} = r \operatorname{cosec} \alpha. \quad ...(2)$$

If we measure the arc length s from the point $r = 0$ in the direction of r increasing, we have

$$s = \int_0^r \frac{r\, dr}{\sqrt{(r^2 - p^2)}} = \int_0^r \frac{r\, dr}{\sqrt{(r^2 - r^2 \sin^2 \alpha)}} = \int_0^r \sec \alpha\, dr$$

$$= \sec \alpha \int_0^r dr = \sec \alpha\, [r]_0^r = r \sec \alpha. \quad ...(3)$$

Eliminating r between (2) and (3), we have

$$\frac{(ds / d\psi)}{s} = \frac{\operatorname{cosec} \alpha}{\sec \alpha} = \cot \alpha$$ [Dividing (2) by (3)]

or $ds / s = \cot \alpha\, d\psi.$

Integrating, $\log s = \psi \cot \alpha + \log a$, where a is constant of integration

or $\log (s / a) = \psi \cot \alpha$

or $s = a\, e^{\psi \cot \alpha}$,

which is the required intrinsic equation of the curve.

Example 17: *Find the intrinsic equation of the curve for which the length of the arc measured from the origin varies as the square root of the ordinate. Find also parametric equations of the curve in terms of any parameter.*

Solution: Let s denote the arc length of the curve measured from the origin to any point $P(x, y)$ such that s increases as y increases. As given $s \propto \sqrt{y}$ so that $s = \lambda \sqrt{y}$, where λ is some constant.

Choosing this constant $\lambda = \sqrt{(8a)}$, we have **(Note)**

$$s = \sqrt{(8ay)} \quad \text{or} \quad s^2 = 8ay. \qquad \text{...(1)}$$

Now differentiating (1) w.r.t. y, we have $2s\,(ds/dy) = 8a$

or $$ds/dy = 4a/s. \qquad \text{...(2)}$$

Now we know that $dy/ds = \sin\psi$.

$\therefore$ $$\sin\psi = dy/ds = s/4a \qquad \text{[From (2)]}$$

or $s = 4a \sin\psi$, which is the required intrinsic equation.

Again from (1), we have

$$y = \frac{s^2}{8a} = \frac{16a^2 \sin^2\psi}{8a} \qquad [\because s = 4a\sin\psi]$$

$$= a(1 - \cos 2\psi). \qquad \text{...(3)}$$

Also $$\frac{ds}{dx} = \frac{ds}{d\psi} \cdot \frac{d\psi}{dx} = 4a\cos\psi \frac{d\psi}{dx} \qquad \left[\because \frac{ds}{d\psi} = 4a\cos\psi\right]$$

or $$\frac{1}{\cos\psi} = 4a\cos\psi \frac{d\psi}{dx} \qquad \left[\because \frac{dx}{ds} = \cos\psi\right]$$

or $$dx = 4a\cos^2\psi\, d\psi = 2a(1 + \cos 2\psi)\, d\psi. \qquad \text{...(4)}$$

If $x = 0$ when $\psi = 0$, then integrating (4), we get

$$\int_0^x dx = 2a\int_0^\psi (1 + \cos 2\psi)\, d\psi$$

or $$x = 2a\left[\psi + \frac{1}{2}\sin 2\psi\right]_0^\psi$$

or $$x = a[2\psi + \sin 2\psi]. \qquad \text{...(5)}$$

So from (3) and (5), the required parametric equations of the curve are

$$x = a(2\psi + \sin 2\psi) \quad \text{and} \quad y = a(1 - \cos 2\psi),$$

which are the parametric equations of a cycloid.

Comprehensive Exercise 4

1. Prove that the intrinsic equation of the parabola $x^2 = 4ay$ is

$$s = a\tan\psi\sec\psi + a\log(\tan\psi + \sec\psi).$$

2. Find the intrinsic equation of the parabola $y^2 = 4ax$. Hence deduce the length of the arc measured from the vertex to an extremity of the latus rectum.

3. Show that the intrinsic equation of the semi-cubical parabola

$$3ay^2 = 2x^3 \text{ is } 9s = 4a(\sec^3 \psi - 1).$$

(Meerut 2005, 09B; Rohilkhand 08B)

4. Find the intrinsic equation of the catenary $y = c \cosh (x / c)$.

(Rohilkhand 2007; Kumaun 08; Kanpur 14)

Hence show that $c\rho = c^2 + s^2$, where ρ is the radius of curvature.

5. Prove that the intrinsic equation of the curve

$$x = a(1 + \sin t), \quad y = a(1 + \cos t) \text{ is } s + a\psi = 0.$$

6. Find the intrinsic equation of the cardioid $r = a(1 - \cos \theta)$.

(Meerut 2007B; Avadh 05, 12; Rohilkhand 12)

7. Find the intrinsic equation of $r = a\, e^{\theta \cot \alpha}$, where s is measured from the point $(a, 0)$.

8. Find the intrinsic equation of the spiral $r = a\theta$, the arc being measured from the pole.

9. Find the intrinsic equation of the curve $p^2 = r^2 - a^2$.

10. In the four-cusped astroid $x^{2/3} + y^{2/3} = a^{2/3}$, show that

(i) $s = \frac{3}{4} a \cos 2\psi$, s being measured from the vertex;

(ii) $s = \frac{3}{2} a \sin^2 \psi$, s being measured from the cusp on x-axis; **(Purvanchal 2014)**

(iii) whole length of the curve is $6a$.

11. Find the cartesian equation of the curve whose intrinsic equation is $s = c \tan \psi$ when it is given that at $\psi = 0, x = 0$ and $y = c$.

Answers 4

2. $s = a \cot \psi \operatorname{cosec} \psi + a \log (\cot \psi + \operatorname{cosec} \psi), a\{\sqrt{2} + \log (1 + \sqrt{2})\}$
4. $s = c \tan \psi$
6. $s = 8a \sin^2 \frac{1}{6} \psi$
7. $s = a \sec \alpha \, [e^{(\psi - \alpha) \cot \alpha} - 1]$
8. $s = \frac{1}{2} a \, [\theta \sqrt{(1 + \theta^2)} + \log \{\theta + \sqrt{(1 + \theta^2)}\}]$, where $\psi = \theta + \tan^{-1} \theta$
9. $s = \frac{1}{2} a\psi^2$
11. $y = c \cosh (x / c)$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If the equations of the curve be given in the parametric form $x = f(t)$, $y = \phi(t)$, and the arc length s is measured in the direction of t increasing, then on integrating between the proper limits, the required length s is given as

(a) $\int_{t_1}^{t_2} \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}\, dt$
(b) $\int_{t_1}^{t_2} \sqrt{\left\{\left(\frac{dx}{dt}\right) + \left(\frac{dy}{dt}\right)\right\}}\, dt$

(c) $\int_{t_1}^{t_2} \sqrt{\left\{\left(\frac{dx}{dt}\right)^3 + \left(\frac{dy}{dt}\right)^3\right\}}\, dt$
(d) $\int_{t_1}^{t_2} \sqrt{\left\{\left(\frac{dx}{dt}\right)^{1/2} + \left(\frac{dy}{dt}\right)^{1/2}\right\}}\, dt$

(Meerut 2003)

2. For the curve $r = a(1 + \cos\theta)$, $ds/d\theta$ is

(a) $2\cos\frac{1}{2}\theta$
(b) $2a\cos\frac{1}{2}\theta$

(c) $a\cos\frac{1}{2}\theta$
(d) $\frac{3}{2}a\cos\frac{1}{2}\theta$

(Kumaun 2011)

3. If $x = a\cos^3 t$, $y = a\sin^3 t$, then $\left(\frac{ds}{dt}\right)^2$ is

(a) $(a\sin t\cos t)^2$
(b) $(\sin t\cos t)^2$

(c) $(3a\sin t\cos t)^2$
(d) $3a\sin t\cos t$

4. The entire length of the cardioid $r = a(1 + \cos\theta)$ is

(a) $8a$
(b) $4a$

(c) $6a$
(d) $2a$

5. Rectification is the process of evaluating the:

(a) double integrals
(b) multiple integrals

(c) the length of arcs of plane curve

(d) the area under plane curve

(Kumaun 2007, 08, 09)

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. The process of finding the length of an arc of a curve between two given points is called …….

2. The arc length of the curve $y = f(x)$ included between two points for which $x = a$ and $x = b$ $(b > a)$ is …… .

3. The arc length of the curve $y = \frac{1}{2} x^2 - \frac{1}{4} \log x$ from $x = 1$ to $x = 2$ is …… .

4. If $r = a\, e^{\theta \cot \alpha}$, then $ds =$ …… . (Meerut 2001, 03)

5. $\frac{ds}{dr} = \sqrt{\ldots\ldots}$. (Meerut 2001)

6. The length of an arch of the cycloid whose equations are

$$x = a(t - \sin t),\; y = a(1 - \cos t) \text{ is } \ldots\ldots$$

True or False

Write 'T' for true and 'F' for false statement.

1. The length of the arc of the curve $x = f(y)$ between $y = a$ and $y = b, (b > a)$ is equal to $\int_a^b \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}\, dy$.

2. The relation between s and ψ for any curve is called its polar equation.

3. If the equation of the curve be $\theta = f(r)$, then the arc length from $r = r_1$ to $r = r_2$ is given by $\int_{r_1}^{r_2} \sqrt{\left\{1 + \left(r \frac{d\theta}{dr}\right)^2\right\}}\, dr$.

4. If the equation of the curve be $r = f(\theta)$, then the arc length from $\theta = \theta_1$ to $\theta = \theta_2$ is given by $\int_{\theta_1}^{\theta_2} \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\, dr$. (Meerut 2003)

5. The whole length of curve $x^{2/3} + y^{2/3} = a^{2/3}$ is $8a$. (Agra 2002)

Answers

Multiple Choice Questions

1. (a) 2. (b) 3. (c) 4. (a) 5. (c)

Fill in the Blank(s)

1. rectification
2. $\int_a^b \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx$
3. $\frac{3}{2} + \frac{1}{4} \log 2$
4. $r \operatorname{cosec} \alpha \, d\theta$
5. $\left\{1 + \left(r \frac{d\theta}{dr}\right)^2\right\}$
6. $8a$

True or False

1. *T* 2. *F* 3. *T* 4. *F* 5. *F*

□

Chapter 9

Volumes and Surfaces of Solids of Revolution

1 Revolution

Solid of revolution: If a plane area is revolved about a fixed line lying in its own plane, then the body so generated by the revolution of the plane area is called a solid of revolution.

Surface of revolution: If a plane curve is revolved about a fixed line lying in its own plane, then the surface generated by the perimeter of the curve is called a surface of revolution.

Axis of revolution: The fixed straight line, say AB, about which the area revolves is called the axis of revolution or axis of rotation.

2 Volumes of Solids of Revolution

(a) The axis of rotation being x-axis.

If a plane area bounded by the curve $y = f(x)$, the ordinates $x = a, x = b$ and the x-axis revolves about the x-axis then the volume of the solid thus generated is

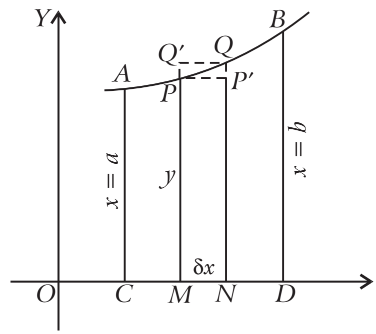
$$\int_a^b \pi y^2 \, dx = \int_a^b \pi [f(x)]^2 \, dx,$$

where $y = f(x)$ is a finite, continuous and single valued function of x in the interval $a \le x \le b$.

Or

The volume of the solid generated by the revolution of the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$, $x = b$ about the x-axis is $\int_a^b \pi y^2 \, dx$.

Proof: Let AB be the arc of the curve $y = f(x)$ included between the ordinates $x = a$ and $x = b$. It is being assumed that the curve does not cut the x-axis and $f(x)$ is a continuous function of x in the interval (a, b).



Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be any two neighbouring points on the curve $y = f(x)$. Draw the ordinates PM and QN. Also draw PP' and QQ' perpendiculars to these ordinates.

Let V denote the volume of the solid generated by the revolution of the area $ACMP$ about the x-axis and let the volume of revolution obtained by revolving the area $ACNQ$ about x-axis be $V + \delta V$, so that volume of the solid generated by the revolution of the strip $PMNQ$ about the x-axis is δV.

Now $PM = y, QN = y + \delta y$ and $MN = (x + \delta x) - x = \delta x$.

Then the volume of the solid generated by revolving the area $PMNP' = \pi y^2 \delta x$

and the volume of the solid generated by revolving the area $Q'MNQ = \pi(y + \delta y)^2 \delta x$.

Also the volume of the solid generated by the revolution of the area $PMNQP$ (*i.e.*, the volume δV) lies between the volumes of the right circular cylinders generated by the revolution of the areas $PMNP'P$ and $MNQQ'$ *i.e.*,

$$\delta V \text{ lies between } \pi y^2 \delta x \text{ and } \pi(y + \delta y)^2 \delta x$$

or $$(\delta V / \delta x) \text{ lies between } \pi y^2 \text{ and } \pi(y + \delta y)^2$$

i.e., $$\pi y^2 < (\delta V / \delta x) < \pi(y + \delta y)^2.$$

In the limiting position as $Q \to P$, $\delta x \to 0$ (and therefore $\delta y \to 0$), we have

$$dV / dx = \pi y^2$$

or $$dV = \pi y^2 dx.$$

Hence $$\int_a^b \pi y^2 dx = \int_a^b dV = [V]_{x=a}^{x=b}$$

$$= (\text{value of } V \text{ for } x = b) - (\text{value of } V \text{ for } x = a)$$

$$= \text{volume generated by the area } ACDB - 0$$

= volume of the solid generated by the revolution of the given area $ACDB$ about the axis of x.

$\therefore$ the required volume $= \pi \int_a^b y^2 dx.$

(Meerut 2003)

(b) The axis of rotation being y-axis:

Similarly, it can be shown that the *volume of the solid generated by the revolution about y-axis of the area between the curve $x = f(y)$, the y-axis and the two abscissae $y = a$ and $y = b$ is given by*

$$\int_a^b \pi x^2 dy.$$

Remarks:

(i) If the given curve is symmetrical about x-axis and we have to find the volume generated by the revolution of the area about x-axis, then in such case we shall revolve only one of the two symmetrical areas and **shall not double it** as in the case of area or length. Obviously each of the two symmetrical parts will generate the same volume.

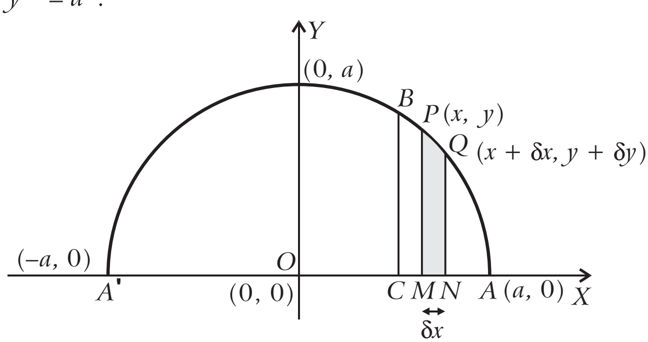
(ii) If the curve is symmetrical about x-axis and it is required to find the volume generated by the revolution of the area about y-axis, then the volume generated **will be twice** the volume generated by half of the symmetrical portion of the curve.

Illustrative Examples

Example 1: *Show that the volume of a sphere of radius a is* $\frac{4}{3} \pi a^3$.

(Bundelkhand 2010; Avadh 10)

Solution: The sphere is generated by the revolution of a semi-circular area about its bounding diameter. The equation of the generating circle of radius a and centre as origin is $x^2 + y^2 = a^2$.



Let AA' be the bounding diameter about which the semi-circle revolves.

Take an elementary strip $PMNQ$ where P is the point (x, y) and Q is the point $(x + \delta x, y + \delta y)$.

We have $\quad PM = y$

and $\quad MN = \delta x$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the diameter AA' is

$$= \pi \,.\, PM^2 . MN = \pi\, y^2\, \delta x = \pi\,(a^2 - x^2)\,\delta x\,.$$

Also the semi-circle is symmetrical about the y-axis and for the portion of the curve lying in the first quadrant x varies from 0 to a .

$\therefore$ the required volume of the sphere

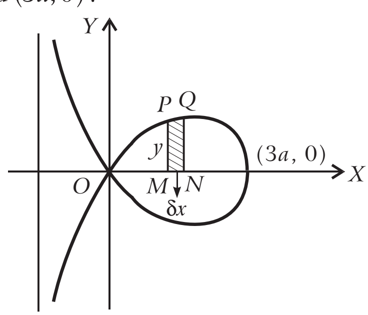
$$= 2\int_0^a \pi\,(a^2 - x^2)\,dx = 2\pi\left[a^2\,x - \frac{1}{3}\,x^3\right]_0^a$$

$$= 2\pi\,[a^3 - \frac{1}{3}\,a^3] = \frac{4}{3}\,\pi a^3.$$

Example 2: *The curve $y^2\,(a + x) = x^2\,(3a - x)$ revolves about the axis of x .Find the volume generated by the loop.* **(Meerut 2004; Bundelkhand 05)**

Solution: The given curve is $y^2\,(a + x) = x^2\,(3a - x)$ . ...(1)

It is symmetrical about x-axis. Putting $y = 0$ in (1), we get $x = 0$ and $x = 3a$ *i.e.*, a loop is formed between $(0, 0)$ and $(3a, 0)$.



The volume generated by the revolution of the whole loop about x-axis is the same as the volume generated by the revolution of the upper half of the loop about x-axis.

Take an elementary strip $PMNQ$ where P is the point (x, y) and Q is the point $(x + \delta x,\, y + \delta y)$. We have $PM = y$ and $MN = \delta x$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the axis of x is $= \pi PM^2\,.\,MN = \pi\,y^2\,\delta x$.

$\therefore$ the required volume generated by the loop

$$= \int_0^{3a} \pi y^2\,dx = \pi\int_0^{3a} \frac{x^2\,(3a - x)}{a + x}\,dx \qquad \text{[From (1)]}$$

$$= \pi\int_0^{3a}\left\{- x^2 + 4ax - 4a^2 + \frac{4a^3}{x + a}\right\} dx, \text{ dividing the Nr. by the Dr.}$$

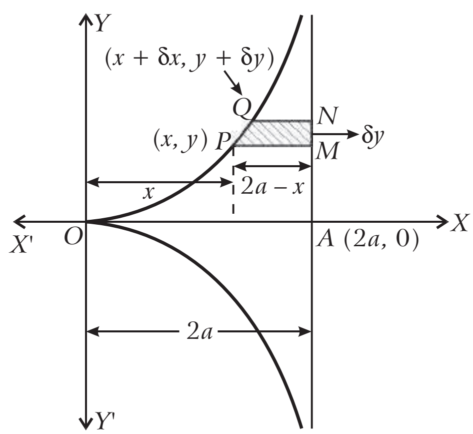
$$= \pi \left[-\frac{x^3}{3} + \frac{4ax^2}{2} - 4a^2 x + 4a^3 \log (x + a) \right]_0^{3a}$$

$$= \pi \left[-9a^3 + 18a^3 - 12a^3 + 4a^3 (\log 4a - \log a) \right]$$

$$= \pi \left[-3a^3 + 4a^3 \log 4 \right] = \pi a^3 \left[8 \log 2 - 3 \right].$$

Example 3: *Find the volume of the solid generated by the revolution of the cissoid* $y^2 (2a - x) = x^3$ *about its asymptote.* **(Meerut 2007; Kanpur 14)**

Solution: The given curve is $y^2 (2a - x) = x^3$. Its shape is as shown in the figure. Equating to zero the coefficient of highest power of y, the asymptote parallel to the axis of y is $x = 2a$. Take an elementary strip $PMNQ$ perpendicular to the asymptote $x = 2a$ where P is the point (x, y) and Q is the point $(x + \delta x, y + \delta y)$.



We have $\quad PM = 2a - x \quad$ and $\quad MN = \delta y$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the line $x = 2a$ is

$$= \pi \, . \, PM^2 \, . \, MN = \pi (2a - x)^2 \, \delta y.$$

The given curve is symmetrical about x-axis and for the portion of the curve above x-axis y varies from 0 to ∞.

$\therefore$ the required volume $= 2 \int_{y=0}^{\infty} \pi (2a - x)^2 \, dy.$ ...(1)

From the given equation of the curve $y^2 (2a - x) = x^3$ we observe that the value of x cannot be easily found in terms of y. Hence for the sake of integration we change the independent variable from y to x. **(Note)**

The curve is $y^2 = \dfrac{x^3}{2a - x}$;

$$\therefore \qquad 2y \frac{dy}{dx} = \frac{(2a - x) \, . \, 3x^2 - x^3 (-1)}{(2a - x)^2} = \frac{2 (3a - x) \, x^2}{(2a - x)^2}$$

or $$dy = \frac{(3a-x)\,x^2}{(2a-x)^2} \cdot \frac{\sqrt{(2a-x)}}{x\sqrt{x}}\,dx = \frac{(3a-x)\sqrt{x}\sqrt{(2a-x)}}{(2a-x)^2}\,dx.$$

Also when $y = 0, x = 0$ and when $y \to \infty, x \to 2a$.

Hence from (1), the required volume

$$= 2\pi \int_{x=0}^{2a} (2a-x)^2 \left[\frac{(3a-x)\sqrt{x}\sqrt{(2a-x)}}{(2a-x)^2}\right] dx$$

$$= 2\pi \int_0^{2a} (3a-x)\sqrt{x}\sqrt{(2a-x)}\,dx.$$

Now put $x = 2a\sin^2\theta$ so that $dx = 4a\sin\theta\cos\theta\,d\theta$. When $x = 0, \theta = 0$ and when $x = 2a$, $\theta = \pi/2$. Therefore the required volume

$$= 2\pi \int_0^{\pi/2} (3a - 2a\sin^2\theta)\sqrt{(2a)}\sin\theta\sqrt{[2a(1-\sin^2\theta)]} \times 4a\sin\theta\cos\theta\,d\theta$$

$$= 16\pi a^3 \int_0^{\pi/2} (3\sin^2\theta\cos^2\theta - 2\sin^4\theta\cos^2\theta)\,d\theta$$

$$= 16\pi a^3 \left[\frac{3\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(3)} - \frac{2\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(4)}\right]$$

$$= 16\pi a^3 \left[\frac{3\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{2.2.1} - \frac{2\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{2.3.2.1}\right]$$

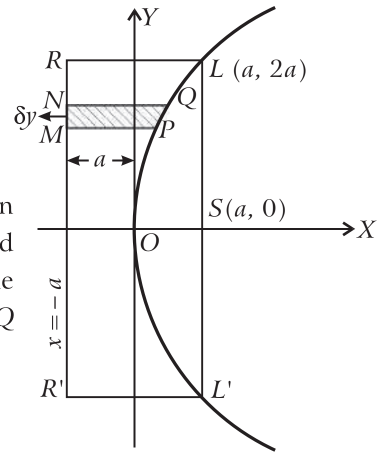
$$= 16\pi a^3 \left[\frac{3\pi}{16} - \frac{\pi}{16}\right] = 2\pi^2 a^3.$$

Note: If the given curve is $y^2(a-x) = x^3$, then the required volume can be obtained by putting a for $2a$ in the above Exercise. The volume so obtained is $\frac{1}{4}\pi^2 a^3$.

Remark: *When we are to revolve an area about a line which is neither the x-axis nor the y-axis we must take an elementary strip which is perpendicular to the line of revolution as explained in the above example.*

Example 4: *The area between a parabola and its latus rectum revolves about the directrix. Find the ratio of the volume of the ring thus obtained to the volume of the sphere whose diameter is the latus rectum.*

Solution: Let the parabola be $y^2 = 4ax$. Then the directrix is the line $x = -a$. Let LL' be the latus rectum. The area $LOL'SL$ is revolved about the directrix. The volume of the ring thus obtained = (the volume V_1 of the cylinder formed by the revolution of the rectangle $LL'R'R$ about the directrix) – (the volume V_2 of the reel formed by the revolution of the arc LOL' about the directrix).



Now the volume V_1 of the cylinder

$$= \pi r^2 h = \pi (LR)^2 . LL'$$

$$= \pi (2a)^2 . 4a = 16\pi a^3.$$

To find the volume V_2 of the reel consider an elementary strip $PMNQ$ where $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ are two neighbouring points on the arc OL and PM, QN are perpendiculars from P and Q on the directrix.

We have $\quad PM = a + x$ and $MN = \delta y$.

$\therefore$ the volume V_2 of the reel

$$= 2\int_0^{2a} \pi (a + x)^2 \, dy \qquad \text{[By symmetry about } x\text{-axis]}$$

$$= 2\int_0^{2a} \pi (a^2 + 2ax + x^2) \, dy = 2\pi \int_0^{2a} \left(a^2 + 2a \cdot \frac{y^2}{4a} + \frac{y^4}{16a^2} \right) dy$$

$$[\because \ x = y^2/4a]$$

$$= 2\pi \left[a^2 y + \frac{1}{2} \frac{y^3}{3} + \frac{1}{16a^2} \cdot \frac{y^5}{5} \right]_0^{2a}$$

$$= 2\pi \left[2a^3 + \frac{4}{3} a^3 + \frac{2}{5} a^3 \right] = 2\pi a^3 \cdot \frac{56}{15} = \frac{112 \pi a^3}{15}.$$

$\therefore$ Volume of the ring

$$= \text{volume of the cylinder} - \text{volume of the reel}$$

$$= V_1 - V_2 = 16\pi a^3 - \frac{112}{15} \pi a^3 = \frac{128}{15} \pi a^3.$$

Volume of the sphere whose diameter is the latus rectum $4a$ *i.e.*, the radius is $2a$

$$= \frac{4}{3} \pi r^3 = \frac{4}{3} \pi (2a)^3 = \frac{32}{3} \pi a^3.$$

$\therefore$ the required ratio $= \dfrac{128\pi a^3 / 15}{32\pi a^3 / 3} = \dfrac{4}{5}.$

Comprehensive Exercise 1

1. (i) Find the volume of a hemisphere.

(ii) Find the volume of a spherical cap of height h cut off from a sphere of radius a.

(Kanpur 2010)

2. (i) A segment is cut off from a sphere of radius a by a plane at a distance $\frac{1}{2}a$ from the centre. Show that the volume of the segment is 5/32 of the volume of the sphere.

(ii) The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the volume of the reel thus generated.

3. Prove that the volume of the solid generated by the revolution of an ellipse round its minor axis is a mean proportional between those generated by the revolution of the ellipse and of the auxiliary circle about the major axis. **(Rohilkhand 2010)**

4. (i) Find the volume of the solid generated by the revolution of an arc of the catenary $y = c \cosh (x / c)$ about the x-axis. **(Meerut 2009B; Purvanchal 11)**

(ii) Find the volume of the solid generated by the revolution of the curve $y = a^3 / (a^2 + x^2)$ about its asymptote. **(Meerut 2009)**

5. If the hyperbola $x^2/a^2 - y^2/b^2 = 1$ revolves about the x-axis, show that the volume included between the surface thus generated, the cone generated by the asymptotes and two planes perpendicular to the axis of x, at a distance h apart, is equal to that of a circular cylinder of height h and radius b.

6. (i) Find the volume formed by the revolution of the loop of the curve $y^2 (a + x) = x^2 (a - x)$ about the axis of x. **(Kanpur 2008)**

(ii) Find the volume of the solid generated by the revolution of the loop of the curve $y^2 = x^2 (a - x)$ about the axis of x. **(Kanpur 2011)**

7. Show that the volume of the solid generated by the revolution of the upper half of the loop of the curve $y^2 = x^2 (2 - x)$ about x-axis is $\frac{4}{3}\pi$. **(Meerut 2005)**

8. The area of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ lying in the first quadrant revolves about x-axis. Find the volume of the solid generated. **(Agra 2014)**

9. Find the volume of the solid obtained by revolving the loop of the curve

$$a^2 y^2 = x^2 (2a - x)(x - a) \text{ about } x\text{-axis.}$$

10. A basin is formed by the revolution of the curve $x^3 = 64y, (y > 0)$ about the axis of y. If the depth of the basin is 8 inches, how many cubic inches of water it will hold?

11. Show that the volume of the solid generated by the revolution of the curve $(a - x) y^2 = a^2 x$, about its asymptote is $\frac{1}{2}\pi^2 a^3$.

(Meerut 2004B, 06B; Kumaun 07, 08, 12; Rohilkhand 12)

12. The figure bounded by a quadrant of a circle of radius a and tangents at its extremities revolves about one of the tangents. Prove that the volume of the solid generated is $\left(\frac{5}{3} - \frac{1}{2}\pi\right)\pi a^3$.

13. The area cut off from the parabola $y^2 = 4ax$ by the chord joining the vertex to an end of the latus rectum is rotated through four right angles about the chord. Find the volume of the solid generated. **(Rohilkhand 2008; Bundelkhand 09)**

Answers 1

1. (i) $\frac{2}{3}\pi a^3$ (ii) $\pi h^2\left[a - \frac{1}{3}h\right]$

2. (ii) $\frac{4}{5}\pi a^3$ 4. (i) $\frac{\pi c^2}{2}\left[x + \frac{c}{2}\sinh\frac{2x}{c}\right]$ (ii) $\frac{\pi^2 a^3}{2}$

6. (i) $2a^3\pi\left[\log 2 - \frac{2}{3}\right]$ (ii) $\frac{1}{12}\pi a^4$ 8. $\frac{16}{105}\pi a^3$

9. $\frac{23}{60}\pi a^3$ 10. $\frac{1536}{5}\pi$ cubic inches 13. $\frac{2}{75}\sqrt{5}\pi a^3$

3 Volume of a Solid Revolution when the Equations of the Generating Curve are given in Parametric Form

(i) If the curve is given by the parametric equations, say $x = \phi(t)$, $y = \psi(t)$, then the volume of the solid generated by the revolution about x-axis of the area bounded by the curve, the axis of x and the ordinates at the points when $t = a$ and $t = b$ is

$$= \int_a^b \pi y^2 \frac{dx}{dt}\, dt = \pi \int_a^b \{\psi(t)\}^2 \phi'(t)\, dt.$$

(ii) The volume of the solid generated by the revolution about y-axis of the area between the curve $x = \phi(t)$, $y = \psi(t)$, the y-axis and the absissae at the points where $t = a, t = b$ is

$$= \int_a^b \pi x^2 \frac{dy}{dt}\, dt = \pi \int_a^b \{\phi(t)\}^2 \psi'(t)\, dt.$$

Illustrative Examples

Example 5: *Find the volume of the solid formed by revolving the cycloid*

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)$$

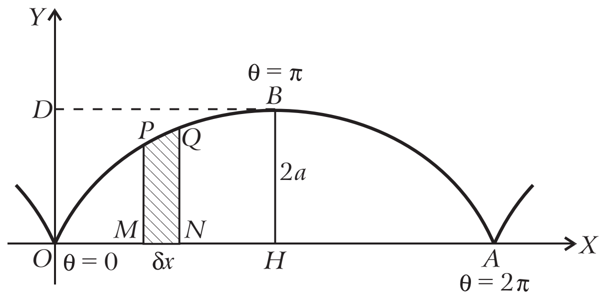
(i) about its base *(ii) about the y-axis.*

Solution: The given equations of the cycloid are

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta)\,. \qquad \ldots(1)$$

(i) The arc OBA is revolved about the base *i.e.*, the x-axis. For the arc OBA, θ varies from 0 to 2π and at B, $\theta = \pi$.

Take an elementary strip $PMNQ$ where P is the point (x, y) and Q is the point $(x + \delta x, y + \delta y)$.



We have $\quad PM = y$ and $MN = \delta x$.

Now the volume of the elementary disc formed by revolving the strip $PMNQ$ about the base (*i.e.*, the x-axis) is

$$= \pi \, PM^2 \, . \, MN = \pi \, y^2 \, \delta x .$$

Now the cycloid is symmetrical about the line BH.

$\therefore$ the required volume $= 2 \int \pi y^2 dx$, the limits of integration being extended from O to B

$$= 2\pi \int_{\theta=0}^{\pi} y^2 \frac{dx}{d\theta} d\theta = 2\pi \int_0^{\pi} a^2 (1 - \cos \theta)^2 \, a (1 - \cos \theta) \, d\theta \text{ [From (1)]}$$

$$= 2\pi \int_0^{\pi} a^3 (1 - \cos \theta)^3 \, d\theta$$

$$= 2\pi \, a^3 \int_0^{\pi} \left(2 \sin^2 \frac{\theta}{2}\right)^3 d\theta = 16\pi \, a^3 \int_0^{\pi} \sin^6 \frac{\theta}{2} \, d\theta$$

$$= 32\pi a^3 \int_0^{\pi/2} \sin^6 \phi \, d\phi, \text{ putting } \frac{\theta}{2} = \phi \text{ so that } d\theta = 2 \, d\phi$$

$$= 32\pi \, a^3 \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{1}{2} \pi = 5\pi^2 a^3 .$$

(ii) When the curve revolves about y-axis, the required volume of the solid generated

= the volume generated by the revolution of the area $OABDO$ about y-axis

– the volume generated by the revolution of the area $OBDO$ about the y-axis. ...(2)

Also at $A, \theta = 2\pi$; at $B, \theta = \pi$ and at $O, \theta = 0$.

Now the area *OABD* is bounded by the arc *AB* of the cycloid and the axis of y. Therefore volume of the solid generated by the revolution of the area *OABDO* about y-axis

$$= \int_{\theta=2\pi}^{\pi} \pi x^2 \, dy = \int_{\theta=2\pi}^{\pi} \pi x^2 \frac{dy}{d\theta} \, d\theta$$

$$= \pi \int_{\theta=2\pi}^{\pi} a^2 (\theta - \sin\theta)^2 \, a \sin\theta \, d\theta \qquad \text{[From (1)]}$$

$$= \pi \int_{\theta=2\pi}^{\pi} a^2 (\theta^2 - 2\theta \sin\theta + \sin^2\theta) \, a \sin\theta \, d\theta$$

$$= \pi a^3 \int_{\theta=2\pi}^{\pi} (\theta^2 \sin\theta - 2\theta \sin^2\theta + \sin^3\theta) \, d\theta$$

$$= \pi a^3 \int_{\theta=2\pi}^{\pi} [\theta^2 \sin\theta - \theta(1 - \cos 2\theta) + \frac{1}{4}(3\sin\theta - \sin 3\theta)] \, d\theta \qquad \textbf{(Note)}$$

$$= \pi a^3 \Big[\theta^2 . (-\cos\theta) - 2\theta(-\sin\theta) + 2\cos\theta - \frac{1}{2}\theta^2 + \theta(\frac{1}{2}\sin 2\theta)$$
$$- 1(-\frac{1}{4}\cos 2\theta) - \frac{3}{4}\cos\theta + \frac{1}{12}\cos 3\theta\Big]_{2\pi}^{\pi},$$

the values of the integrals $\int \theta^2 \sin\theta \, d\theta$ and $\int \theta \cos 2\theta \, d\theta$ have been written after applying integration by parts

$$= \pi a^3 \left[\left(\pi^2 - 2 - \frac{1}{2}\pi^2 + \frac{1}{4} + \frac{3}{4} - \frac{1}{12}\right) - \left(-4\pi^2 + 2 - 2\pi^2 + \frac{1}{4} - \frac{3}{4} + \frac{1}{12}\right)\right]$$

$$= \pi a^3 \left[\frac{13}{2}\pi^2 - \frac{8}{3}\right]. \qquad ...(3)$$

Again volume of the solid generated by the revolution of the area *OBDO* about y-axis

$$= \int_{\theta=0}^{\pi} \pi x^2 dy = \int_{\theta=0}^{\pi} \pi x^2 \frac{dy}{d\theta} \, d\theta$$

$$= \pi \int_0^{\pi} a^2 (\theta - \sin\theta)^2 . a \sin\theta \, d\theta$$

$$= \pi a^3 \int_0^{\pi} (\theta^2 - 2\theta \sin\theta + \sin^2\theta) \sin\theta \, d\theta$$

$$= \pi a^3 \int_0^{\pi} (\theta^2 \sin\theta - 2\theta \sin^2\theta + \sin^3\theta) \, d\theta$$

$$= \pi a^3 \int_0^{\pi} \left[\theta^2 \sin\theta - \theta(1 - \cos 2\theta) + \frac{1}{4}(3\sin\theta - \sin 3\theta)\right] d\theta$$

$$= \pi a^3 \Big[\theta^2 (-\cos\theta) - 2\theta(-\sin\theta) + 2\cos\theta - \frac{1}{2}\theta^2 + \theta(\frac{1}{2}\sin 2\theta)$$
$$- 1(-\frac{1}{4}\cos 2\theta) - \frac{3}{4}\cos\theta + \frac{1}{12}\cos 3\theta\Big]_0^{\pi}$$

$$= \pi a^3 \left[\left(\pi^2 - 2 - \frac{1}{2}\pi^2 + \frac{1}{4} + \frac{3}{4} - \frac{1}{12}\right) - \left(2 + \frac{1}{4} - \frac{3}{4} + \frac{1}{12}\right)\right]$$

$$= \pi a^3 \left(\frac{1}{2}\pi^2 - \frac{8}{3}\right). \qquad ...(4)$$

$\therefore$ from (2), the required volume = (3) – (4)

$$= \pi a^3 \left[\frac{13}{2}\pi^2 - \frac{8}{3}\right] - \pi a^3 \left[\frac{1}{2}\pi^2 - \frac{8}{3}\right] = \pi\, a^3\, [6\pi^2] = 6\pi^3 a^3.$$

Example 6: *Find the volume of the solid generated by the revolution of the tractrix* $x = a \cos t + \frac{1}{2} a \log \tan^2 (t/2),\ y = a \sin t$ *about its asymptote.*

(Meerut 2000, 05B; Garhwal 03; Rohilkhand 06; Avadh 09, 11; Kashi 12; Purvanchal 14)

Solution: The given curve is

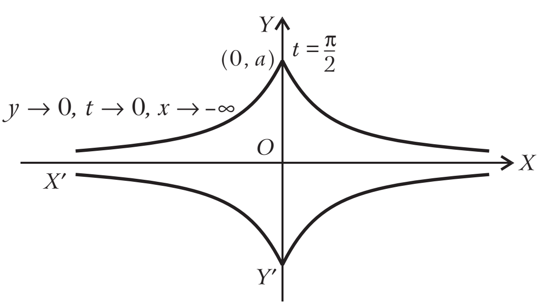
$$x = a \cos t + \frac{1}{2} a \log \tan^2 (t/2),\ y = a \sin t. \qquad ...(1)$$

$$\therefore \qquad \frac{dx}{dt} = -a \sin t + \frac{1}{2} a \cdot \frac{1}{\tan^2 (t/2)} \cdot 2 \tan (t/2) \sec^2 (t/2) \cdot \frac{1}{2}$$

$$= -a \sin t + \frac{a}{2 \sin (t/2) \cos (t/2)} = -a \sin t + \frac{a}{\sin t}$$

$$= a\,\frac{(1 - \sin^2 t)}{\sin t} = a\,\frac{\cos^2 t}{\sin t} \qquad ...(2)$$

Now the given curve is symmetrical about both the axes and the asymptote is the line $y = 0$ *i.e.*, x-axis.



For the portion of the curve lying in the second quadrant y varies from a to 0, t varies from $\pi/2$ to 0 and x varies from 0 to $-\infty$.

$\therefore$ the required volume

$$= 2\int_{-\infty}^{0} \pi\, y^2 dx = 2\int_{0}^{\pi/2} \pi y^2 \frac{dx}{dt} \cdot dt$$

$$= 2\pi \int_{0}^{\pi/2} a^2 \sin^2 t \,.\, \frac{a \cos^2 t}{\sin t}\, dt \qquad \text{[From (1) and (2)]}$$

$$= 2\pi a^3 \int_{0}^{\pi/2} \cos^2 t \sin t\, dt = 2\pi a^3 \frac{1}{3.1} = \frac{2}{3}\pi a^3.$$

Comprehensive Exercise 2

1. Find the volume of the solid generated by the revolution of the cycloid

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta), -\pi \le \theta \le \pi,$$

(i) about the x-axis, (ii) about the base.

2. Show that the volume of the solid generated by the revolution of the cycloid

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta), 0 \le \theta \le \pi.$$

about the y-axis is $\pi a^3 \left(\frac{3}{2}\pi^2 - \frac{8}{3}\right)$.

3. Prove that the volume of the reel formed by the revolution of the cycloid

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$$

about the tangent at the vertex is $\pi^2 a^3$. **(Kumaun 2013)**

4. Prove that the volume of the solid generated by the revolution about the x-axis of the loop of the curve $x = t^2,\ y = t - \frac{1}{3}t^3$ is $\frac{3}{4}\pi$.

5. Find the volume of the spindle shaped solid generated by revolving the astroid

$$x^{2/3} + y^{2/3} = a^{2/3} \text{ about the } x\text{-axis.}$$

6. Find the volume of the solid generated by the revolution of the cissoid

$$x = 2a\sin^2 t,\ y = 2a\sin^3 t / \cos t \text{ about its asymptote.}$$

(Kanpur 2006; Bundelkhand 14)

Answers 2

1. (i) $\pi^2 a^3$ (ii) $5\pi^2 a^3$ 5. $\frac{32}{105}\pi a^3$ 6. $2\pi^2 a^3$

4 Volume of Solid of Revolution when the Equation of the Generating Curve is given in Polar Co-ordinates

If the equation of the generating curve is given in polar co-ordinates, say $r = f(\theta)$, and the curve revolves about the axis of x, the volume generated

$$s = \pi\int_{x=a}^{b} y^2\,dx = \pi\int_{\theta=\alpha}^{\beta} y^2 \frac{dx}{d\theta}\,d\theta,$$

where α and β are the values of θ at the points where $x = a$ and $x = b$ respectively. Now $x = r\cos\theta$ and $y = r\sin\theta$. Therefore the volume

$$= \pi \int_{\theta=\alpha}^{\beta} r^2 \sin^2 \theta \frac{d}{d\theta}(r \cos \theta)\, d\theta,$$

in which the value of r in terms of θ must be substituted from the equation of the given curve.

A similar procedure can be adopted in case the curve revolves about the axis of y.

Alternative method in the case of polar curves:

The volume of the solid generated by the revolution of the area boudned by the curve $r = f(\theta)$ and radii vectors $\theta = \theta_1, \theta = \theta_2$

(i) about the initial line $\theta = 0$ (*i.e.*, the x-axis) is $\int_{\theta_1}^{\theta_2} \frac{2}{3} \pi r^3 \sin \theta \, d\theta,$

(ii) about the line $\theta = \pi / 2$ (*i.e.*, the y-axis) is $\int_{\theta_1}^{\theta_2} \frac{2}{3} \pi r^3 \cos \theta \, d\theta,$

(iii) about any line $(\theta = \gamma)$ is $\int_{\theta_1}^{\theta_2} \frac{2}{3} \pi r^3 \sin (\theta - \gamma) \, d\theta,$

where in each of the above three formulae the value of r in terms of θ must be substituted from the equation of the given curve.

Note: The above results are important and should be committed to memory.

5 Volume of the Solid Generated by the Revolution when the Axis of Rotation being any Line

If, however, the axis of rotation is neither x-axis nor y-axis, but is any other line CD, then *the volume of the solid generated by the revolution about CD of the area bounded by the curve AB, the axis CD and the perpendiculars AC, BD on the axis is*

$$\int_{OC}^{OD} \pi (PM)^2 \, d(OM),$$

where PM is the perpendicular drawn from any point P on the curve to the axis of rotation and O is some fixed point on the axis of rotation.

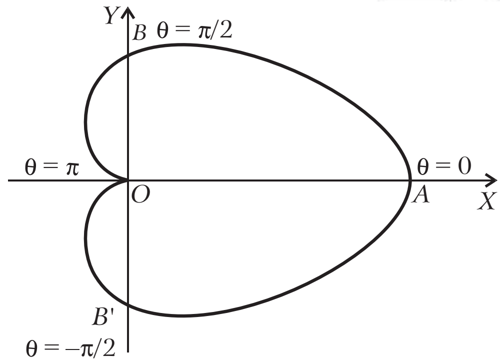
Illustrative Examples

Example 7: *The cardioid $r = a(1 + \cos \theta)$ revolves about the initial line. Find the volume of the solid thus generated.* **(Kumaun 2000, 10; Meerut 01, 03, 07B; Agra 01, 06, 07, 08; Rohilkhand 13, 13B)**

Solution: The given curve is $r = a(1 + \cos \theta)$. ...(1)

It is symmetrical about the initial line. We have $r = 0$ when $\cos \theta = -1$ *i.e.*, $\theta = \pi$.

Also r is maximum when $\cos\theta = 1$ *i.e.*, $\theta = 0$ and then $r = 2a$. As θ increases from 0 to π, r decreases from $2a$ to 0. Hence the shape of the curve is as shown in the figure. For the upper half of the curve, θ varies from 0 to π.



$\therefore$ the required volume

$$= \frac{2}{3}\int_0^{\pi} \pi r^3 \sin\theta \, d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi} a^3 (1+\cos\theta)^3 \sin\theta \, d\theta \qquad \text{[From (1)]}$$

$$= -\frac{2}{3}\pi a^3 \int_0^{\pi} (1+\cos\theta)^3 (-\sin\theta)\, d\theta \qquad \textbf{(Note)}$$

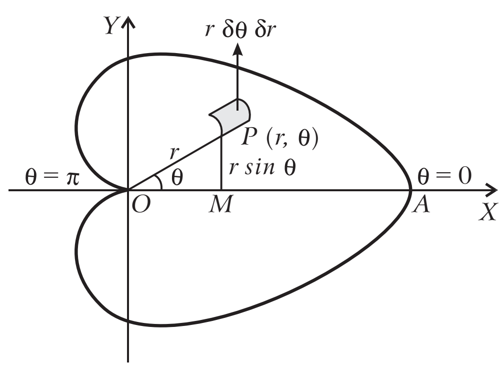
$$= -\frac{2}{3}\pi a^3 \left[\frac{(1+\cos\theta)^4}{4}\right]_0^{\pi}, \text{ using power formula}$$

$$i.e., \int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$= -\frac{1}{6}\pi a^3 (0 - 2^4) = \frac{8}{3}\pi a^3.$$

Aliter: (By double integration)

Take a small element $r\delta\theta\,\delta r$ at any point $P(r,\theta)$ lying within the area of the upper half of the cardioid. Draw PM perpendicular to OX. Then $PM = r\sin\theta$. The volume of the elementary ring formed by revolving the element $r\,\delta\theta\,\delta r$ about OX



$$= 2\pi (r\sin\theta)\, r\,\delta\theta\,\delta r$$

$$= 2\pi r^2 \sin\theta\,\delta\theta\,\delta r.$$

$\therefore$ the required volume formed by revolving the whole cardioid about the initial line

$$= \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} 2\pi r^2 \sin\theta \, d\theta\, dr$$

$$= \int_0^{\pi} 2\pi \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} \sin\theta\, d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi} a^3 (1+\cos\theta)^3 \sin\theta \, d\theta$$

$$= -\frac{2\pi a^3}{3}\int_0^{\pi} (1+\cos\theta)^3 (-\sin\theta)\, d\theta = -\frac{2\pi a^3}{3}\left[\frac{(1+\cos\theta)^4}{4}\right]_0^{\pi}$$

$$= -\frac{2\pi a^3}{3}\cdot\frac{1}{4}[0-2^4] = \frac{2}{3}\cdot\pi a^3\cdot\frac{1}{4}\cdot 16 = \frac{8}{3}\pi a^3.$$

Example 8: *Find the volume of the solid formed by revolving one loop of the curve* $r^2 = a^2\cos 2\theta$ *about the initial line.* **(Rohilkhand 2007B)**

Solution: For the upper half of the loop θ varies from 0 to $\pi/4$. Here the curve is revolving about the initial line (*i.e.*, *x*-axis).

$\therefore$ the required volume $= \frac{2}{3}\pi\int_0^{\pi/4} r^3 \sin\theta\, d\theta$

$$= \frac{2\pi}{3}\int_0^{\pi/4} \{a\sqrt{(\cos 2\theta)}\}^3 \sin\theta\, d\theta \qquad [\because r^2 = a^2\cos 2\theta]$$

$$= \frac{2\pi a^3}{3}\int_0^{\pi/4} (2\cos^2\theta - 1)^{3/2} \sin\theta\, d\theta. \qquad \textbf{(Note)}$$

Put $\sqrt{2}\cos\theta = \sec\phi$ so that $-\sqrt{2}\sin\theta\, d\theta = \sec\phi\tan\phi\, d\phi$.

When $\theta = 0, \phi = \pi/4$ and when $\theta = \pi/4, \phi = 0$.

$\therefore$ the required volume

$$= \frac{2\pi a^3}{3}\int_{\pi/4}^{0} (\sec^2\phi - 1)^{3/2} \frac{(-\sec\phi\tan\phi)}{\sqrt{2}}\, d\phi$$

$$= \frac{\sqrt{2}\pi a^3}{3}\int_0^{\pi/4} \tan^4\phi\sec\phi\, d\phi = \frac{\sqrt{2}\pi a^3}{3}\int_0^{\pi/4} (\sec^2\phi - 1)^2 \sec\phi\, d\phi$$

$$= \frac{\sqrt{2}\pi a^3}{3}\int_0^{\pi/4} (\sec^5\theta - 2\sec^3\phi + \sec\phi)\, d\phi. \qquad \ldots(1)$$

Also we know the reduction formula

$$\int \sec^n\phi\, d\phi = \frac{\sec^{n-2}\phi\tan\phi}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}\phi\, d\phi. \qquad \textbf{[Establish it here]}$$

$$\therefore \qquad \int_0^{\pi/4} \sec^5\phi\, d\phi = \left[\frac{\sec^3\phi\tan\phi}{4}\right]_0^{\pi/4} + \frac{3}{4}\int_0^{\pi/4} \sec^3\phi\, d\phi$$

$$= \frac{\sqrt{2}}{2} + \frac{3}{4}\left\{\left[\frac{\sec\phi\tan\phi}{2}\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \sec\phi\, d\phi\right\}$$

$$= \frac{\sqrt{2}}{2} + \frac{3}{4}\left\{\frac{\sqrt{2}}{2} + \frac{1}{2}[\log(\sec\phi + \tan\phi)]_0^{\pi/4}\right\}$$

$$= \frac{\sqrt{2}}{2} + \frac{3\sqrt{2}}{8} + \frac{3}{8}\log(\sqrt{2}+1) = \frac{7\sqrt{2}}{8} + \frac{3}{8}\log(\sqrt{2}+1)$$

$$\int_0^{\pi/4} \sec^3\phi \, d\phi = \left[\frac{\sec\phi\tan\phi}{2}\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \sec\phi \, d\phi$$

$$= \frac{\sqrt{2}}{2} + \frac{1}{2}\log(\sqrt{2}+1)$$

and $\int_0^{\pi/4} \sec\phi \, d\phi = \log(\sqrt{2}+1)$.

Hence the required volume from (1) is

$$= \frac{\sqrt{2}\pi a^3}{3}\left[\frac{7\sqrt{2}}{8} + \frac{3}{8}\log(\sqrt{2}+1) - 2\left\{\frac{\sqrt{2}}{2} + \frac{1}{2}\log(\sqrt{2}+1)\right\} + \log(\sqrt{2}+1)\right]$$

$$= \frac{\sqrt{2}\pi a^3}{3}\left[\frac{3}{8}\log(\sqrt{2}+1) - \frac{\sqrt{2}}{8}\right]$$

$$= \frac{\pi a^3\sqrt{2}}{24}[3\log(\sqrt{2}+1) - \sqrt{2}].$$

Aliter: The equation of the given curve is

$$r^2 = a^2\cos 2\theta \text{ or } r^4 = a^2 r^2(\cos^2\theta - \sin^2\theta).$$

Changing to cartesians, the equation becomes

$$(x^2+y^2)^2 = a^2(x^2-y^2) \quad \text{or} \quad y^4 + y^2(2x^2+a^2) + x^4 - a^2x^2 = 0$$

Solving for y^2, we have

$$y^2 = [-(2x^2+a^2) \pm \sqrt{\{(2x^2+a^2)^2 - 4(x^4 - a^2x^2)\}}]/2.$$

Neglecting the negative sign because y^2 cannot be –ive, we have

$$y^2 = \frac{-(2x^2+a^2) + \sqrt{(8a^2x^2+a^4)}}{2} = \frac{-(2x^2+a^2) + 2\sqrt{2}a\sqrt{\left(x^2+\frac{1}{8}a^2\right)}}{2}$$

Now for one loop of the given curve x varies from 0 to a.

$\therefore$ the required volume $= \pi\int_0^a y^2 dx$

$$= \frac{\pi}{2}\int_0^a \left[-2x^2 - a^2 + 2\sqrt{2}a\sqrt{\left(x^2+\frac{1}{8}a^2\right)}\right]dx$$

$$= \frac{\pi}{2}\left[-\frac{2}{3}x^3 - a^2x + 2\sqrt{2}a\cdot\frac{x}{2}\sqrt{\left(x^2+\frac{1}{8}a^2\right)} + 2\sqrt{2}a\cdot\frac{1}{16}a^2\log\left\{x+\sqrt{\left(x^2+\frac{1}{8}a^2\right)}\right\}\right]_0^a$$

$$= \frac{\pi}{2}\left[-\frac{2}{3}a^3 - a^3x + 2\sqrt{2}a\cdot\frac{a}{2}\cdot\frac{3a}{2\sqrt{2}} + \frac{1}{8}\sqrt{2}a^3\left\{\log\left(a+\frac{3a}{2\sqrt{2}}\right) - \log\frac{a}{2\sqrt{2}}\right\}\right]$$

$$= \frac{\pi}{2}\left[-\frac{5}{3}a^3 + \frac{3}{2}a^3 + \frac{1}{8}\sqrt{2}a^3 \log\left\{\frac{a(2\sqrt{2}+3)}{2\sqrt{2}} \cdot \frac{2\sqrt{2}}{a}\right\}\right]$$

$$= \frac{\pi}{2}\left[-\frac{1}{6}a^3 + \frac{1}{8}\sqrt{2}a^3 \log(2\sqrt{2}+3)\right]$$

$$= \frac{\pi}{2}\left[-\frac{1}{6}a^3 + \frac{1}{8}\sqrt{2}a^3 \log(\sqrt{2}+1)^2\right]$$

$$= \frac{\pi a^3}{2}\left[2 \cdot \frac{1}{8}\sqrt{2}\log(\sqrt{2}+1) - \frac{1}{6}\right] = \frac{\pi a^3}{2}\left[\frac{1}{4}\sqrt{2}\log(\sqrt{2}+1) - \frac{1}{6}\right]$$

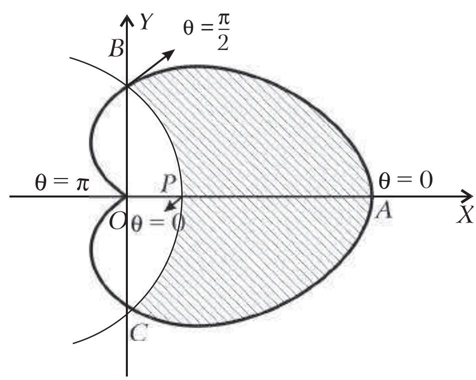
$$= \frac{\pi a^3}{24}[3\sqrt{2}\log(\sqrt{2}+1) - 2] = \frac{\pi a^3\sqrt{2}}{24}[3\log(\sqrt{2}+1) - \sqrt{2}].$$

Example 9: *Show that if the area lying within the cardioid* $r = 2a(1+\cos\theta)$ *and without the parabola* $r(1+\cos\theta) = 2a$ *revolves about the initial line, the volume generated is* $18\pi a^3$. **(Kumaun 2009)**

Solution: The equation of the cardioid is $r = 2a(1+\cos\theta)$, ...(1)

and that of the parabola is $r = 2a/(1+\cos\theta)$. ...(2)

Equating the values of r from (1) and (2), we get



$$2a(1+\cos\theta) = 2a/(1+\cos\theta)$$

or $(1+\cos\theta)^2 = 1$

or $\cos\theta(\cos\theta + 2) = 0.$

Now $\cos\theta \neq -2.$

Therefore $\cos\theta = 0$

i.e., $\theta = \pi/2, -\pi/2.$

Thus the curves (1) and (2) intersect where $\theta = \pi/2$ and $\theta = -\pi/2$.

Also both the curves are symmetrical about the initial line (*i.e.*, x-axis). The required volume is generated by revolving the upper half of the shaded area about the initial line.

$\therefore$ the required volume = (Volume generated by the revolution of the area *OABO* of the cardioid) – (volume generated by the revolution of the area *OPBO* of the parabola)

$$= \frac{2\pi}{3}\int_0^{\pi/2} r^3 \sin\theta\, d\theta - \frac{2\pi}{3}\int_0^{\pi/2} r^3 \sin\theta\, d\theta$$

(for cardioid) (for parabola)

$$= \frac{2\pi}{3}\int_0^{\pi/2}\left[8a^3(1+\cos\theta)^3 - \frac{8a^3}{(1+\cos\theta)^3}\right]\sin\theta\, d\theta$$

$$= \frac{-16\pi a^3}{3}\int_0^{\pi/2}[(1+\cos\theta)^3 - (1+\cos\theta)^{-3}](-\sin\theta)\, d\theta \qquad \textbf{(Note)}$$

$$= \frac{-16\pi a^3}{3} \left[\frac{(1 + \cos\theta)^4}{4} - \frac{(1 + \cos\theta)^{-2}}{-2} \right]_0^{\pi/2}, \text{ using power formula}$$

$$= \frac{-16\pi a^3}{3} \left[\frac{1}{4}(1 - 16) + \frac{1}{2}\left(1 - \frac{1}{4}\right) \right] = \frac{-16}{3} \pi a^3 \left[-\frac{15}{4} + \frac{3}{8} \right]$$

$$= \left(-\frac{16}{3} \pi a^3 \right) \left(\frac{-27}{8} \right) = 18\pi a^3.$$

Comprehensive Exercise 3

1. Find the volume of the solid generated by the revolution of $r = 2a\cos\theta$ about the initial line.
2. The arc of the cardioid $r = a(1 + \cos\theta)$, specified by $-\pi/2 \le \theta \le \pi/2$, is rotated about the line $\theta = 0$, prove that the volume generated is $\frac{5}{2}\pi a^3$.
3. Find hte volume of the solid generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about the initial line. **(Rohilkhand 2010)**
4. Show that the volume of the solid formed by the revolution of the curve, $r = a + b\cos\theta\,(a > b)$ about the initial line is $\frac{4}{3}\pi a(a^2 + b^2)$. **(Meerut 2008)**
5. Find the volume of the solid generated by revolving one loop of the lemniscate $r^2 = a^2\cos 2\theta$ about the line $\theta = \frac{1}{2}\pi$. **(Meerut 2006; Kumaun 07, 11)**

Answers 3

1. $\frac{4}{3}\pi a^3$ 3. $\frac{8}{3}\pi a^3$ 4. $\frac{4}{3}\pi a(a^2 + b^2)$ 5. $(\pi^2 a^3)/4\sqrt{2}$

6 Surfaces of Solids of Revolution

(a) Revolution about the axis of x. *To prove that the curved surface of the solid generated by the revolution, about x-axis, of the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis is*

$$\int_{x=a}^{x=b} 2\pi y \, ds,$$

where s is the length of the arc measured from $x = a$ to any point (x, y).

Or

Show that the area of the surface of the solid obtained by revolving about x-axis the arc of the curve intercepted between the points whose abscissae are a and b is

$$\int_a^b 2\pi y \frac{ds}{dx}\, dx.$$

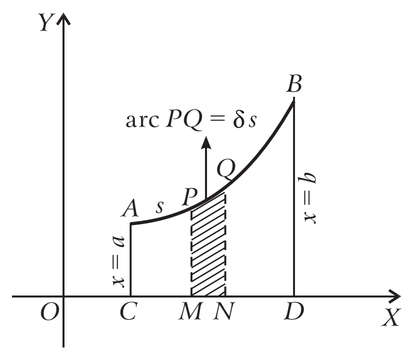
Proof: Let AB be the arc of the curve $y = f(x)$ included between the ordinates $x = a$ and $x = b$. It is being assumed that the curve does not cut x-axis and $f(x)$ is a continuous function of x in the interval (a, b).

Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be any two neighbouring points on the curve $y = f(x)$.

Let the length of the arc AP be s and arc $AQ = s + \delta s$

so that arc $PQ = \delta s$.

Draw the ordinates PM and QN. Let S denote the curved surface of the solid generated by the revolution of the area $CMPA$ about the x-axis. Then the curved surface of the solid generated by the revolution of the area $MNQP = \delta S$.



We shall take it as an axiom that the curved surface of the solid generated by the revolution of the area $MNQP$ about the x-axis lies between the curved surfaces of the right circular cylinders whose radii are PM and NQ and which are of the same thickness (height) δs. There is no loss in assuming so because ultimately Q is to tend to P.

Thus δS lies between $2\pi\, y\, \delta s$ and $2\pi\,(y + \delta y)\,\delta s$

i.e., $\quad 2\pi\, y\, \delta s < \delta S < 2\pi\,(y + \delta y)\,\delta s$

or $\quad 2\pi\, y < (\delta S / \delta s) < 2\pi\,(y + \delta y)$.

Now as Q approaches P *i.e.*, $\delta s \to 0$, δy will also tend to zero. Hence by taking limits as $\delta s \to 0$, we have

$$\frac{dS}{ds} = 2\pi\, y \quad \text{or} \quad dS = 2\pi\, y\, ds.$$

$$\therefore \quad \int_{x=a}^{x=b} 2\pi y\, ds = \int_{x=a}^{x=b} dS = [S]_{x=a}^{x=b}$$

= (the value of S when $x = b$) − (the value of S when $x = a$)

= surface of the solid generated by the revolution of the area $ACDB - 0$.

$\therefore$ the required curved surface = $\displaystyle\int_{x=a}^{x=b} 2\pi\, y\, ds$

$$= \int_{x=a}^{x=b} 2\pi y \frac{ds}{dx}\, dx, \text{ where } \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}.$$

(b) Axis of revolution as y-axis. *Similarly the curved surface of the solid generated by the revolution about the y-axis, of the area bounded by the curve $x = f(y)$, the lines $y = a$, $y = b$ and the y-axis is*

$$2\pi \int_{y=a}^{y=b} x\, ds \quad \text{or} \quad S = 2\pi \int_{y=a}^{b} x \frac{ds}{dy}\, dy,$$

where $\dfrac{ds}{dy} = \sqrt{\left\{1 + \left(\dfrac{dx}{dy}\right)^2\right\}}$.

Remark: If an arc length revolves about x-axis, the basic formula for the surface of revolution in all cases is $\int 2\pi y\, ds$, between the suitable limits. If we want to integrate w.r.t. x, we shall change ds as $(ds/dx)\, dx$ and adjust the limits accordingly.

A similar transformation can be made if we want to integrate w.r.t. y or with respect to θ or w.r.t. some parameter, say t.

Illustrative Examples

Example 10: *Find the curved surface of a hemisphere of radius a.* **(Agra 2005; Kanpur 14)**

Solution: A hemisphere is generated by the revolution of a quadrant of a circle about one of its bounding radii.

Let the equation of the circle be $x^2 + y^2 = a^2$. ...(1)

Let the hemisphere be formed by revolving about x-axis the arc of the circle (1) lying in the first quadrant.

Differentiating (1), w.r.t. x, we get

$$2x + 2y\,(dy/dx) = 0 \quad \text{or} \quad dy/dx = -x/y.$$

Therefore $$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left\{1 + \frac{x^2}{y^2}\right\}} = \sqrt{\left\{\frac{y^2 + x^2}{y^2}\right\}} = \sqrt{\left(\frac{a^2}{y^2}\right)}$$

[From (1)]

$$= a/y.$$

For the arc of the circle (1) lying in the first quadrant x varies from 0 to a.

$\therefore$ the required surface

$$= 2\pi \int_{x=a}^{0} y\, ds = 2\pi \int_0^a y \frac{ds}{dx} \cdot dx$$

$$= 2\pi \int_0^a y \cdot \frac{a}{y}\, dx = 2\pi \int_0^a a\, dx = 2\pi a\, [x]_0^a = 2\pi a \,.\, a = 2\pi a^2.$$

Example 11: *Find the surface generated by the revolution of an arc of the catenary* $y = c \cosh (x/c)$ *about the axis of* x. **(Meerut 2000, 04B, 07, 07B, 10; Rohilkhand 14)**

Solution: The given curve is, $y = c \cosh (x/c)$. ...(1)

Differentiating (1) w.r.t x, we get

$$\frac{dy}{dx} = c \sinh \frac{x}{c} \cdot \frac{1}{c} = \sinh \frac{x}{c} \cdot$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left\{1 + \sinh^2 \frac{x}{c}\right\}} = \cosh \frac{x}{c} \cdot \qquad ...(2)$$

If the arc be measured from the vertex $(x = 0)$ to any point (x, y), then the required surface formed by the revolution of this arc about x-axis

$$= \int_{x=a}^{0} 2\pi\, y \frac{ds}{dx}\, dx = 2\pi \int_0^x c \cosh \frac{x}{c} \cdot \cosh \frac{x}{c}\, dx, \qquad \text{from (1) and (2)}$$

$$= \pi c \int_0^x 2 \cosh^2 \frac{x}{c}\, dx = \pi c \int_0^x \left[1 + \cosh \frac{2x}{c}\right] dx \qquad \textbf{(Note)}$$

$$= \pi c \left[x + \frac{c}{2} \sinh \frac{2x}{c}\right]_0^x = \pi c \left[x + \frac{c}{2} \sinh \frac{2x}{c}\right]$$

$$= \pi c \left[x + c \sinh \frac{x}{c} \cosh \frac{x}{c}\right] \cdot$$

Example 12: *Prove that the surface of the prolate spheroid formed by the revolution of the ellipse of eccentricity* e *about its major axis is equal to* $2 \times$ *area of the ellipse* $\times [\sqrt{(1 - e^2)} + (1/e) \sin^{-1} e]$.

Solution: **[Note. Prolate spheroid is generated by the revolution of an ellipse about its major axis]**

Let the equation of the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, ...(1)

the x-axis being the major axis so that $a > b$.

The parametric equations of (1) are $x = a \cos t$, $y = b \sin t$.

$\therefore \quad dx/dt = -a \sin t$ and $dy/dt = b \cos t$.

We have $\quad \dfrac{ds}{dt} = \sqrt{\left\{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2\right\}} = \sqrt{(a^2 \sin^2 t + b^2 \cos^2 t)}$

$$= \sqrt{\{a^2 \sin^2 t + a^2 (1 - e^2) \cos^2 t\}}, \quad [\because \text{for the ellipse } b^2 = a^2 (1 - e^2)]$$

$$= a \sqrt{(1 - e^2 \cos^2 t)}. \qquad ...(2)$$

Now the ellipse (1) is symmetrical about y-axis and for the arc of the ellipse lying in the first quadrant t varies from 0 to $\pi/2$. At the point $(a, 0)$ we have $t = 0$ and at the point $(0, b)$ we have $t = \pi/2$.

Hence the required surface S formed by the revolution of the ellipse (1) about the x-axis

$$= 2\int 2\pi y \, ds, \text{ between the suitable limits}$$

$$= 4\pi \int_0^{\pi/2} y \frac{ds}{dt} dt = 4\pi \int_0^{\pi/2} b \sin t \, . \, a \sqrt{(1 - e^2 \cos^2 t)} \, dt,$$

$$[\because y = b \sin t \text{ and } ds/dt = a\sqrt{(1 - e^2 \cos^2 t)}, \text{ from (2)}]$$

$$= 4\pi ab \int_0^{\pi/2} \sin t \sqrt{(1 - e^2 \cos^2 t)} \, dt.$$

Put $e \cos t = z$ so that $-e \sin t \, dt = dz$. When $t = 0, z = e$ and when $t = \frac{1}{2}\pi, z = 0$.

$$\therefore \quad S = -4\pi ab \int_e^0 \frac{1}{e} \sqrt{(1 - z^2)} \, dz = \frac{4\pi ab}{e} \int_0^e \sqrt{(1 - z^2)} \, dz$$

$$= \frac{4\pi ab}{e} \left[\frac{z}{2} \sqrt{(1 - z^2)} + \frac{1}{2} \sin^{-1} z \right]_0^e = \frac{4\pi ab}{e} \left[\frac{e}{2} \sqrt{(1 - e^2)} + \frac{1}{2} \sin^{-1} e \right]$$

$$= 2\pi ab \, [\sqrt{(1 - e^2)} + (1/e) \sin^{-1} e]$$

$$= 2 \times \text{area of the ellipse} \times [\sqrt{(1 - e^2)} + (1/e) \sin^{-1} e].$$

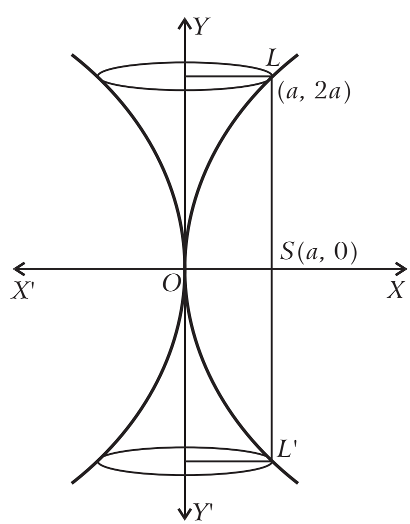
Remark: The solid of revolution formed by revolving an ellipse about its minor axis is called an **oblate spheroid**.

Example 13: *The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the curved surface of the reel thus generated.*

(Kumaun 2010; Bundelkhand 11)

Solution: The given parabola is

$$y^2 = 4ax. \qquad \text{...(1)}$$



Differentiating (1) w.r.t. x, we get

$$dy/dx = 2a/y.$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left\{1 + \frac{4a^2}{y^2}\right\}}$$

$$= \sqrt{\left\{1 + \frac{4a^2}{4ax}\right\}} = \sqrt{\left(\frac{x + a}{x}\right)}.$$

The required curved surface is generated by the revolution of the arc LOL' (LSL' is the latus rectum), about the tangent at the vertex *i.e.*, y-axis. The curve is symmetrical about x-axis and for the arc OL, x varies from 0 to a.

$\therefore$ the required surface $= 2\int_{x=0}^{a} 2\pi\, x \dfrac{ds}{dx}\, dx$

$$= 4\pi \int_0^a x \sqrt{\left(\frac{x+a}{x}\right)}\, dx$$

$$= 4\pi \int_0^a \sqrt{(x^2 + ax)}\, dx$$

$$= 4\pi \int_0^a \sqrt{\left\{\left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2\right\}}\, dx \qquad \textbf{(Note)}$$

$$= 4\pi \left[\frac{1}{2}\left(x + \frac{a}{2}\right)\sqrt{(x^2 + ax)} - \frac{1}{2}\cdot\frac{a^2}{4} \log\left\{\left(x + \frac{a}{2}\right) + \sqrt{(x^2 + ax)}\right\}\right]_0^a$$

$$\left[\because \int \sqrt{(x^2 - a^2)}\, dx = \frac{1}{2} x \sqrt{(x^2 - a^2)} - \frac{1}{2} a^2 \log\{x + \sqrt{(x^2 - a^2)}\}\right]$$

$$= 4\pi \left[\frac{1}{2}\cdot\frac{3}{2} a\, a \sqrt{2} - \frac{1}{8} a^2 \log\left\{\frac{3}{2} a + a\sqrt{2}\right\} + \frac{1}{8} a^2 \log\left(\frac{1}{2} a\right)\right]$$

$$= 4\pi \left[\frac{3}{4} a^2 \sqrt{2} - \frac{1}{8} a^2 \log\left\{\left(\frac{3}{2} a + a\sqrt{2}\right) \Big/ \left(\frac{1}{2} a\right)\right\}\right]$$

$$= \pi a^2 \left[3\sqrt{2} - \frac{1}{2} \log(3 + 2\sqrt{2})\right]$$

$$= \pi a^2 \left[3\sqrt{2} - \frac{1}{2} \log(\sqrt{2} + 1)^2\right] \qquad \textbf{(Note)}$$

$$= \pi a^2 [3\sqrt{2} - \log(\sqrt{2} + 1)].$$

Comprehensive Exercise 4

1. Find the surface of a sphere of radius a. **(Kanpur 2006)**
2. Show that the surface of the spherical zone contained between two parallel planes is $2\pi ah$ where a is the radius of the sphere and h the distance between the planes. **(Kanpur 2009)**
3. Find the area of the surface formed by the revolution of the parabola $y^2 = 4ax$ about the x-axis by the arc from the vertex to one end of the latus rectum. **(Rohilkhand 2011)**
4. Find the surface generated by the revolution of an arc of the catenary $y = c \cosh(x / c)$ about the axis of x, between the planes $x = a$ and $x = b$.
5. For a catenary $y = a \cosh(x / a)$, prove that $aS = 2V = \pi a(ax + sy)$, where s is the length of the arc from the vertex, S and V are respectively the area of the curved surface and volume of the solid generated by the revolution of the arc about x-axis.

6. Find the surface of the solid generated by the revolution of the ellipse $x^2 + 4y^2 = 16$ about its major axis. **(Meerut 2005, 06)**

7. Find the surface of the solid formed by the revolution, about the axis of y, of the part of the curve $ay^2 = x^3$ from $x = 0$ to $x = 4a$ which is above the x-axis.

Answers 4

1. $4\pi a^2$

3. $\frac{8}{3}\pi a^2 [2\sqrt{2} - 1]$

4. $\pi c\left[(b - a) + \frac{c}{2}\sinh\frac{2b}{c} - \frac{c}{2}\sinh\frac{2a}{c}\right]$

6. $8\pi\left[1 + \frac{4\pi}{3\sqrt{3}}\right]$

7. $\frac{128}{1215}\pi a^2 [125\sqrt{(10)} + 1]$

7 Surface Formula for Parametric Equations

Suppose the equation of the curve is given in parametric form $x = f(t)$, $y = \phi(t)$, t being the variable parameter. Then the curved surface of the solid formed by the revolution about the x-axis

$$= \int 2\pi y \frac{ds}{dt}\, dt, \text{ between the suitable limits}$$

where $$\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}.$$

Illustrative Examples

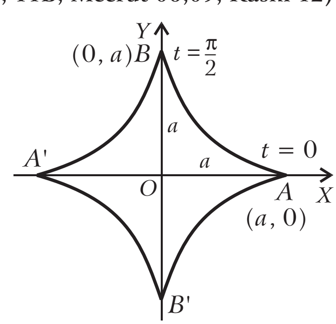
Example 14: *Find the surface of the solid generated by revolution of the astroid*

$$x^{2/3} + y^{2/3} = a^{2/3} \quad \text{or} \quad x = a\cos^3 t,\ y = a\sin^3 t \text{ about the x-axis.}$$

(Kumaun 2000, 13; Agra 01; Rohilkhand 07, 09, 11B; Meerut 06,09; Kashi 12)

Solution: The parametric equations of the curve are

$$x = a\cos^3 t,\ y = a\sin^3 t.$$



$$\therefore \quad \frac{dx}{dt} = -3a\cos^2 t\sin t \text{ and } \frac{dy}{dt} = 3a\sin^2 t\cos t.$$

Hence $$\frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}$$

$$= \sqrt{[9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t]}$$

$$= \sqrt{[9a^2 \sin^2 t \cos^2 t (\cos^2 t + \sin^2 t)]} = 3a \sin t \cos t .$$

Also the given curve (astroid) is symmetrical about both the axes and for the curve in the first quadrant, t varies from 0 to $\pi/2$.

$\therefore$ the required surface $= 2\int_{t=0}^{\pi/2} 2\pi\, y \dfrac{ds}{dt}\, dt$

$$= 4\pi \int_0^{\pi/2} a \sin^3 t \,.\, 3a \sin t \cos t\, dt = 12\,\pi\, a^2 \int_0^{\pi/2} \sin^4 t \cos t\, dt$$

$$= 12\pi a^2 \left[\frac{\sin^5 t}{5}\right]_0^{\pi/2} = 12\pi\, a^2 \left[\frac{1}{5} - 0\right] = \frac{12\pi a^2}{5} .$$

Example 15: *Prove that the surface of the solid generated by the revolution of the tractrix*

$$x = a \cos t + \frac{1}{2} a \log \tan^2 \frac{1}{2} t\,,\; y = a \sin t$$

about its asymptote is equal to the surface of a sphere of radius a .

(Agra 2002; Gorakhpur 06; Meerut 09B)

Solution: The given tractrix is

$$x = a \cos t + \frac{1}{2} a \log \tan^2 \frac{1}{2} t\,,\; y = a \sin t .$$

$\therefore$

$$\frac{dx}{dt} = -a \sin t + a \frac{\sec^2 \frac{1}{2} t}{\tan \frac{1}{2} t} \cdot \frac{1}{2} = a \left(-\sin t + \frac{1}{2 \sin \frac{1}{2} t \cos \frac{1}{2} t} \right)$$

$$= a \left(-\sin t + \frac{1}{\sin t} \right) = a \frac{(-\sin^2 t + 1)}{\sin t} = \frac{a \cos^2 t}{\sin t}$$

and

$$\frac{dy}{dt} = a \cos t .$$

Hence

$$\frac{ds}{dt} = \sqrt{\left\{ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \right\}} = \sqrt{\left\{ \frac{a^2 \cos^4 t}{\sin^2 t} + a^2 \cos^2 t \right\}} = \frac{a \cos t}{\sin t} .$$

The given curve is symmetrical about both the axes and the asymptote is the line $y = 0$ *i.e.*, x-axis. For the arc of the curve lying in the second quadrant t varies from 0 to $\frac{1}{2}\pi$.

$\therefore$ the required surface $= 2 \,.\, \int_0^{\pi/2} 2\pi y \dfrac{ds}{dt}\, dt$ **(Note)**

$$= 4\pi \int_0^{\pi/2} a \sin t \,.\, \frac{a \cos t}{\sin t}\, dt = 4\pi a^2 \int_0^{\pi/2} \cos t\, dt$$

$$= 4\pi a^2 [\sin t]_0^{\pi/2} = 4\pi a^2$$

= the surface of a sphere of radius a .

Comprehensive Exercise 5

1. Find the surface area of the solid generated by revolving the cycloid
$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta) \text{ about the } x\text{-axis.}$$
2. Find the area of the surface generated by revolving an arc of the cycloid.
$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta) \text{ about the tangent at the vertex.}$$
3. The portion between two consecutive cusps of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$ is revolved about the x-axis. Prove that the area of the surface so formed is to the area of the cycloid as 64 : 9.
4. Prove that the surface area of the solid generated by the revolution, about the x-axis of the loop of the curve $x = t^2,\ y = t - \frac{1}{3}t^3$ is 3π.
5. Prove that the surface of the oblate spheroid formed by the revolution of the ellipse of the semi-major axis a and eccentricity e is $2\pi a^2 \left[1 + \frac{1 - e^2}{2e} \log\left(\frac{1 + e}{1 - e}\right)\right]$.

Answers 5

1. $\frac{64}{3}\pi a^2$ 2. $\frac{32}{3}\pi a^2$

8 Surface Formula for Polar Equations

Suppose the equation of the curve is given in the polar form $r = f(\theta)$. Then the curved surface generated by the revolution about the initial line, of the arc intercepted between the radii vectors $\theta = \alpha$ and $\theta = \beta$ is

$$\int_{\theta = \alpha}^{\theta = \beta} 2\pi (r \sin\theta) \frac{ds}{d\theta}\, d\theta, \text{ where } \frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}. \quad [\because y = r\sin\theta]$$

Note: In some cases we may use the formula

$$S = \int 2\pi y \frac{ds}{dr}\, dr, \text{ where } \frac{ds}{dr} = \sqrt{\left\{1 + \left(r\frac{d\theta}{dr}\right)^2\right\}}.$$

9 Curved Surface Generated by Revolution about any Axis

If the given arc AB is revolved about a line CD other than the coordinate axes, then the curved surface thus generated is

$$= 2\pi \int (PM)\, ds, \qquad \text{(between the proper limits of integration)}$$

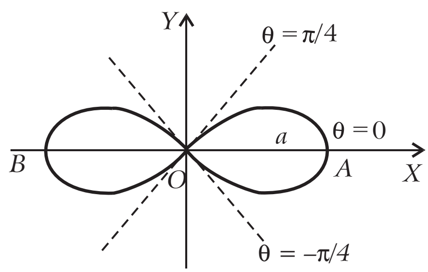
where PM is the perpendicular drawn from any point P on the arc AB to the axis of revolution CD and ds is the length of an element of the arc AB at the point P.

Illustrative Examples

Example 16: *Find the surface of the solid generated by the revolution of the lemniscate* $r^2 = a^2 \cos 2\theta$ *about the initial line.*

(Garhwal 2000, 02; Meerut 04, 10B, 11; Rohilkhand 08B; Agra 14; Purvanchal 14)

Solution: The given curve is $r^2 = a^2 \cos 2\theta$. ...(1)



Differentiating (1) w.r.t. θ, we get

$$2r \frac{dr}{d\theta} = -2a^2 \sin 2\theta$$

or $$\frac{dr}{d\theta} = \frac{-a^2 \sin 2\theta}{r}.$$

$\therefore$ $$\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}$$

$$= \sqrt{\left\{a^2 \cos 2\theta + \frac{a^4 \sin^2 2\theta}{r^2}\right\}}$$

$$= \frac{1}{r} \sqrt{\{r^2 . a^2 \cos 2\theta + a^4 \sin^2 2\theta\}}$$

$$= \frac{1}{r} \sqrt{\{a^4 \cos^2 2\theta + a^4 \sin^2 2\theta\}}, \qquad [\because r^2 = a^2 \cos 2\theta]$$

$$= a^2 / r. \qquad ...(2)$$

The given curve is symmetrical about the initial line and about the pole.

Putting $r = 0$ in (1), we get $\cos 2\theta = 0$ giving $2\theta = \pm \frac{1}{2}\pi$ *i.e.*, $\theta = \pm \frac{1}{4}\pi$.

Therefore one loop of the curve lies between $\theta = -\frac{1}{4}\pi$ and $\theta = \frac{1}{4}\pi$.

There are two loops in the curve and for the upper half of one of these two loops θ varies from 0 to $\frac{1}{4}\pi$.

$\therefore$ the required surface

$$= 2 \times \text{the surface generated by the revolution of one loop}$$

$$= 2 \cdot \int_0^{\pi/4} 2\pi\, y \frac{ds}{d\theta}\, d\theta, \text{ where } y = r \sin\theta$$

$$= 4\pi \int_0^{\pi/4} r \sin\theta \cdot \frac{a^2}{r}\, d\theta \qquad \text{[From (2)]}$$

$$= 4\pi a^2 \int_0^{\pi/4} \sin\theta\, d\theta = 4\pi a^2 \left[-\cos\theta\right]_0^{\pi/4}$$

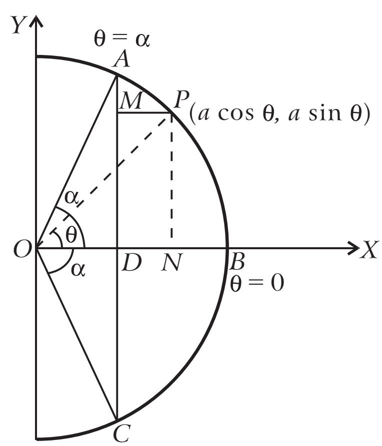
$$= 4\pi a^2 \left[-(1/\sqrt{2}) + 1\right] = 4\pi a^2 \left[1 - (1/\sqrt{2})\right].$$

Example 17: *A circular arc revolves about its chord. Find the area of the surface generated, when 2α is the angle subtended by the arc at the centre.*

Solution: Let the parametric equations of the circle be

$$x = a\cos\theta,\ y = a\sin\theta, \qquad \ldots(1)$$

θ being the parameter.



Take any point $P(a\cos\theta, a\sin\theta)$ on the circular arc ABC which is symmetrical about the x-axis and which subtends an angle 2α at the centre O so that $\angle AOB = \alpha$.

We have $OD = OA\cos\alpha = a\cos\alpha$. Draw PM perpendicular from P to AC, the axis of rotation. Then

$$PM = ON - OD = a\cos\theta - a\cos\alpha. \qquad \ldots(2)$$

For the upper half of the arc to be rotated *i.e.*, for the arc BA, θ varies from 0 to α.

Also $$\frac{ds}{d\theta} = \sqrt{\left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\}}$$

$$= \sqrt{\{a^2 \sin^2 \theta + a^2 \cos^2 \theta\}} = a.$$

$\therefore$ the required surface

$= 2 \times$ surface generated by the revolution of the arc BA about the chord AC

$$= 2 \times \int_0^{\alpha} 2\pi (PM) \frac{ds}{d\theta} d\theta$$

$$= 4\pi \int_0^{\alpha} (a \cos \theta - a \cos \alpha) . a . d\theta \qquad \text{[From (2)]}$$

$$= 4\pi a^2 [\sin \theta - \theta \cos \alpha]_0^{\alpha} = 4\pi a^2 [\sin \alpha - \alpha \cos \alpha].$$

Comprehensive Exercise 6

1. Find the area of the surface of revolution formed by revolving the curve $r = 2a \cos \theta$ about the initial line.
2. Find the surface of the solid formed by the revolution of the cardioid $r = a (1 + \cos \theta)$ about the initial line. **(Purvanchal 2006, 10; Kashi 11)**
3. The arc of the cardioid $r = a (1 + \cos \theta)$ included between $-\frac{1}{2} \pi \leq \theta \leq \frac{1}{2} \pi$ is rotated about the line $\theta = \frac{1}{2} \pi$. Find the area of the surface generated. **(Purvanchal 2010)**
4. A quadrant of a circle of radius a revolves about its chord. Show that the surface of the spindle generated is
$$2\pi a^2 \sqrt{2} \left(1 - \frac{1}{4} \pi\right).$$
5. The lemniscate $r^2 = a^2 \cos 2\theta$ revolves about a tangent at the pole. Show that the surface of the solid generated is $4\pi a^2$. **(Meerut 1993, 2005B; Kumaun 12)**

Answers 6

1. $4\pi a^2$ 2. $\frac{32}{5} \pi a^2$ 3. $\frac{48}{5} \sqrt{2} \pi a^2$

10 Theorems of Pappus and Guldin

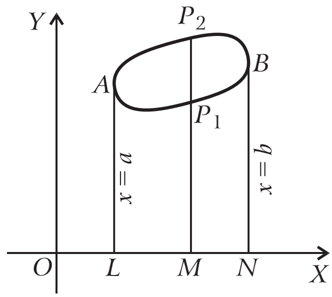
(Agra 2014)

State and prove the theorems of Pappus and Guldin.

Theorem 1: Volume of a Solid of Revolution:

If a closed plane curve revolves about a straight line in its plane which does not intersect it, the volume of the ring thus obtained is equal to the area of the region enclosed by the curve multiplied by the length of the path described by the centroid of the region.

Proof: Let AP_1BP_2A be the closed plane curve and let it rotate about the axis of x.



Let AL $(x = a)$ and BN $(x = b)$ be the tangents to the curve parallel to the y-axis $(a < b)$. Also let any ordinate meet the curve at P_1, P_2 and let $MP_1 = y_1$, $MP_2 = y_2$ so that y_1, y_2 are functions of x.

Now volume of the ring generated by the revolution of the closed curve AP_1BP_2A about the axis of x

= volume generated by the area $ALNBP_2A$

– volume generated by the area $ALNBP_1A$

$$= \pi\int_a^b y_2^2\,dx - \pi\int_a^b y_1^2\,dx = \pi\int_a^b (y_2^2 - y_1^2)\,dx. \qquad ...(1)$$

Also if $\bar{y}$ be the ordinate of the centroid of the area of the closed curve, then

$$\bar{y} = \frac{\int_a^b \frac{1}{2}(y_1 + y_2)(y_2 - y_1)\,dx}{A} = \frac{\frac{1}{2}\int_a^b (y_2^2 - y_1^2)\,dx}{A}, \qquad ...(2)$$

where A is the area of the closed curve. [See the chapter on centre of gravity]

Hence from (1) and (2), the required volume $= 2\pi A\,\bar{y} = A \times 2\pi\,\bar{y}$

= area of the closed curve × circumference of the circle of radius $\bar{y}$

= (area of the curve) × (length of the arc described by the centroid of the region bounded by the closed curve).

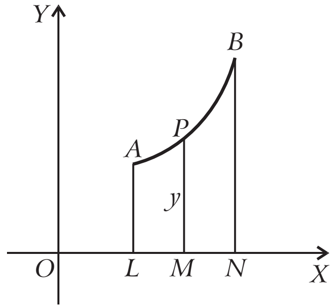
Theorem 2: Surface of a solid of revolution :

If an arc of a plane curve revolves about a straight line in its plane, which does not intersect it, the surface of the solid thus obtained is equal to the arc multiplied by the length of the path described by the centroid of the arc.

Proof: Let l be the length of the arc AB and let it revolve about OX.

Let the abscissae of the extremities A and B of the arc be a and b.

Then the surface generated by the revolution of the arc AB about x-axis is



$$= \int_{x=a}^{x=b} 2\pi\ y\ ds \qquad ...(1)$$

Also we know that (see the chapter on centre of gravity) the ordinate $\bar{y}$, of the centroid of the arc from $x = a$ to $x = b$, of length l, is given by

$$\bar{y} = \frac{\int_{x=a}^{b} y\ ds}{l} \qquad ...(2)$$

From (1) and (2), we get the required surface

$= 2\pi\ \bar{y}\ l = l \times 2\pi\ \bar{y}$

= (length of the arc) × (length of the path described by the centroid of the arc).

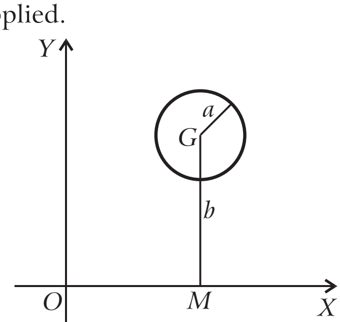
Note 1: The closed curve or arc in the above theorems must not cross the axis of revolution but may be terminated by it.

Note 2: When the volume or surface generated is known, the theorems may be applied to find the position of the centroid of the generating area or arc.

Illustrative Examples

Example 18: *Find the volume and surface-area of the anchor-ring generated by the revolution of a circle of radius a about an axis in its own plane distant b from its centre* $(b > a)$.

Solution: Here the given curve (circle) does not intersect the axis of rotation, so Pappus theorem can be applied.



In this case, A = area of the region of the closed curve

$= \text{area of the circle of radius } a$

$= \pi a^2$

and l = length of the arc of the curve

$= \text{circumference of the circle}$

$= 2\pi a$.

As the centroid of the area of a circle and also of its circumference lies at the centre, so $\bar{y} = b$ in both the cases and hence the length of the path described by the C.G. $= 2\pi b$.

Now by Pappus theorem, the required volume of the anchor-ring

= (area of the circle) × (circumference of the circle generated by the centroid)

$= \pi a^2 \,.\, 2\,\pi b = 2\,\pi^2\, a^2\, b.$

And the surface area of the anchor-ring

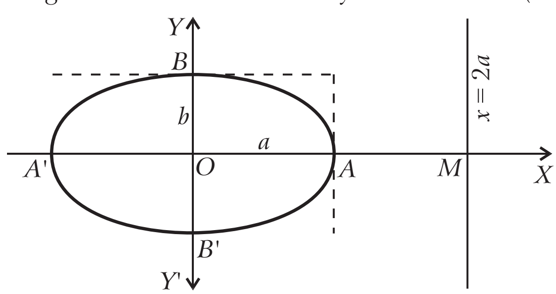
= (arc length of the circle) × (circumference of the circle generated by the centroid)

$= 2\pi a \,.\, 2\pi b = 4\pi^2 ab.$

Example 19: *Show that the volume generated by the revolution of the ellipse* $x^2/a^2 + y^2/b^2 = 1$ *about the line* $x = 2a$ *is* $4\pi^2 a^2 b$.

Solution: Area of the given ellipse is πab.

The C.G. of the ellipse will describe a circle of radius $2a$ when revolved about the line $x = 2a$. Hence the length of the arc described by the C.G. $= 2\pi\,(2a) = 4\pi a$.



∴ by Pappus theorem the required volume

= (area of the ellipse) × (length of the arc described by its C.G.)

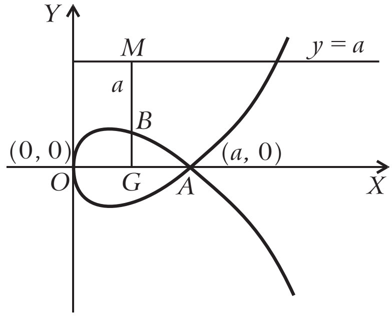
$= \pi ab \,.\, 4\pi a = 4\pi^2 a^2 b.$

Example 20: *The loop of the curve* $2ay^2 = x(x-a)^2$ *revolves about the straight line* $y = a$. *Find the volume of the solid generated.*

Solution: The given curve is $2ay^2 = x\,(x-a)^2$. ...(1)

The curve (1) is symmetrical about the x-axis and the loop lies between $x = 0$ and $x = a$.

Differentiating (1) w.r.t. x, we get



$$4ay\,(dy/dx) = 2x\,(x-a) + (x-a)^2 = 3x^2 - 4ax + a^2.$$

Now $(dy/dx) = 0$ when $3x^2 - 4ax + a^2 = 0$

or when $x = a/3$, which gives from (1),

$$y = (a\sqrt{2})/(3\sqrt{3}),\ i.e., < a,$$

showing that the loop does not intersect the straight line $y = a$.

By symmetry the C.G. of the loop lies on x-axis *i.e.*, the distance of the C.G. from the axis of revolution ($y = a$) is a. When the loop is rotated about $y = a$, its C.G. will describe a circle of radius a whose perimeter is $2\pi a$.

Also the area A of the loop

$$= 2\int_0^a y\,dx = 2\int_0^a \frac{(x-a)\sqrt{x}}{\sqrt{(2a)}}\,dx, \qquad \left[\because \text{ from (1)}, y = \frac{(x-a)\sqrt{x}}{\sqrt{(2a)}}\right]$$

$$= \sqrt{\left(\frac{2}{a}\right)}\int_0^a (x^{3/2} - ax^{1/2})\,dx = \sqrt{\left(\frac{2}{a}\right)}\left[\frac{x^{5/2}}{5/2} - \frac{ax^{3/2}}{3/2}\right]_0^a = \frac{4}{15}\sqrt{2}a^2.$$

$\therefore$ by Pappus theorem, the required volume

$$= 2\pi a \times A = 2\pi a \times \frac{4}{15}\sqrt{2}a^2 = \frac{8}{15}\sqrt{2}\pi a^3.$$

Example 21: *Prove that the volume of the solid formed by the rotation about the line* $\theta = 0$ *of the area bounded by the curve* $r = f(\theta)$ *and the lines* $\theta = \theta_1, \theta = \theta_2$ *is* $\frac{2}{3}\pi\int_{\theta_1}^{\theta_2} r^3 \sin\theta\,d\theta.$

Solution: Let OAB be the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \theta_1$ and $\theta = \theta_2$. We have to find the volume formed by the revolution of area OAB about the initial line OX.

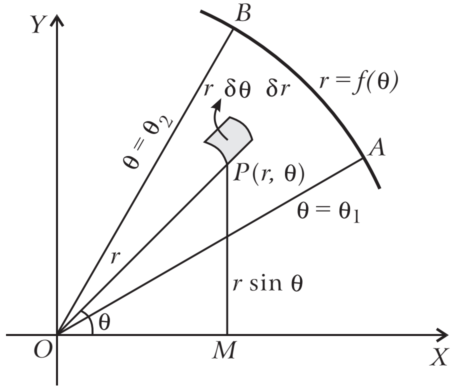
Take any point (r, θ) inside the area OAB and take a small element of the area $r\,\delta\theta\,\delta r$ at the point P. Drop PM perpendicular from P to the axis of rotation OX. We have

$$PM = OP \sin\theta = r\sin\theta,$$

Now the volume of the ring formed by revolving the element of area $r\,\delta\theta\,\delta r$ about OX

$$= 2\pi r\sin\theta \,.\, r\,\delta\theta\,\delta r = 2\pi r^2 \sin\theta\,\delta\theta\,\delta r.$$

Therefore the whole volume formed by revolving the area OAB about OX



$$= \int_{\theta=\theta_1}^{\theta_2} \int_{r=0}^{f(\theta)} 2\pi r^2 \sin\theta \, d\theta \, dr = \int_{\theta=\theta_1}^{\theta_2} 2\pi \sin\theta \left[\frac{r^3}{3} \right]_0^{f(\theta)} d\theta$$

$$= \frac{2}{3}\pi \int_{\theta=\theta_1}^{\theta_2} [f(\theta)]^3 \sin\theta \, d\theta = \frac{2}{3}\pi \int_{\theta_1}^{\theta_2} r^3 \sin\theta \, d\theta.$$

where r is to be replaced from the equation of the curve $r = f(\theta)$.

Note: Proceeding as above we can also show that the volume of the solid formed by the rotation of the above mentioned area about the line $\theta = \frac{\pi}{2}$ is equal to $\frac{2}{3}\pi \int_{\theta_1}^{\theta_2} r^3 \cos\theta \, d\theta$.

Comprehensive Exercise 7

Use Pappus theorem to find:

1. The position of the centroid of a semi-circular area.
2. The volume generated by the revolution of an ellipse having semi-axes a and b about a tangent at the vertex.
3. Find by using Pappus theorem the volume of the ring generated by the revolution of an ellipse of eccentricity $1/\sqrt{2}$ about a straight line parallel to the minor axis and situated at a distance from the centre equal to three times the major axis.
4. Find the volume of the ring generated by the revolution of the cardiod $r = a(1 + \cos\theta)$ about the line $r\cos\theta + a = 0$, given that the centroid of the cardioid is at a distance $5a/6$ from the origin.
5. A semi-circular bend of lead has a mean radius of 8 inches; the initial diameter of the pipe is 4 inches and the thickness of the lead is $\frac{1}{2}$ inch. Applying the theorem of Pappus and Guldin find the volume of the lead and its weight, given that 1 cubic inch of lead weighs 0.4 lb.
   [**Hint:** Internal diameter of pipe = 4 inches.

Thickness of metal $= \frac{1}{2}$ inch

$\therefore$ external diameter of the pipe $= 4 + 1 = 5$ inches.

$\therefore$ area of lead $= \frac{1}{4}\pi(5^2 - 4^2) = \frac{9}{4}\pi.$

The centroid of this area is at a distance of 8 inches from the axis of rotation. Therefore the length of path traced out by its centroid in describing a semi-circle $= 8\pi$ inches.

$\therefore$ volume of the lead $= 8\pi \times \frac{9}{4}\pi = 18\pi^2$ cu. inch.

$\therefore$ weight of the pipe = volume × density $= 18\pi^2 \times 0.4$ lb. $= 71.1$ lb]

6. State the theorems of Pappus and Guldin. **(Meerut 2008)**

Answers 7

1. $4a / 3\pi$
2. $2\pi^2 a^2 b$ or $2\pi^2 ab^2$
3. $6\sqrt{2}\pi^2 a^3$, where a is the semi-major axis
4. $\frac{11}{2}\pi^2 a^2$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \theta_1, \theta = \theta_2$ about any line $(\theta = \gamma)$ is

(a) $\int_{\theta_1}^{\theta_2} \frac{2}{3}\pi r^3 \cos(\theta - \gamma)\, d\theta$

(b) $\int_{\theta_1}^{\theta_2} \frac{2}{3}\pi r^3 \sin(\theta - \gamma)\, d\theta$

(c) $\int_{\theta_1}^{\theta_2} \pi r^2 \sin(\theta - \gamma)\, d\theta$

(d) $\int_{\theta_1}^{\theta_2} \pi r^2 \cos(\theta - \gamma)\, d\theta$

2. The volume of the paraboloid generated by the revolution about the x-axis of the parabola $y^2 = 4ax$ from $x = 0$ to $x = h$ is

(a) $2\pi ah^2$ (b) $2\pi ah$

(c) $\frac{2}{3}\pi ah^2$ (d) $\frac{2}{3}\pi ah$

(Rohilkhand 2005)

3. The curved surface of the solid generated by the revolution about the of y-axis the area bounded by the curve $x = f(y)$, the lines $y = a$, $y = b$ and y-axis is

(a) $\int_a^b \pi x\, ds$ (b) $\int_a^b 2\pi x\, ds$

(c) $\int_a^b \frac{2}{3}\pi x\, ds$ (d) $\int_a^b \pi^2 x\, ds$

Fill in The Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. The volume of the solid generated by the revolution of the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$, $x = b$ about the x-axis is …… .

(Meerut 2003)

2. The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \theta_1, \theta = \theta_2$ about the initial line is …… .

(Meerut 2001)

3. If the equation of the curve in the polar form is $r = f(\theta)$, then the curved surface generated by the revolution about the initial line of the arc intercepted between the radii vectors $\theta = \alpha$ and $\theta = \beta$ is

$$\int_{\theta=\alpha}^{\theta=\beta} 2\pi\,(r\sin\theta)\,\frac{ds}{d\theta}\,d\theta, \text{ where } \frac{ds}{d\theta} = \ldots\ldots$$

4. If the equations of the curve in parametric form are $x = f(t)$, $y = \phi(t)$, t being the variable then the curved surface of the solid formed by the revolution about the x-axis is $\int 2\pi y \frac{ds}{dt}\,dt$, between the suitable limits, where $\frac{ds}{dt} = \ldots\ldots$

True or False

Write 'T' for true and 'F' for false statement.

1. The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \theta_1, \theta = \theta_2$ about the initial line $\theta = 0$ is $\int_{\theta_2}^{\theta_1} \frac{2}{3}\pi r^3 \sin\theta\, d\theta$.

2. If an arc length revolves about x-axis, the basic formula for the surface of revolution in all cases is $\int 2\pi\, y ds$, between the suitable limits.

Answers

Multiple Choice Questions

1. (b) 2. (a) 3. (b)

Fill in the Blank(s)

1. $\int_a^b \pi y^2 dx$

2. $\int_{\theta_1}^{\theta_2} \frac{2}{3} \pi r^3 \sin\theta \, d\theta$

3. $\sqrt{\left\{ r^2 + \left(\frac{dr}{d\theta} \right)^2 \right\}}$

4. $\sqrt{\left\{ \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right\}}$

True or False

1. *T* 2. *T*

□