Krishna's

B.Sc. Objective

Real Analysis

(For B.Sc. III year Students of all Colleges affiliated to Universities in U.P.)

As Per U.P. Unified Syllabus

1600+ MCQs

By

Manoj Garg

M.Sc., M.Phil, Ph.D.
C.S.I.R. NET (JRF)

Associate Professor & **Head**, Dep't of Mathematics
Nehru P.G. College
Chhibramau, Kannauj (U.P.)

KRISHNA Prakashan Media (P) Ltd.

KRISHNA HOUSE, 11, Shivaji Road, Meerut-250 001 (U.P.), India

Syllabus

Real Analysis

B.Sc. III Year; I Paper

As per U.P. UNIFIED Syllabus (w.e.f. 2013-14)

M.M. : 36/75

Unit 1

Axiomatic study of real numbers, Completeness property in R, Archimedean property, Countable and uncountable sets, Neighbourhood, Interior points, Limit points, Open and closed sets, Derived sets, Dense sets, Perfect sets, Bolzano-Weierstrass theorem.

Unit 2

Sequences of real numbers, Subsequences, Bounded and monotonic sequences, Convergent sequences, Cauchy's theorems on limit, Cauchy sequence, Cauchy's general principle of convergence, Uniform convergence of sequences and series of functions, Weierstrass M-test, Abel's and Dirichlet's tests.

Unit 3

Sequential continuity, Boundeness and intermediate value properties of continuous functions, Uniform continuity, Meaning of sign of derivative, Darboux theorem. Limit and continuity of functions of two variables, Taylor's theorem for functions of two variables, Maxima and minima of functions of three variables, Lagrange's method of undetermined multipliers.

Unit 4

Riemann integral, Integrability of continuous and monotonic functions, Fundamental theorem of integral calculus, Mean value theorems of integral calculus, Improper integrals and their convergence, Comparison test, μ-test, Abel's test, Dirichlet's test, Integral as a function of a parameter and its differentiability and integrability.

Unit 5

Definition and examples of metric spaces, Neighbourhoods, Interior points, Limit points, Open and closed sets, Subspaces, Convergent and Cauchy sequences, Completeness, Cantor's intersection theorem.

Brief Contents

Real Analysis

B.Sc. Objective Mathematics 3.1

Book-1

Real Analysis

Chapter 1

Real Number System (Axiomatic Study of Real Numbers)

INTRODUCTION

Real analysis is a development of the set of real numbers which is reached through a series of successive extensions and generalisations starting from the set of natural numbers. The real number system is the foundation on which the whole branch of mathematics known as real analysis rests. The real number system can be described by means of certain axioms which can be divided into three categories :

1. Field axioms
2. Order axioms
3. Completeness axioms

Field Axioms

Let R be the set of real numbers having at least two distinct elements equipped with two algebraic operations denoted by + and ×, and called additon and multiplication respectively. These operations satisfy the following axioms :

Addition Axioms

1. Closure Law : $a+b \in R \ \forall a, b \in R$
2. Associative Law :

   $(a+b)+c = a+(b+c) \ \forall a, b, c \in R$
3. Commutative Law : $a+b = b+a \ \forall a, b \in R$
4. Existence of Additive Identity :

   $a+0 = a = 0+a \ \forall a \in R$, then 0 is called additive identity.
5. Existence of Additive Inverse :

   $a+(-a) = 0 = (-a)+a \ \forall a \in R$, then $-a$ is called additive inverse of a.

Multiplication Axioms

1. Closure : $a, b \in R \ \forall a, b \in R$
2. Associative Law : $a.(b.c) = (a.c).c \ \forall a, b, c \in R$
3. Commutative Law : $a.b = b.a \ \forall a, b \in R$
4. Existence of Multiplicative Identity :

   $a.1 = a = 1.a \ \forall a \in R$, then 1 is called the multiplicative identity.
5. Existence of Multiplicative Inverse :

   $a.a^{-1} = 1 = a^{-1}.a \ \forall a \in R$ then a^{-1} is called the multiplicative inverse of a Distributive Law.
6. $a.(b+c) = a.b + a.c \ \forall a, b, c \in R$
7. $(a+b).c = a.c + b.c \ \forall a, b, c \in R$

   Due to these properties the algebraic structure $(R,+,\cdot)$ is called a field.

Order Axioms

The order relation greater than (>) between pairs of real numbers satisfies the following axioms :

1. Law of Trichotomy : For any two real numbers a, b one and only one of the following is true $a > b, a = b, a < b$
2. Transitivity Law : For $a, b, c \in R$, $a > b, b > c \Rightarrow a > c$
3. Monotone Property for Addition : For all real number a, b and c, $a > b \Rightarrow a+c > b+c$.
4. Monotone Property for Multiplication : For all real numbers a, b and $c, a > b$ and $c > 0 \Rightarrow ac > bc$.

In view of the above axioms, the set R is said to be ordered. Thus R is an ordered field. The system Q of all rational numbers is an ordered field while the system C of all complex numbers is a field which is not ordered.

Some More Relations

1. The order relation less than (<) between the real numbers a and b is defined as $a < b$ if $b > a$.

2. A real number a is said to be greater than or equal to $b (a \geq b)$ if either $a > b$ or $a = b$.
3. A real number a is said to be less than or equal to $b (a \leq b)$ if either $a < b$ or $a = b$.
4. a is said to be negative if $a < 0$.
5. a is said to be positive if $a > 0$.
6. If R^+ and R^- are sets of all positive and negative real numbers then
$$R = R^+ \cup \{0\} \cup R^-$$

The Extended Real Number System

It is often convenient to extend the system of real numbers by the addition of two elements ∞ and $-\infty$. The enlarged set is called the extended real numbers.

If a is any real number, then

$$-\infty < a < \infty,\ a + \infty = \infty + a = -a + \infty = \infty,$$
$$a - \infty = -\infty + a = -\infty - a = -\infty$$
$$\frac{a}{\infty} = 0,\ \frac{\infty}{a} = \infty \times a = a \times \infty = \begin{cases} \infty \text{ if } a > 1 \\ -\infty \text{ if } a < 0 \end{cases}$$

Also $\infty \times \infty = (-\infty) \times (-\infty) = \infty + \infty = \infty,$
$$\infty \times (-\infty) + (-\infty \times \infty) = -\infty - \infty = -\infty$$

The following combinations are meaningless

$$\infty - \infty,\ -\infty + \infty,\ 0 \times \infty,\ \infty \times 0,\ \frac{\infty}{\infty},\ \frac{0}{0}$$

Intervals

A subset S of R is called an interval if

$$a, b \in S,\ x \in R,\ a < x < b \Rightarrow x \in S$$

Open interval : It is defined as

$$(a, b) =]a, b[= \{x \in R : a < x < b\}$$

Thus both the end points a and b do not belong to the interval.

Closed interval : It is defined as

$$[a, b] = \{x \in R : a \leq x \leq b\}$$

Here both the end points a and b belong to the interval.

Semi-open or closed open interval : It is defied as $[a, b) = \{x \in R : a \leq x < b\}$

Semi-closed or open closed interval : It is defined as $(a, b] = \{x \in R : a < x \leq b\}$

Each of the above intervals have length $b - a$ which is a finite positive real number.

Infinite open intervals or open rays : It is defined as

$$]a, \infty] = \{x \in R : x > a\}$$

and $]-\infty, a[= \{x \in R : x < a\}$

Infinite closed intervals or closed rays : It is defined as

$$[a, \infty[= \{x \in R : x \geq a\}$$

and $]-\infty, a] = \{x \in R : x \leq a\}$

The intervals $[a, \infty[,\ ]a, \infty[,\ ]-\infty, a[,\ ]-\infty, a]$ and $]-\infty, \propto[$ are called infinite intervals.

Absolute Value (Modulus of a Real Number)

The absolute value (modulus) of a real number x denoted by $|x|$ is defined as

$$|x| = \begin{cases} x \text{ if } x \geq 0, \\ -x \text{ if } x < 0. \end{cases}$$

It is clear that if $a = b \Rightarrow |a| = |b|$ but if $|a| = |b|$ then it is not necessarily implies that $a = b$.

Properties : For all $x, y \in R$

1. $|x| \geq 0$
2. $|x| = \max.\{-x, x\}$
3. $|x| \geq x$
4. $x \geq -|x|$
5. $|x| = |-x|$
6. $|x|^2 = x^2 = -|x|^2$
7. $|xy| = |x|.|y|$
8. $|x + y| \leq |x| + |y|$
9. $|x - y| \geq ||x| - |y||$
10. $|x - y| \geq |x| - |y|$
11. $|x - y| \geq |y| - |x|$
12. $|x| < \varepsilon \Leftrightarrow -\varepsilon < x < \varepsilon$
13. $|x - a| < \varepsilon \Leftrightarrow a - \varepsilon < x < a + \varepsilon$

Bounded and Unbounded Subsets of Real Numbers

1. **Aggregate:** A non-empty subset S of R is called an aggregate.

2. **Upper bound:** A subset S of R is said to be bounded above if there exists a real number r such that every element of S is less than or equal to r, *i.e.*

$$x \leq r \;\; \forall\, x \in S$$

The number r is called an upper bound of S. If there exists no real number r such that $x \leq r \,\forall\, x \in S$, then the set S is said to be not bounded above or unbounded above.

3. **Least upper bound or supremum:** If r is an upper bound of a subset S of R and any real number less than r is not an upper bound of S then r is called the least upper bound (l.u.b) or supremum (sup) of S.

   There a real number r is supremum of S if

   (i) r is an upper bound of S,

   (ii) $r \leq r$; for every upper bound r of S.

4. **Lower bound :** A subset S of R is said to be bounded below if there exists a real number r such that every element of S is greater than or equal to r, *i.e.*

$$x \geq r \;\; \forall\, x \in S$$

The number r is called a lower bound of S. If there exists no real number r such that $x \geq r \;\forall\, x \in S$, then the set S is said to be not bounded below or unbounded below.

5. **Greatest lower bound (Infimum):** If s is an lower bound of a subset S of R and any real number greater than r is not a lower bound of S, then S is called the greatest lower bound (g.l.b.) or infimum of S.

6. **Bounded set:** A subset S of R is said to be bounded if it is bounded above as well or bounded below.

   Thus a subset S of R is bounded if there exists a positive real number k such that $|x| < k$ for all $x \in S$.

   A subset S of R is said to be unbounded if it is not bounded above or not bounded below.

7. **Properties of supremum and infimum:**

   (i) If $\sup S = K$ and $\inf. S = K$ then $K \leq K$.

   (ii) If K is an upper bound of S and $K \in S$ then $\sup. S = K$.

   (iii) If x is a lower bound of S and $k \in S$, then $\inf. S = k$.

   (iv) If supremum of a non-empty subset of R exists, then it is unique.

   (v) If infimum of a non-empty subset of R exists, then it is unique.

   (vi) $\sup (A \cup B) = \max. \{\sup A, \sup B\}$

   (vii) $\inf (A \cup B) = \min. \{\inf. A, \inf. B\}$

   (viii) If $A \subset B$ then

$$\inf B \leq \inf A \leq \sup A \leq \sup B$$

   (ix) $\sup S = \max. S$ and $\inf S = \min S$.

Completeness Axiom

A system S of numbers is said to be complete if every non-empty subset of S, which is bounded above has a member of S for its supremum.

Complete ordered field: An ordered field F is said to be a complete ordered field if every non-empty subset of F which is bounded above has an element of F for its supremum.

The field R of real numbers is a complete ordered field while the field Q of rational numbers is an ordered field but is not complete.

Result :

1. Any non-empty subset of real numbers which is bounded below has an infimum.
2. The set Q of rational numbers is not a complete ordered field.

Archimedean Property of Real Numbers

Let a be real number and b any positive real number, then there exists a positive integer n such that

$$nb > a$$

Archimedean ordered field : An ordered field F is said to be an archimedian ordered field if $\forall\, x, y \in F$, $y > 0$, there exists some $n \in N$ such that $ny > x$.

The field R of real numbers is an Archimedean ordered field.

Results :

1. For any real number a there exists a positive integer n such that $n > a$.
2. For any positive real numbers x, there exists a positive integer such that $\frac{1}{n} < x$.
3. For any real number x, there exist two integers m and n such that $m < x < n$.
4. For any real number x, there exists a unique integer n such that $n \le x < n + 1$.
5. For any $x \in R$, there exists a unique integer n such that $x - 1 < n \le x$.
6. For any $x \in R$, there exists a unique integer n such that $x - 1 \le n < x$.
7. **Dedixind-Cantor axiom:** To every real number there corresponds a unique point on a directed line and conversely, to every point on a directed line there corresponds a unique real number.
8. Between any two distinct real numbers there always lies a rational number and therefore infinitely may rational numbers.
9. Between any two distinct real numbers, there always lies an rational number and therefore infinitely many rational number.
10. Between any two distinct real numbers, there lie an infinite number of real numbers.

EXERCISE

MULTIPLE CHOICE QUESTIONS

Direction : Each of the following questions has four alternative answers. One of them is correct. Choose the correct answer.

1. The set of natural number N is :

   a. Bounded b. Bounded above
   c. Bounded below d. None of these

2. The supremum and infimum of

   $$S = \{x \in Z : x^2 \le 25\} \text{ are respectively :}$$

   a. 5, – 5 b. –5, 5
   c. –25, 25 d. 25, – 25

3. The set $S = \{x \in R : x = n + 3, n \in N\}$ is :

   a. Bounded below b. Bounded above
   c. Bounded d. None of these

4. The supremum of the set $(1, 2] \cup [3, 8)$ is :

   a. 2 b. 3
   c. 8 d. Not exist

5. The supremum and infimum of the set

   $$S = \left\{x \in Q : x = \frac{(-1)^n}{n}, n \in N\right\} \text{ are respectively :}$$

   a. $\frac{1}{2}, -1$ b. $\frac{-1}{2}, 1$
   c. $-1, \frac{1}{2}$ d. $-1, \frac{-1}{2}$

6. The set Q of rational numbers is :

   a. Field only
   b. Ordered only
   c. Complete ordered field
   d. Ordered field only

7. For any real number x, there exist two integer m and n such that :

   a. $x < m + n$
   b. $x > m + n$
   c. $m < x < n$
   d. None of these

8. The supremum of the set of positive integer z^+ is :

   a. 1 b. 2
   c. ∞ d. Not exists

9. The infimum of the set $\{(-1)^n \cdot n : n \in N\}$ is :

   a. –1 b. .0
   c. ∞ d. Not exist

10. The l.u.b. of the set $\left\{\pi+\frac{1}{2}, \pi+\frac{1}{4}, \pi+\frac{1}{8}, ...\right\}$ is : **[Kanpur 2018]**

a. π b. $\pi+\frac{1}{4}$
c. $\pi+\frac{1}{2}$ d. Not exist

11. The supremum of the set $\left\{1+\frac{(-1)^n}{n} : n \in N\right\}$ is :

a. 1 b. 2
c. $\frac{3}{2}$ d. Not exist

12. The set R of real number is : **[Kanpur 2018]**

a. Complete only
b. Complete ordered only
c. Ordered field only
d. Complete ordered field

13. If a be any real number and b is any positive real number then exist positive integer n such that :

a. $nb > a$ b. $na > b$
c. $b > na$ d. $n > ab$

14. The set $\{x : x = (-2)^n, n \in N\}$ is :

a. Bounded above b. Bounded below
c. Bounded d. Unbounded

15. The set $\left\{1, \frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, ... \frac{1}{4^n} ....\right\}$ is :

a. Bounded above only
b. Bounded below only
c. Bounded
d. Unbounded

16. The supremum of the set $\left\{\frac{3n+2}{2n+1} : n \in N\right\}$ is :

a. $\frac{3}{2}$ b. $\frac{5}{2}$
c. $\frac{5}{3}$ d. Not exist

17. The closed interval $[a, b]$ is :

a. Finite bounded set
b. Infinite bounded set
c. Finite unbounded set
d. None of these

18. The set R is real numbers is :

a. Bounded below b. Bounded above
c. Unbounded d. Bounded

19. The supremum of the set $S = \left\{m+\frac{1}{u} : m, n \in N\right\}$ is:

a. 2 b. 1
c. ∞ d. Does not exist

20. The infimum of the set $\left\{1+\frac{(-1)^n}{n} : n \in N\right\}$ is :

a. 0 b. 1
c. $\frac{3}{2}$ d. Not exist

21. For any $x \in R$ there exists a unique integer n such that : **[Meerut 2017]**

a. $x-1 < n \le x$ b. $x-1 \le n < x$
c. $n < x \le n+1$ d. None of these

22. The supremum of the set $\{(-1)^n n^2 : n \in N\}$ is :

a. -1 b. 2
c. $-\infty$ d. Not exist

23. The set $\{2, 2^2, 2^3, ... 2^{4/n}, ...\}$ is :

a. Bounded above but not below
b. Bounded
c. Bounded below but not above
d. None of these

24. The supremum of the set $\{x \in R : x = 2^{4/n}, n \in N\}$ is :

a. 2 b. 2^{∞}
c. 0 d. Not exist

25. The set $\{-1, -2, -3, -4, ...\}$ is :

a. Bounded above b. Bounded below
c. Bounded d. None of these

26. Between two distinct real numbers there always lies infinitely many :

a. Rational numbers b. Irrational numbers
c. Real numbers d. All are true

27. The set Q of rational numbers is :

a. Archimedean ordered field
b. Complete ordered field
c. Complete field
d. None of these

28. The infimum of the set $\left\{n+\frac{1}{m}: m, n \in N\right\}$ is :

a. 0 b. 1
c. 2 d. Does not exist

29. For any real number x, there exists a unique integer n such that : **[Kanpur 2018]**

a. $n < x \le n+1$
b. $x+ \le n \le x$
c. $n \le x < n+1$
d. None of these

30. The infimum of the set $\{n+3 : n \in N\}$ is :

a. 3 b. 4
c. ∞ d. Not exist

31. The supremum of the null set ϕ is :

a. 0 b. 1
c. Finite d. Does not exist

32. The number $\sqrt{3}$ 13 is :

a. Natural number b. Integer
c. Rational number d. Irrational number

33. The set $\{x : 0 \le x < y\}$ has :

a. Supremum but not infimum
b. Infimum but not supremum
c. Supremum and infimum both
d. Neither supremum nor infimum

34. The supremum of the set $\left\{\sin\frac{n\Pi}{3} : n \in N\right\}$ is :

a. $\frac{-\sqrt{3}}{2}$ b. $\frac{\sqrt{3}}{2}$
c. 1 d. Does not exist

35. If S is a non-empty bounded subset of R such that $\sup S = \inf S$ then the number of elements in S are :

a. One
b. Two
c. Finitely many points
d. Infinite

36. If a set S is bounded then the set $\{|x| : x \in S\}$ is :

a. Bounded
b. Unbounded
c. May be bounded or unbounded
d. None of these

37. The set $S = \left\{\frac{1}{2^n} : n \in N\right\}$ is :

a. Bounded
b. Unbounded
c. Bounded above only
d. Bounded below only

38. The supremum of the set $\left\{-\frac{1}{n} : n \in N\right\}$ is :

a. 1 b. –1
c. ∞ d. 0

39. The infimum of the set $\left\{\sin\frac{n\Pi}{3} : n \in N\right\}$ is :

a. 0 b. $\frac{\sqrt{3}}{2}$
c. $\frac{-\sqrt{3}}{2}$ d. Does not exist

40. The supremum and infimum both exist for :

a. R b. Q
c. Z d. None of these

41. Which of the following is not true ?

a. $|-x| = |x|$
b. $|x| = \max.\{x, -x\}$
c. $|x| = m,n\{x, -x\}$
d. $|x/y| = |x|/|y|$, provided $y \neq 0$

42. Which of the following is not true ?

a. $|x+y| \le |x|+|y|$ b. $|x-y| \le |x|+|y|$
c. $|x-y| \ge |x|-|y|$ d. $|x-y| \ge |x|+|y|$

43. The set $\{x \in R : x \ge a\}$ is represented by :

a. $[a, \infty)$ b. (a, ∞)
c. $(-\infty, a)$ d. $[a, \infty]$

44. To every real number corresponds a unique point on a directed line and to every point on the directed line corresponds a unique real number is known as :

a. Completeness axioms
b. Ordered axioms
c. Field axioms
d. Dedekind-Cantor axioms

45. Which of the following is meaningless ?

a. $\infty \times \infty$ b. $\infty \times (-\infty)$
c. $\infty - \infty$ d. $\frac{\infty}{a}, a \neq \infty$

46. If $a, b \in R$ then one and only one of the following is true $a > b$, $a = b$, $a < b$, this is called :
a. Closure law b. Transitivity law
c. Trichotomy law d. Associative law

47. Every non-empty bounded above subset of R has a supremum in R, is known as :
a. Ordered axiom
b. Field axiom
c. Completeness axiom
d. Dedekind-Cantor axiom

48. $[a, b)$ is defined by :
a. $\{x \in R : a < x < b\}$ b. $\{x \in R : a \le x \le b\}$
c. $\{x \in R : a \le x < b\}$ d. $\{x \in R : a < x \le b\}$

49. The supremum of the set $\{x \in R : -5x < 3\}$ is :
a. $\frac{-3}{5}$ b. $\frac{-5}{3}$
c. 0 d. Does not exist

50. The supremum of the set $\left\{\cos\frac{n\Pi}{2} : n \in N\right\}$ is :
a. 1 b. 0
c. –1 d. Does not exist

51. Every non-empty finite subset of R is :
a. Bounded
b. Unbounded
c. Bounded above only
d. None of these

52. The infimum of the set of all positive even integers is :
a. 0 b. 1
c. 2 d. Not exist

53. How many real numbers are there in [1, 4] :
a. 1 b. 4
c. 3 d. Infinite

54. If $A \le B$ such that A and B are non-empty subset of R then :
a. $\sup A = \sup B$ b. $\sup A \ge \sup B$
c. $\sup A \le \sup B$ d. None of these

55. If A and B non-empty subset S of R then $\inf(A \cup B)$ is equal to :
a. $\max\{\inf A, \inf B\}$ b. $\min\{\inf A, \inf B\}$
c. $\max\{\sup A, \sup B\}$ d. $\min\{\sup A, \sup B\}$

56. If $S = \{1, 3, 5, 7, ... 2n+1\}$ then $\sup S$ is :
a. 1 b. $2n+1$
c. ∞ d. Not exist

57. If A and B are non-empty subsets of R then $\sup(A \cup B)$ is equal to :
a. $\min\{\sup A, \sup B\}$ b. $\min\{\sup A, \inf B\}$
c. $\max\{\inf A, \sup B\}$ d. $\max\{\sup A, \sup B\}$

58. Which is true : **[Meerut 2018]**
a. Q is an open set b. R is not complete
c. Q is not complete d. Both (a) and (c)

59. Every non-empty set of real number which bounded has its :
a. Supremum b. Infimum
c. Both (a) and (b) d. All elements

60. Which one of the following is a complete order :
a. I b. Q
c. C d. R

61. The supremum of the set
$S = \left\{\left(1 - \frac{1}{n}\right)\sin\frac{n\pi}{2}, n \in N\right\}$ is : **[Meerut 2018,19]**
a. 0 b. 1
c. 3 d. –1

62. The supremum of the set $S = \{x : x \in N$ and $|x| < 13.33\}$:
a. 13.33 b. 14
c. 13 d. –13

63. Which is true : **[Meerut 2018]**
a. Supremum is always unique
b. Infimum is always unique
c. Set of upper bound is always infinite, if exist
d. All the above

64. The infimum of set $S = \{x : x \in R^+$ and $|x| < 7.7\}$ is: **[Meerut 2018]**
a. ϕ b. –7.7
c. –7 d. 0

65. The least upper bond of the set $\left\{\frac{1}{4} : n \in N\right\}$ is : **[Meerut 2017]**
a. 1 b. –1
c. 0 d. 2

66. The Null ϕ is : **[Meerut 2018]**

a. Bounded b. Unbounded
c. Both (a) and (b) d. None of these

67. If a and b be any two positive real numbers then from archimedean property $\exists$ a positive integer n such that : **[Meerut 2014]**

a. $a > nb$ b. $na \geq b$
c. $na > b$ d. $na > a$

68. Which set is complete with respect to boundeness :

a. N natural numbers
b. Q rational numbers
c. Z integer
d. Both (a) & (c)

69. Supremum of a bounded set ______ to the set is a limit point of the set : **[Meerut 2014]**

a. Belonging b. Not belonging
c. (a) or (b) d. None of these

70. If $\lim f(z) = n$ then $\lim |f(z)|$ is : **[Meerut 2014]**

a. n b. $|n|$
c. $-n$ d. 0

71. Between two distinct real number there exist at least one rational number is known as : **[Meerut 2014]**

a. Archimedean property
b. B.V. property
c. Density property
d. None of these

72. A is said to be dense in X if : **[Meerut 2014]**

a. $A \subset X$ b. $\bar{A} \subset X$
c. $\bar{A} = X$ d. None of these

73. The function $f(x) = x^n$ decreasing on the interval :

[Meerut 2016]

a. $]0, e[$ b. $\left]0, \frac{1}{e}\right[$
c. $]0, 1[$ d. None of these

74. Let x, y and z are real numbers then if $x < y$ and $z < 0$ then : **[Meerut 2019]**

a. $xz > yz$ b. $xz < yz$
c. $xz = yz$ d. None of these

75. If x and y are two real numbers and $x > 1$ then $\exists$ a natural number n such that : **[Meerut 2014]**

a. $x^n < y$ b. $x^n > y$
c. $x^n = y$ d. None of these

76. The least upper bond of the set $\left\{\frac{1}{n} : n \in N\right\}$ is :

[Meerut 2017]

a. 1 b. -1
c. 0 d. 2

77. The set of all real numbers is : **[Meerut 2017]**

a. An open set b. A closed set
c. Both (a) and (b) d. None of these

78. The Null set ϕ is :

a. Bounded b. Unbounded
c. Both (a) and (b) d. None of these

79. A and B are sets such that $a \in A,\ b \in B \Rightarrow a < b$ then :

[Meerut 2017]

a. $l.u.b. A \leq g.l.b. B$ b. $l.u.b. A \geq g.l.b. B$
c. $l.u.b. < g.l.b. B$ d. None of these

80. The derived set of the closed interval [3,5] is :

a.]3,5[b.]3,5]
c. [3,5] d. [3,5[

81. The limit points of the set $S = \left\{\frac{3n+2}{2n+1} : n \in N\right\}$ is :

a. $\frac{2}{3}$ b. $\frac{3}{2}$
c. $\frac{1}{3}$ d. 3

82. Derived set of the set $S = \{1 + 3^{-n} : n \in N\}$:

a. $\{1,2\}$ b. $\{1,2,3\}$
c. Singleton $\{1\}$ d. None of these

83. The derived set of the set $S = \{1,3,7,11\}$ is :

a. $\{1,3,7,11\}$ b. $\{1,3\}$
c. Empty set ϕ d. None of these

84. If $A =]2, 7[$ then A is equal to : **[Meerut 2018]**

a. [2,7[b. [2,7]
c. [2,7] d.]2,7[

85. Every non empty set of real number which is bounded has its : **[Meerut 2018]**

a. Supremum b. Infimum

c. Both (a) and (b) d. All elements

86. The infimum of the set

$$S = \left\{ x : x \in \text{ and } x = (-1^n) \cdot \left(\frac{1}{n} - \frac{4}{n} \right) n \in N \right\} \text{ is :}$$

[Meerut 2018]

a. $-\frac{2}{3}$ b. $-\frac{3}{3}$

c. $\frac{3}{2}$ d. $\frac{2}{3}$

87. The infimum of the set $S = \{x : x \in N \text{ and } |x| < 5.7\}$:

[Meerut 2018]

a. 0 b. –5.7

c. –5 d. 1

88. Which is true : **[Meerut 2018]**

a. Union of arbitrary family of open set is open

b. Intersection of finite collection of open set is open

c. Both (a) and (b)

d. None of the above

89. Q is equal to : **[Meerut 2018]**

a. Q b. $R - Q$

c. R d. ϕ

90. The infimum of the set

$S = \{x : x \in R^+ \text{ and } |x| > 5.8\}$: **[Meerut 2018]**

a. 5 b. –5

c. –5.8 d. Does not exist

91. The supremum and infimum of the set

$S = \{ |\cos x| \text{ and } x \in R\}$: **[Meerut 2018]**

a. 1 and –1 b. –1 and 1

c. 1 and 0 d. –1 and 0

92. Which is true : **[Meerut 2018]**

a. Q is an open set

b. R is not complete set

c. Q is not complete set

d. Both (a) and (c)

93. If $x > 0$ and $\forall\, y \in R\ \exists n \in N$ such that:

[Meerut 2018]

a. $nx = y$ b. $nx > y$

c. $nx < y$ d. $nx \geq y$

94. Which is true : **[Meerut 2019]**

a. The null set ϕ is unbounded

b. The null set ϕ is bounded

c. A finite subset of R is unbounded

d. None of the above

95. Supremum of a set S is always : **[Meerut 2019]**

a. Belongs to set S

b. Not greatest member of S

c. Exist

d. Greatest member of S

96. Let $S = \{y : y = |\sin x|\ \forall x \in R\}$, then $\sup S$ is equal to : **[Meerut 2019]**

a. –1 b. 1

c. 0 d. ∞

97. Let $A = \{x : x = |\sin y|\ \forall y \in R\}$, then $\inf A$ is equal to : **[Meerut 2019]**

a. –1 b. 1

c. 0 d. $-\infty$

98. If $S = \left\{ x : x = m + \frac{1}{n}, m, n \in N \right\}$ then $D(s)$ is :

[Meerut 2019]

a. ϕ b. $\{0\}$

c. 0 d. None of these

99. Let $S = \{x \in [-1, 4] \text{ and } \sin x > 0\}$, then which is true: **[Meerut 2019]**

a. $\inf S < 0$

b. $\sup S$ does not exist

c. $\sup S = \pi$

d. $\inf S = \pi / 2$

100. ______ set of real number has a non empty derived set : **[Meerut 2019]**

a. Every finite bounded set

b. Every infinite set

c. Every infinite bounded set

d. Every infinite unbounded set

101. Which one is not a perfect set : **[Meerut 2019]**

a. $A = \left\{x : x = \frac{1}{n}, n \in N\right\}$

b. $A = \left\{x : x = m + \frac{1}{n}, n \in N, m \in N\right\}$

c. $A = \left\{x : x = \frac{1}{m} + \frac{1}{n}, n \in N, m \in N\right\}$

d. (b) and (c) and (a) also

102. Sup and Inf of S is, where

$S = \{x : x \in z, |x|^2 \geq [25.99]\}$: **[Meerut 2019]**

a. Sup $S = \infty$ inf $S = -\infty$

b. Sup $S = 26$ inf $S = 25$

c. Sup $S = 0$ inf $S = 5$

d. Does not exist

103. If $A = \{x : x \in R - Q$ and $|x|^2 \leq [81, 99]\}$ then sup A and inf A are : **[Meerut 2019]**

a. Sup $A = 9$, inf $A = -9$

b. Sup $A = 9$, inf $A = 0$

c. Sup $A = 0$, inf $A = -9$

d. Sup $A = 9.9$, inf $A = -9.9$

104. Which of the following sets is bounded below but not bounded above : **[Meerut 2015]**

a. N b. Z

c. Q d. R

105. Every bounded set has its : **[Meerut 2018]**

a. Supremum b. Infimum

c. Both (a) and (b) d. Limit point

106. The domain of a sequence is always **[Meerut 2018]**

a. N b. R

c. R^+ d. Q

107. If $S = \left\{m + \frac{1}{n} : m, n \in N\right\}$ then sup S is : **[Meerut 2019]**

a. ∞ b. $-\infty$

c. Does not exist d. 0

108. Let $S = \left\{r^2 + \frac{1}{s^2}, r, s \in N\right\}$ then inf S will be :

a. 0 b. 1 c. –1 d. 2

109. Let r____ and s____, then ∃t____ such that st ___ r :

a. $\in R, \in R^+, \in I, >$

b. $\in R^+, \in R^+, \in I <$

c. $\in R^+, \in R, \in I, >$

d. None of these

110. Which is true :

a. Q is nbd of all its points

b. $R - Q$ is nbd of all its points

c. $R - Q$ is not nbd of all its points

d. R is not nbd of its points

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(c)	**2.**	(a)	**3.**	(a)	**4.**	(c)	**5.**	(a)	**6.**	(d)	**7.**	(c)	**8.**	(d)	**9.**	(d)	**10.**	(c)
11.	(c)	**12.**	(d)	**13.**	(a)	**14.**	(d)	**15.**	(c)	**16.**	(c)	**17.**	(b)	**18.**	(c)	**19.**	(d)	**20.**	(a)
21.	(b)	**22.**	(d)	**23.**	(c)	**24.**	(d)	**25.**	(a)	**26.**	(d)	**27.**	(a)	**28.**	(b)	**29.**	(c)	**30.**	(b)
31.	(d)	**32.**	(d)	**33.**	(b)	**34.**	(b)	**35.**	(a)	**36.**	(a)	**37.**	(a)	**38.**	(d)	**39.**	(c)	**40.**	(d)
41.	(c)	**42.**	(d)	**43.**	(a)	**44.**	(d)	**45.**	(c)	**46.**	(c)	**47.**	(c)	**48.**	(c)	**49.**	(d)	**50.**	(a)
51.	(a)	**52.**	(c)	**53.**	(d)	**54.**	(c)	**55.**	(b)	**56.**	(b)	**57.**	(d)	**58.**	(d)	**59.**	(c)	**60.**	(d)
61.	(b)	**62.**	(a)	**63.**	(d)	**64.**	(d)	**65.**	(c)	**66.**	(a)	**67.**	(b)	**68.**	(d)	**69.**	(b)	**70.**	(b)
71.	(c)	**72.**	(c)	**73.**	(b)	**74.**	(a)	**75.**	(b)	**76.**	(a)	**77.**	(c)	**78.**	(a)	**79.**	(a)	**80.**	(c)
81.	(b)	**82.**	(c)	**83.**	(c)	**84.**	(c)	**85.**	(c)	**86.**	(b)	**87.**	(d)	**88.**	(c)	**89.**	(c)	**90.**	(d)
91.	(c)	**92.**	(d)	**93.**	(b)	**94.**	(b)	**95.**	(b)	**96.**	(b)	**97.**	(c)	**98.**	(d)	**99.**	(c)	**100.**	(c)
101.	(d)	**102.**	(d)	**103.**	(b)	**104.**	(a)	**105.**	(c)	**106.**	(a)	**107.**	(c)	**108.**	(b)	**109.**	(d)	**110.**	(c)

HINTS AND SOLUTIONS

1. $N = \{1, 2, 3, 4, ...\}$

   Its infimum is 1 so it is bounded below by 1.

2. $S = \{x \in Z : x^2 \le 25\}$

   So, $S = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$

   Thus, supremum = 5 and infimum = –5

5. $S = \left\{x \in Q : x = \dfrac{(-1)^n}{n}, n \in N\right\}$

   or $S = \left\{-1, \dfrac{-1}{3}, \dfrac{-1}{3} .... \dfrac{1}{6}, \dfrac{1}{4}, \dfrac{1}{2}\right\}$

   So, supremum = $\dfrac{1}{2}$, infimum = –1

6. The set Q is not complete but ordered field.

8. $z^+ = \{1, 2, 3, 4, ...\}$

   Hence, infimum = 1 but supremum does not exist.

9. Let $S = \{(-1)^n . n : n \in N\}$

   then $S = \{.... -5, -3, -1, 2, 4, 6.....\}$

   Thus neither supremum nor infimum exists.

11. $S = \left\{1 + \dfrac{(-1)^n}{n} : n \in N\right\}$

    or $S = \left\{1 - 1, 1 + \dfrac{1}{2}, 1 - \dfrac{1}{3}, 1 + \dfrac{1}{4}, 1 - \dfrac{1}{5}.....\right\}$

    or $S = \left\{0, \dfrac{3}{2}, \dfrac{2}{3}, \dfrac{5}{4}, \dfrac{4}{5}, ..., \dfrac{3}{2}\right\}$

    So, its supremum = $\dfrac{3}{2}$

14. $S = \{(-2)^n : n \in N\}$

    or $S = \{.....-8, -2, 4, 16, ...\}$

    Thus, neither supremum nor infimum exists.

    i.e., S is unbounded.

15. $S = \left\{1, \dfrac{1}{4}, \dfrac{1}{4^2}, \dfrac{1}{4^3}, ... \dfrac{1}{4^n}, ...\right\}$

    Here sup = 1 and infimum = 0 but does not belongs to S. Thus the set S is bounded.

16. $S = \left\{\frac{3n+2}{2n+1} : n \in N\right\}$

If $n = 1$ then $S = \frac{5}{3}$

Also $S = \frac{3n+2}{2n+1} = \frac{3+2/n}{2+1/n}$

So, $\lim_{n\to\infty} S = \frac{3}{2}$

Thus, sup $= \frac{5}{3}$ and infimum $= \frac{3}{2}$

19. $S = \left\{m + \frac{1}{n} : m, n \in N\right\}$

$S = \left\{1+1, 1+\frac{1}{2}, 1+\frac{1}{3} \ldots\right\} \cup \left\{2+1, 2+\frac{1}{2}, \ldots\right\} \cup \ldots$

Here infimum $= 1 \notin S$ and supremum does not exist.

20. $S = \left\{1 + \frac{(-1)^n}{n} : n \in N\right\}$

$S = \left\{1-1, 1+\frac{1}{2}, 1-\frac{1}{3}, 1+\frac{1}{4}, \ldots\right\}$

or $S = \left\{0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \ldots\right\}$

Thus, infimum = 0

22. $S = \{(-1)^n n^2 - n \in N\}$

or $S = \{-(1)^2, (2)^2, -(3)^2, (4)^2, -(5)^2 \ldots\}$

or $S = \{\ldots -25, -9, -1, 4, 16, \ldots\}$

Thus, neither supremum nor infimum exist.

23. $S = \{2, 2^2, 2^3, \ldots, 2^n, \ldots\}$

sup $= \lim_{n\to\infty} 2^n = \infty$

and infimum = 2

Thus, S is bounded below but not bounded above.

25. $S = \{-1, -2, -3, -4, \ldots\}$

Here supremum = –1, infimum = not exist.

So, S is bounded above by –1.

30. $S = \{n + 3 = n \in N\}$

or $S = \{4, 5, 6, 7, \ldots\}$

So, sup is not exist but infimum = 4

33. $S = \{x : 0 \le x < 4\}$

Here, infimum of $S = 0$ and supremum of $S = 4$ but supremum does not exist in S.

34. $S = \left\{\sin\frac{n\pi}{3} : n \in N\right\}$

or $S = \left\{\sin\frac{\pi}{3}, \sin\frac{2\pi}{3}, \sin\pi, \sin\frac{4\pi}{3}, \ldots\right\}$

or $S = \left\{\sqrt{3}, \frac{-1}{2}, 0, \frac{\sqrt{3}}{2}, \frac{-\sqrt{3}}{2}\right\}$

Thus, supremum of $S = \frac{\sqrt{3}}{2}$

37. $S = \left\{\frac{1}{2n} : n \in N\right\}$

or $S = \left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots 0\right\}$

Here, supremum $= \frac{1}{2}$, infimum = 0

So, S is bounded set.

38. $S = \left\{\frac{-1}{n} : n \in N\right\} = \left\{-1, \frac{-1}{2}, \frac{-1}{3}, \ldots\right\}$

Here, supremum $= 0 \notin S$ and infimum $= -1 \in S$

39. By the solution of question (34) infimum of S

$= \frac{-\sqrt{3}}{2}$

49. $S = \{x \in R : -5x < 3\}$

or $S = \{x \in R : x > -3/5\}$

Thus, $S =]\frac{-3}{5}, \infty[$

So, its supremum does not exist but infimum $S = \frac{-3}{5}$.

50. $S = \left\{\sin\frac{n\pi}{2} : n \in N\right\}$

or $S = \{1, 0, -1, 0, \ldots\}$

Thus, supremum of $S = 1$ and infimum of $S = -1$

53. Here $S = [1, 4]$ since it is an interval so S contains infinite number of real numbers.

❍❍❍

CHAPTER

2

Countable and Uncountable Sets

COUNTABLE AND UNCOUNTABLE SET

1. **Equivalent sets :** Two sets A and B are said to be equivalent or equipotent if there exists a one-one onto mapping *i.e.* bijective mapping $f : A \to B$ and is denoted by $A \sim B$.
2. **Denumerable set :** A set A is said to be denumerable or countable infinite if there exists a one-one correspondence between the set A and the set N of natural numbers *i.e.* there exists a one-one $f : N \to A$.

   Example: Consider the set $A = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\right\}$.

   Then A is denumerable since $f : N \to A$ is defined by $f(n) = \frac{n}{n+1}$, $\forall n \in N$ is one-one and onto.
3. **Countable set :** A set which is either finite or denumerable is said to be countable.
4. **Uncountable set :** A set which is neither finite nor denumerable is said to be uncountable or non-denumerable.

USEFUL RESULTS

1. A subset of a countable set is countable.
2. Every superset of an uncountable set is uncountable.
3. Union of a finite number of countable set is countable.
4. The union of a countable family of countable sets is countable.
5. The set $N \times N$ is countable.
6. Union of two countable sets is countable.
7. Finite set is countable.
8. If A and B are countable then $A \times B$ is also countable.
9. If one enumerable set is subtracted from the other enumerable set then remaining set will be enumerable.
10. If enumerable set is subtracted from non-enumerable set then the remaining set will be non-enumerable.

EXERCISE

MULTIPLE CHOICE QUESTIONS

Direction : Each of the following questions has four alternative answers. One of them is correct. Choose the correct answer.

1. The set z of all integers is :

   a. Finite b. Uncountable

   c. Countable d. None of these

2. Denumerable set is :

   a. Countable finite b. Uncountable

   c. Countable infinite d. None of these

3. The set $A_n = \left\{\frac{0}{n}, \frac{1}{n}, \frac{-1}{n}, \frac{2}{n}, \frac{-2}{n} \ldots\right\} \forall n \in N$ is :

   a. Bounded

   b. Countable

   c. Uncountable

   d. Unbounded and uncountable both

4. The set $[a, b]$, $a < b$ is always :

   a. Bounded and countable

   b. Unbounded and countable

   c. Bounded and uncountable

   d. Unbounded and uncountable

5. The super set of a countable set is :
   a. Countable
   b. Uncountable
   c. Countable or uncountable
   d. None of these
6. The set $\{\pm 1, \pm 4, \pm 9, \pm 16....\}$ is :
   a. Finite and bounded
   b. Infinite and bounded
   c. Bounded and countable
   d. Unbounded and countable
7. The set of all rational points in the co-ordinate plane R^2 is :
   a. Countable b. Uncountable
   c. Finite d. Bounded
8. The set of all irrational numbers is :
   a. Finite b. Bounded
   c. Countable d. Uncountable
9. The cardinal number of the empty set φ is :
   a. 0 b. 1
   c. Finite d. Infinite
10. A countable set is :
   a. Finite only b. Infinite only
   c. Finite or infinite d. Neither finite nor infinite
11. The set R of real number is :
   a. Bounded b. Countable
   c. Uncountable d. Finite
12. The super set of uncountable set is :
   a. Bounded b. Countable
   c. Uncountable d. None of these
13. The cardinal number of the set $\{\pm 1, \pm 2, \pm 3, \pm 4\}$ is :
   a. Infinite b. 0
   c. 8 d. 4
14. If A_i is enumerable set for $i = 1, 2, 3, ...$ onto $A = \bigcup_{i=1}^{\infty} A_i$ then card A is equal to :
   a. a b. c
   c. ∞ d. Finite
15. A subset of a countable set is always :
   a. Bounded b. Unbounded
   c. Countable d. Uncountable
16. The union of countable family of countable sets is :
   a. Countable
   b. Uncountable
   c. Countable or uncountable
   d. None of these
17. Every denumerable set is equivalent to :
   a. N b. Z
   c. Q d. R
18. If A_i is non-enumerable for $i = 1, 2, 3$ with $A = \bigcup_{i=1}^{\infty} A_i$ then card (A) is equal to :
   a. a b. c
   c. ∞ d. 0
19. The family of all finite subsets of N is :
   a. Countable finite b. Countable infinite
   c. Uncountable d. None of these
20. If a, c and f denote the cardinal numbers of set of all natural numbers, real numbers and set of all real valued functions defined over [0, 1] respectively then :
   a. $c < a < f$ b. $a < c < f$
   c. $f < a < c$ d. $a < f < c$
21. Which of the following set is countable :
   a. C b. R
   c. Q d. $[a, b]$
22. If a is any real number and b is positive real number then there exists a positive integer n, such that :
   a. $nb > a$ b. $nb \geq a$
   c. $nb < a$ d. $nb \leq a$
23. The bounded set is always :
   a. Countable
   b. Uncountable
   c. Infinite
   d. May or may not be countable

24. If a and c are the cardinal numbers of set of all natural numbers and real numbers then :
a. $a+c=c$ b. $a+a=a$
c. $a.c=c$ d. All are true

25. Every isolated set of points is :
a. Countable
b. Uncountable
c. Countable or uncountable
d. None of these

26. The set of all polynomial functions with integer (rational coefficient) is :
a. Countable
b. Uncountable
c. Countable or Uncountable
d. None of these

27. The set of all sequences whose elements are the digits 0 and 1 is :
a. Countable
b. Uncountable
c. Countable or uncountable
d. None of these

28. If a finite set of elements is added to an enumerable set then the resulting set is :
a. Finite b. Enumerable
c. Finite or enumerable
d. None of these

29. If the range of $f: A \to B$ is uncountable the domain of f is :
a. Countable
b. Uncountable
c. Countable or uncountable
d. None of these

30. The set of all characteristic functions on R is :
a. Countable
b. Uncountable
c. Countable or uncountable
d. None of these

31. The set of all real valued functions on [0, 1] has the cardinal number :
a. 2^a b. a^2 c. 2^c d. c^2

32. c^a is equal to :
a. a b. c
c. a^c d. None of these

33. The set of all algebraic numbers is :
a. Finite b. Enumerable
c. Uncountable d. None of these

34. The set of all transcendental numbers in any interval is :
a. Finite b. Enumerable
c. Non-enumerable d. None of these

35. Every infinite set is always :
a. Countable
b. Uncountable
c. May be countable or uncountable
d. None of these

36. The unit interval [0,1] is : **[Kanpur 2018]**
a. Countable b. Denumerable
c. Non-denumerable d. None of these

37. Which is true :
a. The set R is uncountable
b. The set R is countable
c. There is no subset of countable set
d. N is not countable

38. Which is not true : **[Meerut 2018]**
a. The set Q is numerable
b. The set Q is non-numerable
c. The set $N \times N$ is countable
d. The interval [0,1] is uncountable

39. The sequence $< (-1)^n >$ is : **[Meerut 2018]**
a. Convergent b. Divergent
c. Oscillates finitely d. Oscillatory

40. $\lim r^{1/n} = 1$ if : **[Meerut 2018]**
a. $r=0$ b. $r>-1$
c. $r<1$ d. None of these

41. Which is true : **[Meerut 2018]**
a. The set R is uncountable
b. The set R is countable
c. There is no subset of countable set
d N is not countable

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(c)	**2.**	(c)	**3.**	(b)	**4.**	(c)	**5.**	(c)	**6.**	(d)	**7.**	(a)	**8.**	(d)	**9.**	(a)	**10.**	(c)
11.	(c)	**12.**	(c)	**13.**	(c)	**14.**	(a)	**15.**	(c)	**16.**	(a)	**17.**	(a)	**18.**	(b)	**19.**	(b)	**20.**	(b)
21.	(b)	**22.**	(a)	**23.**	(d)	**24.**	(d)	**25.**	(a)	**26.**	(a)	**27.**	(b)	**28.**	(b)	**29.**	(b)	**30.**	(b)
31.	(c)	**32.**	(b)	**33.**	(b)	**34.**	(c)	**35.**	(c)	**36.**	(c)	**37.**	(a)	**38.**	(a)	**39.**	(c)	**40.**	(d)
41.	(a)																		

HINTS AND SOLUTIONS

1. Let N and Z denotes the set of natural number and integer. Consider $f : N \to Z$ by

$$f(r) = \frac{r-1}{2} \quad \text{where } r \text{ is odd and}$$

$$f(r) = -\frac{r}{2} \quad \text{where } r \text{ is even}$$

f is one-one, since $f(1) = 0$, $f(2) = -1$, $f(3) = 1$,

$$f(4) = -2,\ f(5) = 2.....$$

f is onto : Let $y \in Z$. If $y \geq 0$, then

$2y + 1 \in N$ and $f(2y + 1) = \frac{\{(2y+1)-1\}}{2} = y$

If $y < 0$, then, $-2y \in N$ and

$$f(-2y) = \frac{\{-(-2y)\}}{2} = y$$

Thus, $y \in Z$ *i.e.* there exist some $x \in N$ such that $f(x) = y$. Therefore, f is onto. Hence z is countable.

3. Let $A_n = \left\{\frac{0}{n}, \frac{-1}{n}, \frac{1}{n}, \frac{-2}{n}, \frac{2}{n} ....\right\}$ so A_n is the set of all those rationals whose denominator is $n \in N$.

Each A_n is equivalent to N since $f : N \to A_n$ defined by

$$f(r) = \begin{cases} \frac{r-1}{2n}, & \text{when } r \text{ is odd} \\ \frac{-r}{2n}, & \text{when } r \text{ is even} \end{cases}$$

is one-one onto as by the solution of example (1).

Also $\bigcup_{n=1}^{\infty} A_n$ is the set of all rationals. Thus the set of all rationals *i.e.* $\bigcup_{n=1}^{\infty} A_n$ is countable.

6. Let $A = \{\pm 1^2, \pm 2^2, \pm 3^2, \pm 4^2, ...\}$

Consider $f : N \to A$, defined by

$$f(n) = \begin{cases} \left(\frac{n+1}{2}\right)^2, & \text{if } n \text{ is odd} \\ -\left(\frac{n}{2}\right)^2, & \text{if } n \text{ is even} \end{cases}$$

Then f is one-one onto. Hence, A is countable.

8. Since the set R of real numbers is uncountable and the set Q of rational numbers is countable, if follows that the set $R - Q$ of all irrational numbers is uncountable.

11. We know that a subset of a countable set is countable. Hence if R was countable, then [0,1] would also be countable. This contradicts that [0,1] is uncountable. Thus, R is uncountable.

12. Let A be uncountable set and $A \subset B$.

Suppose B is countable. Since $A \subset B$ therefore A must be countable which is against our hypothesis. Hence B must be countable.

15. Let A be any subset (countable) and $B \subset A$.

If B is finite then B is countable.

If B is infinite then A is also infinite. Thus A is countably infinite *i.e.* denumerable so A can be written as a sequence $< a_1, a_2, a_3 .... a_n >$. Let n_1 be the smallest positive integer such that $a_{n_1} \in B$. Again let n_2 be the next smallest positive integer such that $n_2 > n_1$ and $a_{n_2} \in B$ as so on.

Then, $\quad B = \{a_{n_1}, a_{n_2} .....\}$

Thus the mapping $f : N \to B$ defined by $f(k) = a_{n_k}$ is one-one and onto. Hence, B is also countable.

21. Q is countable as proved by the solution of question (3).

23. Consider $A = \{1, 2, 3, 4, 5\}$ then it is bounded and countable also.

Now consider $A = [1, 2]$ then B is bounded but not countable.

26. Consider $P_n(x) = \alpha_n x^n + \alpha_{n-1} x^{n-1} + ... \alpha_1 x$ $+ \alpha_0 (\alpha_n \neq 0)$ with integer (rational) coefficients.

If $|\alpha_n| + |\alpha_n - 1| + ... + |\alpha_0| = m$ then for each pair of natural numbers (m, n), the set P_{mn} of all polynomials of the form $P_n(x) = \alpha_n x^n + 2_{n-1} x^{n-1} + ... + \alpha_1 n + \alpha_0$ is finite and hence countable. Also the sets $P_{(m,n)} = P_k, K = (m,n) \in N \times N$ themselves are countable. Therefore, the set $P = \bigcup_{(m,n) \in N \times N} P_{m,n}$ is also countable.

28. Let cord $A = \alpha$ *i.e.* A is infinite set. Since A is infinite set so $\exists$ a subset B of A such that B is enumerable *i.e.* cord $B = a$.

Now, $A = (A - B) \cup B$

or $A \cup N = (A - B) \cup B \cup N$

or $A \cup N = (A - B) \cup (B \cup N)$

B and N are enumerable sets

$\Rightarrow$ $B \cup N$ is enumerable

$\Rightarrow$ $B \cup N \sim N$

Now, $B \cup N \sim N, N \sim B \Rightarrow B \cup N \sim B$

$= (A - B) \cup (B \cup N) \sim (A - B) \cup B$

i.e. $A \cup N \sim A$

or cord $(A \cup N) =$ cord $A \Rightarrow \alpha + a = \alpha$

33. Consider the algebraic equation,

$\alpha_0 x^n + \alpha_1 x^{n-1} + ... \alpha_n$ of degree n with $\alpha_0 \neq 0$.

Define the rank of this equation by

$$|\alpha_0| + |\alpha_1| + ... |\alpha_n| = m$$

Here, m is positive. Also α, s are integers r_0 rank is an integer greater than 1. Also for a given rank the roots of the equation will be finite and so enumerable.

Now put a one-one correspondence in the set N with algebraic equation arranged with respect to rank and hence the set of all algebraic equation is enumerable. Now each algebraic equation has enumerable number of roots and so the set of all algebraic number is enumerable collection of enumerable sets and hence enumerable.

34. We know that the set of all algebraic numbers and transcendental number is the set of all real numbers which is known to be uncountable. But the set of algebraic numbers is an interval *i.e.* enumerable. Also we know that if an enumerable set is removed from a non-enumerable set the remaining set is non-enumerable so the set of all transcendental number is non-enumerable.

❍❍❍

CHAPTER

3 Point Set Theory

NEIGHBOURHOOD OF A POINT

A subset N of R is a said to be a neighbourhood (nbd.) of a point $p \in R$ if there exists a real number $\varepsilon > 0$ such that

$$p \in (p - \varepsilon,\ p + \varepsilon) \subset N$$

In other words, $N \subset R$ is nbd of a point $p \in R$ if there exists an open interval contained in N whose centre is the point p.

In very simple form the open interval $(p - \varepsilon,\ p + \varepsilon)$ is called an ε-neighbourhood of p and is denoted by $N_\varepsilon(p)$.

Deleted neighbourhood : The set $N - \{p\}$ is called a deleted nbd of p.

PROPERTIES

1. A subset N of R is a nbd of a point $p \in R$ if and only if there exists an open interval $]\,a, b\,[$ containing p and contained in N is

$$p\ \varepsilon\ ]\,a, b\,[\subset N$$

2. A non-empty subset A of R is a *nbd* of $p \in R$ iff there exists a positive integer n such that

$$p\ \varepsilon\ ]\,p - \varepsilon,\ p + \varepsilon\,[\subset A$$

3. Any open interval is a *nbd* of each of its points. For let $]\,a, b\,[$ be any open interval and x be any arbitrary point of $]\,a, b\,[$. We have to show that $]\,a, b\,[$ is a *nbd* of x. Choose $\in$ as the minimum of two positive numbers $x - a$ and $b - x$ then $\varepsilon > 0$ is such that

$$x\ \varepsilon\ ]\,x - \varepsilon,\ x + \varepsilon\,[\subset]\,a, b\,[$$

Hence, $]\,a, b\,[$ is a *nbd* of x. Since $x \in\ ]\,a, b\,[$ so every open interval is a *nbd* of each of its points.

4. Any closed interval $[a, b]$ is a *nbd* of each of its points except the two end points a and b.
5. On the real line R, for each point $p \in R$, there exists at least one *nbd* of p.
6. if N is a *nbd* of any point $p \in R$ then $p \in N$.
7. Any superset of a *nbd* of a point is also a *nbd* of that point.
8. The intersection of two *nbds* of a point is also a *nbd* of that point.
9. On the real line R for each point $p \in R$ and each *nbd* N of p, there exists a *nbd* M of p such that $M \subset N$ and M is a *nbd* of each of its points.
10. The intersection of finitely many nbds of a point is also a nbd of that point.
11. If p and q are any two distinct real numbers, then there exist nbd of p and q which are disjoint. This property is called Hausdarff property.
12. Let be any point of the nbd $N(p, \varepsilon)$, then there exists a nbd of a which is entirely contained in $N(p, \varepsilon)$.
13. Let a be any point of the intersection M of the nbds $N(a_1, \varepsilon_1)$ and $N(a_2, \varepsilon_2)$ of a_1 and a_2. Then there exists a nbd of a which is entirely contained in M.

LIMIT POINTS AND CLOSED SETS

1. **Limit point or limiting point :** A point $p \in R$ is said to be a limit point (or accumulation point or condensation point or cluster point) of a set $A \subset R$ if every neighbourhood (*nbd*)N of p contains a point of A distinct from p *i.e.* $[N - \{p\}] \cap A \neq \phi$.

   Thus p is a limit point of A iff for each $\varepsilon > 0$, the open interval $]\,p - \varepsilon,\ p + \varepsilon\,[$ contains a point of A other than p. The set of all limit points of A is called the derived set of A and is denoted by $D(A)$.

2. **Adherent point :** A point $p \in R$ is said to be an adherent point of a set $A \subseteq R$ if every nbd of p contains a point of A.

The set of all adherent points of A is called the adherence of A and is denoted by $Adh\ A$.

Remark : It is clear that every limit point of $A \subseteq R$ is also an adherent point of A. But the converge is not always true. For example, for the set $A = \{1, 1/2, 1/3, \ldots 1/4 \ldots\}$, adherent point of A is 1 but it is not a limit point of this set.

3. **Closed set :** A set $A \subset R$ is said to be closed if it contains all its limit points. There A is closed set is $D(A) \subseteq A$.

4. **Isolated points :** A point $a \in A$ is said to be an isolated point of $A \subseteq R$ if it is not a limit point of A *i.e.* if there exists a nbd of a which contains no points of A other than a itself.

5. **Discrete set :** A set $A \subseteq R$ is called a discrete set if all its points are isolated points.

Example : Let $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n} \ldots\right\}$ then all the points of A are isolated and so A is a discrete set.

6. **Dense-in-itself set :** A subset A of R is said to be dense-in-itself if it possesses no isolated points *i.e.* every point of $A \subseteq R$ is a limit point of A.

7. **Perfect set :** A subset A of R is said to be perfect if $A = D(A)$ where $D(A)$ is the set of all limit points of A is derived set of A.

In other words a set $A \subseteq R$ is called a perfect set if it is dense-in-itself and if it contain all its limit points.

Example: Let $A = [0, 1]$ then $D(A) = [0, 1]$ so $A = D(A)$ *i.e.* A is a perfect set.

8. **Bolzano-weierstrass theorem :** Every bounded infinite subset of R has at least one limit point.

9. **Closure of a set :** Let $A \subseteq R$, then the closure of A is the intersection of all closed super sets of A and is denoted by $\overline{A}$.

Thus, $\overline{A} = \cap \{F : F \text{ is closed and } A \subset F\}$

Obviouslly $A \subseteq \overline{A}$

Further $\overline{\varphi} = \varphi$ and $\overline{R} = R$

10. **Dense set :** A subset A of R is said to be dense in R *i.e.* everywhere dense if $\overline{A} = R$.

11. **Properties :**

(i) A point $p \in R$ is a limit point of a set $A \subseteq R$ iff every nbd of p contains infinitely many points of A.

(ii) A point $p \in R$ is a limit point of a set $A \subseteq R$ iff for each *nbd* of p

$$(A \cap N) - \{p\} \neq \phi$$

(iii) If a non-empty subset of A of R which is bounded above has no maximum member, then its supremum is a limit point of the set A.

(iv) If a non-empty subset A of R which is bounded below has no minimum member, then its infimum is a limit point of the set A.

(v) If ϕ be the empty set, then its derived set $D(\phi) = \phi$.

(vi) The derived set of any bounded set is again a bounded set.

(vii) Every infinite bounded set has the greatest and the smallest limit points *i.e.* the derived set of any infinite bounded set attains its bounds.

(viii) A set is said to be of first species if it has only a finite number of derived sets and if there exist infinite number of derived sets then the set is of second species.

(ix) A set where n^{th} derived set is a finite set so that its $(n + 1)$ the derived set is empty is called a set of n^{th} order.

(x) The smallest and greatest limit points of an infinite bounded set S, or they do exist are called the lower (inferior) limit and the upper (superior) limit of S and denoted by $p = \underline{\lim}\, S$ and $q = \overline{\lim}\, S$ respectively and satisfying $\underline{\lim}\, S \le \overline{\lim}\, S$.

(xi) The derived set of a set is always closed set.

(xii) If A and B be two subsets of R then

(a) $A \subset B \Rightarrow D(A) \subseteq D(B)$

(b) $D(A \cup B) = D(A) \cup D(B)$

(c) $D(A \cap B) \le D(A) \cap D(B)$

(d) $x \in D(A) \Rightarrow x \in D(A - \{x\})$

(xiii) The intersection of an arbitrary collection of closed sets is closed.

(xiv) The union of a finite collection of closed sets is closed.

(xv) If A any subset of R then

(a) $\overline{A}$ is closed

(b) $\overline{A}$ is the smallest closed superset of A

(c) A is closed $\Leftrightarrow A = \overline{A}$

(d) $\overline{A} = A \cup D(A)$

(xvi) If A and B are two subsets of R then

(a) $A \subset B \quad \Rightarrow \quad \overline{A} \subseteq \overline{B}$

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(c) $\overline{A \cap B} \le \overline{A} \cap \overline{B}$

(d) $\overline{(\overline{A})} = \overline{A}$

INTERIOR POINTS AND OPEN SETS

1. **Interior points :** Let A be a subset of R. A point $p \in A$ is said to be an interior point of A if there exists a *nbd* $]p - \varepsilon, p + \varepsilon[$ of p which is entirely contained in A is

$$p \in]p - \varepsilon, p + \varepsilon[\subset A$$

The set of all interior points of A is called the interior of A and is denoted by int (A) or $A°$. obviously $A° \subset A$.

2. **Open sets :** A subset A of R is called an open set if every point of A is an interior point of A. It is obvious that a set A is open iff $A° = A$ and A is open iff it contains a nbd of each of its points.

3. **Exterior points :** Let A be a subset of R. A point $p \in R$ is said to be an exterior point of A if there exists a *nbd* of p which is contained in A^c. Clearly $p \notin A$.

4. **Frontier points or Boundary points :** Let A be a subset of R. A point $p \in R$ is said to be a frontier or boundary point of A if every nbd of p contains points of A and AC.

The set of all boundary point of A is called the boundary of A and denoted by $B(A)$.

5. **Properties :**

(i) Every point of an open interval is an interior point of the interval.

(ii) Every point of a closed interval except the end points of the interval is an interior point of the interval.

(iii) Every open interval is an open set but not conversely.

(iv) The union of an arbitrary family of open sets is open.

(v) The intersection of a finite collection of open sets is an open set.

(vi) A subset A of R is open iff it is a union of open intervals.

(vii) Let A be a subset of R then

(a) Int (A) equals the union of all open subsets of A.

(b) Int (A) is an open set.

(c) Int (A) is the largest open subset of A.

(d) A is open $\Leftrightarrow$ Int $(A) = A$.

(viii) No point of an open set A can be bound of A.

(ix) A subset A of R is closed if and only if its complement A^C is open.

EXERCISE

MULTIPLE CHOISE QUESTIONS

Direction : Each of the following question has four alternative answers. One of them is correct. Choose the correct answer.

1. The set of all real numbers is : **[Kanpur 2017,18]**
   a. Closed only
   b. Open only
   c. Both open and closed
   d. Neither open nor closed
2. Which of the following set is a nbd of each of its points : **[Meerut 2018]**
   a. N b. Q
   c. Z d. R
3. The set {1, 2, 3, 4} is a nbd of :
   a. 2 b. 3
   c. 1 and 4 d. None of these
4. The finite set is :
   a. Open
   b. Closed
   c. May be open or closed
   d. Neither open nor closed
5. If $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$ then 1 is a :
   a. Limit point
   b. Adherent point
   c. Limit point and adherent point both
   d. None of these
6. If a set contains all its limit points then the set is called :
   a. Open set b. Dense set
   c. Closed set d. Neighbouring
7. A set A is called discrete set if all its points are :
   a. Limit points b. Adherent points
   c. Isolated points d. None of these
8. Which of the following is nbd of each of its points :
   a. Closed interval
   b. The set N of natural number
   c. The set Q of rational nubmers
   d. Open interval
9. A point $p \in R$ will be a limit point of a subset A of R iff :
   a. $N \cap A - \{p\} = \phi$
   b. $(N - \{p\}) \cap = \phi$
   c. $N \cap (A - \{p\}) = \phi$
   d. None of these
10. The set of rational number Q is :
    a. Open
    b. Closed
    c. Either open or closed
    d. Neither open nor closed
11. The intersection of finite number of open set is :
    a. Closed
    b. Open
    c. Neither open nor closed
    d. None of these
12. The set $[1, 3] - \left\{2\frac{1}{2}\right\}$ is a nbd of :
    a. 2 b. $2\frac{1}{2}$
    c. 1 d. 3
13. Which of the following is true :
    a. Every limit point is adherent points
    b. Every adherent point is limit points
    c. Limit points and adherent parts are same
    d. None of these
14. Which of the following set is a nbd of each of its points :
    a. Finite set
    b. Set of natural number N
    c. Set of rational number Q
    d. Empty set
15. If $I_n =]\frac{-1}{n}, 1 + \frac{1}{n}[\ \forall n \in N$ then $\bigcap_{n=1}^{\infty} I_n$ is :
    a. $]\,0\,1\,[$ b. $]-1, 2[$
    c. $[0, 1]$ d. ϕ

16. The set $A =]\ a, b]$ is a nbd of each of its points except at :
a. a b. b
c. Both a and b d. None of these

17. Which of the following is an open set but not interval:
a. (0 1] b. [2 4]
c. $(0, 1) \cup (1\ 2]$ d. $(2,3) \cup (4,5)$

18. Which of the following set is not closed :
a. $]\ a, \infty\ [$ b. $[0, 2] \cup [3, 4]$
c. Q d. Finite subset of R

19. The nbd of 2 is :
a. N b. Z
c. Q d. R

20. Which of the following set is a nbd of 3 :
a. (3,6) b. (3,6]
c. $[2,4]-\{3\}$ d. (2,4)

21. Which of the following is true :
a. $A = Adh\ A$ b. $A \subseteq Adh\ A$
c. $Adh\ A \subseteq A$ d. None of these

22. A set $A \subseteq R$ is said to be a closed set if it contains all its :
a. Isolated points b. Adherent points
c. Interior points d. Limit points

23. The set $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots. \frac{1}{n} \ldots.\right\}$ is a :
a. Closed set b. Open set
c. Discrete set d. Uncountable set

24. If a set A is dense-in-self and contains all its limit points then A is :
a. Open set b. Perfect set
c. Discrete set d. Uncountable set

25. The derived set of (a, b) is :
a. (a, b) b. $[a, b]$
c. $\{a, b\}$ d. Whole real set

26. The union of an arbitrary family of open sets is :
a. Open
b. Closed
c. Not necessary open
d. Both open and closed

27. Which of the following set has no limit points :
a. Set of real numbers
b. Set of rational numbers
c. Set of natural numbers
d. $\left\{\frac{1}{n^2} : n \in N\right\}$

28. If $A =]\ a, b\ [$ then Int (A) is :
a. $\{a, b\}$ b. $[a, b]$
c. $]\ a, b[$ d. ϕ

29. Which of the following is a nbd of 2 :
a. (2,5) b. (2,4)
c. (1,2) d. (1,3)

30. A non empty finite set is a nbd of :
a. Its all points b. Same of its points
c. Only one points d. None of points

31. A point $a \in A$ is said to be an isolated point of A if it B not a :
a. Interior points
b. Adherent points
c. Limit points
d. Isolated points

32. The intersection of an arbitrary collection of open sets is :
a. Open
b. Closed
c. May be open or closed
d. May or may not be open

33. Every finite subset of R is a :
a. Open set
b. Closed set
c. May be open or closed
d. Neither open nor closed

34. The union of a finite collection of closed sets is :
a. Closed
b. Open and closed both
c. Not necessarily closed
d. Neither open nor closed

35. If A is bounded infinite subset of R then its minimum number of limit points are :
a. One b. Two
c. Nil d. Infinite

36. Which of the following sets are closed :
(i) finite set (ii) set of integers
(iii) $[a, b]$ (iv) derived set
(v) $[1, 2] \cup [3, 4[$ (vi) ϕ
a. (i), (ii), (iii), (iv), (vi)
b. (i), (iii), (iv)
c. (ii), (iii), (iv), (v), (vi)
d. (i), (ii), (iii), (iv), (v), (vi)

37. For the set of rational number Q, $\overline{Q}$ is equal to :
a. Q b. Irrational number
c. R d. ϕ

38. $A \subseteq R$ is said to be dense-in-itself every energy point of A is a :
a. Limit point of A b. Isolated point of A
c. Interior point of A d. Frontier point of A

39. Which of the following is a nbd of 3 :
a. $]3,7[$ b. $]3,5]$
c. $[3, 6]$ d. $[2, 4] - \left\{3\frac{1}{4}\right\}$

40. The derived set $D(A)$ of the set $A = \{1, 2, 3, 4\}$ is :
a. $(1, 4)$ b. $[1, 4]$
c. $\{1, 2, 3, 4\}$ d. ϕ

41. A is closed iff :
a. $A \subseteq \overline{A}$ b. $\overline{A} \subseteq A$
c. $\overline{A} = A$ d. $\overline{A} = A^C$

42. If $F_n = \left[\frac{1}{n}, 1\right]$ $\forall n \in N$ then $\cup\{F_n : n \in N\}$ is :
a. Closed only
b. Open only
c. Both open and closed
d. Neither open nor closed

43. The subset of an open set is :
a. Open only
b. Closed only
c. Both open and closed
d. May or may not be open

44. Interior of $[a, b]$ is given by :
a. $[a, b]$ b. $]a, b[$
c. $\{a, b\}$ d. None of these

45. If A is finite set their int(A) is : **[Kanpur 2018]**
a. A b. R
c. ϕ d. A^C

46. The boundary point of a set A is :
a. Point of A
b. Not a point of A
c. May or may not point of A
d. None of these

47. If Q be the set of all rational numbers then Q° or int (Q) is :
a. Q b. R
c. Q^C d. ϕ

48. Superset of a nbd of a point is always a :
a. Closed set b. Open set
c. Nbd d. Not a nbd

49. If A and B are nbds of a point p then which one of the following is a nbd of p :
a. $A \cup B$ b. $A \cap B$
c. $A^C \cup B^C$ d. None of these

50. The intersection of the family of all nbds of an arbitrary point $x \in R$ is :
a. R b. $R - \{x\}$
c. $\{x\}$ d. Uncountable set

51. The set $A =]1, 2[\cup]3, 4[$ is not a nbd of :
a. 1,3 b. 2,4
c. 1,2,3,4 d. None of these

52. The derived set of any set is : **[Kanpur 2018]**
a. Closed
b. Open
c. Closed and open both
d. None of these

53. The intersection an arbitrary collection of closed sets is :
a. Open b. Both open and closed
c. Closed d. Not necessarily closed

54. If A° is the set of all interior points of A then :
a. $A \subset A^\circ$ b. $A = A^\circ$
c. $A^\circ \subset A$ d. $A^\circ \subset A^C$

55. A countable set is :

a. Perfect b. May be perfect

c. Never perfect d. None of these

56. The set $\left\{\frac{1}{2^n} : n \in N\right\}$ is :

a. Open

b. Closed

c. Both open and closed

d. Neither open nor closed

57. The set of all limit points of the set $\left\{\left(\frac{1}{m}\right)+\left(\frac{1}{n}\right) : m,n \in N\right\}$ are :

a. $\left\{\left(\frac{1}{n}\right) : n \in N\right\} \cup \{0\}$

b. $\left\{\frac{1}{n} : n \in N\right\}$ only

c. $\{0\}$ only

d. R

58. If $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right) : n \in N$ then $\bigcap_{n=1}^{\infty} I_n$ is :

a. $(-1, 1)$ b. $\{0\}$

c. ϕ d. $(-\infty, \infty)$

59. The set of non-zero real numbers is :

a. Open b. Closed

c. Open or closed d. Neither open nor closed

60. Which of the following is not a nbd of 1 :

a. (0,2) b. (–1, 2)

c. (–2, 2) d. {0,1,2,3}

61. The intersection of an infinite collection of open sets is :

a. Open set

b. Closed set

c. Open and closed both

d. Not necessarily open set

62. $\overline{A}$ is defined :

a. Union of all closed sets containing A

b. Intersection of all closed sets contained in A

c. Union of all closed sets contained in A

d. Intersection of all closed sets containing A

63. Int (R) is equal to :

a. Q b. N

c. R d. ϕ

64. The exterior point of the set [0,2] is :

a. 0 b. 1

c. 2 d. None of these

65. The interior points of the set {1,2,3,4,5} are :

a. {1} b. {5}

c. {1,2,3,4,5} d. None of these

66. The limit points of the set $\left\{\frac{1}{2n} : n \in N\right\}$ are :

a. 0 b. $\frac{1}{2}$

c. $\left\{\frac{1}{2}, \frac{1}{2^2}, ...\right\}$ d. ϕ

67. The limit points of the set $\left\{(-1)^4\left[1+\left(\frac{1}{n}\right)\right]\right\}$ are :

a. {0,1} b. {0, –1}

c. {0,1, –1} d. {1, –1}

68. The set of non-zero real numbers is :

a. Open set

b. Both open and closed

c. Not necessarily open set

d. Neither open nor closed

69. The empty set is :

a. Open only

b. Closed only

c. Neither open nor closed

d. Both open and closed

70. If $D(A)$ be the derived set of A then which one is true:

a. $\overline{A} \subset A \cup D(A)$ only b. $A \cup D(A) \subset \overline{A}$ only

c. $\overline{A} = A \cup D(A)$ d. $A = \overline{A} \cup D(A)$

71. If $A_n = \left[\frac{1}{n}, 1\right]$ $\forall n \in N$ then $U\{A_n : n \in N\}$ is :

a. [0,1] b.] 0,1 [

c.] 0,1] d. [0,1 [

72. If $A = [a, b]$ then boundary points of A are :

a. $[a, b]$ b. $]\, a, b\, [$

c. $\{a, b\}$ d. R

73. $A \subseteq R$ is open iff :

a. $A \subset A^\circ$ b. $A^\circ \subset A$

c. $A^\circ = A$ d. $\overline{A}^\circ = A$

74. If $A = \left\{\frac{1}{2n} : n \in N\right\}$ then adherent (A) is :

a. $\{0\}$ b. $\left\{\frac{1}{2}, \frac{1}{2^2}, ...\right\}$

c. $A \cup \{0\}$ d. ϕ

75. The set of natural number N is :

a. Dense but not closed

b. Closed and dense both

c. Closed but not dense

d. Neither closed nor dense

76. $A°$ is :

a. Smallest open set contained in A

b. Smallest open set containing A

c. Largest open set containing A

d. Largest open set contained in A

77. The set of all limit points of the set $\left\{(-1)^n + \frac{1}{m} : m, n \in N\right\}$ are :

a. $\{0, 1, -1\}$ b. $\{1, -1\}$

c. R d. ϕ

78. The collection of disjoint intervals of positive length is :

a. Countable

b. Uncountable

c. May be countable or uncountable

d. None of these

79. Every non-empty open set contains :

a. Rational points

b. Irrational points

c. Both rational and irrational points

d. None of these

80. Every closed set in R is the intersection of :

a. Countable collection of open sets

b. Countable collection of closed sets

c. Uncountable collection of closed sets

d. Uncountable collection of open sets

81. Which of the following is not true :

a. $A° \subset A$ b. $(A \cap B)° = A° \cap B°$

c. $A°° = A°$ d. $A°° = A$

82. If Q be the set of all rational numbers then boundary points of Q are :

a. Q b. R

c. Finite set d. ϕ

83. If $A =]\, 0, 1\, [$ then ext (A) is :

a. $]\, 0, 1\, [$ b. $[0, 1]$

c. $\{0, 1\}$ d. $]-\infty, 0\,[\, \cup\,]\, 1, \infty\, [$

84. If the derived set of A is $D(A)$ is closed then A is :

a. Closed

b. Open

c. Either open or closed

d. None of these

85. If A is open and B is closed then $A - B$:

a. Open only

b. Closed only

c. May be open or closed

d. Neither open nor closed

86. Limit point of the set A :

a. Belong to set A

b. Does not belong to set A

c. May or may not belong to set A

d. None of these

87. Which of the following statement is wrong :

a. Every closed interval is closed set

b. Every open interval is open set

c. Intersection of two open set is open

d. Union of finite collection of open set is open

88. Which of the following is a perfect set :

[Kanpur 2018]

a. (a, b) b. Q

c. N d. $[a, b]$

89. The interior points of the set $\{x : 0 \le x \le 1\}$ are :

a. $\{0,1\}$ b. $[0,1]$

c. $]\, 0,1\, [$ d. $\{0\}$

90. If $S = \left\{\frac{1}{m} + \frac{1}{n} : m, n \in N\right\}$ then $D^2(S)$ is equal to :

a. $\left\{\frac{1}{n} : n \in N\right\}$ b. $\{0\}$

c. $\left\{\frac{1}{n} : n \in N\right\} \cup \{0\}$ d. None of these

91. If $S = [0, 1]$ then $D^2(S) =$:

a. S b. $\{0\}$
c. $\{1\}$ d. None of these

92. Which set is the nbd of all its point : **[Meerut 2018]**

a. ϕ b. R
c. Both (a) and (b) d. None of theses

93. The derived set of rational number Q is : **[Kanpur 2018]**

a. ϕ b. φ
c. R d. None of these

94. A set, which is a nbd of each of its points with the exception of two points is : **[Kanpur 2018]**

a. (2,5) b. [2,5]
c. (2,5] d. $(3,4) \cup (5,6)$

95. If a set is closed and bounded, then it is : **[Kanpur 2018]**

a. Covering set b. Compact set
c. Derived set d. Cluster set

96. The null set ϕ is : **[Kanpur 2018]**

a. Open
b. Closed
c. Both open and closed
d. None of these

97. Which of the following sets is not closed : **[Kanpur 2018]**

a. [2,5] b. $\left\{\frac{1}{n} : n \in N\right\}$
c. $\{1,2,3,4\}$ d. R

98. Which one of the following sets is a perfect set : **[Kanpur 2018]**

a. Finite set
b. The set N of natural numbers
c. The set Q of rational numbers
d. The set $E = [0, 1]$

99. If a point $p \in S$ is not a limit point of S then it is called : **[Kanpur 2018]**

a. Perfect point b. Isolated point
c. Adherent point d. Boundary point

100. If $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots. \frac{1}{n} \ldots.\right\}$ then $D(S)$ is equal to : **[Meerut 2018]**

a. 0 b. –1
c. 1 d. ϕ

101. If S is bounded then $D(S)$ is : **[Kanpur 2018]**

a. Unbounded b. ϕ
c. Convergent d. Always bounded

102. Which is true :

a. $A \supset B \Rightarrow D(A) \supset D(B)$
b. $D(A \cap B) = D(A) \cap D(B)$
c. $D(A \cup B) = D(A) \cup D(B)$
d. all the above

103. Which is not true : **[Meerut 2018]**

a. Any open interval is not nbd of all its point
b. Finite set is the nbd of all its point
c. The set N is nbd of 3
d. All of the above

104. $D(R - Q)$ is equal to : **[Meerut 2018]**

a. R b. N
c. Q d. ϕ

105. The set $D(Q)$ is equal to : **[Meerut 2018]**

a. R b. R^+
c. Q d. $R - Q$

106. If $S = \left\{\frac{1}{m} + \frac{1}{n} : m, n \in N\right\}$ then $D(S)$ is : **[Meerut 2018]**

a. ϕ b. $\left\{\frac{1}{m} : m \in N\right\}$
c. $\{0\}$ d. $\left\{\frac{1}{n} : n \in N\right\} \cup \{0\}$

107. Which is not nbd of 2 : **[Meerut 2018]**

a.] 2,4 [b.] 1,4 [
c.] 1,3 [d.] 0,3]

108. The infimum of the set $S = \{x : x \in N \text{ and } |x| < 5.7\}$ is : **[Meerut 2018]**

a. 0 b. –5.7
c. –5 d. 1

109. Which is true :

a. Union of arbitrary family of open set is open

b. Intersection of finite collection of open set is open

c. Both (a) and (b)

d. None of the above

110. Which is true : **[Meerut 2018]**

a. The $R - Q$ is nbd of all its points

b. The set Q is not nbd of all its points

c. Both (a) and (b)

d. The set N is nbd of all its points

111. Which is true : **[Meerut 2018]**

a. If $A \subseteq R$ then $D(A)$ is closed

b. $\overline{A} = A \cup D(A)$

c. A is closed iff $\overline{A} = A$

d. All the above

112. If $A =]\,2, 7\,[$ then $\overline{A}$ is equal to : **[Meerut 2018]**

a. $[\,2, 7\,[$ b. $]\,2, 7\,]$

c. $[2, 7]$ d. $]\,2, 7\,[$

113. Which is not true : **[Meerut 2018]**

a. If $A = [2, 6\,[$ then $D(A) = [2, 6\,[$

b. If $A =]\,2, 6]$ then $D(A) =]\,2, 6]$

c. If $A =]\,2, 6\,[$ then $D(A) = [2, 6]$

d. Both (a) and (b)

114. The derived set of the closed interval [3,5] is : **[Meerut 2017]**

a.] 3,5 [b.] 3,5]

c. [3,5] d. [3,5 [

115. The limit points of the set $S = \left\{\frac{3n+2}{2n+1} : n \in N\right\}$ is : **[Meerut 2016, 17]**

a. $\frac{2}{3}$ b. $\frac{3}{2}$

c. $\frac{1}{3}$ d. 3

116. Derived set of the set $S = \{1 + 3^{-n} : n \in N\}$ is : **[Meerut 2017]**

a. {1,2} b. {1,2,3}

c. Singleton {1} d. None of these

117. The derived set of the set $S = \{1, 3, 7, 11\}$ is :

a. {1,3,7,11} b. {1,3}

c. Empty set ϕ d. None of these

118. The value of :

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)....\left(1+\frac{4n}{n}\right)\right]^{\frac{1}{n}} \text{ is :}$$

[Meerut 2019]

a. $\frac{5}{e^4}$ b. $\frac{5^4}{e^4}$

c. $\frac{5^5}{e^5}$ d. $\frac{5^5}{e^4}$

119. A one are map $f : (x, T) \to (y, U)$ is homomorphism iff : **[Meerut 2014]**

a. $f(\overline{A}) = \overline{f(A)}$ b. $f(A^0) = [f(A)]^0$

c. Both (a) & (b) d. None of these

120. If $\overline{S}$ and S' are closure and derived set of S in R^n then $\overline{S} =$ **[Meerut 2014]**

a. $S \cap S'$ b. $S \pm S'$

c. $S \cup S'$ d. None of these

121. $\bigcup_{n\in N}\left[1+\frac{1}{n}, 3-\frac{1}{n}\right] =$ **[Meerut 2014]**

a. [1, 3] b. (1, 3)

c. [1, 3) d. (1, 3]

122. Least and greatest element of the set

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...\frac{n}{n+1}, ...\right\} \text{ in } Q \text{ are :}$$

[Meerut 2014]

a. $\frac{1}{2}, 0$ b. $\frac{1}{2}, 1$

c. $0, \frac{1}{2}$ d. None

123. Infimum and supremum of the set

$$S = \left\{\frac{1}{n} : n \in N\right\} \text{ are :}$$

[Meerut 2014]

a. 0, 1 b. 1, does not exist

c. 0, does not exist d. Does of exist, 1

124. Every infinite ______ Subset at R has a limit point :

a. Bounded b. Unbounded

c. Finite d. None of these

125. A sequence whose associated set is singleton is called : **[Meerut 2014]**

a. Identity sequence b. Singleton sequence

c. Constant sequence d. None of these

126. $[a, b]$ is a nbd of each of its points except the : **[Meerut 2014]**

a. a b. b

c. a & b d. None of these

127. A non empty subset S of R is a nbd of $x \in R$ iff $\exists$ an open interval (a, b) such that : **[Meerut 2014]**

a. $x \notin (a, b) \subseteq s$ b. $x \in (a, b) \not\subseteq s$

c. $x \in (a, b) \subseteq s$ d. $x \in (a, b) \supseteq s$

128. Let S be any given bounded and

$T = \{|x - y| : x \in S, y \in S\}$ then Sup T = **[Meerut 2014]**

a. $\sup S$ b. $\inf S$

c. $\sup S + \inf S$ d. $\sup S - \inf S$

129. A closed interval $[a, b]$ is not an open set because its end point a and b are not _______ of the interval : **[Meerut 2014]**

a. Llimit point b. Interior point

c. Adherent point d. None of these

130. The set of all limit points of a sert PCR is called the : **[Meerut 2016]**

a. Derived set of P b. Closed set of P

c. Open set of P d. None of these

131. Any set S cannot be neighbourhood of any point of the set : **[Meerut 2016]**

a. $R - S$ b. $R + S$

c. $\frac{R}{S}$ d. None of these

132. The least upper bound of the set

$A = \left\{x : x = \left(1 - \frac{1}{n}\right) n \in N\right\}$ is : **[Meerut 2015]**

a. 0 b. –1

c. 1 d. None of these

133. Neighbourhood of $\frac{1}{2}$ is the set : **[Meerut 2015]**

a. $\left]0, \frac{1}{2}\right[$ b. $\left[-\frac{1}{2}, \frac{1}{2}\right]$

c. R d. None of these

134. If A and B are any subsets of R, then $A \subset B \Rightarrow$ **[Meerut 2015]**

a. $D(A) \subset D(B)$ b. $D(A) \supset D(B)$

c. $D(A) = D(B)$ d. None of these

135. The set of all limit points of a set $A \subset R$ is called : **[Meerut 2015]**

a. Derived set of A b. Closed set of A

c. Open set of A d. None of these

136. If S is a finite set then :

$D(S) =$ **[Meerut 2015]**

a. ϕ b. ϕ'

c. $\bar{\phi}$ d. None of these

137. The set of all limit points is called the :

a. Discreate set b. Closed set

c. Derived set d. Open set

138. Which interval is a neighbourhood of each of its points :

a. Closed interval b. Open interval

c. Both (a) and (b) d. None of these

139. If $S = \{x \in R : x = n + 3, n \in N\}$ the $g.l.b.$ of of S is : **[Meerut 2016]**

a. 3 b. 4

c. 5 d. 6

140. The supremum and infimum of the set

$S = \{x \in R : 2^n, n \in N\}$ will be : **[Meerut 2016]**

a. $\inf S = 2, \sup S = 3$

b. $\inf S = 3, \sup S = 4$

c. $\inf S = 2, \sup S$ – not exist

d. None of these

141. The set of all empty sets ϕ are : **[Meerut 2016]**

a. Closed b. Open

c. Discrete d. None of these

142. If $0 < \theta < 1$ and $|x| < 1$ then $\left|\frac{x(1-\theta)}{1+\theta x}\right|$ is : **[Meerut 2016]**
a. Equal one b. Less than one
c. Equal two d. None of these

143. Which of the following subset of R are nbds of 3 :
a.]2, 4[b.]3, 7[
c.]3, 5[d. None of these

144. The derived set of any bounded set is again a : **[Meerut 2016]**
a. Bounded set b. Not bounded set
c. Closed set d. None of these

145. Every finite subset of R is a : **[Meerut 2016]**
a. Open set b. Closed set
c. Null set d. None of these

146. The derived set of the closed interval [3, 5] is : **[Meerut 2017]**
a.]3, 5[b.]3, 5]
c. [3, 5] d. [3, 5[

147. The limit point of $<\frac{1}{n}>$ is/does not : **[Meerut 2018]**
a. Belong to the range set
b. 0
c. –1
d. Both (a) and (b) true

148. The limit point of $<(-1)^n>$ is : **[Meerut 2018]**
a. A finite set b. [1, – 1]
c. {1, – 1} d. Both (a) and (c) true

149. Which is not true : **[Meerut 2018]**
a. If $A = [2, 6[$ then $D(A) = [2, 6[$
b. If $A = [2, 6[$ then $D(A) =]2, 6]$
c. If $A = [2, 6[$ then $D(A) = [2, 6]$
d. Both (a) and (b)

150. Which is not nbd of 2 : **[Meerut 2018]**
a.]2, 4[b.]1, 4[
c.]1, 3[d.]0, 3[

151. $D(R-q)$ is equal to : **[Meerut 2018]**
a. R b. N
c. Q d. ϕ

152. If $S = \left\{\frac{n}{n+1} : n \in N\right\}$ then $D(S)$ is equal to : **[Meerut 2018]**
a. 1 b. –1
c. 0 d. $\frac{1}{2}$

153. Every bounded sequence has ______ limit point : **[Meerut 2018]**
a. One b. Two
c. At least one d. Infinite

154. The set of limit point of a bounded sequence is : **[Meerut 2018]**
a. ϕ b. Bounded
c. Convergent d. Finite

155. The supremum of the set
$$S = \left\{\left(1-\frac{1}{n}\right)\sin\frac{n\pi}{2}, n \in N\right\}$$ is : **[Meerut 2018]**
a. 0 b. 1
c. 3 d. –1

156. Which is true : **[Meerut 2018]**
a. $A \supset B \Rightarrow D(A) \supset D(B)$
b. $D(A \cap B) = D(A) \cap D(B)$
c. $D(A \cup B) = D(A) \cup D(B)$
d. All the above

157. The set $D(Q)$ is equal to :
a. R b. R^+
c. Q d. $R-Q$

158. The supremum of the set $S = \{x : x \in N\}$ and $|x| < 13.33\}$: **[Meerut 2018]**
a. 13.33 b. 14
c. 13 d. –13

159. The set $D(N)$ is equal to : **[Meerut 2018]**
a. R b. ϕ
c. $R-Q$ d. R^+

160. If S is bounded then $D(S)$ is : **[Meerut 2018]**
a. Unbounded b. ϕ
c. Convergent d. Always bounded

161. Which set is the nbd of all its point : **[Meerut 2018]**
a. ϕ b. R
c. Both (a) and (b) d. N

162. Which is not true : **[Meerut 2018]**

a. Any open interval is not nbd of all its point

b. Finite set is the nbd of all its point

c. The set N is nbd of 3

d. All the above

163. Let $z = e^{\frac{2\pi i}{7}}$ and $\theta = z^4 + z^2 + z$, then **[Meerut 2019]**

a. $\theta \in Q$

b. $\theta \in Q(\sqrt{D})$ for some $D > 0$

c. $\theta \in Q(\sqrt{D})$ for some $D < 0$

d. $\theta \in iR$

164. Which is true :

a. $D(Q) = R$ b. $D(R - Q) = R$

c. $D(R) = D(Q)$ d. All of these

165. The limit point of the set

$$A = \left\{x : x = \frac{1}{r}, r \in N\right\} \text{ is :}$$ **[Meerut 2019]**

a. 1 b. –1

c. 0 d. 2

166. Which is true : **[Meerut 2019]**

a. $D^{n+1}(Q) \neq D(R - Q)$

b. $D^n(Q) \neq D^{n+1}(R - Q)$

c. $D^{n+1}(Q) = D^n(R - Q)$

d. $D^n(R - Q) \neq D^{n+1}(Q)$

167. The limit point of $]-2, 2[$ is :

a. The solution of $|x - 1| < 1$

b. The solution of $|x - 1| = 1$

c. The solution of $|x - 1| \geq 1$

d. None of these

168. The empty set ϕ is : **[Meerut 2019]**

a. Open

b. Closed

c. Open and closed both

d. Finite

169. Intersection of all closed set containing a closed set A, is equal to $(\bar{A})$:

a. ϕ b. A

c. $\{0\}$ d. $\{\phi\}$

170. Which one is not nbd of zero : **[Meerut 2019]**

a. $]-1, 1[$ b. $[-1, 1]$

c. $]-1 [\cup] 0, 1[$ d. $]-1, 2[$

171. If A is a closed set then $D(A)$: **[Meerut 2019]**

a. Does not exist b. $D(A) \subseteq A$

c. $D(A) \supset A$ d. $D(A) = \phi$

172. Let $B \subseteq R$ and every infinite sequence in B has a subsequence which converges in B. The above statement is true if : **[Meerut 2019]**

a. $B = [0, \infty[$ b. $B = [0, 1] \cup [3, 4]$

c. $B = [-1, 1] \cup [1, 2]$ d. $B =]-1, 1]$

173. If S is a finite set them : **[Meerut 2019]**

a. S has at least one limit point

b. S has more than one limit point

c. S has only one limit point

d. S has no limit point

174. The radius of convergence of power series :

$$f(x) = \sum_{n=2}^{\infty} x^n . \log x$$

a. 0 b. 1

c. 3 d. none of these

175. Radius od convergence of power series $\sum_{n=1}^{\infty} z^{n^2}$ is :

a. 0 b. ∞

c. 1 d. 2

176. If f is holomorphic is an open nbd of $z_0 \in C$ and $\Sigma f^n(z_0)$ is absolutely convergent, then :

a. f is constant

b. f is polynomial

c. f can be extended to an entire function

d. $f(x) \in R \ \forall x \in R$

177. Which is true, if

$A \subseteq R$, $p \in R$ is limit point of A

a. $N \cap (A - \{p\}) = \phi \ \forall$ Nbd N of P

b. $N \cup (A - \{p\}) \neq \phi \ \forall$ Nbd N of P

c. $N \cap (A - \{p\}) \neq \phi \ \forall$ Nbd N of P

d. $N \cup (A - \{p\}) = \phi \ \forall$ Nbd N of P

178. Which is true :

a. $D(Q) = D(R - Q) = D(I)$

b. $D(Q) \neq D(R - Q) = D(I)$

c. $D(Q) = D(R - Q)$ and $D(I) = \phi$

d. $D(Q) = \phi, D(R - Q) = \phi, D(I) = \phi$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(c)	**2.**	(d)	**3.**	(d)	**4.**	(b)	**5.**	(b)	**6.**	(c)	**7.**	(c)	**8.**	(d)	**9.**	(d)	**10.**	(d)
11.	(b)	**12.**	(a)	**13.**	(a)	**14.**	(d)	**15.**	(c)	**16.**	(b)	**17.**	(d)	**18.**	(c)	**19.**	(d)	**20.**	(d)
21.	(b)	**22.**	(d)	**23.**	(c)	**24.**	(b)	**25.**	(b)	**26.**	(a)	**27.**	(c)	**28.**	(c)	**29.**	(d)	**30.**	(d)
31.	(c)	**32.**	(d)	**33.**	(b)	**34.**	(a)	**35.**	(a)	**36.**	(a)	**37.**	(c)	**38.**	(a)	**39.**	(d)	**40.**	(d)
41.	(c)	**42.**	(d)	**43.**	(d)	**44.**	(b)	**45.**	(c)	**46.**	(c)	**47.**	(d)	**48.**	(c)	**49.**	(b)	**50.**	(c)
51.	(d)	**52.**	(a)	**53.**	(c)	**54.**	(c)	**55.**	(c)	**56.**	(d)	**57.**	(a)	**58.**	(b)	**59.**	(a)	**60.**	(d)
61.	(d)	**62.**	(d)	**63.**	(c)	**64.**	(d)	**65.**	(d)	**66.**	(a)	**67.**	(d)	**68.**	(a)	**69.**	(d)	**70.**	(c)
71.	(c)	**72.**	(c)	**73.**	(b)	**74.**	(c)	**75.**	(c)	**76.**	(d)	**77.**	(b)	**78.**	(a)	**79.**	(c)	**80.**	(a)
81.	(d)	**82.**	(b)	**83.**	(d)	**84.**	(c)	**85.**	(a)	**86.**	(c)	**87.**	(d)	**88.**	(d)	**89.**	(c)	**90.**	(b)
91.	(a)	**92.**	(c)	**93.**	(c)	**94.**	(b)	**95.**	(b)	**96.**	(c)	**97.**	(b)	**98.**	(d)	**99.**	(b)	**100.**	(a)
101.	(d)	**102.**	(b)	**103.**	(d)	**104.**	(a)	**105.**	(a)	**106.**	(d)	**107.**	(a)	**108.**	(d)	**109.**	(c)	**110.**	(b)
111.	(d)	**112.**	(c)	**113.**	(d)	**114.**	(c)	**115.**	(b)	**116.**	(c)	**117.**	(c)	**118.**	(d)	**119.**	(c)	**120.**	(c)
121.	(b)	**122.**	(d)	**123.**	(a)	**124.**	(a)	**125.**	(c)	**126.**	(c)	**127.**	(c)	**128.**	(d)	**129.**	(b)	**130.**	(a)
131.	(a)	**132.**	(a)	**133.**	(b)	**134.**	(a)	**135.**	(a)	**136.**	(a)	**137.**	(c)	**138.**	(b)	**139.**	(b)	**140.**	(c)
141.	(a)	**142.**	(b)	**143.**	(a)	**144.**	(c)	**145.**	(a)	**146.**	(c)	**147.**	(d)	**148.**	(c)	**149.**	(d)	**150.**	(d)
151.	(a)	**152.**	(a)	**153.**	(c)	**154.**	(b)	**155.**	(c)	**156.**	(d)	**157.**	(a)	**158.**	(c)	**159.**	(b)	**160.**	(c)
161.	(c)	**162.**	(d)	**163.**	(c)	**164.**	(d)	**165.**	(c)	**166.**	(c)	**167.**	(d)	**168.**	(c)	**169.**	(b)	**170.**	(c)
171.	(b)	**172.**	(b)	**173.**	(d)	**174.**	(d)	**175.**	(c)	**176.**	(c)	**177.**	(c)	**178.**	(c)				

HINTS AND SOLUTIONS

1. If $p \in R$ then for every $\varepsilon > 0$ we have $]\, p-\varepsilon, p+\varepsilon\, [\subseteq R$. then R contain a neighbourhood of each of its points. Hence R is an open set. Now $R^C = \phi$ which is open. Since ϕ has no points, the condition that ϕ contains a nbd of each of its point is vacuously satisfied, so ϕ is open *i.e.* R is closed.

4. Let $E = \{x_1, x_2, ... x_n\}$ be a finite subset of R. We can write it as $E = \{x_1\} \cup \{x_2\} \cup .... \{x_n\}$. But every singleton set in R is closed set and the union of finite collection of closed sets is closed. So E is closed set.

8. Let (a, b) be an open interval and $p \in (a, b)$. Since, $a < p < b$, we have $p - a > 0$ and $b - p > 0$. Choose $\varepsilon = \min\{p - a, b - p\}$ so that $\varepsilon > 0$.

Now we show $(p-\varepsilon, p+\varepsilon) \leq (a, b)$

Let $\quad x\ \varepsilon\ (p-\varepsilon, p+\varepsilon)$

$\Rightarrow \quad p - \varepsilon < x < p + \varepsilon$

$\Rightarrow \quad -\varepsilon < x - p < \varepsilon$

Since, $\varepsilon \leq p - a$ or $a - p \leq -\varepsilon$ and $\varepsilon \leq b - p$

we have $a - p < x - p < b - p \Rightarrow a < x < b$

or $\quad x \in (a, b)$

Thus $p\ \varepsilon (p-\varepsilon, p+\varepsilon) \leq (a, b)$ *i.e.* (a, b) is an open set.

10. Since $Q^\circ = \phi$ so $Q^\circ \neq Q$ *i.e.* Q is not an open set.

$\because$ $Q' = R - Q$ = the set of all irrational numbers and Q^1 is not open. For if $p \in Q'$ then for every $\varepsilon > 0$, $]p - \varepsilon, p + \varepsilon[$ contain rational number also and so is not a subset of Q'. Thus if $p \in Q'$ then p is not an interior point of Q' and so Q' is not open. Hence, Q is not closed set.

11. Let $G = \bigcap_{i=1}^{n} G_i$, where each G_i is an open set.

If $G = \phi$ then it is open.

If $G \neq \phi$, let $p \in G$ so $p \in G_i \ \forall i = 1, 2, ..., n$.

Since, each G_i is an open so for every $i = 1, 2, ..., n$ there exists $\varepsilon_i > 0$ such that

$$]p - \varepsilon, p + \varepsilon_i [\subset G_i$$

Let $\varepsilon = m_i n [\varepsilon_1, \varepsilon_2 .... \varepsilon_n]$

Then $]p - \varepsilon, p + \varepsilon[\subset]p - \varepsilon_i, p + \varepsilon_i[\ \forall_i = 1, 2, ..., n]$

$\Rightarrow$ $]p - \varepsilon, p + \varepsilon[\subset \bigcap_{i=1}^{n} G_i$

$\Rightarrow$ $]p + \varepsilon, p + \varepsilon[\subset G$ *i.e.* p is an interior point of G. So G is open set *i.e.* $\bigcap_{i=1}^{n} G_i$ is open.

17. $(2,3) \cup (4,5)$ is an open set, since it is union of two open sets but it is not an interval.

23. Given that $A = \left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\}$. Since all the points of A are its isolated points so it is a discrete set.

26. Let $\{G_\lambda : \lambda \in \Lambda\}$ be an arbitrary family of open sets and $G = \bigcup_{\lambda \in \Lambda} G_\lambda$. Let $x \in G$ then x must belong to G_{λ_0} for some $\lambda_0 \in \Lambda$. Since, G_{λ_0} is open there exists a nbd N of x such that $x \in N \subset G_{\lambda_0}$. But $G_{\lambda_0} \subseteq G$ so $N \subseteq G$. Thus, $x \in N \subseteq G$ *i.e.*, x is an interior point of G. Hence, G is open set.

32. Let $G_n =]-\frac{1}{4}, \frac{1}{4}[: n \in N$. Then each G_n is an open set because every open interval is an open set.

Now $\bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty}]-\frac{1}{a}, \frac{1}{a}[= \{0\}$ which B not open because there exists $h_0 \ \varepsilon > 0$ such that $]-\varepsilon, \varepsilon[\subseteq \{0\}$. Hence the intersection of a infinite collection of open sets is not necessarily an open set.

34. Let $F_1, F_2, ...F_n$ be n closed sets. Let $F_2 \bigcap_{i=1}^{n} F_i$.

Since each F_i is closed so each F_i' is open $\forall i = 1, 2, ..., n$. But intersection of a finite collection of open sets is open *i.e.* $\bigcap_{i=1}^{n} F_i'$ is open.

By De Morgan's law $\left(\bigcap_{i=1}^{n} F_i\right) = \bigcap_{i=1}^{n} F_i'$

So $\left(\bigcap_{i=1}^{n} F_i\right)'$ is an open set. Hence, $\bigcap_{i=1}^{n} F_i = F$ is a closed set.

37. $\because$ $\overline{Q} = Q \cup D(Q)$

But $D(Q) = R$ so $\overline{Q} = Q \cup R = R$

42. Let $F_n = \left[\frac{1}{4}, 1\right] : n \in N$ then each F_n is a closed set in R since each closed interval is a closed set.

$$\bigcup \{F_n : n \in N\} = \{1\} \cup \left[\frac{1}{2}, 1\right] \cup \left[\frac{1}{3}, 1\right] \cup ......$$

$$=]\, 0, 1\,]$$

which is neither open nor closed.

44. Let $A = [a, b]$ then $a \in A$ but there exists $h_0 \ \varepsilon > 0$ such that $]a - \varepsilon, a + \varepsilon[\subset A$. Hence a is not an interior point of A. Similarly b is not an interior point of A. Now let $p \in]a, b[$ so $a < p < b$. If we choose $\varepsilon = \text{mini}\{p - a, b - p\}$ then $\varepsilon > 0$ is such that $]p - \epsilon, p + \epsilon[\subset]a, b[$ so p is an interior point of $]a, b[$. Thus int $[a, b] =]a, b[$.

49. Let A and B be two nbds of a point p then there exists $\varepsilon_1 > 0, \varepsilon_2 > 0$ such that $]p - \varepsilon_1, p + \varepsilon_1[\subset A$ and $]p - \varepsilon_2, p + \varepsilon_2[\subset B$ choose $\varepsilon = \text{mini}\{\varepsilon_1, \varepsilon_2\}$ then

$$]p - \varepsilon, p + \varepsilon[\subseteq]p - \varepsilon_1, p + \varepsilon_1[\subset A$$

and $]p - \varepsilon, p + \varepsilon[\subseteq]p - \varepsilon_2, p + \varepsilon_2[\subset B$

so $]p - \varepsilon, p + \varepsilon[\subset A \cap B$ *i.e.* $A \cap B$ is a nbd of p.

53. Let $[F_\lambda . \lambda \in \Lambda]$ be an arbitrary collection of closed sets. Since each F_λ is a closed set so each F'_λ is an open set. But union of arbitrary collection of open sets is open so $\bigcup \{F'_\lambda : \lambda \in \Lambda\}$ is open.

By De Morgan's law

$$[\bigcap \{F_\lambda : \lambda \in \Lambda\}]' = \bigcup \{F'_\lambda : \lambda \in \Lambda\}$$

so $[\bigcap \{F_\lambda : \lambda \in \Lambda\}]'$ is an open set

i.e. $\bigcap \{F_\lambda : \lambda \in \Lambda\}$ is a closed set.

57. Let $S = \left\{\frac{1}{m} + \frac{1}{n} : m, n \in N\right\}$

Keep m fixed and very n. As we increase n, $\frac{1}{n}$ gets nearer to 0 so that $\frac{1}{m} + \frac{1}{n}$ gets nearer to $\frac{1}{m}$ and consequently $\frac{1}{m}$ is a limit point of S. Since m is any number of N there for all the points of the set $\left\{\frac{1}{m} : m \in N\right\}$ are limit points of S.

Again very both m and n so that $\frac{1}{m} + \frac{1}{n}$ sets nearer and nearer to 0 and consequently 0 is also a limit point of S. Thus

$$D(S) = \left\{\frac{1}{n} : m \in N\right\} \cup \{0\}$$

64. Point does not belongs to the set. Here 0,1,2 are belongs to the set [0, 2).

71. $A_n = \left[\frac{1}{n}, 1\right] \forall n \in N$

$$\cup A_n = \{1\} \cup \left[\frac{1}{2}, 1\right] \cup \left[\frac{1}{3}, 1\right] \cup \ldots \cup]\ 0, 1]$$

$$=]\ 0, 1\ ]$$

72. $A = [a, b]$ then $A^C =]-\infty, a[\ \cup\ ]\ b, \infty\ [$

So boundary of $A = \{a, b\}$

84. Let A be any open or closed subset of R. If $D(A)$ is empty then $D(A)$ is closed.

If $(A) \neq \phi$, Let $x \in [D(A)]^c$ so $x \notin D(A)$. Then there exists a nbd $N_\in(x)$ of x which does not contain any point of A except possibly x. We now show that $N_\in(x)$ does not contain any point of $D(A)$. For let $y \in N_\in(x)$ so $N_\in(x)$ is a nbd of y which contains only a finite number of points of A. So y cannot be a limit point of A *i.e.* $y \notin D(A)$. Thus $N_\in(x)$ does not contain any point of $D(A)$. Thus $N_\in(x) \subseteq [D(A)]^c$ which shows that $[D(A)]^c$ is open. Hence, $D(A)$ is closed.

90. By the solution of question (57) for

$$S = \left\{\frac{1}{m} + \frac{1}{n} = m, n \in N\right\}$$

$$D(S) = \left\{\frac{1}{n} : n \in N\right\} \cup \{0\}$$

So $D^2S = \{0\} \cup \{0\} = \{0\}$

91. If $S = [0, 1]$ then $D([0, 1]) = [0, 1]$

and so $D^2([0, 1]) = [0, 1]$

 OOO

CHAPTER

4

Sequences

SEQUENCE

A sequence is a function f whose domain is the set N of natural numbers and whose range is the set R of real numbers.

Thus a sequence in a set S is a rule which assigns to each natural number a unique element of S.

The value of function f at $n \in N$ is $f(n) = x_n$ say. It is customary to denote the sequence f by the symbol $< f(n) >$ or $< x_n >$ or $< x_1, x_2, \ldots. x_n >$. The range x_n of n is called the n^{th} term of the sequence.

Examples :

1. $< \frac{1}{n} >$ is the sequence $< 1, \frac{1}{2}, \frac{1}{3}, \ldots. \frac{1}{n} >$
2. $< (-1)^{n-1} >$ is the sequence $< 1, -1, 1, -1, \ldots. >$
3. $< \frac{1}{2^n} >$ is the sequence $< \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots.. >$.

Equal sequence : Two sequence $< x_n >$ and $< y_n >$ are said to be equal if $x_n = y_n, \ \forall n \in N$.

Range of a sequence : The set of all distinct terms of a sequence is called its range. The range of the sequence $< x_n >$ is the set $\{x_1, x_2, \ldots. x_n\}$.

Constant sequence : A sequence $< x_n >$ defined by $x_n = a \ \forall n \in N$ is called a constant sequence.

The sequence $< a >$ is a constant sequence and its range is $\{a\}$.

OPERATIONS ON SEQUENCES

Let $< x_n >, < y_n >$ be two sequences then

(i) $x_n + y_n$ is called the sum and denoted by $< x_n + y_n >$

(ii) $x_n - y_n$ is called the difference and denoted by $< x_n - y_n >$

(iii) $x_n \cdot y_n$ is called the product and denoted by $< x_n \cdot y_n >$

(iv) If $x_n \neq 0$ then $\frac{1}{x_n}$ is called the reciprocal and denoted by $< \frac{1}{x_n} >$.

(v) $\frac{x_n}{y_n}$ is called the quotient of the sequence $< x_n >$ by the sequence $< y_n >$ and is denoted by $< \frac{x_n}{y_n} >$ here $y_n \neq 0$.

SUBSEQUENCE

Let $< x_n >$ be any sequence. If $< n_1, n_2, \ldots. n_k >$ be a strictly increasing sequence of positive integers *i.e.* $\lambda > J \Rightarrow n_i > n_J$, then the sequence $< x_{n_1}, x_{n_2}, \ldots x_{n_k}, \ldots. >$ is called a subsequence of $< x_n >$.

Since $i > J \Rightarrow n_i > n_J$ so the order of various terms in the subsequence is the same as it is in the sequence.

Examples :

1. Consider a sequence $< 1, 0, 1, 0, 1, 0, \ldots >$ then $< 1, 1, 1, \ldots. >$ and $< 0, 0, 0 \ldots. 0 >$ are the subsequence.
2. Consider a sequence $< 1, 2, 3, \ldots n >$ then its subsequence are $< 1, 3, 5, \ldots. >$ and $< 2, 4, 6, \ldots. >$.

BOUNDED SEQUENCES

1. **Bounded above sequence :** A sequence $< x_n >$ is said to be bounded above if the range set of $< x_n >$ is bounded above *i.e.* if there exists a real number k such that

   $$x_n \leq k \ \ \forall n \in N$$

   The number k is called an upper bound of the sequence $< x_n >$.

2. **Bounded below sequence :** A sequence $< x_n >$ is said to be bounded below if the range set of $< x_n >$ is bounded below i.e. if there exists a real number k such that

$$x_n \geq k \;\; \forall n \in N$$

The number k is called a lower bound of $< x_n >$.

3. **Bounded sequence :** A sequence $< x_n >$ is said to be bounded if the range set of $< x_n >$ is both bounded above and bounded below *i.e.* there exist real numbers k and K such that

$$k \leq x_n \leq K \;\; \forall n \in N$$

A sequence which is not bounded is called unbounded sequence.

4. **Supremum (or Least upper bound) of a sequence) :** An upper bound of a sequence is called the supremum (or lub) if it is less than or equal to every upper bound of the sequence.

5. **Infimum (or Greatest lower bound) :** A lower bound of a sequence is called infimum (or g.l.b) if it is greater than or equal to every lower bound of the sequence.

6. **Important results :**

   (i) If the range of a sequence is a finite set than the sequence is bounded.

   (ii) Every subsequence of a bounded sequence is bounded.

   (iii) A sequence $< x_n >$ is bounded iff there exists $m \in N, l \in R$ and $a > 0$ such that $|x_n - l| < a \;\; \forall n \geq m$.

CONVERGENT SEQUENCE

A sequence $< x_n >$ is said to converge to a number l, if for any given $\in > 0$ there exists a positive number m such that

$$|x_n - l| < \in \;\; \forall n \geq m$$

The number l is called the limit of the sequence $< x_n >$ and we write $x_n \to l$ as $n \to \infty$ or $\lim\limits_{n\to\infty} x_n = l$ or $\lim x_n = l$.

Here the positive integer m depends on $\in$.

Results :

1. If $< x_n >$ is a sequence of non-negative numbers such that $\lim x_n = l$ then $l \geq 0$.
2. The limit of a sequence is unique.
3. If $< x_n >$ converge to l then any subsequence of $< x_n >$ also converges to l *i.e.* all subsequences of a convergent sequence converge to the same limit.
4. A negative number can not be the limit of a sequence of non-negative numbers.
5. If $\lim x_n = l$ and $l < 0$ then there exists a positive integer m such that $x_n < 0$ for all $n \geq m$.
6. *A* sequence $< x_n >$ is null sequence if $\lim x_n = 0$.
7. If the subsequences $< x_{2n-1} >$ and $< x_{2n} >$ of the sequence $< x_n >$ converge to the same limit l, then the sequence $< x_n >$ converges to l.
8. Every convergent sequence is bounded.

DIVERGENT AND OSCILLATORY SEQUENCE

1. **Divergent sequence :** A sequence $< x_n >$ is said to diverge to $+\infty$ if for any given $k > 0$ (however large) there exist $m \in N$ such that $x_n > k$ for all $n \geq m$. If $< x_n >$ diverge to ∞, we write $x_n \to \infty$ or $n \to \infty$ or $\lim x_n = +\infty$.

   A sequence $< x_n >$ is said to diverge to $-\infty$ if for any given $k > 0$ (however sum all) there exists $m \in N$ such that $x_n < k$ for all $n \geq m$.

   If $< x_n >$ diverges to $-\infty$ then we write $x_n \to -\infty$ as $n \to \infty$ or $\lim x_n = -\infty$.

   A sequence is said to be a divergent sequence if it diverges to either $+\infty$ or $-\infty$.

   Examples : (i) $< 1, 2, 3, \ldots. n \ldots >$ diverge to $+\infty$.

   (ii) $< -2, -4, -6, \ldots. 2n \ldots. >$ diverges to $-\infty$.

   (iii) $< x, x^2, x^3 \ldots. x^n \ldots. >, x > 1$ diverges to $+\infty$.

2. **Oscillatory sequences :** A sequence $< x_n >$ is said to be an oscillatory sequence if it is neither convergent nor divergent.

An oscillatory sequence is said to oscillatory finitely or infinitely according as it is bounded or unbounded.

Examples :

(i) $<(-1)^n>$ oscillator finitely

(ii) $<(-1)^n n>$ oscillator infinetely

3. **Results :**

(i) If a sequence $<x_n>$ diverges to infinitely then any subsequence of $<x_n>$ also diverges to infinitely.

(ii) If $x_{2n} \to \infty$ and $x_{2n} \to \infty$ or $n \to \infty$ then $x_n \to \infty$ or $n \to \infty$.

(iii) If $x_n > 0 \ \forall n \in N$ then $x_n \to \infty$ as $n \to \infty$ $\Leftrightarrow \frac{1}{x_n} \to 0$ as $n \to \infty$.

(iv) If the sequence $<x_n>, <y_n>$ diverge to ∞ then $<x_n + y_n>$ and $<x_n.y_n>$ also diverge to ∞.

(v) If $<x_n>$ diverges to ∞ and $<y_n>$ is bounded then $<x_n + y_n>$ diverges to ∞.

(vi) If $<x_n>$ diverges to ∞ and $<y_n>$ converges then $<x_n + y_n>$ diverges to ∞.

ALGEBRA OF CONVERGENT SEQUENCES

1. If $\lim x_n = l$ and $\lim y_n = l'$ then $\lim (x_n \pm y_n) = l \pm l'$ respectively.
2. If $\lim x_n = l$ and $c \in R$ then $\lim (cx_n) = cl$.
3. If $\lim x_n = l$ and $\lim y_n = l'$ such that $x_n \le y_n \ \forall n \in N$ then $l \le l'$.
4. $\lim x_n = 0$ and the sequence $<y_n>$ is bounded then $\lim (x_n . y_n) = 0$.
5. If $\lim x_n = l$ and $\lim y_n = l'$ thenlim $(x_n . y_n) = ll'$ but not conversely.
6. If $\lim x_n = l$ and $l \ne 0$ then there exists a positive number k and a positive integer m, such that $|x_n| > k \ \forall n \ge m$.
7. If $\lim x_n = l'$ and $l' \ne 0$ with $y_n \ne 0 \, \forall n \in N$ then $\lim \left(\frac{1}{y_n}\right) = \frac{1}{l'}$.
8. If $\lim x_n = l$ and $\lim y_n = l', l' \ne 0, y_n \ne 0$ $\forall n \in N$ then
$$\lim \left(\frac{x_n}{y_n}\right) = \frac{l}{l'}$$
9. **Sandwich Theorem :** If $<x_n>, <y_n>$ and $<z_n>$ are three sequences such that
   (i) For some positive integer
   $$k, x_n \le z_n \le y_n \ \forall n \ge k$$
   (ii) $\lim x_n = \lim y_n = l$ then $\lim z_n = l$
10. If $<x_n>$ and $<y_n>$ are two sequences such that $|x_n| \le |y_n| \ \forall n \ge k$ where $k \in N$ and $\lim y_n = 0$ then $\lim x_n = 0$.
11. **Cauchy's first theorem on limits :** If $\lim x_n = l$ then
$$\ln \frac{x_1 + x_2 + \ldots x_n}{n} = l$$
12. **Cauchy's second theorem on limits :** If $<x_n>$ is a sequence such that $x_n > 0$ for all n and $\lim x_n = l$ then
$$\lim (x_1, x_2, \ldots x_n)^{1/n} = l$$
13. If $<x_n>$ is a sequence such that $x_n > 0$ for all $n \in N$ and $\lim \frac{x_{n+1}}{x_n} = l$ then
$$\lim (x_n)^{1/n} = l$$
14. **Cesaro's theorem :** If $\lim x_n = l$ and $\lim y_n = l'$ then
$$\lim \frac{x_1 y_n + x_2 y_{n-2} + \ldots . x_n y_1}{n} = ll'$$

MONOTONIC SEQUENCES

A sequence $<x_n>$ is said to be

1. Monotonically increasing (non-decreasing) if $x_n \le x_{n+1} \, \forall n \in N$ *i.e.* $x_1 \le x_2 \le x_3 \le \ldots..$
2. Strictly increasing if $x_n < x_{n+1} \ \forall n \in N$ *i.e.* $x_1 < x_2 < x_3 \ldots.$
3. Monotonically decreasing (non-increasing) if $x_n \ge x_{n+1} \ \forall n \in N$ *i.e.* $x_1 \ge x_2 \ge x_3 \ge \ldots.$
4. Strictly decreasing if $x_n > x_{n+1} \ \forall n \in N$ *i.e.* $x_1 > x_2 > x_3 > \ldots.$

5. Monotonic if it is either monotonically increasing or monotonically decreasing.

Examples :

(i) $< 1, 2, 3, 4, \ldots n \ldots >$ is strictly increasing.

(ii) $< 1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{4} \ldots >$ is strictly decreasing.

(iii) $< 2, 2, 4, 4, 6, 6 \ldots >$ is monotonically increasing .

(iv) $< 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3} \ldots >$ is monotonically decreasing.

(v) $< 0, 1, 0, 1, 0, 1, 0 \ldots >$ is not monotonic.

6. **Results :**

(i) **Monotonic convergence theorem:** Every bounded monotonically increasing sequence converges.

(ii) Every bounded monotonically decreasing sequence converges.

(iii) Every bounded monotonic sequence converges.

(iv) A non-decreasing (monotonically increasing) sequence which is not bounded above diverges to infinity.

(v) A non-increasing (monotonically decreasing) sequence which is not bounded below diverges to minor infinity.

(vi) Every sequence has a monotonic subsequence.

LIMIT POINTS OF A SEQUENCE

A real number p is said to be a limit point (cluster point) of a sequence $< x_n >$ if every neighbourhood of p contains infinite number of terms of the sequence.

Examples :

(i) $< \frac{1}{n} >$ has only one limit points *i.e.* 0.

(ii) $< (-1)^n >$ has 1 and -1 as limit points.

Results :

(i) Limit point of a sequence is different from the limit of a sequence. The limit of a sequence is a limit point of the sequence while a limit point of a sequence need not be the limit of the sequence.

(ii) Limit point of sequence need not be a term of the sequence.

(iii) If $x_n = l$ for infinitely many values of n then l is a limit point of $< x_n >$.

(iv) If l is a limit point of the range of a sequence $< x_n >$, then l is a limit point of the sequence $< x_n >$ but not conversely.

(v) If $x_n \to l$ then l is the only limit point of $< x_n >$ but not conversely.

(vi) Bolzano-weistrass theorem for sequences. Every bounded sequence has at least one limit point.

(vii) Every sequence in a closed bounded set of real number has a limit point in closed set.

(viii) Every sequence in a closed interval I has a limit point in I.

(ix) If a sequence $< x_n >$ is bounded and has only one limit point say l then $x_n \to l$.

(x) If a real number p is a limit point of a sequence $< x_n >$ iff there exists a subsequence of $< x_n >$ converging to p.

(xi) The set of limit points of a bounded sequence is bounded.

(xii) Every bounded sequence has the greatest and the least limit points.

CAUCHY SEQUENCE

A sequence $< x_n >$ is said to be a Cauchy sequence if given $\in > 0$ there exists $m \in N$ such that

$$|x_n - x_m| < \in \ \forall n \geq m$$

or $|x_{n+p} - x_n| < \in \ \forall n \geq m$ and $p \geq 0$

or $|x_{m+p} - x_m| < \in \ \forall p \geq 0$

or $|x_p - x_q| < \in \ \forall p, q \geq m$

Results :

(i) Every cauchy sequence is bounded but converge need not be true.

(ii) **Cauchy convergence criterion:** A sequence converges iff it is a cauchy sequence.

LIMIT SUPERIOR AND LIMIT INTERIOR OF A SEQUENCE

1. **Limit Superior :** Let $< x_n >$ be a sequence of real numbers which is bounded above and let

$$\bar{x}_n = \sup\{x_n, x_{n+1}, \ldots\}$$

If $< \bar{x}_n >$ converges then the limit superior (upper limit) of $< x_n >$ is defined by

$$\overline{\lim}\, x_n = \lim_{n\to\infty} \sup x_n = \lim_{n\to\infty} < \bar{x}_n >$$

If $< \bar{x}_n >$ diverges to $-\infty$ then

$$\lim_{n\to\infty} \sup x_n = -\infty$$

If $< x_n >$ is not bounded above then

$$\lim_{n\to\infty} \sup x_n = \infty$$

2. **Limit Interior :** Let $< x_n >$ be a sequence of real number which is bounded below and let

$$\underline{x}_n = \inf\{x_n, x_{n+1}, \ldots\}$$

If $< \underline{x}_n >$ converges then limit interior or lower limit of $< x_n >$ is defined by

$$\underline{\lim}\, x_n = \lim_{n\to\infty} \inf x_n = \lim_{n\to\infty} \underline{x}_n$$

If $< \underline{x}_n >$ diverges to ∞ then $\lim_{n\to\infty} \inf x_n = \infty$.

If $< x_n >$ is not bounded then $\lim_{n\to\infty} \inf x_n = -\infty$.

Remark :

1. $\overline{\lim}\, x_n = \inf\{\bar{x}_1, \bar{x}_2, \ldots \bar{x}_n, \ldots\}$
2. $\underline{\lim}\, x_n = \sup\{\underline{x}_1, \underline{x}_2, \ldots \underline{x}_n \ldots\}$

Examples :

1. Let $x_n = -n\ \forall n \in N$. It is bounded above by -1 but not bounded below.

$$\bar{x}_n = \sup\{-n, -n-1, \ldots\} = -n$$

Since, $\bar{x}_n \to -\infty$ as $n \to \infty$ so $\overline{\lim}\, x_n = -\infty$.

Since, $< x_n >$ is not bounded below so $\underline{\lim}\, x_n = -\infty$.

2. Let $x_n = (-1)^n\ \forall n \in N$

Then it is bounded above by 1 and bounded below by -1.

So, $\bar{x}_n = 1$ and $\underline{x}_n = -1\ \forall n \in N$

Hence, $\overline{\lim}\, x_n = 1$ and $\underline{\lim}\, x_n = -1$

Results :

1. If $< x_n >$ is convergent sequence of real numbers and if $\lim x_n = l$ then, $\overline{\lim}\, x_n = \underline{\lim}\, x_n = l \in R$. Conversely if $\overline{\lim}\, x_n = \underline{\lim}\, x_n = l \in R$ then $< x_n >$ is convergent and $\lim_{n\to\infty} x_n = l$.
2. $< x_n >$ diverges to $+\infty$ iff $\overline{\lim}\, x_n = \underline{\lim}\, x_n = \infty$.
3. $< x_n >$ diverges to $-\infty$ iff $\overline{\lim}\, x_n = \underline{\lim}\, x_n = -\infty$.
4. If $< x_n >$ and $< y_n >$ are bounded sequences of real numbers such that $x_n \le y_n\ \forall n \in N$. then $\overline{\lim}\, x_n \le \overline{\lim}\, y_n$ and $\underline{\lim}\, x_n \le \underline{\lim}\, y_n$
5. If $< x_n >$ and $< y_n >$ are bounded sequences of real numbers, then

$$\overline{\lim}\,(x_n + y_n) \le \overline{\lim}\, x_n + \overline{\lim}\, y_n$$

and $$\underline{\lim}\,(x_n + y_n) \ge \underline{\lim}\, x_n + \underline{\lim}\, y_n$$

6. **Cantor's Intersection Theorem :** For each $n \in N$, let $I_n = [a_n, b_n]$ be a non-empty closed and bounded interval on R, such that $< I_n >$ is a nested sequence with $\lim_{n\to\infty}(b_n - a_n) = 0$, then $\bigcap_{n=1}^{\infty} I_n$ contour precisely the point.

EXERCISE

MULTIPLE CHOICE QUESTIONS

1. If $x_n + 1 \geq x_n \ \forall n \in N$ then the sequence $< x_n >$ is :

[Kanpur 2019]

a. Monotonic decreasing
b. Monotonic increasing
c. Strictly monotonic decreasing
d. None of these

2. The range of the sequence $< (-1)^n >$ is :

[Meerut 2018]

a. $\{1\}$ b. $\{-1\}$
c. $\{1, -1\}$ d. ϕ

3. Which of the following is not a subsequence of the sequence $< n >$:

a. <2,3,8,6,....> b. <1,3,5,7,.....>
c. <2,4,6,8,....> d. <2,3,5,7,....>

4. The sequence $< n^2 >$ is :

a. Bounded set
b. Bounded below but not bounded above
c. Bounded above but not bounded below
d. None of these

5. The range set of the sequence $< 1 + (-1)^n >$ is :

a. $\{0\}$ b. $\{1, 0, -1\}$
c. $\{1, -1\}$ d. $\{0, 2\}$

6. If a sequence $< x_n >$ is convergent then its limit is :

a. Unique b. Finite
c. Infinite d. Not exist

7. The subsequence of a convergent sequence is :

a. Divergent
b. Convergent
c. Convergent or divergent
d. None of these

8. The sequence $< r^n >$ converge to zero if :

a. $|r| > 1$ b. $|r| < 1$
c. $|r| = 1$ d. $\forall \ r \in N$

9. The sequence $< \frac{1}{2n} >$ is convergent to :

a. 1 b. $\frac{1}{2}$
c. 0 d. None of these

10. The range of the sequence $< \frac{1}{n} >$ is :

a. $\{0\}$ b. $\{0, 1\}$
c. N d. Infinite set

11. The sequence $< -n^2 >$ is :

a. Bounded above by 1
b. Bounded below by 1
c. Bounded above by -1
d. Bounded sequence

12. If the sequence is divergent then its subsequence :

a. Convergent
b. Divergent
c. May be convergent or divergent
d. Neither convergent nor divergent

13. Which of the following sequence is not convergent :

a. $< \frac{1}{n} >$ b. Constant sequence
c. $< \frac{3n}{n + 5n^{1/2}} >$ d. $< (-1)^n >$

14. If $\left| \frac{2n}{n+3} - 2 \right| < \frac{1}{5} \ \forall \ n \geq m$ then the least value of $m \in N$ is :

a. 2 b. 3
c. 20 d. 28

15. The sequence $< \frac{3n}{n + 5n^{1/2}} >$ is :

a. Convergent with limit $\frac{1}{5}$
b. Divergent
c. Convergent with limit 3
d. Convergent with limit $\frac{3}{5}$

16. The range set of $<(-1)^{n^2}>$ is :
a. {0,1} b. {1, –1}
c. {–1, 0} d. None of these

17. If the sequence $<x_n>$ is convergent then its subsequence is :
a. Convergent
b. Divergent
c. May be convergent or divergent
d. None of these

18. A monotonic increasing sequence which is not bounded above is :
a. Diverges to $+\infty$ b. Convergent
c. Diverges to $-\infty$ d. Oscillatory

19. The range of a constant sequence has :
a. One element b. Two elements
c. Infinite elements d. None of these

20. The sequence $<(-1)^n n>$ is :
a. Bounded below
b. Bounded above
c. Bounded
d. Neither bounded above nor bounded below

21. If $p>0$ then $\lim <\frac{1}{n^p}>$ is :
a. p b. $\frac{1}{n}$
c. 0 d. ∞

22. The sequence $<\sin n\pi\theta>$ is convergent for :
a. $\theta=0$ only b. $\theta=1$ only
c. $\theta\neq 0$ and $\theta\neq 1$ d. $\theta=0$ and $\theta=1$ both

23. Which of the following sequence is convergent :
a. $<2, 4, 6, ...>$ b. $<3, 3^2, 3^3, ...>$
c. $<-2, -4, -6, ...>$ d. $<1, \frac{1}{2}, \frac{1}{3}, ...>$

24. The sequence $<(-1)^n.n>$ is :
a. Convergent b. Divergent
c. Oscillate infinitely d. Oscillate finitely

25. The sequence $<x_n>$ is bounded if for $k>0$:
a. $x_n \geq k$ b. $x_n \leq k$
c. $|x_n| \geq k$ d. $|x_n| \leq k$

26. If $\frac{x_{n+1}}{x_n} \to l$ and $|l|<1$ then $\lim x_n$ is :
a. l b. 1
c. 0 d. ∞

27. Which of the following is not true :
a. Every bounded and monotonic sequence is convergent
b. A decreasing sequence which is bounded below diverges to $-\infty$
c. Every convergent sequence is bounded
d. Every convergent sequence is cauchy sequence

28. The sequence $<\frac{1}{n}>$ is :
a. Bounded above by 1
b. Bounded below by 1
c. Bounded below by $\frac{1}{2}$
d. Unbounded

29. Which of the following is not true :
a. Every cauchy's sequence in R is convergent
b. Every convergent sequence in R is cauchy's sequence
c. Every convergent sequence is bounded
d. Every bounded sequence is cauchy's sequence

30. A sequence $<x_n>$ is converges to limit l if for given $\varepsilon>0$, $\exists\, m \in N$ such that for $n \geq m$:
a. $|S_n - l| < \varepsilon$ b. $|S_n - l| > \varepsilon$
c. $|S_n - l| \leq \varepsilon$ d. $|S_n - l| \geq \varepsilon$

31. If $x_n = (-1)^n\left(1+\frac{1}{n}\right)$, then lim inf of x_n is :
a. 1 b. 0
c. –1 d. $-\infty$

32. The sequence $<\frac{n}{n+1}>$ is :
a. Bounded above by 1
b. Bounded below by 1
c. Bounded above by $\frac{1}{2}$
d. Unbounded

33. The sequence $< \dfrac{2n^2+1}{2n^2-1} >$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

34. If the sequence $< S_n >$ defined by $S_1 = \frac{1}{2}, S_{n+1} = \frac{2S_n+1}{3}\ \forall n \in N$ is convergent then its limit is :

a. $\frac{1}{2}$ b. $\frac{2}{3}$
c. 1 d. Not exist

35. The sequence $< S_n >$ defined by $S_1 = 2$, $S_n = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor n-1}$ for all $n \geq 2$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

36. The sequence $< (-1)^n >$ is :

a. Bounded above b. Bounded below
c. Bounded d. All the above

37. Every bounded sequence is :

a. Convergent
b. Divergent
c. Cauchy sequence
d. None of these

38. The sequence $< \dfrac{(-1)^{n-1}}{n} >$ converges to :

a. 1 b. –1
c. 0 d. ∞

39. The sequence $< -\log n >$ is :

a. Convergent b. Diverges to 0
c. Diverges to $+\infty$ d. Diverges to $-\infty$

40. If $< x_n >$ diverges to ∞ and $< y_n >$ is bounded then $< x_n + y_n >$ is :

a. Bounded b. Convergent
c. Oscillatory d. Diverges to ∞

41. The sequence $< \dfrac{(-1)^n}{n} >$ is :

a. Bounded above only
b. Bounded below only
c. Bounded
d. Unbounded

42. Every convergent sequence is :

a. Cauchy sequence
b. Bounded
c. Converge to unique limit
d. All of the above

43. If $S_n = (-1)^n\ \forall n \in N$ then, $\overline{\lim}\ S_n$ is :

a. –1 b. 1
c. 0 d. None of these

44. The sequence $S_n = 2 - \dfrac{1}{2^{n-1}}$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

45. Oscillatory sequence is :

a. Not convergent
b. Not divergent
c. Neither convergent nor divergent
d. Both convergent and divergent

46. If sequence diverges to infinity then any of its subsequence is :

a. May be convergent
b. May be oscillatory
c. Diverges to $-\infty$
d. Diverges to $+\infty$

47. If $\lim (x_n + y_n) = l + l'$ then :

a. $\lim x_n = l$
b. $\lim y_n = l'$
c. $< x_n >$ and $< y_n >$ both are convergent
d. None of these

48. If $S_n = -n\ \forall n \in N$, then :

a. $\overline{\lim}\ S_n > \underline{\lim}\ S_n$ b. $\overline{\lim}\ S_n < \underline{\lim}\ S_n$
c. $\overline{\lim}\ S_n = -\underline{\lim}\ S_n$ d. $\overline{\lim}\ S_n = \underline{\lim}\ S_n$

49. If the sequence $< S_n >$ defined by $S_1 = \sqrt{2}$, $S_{n+1} = \sqrt{2S_n}\ \forall n \in N$ is convergent then $\lim S_n$ is :
[Meerut 2017]

a. $\sqrt{2}$ b. 2
c. 0 d. ∞

50. The sequence $< \sqrt{n+1} - \sqrt{n} >$ is convergent to :
[Kanpur 2019; Meerut 2018]

a. 0 b. 1
c. e d. None of these

51. The sequence $<(-1)^n>$ is : **[Meerut 2018]**

a. Convergent b. Divergent

c. Oscillatory finitely d. Oscillates infinitely

52. Consider the following statements :

A. $\overline{\lim}(S_n + t_n) \le \overline{\lim} S_n + \overline{\lim} t_n$

B. $\underline{\lim}(S_n + t_n) \ge \underline{\lim} S_n + \underline{\lim} t_n$

For $< S_n >$ and $< t_n >$ are bounded sequence then,

a. A is true only

b. B is true only

c. A and B both are true

d. Neither A nor B is true

53. $\lim n^{1/n}$ is equal to :

a. e b. 1

c. 0 d. ∞

54. The sequence $< \sin n\pi\theta >$ where $\sigma < \theta < 1$ and is a rational number is :

a. Convergent b. Divergent

c. Oscillatory d. Unbounded

55. Which of the following sequence is convergent :

a. $< \frac{1}{2}, \frac{4}{5}, 1, \ldots \frac{2n}{2n+3} >$

b. $< -3, -3^2, -3^3, \ldots >$

c. $< -x, -x^2, -x^3 \ldots >, x > 1$

d. $< x, x^2, x^3, \ldots >, x > 1$

56. If $S_n = (-1)^n$ and $t_n = (-1)^{n+1}$ $\forall n \in N$ then $S_n + t_n$ $\forall n \in N$ is equal to :

a. 1 b. –1

c. 0 d. Not exist

57. If $< S_n >$ is defined by $S_1 = 1, S_{n+1} = \frac{4+3S_n}{3+2S_n}, n \in N$ is convergent to l then l is :

a. 1 b. $\frac{4}{3}$

c. $\frac{3}{2}$ d. $\sqrt{2}$

58. An oscillatory sequence is :

a. Always bounded

b. Always unbounded

c. May be bounded or unbounded

d. None of these

59. The sequence $< S_n >$ defined by

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \text{ is :}$$ **[Meerut 2018]**

a. Convergent b. Divergent

c. Oscillators d. None of these

60. If $r > 0$ then $\lim r^{1/n}$ is :

a. 0 b. –1

c. 1 d. ∞

61. $\lim \left\{ \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3 \ldots \left(\frac{n+1}{n}\right)^n \right\}^{1/n}$ is equal to :

a. 0 b. 1

c. e d. ∞

62. The number of limit points of the sequence $< 1, -1, 1, 1 \ldots >$ are :

a. 0 b. 1

c. 2 d. ∞

63. Which of the following is not a Cauchy sequence :

a. $< \frac{n}{n+1} >$ b. $< n^2 >$

c. $< \frac{1}{n} >$ d. $< \frac{1}{n^2} >$

64. If $< x_n >$ diverges to ∞ and $< y_n >$ converges then $< x_n + y_n >$ is : **[Meerut 2018]**

a. Convergent b. Bounded

c. Oscillatory d. Diverges to ∞

65. If $x_n = (-1)^n \left(1 + \frac{1}{n}\right)$, then limit superior of x_n is : **[Meerut 2018]**

a. 0 b. –1

c. 1 d. ∞

66. The sequence $< S_n >$ defined by

$$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+n} \text{ is :}$$

a. Convergent b. Divergent

c. Oscillatory d. None of these

67. If $< S_n >$ is defined by

$$S_1 = 1, S_{n+1} = \sqrt{2 + S_n} \; \forall n \in N$$

then $\lim S_n$ is :

a. 1 b. $\sqrt{2}$

c. 2 d. Not exist

68. Which of the following sequence is bounded :

a. $< n >$ b. $(-1)^n n$

c. $(-1)^{n+2}$ d. $< -n >$

69. $\lim \sqrt{\frac{n+1}{n}}$ is :

a. 0 b. 1

c. –1 d. Not exist

70. $\lim \frac{n}{(|\underline{n})^{1/n}}$ is equal to : **[Meerut 2019]**

a. 0 b. 1

c. e d. Not exist

71. The sequence defined by $S_n = 1$ if n is divisible by 3 and $S_n = 0$, therewise is :

a. Bounded above by 0

b. Bounded below by 1

c. Bounded above by 1

d. None of these

72. If the sequence $< S_n >$ converges to l then $|S_n|$ is converges to :

a. l only b. $-l$ only

c. $|l|$ d. Nothing to say

73. If $\lim x_n = l$ and $k \in R$ then $\lim (kx_n)$ is :

a. k b. l

c. kl d. 0

74. If $S_n = n \; \forall n \in N$ then, $\underline{\lim}\; S_n$ is equal to :

a. n b. 0

c. 1 d. ∞

75. The sequence $\left(1+\frac{1}{n}\right)^n$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

76. The sequence $< S_n >$ defined by

$S_1 = \sqrt{2}, S_{n+1} = \sqrt{(2S_n)}$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

77. Which of the following sequence is convergent :

a. $< (-1)^n_n >$ b. $< n >$

c. $< \cos \frac{n\pi}{2} >$ d. $< \frac{(-1)^{n+1}}{n} >$

78. If $< S_n >$ be a sequence of positive numbers defined by $S_n = \frac{1}{2}(S_{n-1} + S_{n-2}) \; \forall n > 2$ is convergent then $\lim S_n$ is :

a. 0 b. $\frac{1}{2}(S_1 + S_2)$

c. $\frac{1}{3}(S_1 + 2S_2)$ d. $\frac{1}{3}(S_1 + S_2)$

79. The sequence $< n^p >$ where $p > 0$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

80. The limit of the sequence $< 1 + \frac{1}{3} + \frac{1}{5} + ... \frac{1}{2n-1} >$ is :

a. 1 b. $\frac{4}{3}$

c. 0 d. Not exist

81. $\lim [\{(n+1)(n+2)....(n+n)\}^{1/n} / n] =$

a. 0 b. e

c. $4e$ d. $\frac{4}{e}$

82. If $S_n = (-1)^n$ then :

a. $< S_n >$ is convergent

b. $<|S_n|>$ is convergent

c. Both $< S_n >$ and $<|S_n|>$ convergent

d. None of them is convergent

83. If $S_n = \frac{(3n-1)(n^4 - n)}{(n^2+2)(n^3+1)}$ then $\lim S_n$ is equal to :

a. 3 b. $\frac{3}{2}$

c. $\frac{1}{2}$ d. 0

84. The sequence $\sqrt{3}, \sqrt{3\sqrt{3}}, \sqrt{3\sqrt{3\sqrt{3}}}$, converges to :

a. 0 b. 3

c. $\sqrt{3}$ d. ∞

85. $\lim \frac{\sin(n\pi/3)}{\sqrt{n}}$ is equal to :

a. 0 b. $\frac{\pi}{3}$

c. $\frac{1}{3}$ d. ∞

86. $\lim \frac{1}{n}(1+2^{1/2}+3^{1/3}+....n^{1/n})$ is :

a. 0 b. −1

c. +1 d. ∞

87. The sequence $\left\langle \left(1+\frac{2}{n}\right)^{n+3} \right\rangle$ converges to : **[Meerut 2017]**

a. e b. e^2+3

c. e^2 d. e^3

88. The limit of the convergent sequence $\frac{3n^2+1}{3n^2-1}$ is :

a. 1 b. 3

c. −1 d. 0

89. If $< Sn^2 >$ converges to zero then :

a. $< S_n >$ converges to zero

b. $< S_n >$ not converges to zero

c. $< S_n >$ may be oscillatory

d. None of these

90. If $< S_n >$ converges to $l \neq 0$ then $< (-1)^n S_n >$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

91. The limit of the sequence

$$\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+....+\frac{1}{\sqrt{n^2+n}} \text{ is :}$$

a. 0 b. 1

c. −1 d. ∞

92. $\lim \left[\frac{\lfloor 3n}{(\lfloor n)^3}\right]^{1/n} =$

a. $\lfloor 3$ b. $e^{\lfloor 3}$

c. 27 d. 0

93. $\lim \left[\frac{(\lfloor n)^{1/n}}{n}\right] =$

a. e b. $\frac{1}{e}$

c. 0 d. ∞

94. If the sequence $< S_n >$ is defined by

$$S_1 = \frac{3}{2}, S_{n+1} = 2 - \frac{1}{S_n} \ \forall n \geq 1$$

then $\lim S_n$ is :

a. $\frac{3}{2}$ b. 2

c. $\frac{1}{2}$ d. 1

95. If x be any real number then $\lim\limits_{n \to \infty} \frac{x^n}{\lfloor n}$ is :

a. 0 b. 1

c. $\lfloor n$ d. Not exist

96. The limit point of the sequence $(-1)^n\left(1+\frac{1}{n}\right)$ is :

a. 1 only b. −1 only

c. 0, ± 1 d. ± 1

97. If the sequence $< S_n >$ is defined by $S_1 = a > 0$, $S_{n+1} = \sqrt{\frac{ab^2+Sn^2}{a+1}}$, $b > a$, $n \geq 1$ then $\lim S_n$ is :

a. $\sqrt{ab^2}$ b. $\sqrt{\frac{ab^2}{a+1}}$

c. a d. b

98. The set of limit points of every sequence is :

a. Open b. Closed

c. Compact d. None of these

99. The sequence $< \sin n\pi >$ converges to :

a. 0 b. 1

c. −1 d. π

100. Which of the following sequence is divergent :

a. $< \sin n\pi >$ b. $< \frac{(-1)^n}{n} >$

c. $< \frac{1}{3^n} >$ d. $< \sin \frac{n\pi}{2} >$

101. A bounded set has :

a. One limit point b. Two limit points

c. No limit points d. Infinite limit points

102. A bounded sequence has : **[Meerut 2017]**

a. One limit point b. Two limit points

c. No limit points d. At least one limit point

103. Every bounded monotonically increasing sequence converges to it's :

a. Supremum b. Infimum
c. 0 d. None of these

104. The limit of sequence $< \frac{1}{n} >$ is: **[Meerut 2017, 2018]**

a. 1 b. 2
c. 3 d. 0

105. A sequence $< S_n >$ is oscillatory sequence if it is : **[Meerut 2017]**

a. Convergent
b. Divergent
c. Neither convergent nor divergent
d. None of these

106. The limit of sequence $< S_n >$ where $S_n = \frac{3n}{n + 5n^{1/2}}$ is: **[Meerut 2014, 17]**

a. 1 b. 2
c. 3 d. 4

107. If limit $S_n = 0$ and the sequence $< t_n >$ is bounded then limit $< S_n t_n >$ is : **[Meerut 2017]**

a. 0 b. 1
c. 2 d. 3

108. If limit $S_n = l_1$ and $\lim t_n = l_2$ then $\lim(s_n t_n)$ is : **[Meerut 2017]**

a. $\frac{l_1}{l_2}$ b. $l_1 l_2$
c. l_1 d. l_2

109. Every cauchy sequence is : **[Meerut 2017]**

a. Oscillatory b. Divergent
c. Unbounded d. Convergent

110. The supremum of the sequence $< \frac{n}{n+1} >$ is : **[Meerut 2017]**

a. 1 b. 2
c. 3 d. 0

111. If $x_1 = \sqrt{7}$ and $x_{n+1} = \sqrt{(7 + x_n)}$ then $< x_n >$ converges to : **[Meerut 2018]**

a. Positive value
b. Positive root of $x^2 - x - 7 = 0$
c. 7
d. All the above

112. The infimum of the set $S = \{x : x \in Q$ and $x = (-1)^n \left(\frac{1}{n} - \frac{4}{n}\right), n \in N\}$ is : **[Meerut 2018]**

a. $\frac{-2}{3}$ b. $\frac{-3}{2}$
c. $\frac{3}{2}$ d. $\frac{2}{3}$

113. The domain of the sequence is always : **[Meerut 2018]**

a. N b. R
c. R^+ d. Q

114. The supremum of the set $S = \left\{\frac{2n+1}{3n+2} : n \in N\right\}$ is : **[Meerut 2018]**

a. $\frac{3}{2}$ b. $\frac{-2}{3}$
c. $\frac{2}{3}$ d. $\frac{-3}{2}$

115. The sequence $< \frac{2^n}{\lfloor n} >$ is : **[Meerut 2018]**

a. Monotonic
b. Monotonic increasing
c. Monotonic decreasing
d. Divergent

116. If $\lim s_n = l$ and $\lim t_n = t$ then $\lim \frac{s_n}{t_n} = \frac{l}{t}$ is : **[Meerut 2018]**

a. True b. False
c. True if $t \neq 0$ d. Always true

117. The limit points of $< (-1)^n >$ is : **[Meerut 2018]**

a. A finite set b. $[-1, 1]$
c. $\{1, -1\}$ d. Both (a) and (c) true

118. Which is not true :

a. $< \frac{1}{n} >$ converges to zero
b. $< \frac{3n}{n + 5n^{1/2}} >$ converges to $\frac{3}{5}$
c. $< n >$ is divergent sequence
d. $< 3^n >$ is divergent sequence

119. The limit point of $< \frac{1}{n} >$ is/does : **[Meerut 2018]**

a. Does not belong to the range set
b. 0
c. –1
d. Both (a) and (b) true

120. If $S_n = \frac{2n-7}{3n+2}$, then $< S_n >$ is : **[Meerut 2018]**

a. Convergent
b. Divergent
c. Monotonic increasing
d. Both (a) and (c)

121. If $S_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + ... \frac{1}{n+n}$, then $< S_n >$ is : **[Meerut 2018]**

a. Monotonic decreasing
b. Monotonic increasing
c. Convergent
d. Both (b) and (c)

122. The sequence $< \sin n\pi\theta >$ is : **[Meerut 2018]**

a. Convergent when $\theta = 0$
b. Convergent when $\theta = 1$
c. Divergent when $\theta \geq 0$
d. Both (a) and (b)

123. Which is not true : **[Meerut 2018]**

a. Every bounded sequence is convergent
b. Every convergent sequence is bounded
c. $\lim \frac{1}{n^p} = 0; p < 0$
d. Both (a) and (c)

124. If $\lim s_n = l_1$ and $\lim t_n = l_2$ then $\lim(s_n + t_n)$ is: **[Meerut 2017]**

a. $l_1 - l_2$ b. $l_1 + l_2$
c. $l_1 l_2$ d. None of these

125. If the sequence $<|S_n|>$ is convergent then the sequence $< S_n >$ is : **[Meerut 2017]**

a. Convergent
b. Not convergent
c. May or may not be convergent
d. None of these

126. A sequence is said to be a divergent sequence if it diverges to : **[Meerut 2017]**

a. ∞ b. $-\infty$
c. Either ∞ or $-\infty$ d. 0

127. If $\left|\frac{1}{2n} - 0\right| \leq \frac{1}{500}$ $\forall n \geq m$, then m is equal to : **[Meerut 2018]**

a. –250 b. –300
c. 300 d. 250

128. The set of limit point of a bounded sequence is : **[Meerut 2018]**

a. ϕ b. Bounded
c. Convergent d. Finite

129. The sequence $< x_n >$, where $x_n = 3^n : n \in N$ is : **[Kanpur 2019]**

a. Divergent b. Convergent
c. Oscillatory d. None of these

130. Every convergent sequence must be : **[Kanpur 2019]**

a. Oscillators b. Unbounded
c. Bounded d. None of these

131. $\lim \frac{\sin\left(\frac{n\pi}{3}\right)}{\sqrt{n}}$ is equal to : **[Meerut 2017]**

a. 0 b. 1
c. –1 d. None of these

132, If $\lim_{n\to\infty} \frac{S_n}{n} = l \neq 0$ then $< S_n >$ is : **[Meerut 2014]**

a. Bounded b. Unbounded
c. Convergent d. None of these

133. The sequence $< a_n >$ defined by $a_n = \frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + ... + \frac{1}{n^n}$ is converged whose limit lies between : **[Meerut 2014]**

a. 0 and 1 b. 1 and 1/2
c. 1 and 3/2 d. 3 and 7/2

134. Sequence $< a_n >$ where $a_n = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{n+n}$ is : **[Meerut 2014]**

a. Convergent b. Divergent
c. Oscillatery d. None

135. The sequence $<a_n>$ defined by $a_n = \frac{a}{1+a_{n-1}}$ $a > 0, a_1 > 0$ converges to the positive root of the equation : **[Meerut 2014]**

a. $x^2 - x + a = 0$ b. $x^2 + x + a = 0$

c. $x^2 - x - a = 0$ d. $x^2 + x - a = 0$

136. Every Cauchy Sequence is : **[Meerut 2014]**

a. Convergent b. Bounded

c. Both (a) and (b) d. None of these

137. Find the least positive integer n such that

$$\left|\frac{2n}{n+3} - 2\right| < \frac{1}{5}$$ **[Meerut 2014]**

a. 20 b. 27

c. 28 d. None of these

138. Sequence $< S_n >$ where $S_n = (-1)^n$ has limit point : **[Meerut 2014]**

a. 1 b. –1

c. –1 and 1 d. None of these

139. If limit $S_n = 0$ and the Sequence $< P_n >$ is bounded then the limit $< S_n P_n >$ is : **[Meerut 2014]**

a. 0 b. 1

c. 2 d. None of these

140. The function $f(x) = \cos x - 2Px$ is Monotonically decreasing for : **[Meerut 2014]**

a. $P > \frac{1}{2}$ b. $P < \frac{1}{2}$

c. $P = \frac{1}{2}$ d. $P \neq \frac{1}{2}$

141. Which of the following is not true for $x > 0$: **[Meerut 2014]**

a. $\lim_{n\to\infty} \frac{1}{n^x} = 0$ b. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

c. $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$ d. All are true

142. The sequence $< S_n > = \frac{\lim}{n \to \infty}\left(1+\frac{1}{n}\right)^n$ lies between : **[Meerut 2015]**

a. 3 and 4 b. 1 and 2

c. 2 and 3 d. None of these

143. The sequence $< S_n >$ where

$S_n = \frac{5n}{n + 3n^{\frac{1}{2}}}$ has the limit : **[Meerut 2015]**

a. 3 b. 1

c. $\frac{1}{3}$ d. 5

144. The sequence $< S_n >$ where $S_n = 3 - \frac{1}{3^{n-1}}$ converges to : **[Meerut 2015]**

a. 1 b. 2

c. 3 d. –3

145. The sequence $< S_n >$ where

$S_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ is : **[Meerut 2015]**

a. A Cauchy sequence

b. Not a Cauchy sequence

c. May or may not be a Cauchy sequence

d. None of these

146. If $S_n = \left(1+\frac{1}{n}\right)^n$ then

$\lim_{n\to\infty} (s_1 s_2 .... s_n)^{\frac{1}{n}} =$ **[Meerut 2015]**

a. e b. e^{-1}

c. ae d. ae^{-1}

147. Every bounded monotonic sequence : **[Meerut 2015]**

a. Divergent b. Convergent

c. Unbounded d. Oscillatory

148. If a sequence $< S_n >$ is convergent then the sequence $<|S_n|>$ is : **[Meerut 2015]**

a. Convergent b. Divergent

c. Oscillatory d. None of these

149. If $\lim S_n = 0$ and the sequence $< t_n >$ is bounded then, **[Meerut 2015]**

$\lim < S_n t_n >$ is :

a. 1 b. ∞

c. 0 d. None of these

150. If $\lim t_n = l', l' \neq 0$ and $t_n \neq 0 \; \forall n$ then $\lim\left(\frac{1}{t_n}\right)$ is : **[Meerut 2015]**

a. 1 b. l'

c. $\frac{1}{l'}$ d. None of these

151. If $\lim S_n = l$ and $\lim t_n = l'$ then

$\lim \frac{S_1 t_n + S_2 t_{n-1} + \ldots + S_n t_1}{n}$ is : **[Meerut 2015]**

a. l b. l'

c. $\frac{l}{l'}$ d. ll'

152. Every bounded sequence has at least : **[Meerut 2015]**

a. Two limit point

b. One limit point

c. Three limit point

d. None of these

153. The Set $S = \left\{1 + \frac{(-1)^n}{2^n} : n \text{ is positive integer}\right\}$ is : **[Meerut 2016]**

a. Bounded

b. Only bounded above

c. Only bounded below

d. None of these

154. Every infinite bounded set of real numbers has : **[Meerut 2016]**

a. Only two limit points

b. At least one limit point

c. No limit points

d. None of these

155. Derived set of the set Q of all rational numbers is : **[Meerut 2016]**

a. Q b. R

c. Z d. None of these

156. Every singleton set in R is : **[Meerut 2016]**

a. Open

b. Closed

c. Neither open nor closed

d. None of these

157. A set which contain all of its limit points is called : **[Meerut 2016]**

a. Ddiscrete set b. Derived set

c. Closed set d. Open set

158. Sequence $< S_n >$ where

$S_n = 1 + \frac{(-1)^n}{n}$ is : **[Meerut 2016]**

a. Bounded b. Not bounded

c. Both (a) and (b) d. None of these

159. Sequence $< S_n >$ where $S_n = \frac{n}{n+1}$ converges to : **[Meerut 2016]**

a. 1 b. 2

c. 3 d. 0

160. If $S_n = \frac{(3n-1)(n^4-n)}{(n^2+2)(n^3+1)}$ than limit of $< S_n >$ is : **[Meerut 2016]**

a. 1 b. 2

c. 3 d. 4

161. Sequence $< S_n >$ where

$S_n = \frac{n^2+3n+5}{2n^2+5n+7}$ converges to : **[Meerut 2016]**

a. $\frac{1}{2}$ b. $\frac{3}{2}$

c. $\frac{5}{2}$ d. $\frac{7}{2}$

162. If $S_n = \left\{\frac{(\lfloor 3n)\, 3}{(\lfloor n)}\right\}^{1/n}$ then, $\lim_{n \to \infty} S_n$ is : **[Meerut 2016]**

a. 25 b. 26

c. 27 d. 28

163. If $S_n = \frac{2n-7}{3n+1}$ then, sequence $< S_n >$ is : **[Meerut 2016]**

a. Monotonic increasing

b. Bounded above

c. Bounded below

d. All of the above

164. If $S_n = \frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{n(n+1)}$ then sequence $< S_n >$ will be : **[Meerut 2016]**

a. Convergent b. Increasing

c. Both (a) and (b) d. None of these

165. Every bounded monotonically increasing sequence converges to : **[Meerut 2016]**

a. Its supremum b. Its infimum
c. 0 d. 1

166. Every subsequence of a convergent sequence is : **[Meerut 2016]**

a. Divergent b. Convergent
c. Oscillatory d. None of these

167. If $\lim S_n = l$ and $\lim t_n = l'$ then $\lim(S_n t_n) =$ **[Meerut 2016]**

a. $\frac{l}{l'}$ b. ll'
c. $l + l'$ d. $l - l'$

168. If $S_n = \left(1+\frac{1}{n}\right)^{n+2}$ then $\lim_{n\to\infty} S_n$ is : **[Meerut 2016]**

a. e b. $\frac{1}{e}$
c. $2e$ d. $\frac{2}{e}$

169. The sequence $S_n = <1, -1, 1, -1, 1, -1....>$ is : **[Meerut 2016]**

a. Bounded b. Unbounded
c. Oscillatory d. Convergent

170. The sequence $< S_n >$ defined by

$$S_n = \sqrt{2}, S_{n+1} = \sqrt{(2S_n)}$$

converges to : **[Meerut 2017]**

a. 1 b. 2
c. 3 d. 4

171. The sequence $< S_n >$ where

$S_n = \left(1+\frac{2}{n}\right)^{n+3}$ converges to : **[Meerut 2017]**

a. e b. e^2
c. $e + 3$ d. $e^2 + 3$

172. Every Cauchy sequence is : **[Meerut 2017]**

a. Oscillatory b. Divergent
c. Unbounded d. Convergent

173. The limit of sequence $< \frac{1}{n} >$ is : **[Meerut 2017]**

a. 1 b. 2
c. 3 d. 0

174. A sequence $< S_n >$ is oscillatory sequence if it is : **[Meerut 2017]**

a. Convergent
b. Divergent
c. Neither convergent nor divergent
d. None of these

175. The sequence $< 2, -2, 2, -2, 2, -2 >$ is : **[Meerut 2016]**

a. Bounded b. Unbounded
c. Oscillatory d. Convergent

176. A sequence is said to be a divergent sequence if it diverges to : **[Meerut 2017]**

a. ∞ b. $-\infty$
c. either ∞ or $-\infty$ d. 0

177. The supremum of the sequence $< \frac{n}{n+1} >$ is : **[Meerut 2017]**

a. 1 b. 2
c. 3 d. 0

178. Every bounded sequence has at least : **[Meerut 2017]**

a. Two limit points
b. One limit point
c. No limit point
d. None of these

179. If the sequence $<|s_n|>$ is convergent then the sequence $< s_n >$ is : **[Meerut 2017]**

a. Convergent
b. Not convergent
c. May or may not be convergent
d. None of these

180. $\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}$ is equal to : **[Meerut 2018]**

a. 0 b. 1
c. –1 d. e

181. The sequence $< \sin n\pi\theta >$ is : **[Meerut 2018]**

a. Convergent when $\theta = 0$
b. Convergent when $\theta = 1$
c. Divergent when $\theta \geq 0$
d. Both (a) and (b)

182. If $S_n = (-1)^n\left(1+\frac{1}{n}\right)$ then $\lim S_n$ is equal to : **[Meerut 2018,19]**

a. 0 b. 1

c. –1 d. ϕ

183. If $x_1 = \sqrt{7}$ and $x_{n+1} = \sqrt{(7+x_n)}$ then $< x_n >$ converges to : **[Meerut 2018]**

a. Positive value

b. Positive root of $x^2 - x - 7 = 0$

c. 7

d. All the above

184. The supremum of the set $S = \left\{\frac{2n+1}{3n+2} : n \in N\right\}$ is : **[Meerut 2018]**

a. $\frac{3}{2}$ b. $-\frac{2}{3}$

c. $\frac{2}{3}$ d. $-\frac{3}{2}$

185. If $S = \{(-1)^n : n \in N\}$ then $D(S)$ is : **[Meerut 2018]**

a. 1 b. –1

c. ϕ d. $\{1, -1\}$

186. The sequence $< \frac{2^n}{n!} >$ is : **[Meerut 2018]**

a. Monotonic

b. Monotonic increasing

c. Monotonic decreasing

d. Divergent

187. If $\left|\frac{2n}{2n+3} - 2\right| < \frac{1}{4} \forall n \geq m$ then m is equal to : **[Meerut 2018,19]**

a. 12 b. 13

c. –12 d. None of above

188. If $S_n = \sqrt{n+1} - \sqrt{n}$, then $l \lim S_n$ is equal to : **[Meerut 2018]**

a. 0 b. 1

c. –1 d. Does not exist

189. If $S_n = (-1)^n \cdot \left(1+\frac{1}{n}\right)$, then $\overline{\lim}\, S_n$ is equal to : **[Meerut 2018]**

a. 0 b. 1

c. –1 d. ϕ

190. The supremum and infimum of the set $S = \left\{m + \frac{1}{m} : m, n \in N\right\}$ is : **[Meerut 2018]**

a. 1 and 0

b. 0 and 1

c. –1 and 0

d. sup does not exist and inf = 1

191. Which is not true : **[Meerut 2018]**

a. $< \frac{1}{n} >$ converges to zero

b. $< \frac{3n}{n+5n^{1/2}} >$ converges to $\frac{3}{5}$

c. $< n >$ is divergent sequence

d. $< 3^n >$ is divergent sequence

192. If $S_n = \{x : x \in R^+\}$ and $\lim S_n = l$, then : **[Meerut 2018]**

a. $l = 0$ b. $l > 0$

c. $l \leq 0$ d. $l \geq 0$

193. Which is not true : **[Meerut 2018]**

a. Every bounded sequence is convergent

b. Every convergent sequence is bounded

c. $\lim \frac{1}{np} = 0; p < 0$

d. Both (a) and (c)

194. The range of the sequence $< (-1)^n >$ is : **[Meerut 2018]**

a. ϕ b. 1

c. –1 d. $\{1, -1\}$

195. If $\left|\frac{1}{2n} - 0\right| \leq \frac{1}{500} \forall n \geq m$, then m is equal to : **[Meerut 2018]**

a. –250 b. –300

c. 300 d. 250

196. If $\lim S_n = \infty$ and $< t_n >$ is bounded then $< S_n + t_n >$ is : **[Meerut 2018]**

a. Convergent b. Divergent

c. Divergent to ∞ d. Divergent to $-\infty$

197. The solution of equation $|S_n - 1| < \in$ is : **[Meerut 2019]**

a. $S_n < 1 + \in$ b. $S_n > 1 - \in$

c. $1 - \in < S_n < 1 + \in$ d. $1 - \in > S_n > 1 + \in$

198. The domain of the sequence $S_n = (-1)^n$ is : **[Meerut 2019]**

a. $<-1, 1>$ b. $\{-1, 1\}$
c. $[-1, 1]$ d. set of natural number, N

199. For the sequence $S_n = \frac{1}{3^n}, \forall \in > 0, |S_n - 0| < \in$ if : **[Meerut 2019]**

a. $n < -\log \in / \log 3$
b. $n > \log \in / \log 3$
c. $n > -\log \in / \log 3$
d. $n < \log \in / \log 3$

200. Which is not true : **[Meerut 2019]**

a. $\lim n^{1/n} = 0$ b. $\lim n^{1/n} = 1$
c. $\lim n^{2/n} = 1$ d. Both (b) and (c)

201. Every cauchy sequence is : **[Meerut 2019]**

a. Bounded
b. Convergent and bounded
c. Divergent
d. May be convergent

202. Every convergent sequence :

a. Has limit point
b. Has limit and limit point
c. Has limit point and bounds
d. All the above

203. Let $< a_n >$ be a real sequence, where

$$\sum_{n=1}^{\infty} |a_n - a_{n-1}| < \infty$$

then the series $\Sigma a_n x^n, x \in R$ is convergent : **[Meerut 2019]**

a. No where on R b. Every where on R
c. On $(-1, 1)$ d. None of these

204. Consider a sequence $< S_n >$ such that $S_n \in (-1, 1)$, then : **(Meerut 2019]**

a. Every limit point of $\{S_n\}$ is in $(-1, 1)$
b. Every limit point of $\{S_n\}$ is in $[-1, 1]$
c. The only limit points are $-1, 0, 1$
d. The limit points are not $-1, 0, 1$

205. Let $\{S_n\}$ be a real sequence such that $S_1 \geq 1$ and $S_{n+1} \geq S_{n+1}$ then which is true :

a. The series $\Sigma S_{n^{-2}}$ converges
b. $< S_n >$ is bounded
c. The series $\Sigma S_{n^{-2}}$ converges
d. None of these

206. Let $S_n = \frac{2n}{n+3}, \in = \frac{1}{5}$ and $\lim S_n = 2$, then using

$\forall \in > 0, |S_n - \in| < l \ \forall n \geq M$, gives :

a. $M = 28$ b. $M \leq 28$
c. $M > 28$ d. Does not exist

207. Domain of a sequence is always :

a. Set of real number
b. Set of integers
c. Set of natural number
d. All the above

208. $\lim_{n \to \infty} \left(1 - \frac{1}{n^2}\right)^n$ equals : **[Meerut 2019]**

a. 1 b. $\frac{1}{\sqrt{e}}$
c. $\frac{1}{e^2}$ d. $\frac{1}{e}$

209. $\lim_{n \to \infty} \left(\frac{(\underline{|n})^3}{(\underline{|3n})} \right)^{\frac{1}{n}}$ is equal to : **[Meerut 2019]**

a. 3 b. -27
c. 27 d. -3

210. $L = \lim \frac{1}{n\sqrt{\underline{|n}}}$, then :

a. $L = 0$ b. $L = 1$
c. $0 < L < \infty$ d. None of these

211. Consider the sequence

$$S_n = \left(1 + (-1)^n \frac{1}{n}\right)^n, \text{ then}$$

a. $\lim\sup S_n = \lim\inf S_n = 1$
b. $\lim\sup S_n = \lim\inf S_n = e$
c. $\lim\sup S_n = \lim\inf S_n = \frac{1}{e}$
d. $\lim\sup S_n = e$ and $\lim\inf S_n = \frac{1}{e}$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(b)	**2.**	(c)	**3.**	(a)	**4.**	(b)	**5.**	(d)	**6.**	(a)	**7.**	(b)	**8.**	(b)	**9.**	(c)	**10.**	(d)
11.	(c)	**12.**	(c)	**13.**	(d)	**14.**	(d)	**15.**	(c)	**16.**	(b)	**17.**	(a)	**18.**	(a)	**19.**	(a)	**20.**	(d)
21.	(c)	**22.**	(d)	**23.**	(d)	**24.**	(c)	**25.**	(d)	**26.**	(c)	**27.**	(b)	**28.**	(a)	**29.**	(d)	**30.**	(a)
31.	(c)	**32.**	(a)	**33.**	(a)	**34.**	(c)	**35.**	(a)	**36.**	(d)	**37.**	(d)	**38.**	(c)	**39.**	(d)	**40.**	(d)
41.	(c)	**42.**	(d)	**43.**	(b)	**44.**	(a)	**45.**	(c)	**46.**	(d)	**47.**	(d)	**48.**	(d)	**49.**	(b)	**50.**	(a)
51.	(c)	**52.**	(c)	**53.**	(b)	**54.**	(b)	**55.**	(a)	**56.**	(c)	**57.**	(d)	**58.**	(c)	**59.**	(b)	**60.**	(c)
61.	(c)	**62.**	(c)	**63.**	(b)	**64.**	(d)	**65.**	(c)	**66.**	(a)	**67.**	(c)	**68.**	(c)	**69.**	(b)	**70.**	(c)
71.	(c)	**72.**	(c)	**73.**	(c)	**74.**	(d)	**75.**	(a)	**76.**	(a)	**77.**	(d)	**78.**	(c)	**79.**	(b)	**80.**	(c)
81.	(d)	**82.**	(b)	**83.**	(a)	**84.**	(b)	**85.**	(a)	**86.**	(c)	**87.**	(c)	**88.**	(a)	**89.**	(a)	**90.**	(c)
91.	(b)	**92.**	(c)	**93.**	(b)	**94.**	(d)	**95.**	(a)	**96.**	(d)	**97.**	(d)	**98.**	(b)	**99.**	(a)	**100.**	(d)
101.	(c)	**102.**	(d)	**103.**	(a)	**104.**	(d)	**105.**	(c)	**106.**	(c)	**107.**	(a)	**108.**	(b)	**109.**	(d)	**110.**	(a)
111.	(b)	**112.**	(b)	**113.**	(a)	**114.**	(c)	**115.**	(d)	**116.**	(c)	**117.**	(d)	**118.**	(b)	**119.**	(d)	**120.**	(d)
121.	(d)	**122.**	(d)	**123.**	(d)	**124.**	(b)	**125.**	(c)	**126.**	(c)	**127.**	(d)	**128.**	(b)	**129.**	(a)	**130.**	(c)
131.	(d)	**132.**	(b)	**133.**	(c)	**134.**	(a)	**135.**	(d)	**136.**	(c)	**137.**	(c)	**138.**	(c)	**139.**	(a)	**140.**	(a)
141.	(d)	**142.**	(c)	**143.**	(d)	**144.**	(c)	**145.**	(a)	**146.**	(a)	**147.**	(b)	**148.**	(a)	**149.**	(c)	**150.**	(c)
151.	(d)	**152.**	(b)	**153.**	(a)	**154.**	(b)	**155.**	(b)	**156.**	(b)	**157.**	(b)	**158.**	(a)	**159.**	(a)	**160.**	(c)
161.	(a)	**162.**	(c)	**163.**	(d)	**164.**	(c)	**165.**	(a)	**166.**	(b)	**167.**	(b)	**168.**	(a)	**169.**	(a)	**170.**	(b)
171.	(b)	**172.**	(d)	**173.**	(d)	**174.**	(c)	**175.**	(a)	**176.**	(c)	**177.**	(a)	**178.**	(b)	**179.**	(b)	**180.**	(d)
181.	(d)	**182.**	(c)	**183.**	(d)	**184.**	(c)	**185.**	(d)	**186.**	(c)	**187.**	(d)	**188.**	(a)	**189.**	(b)	**190.**	(d)
191.	(b)	**192.**	(d)	**193.**	(d)	**194.**	(d)	**195.**	(d)	**196.**	(c)	**197.**	(c)	**198.**	(d)	**199.**	(c)	**200.**	(a)
201.	(b)	**202.**	(d)	**203.**	(d)	**204.**	(b)	**205.**	(d)	**206.**	(d)	**207.**	(c)	**208.**	(a)	**209.**	(c)	**210.**	(d)
211.	(d)																		

HINTS AND SOLUTIONS

2. Let $S_n = \lt (-1)^n \gt$

then $S_n = \lt -1, 1, -1, 1, -1, 1, \ldots \gt$

Thus range of $S_n = \{-1, 1\}$.

4. $S_n = \lt n^2 \gt$

i.e. $S_n = \lt 1^2, 2^2, 3^2, 4^2, \ldots \gt$

Supremum not exist but infimum $= 1$ so s_n is bounded below by 1.

5. $S_n = 1 + (-1)^n$

or $S_n = \{1-1, 1+1, 1-1, 1+1 \ldots\}$

i.e. $S_n = \{0, 2\}$

Thus the range of $S_n = \{0, 2\}$.

9. Given, sequence is $S_n = \frac{1}{2^n}$

$$\lim S_n = \lim_{n \to \infty} \left(\frac{1}{2^n}\right) = 0$$

So, $< S_n >$ is a convergent sequence which is convergent to 0.

11. $S_n = < -n^2 >$

or $S_n = < -1, -4, -9, -16, \ldots >$

Supremum = –1, infimum does not exist.

Thus S_n is bounded above by –1 but not bounded below.

13. $\lim \frac{1}{n} = 0, \lim < k >= k$

$$\lim \frac{3n}{n + 5n^{1/2}} = 3$$

$$\lim (-1)^n = < 1, -1, 1, -1, \ldots >$$

14. We have, $\left|\frac{2n}{n+3} - 2\right| < \frac{1}{5}$

$\Rightarrow \left|\frac{2n - 2n - 6}{n+3}\right| < \frac{1}{5}$

$\Rightarrow \frac{6}{n+3} < \frac{1}{5} \Rightarrow n > 27$

If we take a positive integer $m > 27$, we have

$$\left|\frac{2n}{n+3} - 2\right| < \frac{1}{5} \ \forall n \ge m$$

Hence, for $\in = \frac{1}{5}$ the required last value of $m = 28$.

20. $S_n = (-1)^n . n$

or $S_n = \{-1, 2, -3, 4, -5, \ldots\}$

or $S_n = \{\ldots -5, -3, -1, 2, 4, \ldots\}$

Thus neither supremum nor infimum exist *i.e.* S_n *is* neither bounded above nor bounded below.

22. $S_n = < \sin n\pi\theta >$

If $\theta \to 0$ then $n\pi\theta \to 0$ *i.e.* $\sin n\pi\theta = 0$

If $\theta \to 1$ then $n\pi\theta \to n\pi$ *i.e.* $\sin n\pi\theta = 0$

So $< S_n >$ is a convergent sequence and it is converge to 0.

23. Let $S_n = \frac{1}{n}$

then, $\lim S_n = \lim \frac{1}{n} = 0$

i.e. S_n is a convergent sequence.

31. $x_n = (-1)^n \left(1 + \frac{1}{n}\right)$

Then, $x_n = < -2, \frac{3}{2}, \frac{-4}{3}, \frac{5}{4}, \frac{-6}{5}, \frac{7}{6} \ldots >$

So $\overline{x_1} = \frac{3}{2}, \overline{x_2} = \frac{3}{2}, \overline{x_3} = \frac{5}{4}, \overline{x_4} = \frac{5}{4} \ldots$

and $\underline{x_1} = -2, \underline{x_2} = -\frac{4}{3}, \underline{x_3} = \frac{-4}{3}, \underline{x_4} = \frac{-6}{5} \ldots$

So $\underline{\lim}\, x_n = \sup\{-2, -4/3, -6/5, \ldots\} = -1$

34. $S_1 = \frac{1}{2}, S_{n+1} = \frac{2S_n + 1}{3} \ \forall n \in N$

If $\lim S_n = \lim S_{n+1} = l$

then, $l = \frac{2l + 1}{3} \Rightarrow 3l = 2l + 1$

$\Rightarrow l = 1$

35. $S_{n+1} - S_n = \frac{1}{\lfloor n} > 0 \ \forall n \in N$

So $< S_n >$ is monotonically increasing.

For $n \ge 2, \lfloor n = 1, 2, 3, \ldots n$ contains $(n-1)$ factors each of which is greater than or equal to 2.

Hence, $\lfloor n \ge 2^{n-1}$ for all $n \ge 2$.

$\therefore \frac{1}{\lfloor n} \le \frac{1}{2^{n-1}}$ for all $n \ge 2$

Thus, $S_n = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \ldots + \frac{1}{\lfloor n-1}$

$$\le 1 + \frac{1}{\lfloor 1} + \frac{1}{2} + \frac{1}{2^2} + \ldots \frac{1}{2^{n-1}}$$

$$= 1 + \frac{1 - \left(\frac{1}{2}\right)^{n-1}}{1 - \frac{1}{2}} < 3 \ \forall n \ge 2$$

Also, $S_1 = 2 < 3, 2 \le S_n < 3 \ \forall n \in N$

i.e., $< S_n >$ is bounded.

Since, $< S_n >$ is bounded monotonically increasing sequence so it is convergent.

39. $< S_n > = < -\log n > \ \forall n \in N$

So, $\lim S_n = -\log \infty = -\infty$

So it is divergent sequence and diverge to $-\infty$.

41. $S_n = \frac{(-1)^n}{n} \ \forall n \in N$

or $S_n = -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5} .....$

i.e., $< S_n > = < -1, \frac{-1}{3}, \frac{-1}{5} .... 0, ... \frac{1}{4}, \frac{1}{2} >$

Thus, $\sup S_n = \frac{1}{2}$ and $\inf S_n = -1$

i.e., $< S_n >$ is bounded.

43. $S_n = (-1)^n \ \forall n \in N$

or $< S_n > = < -1, 1, -1, 1, ...>$

Thus, $\overline{S_n} = 1$ and $\underline{S_n} = -1 \ \forall n \in N$

Hence, $\overline{\lim} S_n = 1$ and $\underline{\lim} S_n = -1$

48. Let $S_n = -n \ \forall n \in N$

i.e. $< S_n > = < -1, -2, -3, -4, ...>$

So it is bounded above by -1 but not bounded below.

$$\overline{S_n} = \sup \{-n, -n-1, ...\} = -n$$

Since, $\overline{S_n} \to -\infty$ as $n \to \infty$. Hence, $\overline{\lim} S_n = -\infty$.

Also, $< S_n >$ is not bounded below so by definition $\underline{\lim} S_n = -\infty$.

Thus, $\overline{\lim} S_n = \underline{\lim} S_n = -\infty$

49. $S_1 = \sqrt{2}, S_{n+1} = \sqrt{2S_n} \ \forall n \in N$

Let $\lim S_n = l$

then, $l = \sqrt{2l} \Rightarrow l^2 = 2l$

$l(l-2) = 0 \Rightarrow l = 2$ since, $l \neq 0$

So, $\lim S_n = 2$

53. Let $n^{1/n} = 1 + h_n$ for $h_n \geq 0$

$\therefore \quad n = (1 + h_n)^n$

$$= 1 + nh_n + \frac{n(n-1)}{\lfloor 2} h_n^2 + ... + h_n^n$$

$$> \frac{n(n-1)}{2} h_n^2 \ \forall n \qquad \because h_n \geq 0$$

$\therefore \quad h_n^2 < \frac{2}{n-1}$ for $n \geq 2$

i.e., $|h_n| < \sqrt{\frac{2}{n-1}}$ for $n \geq 2$

Let $\in > 0$ then, $|h_n| < \sqrt{\left(\frac{2}{n-1}\right)} < \in$ provided

$$\frac{2}{n-1} < \in^2 \text{ or } n > \frac{2}{\in^2} + 1$$

If we choose $m \in N$ such that $m > \frac{2}{\in^2} + 1$ then,

$$|h_n| < \in \ \forall n \geq m$$

i.e., $|n^{1/n} - 1| < \in \ \forall n \geq m$

$\therefore \quad \lim n^{1/n} = 1$

55. Given, $S_n = \frac{2n}{2n+3}$ so, $\lim S_n = \lim \frac{2}{2 + \frac{3}{n}} = 1$ so convergent.

If $S_n = -3^n$ then, $\lim S_n = -\infty$ so divergent

If $S_n = -x^n, x > 1$ then, $\lim S_n = -\infty$ so divergent

If $S_n = x^n, x > 1 \lim S_n = \infty$ so divergent

57. $S_1 = 1, S_{n+1} = \frac{4 + 3S_n}{3 + 2S_n} \ \forall n \in N$

Let $\lim S_n = l$

then, $l = \frac{4 + 3l}{3 + 2l} \Rightarrow 3l + 2l^2 = 4 + 3l$

or $2(l^2 - 2) = 0 \Rightarrow l = \sqrt{2}$

Since, l cannot be equal to $-\sqrt{2}$ so $\lim S_n = \sqrt{2}$.

58. Let $S_n = (-1)^n \ \forall n = N$ then, it is oscillatory sequence and is bounded.

Let $S_n = (-1)^n \ \forall n \in N$ then, it is oscillatory sequences but unbounded.

59. Let $S_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$

If $|S_n - S_m| < \in \ \forall n \geq m$ then $< S_n >$ is Cauchy sequence

Choose $n = 2m$ then, $n > m$ and

$$|S_n - S_m| = |S_{2n} - S_m|$$

$$= \frac{1}{m+1} + \frac{1}{m+2} + ... \frac{1}{2m}$$

$$> \frac{1}{2m} + \frac{1}{2m} + \frac{1}{2m} + \text{.... upto } m \text{ terms}$$

$$= m \cdot \frac{1}{2m} = \frac{1}{2}$$

Thus if we take $\in = \frac{1}{2}$ then, whatever positive integer m we take, we have

$$n = 2m > m$$

and $|S_n - S_m| = |S_{2m} - S_m| > \frac{1}{2}$

i.e. $|S_n - S_m| > \in$

Thus for $\in = \frac{1}{2} > 0$, there exists no positive integer m such that $|S_n - S_m| < \in \forall n \geq m$

i.e. $< S_n >$ is not a Cauchy sequence. So $< S_n >$ is not convergent *i.e.* it is divergent.

61. Given that $\lim \left\{\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3 \ldots\left(\frac{n+1}{n}\right)^n\right\}^{1/n}$

Let $S_n = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$

then, $\lim S_n = e$

Here, $S_n > 0 \ \forall n \in N$

we know that $\lim (S_1, S_2 \ldots S_n)^{1/n} = e$

since,

$$S_1 = \frac{2}{1}, S_2 = \left(\frac{3}{2}\right)^2, S_3 = \left(\frac{4}{3}\right)^3, \ldots S_n = \left(\frac{n+1}{n}\right)^n$$

$$\therefore \quad \lim \left\{\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3 \ldots\left(\frac{n+1}{n}\right)^n\right\}^{1/n} = e$$

63. Let $S_n = n^2$

If $m > n$ then, $S_n - S_m = n^2 - m^2 = (n-m)(n+m)$ $> 2m > 1$ for any value of m. Taking $\in = 1$, we can not find a positive integer m such that $|n^2 - m^2| < \in$ for all $n \geq m$. Thus $< n^2 >$ is a Cauhy sequence.

65. Let $S_n = (-1)^n\left(1+\frac{1}{n}\right)$

$$< S_n > = < -2, \frac{3}{2} >, -\frac{4}{3}, \frac{5}{4}, -\frac{6}{5}, \frac{7}{6}, \ldots >$$

$$\overline{S_1} = \frac{3}{2}, \overline{S_2} = \frac{3}{2}, \overline{S_3} = \frac{5}{4}, \overline{S_4} = \frac{5}{4}, \overline{S_5} = \frac{7}{6} \ldots$$

$$\therefore \quad \overline{\lim} S_n = \inf. \left\{\frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \ldots\right\} = 1$$

66. $S_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+n}$

$$S_{n+1} - S_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2n+1} - \frac{1}{2n+2} > 0 \ \forall n \in N$$

Hence, the sequence $< s_n >$ is monotonically increasing.

Now, $|s_n| = s_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots \frac{1}{n+n}$

$$< \frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n} \quad \text{(upto } n \text{ terms)}$$

$$= n \cdot \frac{1}{n} = 1$$

$$\therefore \quad |s_n| < 1 \ \forall n \in N$$

Hence, the sequence $< s_n >$ is bounded.

Since, $< s_n >$ is a bounded, monotonically increasing sequence, hence it is converges.

70. Let $S_n = \frac{n^n}{\lfloor n}$

then, $S_{n+1} = \frac{(n+1)^{n+1}}{\lfloor n+1}$

$$\frac{S_{n+1}}{S_n} = \frac{(n+1)^{n+1}}{n+1} \cdot \frac{1}{n^n}$$

$$= \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

Also, $S_n > 0 \ \forall n \in N$

Hence, $\lim (S_n)^{1/n} = \lim \frac{S_{n+1}}{S_n} = \lim \left(1+\frac{1}{n}\right)^n = e$

$$\therefore \quad \lim \frac{n}{(\lfloor n)^{1/n}} = e$$

74. Let $S_n = n \ \forall n \in N$

It is bounded below but not bounded above.

$$\underline{S_n} = \inf \{n, n+1, n+2, \ldots\} = n$$

Since, $S_n \to \infty$ as $n \to \infty$, hence $\underline{\lim} S_n = \infty$.

Also, $< S_n >$ is not bounded above so $\overline{\lim} S_n = \infty$.

75. Since $\lim\left(1+\frac{1}{n}\right)^n = e$

so it is convergent sequence.

78. Given, $S_n = \frac{1}{2}(S_{n-1} + S_{n-2})$, put $n = 3, 4, 5, ...k$

$$S_3 = \frac{1}{2}(S_2 + S_1)$$

$$S_n = \frac{1}{2}(S_3 + S_2)$$

$$\vdots \qquad \vdots$$

$$S_{k-1} = \frac{1}{2}(S_{k-2} + S_{k-3})$$

$$S_k = \frac{1}{2}(S_{k-1} + S_{k-2})$$

Adding these, $S_k + \frac{S_{k-1}}{2} = \frac{1}{2}(S_1 + 2S_2)$

Let this $k \to \infty$ we get, (if $\lim S_n = l$)

$$\frac{3}{2}l = \frac{1}{2}(S_1 + 2S_2)$$

$$\Rightarrow \quad l = \frac{1}{3}(S_1 + 2S_2)$$

81. Let $S_n = \frac{(n+1)(n+2)....(n+n)}{n^n}$

$$\frac{S_{n+1}}{S_n} = \frac{2(2n+1)}{(n+1)} \cdot \left(\frac{n}{n+1}\right)^n$$

So, $\lim \frac{S_{n+1}}{S_n} = \lim \left[\frac{2(2n+1)}{n+1} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n}\right]$

$$= 4 \cdot \frac{1}{e} = \frac{4}{e}$$

So, $\lim S_n^{1/n} = \lim \frac{S_{n+1}}{S_n} = \frac{4}{e}$

83. $S_n = \frac{(3n-1)(n^4-n)}{(n^2+2)(n^3+1)}$

$$\lim S_n = \lim \frac{\left(3-\frac{1}{n}\right)\left(1-\frac{1}{n^3}\right)}{\left(1+\frac{2}{n^2}\right)\left(1+\frac{1}{n^3}\right)} = 3$$

84. $S_n = \sqrt{3\sqrt{3\sqrt{3.....}}}$

If $\lim S_n = l$ then,

$$l = \sqrt{3l} \quad \Rightarrow \quad l^2 = 3l$$

$$l(l-3) = 0 \quad \Rightarrow \quad l = 3$$

$\because \quad l = 0$ is not possible.

86. Let $S_n = n^{1/n}$ and we know $\lim n^{1/n} = 1$

So by Cauchy's first theorem on limits

$$\lim \frac{1}{n}(S_1 + S_2 + ... + S_n) = 1$$

or $\lim \frac{1}{n}(1 + 2^{1/2} + 3^{1/3} + ... + n^{1/n}) = 1$

87. $\lim\left(1+\frac{2}{n}\right)^{n+3}$

$$= \lim\left(1+\frac{2}{n}\right)^n . \left(1+\frac{2}{n}\right)^2$$

$$= e.1 = e$$

91. $S_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + ... \frac{1}{\sqrt{n^2+n}}$

$$\forall n > 1, S_n > \frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+n}} + ... \frac{n}{\sqrt{n^2+n}}$$

or $S_n > \frac{n}{\sqrt{n^2+n}} = \frac{1}{\sqrt{1+\frac{1}{n}}}$

and $S_n < \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + ... + \frac{n}{\sqrt{n^2+1}}$

$$= \frac{n}{\sqrt{n^2+1}} \text{ or } S_n < \frac{1}{\sqrt{1+\frac{1}{n^2}}}$$

Thus, $\frac{1}{\sqrt{1+\frac{1}{n}}} < S_n < \frac{1}{\sqrt{1+\frac{1}{n^2}}}, \forall n > 1$

But, $\lim \frac{1}{\sqrt{1+\frac{1}{n}}} = 1$

and $\lim \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1$

So by sandwitch theorem $\lim S_n = 1$

92. $S_n = \frac{\lfloor 3n}{(\lfloor n)^3}, S_n > 0 \ \forall n \in N$

If $S_n > 0 \ \forall n \in N$ then,

$$\lim (S_n)^{1/n} = \lim \frac{S_{n+1}}{S_n}$$

provided the letter limit exists.

Now, $\frac{S_{n+1}}{S_n} = \frac{\lfloor 3n+3}{(\lfloor n+1)^3} \cdot \frac{(\lfloor n)^3}{\lfloor 3n}$

$$= \frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3}$$

or $\lim \frac{S_{n+1}}{S_n} = \lim \frac{\left(3+\frac{3}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^3}$

or $\lim \frac{S_{n+1}}{S_n} = 27$

So $\lim (S_n)^{1/n} = \lim \frac{S_{n+1}}{S_n} = 27$

93. Let $S_n = \frac{\lfloor n}{n^n}$, then, $S_n > 0 \ \forall n \in N$

Also, $\lim (S_n)^{1/n} = \lim \frac{S_{n+1}}{S_n}$ provided the limit exists

Now, $\frac{S_{n+1}}{S_n} = \frac{\lfloor n+1}{(n+1)^{n+1}} \cdot \frac{n^n}{\lfloor n} = \frac{n^n}{(n+1)^n}$

or $\lim \frac{S_{n+1}}{S_n} = \lim \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$

or $\lim (S_n)^{1/n} = \lim \frac{S_{n+1}}{S_n} = \frac{1}{e}$

94. $S_1 = \frac{3}{2}, S_{n+1} = 2 - \frac{1}{S_n} \ \forall n \geq 1$

Let, $\lim S_n = l$

so, $l = 2 - \frac{1}{l} \Rightarrow l^2 = 2l - 1$

or, $l^2 - 2l + 1 = 0 \Rightarrow (l-1)^2 = 0 \Rightarrow l = 1$

so, $\lim S_n = 1$

97. Given, $S_1 = a > 0$, $S_{n+1} = \sqrt{\frac{ab^2 + Sn^2}{a+1}}, b > a, n \geq 1$

Let, $\lim S_n = l$ then,

$$l = \sqrt{\frac{ab^2 + l^2}{a+1}}$$

$\Rightarrow$ $l^2 = \frac{ab^2 + l^2}{a+1}$

$$l^2 a + l^2 = ab^2 + l^2$$

$\Rightarrow$ $al^2 = ab^2 \Rightarrow l^2 = b^2$

or $l = \pm b$

$\because$ $l \pm -b$

Since, $S_n \geq a > 0 \ \forall n \in N$

So, $l = \lim S_n = b$

99. Let, $S_n = \sin n\pi$

then, $< S_n > = < 0, 0, 0, \ldots >$

So $\lim S_n = 0$ *i.e.* $< S_n >$ converges to 0.

100. Let $S_1 = \sin n\pi$ then, $\lim S_1 = 0$ *i.e.* convergent

Let $S_2 = \frac{(-1)^n}{n}$ then, $\lim S_2 = 0$ *i.e.* convergent

Let $S_3 = \frac{1}{3^n}$ then, $\lim S_3 = 0$ *i.e.* convergent

Let $S_4 = \sin \frac{n\pi}{2}$ then, $< S_n > = < 1, 0, 1, 0 \ldots >$

This is an oscillatory sequence *i.e.* not a convergent sequence.

OOO

CHAPTER

5

Uniform Convergence of Sequences and Series of Functions

UNIFORMLY BOUNDED SEQUENCE

A sequence $< f_n >$ is said to be uniformly bounded on an interval I if

$$|f_n(x)| < M \text{ for every } x \in I$$

and for every positive integer n.

Examples :

1. The sequence $< f_n >$, where $f_n = \sin nx$ or $\cos nx$ is uniformly bounded on R since

   $|\sin nx|$ or $|\cos x| \leq 1 \ \forall x \in R$ and $\forall n \in N$.

2. The sequence $< f_n >$ where $f_n = \dfrac{1}{nx}$ is bounded in (0, 1] but not uniformly bounded because there does not exist any positive real number M such that

   $$\left(\frac{1}{nx}\right) \leq m \ \forall x \in (0, 1] \text{ and } \forall n \in N.$$

   However $< f_n >$ is uniformly bounded in $[a,1]$ where $a > 0$.

POINTWISE CONVERGENCE OF A SEQUENCES OF FUNCTIONS

A sequence $< f_n >$ of real valued functions defined over an internal I is said to be pointwise convergent if for each $x \in I$, the sequence $< f_n(x) >$ of real numbers is convergent.

Thus if $< f_n >$ converges pointwise on I, then define a function $f : I \to R$ by

$$f(x) = \lim_{n \to \infty} f_n(x), \ \forall x \in I$$

Here f is called the limit function of $< f_n >$.

Sum function of a series :

Let $u_1(x) + u_2(x) + \ldots + u_n(x)$ ...(1)

be the series of real valued functions defined on the interval I. Define,

$$f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$$

then series (1) is convergent is the sequence $< f_n(x) >$ is convergent and the limit function f of the sequence is called the sum function of the series.

UNIFORM CONVERGENCE OF SEQUENCES

Let $< f_n >$ be a sequence of functions defined on an interval I. The sequence $< f_n >$ is said to converge uniformly to the function f on I if for every $\varepsilon > 0$, there can be found a positive integer m such that

$$|f_n(x) - f(x)| < \in$$

for all $n \geq m$ and for all $x \in I$.

The function f is called uniform limit of the sequence $< f_n >$ on I.

Results :

1. The sequence $< f_n >$ does not converge uniformly to f on an interval I iff there exists $\varepsilon > 0$ such that there is no positive integer m for which

   $$|f_n(x) - f(x)| < \in \ \forall n \geq m \text{ and } x \in I$$

2. Uniform convergence of a sequence $< f_n >$ on I implier pointwise convergence of the sequence $< f_n >$ at every point of I but pointwise convergence does not necessarily ensure its uniform convergence on I.

3. The point of non-uniform convergence of the sequence is a point, such that the sequence does not converge uniformly in any nbd of it, how ever small.

4. Cauchy's general principle of uniform convergence. Let $< f_n >$ be a sequence of real-valued function defined on an interval I. Then $< f_n >$ converges uniformly on I iff for every $\varepsilon > 0$, there exists a positive integer in such that

   $|f_n(x) - f_p(x)| < \varepsilon$ for all $n, p \geq m$ and $\forall x \in I$.

5. A sequence $< f_n >$ is uniformly convergent on I if given $\varepsilon > 0$ there exists a positive integer in such that
$$|f_{n+p}(x) - f_n(x)| < \varepsilon \ \forall n \geq m$$
$\forall x \varepsilon I$ and $\forall p \in N$.

6. A series $\Sigma u_n(x)$ will converge uniformly on I iff for every $\varepsilon > 0$, there exists a positive integer m such that
$$|u_{n+1}(x) + u_{n+2}(x) + \ldots + u_{n+p}(x)| < \varepsilon$$
$\forall n \geq m$, for all $x \in I$ and for all $p = 1, 2, 3 \ldots$.

TEST FOR UNIFORM CONVERGENCE

1. **Uniform convergence of a series of functions**

Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of functions defined on the interval I and let $f_n(x) = u_1(x) + u_2(x) + \ldots u_n(x) \ \forall n \in N$. Then the series Σu_n is said to be converge uniformly on I if the sequence $< f_n >$ converges uniformly on I.

2. **M_n-Test**

Let $< f_n >$ be a sequence of functions defined on an interval I.

Let $\lim_{n \to \infty} f_n(x) = f(x) \ \forall x \in I$

Set $M_n = \sup \{|f_n(x) - f(x)| : x \in I\}$

Then $< f_n >$ converges uniformly to f iff $M_n \to 0$ as $n \to \infty$.

3. **Weierstrass's M-test**

A series $\sum_{n=1}^{\infty} u_n(x)$ of functions will converge uniformly on I if there exists a convergent series $\sum_{n=1}^{\infty} M_n$ of positive constants such that $|u_n(x)| \leq M_n$ for all n and for all $x \in I$.

4. **Abel's Test**

The series $\sum u_n(x).v_n(x)$ will converge uniformly on $[a, b]$ if

(i) $\sum u_n(x)$ is uniformly convergent on $[a, b]$.

(ii) The sequence $< v_n(x) >$ is monotonic for every x in $[a, b]$.

(iii) $< v_n(x_1) >$ is uniformly bounded on $[a, b]$ *i.e.* there is a positive number k, independent of x and n, such that $|v_n(x)| < k$ for every value of x in $[a, b]$ and every positive integer n.

5. **Dirichlet's Test**

The series $\sum u_n(x).v_n(x)$ will be uniformly convergent on $[a, b]$ if

(i) $< v_n(x) >$ is a positive monotonic decreasing sequence converging uniformly to zero on $[a, b]$.

(ii) $|f_n(x)| = \left|\sum_{r=1}^{n} u_r(x)\right| < k$

6. **Interchange of hints**

Let $< f_n >$ be a sequence of real valued functions defined on $I = [a, b]$ and let $< f_n >$ converges uniformly on I. Let $x_0 \in I$ such that
$$\lim_{x \to x} f_n(x) = a_n, \ n = 1, 2, 3 \ldots.$$
Then the sequence $< a_n >$ of real constants converges and
$$\lim_{x \to x_0} f(x) = \lim_{n \to \infty} a_n$$
i.e. $\lim_{x \to c} \{\lim_{n \to \infty} f_n(x)\} = \lim_{n \to \infty} \{\lim_{x \to x_0} f_n(x)\}$

UNIFORM CONVERGENCE AND CONTINUITY

Result :

1. Let $< f_n >$ be a sequence of real-valued functions on $[a, b]$ which converges uniformly to the function of f on $[a, b]$. If each f_n $(n = 1, 2, 3 \ldots)$ is continuous on $[a, b]$, then f is also continuous on $[a, b]$.

2. Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of real valued continuous functions defined on $[a, b]$, if the series converges uniformly to the function f on $[a, b]$, then f is continuous on $[a, b]$. Thus the sum function of a uniformly convergent series of continuous functions is itself continuous.

3. If the sum function of a series, whose terms are continuous functions on an interval I, is a discontinuous function, then the series cannot be uniformly convergent on I.

UNIFORM CONVERGENCE AND INTEGRATION

Results :

1. Let $< f_n >$ be a sequence of real-valued functions defined on the closed and bounded interval $[a, b]$ and let $f_n \in R[a, b]$, for $n = 1, 2, 3, \ldots$. If $< f_n >$ converges uniformly to the function f on $[a, b]$, then $f \in R[a, b]$ and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx$$

2. Let $< f_n >$ be a sequence of real-valued continuous functions defined on $[a, b]$ such that $f_n \to f$ uniformly on $[a, b]$. Then $f \in R[a, b]$ and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

3. **Term by term integration**

Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of real-valued functions defined on $[a, b]$ such that $u_n(x) \leftarrow R[a, b]$, for $n = 1, 2, 3, \ldots$. If the series converges uniformly to f on $[a, b]$, then $f \in R[a, b]$ and

$$\int_a^b \left[\sum_{n=1}^{\infty} u_n(x)\right] dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx$$

UNIFORM CONVERGENCE AND DIFFERENTIATION

Results :

1. Term by term differentiation in a sequence of functions :

Let $< f_n >$ be a sequence of real-valued functions defined on an interval $[a, b]$ such that

(i) Each f_n is differentiable on $[a, b]$

(ii) $< f_n(x) >$ converges for some point $x_0 \in [a, b]$

(iii) $< f_n' >$ converges uniformly on $[a, b]$. Then the sequence $< f_n >$ converges uniformly to a differentiable function f (called limit function) and

$$\lim_{n\to\infty} f_n'(x) = f'(x) \ \forall\, x \in [a, b]$$

2. Term by differentiation in a series of functions :

Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of real-valued functions defined on an interval $[a, b]$ such that

(i) Each $u_n(k)$ is differentiable on $[a, b]$

(ii) $\sum_{n=1}^{\infty} u_n(x)$ converges for some point $x_0 \in [a, b]$

(iii) $\sum_{n=1}^{\infty} u_n'(x)$ converges uniformly on $[a, b]$

Then the series $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on $[a, b)$ to a differentiable sum function $s(x)$ and

$$s'(x) = \lim_{n\to\infty} \sum_{r=1}^{n} u_r'(x), \ \forall\, x \in [a, b]$$

In other words if $n \in [a, b]$ then

$$\frac{d}{dx}\left\{\sum_{n=1}^{\infty} u_n(x)\right\} = \sum_{n=1}^{\infty}\left[\frac{d}{dx} u_n(x)\right]$$

3. Let $< f_n >$ be a sequence of real-valued functions defined on $[a, b]$ such that

(i) f_n is differentiable on $[a, b]$ for $n = 1, 2, 3, \ldots$

(ii) $< f_n >$ converges to f on $[a, b]$

(iii) $< f_n' >$ converges uniformly to g on $[a, b]$

(iv) Each f_n' is continuous on $[a, b]$

Then $\quad g(x) = f'(x)\ (a \le x \le b)$

i.e. $\quad \lim_{n\to\infty} f_n'(x) = f'(x),\ (a \le x \le b)$

4. Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of function defined on $[a, b]$ such that

(i) $u_n(x)$ is differentiable on $[a, b]$ for $n = 1, 2, 3, \ldots$

(ii) The series $\sum_{n=1}^{\infty} u_n(x)$ converges to f on $[a, b]$

(iii) The series $\sum_{n=1}^{\infty} u_n'(x)$ converges uniformly to g on $[a, b]$

(iv) Each u_n' is continuous on $[a, b]$

Then $f'(x) = g(x)$ $(a \le x \le b)$

i.e., $f'(x) = \sum_{n=1}^{\infty} u_n'(x)$

EXERCISE

MULTIPLE CHOICE QUESTIONS

Direction : Each of the following questions has four alternative answers. One of them is correct. Choose the correct answer.

1. The sequence $< f_n >$ where $f_n(x) = \frac{1}{nx}$ in (0, 1] is :
   a. Bounded b. Uniformly bounded
   c. Unbounded d. None of these

2. The sequence $< x^n >$ over (0,1) converges pointwise to :
   a. 0 b. 1
   c. Both 0 and 1 d. (0,1)

3. If the sequence $< f_n >$ defined over an interval is uniformly convergent then it is :
   a. Pointwise convergent
   b. Not pointwise convergent
   c. May or may not be pointwise convergent
   d. None of these

4. If $f_n(x) = x^n \ \forall x \in [0, 1]$ then the limit function f is :
   a. 0 for all $x \in [0, 1]$ b. 1 for all $x \in [0, 1]$
   c. 0 for all $x \in [0, 1[$ and 1 for $x = 1$
   d. None of these

5. A sequence $< f_n >$ is said to be uniformly bounded on an interval I if for every $x \in I$ and for every positive integer n :
   a. $|f_n(x)| \ge M$ b. $|f_n(x)| \ge M$
   c. $|f_n(x)| \le M$ d. None of these

6. The sequence of functions $< f_n >$ defined by
   $$f_n(x) = \frac{1 - x^n}{1 - x} \ \forall x \in (-1, 1) \text{ is :}$$
   a. Pointwise convergent only
   b. Uniformly convergent only
   c. Both pointwise and uniformly convergent
   d. None of these

7. The sequence $< \sin nx >$ over the interval R is :
   a. Unbounded
   b. Uniformly bounded
   c. Not uniformly bounded
   d. None of these

8. The sequence $< f_n >$ of functions defined by
   $$f_n(x) = x^n \ \forall x \in \{0, 1\} \text{ is :}$$
   a. Uniformly convergent only
   b. Pointwise convergent only
   c. Uniformly convergent and pointwise convergent both
   d. None of these

9. M_n-test is applied over the sequence of functions $< f_n >$ to check its :
   a. Uniform continuity
   b. Uniform convergence
   c. Pointwise convergence
   d. Pointwise continuity

10. The sequence $< f_n >$ where $f_n(x) = \frac{nx}{1 + n^2 x^2}$ is :
    a. Uniformly convergent on R
    b. Uniformly convergent on (0,1)
    c. Uniformly convergent on [0,1]
    d. None of these

11. Let $< f_n >$ be a sequence of real-valued functions on I which converges uniformly to f on I. If each f_n is continuous on I, then the limit function f is :
    a. Continuous on I
    b. Discontinuous on I

c. Neither continuous nor discontinuous on I
d. None of these

12. The series $\{u_n(x)n^{-x}\}$ is uniformly convergent on $[0,1]$ if $\sum u_n(x)$ is :
a. Converges on [0,1]
b. Continuous on [0,1]
c. Uniformly converges on [0,1]
d. None of these

13. The sequence $< f_n >$ defined over I is said to be pointwise convergence to f if for $\varepsilon > 0$ and $x \in I$, there exists $n_0 \in N$ such that

$$n \geq n_0 \Rightarrow |f_n(x) - f(x)| < \varepsilon$$

then no depends on :
a. x only
b. ε only
c. x and ε both
d. None of these

14. If the sequence $< f_n >$ is pointwise convergent in I then it is :
a. Uniformly convergent
b. Not uniformly convergent
c. May or may not be uniformly convergent
d. None of these

15. The point of non-uniform convergence for the sequence $f_n(x) = x^n \ \forall x \in \{0, 1\}$ is :
a. 0
b. 1
c. All the points of [0,1]
d. None of these

16. The series $\sum u_n(x)v_n(x)$ will be uniformly convergent on $[a,b]$ if $< v_n(x) >$ is positive monotonic decreasing sequence converging uniformly to zero on $[a,b]$ and $|f_n(x)| = \left|\sum_{r=1}^{n} u_r(x)\right| < k$ for every value of x in $[a,b]$ and for all integral value of n, where k is fixed number independent of x. This is called :
a. Abel's Test
b. Dirichlet's Test
c. Weierstrass's M-Test
d. M_n-Test

17. The sequence $< f_n(x) >$ on R where

$$f_n(x) = 1 - (1 - x^2)^n \text{ is :}$$

a. Uniformly convergent
b. Not uniformly convergent
c. Converges pointwise to a continuous function
d. None of these

18. The number of positive integer n_0 for a given $\varepsilon > 0$ such that $|x^4 - 0| < \varepsilon$ for $n \geq n_0$ and $\forall x \ \varepsilon \ (0, 1)$ is :
a. 0
b. 1
c. Both 0 and 1
d. None of these

19. A series $\sum_{n=1}^{\infty} u_n(x)$ of functions will converge uniformly on I if there exists a convergent series $\sum_{n=1}^{\infty} M_n$ of positive constants such that $|u_n(x)| \leq M_n$, for all n and for all $x \in I$. It is called :
a. M_n-Test
b. Abel's Test
c. Dirichlet's Test
d. Weiestrass's M-Test

20. The sum of n terms of a series

$$f_n(x) = \frac{n^2 x}{1 + n^4 x^2} \text{ over } [0,1] \text{ is :}$$

a. Converges uniformly
b. Converges non-uniformly
c. Not uniformly bounded
d. None of these

21. The series $\sum_{n=0}^{\infty} x e^{-nx}$ in the closed interval [0,1] is :
a. Uniformly convergent
b. Pointwise convergent
c. Not uniformly convergent
d. None of these

22. If M_n is supremum of $u_n(x)$ where the series

$$\sum u_n(x) = \sum \frac{x}{(n + x^2)^2}$$

then M_n is equal to :
a. $\frac{3}{16n^{3/2}}$
b. $\frac{\sqrt{3}}{16n^{3/2}}$
c. $\frac{3\sqrt{3}}{16n^{3/2}}$
d. $\frac{3}{16n^{1/2}}$

23. The series $\sum \frac{x}{n(1+nx^2)}$ is :
a. Uniformly convergent
b. Not uniformly convergent
c. Divergent
d. Not pointwise convergent

24. A real-valued function f_n over an interval I is uniformly convergent on I iff for every $\varepsilon > 0$, there exists a positive integer in such that
$|f_n(x) - f_p(x)| < \varepsilon$ for all $n, p \geq m$ and $\forall x \in I$ then it is called :
a. M_n-test
b. Cauchy general principle of uniform convergence
c. Abel's test
d. Dirichlet's test

25. The series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ is :
a. Divergent on R
b. Uniformly convergent on R
c. Not uniformly convergent on R
d. None of these

26. The sequence $< f_n(x) >$ where $f_n(x) = x^n$ on [0,1] is convergent pointwise to a function which is :
a. Continuous
b. Discontinuous at $x = 0$
c. Discontinuous at $x = 1$
d. Continuous on [0,1]

27. The sum function of the series for which
$$f_n(x) = \frac{1}{1+nx} (0 \leq x \leq 1) \text{ is :}$$
a. Continuous on [0,1]
b. Discontinuous at $x = 1$
c. Discontinuous at $x = 0$
d. Discontinuous on [0,1]

28. The sum function of the series for which $f_n(x) = nx(1-x)^n$ $(0 \leq x \leq 1)$ is :
a. Continuous and uniformly convergent both
b. Continuous only
c. Uniformly convergent only
d. Neither continuous nor uniformly convergent

29. In M_n-test for uniform convergence of sequence of functions, M_n is defined by :
a. Supremum $|f_n(x) - f(x)|$
b. Supremum $|f_n(x) + f(x)|$
c. Infimum $|f_n(x) - f(x)|$
d. None of these

30. Let $< f_n >$ be a sequence of function defined on I such that
$$\lim_{n\to\infty} f_n(x) = f(x)\ \forall x \in I.$$
Define $M_n = \sup\{|f_n(x) - f(x)| : x \in I\}$
then for $n \to \infty < f_n >$ is uniformly convergent iff :
a. $M_n \to \infty$
b. $M_n \to 0$
c. M_n tents to any finite number
d. None of these

31. The point of non-uniformly convergence of $< f_n(x) >$ defined by $f_n(x) = \frac{nx}{1+n^2x^2}$ is :
a. 0
b. 1
c. (0,1)
d. None of these

32. A sequence $< f_n >$ is uniformly convergent on I, if given $\varepsilon > 0$ there exists a positive integer m such that $\forall n \geq m$, $\forall x \in I$ and $\forall p \in N$.
a. $|f_{n+p}(x) + f_n(x)| < \varepsilon$
b. $|f_{n+p}(x) - f_n(x)| < \varepsilon$
c. $|f_{n+p}(x) - f_n(x)| > \varepsilon$
d. $|f_{n+p}(x) + f_n(x)| > \varepsilon$

33. The sum function of a uniformly convergent series of continuous functions is :
a. Continuous
b. Uniformly continuous
c. Discontinuous
d. Not uniformly continuous

34. The series $\sum \frac{1}{n^3 + n^4x^2}$ is term by term :
a. Differentiable
b. Integrable
c. Both differentiable and integrable
d. None of these

35. The series $\sum \frac{(-1)^{n-1}}{n} x^n$ is uniformly convergent :

(Kanpur 2018)

a. $(-1, 1)$ b. $[-1, 1]$
c. $[0,1]$ d. $[-1,0]$

36. The series $\sum u_n(x)$ will converge uniformly on I, iff for every $\varepsilon > 0$, there exists a positive integer m such that ...... $\forall n \geq m$, $x \in I$ and for all $p = 1, 2, ...$.
a. $|u_{n+1}(x) + u_{n+2}(x) + ... + u_{n+p}(x)| < \varepsilon$
b. $|u_{n+1}(x) + u_{n+2}(x) + ... + u_{n+p}(x)| > \varepsilon$
c. $|u_{n+1}(x) + u_{n+2}(x) + ... + u_{n+p}(x)| = \varepsilon$
d. None of these

37. The point of non-uniformly convergence of the sequence $< f_n >$ where $f_n(x) = 1 - (1 - x^2)^n$ is :
a. 0 b. 1
c. –1 d. None of these

38. The series $\sum \frac{x^4}{\lfloor n}$ is converge uniformly on :
a. R b. $[\, 0, \infty \,[$
c. $]\, -\infty, 0\,]$
d. Every bounded subset of R

39. The sum function of the series for which

$$f_n(x) = nx(1-x)^n, (0 \leq x \leq 1) \text{ is :}$$

a. Continuous at $x = 0$ only
b. Continuous at $x = 1$ only
c. Continuous for all $x \in [0, 1]$
d. Discontinuous

40. In M_n-test the sequence of function $< f_n(x) >$ is uniformly convergent only when :
a. $\lim_{n\to\infty} M_n = \infty$ b. $\lim_{n\to\infty} M_n = 0$
c. $\lim_{n\to\infty} M_n = 1$ d. None of these

41. If $< f_n >$ be a sequence of real valued functions defined on closed and bounded interval $[a, b]$ such that $f_n \in R[a, b]$, then $f \in R[a, b]$ only when $< f_n >$ is :
a. Convergent
b. Uniformly continuous
c. Uniformly convergent
d. Continuous

42. The series $\sum u_n(x) v_n(x)$ converge uniformly on $[a, b]$ of :
(i) $\sum u_n(x)$ is uniformly convergent on $[a, b]$
(ii) $< v_n(x) >$ is monotonic for every x in $[a, b]$
(iii) $< v_n(x) >$ is uniformly bounded in $[a, b]$
This is called :
a. Weierstrass's M-Test
b. M_n-Test
c. Abel's Test d. Dirichlet's Test

43. The point of non-uniform convergence of the series $\sum_{n=0}^{\infty} xe^{-nx}$ in the closed interval [0,1] is :
a. 0 b. 1
c. –1 d. ∞

44. The series $\sum_{n=0}^{\infty} \frac{x^4}{(1+x^4)^n}$ on the interval [0,1] is :
a. Uniformly convergent
b. Divergent
c. Not uniformly convergent
d. None of these

45. Let $< f_n >$ be a sequence of real valued functions on $[a, b)$ which converges uniformly to the function f on $[a, b]$. If each $f_n (n = 1, 2, 3, ...)$ is continuous on $[a, b]$, then f is :
a. Uniformly continuous
b. Continuous
c. Uniformly convergent
d. None of these

46. $\int_0^1 \left(\sum_1^{\infty} \frac{x^4}{n^2} \right) dx$ is equal to :
a. $\sum_1^{\infty} \frac{n}{n+1}$ b. $\sum_1^{\infty} \frac{1}{n(n+1)}$
c. $\sum_1^{\infty} \frac{1}{n^2(n+1)}$ d. None of these

47. If $\sum u_n(x)$ converges to f and $\sum u_n'(x)$ converges uniformly to $g\,[a, b]$ such that $u_n(x)$ is differentiable and each u_n' is continuous on $[a, b]$ then ;
a. $f = g$ b. $f = g'$
c. $f' = g$ d. None of these

48. If the series of continuous functions defined on $[a,b)$ has discontinuous sum then the series is :
a. Uniformly convergent
b. Not uniformly convergent
c. May or may not be uniformly convergent
d. None of these

49. The sum function $s(x)$ of the series for which
$$f_a(x) = nx(1-x)^n, (0 \le x \le 1) \text{ is :}$$
a. 0 b. 1
c. 0 and 1 both d. None of these

50. The series of function which is continuous on I can be integrated term by term only when the series is :
a. Convergent
b. Uniformly convergent
c. Uniformly continuous
d. Continuous

51. The value of $\lim_{n\to 1} \sum_1^{\infty} \frac{nx^2}{n^3+x^3}$ is :
a. $\sum \frac{1}{n^3+1}$ b. $\sum \frac{n}{n^3+1}$
c. $\sum \frac{1}{n^2+1}$ d. $\sum \frac{n}{n^2+1}$

52. The sequence $< f_n(x) >$ which $f_n(x) = \frac{x^n}{n}, 0 \le x \le 1$ converges uniformly to : **[Kanpur 2018]**
a. 0 b. 1
c. [0,1] d. None of these

53. If $\sum_0^{\infty} a_n$ converges absolutely then $\sum_0^{\infty} a_n x^n$ converges uniformly on :
a. [0,1] b. [–1,1]
c. R d. None of these

54. The series $\sum \frac{x}{n(1+nx^2)}$ is uniformly convergent in :
a. [0,1] only b.] 0,1[only
c. [0, ∞. only d. R

55. The sum function of the series $\sum_{n=0}^{\infty} xe^{-nx}$ is :
a. Continuous at $x = 0$
b. Discontinuous at $x = 0$
c. Continuous over R
d. Discontinuous over R

56. If the sum function of a series on I is discontinuous function then the series is :
a. Uniformly convergent
b. Not uniformly convergent
c. May or may not be uniformly convergent
d. None of these

57. The condition of uniform convergence of the series for continuity of sum function is :
a. Necessary
b. Sufficient
c. Both necessary and sufficient
d. None of these

58. The series $\sum u_n(x)$ for which
$$f_n(x) = \frac{1}{2n^2}\log(1+n^4x^2) \text{ is :}$$
a. Term by term differentiable
b. Term by term integrable
c. Both term by term diff. and integrable
d. None of these

59. The series $\sum \frac{1}{1+n^2x}$ is : **[Kanpur 2018]**
a. Converges in [–1, 0]
b. Converges in [1, ∞)
c. Diverges in [1, ∞)
d. None of these

60. The series $\sum_1^{\infty} \frac{1}{n^2+x^2}, 0 \le x \le \infty$ is uniformly convergent on :
a. R b. [0, ∞)
c. (–∞, 0] d. None of these

61. The series for which
$$f_n(x) = \frac{nx}{1+n^2x^2}, 0 \le x \le 1 \text{ is :}$$
a. Integrable term by term at $x = 0$
b. Integrable term by term at $x = 1$
c. Differentiable term by term at $x = 0$
d. None of these

62. The sequence $< f_n(x) >$ where $f_n(x) = \dfrac{nx}{1+n^2x^2}$ is :

a. Uniformly convergation on [0, 1)

b. Uniformly convergent on R

c. Not uniformly convergent on R

d. None of these

63. If $\sum u_n$ is a convergent series of positive constant then it is :

a. Uniformly convergent

b. Not uniformly convergent

c. May or may not be uniformly convergent

d. None of these

64. If $< f_n >$ be a sequence of real valued continuous functions defined on $[a, b]$ such that $f_n \to f$ uniformly on $[a, b]$ then f is :

a. Riemann integrable

b. Not a riemann integrable

c. May or may not be a riemann integrable

d. None of these

65. $\dfrac{d}{dx}\left[\sum_1^{\infty}\dfrac{\sin nx}{n^3}\right]$ is equal to :

a. $\sum_1^{\infty}\dfrac{\cos nx}{n^3}$ b. $\sum_1^{\infty}\dfrac{\sin nx}{n^3}$

c. $\sum_1^{\infty}\dfrac{\cos nx}{n^2}$ d. $\sum_1^{\infty}\dfrac{1}{n^2}$

66. The series for which $f_n(x) = \dfrac{1}{1+nx}$, $0 \le x \le 1$ is :

a. Term by term differentiable

b. Uniformly convergent

c. Term by term differentiable and uniformly convergent both

d. None of these

67. The sequence $f_n(x) = \dfrac{x}{1+nx}$, $(0 \le x < \infty)$ is :

a. Uniformly convergent to 0

b. Not uniformly convergent to 0

c. Uniformly convergent to ∞

d. None of these

68. If the terms of the series are continuous on I such that its sum function is discontinuous then the series is :

a. Uniformly convergent on I

b. Not uniformly convergent on I

c. May or may not be uniformly convergent

d. None of these

69. The series $\sum_1^{\infty}\dfrac{x}{n(n+1)}$ on $(0, \infty)$ is :

a. Uniformly convergent

b. Non-uniformly convergent

c. May or may not be uniformly convergent

d. None of these

70. Let $\sum u_n(x)$ be a series of real valued functions over $[a, b]$ such that $u_n(x) \in R[a, b]$. If the series converges uniformly to f on $[a, b]$ then f is :

a. R-integrable only b. I-integrable only

c. Not R-integrable d. None of these

71. For the validity of term by term integration, the condition of uniform convergence of the series is :

a. Necessary

b. Sufficient

c. Both necessary and sufficient

d. None of these

72. If the sum of first n terms of a series is

$n^2x(1-x)^n$, $(0 \le x \le 1)$ then it is :

a. Term by term integration

b. Not term by term integration

c. Uniformly convergent

d. None of these

73. The series for which

$$f_n(x) = n^2xe^{-n^2x^2} - (n-1)^2xe^{-(n-1)^2x^2}$$

these sum function $f(x)$ is :

a. Continuous in [0,1]

b. Discontinuous in [0,1]

c. Continuous in R

d. None of these

74. The point of non-uniformly convergence of a series which $f_n(x) = n^2 x e^{-n^2 x^2}$ is :

a. 0 b. 1
c. –1 d. ∞

75. The series for which $f_n(x) = x^{1/(2n-1)}$, $x = 0$ is a :

a. Point of non-uniform convergence only
b. The sum function $f(x)$ is discontinuous
c. Both (a) and (b) are true
d. Neither (a) nor (b) is true

76. The series $\sum_{0}^{\infty} \frac{xn}{\lfloor n}$ on bounded subset of R is :

a. Uniformly convergent
b. Not uniformly convergent
c. Divergent
d. None of these

77. If the series $\sum u_n(x)$ is convergent such that $\sum u_n'(x)$ converges uniformly on I then the series is :

a. Continuous
b. Term by term differentiable
c. Term by term integrable
d. None of these

78. If $< f_n >$ is converge to f and uniformly converge to g such that f_n is differentiable and f_n' is continuous on $[a, b]$ then :

a. $g = f$ b. $g' = f$
c. $g = f'$ d. $g' = f'$

79. The integral in which term by term integration hold for a series for which $f_n(x) = nxe^{-nx^2}$ is :

a. $[0\ 1)$ b. $[k, 1]$ where $0 < k < 1$
c. $[0\ \infty)$ d. $[k, \infty]$ where $0 < k < 1$

80. The series $\sum \frac{(-1)^{n-1} x^n}{n}$ is uniformly convergent in :

[Kanpur 2018]

a. $(0, 1)$ b. $[0, 1]$
c. $[-1, 0]$ d. $(-1, 0)$

81. The property associated with the whole domain is :

a. Continuity b. Differentiability
c. Convergence d. Uniform convergence

82. The sequence $< f_n >$ where $f_n(x) = \frac{x}{1 + nx^2}$ converges uniformly on :

a. $[0,1]$ b. $[-1,1]$
c. 0,1. d. R

83. The sequence $< f_n >$ where $f_n(x) = \frac{n}{n + x}$ converges uniformly on :

a. $x \geq 0$ b. $x \leq 0$
c. $x \in [0, 1]$ d. R

84. The series for which $f_n(x) = nx(1 - x)^n$ over $[0,1]$ is :

a. Uniformly convergent
b. Integrated term by term
c. Both uniformly convergent and term by term integrated
d. None of these

85. The series $\sum x^{n-1}(1 - 2x^n)$ in $[0,1]$ is :

a. Term by term differentiable
b. Term by term integrable
c. Not term by term integrable
d. None of these

86. The series $\sum x^{n-1}(1 - x)\ \forall x \in [0, 1]$ is:

[Kanpur 2018]

a. Uniformly convergent
b. Not uniformly convergent
c. May or may not be uniform convergent
d. None of these

87. $\lim_{x \to 1} \sum_{n=1}^{\infty} \frac{nx^2}{n^3 + x^3}$ is equal to :

a. $\sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$ b. $\sum_{n=1}^{\infty} \frac{1}{n^3 + 1}$
c. $\sum_{n=1}^{\infty} \frac{1}{2n^3}$ d. None of these

88. The series $\sum_{n=1}^{\infty} \frac{1}{n^2 + x^2}$, $0 \leq x < \infty$ is uniformly convergent :

a. $]-\infty, \infty[$ b. $[0, \infty[$
c. 0 d. 1

89. The series $\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + ...$ converges uniformly on :
a. $-\infty < a \le x \le b < \infty$
b. $0 < a \le x \le b < \infty$
c. $0 < a \le x \le b < 2\pi$
d. $0 < a \le x \le b < 4\pi$

90. The sequence $< f_n >$ of functions, defined by $f_n(x) = x^n \forall x \in [0, 1)$ is :
a. Uniformly convergent
b. Not a pointwise convergent
c. Non-uniformly convergent
d. None of these

91. If $|\delta| < 1$, then $\sum \frac{x^n}{n+1}$ is uniformly convergent in :
a. $(-\delta, 0)$ b. $(0, \delta)$
c. $(-\delta, \delta)$ d. None of these

92. If $\sum a_n$ converges uniformly on [0,1] then the series $\sum a_n n^{-x}$ on [0,1] :
a. Not converges
b. Uniformly converges
c. Not uniformly convergent
d. None of these

93. If $\sum_{n=0}^{\infty} a_n$ converges absolutely then $\sum_{n=0}^{\infty} a_n x^n$ is :
a. Convergent uniformly in R
b. Converges uniforms in [0,1]
c. Non-uniform convergent
d. None of these

94. If $\sum a_n$ is absolutely convergent then $\sum \frac{a_n x^n}{1+x^{2n}}$ is :
a. Non-uniformly convergent
b. Uniformly convergent on R
c. Uniformly convergent on [0,1]
d. None of these

95. The series $\sum_{n=1}^{\infty} (-1)^{n-1} x^n$ is :
a. Converges uniformly in R
b. Converges uniformly in $0 \le x < 1$
c. Non-uniformly convergence
d. None of these

96. The sequence $< f_n >$ of functions defined by $f_n(x) = \frac{1}{x+n} \forall x \in [0, b], b > 0$ is: **[Kanpur 2018]**
a. Uniformly convergent
b. Not uniformly convergent
c. Not pointwise convergent
d. None of these

97. The series $\sum_{n=1}^{\infty} (1-x) x^n$ is : **[Kanpur 2018]**
a. Continuous at $x = 0 \in [0, 1)$
b. Discontinuous at $x = 0 \in [0, 1]$
c. Uniformly convergent on [0,1]
d. None of these

98. The sequence $< \delta_n >$ where $\delta_n = n_x(1-x)^n$ is uniformly convergent on : **[Meerut 2018]**
a. [0,1] b. [0,1[
c.]0,1] d.]0,1[

99. $f_n(x) = x^n (0 \le x \le 1)$ is uniformly convergent on : **[Meerut 2014]**
a. [0, 1] b. $[0, c], c > 1$
c. $[0, c], c < 1$ d. None of these

100. Sequence $< f_n >$ where $f_n(x) = \frac{n^2 x}{1 + n^3 x^2}$ does not converges uniformly in : **[Meerut 2014]**
a. (a, b) b. [0, 1]
c. (0, 1. d. 0, 1]

101. The series $\sum_{n=1}^{\infty} r^n \cos n\theta$ is uniformly convergent for all real value of θ and **[Meerut 2014]**
a. $r < 1$ only b. $r > 1$ only
c. $0 < r < 1$ only d. $0 < r$ only

102. Transcendental number is : **[Meerut 2014]**
a. Algebric b. Not algebric
c. Both (a) and (b) d. None of these

103. The point of non-uniform convergence of the sequence $< S_n(x) >$ where $S_n(x) = 1-(1-x^2)^n$ is : **[Meerut 2014]**
a. $x = \infty$ b. $x = 0$
c. $x = n$ d. None of these

104. The series $\Sigma \frac{x}{(n+x^2)^2}$ is : **(Meerut 2018)**

a. Convergent

b. Divergent

c. Uniform convergent

d. Unbounded

105. The sequence $< S_n >$, where $S_n = nx(1-x)^n$ is uniformly convergent on : **[Meerut 2018]**

a. [0, 1] b. [0, 1[

c.]0, 1] d.]0, 1[

106. Let $f : (0, \infty) \to R$ is uniformly continuous, then : **[Meerut 2019]**

a. $\lim_{x\to 0_+} f(x)$ and $\lim_{x\to\infty} f(x)$ exist

b. $\lim_{x\to 0_+} f(x)$ exist but $\lim_{x\to\infty} f(x)$ does not exist

c. $\lim_{x\to 0_+} f(x)$ does not exist, but $\lim_{x\to\infty} f(x)$ exist

d. None of these

107. Let $F_n(x) = xe^{-nx^2}$, where $n \geq 1$ and $x \in R$, the $< F_n(x) >$ is : **[Meerut 2019]**

a. Uniformly convergent on R

b. Uniformly convergent of subset of R

c. Bounded and not uniformly convergent on R

d. Unbounded functions

108. The transformation $\omega = \left(\frac{z+z^{-1}}{2}\right)$ is : **[Meerut 2019]**

a. Conformal everywhere

b. Not conformal

c. Conformal except at $z = \pm 1$

d. Conformal at $z = 1$

109. Consider $f(z) = \frac{1}{z}$ a mobius transformation, $z \in c$ and $z \neq 0$, then f map $(c \setminus \{0\})$ to a ....., where c is a circle with positive radius passing through the origin: **[Meerut 2019]**

a. Circle

b. Line

c. Line passing through (0, 0.

d. Line not passing through (0, 0)

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(a)	**2.**	(a)	**3.**	(c)	**4.**	(c)	**5.**	(c)	**6.**	(a)	**7.**	(b)	**8.**	(b)	**9.**	(b)	**10.**	(b)
11.	(a)	**12.**	(c)	**13.**	(b)	**14.**	(c)	**15.**	(b)	**16.**	(b)	**17.**	(b)	**18.**	(d)	**19.**	(d)	**20.**	(b)
21.	(c)	**22.**	(c)	**23.**	(a)	**24.**	(b)	**25.**	(b)	**26.**	(c)	**27.**	(c)	**28.**	(b)	**29.**	(a)	**30.**	(b)
31.	(a)	**32.**	(b)	**33.**	(a)	**34.**	(a)	**35.**	(c)	**36.**	(a)	**37.**	(a)	**38.**	(d)	**39.**	(c)	**40.**	(b)
41.	(c)	**42.**	(c)	**43.**	(a)	**44.**	(c)	**45.**	(b)	**46.**	(c)	**47.**	(c)	**48.**	(b)	**49.**	(a)	**50.**	(b)
51.	(b)	**52.**	(a)	**53.**	(a)	**54.**	(d)	**55.**	(b)	**56.**	(b)	**57.**	(b)	**58.**	(a)	**59.**	(b)	**60.**	(b)
61.	(d)	**62.**	(c)	**63.**	(a)	**64.**	(a)	**65.**	(c)	**66.**	(a)	**67.**	(a)	**68.**	(b)	**69.**	(b)	**70.**	(a)
71.	(b)	**72.**	(b)	**73.**	(a)	**74.**	(a)	**75.**	(c)	**76.**	(a)	**77.**	(b)	**78.**	(c)	**79.**	(b)	**80.**	(a)
81.	(d)	**82.**	(d)	**83.**	(a)	**84.**	(b)	**85.**	(c)	**86.**	(b)	**87.**	(a)	**88.**	(b)	**89.**	(c)	**90.**	(c)
91.	(c)	**92.**	(b)	**93.**	(b)	**94.**	(b)	**95.**	(b)	**96.**	(a)	**97.**	(c)	**98.**	(d)	**99.**	(c)	**100.**	(c)
101.	(c)	**102.**	(b)	**103.**	(b)	**104.**	(c)	**105.**	(a)	**106.**	(d)	**107.**	(a)	**108.**	(c)	**109.**	(d)		

HINTS AND SOLUTIONS

1. The sequence $< f_n >$ where $f_n(x) = \dfrac{1}{nx}$ is bounded in $(0,1]$ since for each $x \in (0, 1]$ the sequence $< \dfrac{1}{x}, \dfrac{1}{2x}, \dfrac{1}{3x} .... >$ is bounded below by 0 and bounded above by the first term $\dfrac{1}{x}$. This sequence is not uniformly bounded in $(0,1]$ since there does not exist any positive real number μ such that

$$\left|\frac{1}{nx}\right| \le \mu \ \forall x \in (0, 1] \text{ and } \forall n \in N$$

However it is uniformly bounded in $[a, 1]$ where $a > 0$.

2. $f_n(x) = x^n \forall x \in (0, 1)$

then $\quad f(x) = \lim_{n\to\infty} x^4 = 0$

Let $x \in (0, 1)$ and $\varepsilon > 0$ then

$$|f_n(x) - f(x)| = |x^n - 0| = x^n$$

so $\quad |f_n(x) - f(x)| < \varepsilon$

$$\Leftrightarrow \quad x^n < \varepsilon \quad \Leftrightarrow \quad \left(\frac{1}{x}\right)^n > \frac{1}{\varepsilon}$$

$$\Leftrightarrow \quad n > \frac{\log(1/\varepsilon)}{\log(1/x)}$$

If we choose positive integer n_0 such that

$$n_0 > \frac{\log(1/\varepsilon)}{\log(1/x)}$$

then $\quad |f_n(x) - f(x)| < \varepsilon \ \forall n \ge n$

So $< f_n >$ converges pointwise to 0 on $(0,1)$.

6. Given $\quad f_n(x) = \dfrac{1 - x^n}{1 - x} \ \forall x \ \varepsilon \ (-1, 1)$

then $\quad f(x) = \lim\limits_{n\to\infty} f_n(x) = \lim_{n\to\infty} \dfrac{1 - x^n}{1 - x}$

$$= \frac{1}{1 - x} \forall x \in (-1, 1)$$

Hence, $< f_n >$ is pointwise convergent.

7. $f_n(x) = \sin nx \ \forall x \in R$ and $n \in N$

Since, $|\sin nx| \le 1 \ \forall x \in R$ and $n \in N$ so $< f_n(x) >$ is uniformly bounded.

10. $f_n(x) = \dfrac{nx}{1 + n^2x^2} \ \forall x \in (0, 1)$

$$f(x) = \lim_{n\to\infty} f_n(x)$$

$$= \lim_{n\to\infty} \frac{nx}{1 + n^2x^2} = 0 \ \forall x \in R$$

$$|f_n(x) - f(x)| = \frac{n|x|}{1 + n^2x^2}$$

If $x = \dfrac{1}{n}$ then

$$|f_n(x) - f(x)| = \frac{n \cdot \frac{1}{n}}{1 + \frac{n^2}{n^2}} = \frac{1}{2}$$

Here $x = 0$ is the point of non-uniform convergent so $f_n(x)$ is uniformly convergent in (0,1) which contains no point of non-uniform convergent.

15. $f_n(x) = x^n \ \forall x \in [0, 1]$

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0, \text{ if } 0 \le x < 1 \\ 1, \text{ if } x = 1 \end{cases}$$

The sequence $< f_n(x) >$ converge for all $x \ \varepsilon \ [0, 1]$

Now $\quad |f_n(x) - f(x)| < \varepsilon$

$$\Rightarrow \quad |x^4| < \varepsilon \quad \Rightarrow \quad x^n < \varepsilon$$

$$\left(\frac{1}{x}\right)^n > \frac{1}{\varepsilon} \quad \Rightarrow \quad n\log\frac{1}{x} > \log\frac{1}{\varepsilon}$$

$$\Rightarrow \quad n > \frac{\log(1/\varepsilon)}{\log(1/x)}$$

Thus $m(x, \varepsilon)$ is an integer. That greater than

$$\frac{\log(1/\in)}{\log(1/x)} \text{ for } x \ne 1$$

At $n \to \infty$ as x starting from 0, increases and approaches 1 and hence it is not possible to find a positive integer m such that

$|f_n(x) - f(x)| < \varepsilon \ \forall n \geq m$ and $\forall x \in [0, 1[$

So $< f_n >$ is not uniformly convergent in $[0, 1[$ -1 is the point of m uniformly convergent.

17. $f_n(x) = 1 - (1 - x^2)^n$

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 \text{ when } x = 0 \\ 1 \text{ when } 0 < |x| < \sqrt{2} \end{cases}$$

$$M_n = \sup\{|f_n(x) - f(x)| : x \in]0, \sqrt{2}[\}$$

$$\sup\{(1 - x^2)^n : x \in]0, \sqrt{2}[\}$$

$$\geq \left(1 - \frac{1}{n}\right)^n = \frac{1}{e} \quad \text{or} \quad n \to \infty$$

Hence, M_n does not tend to zero as $n \to \infty$ so the sequence $< f_n(x) >$ is non-uniformly convergent and 0 is a point of non-uniform convergent.

20. $f_n(x) = \dfrac{n^2 x}{1 + n^4 x^4} \ \forall x \in [0, 1]$

$$f(x) = \lim_{n\to\infty} f_n(x)$$

$$= \lim_{n\to\infty} \frac{n^2 x}{1 + n^4 x^2} = 0 \ \forall x \in [0, 1]$$

$$|f_n(x) - f(x)| = \frac{n^2|x|}{1 + n^4 x^2}$$

$$M_n = \sup\{|f_n(x) - f(x)| : x \in R\}$$

$$= \sup\left\{\frac{n^2(x)}{1 + n^4 x^2} : x \in R\right\}$$

$$\geq \frac{n^2 - \frac{1}{n^2}}{1 + n^4 \cdot \frac{1}{n^4}} = \frac{1}{2}$$

Since, M_n cannot tend to zero as $n \to \infty$. Hence by M_n-test the sequence is non-uniformly convergent.

22. $u_n(x) = \dfrac{x}{(n + x^2)^2}$

For maxima or minima $\dfrac{du_n(x)}{dx} = 0$

$$(n + x^2)^2 - 4x^2(n + x^2) = 0$$

or $$3x^4 + 2nx^2 - n^2 = 0$$

$$x^2 = \frac{n}{3} \quad \text{or} \quad x = \sqrt{\frac{n}{3}}$$

and then $\dfrac{d^2 u_n}{dx^2}$ is negative when $x = \sqrt{\dfrac{n}{3}}$ so

$$M_n = \max|u_n(x)| = \frac{\sqrt{n/3}}{\left(n + \frac{n}{3}\right)^2} = \frac{3\sqrt{3}}{16 n^{3/2}}$$

25. $\because \quad \left|\dfrac{\cos nx}{n^2}\right| \leq \dfrac{1}{n^2} \ \forall x \in R$

and $\sum \dfrac{1}{n^2}$ is convergent so by Weierstrass's M-test the given series is uniformly convergent on R.

27. $f_n(x) = \dfrac{1}{1 + nx}$

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0, \text{ if } 0 < x \leq 1 \\ 1, \text{ if } x = 0 \end{cases}$$

so sum function f is discontinuous at $x = 0$.

28. $f_n(x) = nx(1 - x)^n, (0 \leq x \leq 1)$

If $0 < x < 1$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx(1 - x)^n$

$$f(x) = \lim_{n\to\infty} \frac{nx}{(1 - x)^{-n}}$$

$$= \lim_{n\to\infty} \frac{x}{-(1 - x)^{-n}\log(1 - x)}$$

$$f(x) = 0$$

Also $f_n(x) = 0$ if $x = 0$ or 1

Hence, $\quad f(x) = \lim_{n\to\infty} f_n(x) = 0 \ \forall x \in [0, 1]$

Thus the sum function $f(x)$ is continuous for all $x \in [0, 1]$.

34. Given that $\quad u_n(x) = \dfrac{1}{n^3 + n^4 x^2}$

$$u_n'(x) = -\frac{2x}{n^2(1 + nx^2)^2}$$

$u_n'(x)$ is maximum when $\dfrac{d}{dx}u_n'(x) = 0$

i.e. $\quad (1 + nx^2)^2 - 4nx^2(1 + nx^2)$

$$1 - 3nx^2 = 0 \quad \text{or} \quad x = \pm\frac{1}{\sqrt{3n}}$$

Also $\dfrac{d^2}{dx^2}u_n'(x)$ is negative at $x = -\dfrac{1}{\sqrt{3n}}$

so max. $|u_n'(x)| = \dfrac{2}{\sqrt{3}n^{5/2}\left(1+\frac{1}{3}\right)^2}$

$$= \frac{3\sqrt{3}}{8n^{5/2}}$$

so that $|u_n'(x)| < \dfrac{1}{n^{5/2}} \ \forall x$

but $\sum \dfrac{1}{n^{5/2}}$ is convergent so by Weierstrass's M-test

$\sum u_n'(x)$ convergent uniformly for all value of x. Hence $\sum u_n(x)$ can be differentiated term by term.

35. Given series is $\sum \dfrac{(-1)^{n-1}}{n} x^n \ \forall x \in [0, 1]$

Let $u_n(x) = \dfrac{(-1)^{n-1}}{n}$ and $v_n(x) = x^n$

Then $< v_n(x) >$ is uniformly bounded and monotonic decreasing on [0,1]. Also $\sum \dfrac{(-1)^{n-1}}{n}$ is convergent by Leibnitz test. Hence, by Abel's test the given series is uniformly convergent on [0,1].

38. Let $[-a, a]$ be a bounded subset of R then

$$\left|\frac{x^n}{\lfloor n}\right| \le \left(\frac{a^n}{\lfloor n}\right) < \frac{\varepsilon}{p} \ \forall n \ge m \text{ and } x \in [-a, a]$$

So $\forall n \ge m$ and for all $p \in N$ set.

$$\left|\frac{x^{n+1}}{\lfloor n+1} + \frac{x^{n+2}}{\lfloor n+2} + ... + \frac{x^{n+p}}{\lfloor n+p}\right| \le \left|\frac{x^{n+1}}{\lfloor n+1}\right| + ... + \left|\frac{x^{n+p}}{\lfloor n+p}\right|$$

$< \dfrac{\varepsilon}{p} \cdot p = \varepsilon$. Thus fro $\varepsilon > 0$ there exist $m \ \varepsilon N$ such that for any $p \in N$ and $\forall x \in [-a, a]$.

$$|f_{n+p} - f_n| < \varepsilon \text{ where } f_n = 1 + \frac{x}{\lfloor 1} + ... + \frac{x^4}{\lfloor n}$$

Hence by Cauchy's principle $\sum u_n(x)$ converges uniformly on $[-a, a]$ *i.e.* $\sum u_n(x)$ convergent uniformly or any bounded subset of R.

43. Given series is $\sum_{n=0}^{\infty} x e^{-nx}$ in [0,1]

$$f_n(x) = u_1(x) + u_2(x) + ... + u_n(x)$$

$$= \sum_{n=0}^{n-1} xe^{-nx} = \frac{x\left(1 - \frac{1}{e^{nx}}\right)}{1 - \frac{1}{e^x}}$$

$$= \frac{xe^x}{e^x - 1}\left(1 - \frac{1}{e^{nx}}\right)$$

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & , x = 0 \\ \dfrac{xe^x}{\rho^x - 1}, & 0 < x \le 1 \end{cases}$$

Consider the interval]0,1]

$$M_n = \sup\{|f_n(x) - f(x)| : x \in]0,1]\}$$

$$= \sup\left\{\frac{xe^x}{(e^x - 1)e^{nx}} : x \in]0,1]\right\}$$

$$\ge \frac{\frac{1}{n} \cdot e^{-1/n}}{(e^{1/n} - 1)e} \qquad \text{by putting } x = \frac{1}{n}$$

$M_n \to \dfrac{1}{e}$ as $n \to \infty$, so the sequence is non-uniformly convergent by M_n-test and 0 is a point of non-uniform convergence.

44. Given series is $\sum \dfrac{x^4}{(1+n^4)^n}$ on [0,1]

$$S_n(x) = x^4 \left[\frac{1 - \left(\frac{1}{1+x^4}\right)^n}{1 - \frac{1}{1+x^4}}\right] \text{provided } 0 < x \le 1$$

So $S(x) = \lim_{n\to\infty} S_n(x)$

$$= x^4 \left[\frac{1 - 0}{1 - \frac{1}{1+x^4}}\right] = 1 + x^4 \ \ \forall 0 < x \le 1$$

Thus $S(x) = \begin{cases} 0 & x = 0 \\ 1 + x^4 & 0 < x \le 1 \end{cases}$

Each term of the series is continuous on [0,1]. But $S(x)$ is not continuous at $x = 0$. Hence, the given series does not uniformly convergent on [0,1].

46. The series $\sum \dfrac{x^4}{n^2}$ is uniformly convergent for $0 \le x \le 1$, by weierstrass's M-test so it can be integrated term by term. Hence,

$$\int_0^1\left(\sum_1^\infty \frac{x^n}{n^2}\right)d_k = \sum_1^\infty \int_0^1 \frac{x^n}{n^2}dx$$

$$= \sum_1^\infty \left\{\frac{x^{n+1}}{(n+1)n^2}\right\}_0^1 = \sum_1^\infty \frac{1}{n^2(n+1)}$$

51. The series $\sum \frac{nx^2}{n^3+x^3}$ is uniformly convergent on $[0, k]$ for $k > 0$. For

$$u_n(x) = \frac{1}{n^3+x^3} \text{ and } v_n(x) = nx^2$$

$$|u_n(x)| \le \frac{1}{n^3}\ \forall x \in [0,k]$$

But $\sum \frac{1}{n^3}$ is convergent so by Weierstrass's M-test $\sum u_n(x)$ is uniformly convergent on $[0,k]$.

Also $< v_n(x) >$ is monotonically increasing in $[0,k]$ so by Abel's test $\sum u_n(x)v_n(x) = \sum \frac{nx^2}{n^3+x^3}$ converges uniformly on $[0, k]$.

Hence, $\lim_{x\to 1}\left[\sum_{n=1}^\infty \frac{nx^2}{n^3+x^3}\right] = \sum_1^\infty \left(\lim_{x\to 1}\frac{nx^2}{n^3+x^3}\right)$

$$= \sum_1^\infty \frac{n}{n^3+1}$$

53. Let $u_n(x) = a_n x^4$

then $|u_n(x)| = |a_n x^n| \le |a_n|$

$\forall x \in [0, 1]$ and $\forall n \in N$

But $\sum u_n = \sum |a_n|$

converges so by Weierstrass's M-test $\sum u_n(x)$ converges uniformly on $[0,1]$.

54. Given that $u_n(x) = \frac{x}{n(1+nx^2)}$

For maxima, $n(1+nx^2) - 2n^2x^2 = 0$

$\Rightarrow \quad x = \pm\frac{1}{\sqrt{n}}$

Thus, $M_n = \sup u_n(x) = \frac{1/\sqrt{n}}{n(1+1)} = \frac{1}{2n^{3/2}}$

but $\sum \frac{1}{n^{3/2}} = \sum \mu_n$ is convergent so by Weierstrass's M-test the given series is uniformly convergent for all value of x *i.e.* in R.

58. $f_n(x) = \frac{1}{2n^2}\log(1+n^4x^2)$

So, $f(x) = \lim_{n\to\infty} f_n(x)$

$$= \lim_{n\to\infty}\frac{\log(1+n^4x^2)}{2n^2}$$

$$f(x) = \lim_{n\to\infty}\frac{\frac{4n^3x^2}{1+n^4x^2}}{4n}$$

$$= 0\ \forall x \in [0, 1]$$

Hence, $f'(x) = 0$

Also $\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty}\frac{n^2x}{1+n^4x^2} = 0\ \forall x \in [0, 1]$

Since, $f'(x) = \lim_{n\to\infty} f_n'(x)$

So given series can be differentiated term by term but $\sum u_n'(x)$ does not converse uniformly in $[0,1]$.

Since $< f_n'(x) >$ has 0 as a point at of non-uniform converges.

61. Given $f_n(x) = \frac{nx}{1+n^2x^2}\ \forall x \in [0, 1]$

$$f(x) = \lim_{n\to\infty} f_n(x)$$

$$= \lim_{n\to\infty}\frac{ux}{1+n^2x^2} = 0\ \forall x \in [0, 1]$$

$\therefore \quad f'(r) = 0$

Also $f_n'(r) = \lim_{h\to\infty}\frac{f_n(0+h)+f_n(0)}{h}$

$$= \lim_{h\to 0}\frac{\frac{nh}{1+n^2h^2} - 0}{h}$$

$$= \lim_{h\to 0}\frac{n}{1+n^2h^2} = n \to \infty \text{ as } h \to 0$$

Thus $f'(r) \ne \lim_{n\to R} f_n'(r)$

So given series can not be differentiable term by term at $x = 0$.

65. Let $f(x) = \sum_{1}^{\infty} \frac{\sin nx}{n^3}$

and $u_n(x) = \frac{\sin nx}{n^3}$

Then $u_n'(x) = \frac{\cos nx}{n^2}$

So, $\sum_{1}^{\infty} u_n'(x) = \sum \frac{\cos nx}{n^2}$

Since, $\left|\frac{\cos nx}{n^2}\right| \le \frac{1}{n^2} \forall x$

and $\sum \frac{1}{n^2}$ is convergent.

Hence, $\sum u_n'(x)$ converges uniformly for all value of x by weierstrass's M-test. Thus the series $\sum u_n(x)$ can be differentiated term by term.

Thus, $f'(x) = \sum_{1}^{\infty} u_n'(x) = \sum_{1}^{\infty} \frac{\cos nx}{n^2}$

69. Given $\sum u_n(x) = \sum_{1}^{\infty} \frac{x}{n(n+1)}$ in $]0, \infty[$

$$|u_n(x)| = \frac{|x|}{n(n+1)} \le \frac{k}{n(n+1)} \le \frac{k}{n^2} \forall x \in]0, k[$$

Since, $M_n = \sum \frac{k}{n^2} = k \sum \frac{1}{n^2}$ is convergent so by weierstrass's M-test $\sum u_n(x)$ is uniformly convergent on $]0, k[$. Let $\sum u_n(x)$ is uniformly convergent $]0, \infty[$, take $\in = \frac{1}{4}$

$$\left|\frac{x}{(m+1)(m+2)} + \frac{x}{(m+2)(m+3)} + \ldots + \frac{x}{2m(m+1)}\right| < \frac{1}{4}$$

for $n = p = m$

or $\frac{m_n}{2m(2n+1)} < \frac{1}{4} \forall x \in]0, \infty[$

70. Let $\sum_{n=1}^{n} u_n(x) = f_n(x)$

Then $f_n \in R[a, b]$ for each fixed n, since the sum of a finite number of R-integrable function S is R-integration. Also we know that the uniform convergence of the series $\sum u_n(x)$ is the same thing as the inform convergence of the sequence $< f_n >$ so that $< f_n >$ converges uniformly to f on $[a, b]$. Hence, $f \in R[a, b]$.

72. Here $f_n(x) = n^2 x(1-x)^n \;\forall x \in [0, 1]$

Obviously $f_n(x) = 0$ when $x = 0$ or 1

when $0 < x < 1$, we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n^2 x}{(1-x)^{-n}} \quad \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{2nx}{-(1-x)^{-n}\log(1-x)} \quad \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{2x}{(1-x)^{-n}[\log(1-x)]^2} = 0$$

Hence, $f(x) = \lim_{n\to\infty} f_n(x) = 0 \quad \forall\, 0 \le x \le 1$

$\therefore \int_0^1 f(x)\,dx = 0$

But $\int_0^1 f_n(x)\,dx = \int_0^1 n^2 x(1-x)^n\,dx$

$$= n^2\left[\frac{-x(1-x)^{n+1}}{n+1} - \frac{(1-x)^{n+2}}{(n+1)(n+2)}\right]_0^1$$

$$= \frac{n^2}{(n+1)(n+2)} \to 1 \text{ as } n \to \infty$$

Hence, term by term integration over $[0, 1]$ is not justified.

73. $u_n(x) = n^2 x e^{-n^2x^2} - (n-1)^2 x e^{-(n-1)^2x^2}$

$$u_1(x) = xe^{-x^2} - 0$$

$$u_2(x) = 2^2 xe^{-2^2x^2} - xe^{-x^2}$$

$$u_3(x) = 3^2 x e^{-3^2x^2} - 2^2 e^{-2^2x^2}$$

$$f_n(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$$

$$= n^2 xe^{-n^2x^2}$$

$\therefore f(x) = \lim_{n\to\infty} f_n(x) = 0 \;\forall x \in [0, 1]$

Thus the sum function $f(x)$ is continuous for all values of x in $[0,1]$.

75. Given that $f_n(x) = x^{1/(2n-1)}$

So, $f(x) = \lim_{n\to\infty}$

$$f_n(x) = \begin{cases} 0, \text{ for } x = 0 \\ 1 \text{ for all other values of } x \end{cases}$$

The function f is discontinuous at $x = 0$ and so zero is a point of non-uniform convergence of the series.

79. Given that $f_n(x) = nxe^{-nx^2}$

then $f(x) = \lim_{n\to\infty} nxe^{-nx^2} = 0$

Consider the interval $0 \le x \le 1$, we have

$$\int_0^1 f(x)\,dx = \int_0^1 0.dx = 0$$

and $\int_0^1 f_n(x)\,dx = \int_0^1 nxe^{-nx^2}\,dx$

$$= \left[-\frac{1}{2}e^{-nx^2}\right]_0^1$$

$$= \frac{1}{2}[1 - e^{-n}] \to \frac{1}{2} \text{ as } n \to \infty$$

So term by term integration is not justified in [0,1]. However term by term integration is justified over $[k, 1]$ where $0 < k < 1$, for we have

$$\int_k^1 f_n(x)\,dx = \int_k^1 nx\,e^{-nx^2}\,dx$$

$$= \frac{1}{2}[e^{-nx^2} - e^{-n}] \to 0 \text{ as } n \to \infty$$

Thus, $\int_k^1 f(x)\,dx = \lim_{n\to\infty} \int_x^1 f_n(x)\,dx$

82. $f_n(x) = \dfrac{x}{1+nx^2}$

So, $f(x) = \lim_{n\to\infty}$

$f_n(x) = \lim_{n\to\infty} \dfrac{x}{1+nx^2} = 0 \;\forall x \in R$

Let $y = f_n(x) - f(x) = \dfrac{x}{1+nx^2}$

For maximum or minimum values of y we put

$$\frac{dy}{dx} = 0$$

i.e., $\dfrac{(1+nx^2) - 2nx^2}{(1+nx^2)^2} = 0$

i.e., $\dfrac{1-nx^2}{(1+nx^2)^2} = 0$

which gives $x = \pm\dfrac{1}{\sqrt{n}}$

Also $\dfrac{dy^2}{dx^2} = \dfrac{-2nx(1+nx^2)(3-nx^2)}{(1+nx^2)^4}$

$$< 0 \text{ when } x = \frac{1}{\sqrt{n}}$$

Further $y_{\max} = \dfrac{1/\sqrt{n}}{1+n\left(\frac{1}{n}\right)} = \dfrac{1}{2\sqrt{n}}$

So, $M_n = \sup\limits_{x\in R} |f_n(x) - f(x)|$

$$= y_{\max} = \frac{1}{2\sqrt{n}}$$

Clearly, $\lim_{n\to\infty} M_n = 0$

Hence by M_n-test, the given sequence is uniformly convergent on R.

84. Given that $f_n(x) = nx(1-x)^n$ over [0,1]

Then $f(x) = \lim_{n\to\infty} f_n(x) = 0$

So, $\int_0^1 f(x)\,dx = 0$

and $\int_0^1 f_n(x)\,dx = \int_0^1 nx(1-x)^n\,dx$

$$= n\left[\frac{-x(1-x)^{n+1}}{n+1} - \frac{(1-x)^{n+2}}{(n+1)(n+2)}\right]_0^1$$

$$= \frac{n}{(n+1)(n+2)} \to 0 \text{ as } n \to \infty$$

Hence, the given series can be integrated term by term in $0 \le x \le 1$.

86. We have $u_n(x) = x^{n-1}(1-x)$

So, $S_n(x) = u_1(x) + u_2(x) + ... + u_n(x)$

$$= (1 + x + x^2 + ... x^{n-1})(1-x)$$

$$= \left(\frac{1-x^n}{1-x}\right)(1-x) = 1 - x^n$$

So, $S(x) = \lim_{n\to\infty} S_n(x)$

$$= \lim_{n\to\infty} (1-x^n) = \begin{cases} 0, & \text{if } x = 1 \\ 1, & \text{if } 0 \le x < 1 \end{cases}$$

For each value of n, the function $u_n(k)$ is continuous on [0,1]. But $S(x)$ is not continuous at $x = 1$ as

$$\lim_{x\to 1-0} S(x) = 1 \ne S(1)$$

Hence, given series does not converges uniformly on [0,1].

87. First we show that $\sum_{n=1}^{\infty}\frac{nx^2}{n^3+x^3}$ is uniformly convergent on $[0, x]$ for any $k > 0$.

Let $u_n(x) = \frac{1}{n^3+x^3}$

and $v_n(x) = nx^2$

Then $|u_n(x)| \le \frac{1}{n^3}\ \forall x \in [0, k]$

But $\sum \frac{1}{n^3}$ is convergent. Hence by Weierstrass's M-test $\sum u_n(x)$ is uniformly convergent on $[0, k]$. Also, for every $x \in [0, k], < v_n(x) >$ is monotonically increasing. So by Abel's test.

$$\sum \frac{nx^2}{n^3+x^3} = \sum u_n(x).v_n(x)$$

converges uniformly on $[0, k]$. Thus, we get

$$\lim_{x\to 1}\left(\sum_{n=1}^{\infty}\frac{nx^2}{n^3+x^3}\right) = \sum_{n=1}^{\infty}\left(\lim_{x\to 1}\frac{nx^2}{n^3+x^3}\right)$$

$$= \sum_{n=1}^{\infty}\frac{n}{n^3+1}$$

89. Given series is $\sum \frac{1}{n}\sin nx$

Let $u_n(x) = \sin x, v_n(x) = \frac{1}{n}$

$$f_n(x) = \sin x + \sin 2x + \ldots + \sin nx$$

$$= \frac{\sin\left(x + \frac{n-1}{2}x\right)\sin\frac{nx}{2}}{\sin\frac{x}{2}}$$

$$= \frac{\sin\left(\frac{n+1}{2}\right)x \cdot \sin\frac{nx}{2}}{\sin\frac{x}{2}}$$

$\therefore$ $$|f_n(x)| = \frac{\left|\sin\left(\frac{n+1}{2}\right)x\right|\left|\sin\frac{nx}{2}\right|}{\left|\sin\frac{x}{2}\right|} \le \frac{1}{|\sin x/2|}$$

$\therefore$ $|f_n(x)| \le |\text{cosec}\, x/2|$

But cosec $x/2$ is bounded for all values of $x + n$. $0 < a \le x \le b < 2\pi$. If k be its least upper bond in this interval then $|f_n(x)| < k$ for all values of x in this interval.

Again $< \frac{1}{n} >$ is a positive monotonic decreasing sequence converging to zero. Hence by Dirichlet's test the given series is uniformly convergent in $0 < a \le x \le b < 2\pi$.

91. Consider that

$$u_n(x) = x^n, v_n(x) = \frac{1}{n+1}$$

Since, $\forall x$ in $[-\delta, \delta]$, we have $|x| \le \delta < 1$.

So, $$|f_n(x)| = |x + x^2 + \ldots + x^n|$$

$$\le |x| + |x|^2 + \ldots + |x|^n$$

$$< \delta + \delta^2 + \ldots + \delta^n = \frac{\delta(1-\delta^n)}{1-\delta} < \frac{\delta}{1-\delta}$$

Also $< v_n >$ is a positive monotonic decreasing sequence converging to zero.

Hence, by Dirichlet's test the given series is uniformly convergent in $(-\delta, \delta)$.

92. Let $u_n(x) = a_n x^n$

then $|u_n(x)| = |a_n x^n| \le |a_n|\ \forall x \in [0, 1]$

$\therefore$ $|u_n(x)| \le |a_n|\ \forall x \in [0, 1]$ and $\forall n \in N$

But $\sum M_n = \sum |a_n|$ convergent and hence by Weierstrass's M-test $\sum u_n(x)$ converges uniformly on $[0,1]$.

94. Given that $u_n(x) = \frac{a_n x^n}{1+x^{2n}}$

Then $\frac{du_n}{dx} = 0$

i.e. $$\frac{(1+x^{2n})\, na_n x^{n-1} - 2nx^{2n-1}.a_n x^n}{(1+x^{2n})^2} = 0$$

i.e. $$\frac{na_n x^{n-1}(1-x^{2n})}{(1+x^{2n})} = 0$$

This gives $x = 0, 1$

It can be shown that $\frac{d^2u_n}{dx^2} < 0$ when $x = 1$

provided $a_n > 0$. So $u_n(x)$ is maximum when $x = 1$, provided $a_n > 0$.

Also $\max_{n \in R} u_n(x) = \dfrac{a_n \times 1^n}{1 + 1^{2n}} = \dfrac{1}{2} a_n$

So that $|u_n(x)| \le \dfrac{1}{2} a_n \;\forall\; x \in R$, provided $a > 0$.

Even when $a_n < 0$, we have

$$|u_n(x)| \le \frac{1}{2}|a_n| \;\forall x \in R$$

Thus, $|u_n(x)| \le \dfrac{1}{2}|a_n| \;\forall x \in R$

(whatever $a_n > 0$ or <0)

Take $M_n = \dfrac{1}{2}|a_n|$

Then $\sum M_n$ *i.e.* $\dfrac{1}{2}\sum |a_n|$ is convergent as $\sum a_n$ is given to be absolutely convergent.

Hence by Weierstrass's M-test, the given series is absolutely convergent.

95. Let $u_n = (-1)^{n-1}, v_n(x) = x^n$

Since, $f_n(x) = \sum_{r=1}^{n} u_r = 0$ or 1

according as n is even or odd, $f_n(x)$ is bounded for all n. Also $\{v_n(x)\}$ is a positive monotonic decreasing sequence, converging to zero for all values of x in $0 \le x \le k < 1$. Hence, by Dirichlet's test, the given series is uniformly convergent in $0 \le x \le k \le 1$.

❍❍❍

CHAPTER

6 Sequential Continuity, Boundedness and Intermediate Value Properties of Continuous Functions

LIMITS

1. **Limit**

Let f be a function defined on some nbd of a point a except possibly at a itself. Then $l \in R$ is said to be the limit of f at a if for each $\varepsilon > 0$, however small, there exists a positive number δ (depends on ε) such that

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - l| < \varepsilon$$

and written by $\lim_{x \to a} f(x) = l$

2. **Left hand limit**

A function f is said to approach a number l as $x \to a$ from the left if corresponding to an arbitrary number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$x \in (a - \delta, a) \quad \Rightarrow \quad |f(x) - l| < \varepsilon$$

and denoted by $\lim_{x \to a-0} f(x) = l$

or $f(a - 0) = l$

3. **Right hand limit**

A function f is said to approach a number l as $x \to a$ from the right if corresponding to an arbitrary number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$x \in (a, a + \delta) \quad \Rightarrow \quad |f(x) - l| < \varepsilon$$

and denoted by

$$\lim_{x \to a+0} f(x) = l$$

or $f(a + 0) = l$

4. **Four functional limits at a point**

Let f be defined on (a, b) with $c \in (a, b)$. For $h > 0$, consider a sequence $h > h_2 > h_3 > \ldots..$ converging to zero then for $(c, c + h)$, put $\mu(h_n) =$ supremum of $f(x)$ on $(c, c + h_n)$ and $\mu(h_n) =$ infimum of $f(x)$ on $(c, c + h_n)$.

Then clearly, $\mu(h_1) \geq \mu(h_2) \geq \ldots..$

and $\mu(h_1) \leq \mu(h_2) \leq \ldots..$

Thus, $<\mu(h_n)>$ and $<\mu(h_n)>$ are decreasing and increasing sequences respectively so both are convergent.

Define, $\overline{f(c+0)} = \lim_{n \to \infty} \mu(h_n)$

and $\underline{f(c+0)} = \lim_{n \to \infty} \mu(h_n)$

These limits are called upper and lower limits of f and c on the right.

If $\overline{f(c+0)} = \underline{f(c+0)}$

then right hand limit exists and denoted by $f(c + 0)$.

Again for $(c - h, c)$, put

$\mu'(h_n) =$ supremum of $f(x)$ on $(c - h_n, c)$

$\mu'(h_n) =$ infimum of $f(x)$ on $(c - h_n, c)$

Define $\overline{f(c-0)} = \lim_{n \to \infty} \mu'(h_n)$

and $\underline{f(c-0)} = \lim_{n \to \infty} \mu'(h_n)$

These limits are called upper and lower limits of f at c respectively.

If $\overline{f(c-0)} = \underline{f(c-0)}$

then left hand limit exist and denoted by $f(c - 0)$.

Here $\overline{f(c+0)}, \underline{f(c+0)}, \overline{f(c-0)}, \underline{f(c-0)}$ are called the four functional limits of f at $x = c$.

Example :

Let $f(x) = \dfrac{1}{x-a} \sin\left(\dfrac{1}{x-a}\right), x \neq a$ then

$$\overline{f(a+0)} = \infty, \underline{f(a+0)} = -\infty, \overline{f(a-0)} = \infty$$

$$\underline{f(a-0)} = -\infty$$

5. **Algebra of limits**

If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$ then

(i) $\lim_{x\to a} (f+g)(x) = l + m$

(ii) $\lim_{x\to a} (f-g)(x) = l - m$

(iii) $\lim_{x\to a} (fg)(x) = lm$

(iv) $\lim_{x\to a} \left(\frac{f}{g}\right)(x) = \frac{l}{m}$, provided $g(x) \neq 0$ and $m \neq 0$

(v) $\lim_{x\to a} (kf)(x) = kl$, where k is a constant.

CONTINUOUS FUNCTIONS

1. **Continuity**

A function $f(x)$ is said to be continuous at a point a if for each positive number ε, however small, there exists a positive number δ such that

$|x-a| < \delta \;\Rightarrow\; |f(x) - f(a)| < \varepsilon$

Here δ depends both ε and a.

or

A function $f(x)$ is said to be continuous at a if

$$\lim_{x\to a-0} f(x) = f(a) = \lim_{x\to a+0} f(x)$$

or $f(a-0) = f(a) = f(a+0)$

A function $f(x)$ is said to be continuous in (a, b) if it is continuous at each point of (a, b).

A function $f(x)$ is said to be continuous in the $[a, b]$ if it is :

(i) Continuous in (a, b)

(ii) Continuous from the right at a *i.e.*

$$f(a+0) = f(a)$$

(iii) Continuous from the left at b *i.e.*

$$f(b-0) = f(a)$$

Results :

1. A function f defined on ICR is continuous at a point $a \in I$ iff for every sequence $< x_n >$ in I, converging to a, the sequence $< f(x_n) >$ converges to $f(a)$.

2. A function $f : R \to R$ is continuous iff for every open set G in R, the inverge $f^{-1}(G)$ is an open set in R.

3. A function $f : R \to R$ is continuous on R iff for every closed set H in R, $f^{-1}(H)$ is closed in R.

4. If f and g defined on an interval I are continue at a point $a \in I$ then the following are also continuous.

(i) $f+g$ (ii) fg (iii) $f-g$

(iv) kf, k is constant

(v) $\frac{f}{g}$, provided $g(a) \neq 0$

(vi) max.$\{f, g\}$ (vii) (min.$\{f, g\}$

5. If $f(x)$ is continuous and $g(x)$ is discontinuous at a then $f(x) + g(x)$ is discontinuous at a.

6. If f is continuous at a then $|f|$ is also continuous at a but not convergels.

7. If f and g be defined on intervals I and J with $f(I)\,CJ$. If f is continuous at $a \in I$ and g is continuous at $f(a)$, then the composite mapping $g_r f$ is continuous at a.

8. **Borel's theorem**

If f is a continuous function on the closed interval $[a, b]$ then the interval can always divided up into a finite number of subintervals such that, given $\varepsilon > 0, |f(x_1) - f(x_2)| < \varepsilon$, where x_1 and x_2 are any two points in the same subinterval.

9. **Boundedness theorem**

If a function $f(x)$ is continuous in a closed interval $[a, b]$, then it is bounded in that interval but converge need not be true.

10. **The mostest theorem**

If a function $f(x)$ is continuous in $[a, b]$, then it attains its supremum and infimum at least once in $[a, b]$.

11. If $f(x)$ is continuous at $x = x_0$ where $f(x_0) \neq 0$, then a positive number δ can be found such that $f(x)$ has the same sign as $f(x_0)$ for every value of x in $]x_0 - \delta, x_0 + \delta[$.

12. **Bolzano's theorem**

If $f(x)$ is continuous in $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then there is at least one value of x for which $f(x)$ vanishes.

13. **The intermediate value theorem**

If a function f is continuous in the closed interval $[a, b)$, then $f(x)$ must take at least once all values between $f(a)$ and $f(b)$.

14. Let f be continuous on $[a, b]$ and let $k \in [m, M]$ where $m = \inf f$ and $M = \sup f$ on $[a, b]$ then there exists $c \in [a, b]$ such that $f(c) = k$.

15. Let f be continuous on $[a, b]$. Then

$$f([a, b]) = [m, M] \text{ where } m = \inf f$$

and $M = \sup f$ on $[a, b]$ and thus $f([a, b])$ is a closed set.

EXERCISE

MULTIPLE CHOICE QUESTIONS

Direction : Each of the following questions has four alternative answers. One of them is correct. Chosse the correct answer.

1. The function $f(x)$ is continuous at $x = a$ if :

a. $\lim_{x\to a} f(x)$ exists

b. $\lim_{x\to a} f(x)$ exist and unique

c. $\lim_{x\to a} f(x) = f(a)$

d. $\lim_{x\to a} f(x) = f'(a)$

2. $\lim_{x\to a} \frac{\tan x}{x}$ is equal to :

a. 0 b. 1

c. ∞ d. Not exist

3. If f and g continuous on interval I then which one of the following is not continuous :

a. $f + g$ b. $f - g$

c. fg d. $\frac{f}{g}$

4. If for every sequence $< a_n >$ in interval I converging to a, $\lim_{x\to a} f(a_n) = f(a)$ then f is :

a. Continuous everywhere

b. Differentiable at a

c. Discontinuous

d. Continue at a

5. If $f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 1-x & \text{if } x \text{ is irrational} \end{cases}$

then f takes every value between :

a. 0 and 2 b. 0 and ∞

c. 0 and 1 d. $-\infty$ and ∞

6. If $f(c) = 0$ for at least one $c \in [a,b]$ then $f(x)$ is continuous in $[a,b]$ only when :

a. $f(a) = f(b)$ b. $f(a) - f(b) > 0$

c. $f(a).f(b) < 0$ d. $f(a).f(b) = 0$

7. Which of the following function is not continuous on R :

a. $f(x) = k$ b. $f(x) = x$

c. $f(x) = \sin x$ d. $f(x) = \frac{1}{x}$

8. If for every open set h in R, the inverge image $f^{-1}(h)$ is over set in R then $f : R \to R$ is always :

a. Continuous

b. Differentiable

c. Continuous but not differentiable

d. None of these

9. $\lim_{x\to a} \frac{a^x - 1}{x}$, for $a > 0$ is :

a. a b. $a - 1$

c. $\log a$ d. 0

10. The signum function $f(x) = \begin{cases} x/|x|, x \neq 0 \\ 0, x = 0 \end{cases}$ is continuous at :

a. Everywhere in R

b. $x = 0$

c. Everywhere in R except at $x = 0$

d. Nowhere

11. If f is continuous in $\{a,b\}$ and $f(a)$ and $f(b)$ have opposite signs, then there is at least one value of x for which $f(x)$ vanishes, then it is known as :
   a. Mostest theorem   b. Bolzano's theorem
   c. Borel's theorem   d. Boundedness theorem

12. If $f: R \to R$ be defined by :

$$f(x) = \begin{cases} x \text{ when } x \text{ is rational} \\ -x \text{ when } x \text{ is irrational} \end{cases}$$

then $f(x)$ is continuous at :
   a. $x = 0$
   b. Rational number
   c. Irrational numbers
   d. Whole real number

13. If $\lim\limits_{x\to 0} \frac{a^x - 1}{x} = \log a$ then $\lim\limits_{x\to 0} \frac{2^x - 1}{(1+x)^{1/2} - 1}$ is equal to :
   a. $\log 2$ b. 2
   c. $1 + \log 2$ d. $2\log 2$

14. If $f(x) = x^p \cos\left(\frac{1}{x}\right), x \neq 0$ and $f(x) = 0$ then the value of p for which $f(x)$ is continuous at $x = 0$ is :
   a. $p = 0$ b. $p > 0$
   c. $p < 1$ d. $-1 < p < 1$

15. If f and g be continuous on $[a,b]$ and $f(a) < g(a)$ with $f(b) > g(b)$ then for some $c \in (a,b)$:
   a. $f(b) > g(c)$ b. $f(c) < g(c)$
   c. $f(c) = g(c)$ d. $f(c) \neq g(c)$

16. The function $f(x) = |x|$ at $x = 0$ is :
   a. Continuous and differentiable
   b. Differentiable
   c. Continuous   d. Bounded

17. $\lim\limits_{x\to 0} \sin\frac{1}{x}$ is equal to :
   a. 0   b. 1
   c. $\sin\infty$ d. Does not exist

18. Which of the following function is continuous on R :
   a. $f(x) = \log x$ b. $f(x) = \sin\frac{1}{x}$
   c. $f(x) = \cos\frac{1}{x}$ d. $f(x) = e^x$

19. The value of $\lim\limits_{x\to\infty} \frac{\sin x}{x}$ is :
   a. 0   b. 1
   c. ∞ d. Does not exist

20. If $f(x) = x$ is rational and $1 - x$ if x is irrational then $f(x)$ is continuous at :
   a. 0   b. 1
   c. $\frac{1}{2}$ d. Everywhere in R

21. If $f: R \to R$ be continuous and $f(x) = 0$ on a dense set then f is :
   a. Identically zero   b. Non-zero
   c. Any real number   d. None of these

22. Every continuous function f is :
   a. Differentiable   b. Uniformly continuous
   c. Bounded   d. None of these

23. The limit of $f(x) = \frac{x}{|x|}, x \neq 0$ and $f(0) = 0$ at $x = 0$ is :
   a. 1   b. -1
   c. 0   d. Does not exist

24. If f is continuous at a and $c \in R$ then cf is continuous at :
   a. ca b. a
   c. R d. Nowhere

25. If f is not continuous then f^2 is :
   a. Continuous
   b. Discontinuous
   c. May or may not be continuous
   d. None of these

26. If f is differentiable at x_0 and $f'(x_0) \neq 0$ then :
   a. $\left(\frac{1}{f}\right)'(x_0) = -f'(x_o)/f(x_0)^2$
   b. $\left(\frac{1}{f}\right)'(x) = -\frac{1}{f'(x_0)}$
   c. $\left(\frac{1}{f}\right)'(x_0) = \frac{-f'(x_0)}{[f(x_0)]}$
   d. $\left(\frac{1}{f}\right)'(x_0) = \frac{f(x_0)}{[f'(x_0)]^2}$

27. If $f(x) = [x], \forall x \in R$ be the greatest integer function then $\lim_{x\to 1/2} f(x)$ is equal to :

a. $\frac{1}{2}$ b. 0
c. 1 d. does not exist

28. The value of p for which $f(x) = \begin{cases} \frac{\tan(px)}{x}, x < 0 \\ 3x^2 + 2p^2, x \geq 0 \end{cases}$ will be continuous at $x = 0$ is :

a. 2 b. 5
c. 3 d. $\frac{1}{2}$

29. If $f(x)$ is continuous in $[a, b]$ then $f(x)$ must take at least once all values between $f(a)$ and $f(b)$ then it is called :

a. Bolzano's theorem
b. Mostest theorem
c. Intermediate value theorem
d. Borel's theorem

30. If $f(x) = x$ if x is irrational and 0 if x is rational on $[-1, 1]$ then f is continuous at : **[Meerut 2017]**

a. −1 b. +1
c. 0 d. All real numbers

31. $\lim_{x\to 2} \frac{|x-2|}{x-2}$ is equal to :

a. 0 b. 1
c. −1 d. Does not exist

32. The function $f(x) = x - [x]$ where x is positive variable and $[x]$ be the integral part of x then $f(x)$ is discontinuous at :

a. R
b. Q
c. Integral values of x
d. None of these

33. If f is continuous on $[a, b]$ such that $f(c) = k$ where $k \in [m, M]$ and $m = \inf f$, $M = \sup f$ on $[a, b]$ then :

a. $c \in (a, b)$
b. $c \in R$
c. $c \in [a, b]$
d. $c \in R$ but never equal to a or b

34. If $f(x) = 2^{1/x}$ then the value of $f(0+0)$ is :

a. 0 b. 2
c. ∞ d. Does not exist

35. If $f(x) = 1$, when x is rational and −1 when x is irrational then $f(x)$ is continuous at :

a. Whole R
b. The real numbers
c. Negative real numbers
d. Nowhere

36. If the inverse image of every closed set is closed then function is :

a. Continuous b. Discontinuous
c. Differentiable d. None of these

37. If $f(x) = \begin{cases} 2x + 1 \text{ for } x \leq 1 \\ ax^2 + b \text{ for } 1 < x < 3 \\ 5x + 2a \text{ for } x \geq 3 \end{cases}$ is continuous everywhere then :

a. $a = 1, b = 2$ b. $a = 1, b = 1$
c. $a = 2, b = 2$ d. $a = 2, b = 1$

38. A polynomial function is :

a. Continuous in R
b. Not continuous in R
c. May or may not be continuous in R
d. None of these

39. If $|f|$ is continuous at a then at a f is :

a. Continuous
b. Discontinuous
c. May be continuous or discontinuous
d. None of these

40. Which of the following is continuous at a, if f and g are continuous at $a \in I$:

a. $\max\{f, g\}$ b. $\min\{f, g\}$
c. f/g if $g \neq 0$ d. All the above

41. The function $f(x) = \sin\left(\frac{1}{x}\right)$ for $x \neq 0$, $f(0) = 0$ over $[0, 1]$ is :

a. Continuous but not bounded
b. Bounded but not continuous
c. Both continuous and bounded
d. None of these

42. If $f(x)$ is continuous at $x = a$ in (ε, δ) definition then δ depends on :

a. ε only b. a only
c. Both a and ε d. None of these

43. If f and g are continuous at $a \in I$ then at a, max. $\{f, g\}$ is :

a. Continuous b. Discontinuous
c. Differentiable d. None of these

44. If $f(x)$ is continuous in a closed interval $I = [a, b]$ then it is always :

a. Differentiable in I
b. Bounded in I
c. Bounded but not differentiable in I
d. None of these

45. If a function $f(x)$ is continuous in $[a, b]$ then it attains its supremum and infimum :

a. At one time b. At two time
c. At least once d. None of these

46. If $f(x)$ is an even function and $f'(0)$ exist then the value of $f'(0)$ is :

a. 0 b. 1
c. –1 d. Infinite

47. If $f(x) = \begin{cases} \dfrac{x-1}{2x^2 - 7x + 5} & \text{when } x \neq 1 \\ \dfrac{-1}{3} & \text{when } x = 1 \end{cases}$

then $f'(1)$ is equal to :

a. 1 b. $\frac{2}{9}$
c. $\frac{9}{2}$ d. $\frac{-2}{9}$

48. $f(x) = x^p \cos\left(\frac{1}{x}\right), x \neq 0$ and $f(0) = 0$, then the value of p for which $f(x)$ is differentiable at $x = 0$ is :

a. $p > 0$ b. $0 < p < 1$
c. $p > 1$ d. for all $p \in I$

49. The function $f(x) = \begin{cases} x & 0 < x \leq 1 \\ x - 1 & 1 < x \leq 2 \end{cases}$ at $x = 1$ is :

a. Continuous b. Differentiable
c. Bounded d. None of these

50. If f is continuous then $|f|$ is : **[Meerut 2017]**

a. Differentiable
b. Continuous
c. Not necessarily continuous
d. None of these

51. Every bounded function is :

a. Continuous
b. Differentiable
c. Continuous but not differentiable
d. None of these

52. If f is continuous on $[a, b]$ such that $f([a, b]) = [m, M]$ where $m = \inf f$ and $M = \sup f$ then $f([a, b])$ is :

a. Closed set b. Open set
c. Dense set d. None of these

53. The function $f(x) = |x - 1|$ at $x = 1$ is :

a. Continuous
b. Differentiable
c. Neither continuous nor differentiable
d. Bounded

54. The function $f(x) = \tan x$ defined on $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is : **[Kanpur 2018]**

a. Bounded
b. Continuous
c. Bounded and continuous both
d. None of these

55. If f and g are continuous and differentiable functions such that

$$f'(x) = g'(x) \ \forall x \in]a, b[\text{ then :}$$

a. $f(x) = g(x) \ \forall x \in]a, b[$
b. $f(x) \neq g(x) \ \forall x \in]a, b[$
c. $f(x) = g(x)$ differ only by a constant
d. None of these

56. The function $f(x) = x \tan^{-1}\left(\frac{1}{x^2}\right), x \neq 0$ is :

a. Continuous at $x = 0$
b. Differentiable at $x = 0$
c. Continuous and differentiable at $x = 0$
d. None of these

57. If $f(x)$ is differentiable function such that

$f(x) < f(2)$ then :

a. $f(2) = 0$ b. $f'(2) = 0$

c. $1 \le x \le 3$ d. $0 \le x \le 3$

58. If f is continuous on $[a, b]$ then the interval can be divided up into finite number of sub intervals such that $|f(x_1) - f(x_2)| < \varepsilon$, where x_1 and x_2 are two points in the same sub interval, it is called :

a. Mostest theorem b. Bolzano's theorem

c. Borel's theorem d. Boundedness theorem

59. The greatest integer function $[x]$ is : **[Kanpur 2018]**

a. Continuous at $x = 1$

b. Differentiable at $x = 1$

c. Not differentiable at $x = 1$

d. None of these

60. If $f + g$ is differentiable then f and g are :

a. Differentiable

b. Not necessarily differentiable

c. May or may not be differentiable

d. None of these

61. If f is continuous on $[a, b]$ and $f'(x) \ge 0$ on $]a, b[$ then f is :

a. Increasing in $[a, b]$ b. Increasing in R

c. Decreasing in $[a, b]$ d. Decreasing in R

62. The function $f(x) = \begin{cases} \frac{|x|}{x}, x \ne 0 \\ 1, x = 0 \end{cases}$: **[Kanpur 2018]**

a. Has removable discontinuity at $x = 0$

b. Has discontinuity of first kind at $x = 0$

c. Has discontinuity of second kind at $x = 0$

d. Discontinuous at $x = 0$

63. The function $f(x) = [x] - [-x]$ at $x = 0$ has :

a. Discontinuity of first kind

b. Discontinuity of second kind

c. Removable discontinuity

d. Continuity

64. If $f(x)$ is continuous in $[a, b]$ then it attains its supremum and infimum at least once $[a, b]$, it is know as :

a. Bolzano's theorem b. Mostest theorem

c. Borel's theorem d. None of these

65. If f is diffrentiable in I then $f'(I)$ on I is either an interval or a :

a. Null set b. Infinite set

c. Singleton set d. None of these

66. If $f'(x)$ is positive at $x = a$, then in the neighbourhodd of $x = a$, the function $f(x)$ is :

a. Increasing b. Decreasing

c. Positive d. Negative

67. If f is continuous in $[a, b]$ and $f(a).f(b) < 0$, then for at least one point $c \in [a, b]$: **[Kanpur 2018]**

a. $f(a) = f(b) = f(c)$ b. $f(c) = 0$

c. $f'(c) = 0$ d. All of these

68. If $f(x) = \frac{\sin 5x}{2x}, x \ne 0$ then the value of k for which $f(x)$ is continuous at $x = 0$ is :

a. $\frac{1}{2}$ b. $\frac{3}{2}$

c. $\frac{5}{2}$ d. 0

69. If $f(x) = [x] + [-x]\ \forall x \in R$ then for all integral values of x, $f(x)$ is :

a. Continuous

b. Discontinuity of first kind

c. Discontinuity of second kind

d. Removable discontinuity

70. If f, g and h are defined on a deleted neighbourhood D of a point a such that

$h(x) \le g(x) \le f(x)\ \forall x \in D$

and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = l$

then $\lim_{x \to a} g(x)$ is :

a. $\le l$ b. $\ge l$

c. l d. 0

71. If f and g are continuous at a and $f(a)$ respectively then at $x = a$ gof. is :

a. Continuous b. Discontinuous

c. May or may not be continuous

d. None of these

72. The function $f(x)$ is piecewise continuous if it has :

a. Finite number of jumps

b. No jump

c. Infinite number of jumps

d. None of these

73. If a function $f(x)$ is continuous in a closed interval then $f(x)$ is :

a. Bounded above only
b. Bounded below only
c. Bounded
d. Unbounded

74. If f is continuous on $[a,b]$ such that $\inf f \le k \le \sup f$ then there exists $c \in [a,b]$ such that :

a. $f(c)=0$ b. $f(c)=k$
c. $f'(c)=k$ d. None of these

75. $\lim_{x\to a} \sin\frac{1}{x-a}$ is equal to : **[Kanpur 2018]**

a. 0 b. a
c. $-a$ d. Does not exist

76. The function $f(x)=\begin{cases} e^{1/x}, x\neq 0 \\ 0, x=0 \end{cases}$ is : **[Kanpur 2018]**

a. Continuous at $x=0$
b. Discontinuous at $x=0$
c. Continuous everywhere
d. None of these

77. The function $f(x)=\begin{cases}(x-a)\sin\left(\frac{1}{x-a}\right), & x\neq a \\ 0, & x=a\end{cases}$ continuous at : **[Kanpur 2018]**

a. $x=0$ b. $x=a$
c. $x=-a$ d. $x=\frac{1}{a}$

78. The function $f(x)=\cos\left(\frac{1}{x}\right), x\neq 0$ and $f(0)=1$ then f is :

a. Continuous everywhere
b. Dontinuous at $x=0$
c. Discontinuous at $x=0$
d. Differentiable at $x=0$

79. The function $f(x)=\frac{\sin x}{x}, x\neq 0$ and $f(0)=1$ is :

a. Continuous everywhere
b. Continuous at $x=0$
c. Discontinuous at $x=0$
d. None of these

80. If $f(x)$ is continuous at $x=0$ then $x\,f(x)$ is :

a. Differentiable at $x=0$
b. Not differentiable in R
c. Differentiable everywhere except at $x=0$
d. None of these

81. If f and g are continuous on $[a,b]$ such that

$$f(a)<g(a) \text{ and } f(b)>g(b)$$

then for $c\in]a,b[$:

a. $f(c)=g(c)=0$ b. $f'(c)+g'(c)=0$
c. $f(c)-g(c)=0$ d. $f'(c)-g'(c)=0$

82. If $f: R\to R$ is continuous at $x=a$ such that

$$f(x+y)=f(x)+f(y)\ \forall x,y\in R$$

then f is also continuous at :

a. $a+1$ only b. $a-1$ only
c. $2a$ only d. Whole real number

83. The function

$$f(x)=\lim_{n\to\infty}\frac{e^x-x^n\sin x}{1+x^n}\left(0\le x\le\frac{\pi}{2}\right)$$

at $x=1$ is :

a. Continuous
b. Discontinuous
c. Uniformly continuous
d. None of these

84. The function $f(x)=\tan^{-1}\left(\frac{1}{x}\right)$ at $x=0$ is :

a. Continuous
b. Discontinuity of first kind
c. Discontinuity of second kind
d. Removable discontinuity

85. The function $f(x)=x\log x$ for $x>0$ and $f(0)=0$ at $x=0$ is :

a. Continuous
b. Discontinuous of first kind
c. Removable discontinuity
d. Discontinuity of second kind

86. Let $f(x+y)=f(x).f(y)\ \forall x,y\in R$ and $f(5)=-2$ and $f'(0)=3$ then $f'(5)$ is :

a. 1 b. 3
c. 6 d. –6

87. A function defined on [0,1] and given

$$f(x) = \begin{cases} x : x \text{ is rational} \\ 1-x : x \text{ is irrational} \end{cases} \text{ is :}$$ **[Meerut 2017]**

a. Discontinuous at $x = \frac{1}{2}$

b. Continuous at $x = \frac{1}{2}$

c. Uniformly continuous at $x = \frac{1}{2}$

d. None of these

88. If function f defined by

$$f(x) = \lim_{n\to\infty} \frac{\log(2+x) - x^{2n}\sin x}{1+x^{2n}}$$

then at $x = 1$ the function f is : **[Meerut 2017]**

a. Continuous

b. Continuity of first kind

c. Discontinuity of first kind

d. None of these

89. The function $f(x) = \frac{x}{(x-1)(x-2)}$ is unbounded at the points :

a. $x = 0, x = 1$ b. $x = 1, x = 2$

c. $x = -1, x = 2$ d. $x = 1, x = -2$

90. $\cos x$ is a continuous function when: **[Kanpur 2019]**

a. $x \in R$ b. $x \in Q$

c. $x \in I$ d. None of these

91. If $f(x) = \frac{\sin x}{x}$, then $f(0-0)$ is : **[Kanpur 2019]**

a. –2 b. 2

c. –1 d. 1

92. $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$ is equal to : **[Kanpur 2019]**

a. 0 b. 1

c. –1 d. ∞

93. The function $f(x) = \begin{cases} x^2 \text{ when } x < 0 \\ 5x - 4 \text{ when } x \geq 0 \end{cases}$ is :

[Kanpur 2019]

a. Continuous at $x = 0$

b. Continuous at $x = 1$

c. Continuous at $x = 2$

d. Not continuous at $x = 0$

94. The function $f(x) = \sin\left(\frac{1}{x}\right)$ at $x = 0$ is :

[Kanpur 2019]

a. Mixed discontinuity

b. Removable discontinuity

c. Discontinuity of first kind

d. Discontinuity of second kind

95. A function f is said to be continuous at point $x = a$, if for each $\varepsilon > 0$ there exists $a, \delta > 0$ such that :

[Kanpur 2019]

a. $|f(x) - f(a)| < \varepsilon \Rightarrow |x - a| < \delta$

b. $|f(x) - f(a)| < \varepsilon \Rightarrow |x + a| < \delta$

c. $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

d. $|x - a| > \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

96. The function $f(x)$ defined by $f(x) = \begin{cases} \frac{\tan x}{x}, x < 0 \\ 3x + 2x^2, x \geq 0 \end{cases}$

will be continuous at $x = 3$ then non-zero value for the constant k is :

[Kanpur 2019]

a. $\frac{1}{5}$ b. $\frac{1}{4}$

c. $\frac{1}{3}$ d. $\frac{1}{2}$

97. The function

$$f(x) = \lim_{n\to\infty} \frac{e^x - x^n \sin x}{1+x^n} \text{ is}$$

is where $x \geq 0$ and $x \leq \frac{\pi}{2}$. **[Meerut 2015]**

a. Continuous at $x = 1$

b. Discontinuous at $x = 1$

c. Uniformly continuous at $x = 0$

d. None of these

98. If f and g are continuous at $a \in I$ then $f + g$ is :

[Meerut 2015]

a. Discontinuous at a

b. May or may not be continuous at a

c. Continuous at a

d. None of these

99. A function $f(x)$ is said to be bounded over the interval I if there exist two real number a and $b (b > a)$ such that : **[Meerut 2015]**

a. $a < f(x) < b \; \forall x \in I$

b. $a \le f(x) \le b \; \forall x \in I$

c. $a < f(x) \le b \; \forall x \in I$

d. $a \le f(x) < b \; \forall x \in I$

100. The function $f : R \to R$ defined by

$$f(x) = \begin{cases} 1, \text{ when } x \text{ is rational} \\ -1, \text{ when } x \text{ is irrational} \end{cases}$$

then at every point of R, f will be : **[Meerut 2016]**

a. Continuous

b. Discontinuous

c. Totally discontinuous

d. None of these

101. If f is continuous at point a and $C \in R$ then C_f is : **[Meerut 2016]**

a. Continuous at a

b. Discontinuous at a

c. Discontinuous at C

d. None of these

102. If f continuous at a then $|f|$ is also : **[Meerut 2016]**

a. Discontinuous at a

b. Continuous at a

c. May be continuous at a

d. None of these

103. If a function $f(x)$ is continuous in closed interval $[a, b]$, then $f(x)$ in $[a, b]$ will be :

a. Bounded

b. Unbounded

c. Only bounded above

d. None of these

104. A function defined on [0, 1] and given

$$f(x) = \begin{cases} x \; : \text{If } x \text{ is rational} \\ 1 - x : \text{If } x \text{ is irrational} \end{cases}$$ is : **[Meerut 2017]**

a. Discontinuous at $x = \frac{1}{2}$

b. Continuous at $x = \frac{1}{2}$

c. Uniformly continuous at $x = \frac{1}{2}$

d. None of these

105. If a function f is continuous at a, then $|f|$ is : **[Meerut 2017]**

a. Continuous at a

b. Discontinuous at a

c. Uniformly continuous at a

d. None of these

106. If function f defined on $[-1, 0]$ and

$$f = \begin{cases} 0 : x \text{ rational} \\ x : x \text{ irrational} \end{cases}$$

then f is continuous at : **[Meerut 2017]**

a. $x = 0$ b. $x = 1$

c. $x = -1$ d. $x = \pm 2$

107. The function $f : R \to R$ defined by

$f(x) = \lim_{m \to \infty} \{ \lim_{n \to \infty} (\cos m! \pi x)^{2n} \}$ is : **[Meerut 2017]**

a. Totally continuous b. Continuous

c. Discontinuous d. None of these

108. At $x = 1$ the function

$fx = \lim_{n \to \infty} \dfrac{x^n}{1 + x^n e^x}$ is/has : **[Meerut 2017]**

a. Continuous

b. Continuity of first kind

c. Discontinuity of first kind

d. None of these

109. Find $\lim_{z \to 0} \dfrac{x^3 y(y - ix)}{x^6 + y^2}$, where $z \to 0$ along the curve $y = x^3$: **[Meerut 2018]**

a. $i/2$ b. $-i/2$

c. i d. $-i$

110. Find $\lim_{z \to 0} \dfrac{x^3 y(y - ix)}{x^6 + y^2}$, when $z \to 0$ along any radius vector : **[Meerut 2018]**

a. 0 b. $-i$

c. $-i/2$ d. $i/2$

111. $\lim \frac{\sin n\pi / 3}{\sqrt{n}}$ is equal to : **[Meerut 2018]**

a. 0 b. 1

c. –1 d. None of these

112. Which is not true : **[Meerut 2019]**

a. $f(x) = \sin\frac{1}{x}$ is continuous $\forall x > 0$

b. $f(x) = \sin\frac{1}{x}$ is uniformly continuous $\forall x > 0$

c. $f(x) = \sin\frac{1}{x}$ is continuous, but not uniformly continuous $\forall x \in R^+$

d. All the above

113. Which is true, for

$$f(x) = x^2 \sin\frac{1}{x^2} \forall x \in [-1, 1] :$$

a. $f(x)$ is continuous

b. $f(x)$ is not continuous $\forall x \in [-1, 1]$

c. $f(x)$ is not bounded

d. All the above

114. Let $S = \{f : R \to R\}\ \exists\ \varepsilon > 0$ such that

$$\forall \delta > 0, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon\}$$

then : **[Meerut 2019]**

a. $S = \{f : R \to R |\ f$ is continuous$\}$

b. $S = \{f : R \to R |\ f$ is uniformly continuous$\}$

c. $S = \{f : R \to R |\ f$ is bounded$\}$

d. $S = \{f : R \to R |\ f$ is constant$\}$

115. Which is true for the function

$$f(x) = \sin x . \sin\frac{1}{x} \forall x \in]0, 1[$$ **[Meerut 2019]**

a. $\underline{\lim}_{x\to 0} f(x) = \overline{\lim}_{x\to 0} f(x)$

b. $\overline{\lim}_{x\to 0} f(x) < \underline{\lim}_{x\to 0} f(x)$

c. $\underline{\lim}_{x\to 0} f(x) = 1$

d. $\overline{\lim}_{-x\to 0} f(x) = -1$

116. Improper Riemann Integral $\int_0^x y^{-1/2}.dy$ is :

[Meerut 2019]

a. Continuous in $[0, \infty[$

b. Continuous only in $(0, \infty)$

c. Discontinuous in $(0, \infty)$

d. Discontinuous only in $\left(\frac{1}{2}, \infty\right)$

117. Let $f : R \to R$ be a continuous function and

$f(x + 1) = f(x)\ \forall x \in R$, then :

a. f is bounded above, but not bounded below

b. f is bounded but not attain its bounds

c. f is bounded and attain its bounds

d. f is not uniformly continuous

118. If $f(x) = \begin{cases} 0 & x \in Q \\ x & x \in R - Q \end{cases}$ then f is continuous at $x =$

[Meerut 2019]

a. 1 b. –1

c. 0 d. None of these

119. Let $f : R \to R$ is a monotone function, then :

a. f is continuous

b. f has finite point of discontinuity

c. f has countable point of discontinuity

d. f has uncountable many point of discontinuity

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(c)	**2.**	(b)	**3.**	(d)	**4.**	(d)	**5.**	(c)	**6.**	(c)	**7.**	(d)	**8.**	(a)	**9.**	(c)	**10.**	(c)
11.	(b)	**12.**	(a)	**13.**	(d)	**14.**	(b)	**15.**	(c)	**16.**	(c)	**17.**	(d)	**18.**	(d)	**19.**	(a)	**20.**	(c)
21.	(a)	**22.**	(c)	**23.**	(d)	**24.**	(b)	**25.**	(c)	**26.**	(a)	**27.**	(b)	**28.**	(d)	**29.**	(c)	**30.**	(c)
31.	(d)	**32.**	(c)	**33.**	(c)	**34.**	(c)	**35.**	(d)	**36.**	(a)	**37.**	(d)	**38.**	(c)	**39.**	(c)	**40.**	(d)
41.	(b)	**42.**	(c)	**43.**	(a)	**44.**	(b)	**45.**	(c)	**46.**	(a)	**47.**	(d)	**48.**	(c)	**49.**	(c)	**50.**	(b)
51.	(d)	**52.**	(a)	**53.**	(a)	**54.**	(b)	**55.**	(c)	**56.**	(a)	**57.**	(c)	**58.**	(c)	**59.**	(c)	**60.**	(b)
61.	(a)	**62.**	(b)	**63.**	(a)	**64.**	(b)	**65.**	(c)	**66.**	(a)	**67.**	(b)	**68.**	(c)	**69.**	(d)	**70.**	(c)
71.	(a)	**72.**	(a)	**73.**	(c)	**74.**	(b)	**75.**	(d)	**76.**	(b)	**77.**	(b)	**78.**	(c)	**79.**	(b)	**80.**	(a)
81.	(c)	**82.**	(d)	**83.**	(b)	**84.**	(b)	**85.**	(a)	**86.**	(d)	**87.**	(b)	**88.**	(c)	**89.**	(b)	**90.**	(a)
91.	(d)	**92.**	(a)	**93.**	(d)	**94.**	(d)	**95.**	(c)	**96.**	(d)	**97.**	(a)	**98.**	(c)	**99.**	(b)	**100.**	(b)
101.	(a)	**102.**	(b)	**103.**	(a)	**104.**	(b)	**105.**	(a)	**106.**	(a)	**107.**	(c)	**108.**	(b)	**109.**	(b)	**110.**	(a)
111.	(d)	**112.**	(b)	**113.**	(a)	**114.**	(c)	**115.**	(a)	**116.**	(a)	**117.**	(c)	**118.**	(d)	**119.**	(a)		

HINTS AND SOLUTIONS

2. $\lim_{x\to 0} \frac{\tan x}{x}$

$$= \lim_{x\to 0} \frac{x + \frac{x^3}{3} + ...}{x}$$

$$= \lim_{x\to 0} \left\{1 + \frac{x^2}{3} + ...\right\} = 1$$

3. We know that if f and g be defined and continuous on an interval I and $g(a) \neq 0$ then $\frac{f}{g}$ is continuous at a. Here $g(a) \neq 0$ is not given so $\frac{f}{g}$ is not necessarily continuous.

5. Let $c \in [0, 1]$. If c is rational then $f(c) = c$. If c is irrational then $1 - c$ also irrational and

$$\partial < 1 - c < 1 \ \ i.e.\ \ 1 - c \in [0, 1]$$

we have $f(1-c) = 1 - (1-c) = c$

Thus, f takes every value c in $[0, 1]$.

7. Here $f(x) = \frac{1}{x}$ is not defined on $x = 0$ so it is not continuous at $x = 0$ *i.e.* $f(x) = \frac{1}{x}$ is not continuous on R.

9. $\lim_{x\to 0} \frac{a^x - 1}{x}, a > 0$

$$= \lim_{x\to 0} \left[\frac{1 + x\log a + \frac{x^2}{\lfloor 2}(\log a)^2 + ... 1}{x}\right]$$

$$= \lim_{x\to 0} \left[\log a + \frac{x}{\lfloor 2}(\log a)^2 + ...\right] = \log a$$

10. $f(x) = \begin{cases} \frac{x}{|x|}, x \neq 0 \\ 0, x = 0 \end{cases} = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{cases}$

Here $f(0+0) = 1$, $f(0-0) = -1$ and $f(0) = 0$ so $f(x)$ is not continuous at $x = 0$ only.

12. Let $\varepsilon > 0$ be given and choose $\delta = \frac{\varepsilon}{2}$ then if x is rational we have

$$|x - 0| < \delta$$

$$\Rightarrow \quad |f(x) - f(0)| = |-x - 0| = |-x|$$

$$= |x| < \delta = \frac{\varepsilon}{2} < \varepsilon$$

If x is irrational, we have

$$|x - 0| < \delta$$

$$\Rightarrow \quad |f(x) - f(0)| = |x - 0| = |x| < \delta < \varepsilon$$

Thus, we get $|x| < \delta \ \Rightarrow \ |f(x) - f(0)| < \varepsilon$

So f is continuous at $x = 0$.

Now consider $x \neq 0$ and x be any rational number. For each positive integer n, let x_n be an irrational number such that $|x_n - x| < \frac{1}{n}$.

Then $< x_n >$ is a sequence of irrational numbers such that

$$\lim_{n\to\infty} x_n = x$$

Given that $\quad f(x_n) = x_n \ \forall n$

So, we have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n$$

$$= x \neq -x = f(x)$$

Thus, $f(x)$ is not continuous at any non-zero rational number. Similarly $f(x)$ is not continuous at any irrational number.

14. Given that $f(x) = x^p \cos\left(\frac{1}{x}\right)$

$$x \neq 0 \ \text{and} \ f(0) = 0$$

For continuity at $x = 0$

$$f(0+0) = \lim_{h\to\infty} (0+h)$$

$$= \lim_{h\to\infty} h^p \cos\left(\frac{1}{x}\right) = 0$$

only when $p > 0$

Similarly, $f(0-0) = \lim_{h\to\infty} (0-h)$

$$= \lim_{h\to\infty} (-h)^p \cos\left(\frac{1}{h}\right) = 0$$

only when $p > 0$

So $f(0+0) = f(0-0) = f(0)$

exists only when $p > 0$.

17. Given $f(x) = \sin\frac{1}{k}, x \neq 0$

$$f(0+0) = \lim_{h\to\infty} f(0+h)$$
$$= \lim_{h\to\infty} \sin\frac{1}{h}$$

which does not exist.

Similarly, $f(0-0) = \lim_{h\to\infty} f(0-h)$

$$= \lim_{h\to\infty} \sin\left(-\frac{1}{h}\right)$$

which does not exist.

So given limit does not exist.

19. Put $x = \frac{1}{y}$

we get, $\lim_{y\to 0} y\cdot\frac{1}{y} = 0$

27. Given that $f(x) = [x]\ \forall x \in R$

$$f\left(\frac{1}{2}+0\right) = \lim_{h\to 0} f\left(\frac{1}{e}+h\right)$$
$$= \lim_{h\to 0}\left[\frac{1}{2}+h\right] = 0$$
$$f\left(\frac{1}{2}-0\right) = \lim_{h\to 0} f\left(\frac{1}{2}-h\right)$$
$$= \lim_{h\to 0}\left[\frac{1}{2}-h\right] = 0$$

and $f\left(\frac{b}{2}\right) = \left[\frac{b}{2}\right] = 0$

So, $f\left(\frac{1}{2}+0\right) = f\left(\frac{1}{2}-0\right)$

$$f\left(\frac{1}{2}\right) = 0$$

i.e. f is continuous at $x = \frac{1}{2}$.

28. $f(x)$ will be continuous at $x = 0$ only when

$$f(0+0) = f(0-0) = f(0)$$

or $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(0-h)$

or $\lim_{h\to 0} (3h^2 + 2p^2) = \lim_{h\to 0} \frac{\tan(-ph)}{-h}$

or $2p^2 = \lim_{h\to 0} \frac{\sin ph}{h\cos ph} \times \frac{p}{p}$

or $2p^2 = \lim_{h\to 0}\left(\frac{p}{\cos ph}\right)$

$\because$ $\lim_{h\to 0} \frac{\sin ph}{ph} = 1$

or $2p^2 = p \Rightarrow p = 0$ or $\frac{1}{2}$

i.e., $p = \frac{1}{2}$

31. $f(x) = \frac{|x-2|}{x-2} = \begin{cases} 1 \text{ if } x > 2 \\ -1 \text{ if } x < 2 \end{cases}$

So, $f(2+0) = \lim_{h\to 0} f(2+h) = 1$

and $f(2-0) = \lim_{h\to 0} f(2-h) = -1$

So, $\lim_{x\to 2} \frac{|x-2|}{x-2}$ does not exist.

32. Given $f(x) = x - [x]$

or $f(x) = \begin{cases} x-(n-1) \text{ for } n-1 \le x < n \\ x - n \text{ for } n \le x < n+1 \end{cases}$

So, $f(n) = 0$

Also $f(n+0) = \lim_{h\to 0} f(n+h)$

$$= \lim_{h\to 0} \{(n+h)-n\}$$
$$= \lim_{h\to 0} h = 0$$

and $f(n-0) = \lim_{h\to 0} f(n-h)$

$$= \lim_{h\to 0} \{(n-h)-(n-1)\}$$
$$= \lim_{h\to 0} (1-h) = 1$$

$\because$ $f(n-0) \neq f(n+0)$

So, f is discontinuous at $x = n$

i.e. all integral values of x.

34. $f(x) = 2^{1/x}$

$$f(0+0) = \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} 2^{1/h+0} = \lim_{h\to 0} 2^{1/h}$$
$$= 2^{1/0} = 2^{\infty} = \infty$$

37. $f(x) = \begin{cases} 2x+1 & x \le 1 \\ ax^2 + b & 1 < x < 3 \\ 5x + 2a & x \ge 3 \end{cases}$

If $f(x)$ is continuous at $x = 1$

then $\quad 2\times 1 + 1 = a + b$

or $\quad a + b = 3 \quad ...(1)$

If $f(x)$ is continue at $x = 3$

then $\quad 9a + b = 15 + 2a$

or $\quad 7a + b = 15 \quad ...(2)$

Solving (1) and (2), we get $a = 2, b = 1$.

41. $f(x) = \sin\left(\frac{1}{x}\right), x \ne 0$

and $\quad f(0) = 0$ over $[0,1]$

$$|f(x)| = \left|\sin\left(\frac{1}{x}\right)\right| \le 1$$

So $f(x)$ is bounded $\forall x \in [0, 1]$.

Now $\quad f(0+0) = \lim_{h\to 0} f(0+h)$
$$= \lim_{h\to 0} \sin\left(\frac{1}{h}\right)$$

A value between −1 to +1.

$$f(0-0) = \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} \sin\left(\frac{-1}{h}\right)$$

A value lies between −1 ot +1.

So, $f(x)$ does not exist at $x = 0$ *i.e.* not a continuous function.

43. If f and g are continue at $a \in I$ then

$$f+g,\ f-g,\ |f-g|,\ \frac{1}{2}(f+g),\ \frac{1}{2}|f-g|$$

are all continues at $a \in I$.

Also max. $\{f,g\} = \frac{1}{2}(f+g) + \frac{1}{2}[f-g]$

So, max. $\{f,g\}$ is also continuous at $a \in I$.

47. $f(x) = \dfrac{x-1}{2x^2 - 7x + 5}$

when $x \ne 1$ and $f(1) = \dfrac{-1}{3}$

Then, $f'(1+0) = \lim_{h\to 0} \dfrac{f(1+h) - f(1)}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{1+h-1}{2(1+h)^2 - 7(1+h) + 5} + \dfrac{1}{3}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{h}{2h^2 - 3h} + \dfrac{1}{3}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{1}{2h-3} + \dfrac{1}{3}}{h}$$

$$= \lim_{h\to 0} \frac{2h}{3h(2h-3)} = \frac{-2}{9}$$

Similarly, $\quad f'(1-0) = \dfrac{-2}{9}$

So, $\quad f'(1) = \dfrac{-2}{9}$

49. $f(x) = \begin{cases} x, & 0 < x \le 1 \\ x-1, & 1 < x \le 2 \end{cases}$

$$f(1) = 1$$

and $\quad f(1+0) = \lim_{h\to 0} f(1+h)$
$$= \lim_{h\to 0} \{(1+h) - 1\}$$

and $\quad f(1-0) = \lim_{h\to 0} f(1-h)$
$$= \lim_{h\to 0} \{1-h\} = 1$$

So, $\quad f(1+0) \ne f(1-0)$

i.e., $f(x)$ is not continues and so not differentiable at $x = 1$. But $f(x)$ is bounded at $x = 1$.

50. Let $< a_n >$ be a sequence converging to a. Since f is continuous at a so $\lim_{h\to 0} f(a_n) = f(a)$.

Now, $\quad \lim_{n\to\infty} |f|(a_n) = \lim_{n\to\infty} |f(a_n)|$
$$= |\lim_{n\to\infty} f(a_n)| = |f(a)| = |f|(a)$$

Hence, $|f|$ is also continuous at $x = a$.

52. Since f is continuous on $[a,b]$ then $f(x)$ takes at least once all values between its infimum and supremum so $f(x)$ takes all values between m and M and hence

$$[m, M] \subseteq f([a,b])$$

Again since every value of $f(x)$ on $[a,b]$ lies between m and M we have

$$f([a,b]) \leq [m, M]$$

Thus, we get $f([a,b]) = [m, M]$ and then $f([a,b])$ is a closed set.

56. $f(x) = x\tan^{-1}\left(\frac{1}{x^2}\right), x \neq 0$

$$f(0+0) = \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} h\tan^{-1}\left(\frac{1}{h^2}\right)$$
$$f(0+0) = 0$$
$$f(0-0) = \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} -h\tan^{-1}\left(\frac{1}{h^2}\right) = 0$$

Also $f(0) = 0$

So, $f(0+0) = f(0-0) = f(0)$

i.e., f is continuous at $x = 0$.

59. $f(x) = [x] = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & 1 \leq x < 2 \end{cases}$

$$f(1+0) = \lim_{h\to 0} f(1+h)$$
$$= \lim_{h\to 0} 1 = 1$$
$$f(1-0) = \lim_{h\to 0} f(1-h)$$
$$= \lim_{h\to 0} 0 = 0$$

$\because$ $f(1+0) \neq f(1-0)$

So, $f(x)$ is not continuous at $x = 1$.

i.e., $f(x)$ is not differentiable at $x = 1$.

62. $f(x) = \begin{cases} \frac{|x|}{x}, x \neq 0 \\ 1, x = 0 \end{cases} = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

$$f(0+0) = \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} 1 = 1$$
$$f(0-0) = \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} (-1) = -1$$

So, $f(x)$ has discontinuity of first kind at $x = 0$.

63. $f(x) = [x] - [-x]$ at $x = 0$

Here, $f(0) = [0] - [-0]$
$$= 0 - 0 = 0$$

Also $f(0+0) = \lim_{h\to 0} f(0+h)$
$$= \lim_{h\to 0} ([h] - [-h])$$
$$= \lim_{h\to 0} [0 - (-1)] = 1$$

and $f(0-0) = \lim_{h\to 0} f(0-h)$
$$= \lim_{h\to 0} ([-h] - [h])$$
$$= \lim_{h\to 0} \{(-1) - 0\} = 1$$

Thus, $f(0+0) = f(0-0) \neq f(0)$

So, $f(x)$ has a discontinuity of the first kind at $x = 0$.

68. Given that $f(x) = \begin{cases} \frac{\sin 5x}{2n} & , x \neq 0 \\ k & , x = 0 \end{cases}$ and $f(x)$ is continuous at $x = 0$

$$f(0+0) = \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} \frac{\sin 5h}{2h}$$
$$= \lim_{h\to 0} \frac{\sin 5h}{5h} \cdot \frac{5h}{2h}$$
$$= 1 \cdot \frac{5}{2} = \frac{5}{2}$$
$$f(0-0) = \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} \frac{\sin(-5h)}{-2h} = \frac{5}{2}$$

For continuity,

$$f(0+0) = f(0-0) = f(0)$$

So, $k = \frac{5}{2}$

69. Given that $f(x) = [x] + [-x]\ \forall x \in R$

$$f(x) = \begin{cases} 0, \text{ if } x \text{ is an integer} \\ -1, \text{ if } x \text{ is not an integer} \end{cases}$$

Let $x = n$, where n is an integer then $f(n) = 0$

Also $f(n+0) = \lim_{h\to 0} f(n+h) = -1$

and $f(n-0) = \lim_{h\to 0} f(n-h) = -1$

So, $\lim_{x\to n} f(x) = -1$

Thus, $f(n+h) = f(n-h) \neq f(x)$

i.e., f has removable discontinuity at every integer value of x.

75. $f(a+0) = \lim_{h\to 0} f(a+h)$

$$= \lim_{h\to 0} \sin\frac{1}{a+h-a}$$

$$= \lim_{h\to 0} \sin\frac{1}{h}$$

which does not exist so $\lim_{x\to a} \sin\frac{1}{x-a}$ does not exist.

76. $f(x) = \begin{cases} e^{1/n}, x \neq 0 \\ 0, x = 0 \end{cases}$

$$f(0+0) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} e^{1/h} = e^{\infty} = \infty$$

$$f(0-0) = \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0} e^{-1/h} = e^{-\infty} = \infty$$

Since, $f(0+0) \neq f(0-0)$

So, $f(x)$ is discontinuous at $x = 0$.

77. Given that

$$f(x) = \begin{cases} (x-a)\sin\left(\frac{1}{x-a}\right) & , \ x \neq a \\ 0 & , \ x = a \end{cases}$$

$$f(a+0) = \lim_{h\to 0} f(a+h)$$

$$= \lim_{h\to 0} (a+h-a)\sin\left(\frac{1}{a+h-a}\right)$$

$$= \lim_{h\to 0} h\sin\frac{1}{h} = 0$$

Also $f(a-0) = \lim_{h\to 0} f(a-h)$

$$= \lim_{h\to 0} (a-h-a)\sin\left(\frac{1}{a-h-a}\right)$$

$$= \lim_{h\to 0} h\sin\frac{1}{h} = 0$$

Thus, $f(a+0) = f(a-0) = f(a)$

Hence, $f(x)$ is continuous at $x = a$.

79. Given that

$$f(x) = \frac{\sin x}{x}, x \neq 0 \text{ and } f(0) = 1$$

$$f(0+0) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} \frac{\sin h}{h} = 1$$

$$f(0-0) = \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0} \frac{\sin(-h)}{-h} = 1$$

So, $f(0+0) = f(0-0) = f(0)$

i.e., $f(x)$ is continuous at $x = 0$.

84. Given that $f(x) = \tan^{-1}\left(\frac{1}{x}\right)$

$$f(0+0) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} \tan^{-1}\left(\frac{1}{h}\right)$$

$$= \tan^{-1}\infty = \frac{\pi}{2}$$

$$f(0-0) = \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0} \tan^{-1}\left(\frac{-1}{h}\right)$$

$$= -\lim_{h\to 0} \tan^{-1}\left(\frac{1}{h}\right)$$

$$= -\tan^{-1}\infty = \frac{-\pi}{2}$$

Thus both limits exist but not equal. Hence, $f(x)$ has a discontinuity of the first kind at $x = 0$.

85. Given that

$$f(x) = x\log x$$

For $x > 0$ and $f(0) = 0$

$$f(0+0) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} h\log h$$

$$= \lim_{h\to 0} \frac{\log h}{1/h}, \ \frac{\infty}{\infty} \text{ form}$$

$$= \lim_{h\to 0} \frac{1/h}{-1/h^2}, \text{ by L'Hospital rule}$$

$$= \lim_{h\to 0} (-h) = 0$$

Since, $f(0+0) = 0 = f(0)$

So, f is continuous at $x = 0$ from the right. Here f is defined in $[0, \infty)$ *i.e.* only on the right hand side of 0. Hence, continuity at $x = 0$ from the right implies that f is continuous at $x = 0$.

86. $\because$ $f(x+y) = f(x).f(y)$

So, $f(5+0) = f(5).f(0)$

$\Rightarrow$ $f(5) = f(5).f(0)$

So, $f(0) = 1$

Also $f'(0) = 3 \Rightarrow \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = 3$

$$\Rightarrow \lim_{h\to 0}\left[\frac{f(h)-1}{h}\right] = 3 \qquad ...(1)$$

Also $f'(5) = \lim_{h\to 0} \frac{f(5+h)-f(5)}{h}$

$$= \lim_{h\to 0} \frac{f(5).f(h)-f(5)}{h}$$

$$= f(5).\lim_{h\to 0}\left(\frac{f(h)-1}{h}\right)$$

$= f(5).3$ by (1)

$= (-2)\times 3 = -6$

So, $f'(5) = -6$

❍❍❍

CHAPTER

7 Uniform Continuity

DEFINITION

A function f defined on an interval I is said to be uniformly continuous on I if given $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta$$

where $x, y \in I$ and δ depends on ε only.

It should be noted that uniform continuity is a property associated with an interval and not with a single point.

RESULTS

1. The concept of continuity is local in character whereas the concept of uniform continuity is global in character.
2. If f is uniform continuous on an interval I, then it is continuous on I.
3. A function which is continuous in closed and bounded interval $I = [a, b]$ is uniformly continuous in $[a, b]$.

DIFFERENCE BETWEEN CONTINUITY AND UNIFORM CONTINUITY

1. In continuity, the positive number δ depends upon ε as well as the point a *i.e.* $s = s(a, \varepsilon)$ whereas in uniform continuity, δ depends on ε only *i.e.* $s = s(\varepsilon)$.
2. Continuity is basically defined at a particular point whereas uniform continuity is define on a set and cannot be defined at a point. Thus continuity is local while uniform continuity is global in nature.

NON-UNIFORM CONTINUITY CRITERION

A function f is not uniformly continuous on I, if there exists some $\varepsilon > 0$ for which *no* $\delta > 0$ server *i.e.* for any $\delta > 0$ there exist $x, y \in I$ such that

$$|x - y| < \delta \text{ but } |f(x) - f(y)| \geq \varepsilon$$

EXERCISE

MULTIPLE CHOICE QUESTIONS

Direction : Each of the following questions has four alternative answers. One of them is correct. Choose the correct answer.

1. If f is uniformly continuous in $I = [a, b]$ then in I, f is :
   a. Continuous only
   b. Bounded only
   c. Continuous and bounded both
   d. Continuous but not bounded
2. If f is continuous in $I = [a, b]$ then in I, f is :
   a. Uniformly continuous
   b. Not uniformly continuous
   c. May or may not be uniformly continuous
   d. None of these
3. If $f(x) = \frac{1}{x}$ $\forall x \in]0,1[$, then $f(x)$ is :
   a. Uniformly continuous
   b. Bounded only
   c. Continuous
   d. None of these

4. If $f(x) = \sin\left(\frac{1}{x}\right) \forall x \in R^+, x > 0$ then $f(x)$ is :
   a. Uniformly continuous
   b. Continuous
   c. Unbounded
   d. None of these

5. If f and g are uniformly continuous on an interval I then on I, $f.g$ is :
   a. Uniformly continuous
   b. Not uniformly continuous
   c. Continuous but not uniformly continuous
   d. None of these

6. If $f : R \to R$ be defined by $f(x) = x \ \forall x \in R$ then f is :
   a. Unbounded
   b. Discontinuous
   c. Uniformly continuous
   d. Continuous but not uniformly continuous

7. If $f : R \to R$ be defined by $f(x) = 1 \ \forall x \in R$ then over R, f is :
   a. Uniformly continuous
   b. Continuous only
   c. Bounded only
   d. None of these

8. If f is uniformly continuous in $[a, b] = I$ then in I, f is :
   a. Continuous but not bounded
   b. Bounded but not continuous
   c. Continuous and bounded both
   d. Neither continuous nor bounded

9. If $f : R \to R$ be given by $f(x) = x^2$ then in R, $f(x)$ is :
   a. Bounded
   b. Continuous
   c. Uniformly continuous
   d. None of these

10. A function f define on $[a, b]$ is said to be uniformly continuous on $[a, b]$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|x_1 - x_2| < \delta \ \Rightarrow \ |f(x_1) - f(x_2)| < \varepsilon$$

   then δ depend upon :
   a. x_1 b. x_2
   c. x_1 and x_2 d. ε

11. Uniform continuity define over :
   a. A point
   b. An interval
   c. A point and interval both
   d. None of these

12. If $f(x) = x^3$ is uniformly continuous in $[-2, 2]$, then maximum value of δ is :
   a. $\frac{\varepsilon}{12}$ b. $\frac{\varepsilon}{9}$
   c. $\frac{\varepsilon}{6}$ d. $\frac{\varepsilon}{3}$

13. If $f(x) = g(x) = \sqrt{x} \ \forall x \in [0, \infty[$ are two unbounded functions then fg over the same interval is :
   a. Bounded
   b. Uniformly continuous
   c. Not uniformly continuous
   d. Continuous but not uniformly continuous

14. If $f : R \to R$ be defined by

$$f(x) = \lim_{m \to \infty} \{\lim_{n \to \infty} (\cos m\pi x)^{2n}\}$$

   then over R, f is :
   a. Uniformly continuous
   b. Continuous
   c. Discontinuous
   d. None of these

15. If f is continuous in a closed and bounded interval $I = [a, b]$ then in I, f is :
   a. Uniformly continuous
   b. Not uniformly continuous
   c. May or may not be uniformly continuous
   d. None of these

16. If $f(x) = x^3$ on $[-2, 2]$ then $f(x)$ is : **[Meerut 2016]**
   a. Continuous
   b. Uniformly continuous
   c. Bounded
   d. All the above

17. If $f(x) = x^2 \forall x \in [-2, 2]$ is uniformly continuous then maximum value of δ is :
   a. ε b. $\frac{\varepsilon}{2}$
   c. $\frac{\varepsilon}{4}$ d. $\frac{\varepsilon}{8}$

18. The function $f(x) = \sin x \ \forall x \in [0, \infty[$ then $f(x)$ is :
   a. Unbounded
   b. Discontinuous
   c. Uniformly continuous
   d. Not uniformly continuous

19. If $f(x) = \frac{1}{x} \ \forall x \in (0, 1)$ then $f(x)$ is : **[Kanpur 2018]**
   a. Unbounded
   b. Discontinuous
   c. Uniformly continuous
   d. Not uniformly continuous

20. If $f(x) = \frac{1}{x^2} \ \forall x \in [a, \infty[$ and $a > 0$ then the maximum value of δ is :
   a. $\frac{a^3\varepsilon}{2}$ b. $\frac{a^2\varepsilon}{2}$
   c. $\frac{a\varepsilon}{2}$ d. $\frac{a^3\varepsilon}{3}$

21. The function $f(x) = \frac{x}{x+1} \ \forall x \in [0, 2]$ is :
   a. Unbounded
   b. Discontinuous
   c. Uniformly continuous
   d. None of these

22. The function $f(x) = \frac{1}{x^2} \ \forall x \in [-1, 0]$ is :
   a. Continuous
   b. Uniformly continuous
   c. Continuous and uniformly continuous both
   d. Neither continuous nor uniformly continuous

23. If $f(x) = x^2 \ \forall x \in [-1, 1]$ uniformly continuous then $\delta_{maximum}$ is :
   a. $\frac{\varepsilon}{8}$ b. $\frac{\varepsilon}{4}$
   c. $\frac{\varepsilon}{2}$ d. ε

24. If $f(x) = 2x^2 - 3x + 5$ is uniformly continuous in $[-2, 2]$ then maximum value of δ is :
   a. ε b. $\frac{\varepsilon}{3}$
   c. $\frac{\varepsilon}{7}$ d. $\frac{\varepsilon}{11}$

25. If $f(x) = \sin x$ is uniformly continuous in $[0, \infty[$ then maximum value of δ may be :
   a. ε b. $\frac{\varepsilon}{2}$
   c. $\frac{\varepsilon}{4}$ d. $\frac{\varepsilon}{8}$

26. The function $f(x) = \frac{x}{x+1}$ in $[0, \infty[$ is :
   a. Uniformly continuous
   b. Not uniformly continuous
   c. May or may not be uniformly continuous
   d. None of these

27. The function $f(x) = x^2$ over the interval $(0, \infty)$ is :
   a. Bounded and continuous
   b. Continuous and uniformly continuous
   c. Bounded and uniformly continuous
   d. Continuous but not uniformly continuous

28. The function $f(x) = 2x^4 + 3 \ \forall x \in [-2, 1]$ is :
   a. Continuous
   b. Bounded
   c. Uniformly continuous
   d. All the above

29. If $f(x) = 2x^2 + 1 \ \forall x \in [-2, 3]$ is uniformly continuous over then maximum value of δ may be :
   a. ε b. $\frac{\varepsilon}{5}$
   c. $\frac{\varepsilon}{10}$ d. $\frac{\varepsilon}{20}$

30. If $f(x) = x^2 + 3x \ \forall x \in [-1, 1]$ is uniformly continuous then the maximum value of δ may be :
   a. $\frac{\varepsilon}{10}$ b. $\frac{\varepsilon}{5}$
   c. $\frac{\varepsilon}{3}$ d. ε

31. The product of the uniformly continuous function is :
   a. Uniformly continuous
   b. Not uniformly continuous
   c. May not be uniformly continuous
   d. None of these

32. If $f(x) = x^2 \sin\left(\frac{1}{x^2}\right)$ for $x \neq 0$ and $f(0) = 0$ in the interval $I = [-1, 1]$ then over I, f is : **[Meerut 2016]**
a. Bounded
b. Continuous
c. Uniformly continuous
d. All the above

33. If $f(x) = x^2$ is uniformly continuous in [0, 1] then maximum value of δ may be :
a. ε b. $\frac{\varepsilon}{2}$
c. $\frac{\varepsilon}{3}$ d. $\frac{\varepsilon}{4}$

34. If $f(x) = \frac{x}{1+x}$ is uniformly continuous in [0,2] then maximum value of δ is :
a. $\frac{\varepsilon}{2}$ b. $\frac{\varepsilon}{3}$
c. $\frac{\varepsilon}{4}$ d. ε

35. The function $f(x) = \frac{1}{1-x}$ $\forall x \in (0, 1)$ is :
a. Continuous but not uniformly continuous
b. Uniformly continuous
c. Uniformly continuous but not bounded
d. Continuous but not bounded

36. The function $f(x) = e^x$ $\forall x \in (0, \infty)$ is :
a. Continuous
b. Uniformly continuous
c. Continuous and uniformly continuous both
d. Neither continuous nor uniformly continuous

37. The function $f(x) = \sin\left(\frac{1}{x}\right)^2$ on R^+ is :
a. Bounded and uniformly continuous
b. Bounded and continuous
c. Continuous and uniformly continuous
d. None of these

38. The function $f(x) = 2x^2 + 1$ $\forall x \in [-2, 3]$ is :
a. Continuous
b. Bounded
c. Uniformly continuous
d. All the above

39. If the function $f(x) = 2x^4 + 3$ $\forall x \in [-2, 1]$ is uniformly continuous in [–2, 1] then the maximum value of δ may be :
a. ε b. $\frac{\varepsilon}{10}$
c. $\frac{\in}{20}$ d. $\frac{\in}{30}$

40. The function $f(x) = x^2 + 3x$, $x \in [-1, 1]$ is :
a. Bounded and continuous
b. Continuous and uniformly continuous
c. Bounded and uniformly continuous
d. All the above

41. If $f : A \to R$ is continuous mapping then f is :
a. Uniformly continuous always
b. Uniformly continuous if A is bounded
c. Uniformly continuous if A is compact
d. None of these

42. If f is uniformly continuous in an interval I and $< x_n >$ is a Cauchy sequence of elements in I then $< f(x_n) >$ is :
a. Convergent sequence only
b. Cauchy sequence only
c. Both convergent and Cauchy sequence
d. None of these

43. If f and g are uniformly continuous on an interval I then $f \to g$ over I is :
a. Uniformly continuous
b. Not uniformly continuous
c. May or may not be uniformly continuous
d. None of these

44. Which one is uniformly continuous in $[0, \infty[$:
a. x^2 b. $\sin x$
c. $\sin x^2$ d. x^3

45. Which one is uniformly continuous in $[0, \infty[$:
a. $\sqrt{x}$ b. $\sin^2 x$
c. e^x d. All the above

46. The function $f(x) = \sin^2$ $\forall x \in [0, \infty[$ is :
a. Uniformly continuous
b. Not uniformly continuous
c. Continuous but not bounded
d. None of these

47. If $f(x)$ is differentiable on I such that
$|f'(x)| \le l \;\forall\; l > 0$ on I then $f(x)$ is :
a. Continuous but not uniformly continuous
b. Uniformly continuous but not continuous
c. Continuous and uniformly continuous both
d. None of these

48. The function $f(x) = \frac{1}{x}, x \ge 1$ is :
a. Not continuous at $x = 1$
b. Not differentiable
c. Uniformly continuous
d. Not uniformly continuous

49. If f and g are uniformly continuous with $g(x) > 0 \;\forall x \in R$ then which of the following is uniformly continuous :
a. $f + g$ b. $\frac{f}{g}$
c. $f.g$ d. None of these

50. If $f(x) = 3x^2 + 2 \;\forall x \in [-1, 2]$ is uniformly continuous then maximum value of δ may be :
a. $\frac{\varepsilon}{3}$ b. $\frac{\varepsilon}{6}$
c. $\frac{\varepsilon}{9}$ d. $\frac{\varepsilon}{12}$

51. The function $f(x) = \sin x$ in $[0, \infty[$ is :
a. unbounded
b. discontinuous
c. uniformly continuous
d. non uniformly continuous

52. The function $f(x) = x^2$ is : **[Kanpur 2018]**
a. uniformly continuous on $[-2, 2]$
b. uniformly continuous on $[-1, 1]$
c. continuous on $[-1, 1]$
d. all the above

53. The function
$$f(x) = \lim_{n\to\infty} \frac{e^x - x^n \sin x}{1 + x^n} \;\forall x \in \left[0, \frac{\pi}{2}\right] \text{ is :}$$
a. Continuous at $x = 1$
b. Uniformly continuous at $x = 1$
c. Discontinuous at $x = 1$
d. None of these

54. The function $f(x) = \frac{1}{x}, x > 0$ is : **[Kanpur 2018]**
a. Discontinuous in (0,1)
b. Continuous in (0,1)
c. Uniformly continuous in (0,1)
d. None of the above

55. For all real values of x, the function $f(x) = x^2$ is : **[Meerut 2015]**
a. Continuous
b. Discontinuous
c. Uniformly continuous
d. None of these

56. If f is uniformly continuous on an interval I, then f will be ..... on I : **[Meerut 2016]**
a. Discontinuous
b. Continuous
c. Uniformly discontinuous
d. None of these

57. If f is continuous in closed interval I and bounded in closed interval I then it is : **[Meerut 2017]**
a. Continuous on I
b. Discontinuous on I
c. Uniformly continuous
d. None of these

58. The series $\sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}$ converges uniformly if $x \in$: **[Meerut 2018]**
a. $]1, \infty[$ b. $[1, \infty]$
c. $[1, \infty[$ d. $]1, \infty]$

59. The series $\Sigma \frac{x}{n(1 + nx^2)}$ is : **[Meerut 2018]**
a. Convergent
b. Divergent
c. Converges uniformly
d. Unbounded

60. The series $\Sigma \frac{(-1)^{n-1}}{n} x^n$ converges uniformly if $x \in$: **[Meerut 2018]**
a. $[0, 1[$ b. $]0, 1[$
c. $[0, 1]$ d. $]0, 1]$

61. Which function is not uniformly continuous on $]0, 1[$:

a. $f(x) = x$ b. $f(x) = e^x$ c. $f(x) = \sin x$ d. $f(x) = \tan\left(\frac{\pi x}{2}\right)$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(c)	**2.**	(c)	**3.**	(c)	**4.**	(b)	**5.**	(a)	**6.**	(c)	**7.**	(a)	**8.**	(c)	**9.**	(b)	**10.**	(d)
11.	(b)	**12.**	(a)	**13.**	(b)	**14.**	(c)	**15.**	(a)	**16.**	(d)	**17.**	(c)	**18.**	(c)	**19.**	(d)	**20.**	(a)
21.	(c)	**22.**	(a)	**23.**	(c)	**24.**	(d)	**25.**	(a)	**26.**	(a)	**27.**	(d)	**28.**	(c)	**29.**	(c)	**30.**	(b)
31.	(c)	**32.**	(d)	**33.**	(a)	**34.**	(d)	**35.**	(a)	**36.**	(a)	**37.**	(b)	**38.**	(c)	**39.**	(d)	**40.**	(d)
41.	(c)	**42.**	(c)	**43.**	(a)	**44.**	(b)	**45.**	(d)	**46.**	(b)	**47.**	(c)	**48.**	(c)	**49.**	(c)	**50.**	(c)
51.	(c)	**52.**	(d)	**53.**	(c)	**54.**	(b)	**55.**	(c)	**56.**	(b)	**57.**	(c)	**58.**	(c)	**59.**	(c)	**60.**	(d)
61.	(d)																		

HINTS AND SOLUTIONS

1. Let $x_0 \in I$ and $\varepsilon > 0$ be given. Since f is uniformly continuous on I so there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta \ \forall x, y \in I$$

put $y = x_0$, we get

$$|f(x) - f(x_0)| < \varepsilon \text{ whenever } |x - x_0| < \delta \ \forall x \in I$$

So, f is continuous at x_0. Since every continuous function is bounded so f is bounded also.

3. Given $f(x) = \frac{1}{x} \ \forall x \in]0, 1[$

Since, $\lim_{x \to c} f(x)$

$$= \lim_{x \to c} \frac{1}{x} = \frac{1}{c} = f(0)$$

So, $f(x)$ is continuous at each point c in $]0, 1[$.

Let $x_1 = \frac{1}{m}$ and $x_2 = \frac{1}{2m}$

then $x_1, x_2 \in]0, 1[$ if m is a positive integer.

We have $|x_1 - x_2| = \left|\frac{1}{m} - \frac{1}{2m}\right|$

$$= \left|\frac{1}{2m}\right| = \frac{1}{2m}$$

or $|x_1 - x_2| < \frac{1}{m} < \delta$

If we choose $\frac{1}{m} < \delta$ and $\delta > 0$

Also $|f(x_1) - f(x_2)| = \left|\frac{1}{x_1} - \frac{1}{x_2}\right|$

$$= |m - 2m| = |-m| = m > \frac{1}{2}$$

Thus, if we take $\varepsilon = \frac{1}{2} > 0$, then whatever $\delta > 0$ we try $\exists x_1, x_2 \in]0,1[$ such that $|x_1 - x_2| < \delta$ but

$$|f(x_1) - f(x_2)| > \varepsilon = \frac{1}{2}$$

So for $\varepsilon = \frac{1}{2} > 0$ there exists no $\delta > 0$ such that

$|f(x_1) - f(x_2)| < \varepsilon$ whenever $|x_1 - x_2| < \delta$, $x_1, x_2 \in]0,1[$.

Hence, $f(x)$ is not uniformly continuous in $]0, 1[$.

4. Given $f(x) = \sin\frac{1}{x} \ \forall x > 0$ and $x \in R^+$. Let $a \in R^+$ then we have

$$f(a + 0) = \lim_{h \to 0} f(a + h)$$

$$= \lim_{h\to 0} \sin\left(\frac{1}{a+h}\right) = \sin\frac{1}{a}$$

$$f(a-0) = \lim_{h\to 0} f(a-h)$$

$$= \lim_{h\to 0} \sin\left(\frac{1}{a-h}\right) = \sin\frac{1}{a}$$

Also $\quad f(a) = \sin\frac{1}{a}$

Thus, $\quad f(a+0) + f(a-0) = f(a)$

i.e. f is continuous at a and so continuous on R^+.

Let $\delta > 0$ and $x_1 = \frac{1}{n\pi}$,

$$x_2 = \frac{1}{n\pi + \pi/2} = \frac{2}{(2n+1)\pi}$$

where n is a positive integer such that

$$x_1 - x_2 = \frac{1}{n\pi} - \frac{2}{(2n+1)\pi} < \delta$$

Now $|x_1 - x_2| < \delta$ but

$$|f(x_1) - f(x_2)| = \left|\sin n\pi - \sin\frac{(2n+1)\pi}{2}\right| = 1 > \varepsilon$$

If we choose $\varepsilon = \frac{1}{2} > 0$

Thus, for $\varepsilon = \frac{1}{2}$ we are unable to find $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \varepsilon$$

whenever $\quad |x_1 - x_2| < \delta \ \forall x_1, x_2 \in R^+$

Hence, f is not uniformly continuous on R^+.

9. By Archimedes property for any $\delta > 0$ there exists a positive integer n such that

$$n\delta^2 > \varepsilon \qquad \text{...(1)}$$

Choose $x_1 = n\delta$ and $x_2 = n\delta + \frac{\delta}{2}$ then

$$|x_1 - x_2| = \frac{\delta}{2} < \delta$$

and $\quad |f(x_1) - f(x_2)| = |x_1^2 - x_2^2|$

$$= |x_1 - x_2||x_1 + x_2|$$

$$= \frac{\delta}{2}(2n\delta + \delta/2)$$

$$= n\delta^2 + \frac{\delta^2}{4} > \varepsilon \text{ by (1)}$$

Hence, for $x_1, x_2 \in R, |f(x_1) - f(x_2)| > \varepsilon$ whatever $\delta > 0$ we take. Thus f is not uniformly continuous on R.

12. Here, $\quad f(x) = x^3 \ \forall x \in [-2, 2]$

Let $x_1, x_2 \in [-2, 2]$ then we have

$$|f(x_2) - f(x_1)| = |x_2^3 - x_1^3|$$

$$= |(x_2 - x_1)(x_2^2 + x_1^2 + x_1x_2)|$$

$$\le |x_2 - x_1|\{|x_2|^2 + |x_1|^2 + |x_1||x_2|\}$$

$$\le 12|x_2 - x_1|$$

$\because \quad |x_1| \le 12$ and $|x_2| \le 12$

$\therefore \quad |f(x_2) - f(x_1)| < \varepsilon$ whenever $|x_2 - x_1| < \frac{\varepsilon}{12}$

Thus given $\varepsilon > 0$ there exists $\delta = \frac{\varepsilon}{12}$ such that $|x_2 - x_1| < \delta$. So $f(x)$ is uniformly continuous in $[-2, 2]$.

14. Let $x = \frac{p}{q}$ be a rational number. If m is large then

$$\cos(\underline{|m}\,\pi x) = \pm 1$$

$$\therefore \quad \lim_{n\to\infty} (\cos\underline{|m}\,\pi x)^{2n} = 1$$

Hence, $f(x) = 1$ when x is rational.

If x is irrational then $(\cos\underline{|m}\,\pi n)^{2n} = (r_m)^{2n}$

where $|r_m| < 1$, for a fixed value of m.

Hence, $\quad f(x) = \lim_{m\to\infty}\lim_{n\to\infty} (r_m)^{2n} = 0$

Thus, $f(x)$ is discontinuous for all values of x.

17. Let $x_1, x_2 \in [-2, 2]$

then $|x_1| \le 2$ and $|x_2|\ \varepsilon 2$

Also $\quad |f(x_1) - f(x_2)| = |x_1^2 - x_2^2|$

$$= |(x_1 - x_2)(x_1 + x_2)|$$

$$= |x_1 - x_2|\,|x_1 + x_2|$$

$$\le |x_1 - x_2|(2+2)$$

$$= 4|x_1 - x_2| = \varepsilon$$

provided $|x_1 - x_2| < \frac{\varepsilon}{4}$. Thus, choose $\delta = \frac{\varepsilon}{4}$

We find $\quad |x_1 - x_2| < \delta \ \Rightarrow \ |f(x_1) - f(x_2)| < \varepsilon$

and so $f(x)$ is uniformly continuous.

20. Given $f(x) = \dfrac{1}{x^2} \forall x \in [a, \infty[$ and $a > 0$

$$|f(x_1) - f(x_2)| = \left|\frac{1}{x_1^2} - \frac{1}{x_2^2}\right| = \left|\frac{x_2^2 - x_1^2}{x_1^2 x_2^2}\right|$$

$$= \left|(x_2 - x_1)\left(\frac{x_1 + x_2}{x_1^2 x_2^2}\right)\right|$$

$$= \left|(x_2 - x_1)\left(\frac{1}{x_1^2 x_2} + \frac{1}{x_1 x_2^2}\right)\right|$$

$$= |x_1 - x_2|\left(\frac{1}{x_1^2 x_2} + \frac{1}{x_1 x_2^2}\right)$$

$\because x_1, x_2 > 0$

$$\le |x_1 - x_2|\left(\frac{1}{a^3} + \frac{1}{a^3}\right)$$

$\because x_1, x_2 \ge a > 0$

$$\le \frac{2}{a^3}|x_1 - x_2| = \varepsilon$$

provided $|x_1 - x_2| < \dfrac{a^3 \varepsilon}{2} = \delta$

Thus, we get

$$|x_1 - x_2| < \delta$$

$\Rightarrow \quad |f(x_1) - f(x_2)| < \varepsilon$

Hence, f is uniformly continuous on $[a, \infty[$.

21. $f(x) = \dfrac{x}{x+1} \forall x \in [0, 2]$

$$|f(x_1) - f(x_2)| = \left|\frac{x_1}{x_1 + 1} - \frac{x_2}{x_2 + 1}\right|$$

$$= \frac{|x_1 - x_2|}{|(x_1 + 1)(x_2 + 1)|}$$

$\because \quad x_1 \ge 0$

i.e., $\quad x_1 + 1 \ge 1$ and $x_2 + 1 \ge 1$

so $\dfrac{1}{(x_1 + 1)(x_2 + 1)} \le 1$

$\therefore \quad |f(x_1) - f(x_2)| \le |x_1 - x_2| = \varepsilon$

provided $|x_1 - x_2| < \varepsilon$.

Choose $\delta = \varepsilon$ we get

$$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$$

Hence, f is uniformly continuous on [0,2].

24. $f(x) = 2x^2 - 3x + 5 \forall x \in [-2, 2]$

If $x_1, x_2 \in [-2, 2]$ so $|x_1| \le 2, |x_2| \le 2$

$$|f(x_1) - f(x_2)| = |(2x_1^2 - 3x_1 + 5) - (2x_2^2 - 3x_2 + 5)|$$

$$= |2(x_1^2 - x_2^2) - 3(x_1 - x_2)|$$

$$= |(x_1 - x_2)(2x_1 + 2x_2 - 3)|$$

$$\le |x_1 - x_2|(2|x_1| + 2|x_2| + 3)$$

$$\le |x_1 - x_2|(4 + 4 + 3)$$

$$= 11(x_1 - x_2) = \varepsilon$$

provided $|x_1 - x_2| < \dfrac{\varepsilon}{11}$

Choose $\delta = \dfrac{\varepsilon}{11}$, we find that

$$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$$

So, f is uniformly continuous in $[-2, 2]$.

29. $f(x) = 2x^2 + 1 \; \forall x \in [-2, 3]$

If $x_1, x_2 \in [-2, 3]$ then $|x_1| \le 3$ and $|x_2| \le 3$

$$|f(x_1) - f(x_2)| = |(2x_1^2 + 1) - (2x_2^2 + 1)|$$

$$= |2(x_1 + x_2)(x_1 - x_2)|$$

$$\le 2|x_1 - x_2|(|x_1| + |x_2|)$$

$$= 12|x_1 - x_2| = \varepsilon$$

provided $|x_1 - x_2| < \dfrac{\varepsilon}{12}$

choose $\delta = \dfrac{\varepsilon}{12}$ we get

$$|f(x_1) - f(x_2)| < \varepsilon \; \forall |x_1 - x_2| < \delta$$

So f is uniformly continuous.

32. $f(x) = x^2 \sin\left(\dfrac{1}{x^2}\right)$

$x \ne 0$ and $f(0) = 0 \forall x \in [-1, 1]$

$$|f(x)| = \left|x^2 \sin\left(\frac{1}{x^2}\right)\right|$$

$$= |x^2| \left|\sin\left(\frac{1}{x^2}\right)\right|$$

$$\le 1.1 = 1 \text{ and } f(0) = 0$$

$\Rightarrow \quad |f(0)| \le 0 < 1$

So f is bounded in $[-1, 1]$.

Let $c \in [-1, 1]$ and $c \neq 0$

$$\lim_{x \to c} f(x) = \lim_{x \to c} x^2 \sin\frac{1}{x^2}$$

$$= c^2 \sin\frac{1}{c^2} = f(c)$$

So, $f(x)$ is continuous for all $c \in [-1, 1]$ except at $c \neq 0$

at $x = 0$, we have

$$f(0-0) = \lim_{h \to 0} f(0-h)$$
$$= \lim_{h \to 0} f(-h)$$
$$= \lim_{h \to 0} (-h)^2 \sin\left(\frac{1}{-h^2}\right) = 0$$

and $\quad f(0+0) = \lim_{h \to 0} f(0+h)$

$$= \lim_{h \to 0} h^2 \sin\frac{1}{h^2} = 0$$

Also $\quad f(0) = 0$

So, $\quad f(0+0) = f(0-0) = f(0)$

i.e. $f(x)$ is continues of at $x = 0$.

Thus $f(x)$ is continues in $[-1, 1]$.

Since, $f(x)$ is continue in closed interval $[-1, 1]$ so, it is uniformly continuous in $[-1, 1]$

34. See the solution of question 21.

36. $f(x) = e^x \ \forall x \in [0, \infty[$

Let $c \in [0, \infty[$ then

$$f(c) = e^c = \lim_{x \to c} f(x)$$

So, $f(x)$ is continuous in $[0, \infty[$.

Let $x_1, x_2 \in [0, \infty[$ then

$$|f(x_1) - f(x_2)| = |e^{x_1} - e^{x_2}|$$

$$= \left|\left(1 + x_1 + \frac{x_1^2}{\lfloor 2} + ...\right) - \left(1 + x_2 + \frac{x_2^2}{\lfloor 2} + ...\right)\right|$$

$$= \left|(x_1 - x_2) + \left(\frac{x_1^2 - x_2^2}{\lfloor 2}\right) + ...\right|$$

$$= \left|(x_1 - x_2) + \left(1 + \frac{x_1 + x_2}{\lfloor 2} + ...\right)\right|$$

$$\leq \delta \left|1 + \frac{x_1 + x_2}{\lfloor 2} + ...\right|$$

If $|x_1 - x_2| < \delta$

This term cannot be finite when $x_1, x_2 \to \infty$ so

$|f(x_1) - f(x_2)| \not< \varepsilon$ whenever $|x_1 - x_2| < \delta$

Thus, $f(x)$ is not uniformly continuous in $[0, \infty[$

44. Let $\varepsilon > 0$ be given and $x_1, x_2 \in [0, \infty)$ then

$$|f(x_1) - f(x_2)| = |\sin x_1 - \sin x_2|$$

$$= \left|2\cos\left(\frac{x_1 + x_2}{2}\right)\sin\left(\frac{x_1 - x_2}{2}\right)\right|$$

$$= 2\left|\cos\left(\frac{x_1 + x_2}{2}\right)\right|\left|\sin\left(\frac{x_1 - x_2}{2}\right)\right|$$

$$\leq 2 \times 1 \times \frac{1}{2}|x_1 - x_2|$$

Since, $\quad |\cos\theta| \leq 1$ and $\left|\frac{\sin\theta}{\theta}\right| \leq 1$

$$= |x_1 - x_2|$$

$< \varepsilon$, provided $|x_1 - x_2| < \varepsilon$

Now selecting $\delta = \varepsilon$ we get

$$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$$

Hence, $f(x) = \sin x$ is uniformly continuous on $[0, \infty]$.

46. Given that $\quad f(x) = \sin x^2 \ \forall x \in [0. \infty)$

Let $\varepsilon > 0$ be given and $x_1, x_2 \in [0, \infty)$ then

$$|f(x_1) - f(x_2)| = |\sin x_1^2 - \sin x_2^2|$$

$$= \left|2\cos\left(\frac{x_1^2 + x_2^2}{2}\right)\sin\left(\frac{x_1^2 - x_2^2}{2}\right)\right|$$

$$= 2\left|\cos\left(\frac{x_1^2 + x_2^2}{2}\right)\right|\left|\sin\left(\frac{x_1^2 - x_2^2}{2}\right)\right|$$

$$\leq 2 \times 1 \times \frac{1}{2}|x_1^2 - x_2^2|$$

Since, $\quad |\cos\theta| \leq 1$ and $\left|\frac{\sin\theta}{\theta}\right| \leq 1$

$$\leq |x_1 - x_2||x_1 + x_2|$$

Here $x_1, x_2 \in [0, \infty)$ so $|x_1 + x_2|$ always has no bounded value so for $|x_1 - x_2| < \varepsilon$.

We cannot always find $|f(x_1) - f(x_2)| < \varepsilon$ *i.e.* $f(x)$ is not uniformly continuous in $[0, \infty]$.

48. Given that $f(x) = \frac{1}{x}, x \geq 1$

Let $\varepsilon > 0$ be given and $x_1, x_2 \geq 1$ then

$$|f(x_1) - f(x_2)| = \left|\frac{1}{x_1} - \frac{1}{x_2}\right| = \left|\frac{x_2 - x_1}{x_1 x_2}\right|$$

$$= |x_1 - x_2| \cdot \left|\frac{1}{x_1 x_2}\right|$$

$\because \quad x_1 \geq 1$ so $\frac{1}{x_1} \leq 1$ similarly $\frac{1}{x^2} \leq 1$

So, $\quad |f(x_1) - f(x_2)| \leq |x_1 - x_2|$

$< \varepsilon$ provided $|x_1 - x_2| < \varepsilon$

Thus, we get

$|x_1 - x_2| < \delta \quad \Rightarrow \quad |f(x_1) - f(x_2)| < \varepsilon$

Here, f is uniformly continuous on $x \geq 1$.

50. $f(x) = 3x^2 + 2 \; \forall x \in [-1, 2]$

Let $\varepsilon > 0$ be given and $x_1, x_2 \in [-1, 2]$ then

$$|f(x_1) - f(x_2)| = |(3x_1^2 + 2) - (3x_2^2 + 2)|$$

$$= |3x_1^2 - 3x_2^2| = 3|x_1 + x_2||x_1 - x_2|$$

or $\quad |f(x_1) - f(x_2)| \leq 3(1 + 2)|x_1 - x_2|$

$= 9|x_1 - x_2|$

$< \varepsilon$ provided $|x_1 - x_2| < \frac{\varepsilon}{9}$

Choose $\delta = \frac{\varepsilon}{9}$ we find that

$|x_1 - x_2| < \delta \quad \Rightarrow \quad |f(x_1) - f(x_2)| < \varepsilon$

So maximum value of δ is $\frac{\varepsilon}{9}$ when $f(x)$ is uniformly continuous.

52. $f(x) = x^2$

Let $\varepsilon > 0$ be given and $x_1, x_2 \; \varepsilon \; [-2, 2]$ then

$$|f(x_1) - f(x_2)| = |x_1^2 - x_2^2|$$

$$= |x_1 - x_2||x_1 + x_2|$$

$$\leq |x_1 - x_2|(2 + 2)$$

$$= 4|x_1 - x_2|$$

Thus, $|f(x_1) - f(x_2)| < \varepsilon$, provided $4\,|x_1 - x_2| < \varepsilon$

i.e. $\quad |x_1 - x_2| < \frac{\varepsilon}{4}$

Choose $\delta = \frac{\varepsilon}{4}$ we get

$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$

Thus, $f(x)$ is uniformly continuous in $[-2, 2]$.

It can be easily seen that if $f(x)$ is uniformly continuous in $[-2, 2]$ then it is also uniformly continuous in $[-1, 1]$. Again every uniformly continuous function is continuous so $f(x)$ is continuous in $[-1, 1]$.

OOO

CHAPTER

8

Meaning of Sign of Derivatives and Darboux Theorem

DIFFERENTIABILITY

1. **Differentiable function**

   Let f be defined on I and $c \in I$, then f is said to be differentiable or derivable at c if

   $$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

   or $$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

   exists finitely and denoted by $f'(c)$.

2. **Right hand derivative**

   It is defined by

   $$Rf'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}, h > 0$$

   provided the limit exists finitely. It is also known as progressive derivative and denoted by $f'(c+0)$.

3. **Left hand derivative**

   It is defined by

   $$f'(c-0) = Lf'(c)$$
   $$= \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}, h > 0$$

   provided the limit exists finitely and also know as regressive derivative.

   Thus, function $f(x)$ is said to be differentiable at $x = c$ iff

   $$Rf'(c) = Lf'(c) = f'(c)$$

4. **Differentiability in an open interval**

   A function f defined on $I = (a, b)$ is said to be differentiable in (a, b) if it is differentiable at every point in (a, b).

5. **Differentiability in a closed interval**

   A function f defined on a closed interval $[a, b]$ is differentiable if

   (i) f is differentiable in (a, b)

   (ii) f is differentiable from the right at a

   (iii) f is differentiable from the left at b

MEANING OF THE SIGN OF DERIVATIVE

Let f be differentiable in $[a, b]$ and $c \in (a, b)$ and

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ then}$$

1. If $f'(c) > 0$ then

   $f(x) > f(c), \forall x \in (c, c + \delta)$

   and $f(x) < f(c), \forall x \in (c - \delta, c)$

2. If $f'(c) > 0$ then

   $f(x) > f(c), \forall x \in (c - \delta, c)$

   and $f(x) < f(0) \forall x \in (c, c + \delta)$

3. If $f'(a) > 0$ then

   $f(x) > f(a), \forall x \in (a, a + \delta)$

4. If $f'(a) < 0$ then

   $f(x) < f(a), \forall x \in (a, a + \delta)$

5. If $f'(b) > 0$ then

   $f(x) < f(b), \forall x \in (b - \delta, b)$

6. If $f'(b) < 0$ then

   $f(x) > f(b), \forall x \in (b - \delta, b)$

INTERMEDIATE VALUE THEOREM OR DARBOUX THEOREM

If f is finitely differentiable in a closed interval $[a, b]$ and $f'(a), f'(b)$ are of opposite signs, then there exists at least one point $c \in]a, b[$ such that $f'(c) = 0$

Results :

1. If f is, finitely differentiable on $[a, b]$ and $f'(a) \neq f(b)$, then $f'(x)$ takes all values between $f'(a)$ and $f'(b)$ at least once in (a, b).

2. If f is finitely differentiable on $[a, b]$ and $f'(x) \neq 0$ for any $x \in (a, b)$ then $f'(x)$ returns the same sign, positive or negative in (a, b).
3. If f is finitely differentiable on $I = [a, b]$, then the range $f'(I)$ of f' on I is either an interval or a singleton.
4. Continuity is necessary but not sufficient condition for differentiability.

ALGEBRA OF DERIVATIVES

If f and g are derivable at $x \in I$ then

1. $f(f+g)'(x) = f'(x) + g'(x) \ \forall x \in I$
2. $f(f-g)'(x) = f'(x).g(x) + f(x).g'(x)$
3. $(kf)'(x) = kf'(x)$
4. $\left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{(f'(x))^2}$
5. $\left(\frac{f}{g}\right)'(x) = \frac{g(x).f'(x) - f(x).g'(x)}{[g(x)]^2}$
6. $(gof)'(x) = g'(f(x)).f'(x)$
7. If f is derivable at x and is one-one on nbd of x then $(f^{-1})'(f(x)) = \frac{1}{f'(x)}$ provided $f'(x) \neq 0$

EXERCISE

MULTIPLE CHOICE QUESTIONS

Direction : Each of the following questions has four alternative answers. One of them is correct. Choose the correct answer.

1. The function $f(x) = |x|$ at $x = 0$ is :
   a. Continuous
   b. Differentiable
   c. Continuous and differentiable both
   d. None of these
2. If f is derivable on $[a,b]$ and $f'(a) \neq f(b)$ with $f'(a) < k < f'(b)$ then there exists a point $c \in (a,b)$ such that :
   a. $f(c) = k$ b. $f'(c) = k$
   c. $f(k) = c$ d. $f'(k) = c$
3. The sum and difference of two differentiable function :
   a. Differentiable
   b. May or may not be differentiable
   c. Non-differentiable
   d. None of these
4. A function $f(x)$ is differentiable at $x = c$ if i
   a. $f'(c+0)$ exist
   b. $f'(c-0)$ exist
   c. $f'(c+0) = f'(c-0)$
   d. $f'(c+0) = f'(c-0)$ and both exist
5. For differentiability, continuity is :
   a. Necessary but not sufficient
   b. Sufficient but not necessary
   c. Necessary and sufficient both
   d. None of these
6. If $f(x) = (x) - 1 \ \forall x \in R$ then $Rf'(0)$ is equal to :
   a. 0 b. 1
   c. –1 d. does not exist
7. If $f'(c) > 0$ then for any $\delta > 0$
   a. $f(x) > f(c) \ \forall x \in (c-\delta, c+\delta)$
   b. $f(x) < f(c) \ \forall x \in (c-\delta, c+\delta)$
   c. $f(x) > f(c) \ \forall x \in (c, c+\delta)$
   d. $f(x) < f(c) \ \forall x \in (c, c+\delta)$
8. If $f(x)$ be an decreasing function then :
   a. $f(x) < 0$ b. $f'(x) < 0$
   c. $f'(x) > 0$ d. $f(x) > 0$
9. If $f(x) = \begin{cases} x^2 + 3x + a & x \leq 1 \\ bx + 2 & x > 1 \end{cases}$ is differentiable at $x = 1$ then a and b are respectively :
   a. 5,3 b. 2,3
   c. 3,5 d. 3,2
10. If f is derivable on $[a,b]$ and $f'(a).f'(b) < 0$ then there exists $c \in (a,b)$ such that
   a. $f'(c) = 0$ b. $f'(a) = 0$
   c. $f'(b) = 0$ d. $f(c) = 0$

11. If $f'(c) < 0$ then for $\delta > 0$:
 a. $f(x) > f(c)\ \forall x \in (c-\delta, c)$
 b. $f(x) > f(c)\ \forall x \in (c, c+\delta)$
 c. $f(x) < f(c)\ \forall x \in (c, c+\delta)$
 d. $f(x) > f(c)\ \forall x \in (c-\delta, c+\delta)$

12. The function $f(x) = |x-1| + |x+1|\ \forall x \in R$ is :
 a. Derivable at $x = 1$
 b. Derivable at $x = -1$
 c. Derivable at $x = 1$ and -1 both
 d. Not derivable at $x = 1$ and $x = -1$

13. The function $f(x) = x^3 - 6x^2 + 12x - 4\ \forall x \in R$ is :
 a. Increasing
 b. Decreasing
 c. Neither increasing nor decreasing
 d. Bounded

14. If f is derivable on an interval $\pm$, then $f'(I)$ is :
 a. An interval only
 b. Singleton only
 c. Both interval and singleton
 d. Either an interval or singleton

15. Let f be the function defined on R by

$$f(x) = \begin{cases} x^2 \sin 1/x & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases} \quad \text{then :}$$

 a. f is derivable $\forall x \in R$
 b. f is not derivable $\forall x \in R$
 c. f' is continuous at $x = 0$
 d. None of these

16. If f and g are continuous and differentiable with $f'(x) = g'(x)$ on (a,b) then :
 a. $f'(x) \neq g'(x)\ \forall x \in (a,b)$
 b. $f(x) = g(x)\ \forall x \in (a,b)$
 c. f and g differ only by a constant
 d. None of these

17. If $f'(x) \neq 0\ \forall x \in (a,b)$ then the sign of $f'(x)$ is :
 a. Positive only
 b. Negative only
 c. Retains the same sign
 d. Positive and negative both

18. If $f'(c) > 0$ then $f(c)$ is :
 a. Negative
 b. Positive
 c. May be positive or negative
 d. None of these

19. If $f'(a) > 0$ then for $\delta > 0$:
 a. $f(x) > f(a)\ \forall x \in (a, a+\delta)$
 b. $f(x) > f(a)\ \forall x \in (b-\delta, b)$
 c. $f(x) < f(b)\ \forall x \in (b-\delta, b)$
 d. $f(x) > f(a)\ \forall x \in (a,b)$

20. The function $f(x) = x\,|x|$ is :
 a. Not monotonic
 b. Differentiable $\forall x \in R$
 c. Strictly decreasing function
 d. Differentiable $\forall x \in R$ except at $x = 0$

21. If $f(x) = x^2\ \forall x \in Q$ and $f(x) = 0$ otherwise then :
 a. f is differentiable at $x = 0$
 b. f is differentiable at $x = 1$
 c. f is continue but not differentiable at $x = 0$
 d. None of these

22. If f is derivable on $[a,b]$, $f(a) = f(b) = 0$ and $f'(a).f'(b) > 0$ then $\exists\, c \in (a,b)$ such that :
 a. $f(c) = 0$ b. $f(c) > 0$
 c. $f(c) < 0$ d. $f'(c) = 0$

23. If $f(x)$ is continuous at a point then at that point $f(x)$ is :
 a. Differentiable
 b. Not differentiable
 c. May be differentiable or not differentiable
 d. None of these

24. If $f'(b) > 0$ then for $\delta > 0$:
 a. $f(x) > f(b)\ \forall x \in (b-\delta, b)$
 b. $f(x) < f(b)\ \forall x \in (b-\delta, b)$
 c. $f(x) > f(b)\ \forall x \in (a,b)$
 d. $f(x) < f(b)\ \forall x \in (a,b)$

25. The function $f(x) = -2x^3 - ax^2 - 12x + 1$ is an increasing function in :
 a. $(-2, -1)$ b. $(-2, 1)$
 c. $(-1, 2)$ d. $(1,2)$

26. If $f(x)$ an increasing function then :
a. $f'(x) > 0$ b. $f'(x) > 0$
c. $f'(x) = 0$ d. $f(x) > 0$

27. The function $f(x) = |x|^3 \ \forall x \in R$, then at $x = 0$ $f(x)$ is :
a. Continuous but not differentiable
b. Twice differentiable
c. Thrice differentiable
d. None of these

28. If $f'(x) < 0$ then $f(x)$ is :
a. Decreasing function
b. Increasing function
c. Oscillatory function
d. Bounded function

29. The function $f(x) = e^x \ \forall x \in R$ is :
a. Strictly decreasing
b. Strictly increasing
c. Bounded
d. Oscillatory

30. If $f(x) = \begin{cases} a + \sin^{-1}(x+b) & x \geq 1 \\ x & x \leq 1 \end{cases}$ is differentiable at $x = 1$ then :
a. $a = -1, b = -1$ b. $a = 1, b = -1$
c. $a = 1, b = 1$ d. None of these

31. If $f(x) = |x|$ is defined on $[-2, 2]$ then the point at which f is differentiable are :
a. 0,1 b. −1, 0, 1
c. 0,1,2 d. None of these

32. The function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is decreasing in :
a. (2,3) b. $(-\infty, 2)$
c. $(3, \infty)$ d. None of these

33. $f(x) = |x + 2|$ is not differentiable at :
a. $x = 0$ b. $x = 2$
c. $x = -2$ d. $x = -1$

34. If f is an increasing function then :
a. f is decreasing b. $-f$ is increasing
c. f is constant d. $-f$ is constant

35. If $f : R \to R$ be define by $f(x) = 2x^2 + 3x + 4$ if $x \in]-\infty, 1[$ and $f(x) = px + 9 - x$ if $x \in [1, \infty[$ is differentiable on R then p must be :
a. 7 b. 5
c. 3 d. 1

36. The function $f(x)$ is strictly decreasing on R if $x, y \in R$:
a. $x < y \Rightarrow f(y) < f(x)$
b. $x < y \Rightarrow f(y) \geq f(x)$
c. $x < y \Rightarrow f(y) < f(x)$
d. $x < y \Rightarrow f(x) < f(y)$

37. The function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is increasing in :
a. $[0, 1] \cup [2, \infty[$ b. $]-\infty, 1] \cup [2, \infty[$
c. $[-\infty, 2] \cup [3, \infty[$ d. None of these

38. The derivative of the function $f(x) = x^{2n-1}$ is :
a. Constant function b. Even function
c. Odd function d. None of these

39. If f is increasing function on R then for $x, y \in R$ we have :
a. $x > y \Rightarrow f(x) \geq f(y)$
b. $x < y \Rightarrow f(x) \geq f(y)$
c. $x < y \Rightarrow f(x) > f(y)$
d. None of these

40. The derivative of the function $f(x) = \sin nx$ is :
a. Even function b. Odd function
c. Constant function d. None of these

41. Which of the following statement is not true :
a. Every differentiable function is continuous
b. Every constant function is differentiable
c. If $f'(x) = 0$ at each point in (a, b) then $f(x) = c$
d. If f is differentiable in $[a, b]$ and $f'(a), f'(b)$ have opposite signs then there exist no any point $c \in (a, b)$ such that $f'(c) = 0$

42. If $f(a) = f(b)$ then between a and b $f(x)$ is :
a. Maximum
b. Minimum
c. Maximum or minimum
d. Neither maximum nor minimum

43. Which of the following is not true ?
 a. The polynomial function is differentiable
 b. If $f.g$ is differentiable at $c \in D$ then f is also differentiable at c
 c. If $f'(x) \neq f'(b)$ for differentiable function f then $f'(x)$ takes all values between $f'(a)$ and $f'(b)$ at least once in (a, b)
 d. The identity function is differentiable function

44. If $f(x) = x\tan^{-1}\left(\frac{1}{x}\right), x \neq 0$ and $f(0) = 0$ then the value of $f'(0+0)$ is :
 a. $\frac{-\pi}{2}$ b. $\frac{\pi}{2}$
 c. 1 d. 0

45. If $f(x) = x^2\sin\left(\frac{1}{x}\right), x \neq 0$ and $f(0) = 0$ then :
 a. f is differentiable at $x = 0$
 b. $f'(0+0) = 0$
 c. $f'(0-0) = 0$
 d. All the above

46. If $f(x) = x^n$ then :
 a. $f(x)$ is not differentiable at $x = 0$
 b. $f'(0+0) \neq f'(0-0)$
 c. $f(x)$ is differentiable at $x = 0$
 d. None of these

47. If f and g are differentiable function (a, b) and
$$f'(x) = g'(x) \ \forall x \in (a, b) \text{ then :}$$
 a. $f(x) = cg(x)$ b. $f(x) = g(x)/c$
 c. $f(x) = g(x) + c$ d. None of these

48. If $|y| = 2y - x$ then for negative value of y, $\frac{dy}{dx}$ is :
 a. 2 b. 3
 c. $\frac{1}{2}$ d. $\frac{1}{3}$

49. If $f(x) = x^{\frac{1}{a}-1}\cos\frac{1}{x}, x \neq 0$ and $f(0) = 0$ then $f(x)$ is differentiable at $x = 0$ when :
 a. $a < \frac{1}{2}$ b. $a \leq \frac{1}{2}$
 c. $a \geq \frac{1}{2}$ d. $a > \frac{1}{2}$

50. If $f(x)$ is differentiable in $[a, b]$ and $f(a) = f(b) = 0$ then :
 a. $f(x) = 0 \ \forall x \in [a, b]$
 b. $f(x) = 0 \ \forall x \in]a, b[$
 c. $f'(x) = 0 \ \forall x \in]a, b[$
 d. $f'(x) = 0$ for some $x \in]a, b[$

51. If $f(0) = 0$ and $f''(x) > 0 \ \forall x \in (0, \infty)$ then $\frac{f(x)}{x}$ is :
 a. Decreasing in $(0, \infty)$
 b. Increasing in $(0, \infty)$
 c. Oscillatory
 d. None of these

52. If $f(x + y) = f(x).f(y) \ \forall x \in R$ and $f(x)$ is continuous at $x = 0$ then $f(x)$ is :
 a. Differentiable at $x = 0$
 b. Not differentiable at $x = 0$
 c. Differentiable for all $x \in R$
 d. None of these

53. If $f'(a).f'(b) < 0$ and $f(x)$ is differentiable in $[a, b]$ then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$, then c is called a point of :
 a. Minima
 b. Maxima
 c. Either minima or maxima
 d. None of these

54. The function $f(x) = \tan x$ on $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ is :

[Kanpur 2018]

 a. Continuous
 b. Bounded
 c. Continuous are bounded
 d. Discontinuous

55. If $f(x) = 0 \ \forall x \in (a, b)$ then f is :
 a. Constant
 b. Strictly increasing
 c. Strictly decreasing
 d. None of these

56. The function $f(x) = \begin{cases} \frac{|x|}{x} \text{ when } x \neq 0 \\ 1 \text{ when } x = 0 \end{cases}$ has :

[Kanpur 2018]

a. Removable discontinuity at $x = 0$

b. Discontinuity of first kind at $x = 0$

c. Discontinuity of second kind at $x = 0$

d. Discontinuous at $x = 0$

57. If $f(x)$ is continuous in $[a, b]$ then it is bounded in that interval is called :

a. Reimann theorem

b. Rolle's theorem

c. Lagrange's theorem

d. Boundedness theorem

58. If f is continuous in $[a, b]$ and $f(a).f(b) < 0$ then for at least one point $c \in [a, b]$: **[Kanpur 2018]**

a. $f(a) = f(c) = f(b)$ b. $f(c) = 0$

c. $f'(c) = 0$ d. All the above

59. If $f(x) = x^4 - 62x^2 + px + 9$ attains its maximum value at $x = 1$ in [0, 2] then p is equal to :

a. 100 b. 62

c. 120 d. 130

60. The function $f(x) = \begin{cases} x^2 + 3x + a, \text{ if } x \leq 1 \\ bx + 2, \text{ if } x > 1 \end{cases}$ is differentiable at $x = 1$ then the value of a and b are :

[Kanpur 2019]

a. $a = 2, b = 3$ b. $a = 5, b = 3$

c. $a = -3, b = -5$ d. $a = 3, b = 5$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(a)	**2.**	(b)	**3.**	(a)	**4.**	(d)	**5.**	(a)	**6.**	(b)	**7.**	(c)	**8.**	(b)	**9.**	(c)	**10.**	(a)
11.	(c)	**12.**	(d)	**13.**	(a)	**14.**	(d)	**15.**	(a)	**16.**	(c)	**17.**	(c)	**18.**	(c)	**19.**	(a)	**20.**	(b)
21.	(a)	**22.**	(a)	**23.**	(c)	**24.**	(b)	**25.**	(a)	**26.**	(a)	**27.**	(b)	**28.**	(a)	**29.**	(b)	**30.**	(b)
31.	(d)	**32.**	(a)	**33.**	(c)	**34.**	(a)	**35.**	(a)	**36.**	(c)	**37.**	(c)	**38.**	(b)	**39.**	(a)	**40.**	(a)
41.	(d)	**42.**	(c)	**43.**	(b)	**44.**	(b)	**45.**	(d)	**46.**	(c)	**47.**	(c)	**48.**	(d)	**49.**	(a)	**50.**	(d)
51.	(b)	**52.**	(a)	**53.**	(c)	**54.**	(a)	**55.**	(a)	**56.**	(b)	**57.**	(d)	**58.**	(b)	**59.**	(c)	**60.**	(d)

HINTS AND SOLUTIONS

1. $f(x) = |x|$ at $x = 0$

$$f(0+0) = \lim_{h\to 0} f(0+h)$$
$$= \lim_{h\to 0} |h| = 0$$
$$f(0-0) = \lim_{h\to 0} f(0-h)$$
$$= \lim_{h\to 0} |-h| = 0$$
$$f(0) = 0$$

Since, $f(0+0) = f(0-0) = f(a)$

So, $f(x)$ is continuous at $x = 0$

$$f'(0+0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$
$$= \lim_{h\to 0} \frac{h-0}{h} = 1$$
$$f'(0-0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h}$$
$$= \lim_{h\to 0} \frac{h-0}{-h} = l$$

$\because$ $f'(0+0) \neq f'(0-0)$

So, $f(x)$ is not differentiable at $x = 0$.

6. $f(x) = |x| - 1 \ \forall x \in R$

So, $f(0) = -1$

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$
$$= \lim_{h\to 0} \frac{|h| - 1 + 1}{h}$$
$$= \frac{h}{h} = 1$$

9. Here $f(x) = \begin{cases} x^2 + 3x + a & , \ x \le 1 \\ bx + 2 & , \ x > 1 \end{cases}$

is differentiable at $x = 1$ so

$$f'(1+0) = f'(1-0)$$

Now $f'(1+0) = b$

and $f'(1-0) = 2 + 3 = 5$

So, $b = 5$

Also $f(x)$ is continue at $x = 1$

So $f(1+0) = f(1-0)$

or $b + 2 = 4 + a \Rightarrow b - a = 2$

or $a = b - 2 = 5 - 2 = 3$

$\therefore$ $a = 3, b = 5$

12. $f(x) = |x-1| + |x+1| \ \forall x \in R$

or $f(x) = \begin{cases} -2x & x < -1 \\ +2 & -1 \le x < 1 \\ 2x & x \ge 1 \end{cases}$

so $f'(1+0) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$

$= \lim_{h\to 0} \frac{2(1+h)-2}{h}$

$f'(1+0) = 2$

Similarly, $f'(1-0) = 0$

Since, $f'(1+0) \neq f'(1-0)$

So, f is not differentiable at $x = 1$.

Now at $x = -1$

$f'(-1+0) = \lim_{h\to 0} \frac{f(-1+h)-f(-1)}{h}$

$= \lim_{h\to 0} \frac{+2-2}{h}$

$f'(-1+0) = 0$

Similarly $f'(-1-0) = -2$

Since, $f'(-1+0) \neq f'(-1-0)$

So, f is not differentiable at $x = -1$.

13. Given that

$$f(x) = x^3 - 6x^2 + 12x - 4 \ \forall x \in R$$

then $f'(x) = 3x^2 - 12x + 12$

$= 3(x^2 - 4x + 4)$

$= 3(x-2)^2 \ \forall x \in R$

Thus, $f'(x) > 0$ when $x \neq 2$ and $f'(2) = 0$. Consider $[c, 2], c \in R$ then f is continuous in $[c, 2]$ and $f'(x_1) > 0$ for all $x \in]c,2[$. So f is strictly increasing in $[c, 2[$ similarly f is strictly increasing in $[2, d] : d \in R$.

20. $f(x) = x|x| = \begin{cases} x.x = x^2 & x > 0 \\ x(-x) = -x^2 & x < 0 \end{cases}$

$f'(0+0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0} \frac{h^2-0}{h} = 0$

$f'(0-0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$

$= \lim_{h\to 0} \frac{-h^2-0}{-h} = 0$

$\because$ $f'(0+0) = f'(0-0)$

So, $f(x)$ is differentiable at $x = 0$ *i.e.* $\forall x \in R$.

25. $f(x) = -2x^3 - 9x^2 - 12x + 1$

$f'(x) = -6x^2 - 18x - 12$

$= -6(x^2 + 3x + 2)$

or $f'(x) = -6(x+2)(x+1)$

Here $f'(x) > 0$ only when $x \in (-2, -1)$

So, $f(x)$ increasing function.

27. $f(x) = |x|^3 = \begin{cases} x^3 & \text{if } x > 0 \\ -x^3 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$

$f'(x) = \begin{cases} 3x^2 & \text{if } x > 0 \\ -3x^2 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$

and $f''(x) = \begin{cases} 6x & \text{if } x > 0 \\ -6x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$

and $f'''(x) = \begin{cases} 6 & \text{if } x > 0 \\ -6 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$

So, $f(x)$ is twice differentiable at origin but not trice differentiable.

30. $f(x) = \begin{cases} a + \sin^{-1}(x+b) & x \geq 1 \\ x & x \leq 1 \end{cases}$ and $f(x)$ is :

differentiable *i.e.* continues at $x = 1$ so

$a + \sin^{-1}(1+b) = 1$

due to continuity at $x = 1$

and $\frac{1}{\sqrt{1-(1+b)^2}} = 1$

due to differentiability at $x = 1$

So, $\sqrt{1+(1+b)^2} = 1$

$\Rightarrow$ $1-(1+b)^2 = 1$

$\Rightarrow$ $b = -1$

and by first equation $a = 1$.

32. $f(x) = 2x^3 - 15x^2 + 36x + 1$

$f'(x) = 6x^2 - 30x + 36$

$= 6(x^2 - 5x + 6)$

or $\quad f'(x) = 6(x-2)(x-3)$

Thus, $f(x)$ is decreasing when $f'(x) < 0$ *i.e.* when $x \in (2, 3)$.

35. $f(x) = 2x^2 + 3x + 4 \;\forall x \in]-\infty, 1[$

and $\quad f(x) = px + 9 - k \;\forall x \in]1, \infty[$

and f is differentiable on R *i.e.* at $x = 1$ so

$$\text{L.H.D} = \text{R.H.D at } x = 1$$

i.e., $\quad 4.1 + 3 = p \Rightarrow p = 7$

37. $f(x) = 2x^3 - 19x^2 + 36x + 1$

then $\quad f'(x) = 6x^2 - 30x + 36$

$$= 6(x-2)(x-3)$$

So, $\quad f'(x) > 0$ for $x < 2$,

$f'(x) < 0$ for $2 < x < 3$

$f'(x) > 0$ for $x > 3$

$f'(x) = 0$ for $x = 2$ and 3

Hence, $f'(x)$ is positive in $]-\infty, 2[$ and $]3, \infty[$ and negative in $]2,3[$. This f is monotonically increasing in $]-\infty, 2[$ and $]3, \infty[$ and monotonically decreasing in $]2,3[$

44. $f(x) = x\tan^{-1}\left(\dfrac{1}{x}\right)$

$$x \neq 0 \text{ and } f(0) = 0$$

Since, $\quad f'(0+0) = \lim\limits_{h\to 0}\dfrac{f(0+h) - f(0)}{h}$

$$= \lim_{h\to 0}\frac{f(h) - f(0)}{h}$$

$$= \lim_{h\to 0}\frac{h\tan^{-1}\left(\frac{1}{h}\right) - 0}{h}$$

$$= \lim_{h\to 0}\tan^{-1}\left(\frac{1}{h}\right)$$

$$= \tan^{-1}(\infty) = \frac{\pi}{2}$$

45. $f(x) = x^2\sin\left(\dfrac{1}{x}\right)$

$$x \neq 0 \quad \text{and} \quad f(0) = 0$$

$$f'(0+0) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0}\frac{f(h) - f(0)}{h}$$

$$= \lim_{h\to 0}\frac{h^2\sin\left(\frac{1}{h}\right) - 0}{h}$$

$$= \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$$

$$f'(0-0) = \lim_{h\to 0}\frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{h^2\sin\left(\frac{-1}{h}\right) - 0}{-h}$$

$$= \lim_{h\to 0} -h\sin\left(\frac{-1}{h}\right) = 0$$

Since, $\quad f'(0+0) = f'(0-0)$

So, $f(x)$ is differentiable at $x = 0$.

48. Given that $\quad |y| = 2y - x$

If $y > 0$ then $|y| = y$

i.e. $\quad y = 2y - x \quad\Rightarrow\quad y = x$

If $y < 0$ then $|y| = -y$

i.e. $\quad -y = 2y - x \quad\Rightarrow\quad y = \dfrac{x}{3}$

So when $y < 0$, $\dfrac{dy}{dx} = \dfrac{1}{3}$

54. Given that

$$f(x) = \tan x \;\forall x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

Let $a \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$ then

$$f(a+0) = \lim_{h\to 0} f(a+h)$$

$$= \lim_{h\to 0}\tan(a+h) = \tan a$$

$$f(a-0) = \lim_{h\to 0} f(a-h)$$

$$= \lim_{h\to 0}\tan(a-h) = \tan a$$

Thus, $\quad f(a+0) = f(a-0) = \tan a$

i.e., $\;f(x)$ is continuous at $x = a \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$. But a is arbitrary so $f(x)$ is continuous in whole interval $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$.

56. $$f(x) = \begin{cases} \frac{(x)}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases} = \begin{cases} 1 & , x > 0 \\ -1 & , x < 0 \\ 0 & , x = 0 \end{cases}$$

At $x = 0$

$$f(0+0) = \lim_{h \to 0} f(0+h) = 1$$

$$f(0-0) = \lim_{h \to 0} f(0-h) = -1$$

Thus, $f(0+0) \neq f(0-0)$

So, $f(x)$ is discontinuity at $x = 0$. Also $f(x)$ has discontinuity of first kind at $x = 0$.

59. $f(x) = x^4 - 62x^2 + px + 9$

then $f'(x) = 4x^3 - 124x + p$

or $f'(1) = 4 - 124 + p = 0$

$\Rightarrow$ $p = 120$

❍❍❍

CHAPTER

9

The Riemann Integral

INTRODUCTION

The German mathematician G.F.B Riemann gave the first rigorous arithmetic treatment of definite integral which is free from geometrical concepts. It covered only bounded functions. After that Cauchy extended this definition to unbounded functions. Later that Lebesgue introduced the integral on a firm foundation with many refinements and generalisations.

PARTITIONS AND RIEMANN SUMS

1. **Partition :** If $I = [a, b]$ is a closed and bounded interval in R, then by a partition of I we mean a finite set of real numbers $P = \{x_0, x_1, x_2, \ldots x_n\}$ such that $a = x_0 < x_1 < x_2 < \ldots . < x_n = b$

   The closed sub-intervals

   $I_1 = [x_0, x_1], I_2 = [x_1, x_2], \ldots I_n = [x_{n-1}, x_n]$

   determined by P constitute the segments of the partition.

   The length of the segment $[x_{r-1}, x_r]$ is defined by $\Delta x_r = x_r - x_{r-1}$ for $r = 1, 2, \ldots, n$.

   The norm of a partition P is the greatest of the lengths of the segments of a partition P and it is denoted by $||p||$ *i.e.*,

   $$||p|| = \max.(\Delta x_r : r = 1, 2, \ldots n)$$

2. **Refinement :** The partition P^* is called a refinement of another partition P or P^* is finer than P iff $P^* \supset P$, *i.e.* every point of P is used in the construction of P^*.

   P^* is called the common refinement of two partitions P_1 and P_2 if $P^* = P_1 \cup P_2$.

3. **Lower and upper Riemann sums**

   Let f be a bounded function defined on a bounded interval $[a, b]$ and

   $$P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$$

   be any partition of $[a, b]$. Also let M_r and m_r be the supremum and infimum of the function f or $I_r = [x_{r-1}, x_r]$ for $r = 1, 2, \ldots, n$ then

   $$M_r = \sup\{f(x) : x_{r-1} \le x \le x_r\}$$

   and $$m_r = \inf\{f(x) : x_{r-1} \le x \le x_r\}$$

   Then the sums $U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r$

   and $L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r$ are called the upper Riemann sum (upper Darboux sum) and Lower Riemann sum (lower Darboux sum) of f on $[a, b]$ respectively.

Results :

1. $L(P, f) \le U(P, f)$
2. Let f be a bounded function defined on $[a, b]$ and let m and M be the infimum and supremum of $f(x)$ in $[a, b]$. Then for any partition P of $[a, b]$

   $m(b - a) \le L(P, f) \le U(P, f) \le M(b - a)$
3. If $f : [a, b] \to R$ is bounded function then

   $$U(P, -f) = -L(P, f)$$

   and $$L(P, -f) = -U(P, f)$$
4. Let f be a bounded function defined on $[a, b]$ and let P be a partition of $[a, b]$. If P^* is a refinement of P, then

   $$L(P^*, f) \ge L(P, f)$$

   and $$U(P^*, f) \le U(P, f)$$
5. If p_1 and p_2 be any two partitions of $[a, b]$ then

   $$U(P, f) \ge L(P_2, f)$$

6. Let f and g be bounded functions defined on $[a, b]$ and let P be any partition of $[a, b]$ then

$$L(P, f+g) \geq L(P, f) + L(P, g)$$

and $$U(P, f+g) \leq U(P, f) + U(P, g)$$

UPPER AND LOWER RIEMANN INTEGRALS

Let f be a real valued bounded function defined on $[a, b]$ then

1. The upper Riemann integral (upper R-integral) of f over $[a, b]$ is the infimum of $U(P, t)$ over all partitions $P \in P[a, b]$ and denoted by $\int_a^{-b} f(x)\,dx$.

   Thus, $\int_a^{-b} f(x)\,dx = \inf\{U(P, f) : P$ is partition of $[a, b]\}$

2. The lower Riemann integral (lower R-integral) of f over $[a, b]$ is the supremum of $L(P, f)$ over all partitions $P \in P[a, b]$ and denoted by $\int_a^{b} f(x)\,dx$.

   Thus, $\int_a^{b} f(x)\,dx = \sup\{L(P, f) : P$ is a partition of $[a, b]\}$.

Results :

1. $\int_{-a}^{b} f(x)\,dx \leq \int_a^{-b} f(x)\,dx$

2. $\int_{-a}^{b} (-f) = -\int_a^{-b} f$ and $\int_a^{-b} (-f) = -\int_{-a}^{b} f$

3. **Darboux Theorem :**

   Let f be a bounded function defined on $[a, b]$. Then to every $\varepsilon > 0$ there corresponds $\delta > 0$ such that

   $$U(P, f) < \int_a^{-b} f + \varepsilon \text{ and } L(P, f) > \int_{-a}^{b} f - \varepsilon$$

   for all partitions P with $||P|| \leq \delta$.

4. If f be bounded on $[a, b]$ and P is a partition of $[a, b]$ then

   $$\lim_{||P|| \to 0} L(P, f) = \int_{-a}^{b} f$$

   and $$\lim_{||P|| \to 0} U(P, f) = \int_a^{-b} f.$$

R-INTEGRABILITY

Let f be a bounded function defined on the bounded interval $[a, b]$, then f is called Riemann integrable (R-integrable) on $[a, b]$ iff

$$\int_{-a}^{b} f = \int_a^{-b} f$$

Their common value is called the R-integrable of f on $[a, b]$ and denoted by $\int_a^b f$.

The class of all Riemann integrable functions on $[a, b]$ is denoted by $R[a, b]$. The numbers a and b are called the lower and upper limits of integration.

Second definition of R-integral

A function f defined on $[a, b]$ is said to be R-integrable over $[a, b]$ iff for energy $\varepsilon > 0$, $\exists \delta > 0$ and a number I such that for every partition

$$P = \{a = x_0, x_1, x_2, \ldots x_n = b\}$$

with $||P|| \leq \delta$ and for every choice of $\xi_r \in [x_{r-1}, x_r]$,

$$\left| \sum_{r=1}^{n} f(\xi_r)(x_r - x_{r-1}) - I \right| < \varepsilon$$

and I is said to be R-integral of f over $[a, b]$ *i.e.,*

$$I = \int_a^b f(x)\,dx$$

Results :

1. Every constant function is R-integrable.
2. Existence of R-integral.

   A necessary and sufficient condition for R-integrability of a bounded function $f : [a, b] \to R$ over $[a, b]$ is that for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that for P and all its refinements.

   $$0 < U(P, f) - L(P, f) < \varepsilon$$

3. If f is continuous on $[a, b]$ then, $f \in R[a, b]$.
4. If f is monotonic on $[a, b]$, then $f \in R[a, b]$.
5. If the set of points of discontinuity of a bounded function f defined on $[a, b]$ is finite then $f \in R[a, b]$.
6. If the set of points of discontinuity of a bounded function f defined on $[a, b]$ has only a finite number of limit points then $f \in R[a, b]$.

ALGEBRAIC PROPERTIES OF THE R-INTEGRAL

1. Let $f:[a,b]\to R$ be an R-integrable function on $[a, b]$. If k is any constant in R, then kf is also R-integrable and
$$\int_a^b kf = k\int_a^b f$$
2. If f and g are R-integrable functions on $[a, b]$ then the function $f \pm g$ is also R-integrable in $[a, b]$ and
$$\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g$$
3. If f, g are R-integrable on $[a, b]$ and k_1, k_2 are any two constants, then the function $k_1 + k_2 g$ is also R-integrable on $[a, b]$ and
$$\int_a^b (k_1 f + k_2 g) = k_1\int_a^b f + k_2\int_a^b g$$
4. Let $I = [a, b]$ and $f: I \to R$ be R-integrable on I. If $f(x) \geq 0 \ \forall x \in I$ then
$$\int_a^b f \geq 0$$
5. Let $f, g: I \to R$ be R-integrable on I.
If $f(x) \geq g(x) \ \forall x \in I$, then
$$\int_a^b f \geq \int_a^b g$$
6. Let $f: I \to R$ be R-integrable on $I = [a, b]$.
If $m \leq f(x) \leq M \ \forall x \in I$, then
$$m(b-a) \leq \int_a^b f \leq \mu(b-a)$$
7. Let $f: I \to R$ be R-integrable on $I = [a, b]$.
If $a < c < b$, then f is R-integrable on $[a, c]$ and $[c, b]$ and
$$\int_a^b f = \int_a^c f + \int_c^b f$$
8. If f is R-integrable on $I = [a, b]$, then $|f|$ is also R-integrable on $[a, b]$ and
$$\left|\int_a^b f\right| \leq \int_a^b |f|$$
9. If f is an R-integrable function on $[a, b]$, then f^2 is also R-integrable on $[a, b]$.
10. If f and g are R-integrable functions on $[a, b]$ then fg is also R-integrable on $[a, b]$.
11. If f, g are R-integrable on $[a, b]$ and $|g(x)| \geq k, k > 0, \ \forall x \in [a, b]$, then f/g is R-integrable on $[a, b]$.

FUNDAMENTAL AND MEAN VALUE THEOREM OF INTEGRAL CALCULUS

1. **Fundamental theorem of integral calculus**

   If f is bounded and Riemann integrable on $[a, b]$ and if there is a differential function F on $[a, b]$ such that $F' = f$, then
$$\int_a^b f(x)\,dx = F(b) - F(a)$$
2. **First mean value theorem**

   If $f \in R[a, b]$, then there exists a number μ lying between the bounds m and M of f on $[a, b]$ such that $\int_a^b f(x)\,dx = \mu(b-a)$

   Moreover if first continuous on $[a, b]$, then
$$\int_a^b f(x)\,dx = (b-a)f(c),\ a \leq c \leq b$$
3. **Second mean value theorem**

   If $f, g \in R[a, b]$ and $g(x) \geq 0$ or $\leq 0, \forall x \in [a, b]$, then there exists a number in with $m \leq \mu \leq M$ such that
$$\int_a^b f(x)g(x)\,dx = \mu\int_a^b g(x)\,dx,$$
   where m, M are the bounds of f on $[a, b]$.
4. If $f, g \in R[a, b], f$ is continuous on $[a, b]$ and $g(x) \geq 0$ or $\leq 0, \ \forall x \in [a, b]$, then there exists a point $c \in [a, b)$ such that
$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$$
5. **Bonnet's mean value theorem**

   Let $g \in R[a, b]$ and f be monotonic and non-negative on $[a, b]$. Then for some ξ or $n \in [a, b]$.
$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^\xi g(x)\,dx$$

or $\int_a^b f(x)g(x)\,dx = f(b)\int_\eta^b g(x)\,dx$

according as f is monotonically non-increasing or non-decreasing on $[a, b]$.

6. Let $g \in R[a, b]$ and f is bounded and monotonic on $[a, b]$ then

$$\int_a^b fg = f(a) = \int_a^\xi g + f(b)\int_\xi^b g$$

EXERCISE

MULTIPLE CHOICE QUESTIONS

Direction : Each of the following questions has four alternative answers. One of them is correct. Choose the correct answer.

1. The partition P^* is the refinement of another partition P iff :
   a. $P^* \subset P$ b. $P^* \supset P$
   c. $P^* = P$ d. None of these
2. If f is bounded and P be any partition of $[a,b]$ with P^* is the refinement of p then : **[Meerut 2017]**
   a. $U(P^*,f) = U(P,f)$ b. $U(P^*,f) \geq U(P,f)$
   c. $U(P^*,f) \leq U(P,f)$ d. $L(P^*,f) \leq L(P,f)$
3. If $f:[a,b] \to R$ is a bounded function then :
   a. $U(P,-f) = L(P,f)$
   b. $U(P,f) = -L(P,f)$
   c. $U(P,-f) = L(P,-f)$
   d. $U(P,-f) = -L(P,f)$
4. If Δx_r is the length of the segment. $[x_{r-1}, x_r]$ then the norm of partition P *i.e.* $||P||$ is defined as :
   a. max. $\{\Delta x_r : r = 1, 2, \ldots n\}$
   b. min. $\{\Delta x_r ; r = 1, 2, \ldots n\}$
   c. $\{\Delta x_r : r = 1, 2, \ldots n\}$
   d. Both (a) and (b)
5. If f is bounded function and P be any partition of $[a,b]$ then :
   a. $U(P,f) \geq L(P,f)$ b. $U(P,f) = L(P,f)$
   c. $U(P,f) \leq L(P,f)$ d. $U(P,f) = -L(P,f)$
6. Which of the following is true :
   a. $\int_{-a}^b f \geq \int_a^{-b} f$ b. $\int_{-a}^b f = \int_a^{-b} f$
   c. $\int_{-a}^{-b} f \geq \int_{-a}^{-b} f$ d. None of these
7. If f is bounded and P is a partition of $[a,b]$ then $\lim_{||P|| \to 0} L(P,f)$ is :
   a. $\int_{-a}^b f$ b. $\int_a^{-b} f$
   c. $\int_a^b f$ d. $-\int_a^b f$
8. P^* is common refinement of two partitions P_1 and P_2 if :
   a. $P^* = P_1 + P_2$ b. $P^* = P_1 \cap P_2$
   c. $P^* = P_1 \cup P_2$ d. $P^* = P_1 - P_2$
9. sup $\{L(P,f) : P$ is a partition of $[a,b]\}$ is defined by :
   a. $\int_a^b f\,dx$ b. $\int_{-a}^b f\,dx$
   c. $\int_a^{-b} f\,dx$ d. None of these
10. The statement that $\int_a^b f$ exists indicate that f is :
    a. Bounded only
    b. Integrable only
    c. May be bounded or integrable
    d. Bounded and integrable both
11. If f is bounded and P be the partition of $[a,b]$ then for supremum M and infimum m the form result is :
    a. $m(b-a) \geq M(b-a)$
    b. $m(b-a) \leq M(b-a)$
    c. $m(b-a) = M(b-a)$
    d. None of these
12. If f is defined on $[a,b]$ by $f(x) = k\ \forall x \in [a,b]$ where k is constant then :
    a. $f \in R[a,b]$
    b. $f \notin R[a,b]$
    c. f may or may not be R integrable
    d. None of these

13. The oscillatory sum $\omega(P, f)$ of a bounded function over a partition p is defined by :

a. $\sum_{r=1}^{n} \mu_r \Delta x_r$ b. $\sum_{r=0}^{n} \mu_r \Delta x_r$

c. $U(P, f) + L(P, f)$ d. $U(P, f) - L(P, f)$

14. If f is bounded over $[a, b]$ then for $\varepsilon > 0$ there exists a $\delta > 0$ over all partitions P with $||P|| < \delta$ such that :

a. $L(P, f) > \int_{-a}^{b} f - \varepsilon$ b. $L(P, f) > \int_{a}^{-b} f - \varepsilon$

c. $L(P, f) < \int_{-a}^{b} f - \varepsilon$ d. None of these

15. If f is bounded and P^* is the refinement of P which is a partition of $[a, b]$ then : **[Meerut 2017]**

a. $L(P, f) \le L(P^*, f)$ b. $L(P, f) \ge L(P^*, f)$

c. $L(P, f) = L(P^*, f)$ d. None of these

16. If $f : [a, b] \to R$ is a bounded function then :

a. $L(P_1 - f) = -L(P, f)$ b. $L(P, -f) = -U(P, f)$

c. $L(P, -f) = U(P, f)$ d. $L(P, f) = -U(P, f)$

17. If f is bounded on $[a, b]$ and P is a partition of $[a, b]$, then $\lim_{||p|| \to 0} U(P, f)$ is equal to :

a. $\int_{a}^{b} f$ b. $\int_{-a}^{b} f$

c. $\int_{a}^{-b} f$ d. $-\int_{a}^{-b} f$

18. If $f : [a, b] \to R$ is bounded then f is R-integrable on $[a, b]$ iff for every $\varepsilon > 0$ there is a partition P of $[a, b]$ such that $U(P, f) - L(P, f)$ is :

a. Equal to ε b. Less then ε

c. Greater then ε d. None of these

19. If f is bounded then for $\varepsilon > 0$ there exists $\delta > 0$ over all partitions P with $||P|| \le \delta$ such that :

a. $U(P, f) > \int_{a}^{-b} f + \varepsilon$ b. $L(P, f) < \int_{-a}^{b} f - \varepsilon$

c. $U(P, f) < \int_{a}^{-b} f + \varepsilon$ d. $U(P, f) < \int_{-a}^{b} f + \varepsilon$

20. If $f \in R[a, b]$ and F is primitive of f on $[a, b]$ then

$$\int_{a}^{b} f(x)\, dx = F(b) - F(a)$$

This is called : **[Kanpur 2018]**

a. Fundamental theorem of integral calculus

b. First mean value theorem

c. Second mean value theorem

d. None of these

21. If P_1 and P_2 be any two partitions of $[a, b]$ then : **[Meerut 2017]**

a. $U(P_1, f) \le L(P_2, f)$ b. $U(P_1, f) \ge L(P_2, f)$

c. $U(P_1, f) = L(P_2, f)$ d. $U(P_2, f) \le L(P_1, f)$

22. Which of the following statement is wrong :

a. $U(P, f + g) \le U(P, f) + U(P, g)$

b. $L(P, f + g) \ge L(P, f) + L(P, g)$

c. $U(P, f + g) \ge U(P, f) + U(P, g)$

d. $L(P, f + g) \le L(P^*, f) \le U(P^*, f) \le U(P, f)$ where P^* is the refinement of P

23. If $f(x) = \begin{cases} x + x^2, & x \text{ is rational} \\ x^2 + x^3, & x \text{ is irrational} \end{cases}$ $\forall x \in (0, 2)$ then $\int_{0}^{-2} f(x)\, dx$ is :

a. $\frac{53}{12}$ b. $\frac{73}{12}$

c. $\frac{83}{12}$ d. $\frac{7}{12}$

24. The oscillatory sum for the function f on the interval $[a, b]$ is :

a. $\sum_{1}^{n} \mu_r \Delta x_r$ b. $\sum_{1}^{n} m_r \Delta x_r$

c. $\sum_{1}^{n} (\mu_r - m_r) \Delta x_r$ d. $\sum_{1}^{n} (M_r - \mu_r) \Delta x_r$

25. A bounded function f is integrable on $[a, b]$ iff for each $\varepsilon > 0$, there exists a partition p of $[a, b]$ such that: **[Kanpur 2018]**

a. $U(P, f) - L(P, f) > \varepsilon$ b. $U(P, f) - L(P, f) = \varepsilon$

c. $U(P, f) - L(P, f) < \varepsilon$ d. None of these

26. If f is defined on $[a, b)$ by $f(x) = 2\ \forall x \in [a, b]$ then over $[a, b]$ f is :

a. R-integrable b. Not R-integrable

c. Not bounded d. Not continuous

27. If $\{U(P, f) : P$ is a partition of $[a, b]\}$ is equal to :

a. $\int_{a}^{b} f\, dx$ b. $\int_{-a}^{b} f\, dx$

c. $\int_{a}^{-b} f\, dx$ d. None of these

28. Which of the following is wrong :

a. $\int_{-a}^{b}(-f) = -\int_{a}^{-b} f$ b. $\int_{-a}^{b} f = \int_{a}^{-b} f$

c. $\int_{a}^{-b}(-f) = -\int_{-a}^{b} f$ d. $\int_{-a}^{b} f \geq \int_{a}^{-b} f$

29. A bounded function f is R-integrable on $[a,b]$ if :

a. $\int_{a}^{-b} f$ exists only b. $\int_{-a}^{b} f$ exists only

c. $\int_{a}^{-b} f = \int_{-a}^{b} f$ d. $\int_{a}^{-b} f \neq \int_{-a}^{b} f$

30. If f is R-integrable on $[a,b]$ then :

a. $\left|\int_a^b F\,dx\right| \geq \int_a^b |f|\,dx$ b. $\left|\int_a^b F\,dx\right| = -\int_a^b |f|\,dx$

c. $\left|\int_a^b F\,dx\right| \leq \int_a^b |f|\,dx$ d. $\left|\int_a^b F\,dx\right| \leq |f|\,dx$

31. If f is defined on $[0,1]$ by $f(x) = x$ then $\int_{-0}^{1} f(x)\,dx$ is equal to :

a. 0 b. 1

c. $\frac{1}{2}$ d. None exist

32. The sup $\{L(p, f)\}$, where P is a partition of $[a, b]$ is called : **[Kanpur 2018]**

a. Upper R-integrable

b. Lower R-integrable

c. R-integrable

d. None of these

33. If f is continuous on $[a,b]$ then f is :

a. Differentiable on $[a,b)$

b. R-integrable on R

c. R-integrable on $[a,b]$

d. Not R-integrable on $[a,b]$

34. If P_1 and P_2 are two partitions of $[a,b]$ and $P_1 \subseteq P_2$ then : **[Kanpur 2018]**

a. $||P_2|| \leq ||P_1||$ b. $||P_2|| \geq ||P_1||$

c. $|P_1| = |P_2|$ d. $|P_2| \geq |P_1|$

35. If f is bounded on $[a, b]$ and M be the supremum of $f(x)$ in $[a, b]$ then for partition P of $[a, b]$ which one is true :

a. $U(P,f) = \mu(b-a)$

b. $U(P,f) = \mu(a-b)$

c. $U(P,f) \geq \mu(b-a)$

d. $U(P,f) \leq \mu(b-a)$

36. If $f(x) = \begin{cases} x + x^2, & x \text{ is rational} \\ x^2 + x^3, & x \text{ is irrational} \end{cases}$ $\forall x \in (0, 2)$ then $\int_{\underline{0}}^{2} f(x)\,dx$ is :

a. $\frac{7}{12}$ b. $\frac{53}{12}$

c. $\frac{83}{12}$ d. None of these

37. If f is monotonic on $[a,b]$ then f is :

a. R-integrable b. Continuous

c. Differentiable d. Not R-integrable

38. For the function

$$f(x) = \begin{cases} 1, & x \text{ is rational} \\ -1, & x \text{ is irrational} \end{cases}, \quad \int_0^{\overline{1}} f \text{ is :}$$

a. 1 b. −1

c. 0 d. None of these

39. The function $f(x) = \sin x$ over $\left[0, \frac{\pi}{2}\right]$ is :

a. Bounded only

b. R-integrable only

c. Bounded and R-integrable both

d. None of these

40. If f is bounded on $[a,b]$ and P is a partition of $[a,b]$ then :

a. $\lim_{||P|| \to 0} L(P,f) = \int_a^{\overline{b}} f$

b. $\lim_{||P|| \to 0} U(P,f) = \int_{\underline{a}}^{b} f$

c. $\lim_{||P|| \to 0} L(P,f) = \int_{\underline{a}}^{b} f$

d. None of these

41. If $f : [a,b] \to R$ is a bounded function and P is a partition of (a,b) then which one of the following is correct :

a. $L(P,f) \leq U(P,f)$ b. $L(P,-f) = -U(P,f)$

c. $U(P,-f) = -L(P,f)$ d. All the above

42. The oscillation of a bounded function f on an interval $[a,b]$ is :

a. $|U(P,f) - L(P,f)|$

b. $\sup\{|U(P,f) - L(P,f)|\}$

c. $\inf\{|U(P,f) - L(P,f)|\}$

d. None of these

43. The function $f(x)$ defined by $f(x) = x \ \forall x \in [0, 1]$ is :
a. Discontinuous b. Unbounded
c. R-integrable d. Not a R-integrable

44. The function f defined on [0,1] by :
$$f(x) = \begin{cases} 0, x \text{ is irrational} \\ 1, x \text{ is rational} \end{cases}$$
is not R-integrable over [0,1] then $\int_0^{\bar{1}} f(x)\,dx$ is equal to :
a. 0 b. 1
c. $\frac{1}{2}$ d. Does not exist

45. For the function $f(x) = \begin{cases} 1 & x \text{ is rational} \\ -1 & x \text{ is irrational} \end{cases}$, $L(P, f)$ over the interval [0,1] is : **[Meerut 2018]**
a. 1 b. −1
c. 0 d. None of these

46. If the set of points of discontinuity of a bounded function f defined on $[a, b]$ is finite then f is :
a. R-integrable
b. Not R-integrable
c. May or may not be R-integrable
d. None of these

47. If $f \in R[a, b)$ and $f(x) \geq 0 \ \forall x \in [a, b]$ then for $b \geq a$, $\int_a^b f(x)\,dx$ is :
a. $= 0$ b. ≥ 0
c. ≤ 0 d. Does not exist

48. If $f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational} \end{cases}$ then :
a. f and $|f|$ both R-integrable
b. f is R-integrable
c. $|f|$ is R-integrable
d. None of these

49. A necessary and sufficient condition for a bounded function f to be integrable over $[a, b]$ is :
a. $U(P, f) - L(P, f) < \varepsilon$
b. $\omega(P, f) < \varepsilon$
c. $\lim \omega(P, f) = 0$ as $||P|| \to 0$
d. All the above

50. If f be function defined on [0,1] by
$$f(x) = \begin{cases} 0, x \text{ is irrational} \\ 1, x \text{ is rational} \end{cases} \text{ then } \int_{-0}^{1} f \text{ is equal to :}$$
a. 0 b. 1
c. $\frac{1}{2}$ d. Does not exist

51. If $f(x) = x \ \forall x \in [0, 1]$ is R-integrable then $U(P, f)$ and $L(P, f)$ are respectively given by :
a. $\frac{n+1}{n}, \frac{n-1}{n}$ b. $\frac{n-1}{n}, \frac{n+1}{n}$
c. $\frac{n+1}{2n}, \frac{n-1}{2n}$ d. $\frac{n-1}{2n}, \frac{n+1}{2n}$

52. If the set of points of discontinuity of a bounded function f defined on $[a, b]$ has only a finite number of limit points then f is :
a. R-integrable
b. Not R-integrable
c. May or may not be R-integrable
d. None of these

53. If f is R-integrable w.r.t. α on $[a, b]$ then :
a. f and α are bounded
b. f and α are increasing
c. f is increasing and α is bounded
d. None of these

54. If $f(x) = x$ over [0,3) and $P = \{0, 1, 3, 4\}$ be the its partition then $U(P, f)$ is :
a. 6 b. 3
c. 0 d. None of these

55. Let f be continuous on $[a, b]$ such that
$$f(x) = \int_a^x f(t)\,dt \ \ \forall x \in [a, b] \text{ then :}$$
a. $F(x) = f'(x) \ \forall x \in [a, b]$
b. $F'(x) = f(x) \ \forall x \in [a, b]$
c. $f(x) = f(x) \ \forall x \in [a, b]$
d. None of these

56. The function f defined by $f(x) = x^3 \ \forall x \in [0, a]$, $a > 0$ is :
a. Bounded on [0, a]
b. Continuous on $[0, a]$
c. R-integrable on $[0, a]$
d. All the above

57. If $f(x) = \sin x$ over $\left[0, \frac{\pi}{2}\right]$ then $\int_{-0}^{\pi/2} f$ is equal to :

a. 1 b. −1

c. 0 d. $\frac{\pi}{2}$

58. If $f \in R[a,b]$ and $m < \mu < M$ for f then $\int_a^b f$ is equal to :

a. $(b-a)$ b. $(a-b)$

c. $\mu(b-a)$ d. $\mu(a-b)$

59. The integral of integrable function is :

a. Continuous b. Differentiable

c. Not integrable d. None of these

60. If f is continuous on $[a,b]$ such that

$$F(x) = \int_a^x f(t)\,dt \ \ \forall x \in [a,b]$$

then over the interval $[a,b]$:

a. $F(x) = f'(x)$ b. $F'(x) = f(x)$

c. $F(x) = f(x)$ d. None of these

61. The function f defined on [0,1] by

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ -1, & \text{when } x \text{ is irrational} \end{cases}$$

then f is :

a. Bounded only

b. R-integrable only

c. Bounded but not R-integrable

d. Bounded and R-integrable

62. If $b < a$ then $\int_a^b f = -\int_b^a f$ provided f is :

a. Continuous

b. Differentiable

c. Limit exist at a and b both

d. R-integrable

63. If $f(x) = x$ for $x \in [0, 1]$ and $P = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ be a partition of then $L(P, f)$ is : **[Kanpur 2018]**

a. $\frac{2}{3}$ b. $\frac{1}{3}$

c. $\frac{1}{2}$ d. $\frac{3}{2}$

64. If $f \in R(a,b)$ then the function F defined on $[a,b]$ by

$$F(x) = \int_a^x f(t)\,dt \text{ is :}$$

a. Continuous on $[a,b)$

b. Differentiable on $[a,b]$

c. R-integrable on $[a,b]$

d. None of these

65. If $f(x) = 1$ when $x \neq \frac{1}{2}$ and 0 when $x = \frac{1}{2}$ then $\int_0^1 f$ is :

a. 0 b. $\frac{1}{2}$

c. 1 d. Does not exist

66. Consider the following statements :

(I) If $f, g \in R[a,b)$ then $f \neq g \in R[a,b]$

(II) If $f, g \in R[a,b)$ then $f.g \in R[a,b)$ then

a. I and II are true

b. I is true but II is false

c. II is true but I is false

d. None of these

67. If f is continuous on $[a,b]$ then there exists a point $c \in [a,b)$ such that $\int_a^b f(x)\,dx =$

a. $(b-a)f'(c)$ b. $b-a$

c. $(b-a)f(c)$ d. $c[f(b)-f(a)]$

68. If $\int_a^b f(x)\,dx = \varphi(b) - \varphi(a)$, where φ be a differential function on $[a,b]$ such that

$$\varphi'(x) = f(x) \ \forall x \in [a,b]$$

then f must be :

a. Continuous only

b. R-integrable only

c. Either continuous or R-integrable

d. None of these

69. Function $f(x) = \begin{cases} \cos x, & \text{if } x \text{ is rational} \\ \sin x, & \text{if } x \text{ is irrational} \end{cases}$ over $\left[0, \frac{\pi}{4}\right]$ is :

a. Continuous b. R-integrable

c. Not R-integrable d. None of these

70. If $f(x) = 2rx$ when $\frac{1}{r+1} < x \le \frac{1}{r}$, $r = 1,2,3,....$ over $[0,1]$ then f is :

a. Continuous

b. R-integrable

c. Continuous and R-integrable both

d. None of these

71. If $f, g \in R[a,b]$ and $f \ge g$ then for $b \le a$:

a. $\int_a^b f \ge \int_a^b g$ b. $\int_a^b f \le \int_a^b y$

c. $\int_a^b f = \int_a^b g$ d. None of these

72. If $f(x) = \begin{cases} \sqrt{1-x^2} & x \text{ is rational} \\ 1-x & x \text{ is irrational} \end{cases}$ on $[0,1]$ then $\int_0^{-1} f$ is :

a. $\frac{1}{2}$ b. 1

c. π d. $\frac{\pi}{4}$

73. If $f \in R[a,c]$ and $f \in R[c,b]$ where $c \in (a,b)$ then over $[a,b]$ f is :

a. R-integrable

b. Not a R-integrable

c. May or may not be R-integrable

d. None of these

74. If $f \in R[a,b]$ and $|f(x)| \le k \ \forall x \in [a,b]$ then :

a. $\left|\int_a^b f\right| \le |f(b) - f(a)|$

b. $\left|\int_a^b f\right| \le k|f(b) - f(a)|$

c. $\left|\int_a^b f\right| \le k|b-a|$

d. None of these

75. If $f(x) = \cos x$ when x is rational and $\sin x$ when x irrational over $\left[0, \frac{\pi}{4}\right]$ then $\int_0^{-\pi/4} f$ is :

a. $1 - \frac{1}{\sqrt{2}}$ b. $1 + \frac{1}{\sqrt{2}}$

c. $-\frac{1}{\sqrt{2}}$ d. $\frac{1}{\sqrt{2}}$

76. If $f(x) = x$ and $g(x) = e^x$ over $[-1, 1]$ then :

a. $f \in R[-1, 1]$ only b. $g \in R[-1, 1]$ only

c. $f, g \in R[-1, 1]$ d. $f, g \notin R[-1, 1]$

77. The function $f(x) = x^4$ over $[0,1]$ is :

a. Continuous b. Differentiable

c. R-integrable d. All the above are true

78. If $f(x) = \cos x \ \forall x \in \left[0, \frac{\pi}{2}\right]$ then $\int_0^{-\pi/2} f(x)\,dx$ is :

a. 0 b. 1

c. $\frac{\pi}{2}$ d. -1

79. If $f \in R[a,b]$ and $k \in R$ then $k.f$ is :

a. R-integrable

b. Not a R-integrable

c. May or may not be R-integrable

d. None of these

80. If $f, g \in R[a,b]$ and f is continuous on $[a,b]$ with $a \le c \le b$ then $\int_a^b fg =$

a. $f(c)\int_c^b g$ b. $g(c)\int_a^b f$

c. $c\int_a^b fg$ d. None of these

81. If $f(x) = \begin{cases} 1 & x \ne \frac{1}{2} \\ 0 & x = \frac{1}{2} \end{cases}$ over $[0,1]$ then f is :

a. Continuous b. Differentiable

c. R-integrable d. None of these

82. If $g \in R[a,b]$ and f be bounded and monotonic on $[a,b]$ then for $c \in [a,b]$, $\int_a^b fg =$

a. $f(a)\int_a^b g$ b. $g(b)\int_a^b f$

c. $f(a)\int_a^c g + f(b)\int_c^b g$ d. $f(c)\left[\int_a^c g + \int_c^b g\right]$

83. If $f(x) = \begin{cases} \cos x & x \text{ is rational} \\ \sin x & x \text{ is irrational} \end{cases}$ and the interval $\left[0, \frac{\pi}{4}\right]$ be divided into n equal parts then M_r is equal to :

a. $\sin\frac{(r-1)\pi}{n}$ b. $\cos(r-1)\frac{\pi}{n}$

c. $\cos(r-1)\frac{\pi}{4n}$ d. $\sin(r-1)\frac{\pi}{4n}$

84. If $f(x)=\frac{1}{2^n}$ for $\frac{1}{2^{n+1}}\le x\le\frac{1}{2^n}$, $n=0,1,2,....$ and $f(0)=0$ over $[0,1]$ then f is :

a. Continuous

b. R-integrable

c. Continuous and R-integrable both

d. None of these

85. If [0,1] be divided into n subintervally for $f(x)=x^4$ then for rth sub-interval M_r is :

a. $\frac{r}{n}$ b. $\frac{r-1}{n}$

c. $\frac{r^4}{n^4}$ d. $\frac{(r-1)^4}{n^4}$

86. If f and g are R-integrable on $[a,b)$ and $|g(x)|\ge k$, $k>0\ \forall x\in[a,b]$ then $\frac{f}{g}$ is :

a. R-integrable

b. Not R-integrable

c. May or may not be R-integrable

d. None of these

87. If $f(x)=\begin{cases}\sqrt{1-x^2} & x \text{ is rational}\\ 1-x & x \text{ is irrational}\end{cases}$ over $[0,1]$ then $\int_{-0}^{1} f$ is :

a. $\frac{\pi}{4}$ b. $\frac{1}{2}$

c. π d. $-\frac{1}{2}$

88. If $f(x)=\begin{cases}\cos x & x \text{ is rational}\\ \sin x & x \text{ is irrational}\end{cases}$ over $\left[0,\frac{\pi}{4}\right]$ then $\int_{-0}^{\pi/4} f(x)\,dx$ is :

a. $1+\frac{1}{\sqrt{2}}$ b. $\frac{1}{\sqrt{2}}$

c. $1-\frac{1}{\sqrt{2}}$ d. $-\frac{1}{\sqrt{2}}$

89. If $f(x)=x^4$ over [0,1] then $\int_0^{-1} f(x)\,dx$ is equal :

a. 0 b. 1

c. $\frac{1}{4}$ d. $\frac{1}{5}$

90. If $\left[0,\frac{\pi}{2}\right]$ be divided into n parts for $f(x)\cos x$ then m_r is equal to :

a. $\cos\frac{r\pi}{n}$ b. $\cos\frac{(r-1)\pi}{n}$

c. $\cos\frac{r\pi}{2n}$ d. $\cos\frac{(r-1)\pi}{2n}$

91. If $f(x)=3x+1\ \forall x\in[1,2]$ then f is :

a. Continuous only b. Bounded only

c. R-integrable only d. All the above

92. If $f(x)=\frac{n}{n+1}$ when $\frac{1}{n+1}<x\le\frac{1}{n}$, $n=1,2,3,....$ over $[0,1]$ and $f(x)=1$ at $x=0$ then f is :

a. Continuous at $x=\frac{1}{n}$

b. f is R-integrable

c. Continuous and R-integrable

d. None of these

93. If $f(x)=x\ \forall x\in[0,1]$ and $P=\left\{0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1\right\}$ is a partition of [0,1] then $U(P,f)$ is :

a. $\frac{3}{8}$ b. $\frac{5}{8}$

c. $\frac{7}{8}$ d. $\frac{9}{8}$

94. If $f,g\in R[a,b]$ then :

a. $\int_a^b(f+g)\le\int_a^b f+\int_a^b g$

b. $\int_a^b(f+g)\ge\int_a^b f+\int_a^b g$

c. $\int_a^b(f+g)=\int_a^b f+\int_a^b g$

d. None of these

95. If $f(x)$ is bounded on $[a,b]$ then it is : **[Kanpur 2018]**

a. R-integrable

b. Not R-integrable

c. May or may not be R-integrable

d. None of these

96. Let $f(x)=\frac{1}{a^{r-1}}$ when $\frac{1}{a^r}<x<\frac{1}{a^{r-1}}$, $r=1,2,3,...$ where a is an integer greater then one over [0,1]. Consider the following statements :

(A) f is continuous at $x=\frac{1}{a^r}$, $r=1,2,3$

(B) f is R-integrable

a. A is true only
b. B is true only
c. A and B both are true
d. Neither A nor B is true

97. The norm or the partition is defined as the length of the segments of partition which is :
a. Maximum
b. Minimum
c. Either maximum or minimum
d. None of these

98. If $f(x)=[x]$ the greatest integer function on $[0,4]$ then $\int_0^4 [x]\,dx$ is equal to : **[Kanpur 2018]**
a. 0 b. 4
c. 6 d. 8

99. If $f(x)=\frac{n}{n+1}$ when $\frac{1}{n+1}<x\le\frac{1}{n}$, $n=1,2,3,\ldots$ over $[0,1]$ then $\int_0^1 f(x)\,dx$ is :
a. $\sum\frac{1}{n+1}$ b. $\sum\frac{1}{(n+1)^2}$
c. $\sum\frac{1}{(n+1)^3}$ d. None of these

100. If $f(x)=\frac{1}{a^{r-1}}$ when $\frac{1}{a^r}<x<\frac{1}{a^{r-1}}$ for $r=1,2,3,\ldots$, $a>1$ and a is an integer then $\int_0^1 f(x)\,dx$ is :
a. $\frac{1}{a}$ b. $\frac{a+1}{a}$
c. $\frac{a}{a+1}$ d. a

101. If $f(x)$ is bounded and integrable such that
$$f(x)=\int_a^x f(x)\,dx \text{ then } f(x) \text{ is :}$$
a. Continuous on $[a,b]$
b. Continuous everywhere
c. Discontinuous on $[a,b)$
d. None of these

102. A bounded function f is R-integrable in $[a,b]$ if the set of its points of discontinuity are :
a. Unique b. Finite
c. Infinite d. None of these

103. The greatest integer function $f(x)=[x]$ over $[0,4]$ is :
a. R-integrable
b. Not R-integrable
c. May or may not be R-integrable
d. None of these

104. If $f(x)=\begin{cases} x^2 & \text{when } x \text{ is rational} \\ x^3 & \text{when } x \text{ is irrational}\end{cases}$ on $[0,2]$ then $\int_0^{-2} f(x)\,dx =$
a. $\frac{31}{12}$ b. $\frac{41}{12}$
c. $\frac{49}{12}$ d. $\frac{53}{12}$

105. If $f:[a,b]\to R$ be bounded such that $f(x)\ge 0$ for all $x\in[a,b]$ then $\int_a^b f$:
a. Equal to zero b. Greater then zero
c. Less then zero d. None of these

106. If $f(x)$ is R-integrable on $[a,b)$ then $f(x)$ is :
a. Bounded
b. Unbounded
c. May or may not be bounded
d. None of these

107. If $f(x)=\begin{cases} 0 & 0\le x\le\frac{1}{2} \\ 1 & \frac{1}{2}<x<1\end{cases}$ is R-integrable on $[0,1]$ then $\int_0^1 f\,dx$ is equal to :
a. 0 b. $\frac{1}{2}$
c. 1 d. $\frac{3}{2}$

108. Every constant function is :
a. R-integrable
b. Not R-integrable
c. Improper integral
d. Proper integral

109. The maximum of the length of the subintervals of a partition P is called the :
a. Norm b. Net
c. Dissection d. None of the

110. If $f(x) = k\ \forall x \in [a,b]$ where k is constant then $\int_a^b f(x)\,dx$ is equal to :

a. 0 b. k

c. $k(b-a)$ d. $(b-a)$

111. If $f, g \in R[a,b)$ and $|g(x)| \geq k, k > 0\ \forall x \in [a,b]$ then $\frac{f}{g}$ is :

a. R-integrable

b. Not R-integrable

c. May or may not be R-integrable

d. None of these

112. If $f(x) = \begin{cases} x^2 & x \text{ is rational} \\ x^3 & x \text{ is irrational} \end{cases}$ on [0,2] then

$\int_{-0}^{2} f(x)\,dx =$

a. $\frac{49}{12}$ b. $\frac{47}{12}$

c. $\frac{31}{12}$ d. 0

113. If $f, g : [0, 1] \to R$ defined by

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{1}{n}, n \in N \\ 0 & \text{otherwise} \end{cases}$$

and $g(x) = \begin{cases} n & \text{if } x = n, n \in N \\ 0 & \text{otherwise} \end{cases}$ then :

a. f is R-integrable only

b. g is R-integrables only

c. f and g both are R-integrable

d. None of these

114. $f(x)$ is called a primitive of $F(x)$ for all $x \in [a,b]$ if :

a. $F(x) = f(x)$ b. $f'(x) = F(x)$

c. $F'(x) = f(x)$ d. $F'(x) = f'(x)$

115. Consider the following statements :

(I) If f is continuous and non-negative on $[a,b]$ then $\int_a^b f \geq 0$.

(II) If the set of points of discontinuity of a bounded function f defined on $[a,b]$ has only a finite number of limit points then f is R-integrable :

a. I is true only

b. II is true only

c. I and II both are true

d. I and II both are false

116. Which one of the following is wrong :

a. The functions $f(x) = e^x$, $g(x) = x\ \forall x \in [-1, 1]$ hold Bonnet's mean value theorem

b. If f is continuous in $[a,b]$ such that $c \in [a,b]$ then $\int_a^b f(x)\,dx = f(c)(b-a)$

c. Every constant function is R-integrable

d. If the set of points of discontinuity of f on $[a,b]$ is infinite then $f \in R\,[a,b]$

117. Which of the following is true :

a. If a function is continuous in closed interval then it is bounded and uniformly continuous

b. Continuous function is not a R-integrable

c. If $f(x) = \begin{cases} 0, & x \text{ is irrational} \\ 1, & x \text{ is rational} \end{cases}$ then $\int_0^{-1} f = 0$

d. $\int_0^{\pi/2} \sin x\,dx = 0$

118. Consider the statements :

(I) $\iint f(x,y)\,dA = \lim_{\max.\Delta A_i \to 0} \sum_{i=1}^{n} f(x_i^*, y_k^*)\,\Delta A_i$

(II) $\int_{-a}^{b} f(x) = dx = \text{glb}\,\{L(p,f)\}$

a. I is true

b. II is true

c. I and II both are true

d. None of these

119. Consider the statements :

(I) Every R-integrable function on $[a,b]$ is bounded

(II) $\int_{-a}^{b} f = \sup.\,L(p,f)$

a. I is true only

b. II is true only

c. I and II both are true

d. None of these

120. Consider the following statements :

(I) If $f \in R[a,b]$ then $f^2 \in R[a,b]$

(II) If $|f| \in R[a,b]$ then $f \in R[a,b]$

a. I and II are true

b. I is true but II is false

c. II is true but I is false

d. I and II are false

121. If $f, g \in R[a,b)$ where $g(x) \geq 0$ or $\leq 0 \; \forall x \in [a,b]$ and $m \leq \mu \leq M$ then $\int_a^b fg$ is equal to :

a. $\mu \int_a^b f\,dx$ b. $\mu \int_a^b g\,dx$

c. $\mu(b-a)$ d. $\mu(f(b)-f(a))$

122. Which one of the following is true :

a. If the function f is monotonic then it is R-integrable

b. Every bounded function f is R-integrable in $[a,b]$

c. If $f \in R[a,b]$ such that $|f(x)| \leq k \; \forall x \in [a,b]$ then $\left|\int_a^b f\right| \geq k|b-a|$

d. The integral of an integrable function is continuous

123. Which of the following is true ?

a. The function

$$f(x) = \begin{cases} \sqrt{1-x^2} & \text{when } x \text{ is rational} \\ 1\text{-}x & \text{when } x \text{ is irrational} \end{cases}$$

over $[0,1]$ is not R-integrable

b. If $f \in R[a,b]$ then $m(b-a) \geq \int_a^b f \geq \mu(b-a)$ if $b \geq a$

c. If f is bounded and R-integrable over $[a,b]$ and F is differentiable on $[a,b]$ such that $F' = f$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

d. All the above are true

124. If $f \in R[a,b)$ then :

a. $\left|\int_a^b f\right| \geq \int_a^b |f|$ b. $\left|\int_a^b f\right| = \int_a^b |f|$

c. $\left|\int_a^b f\right| \leq \int_a^b |f|$ d. None of these

125. If $g \in R[a,b]$ and f be monotonic non-increasing non-negative on $[a,b]$ then for some $c \in [a,b]$, $\int_a^b fg =$

a. $f(a)\int_a^c g$ b. $g(a)\int_a^c f$

c. $f(b)\int_c^b g$ d. $g(b)\int_c^b f$

126. If $I = [a,b]$ and $f: I \to R$ be bounded such that $f(x) \geq 0$ for all $x \in I$ then :

a. $\int_a^b f = 0$ b. $\int_a^b f \leq 0$

c. $\int_a^b f \geq 0$ d. None of these

127. If $f:[a,b] \to R$ is a bounded function such that $f(x) = 0$ except for x in $\{k_1, k_2, ..., k_n\}$ in $[a,b]$ then $\int_a^b f$ is equal to :

a. $k_1 + k_2 + ... + k_n$ b. $b-a$

c. 0 d. $a-b$

128 A bounded function f is R-integrable in $[a,b]$, if the set of its points of discontinuity is : **[Meerut 2017]**

a. φ b. Finite

c. Both (a) and (b) d. None of these

129. Let $f(x) = \begin{cases} 0 & \text{when } x \notin \varphi \\ 1 & \text{when } x \in \varphi \end{cases}$ be a function defined on $[0,1]$ then the value of $\int_{-0}^{1} f\,dx$ and $\int_0^{-1} f\,dx$ is : **[Meerut 2017]**

a. 1 and 1 b. 0 and 0

c. 1 and 0 d. 0 and 1

130. If $f:[0,1] \to R$ such that

$$f(x) = \begin{cases} 0 & x \text{ is irrational} \\ 1 & x \text{ is rational} \end{cases}$$ then : **[Meerut 2017]**

a. Upper an lower integral of f not exist

b. f is R-integrable

c. f is not R-integrable

d. None of these

131. Let $f(x) = x\,(0 \leq x \leq 1)$ and P be the partition $\left(0, \frac{1}{3}, \frac{2}{3}\right)$ of $[0,1]$ then value of $L(P,f)$ is : **[Meerut 2017]**

a. $\frac{2}{3}$ b. $\frac{1}{3}$

c. $\frac{3}{2}$ d. 0

132. If $f(x) = \cos x \ \forall x \in \left[0, \frac{\pi}{2}\right)$ then f is : **[Meerut 2017]**

a. Integrable on $\left[0, \frac{\pi}{2}\right]$

b. Not integrable on $\left[0, \frac{\pi}{2}\right]$

c. Noth (a) and (b)

d. None of these

133. If $f \in R[a,b]$, $g \in R[a,b]$ then : **[Meerut 2017, 19]**

a. $fg \notin R[a,b]$

b. $fg \in R[a,b]$

c. (a) is true (b) is false

d. None of these

134. Which is not true if $P = P_1 \cup P_2$ and $f : [a,b] \to R$: **[Meerut 2018]**

a. $L(P_2, f) \le U(P_1, f)$ b. $L(P_1, f) \le U(P_2, f)$

c. $U(P, f) \le L(P_2, f)$ d. All of these

135. If $f(x) = \sin x \ \forall x \in \left[0, \frac{\pi}{2}\right]$, then $L(p, f)$ is equal to : **[Meerut 2018]**

a. $\frac{\pi}{4n}\left(\tan\frac{n\pi}{4} - 1\right)$ b. $\frac{n\pi}{4}\left(\tan\frac{n\pi}{4} - 1\right)$

c. $\frac{\pi}{4n}\left(\tan\frac{4}{n\pi} - 1\right)$ d. $\frac{\pi}{4n}\left(\cot\frac{\pi}{4n} - 1\right)$

136. Which is not true if p^* is refinement of p : **[Meerut 2018]**

a. $L(P^*, f) \le L(P, f)$

b. $U(P^*, f) \le U(P, f)$

c. $L(P^*, f) = L(P^*, -f)$

d. Both (a) and (c)

137. If P^* is refinement of P then which is true : **[Meerut 2018]**

a. $L(P^*, f) \ge L(P, f)$ b. $L(P^*, f) \le L(P, f)$

c. $U(P^*, f) \le U(P, f)$ d. Both (a) and (c)

138. If $f : [a,b] \to R$ is bounded function then $-L(P, f)$ is equal to : **[Meerut 2018]**

a. $-U(P, f)$ b. $U(P, -f)$

c. $L(-P, f)$ d. $-L(-P, f)$

139. If $f(x) = \begin{cases} \cos x & \text{if} \quad x \in Q \\ \sin x & \text{if} \quad x \in RQ \end{cases}$ and $f - \left[0, \frac{\pi}{4}\right] \to R$, then $U(P, f)$ is equal to : **[Meerut 2018]**

a. $\frac{\pi/8m}{\sin\left(\frac{\pi}{8n}\right)} \cdot 2\sin^2\frac{\pi}{8}$

b. $\frac{8/\pi}{\sin\left(\frac{8}{\pi}\right)} \cdot \sin^2\frac{8}{11} \cdot \cos\frac{(n-1)}{8n}$

c. $\frac{8/\pi}{\cos 8/\pi} \cdot \sin^2\frac{8}{11} \cdot \sin\frac{\pi}{8}$

d. $\frac{\pi/8n}{\sin\left(\frac{\pi}{8n}\right)} \cdot 2\cos\frac{(n-1)}{8n} \cdot \sin\frac{\pi}{8}$

140. If $f(x) = x \ \forall x \in [0, 1]$, then $U(P, f)$ is equal to : **[Meerut 2018]**

a. $\frac{n-1}{2n}$ b. $\frac{n+1}{2n}$

c. $\frac{n+1}{n}$ d. $\frac{n-1}{n}$

141. If $f(x) = x^2 \ \forall x \in [0, 1]$ then $L(P, f)$ is equal to : **[Meerut 2018]**

a. $\frac{n(n+1)(2n+1)}{6n^3}$ b. $\frac{n(n-1)(2n+1)}{6n^3}$

c. $\frac{n(n-1)(2n-1)}{6n^3}$ d. $\frac{n(n+1)(2n-1)}{6n^3}$

142. If P^* refinement of P then : **[Meerut 2014]**

a. $L(P, f, \alpha) \le L(P^*, f, \alpha)$

b. $U(P^*, f, \alpha) \le U(P, f, \alpha)$

c. Both (a) and (b)

d. None of these

143. Let $f(x) = \sin x$ for $x \in [0, \pi/2]$ then value of $U(P, f)$ is : **[Meerut 2014]**

a. $\frac{\pi}{4n}\left(\cot\frac{\pi}{4n} + 1\right)$ b. $\frac{\pi}{4n}\left(1 - \cot\frac{\pi}{4n}\right)$

c. $\frac{4n}{\pi}\left(\cos\frac{\pi}{4n}\right)$ d. $\frac{\pi}{4n}\left(\cot\frac{\pi}{4n} - 1\right)$

144. Let $f(x) = x, x \in [0, 1]$ and $P = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ is any partition of [0, 1] then value of $L(P, f)$ is : **[Meerut 2014]**

a. 5/8 b. 3/8

c. 4/8 d. 1/8

145. If f is continuous in $[a,b]$ then there is a $c \in [a,b]$ such that : **[Meerut 2014]**

a. $\int_a^b f(x)\,dx = f'(c)$ b. $\int_a^b f(x)\,dx = f'(c)(b-a)$

c. $\int_a^b f(x)\,dx = f'(c)$ d. $\int_a^b f(x)\,dx = f'(c)(b-a)$

146. Let $f:[a,b] \to R$ be bounded function. If P is a partition of $[a,b]$ and if Q is refinement of P then : **[Meerut 2014]**

a. $L(Q.f) \geq L(P,f), U(Q,f) \geq U(P.f)$

b. $L(Q.f) \leq L(P.f), U(Q,f) \leq U(P.f)$

c. $L(Q.f) \geq L(P.f), U(Q,f) \leq U(P.f)$

d. $L(Q.f) > L(P.f), U(Q,f) = U(P.f)$

147. Let $f:[a,b] \to R$ be bounded function. Then for all partitions P over $[a,b]$ with $||P|| < \delta$ and every $\varepsilon > 0$ there correspond $\delta > 0$ such that : **[Meerut 2014]**

a. $U(P.f) < \int_a^{\bar{b}} f\,dx + \varepsilon$

b. $U(P.f) > \int_{\underline{a}}^b f\,dx - \varepsilon$

c. Both (a) and (b)

d. None of above

148. f is defined as

$$f(x) = \begin{cases} 0 & \text{when } x \text{ is irrational} \\ 1 & \text{when } x \text{ is rational} \end{cases}$$

an $[a,b]$ then value of $L(P,f)$ & $U(P,f)$ are : **[Meerut 2014]**

a. 0 and –1 b. 1 and –1

c. 1 and 0 d. 0 and 1

149. Let $f(\alpha):[0,1] \to R$ defined such that

$f(x) = \alpha(x) = x^2$ then $\int_0^1 f\,dx$ is : **[Meerut 2014]**

a. Does not exist b. 1

c. 1/2 d. None

150. Let $f(x) = 2rs$ for $\frac{1}{r+1} < x \leq \frac{1}{r}$ on $]0,1[$ then f is : **[Meerut 2014]**

a. Non-negative b. Non-positive

c. 0 d. None of these

151. If f is R-integrable and Non-negative on $[a,b]$ then $\int_a^b f$ is : **[Meerut 2014]**

a. Non-negative b. Non-positive

c. 0 d. None of these

152. Every monotonic function is :

a. R-integrable b. Non R-integrable

c. Increasing d. None of these

153. If $f(x) = x^2$ on $[0, a]$, $a > 0$ then : **[Meerut 2014]**

a. $f \in R[0, a]$ b. $f \notin R[0, a]$

c. f is discontinuous d. None of these

154. The value of $\lim_{||P|| \to 0} U(P,f)$ is :

a. $\int_{-a}^b f(dx)$ b. $\int_a^{-b} f(dx)$

c. $\int_a^b f(dx)$ d. None of these

155. The oscillation of a bounded function f on an interval $[a,b]$ is given by : **[Meerut 2015]**

a. $\sup\{|U(P,f) - L(P,f) : x_1, x_2 \in [a,b]|\}$

b. $\inf\{|U(P,f) - L(P,f) : x_1, x_2 \in [a,b]|\}$

c. Both (a) and (b)

d. None of these

156. Let A and A^* be partition of a closed and bounded interval $[a,b]$. Then A^* is called a refinement of A if : **[Meerut 2015]**

a. $A^* \subset A$ b. $A^* \supset A$

c. $A^* \subseteq A$ d. None of these

157. The upper and lower Riemann integrals for the function f defined on [0, 1] as follows :

$$f(x) = \begin{cases} \sqrt{1-x^2} & \text{if } x \text{ is rational} \\ (1-x) & \text{if } x \text{ is irrational} \end{cases} \text{ is :}$$

[Meerut 2015]

a. $\frac{1}{3}$ and $\frac{\pi}{4}$ b. $\frac{1}{4}$ and $\frac{\pi}{4}$

c. $\frac{1}{2}$ and $\frac{\pi}{4}$ d. None of these

158. Let f be a bounded function at defined on $[a,b]$ and let P be partition of $[a,b]$. If P_1 be the refinement of P then : **[Meerut 2015]**

a. $L(P_1,f) \leq L(P,f)$ b. $U(P_1,f) \leq U(P,f)$

c. $U(P_1,f) \geq U(P,f)$ d. $L(P_1,f) = U(P,f)$

159. The function $f: R \to R$ defined by

$$f(x) = \begin{cases} x & \text{when } x \text{ is irrational} \\ -x & \text{when } x \text{ is rational} \end{cases}$$

is continuous at $x =$ **[Meerut 2015]**

a. 1 b. −1
c. 0 d. None of these

160. If $f(x)$ be a function defined on $\left(0, \frac{\pi}{4}\right)$ by

$$f(x) = \begin{cases} \cos x & ; \text{ if } x \text{ is rational} \\ \sin x & ; \text{ if } x \text{ is irrational} \end{cases}$$

then in interval $\left[0, \frac{\pi}{4}\right] f$ is : **[Meerut 2015]**

a. R-integrable b. Not R-integrable
c. Discontinuous d. None of these

161. If P_1 and P_2 be any two partitions of $[a,b]$ then : **[Meerut 2015,17,19]**

a. $U(P_1, f) \geq L(P_2, f)$ b. $U(P_1, f) \leq L(P_2, f)$
c. $U(P_1, f) \leq U(P_2, f)$ d. $U(P_1, f) = L(P_2, f)$

162. Let f be a real valued bounded function defined on $[a,b]$. The lower Riemann integral of over $[a,b]$ is the .... of $L(P,f)$ over all partition $P \in P[a,b]$: **[Meerut 2015]**

a. Infimum
b. Supremum
c. May or may not be infimum
d. None of these

163. For function $f(y) = y$ in the interval [0, 3]. Let $P = \{0, 1, 2, 3\}$ be the partition of [0, 3], then the value of $L(P, f)$ is : **[Meerut 2015]**

a. 0 b. 1
c. 2 d. 3

164. The greatest of the length of the segments of partition P is called : **[Meerut 2015]**

a. Norm b. Mesh
c. Both (a) and (b) d. None of these

165. Let $f(x) = x$ on [0,1] then the value of $\int_{-0}^{1} x\, dx$ is : **[Meerut 2015]**

a. 1 b. 0
c. $\frac{1}{2}$ d. None of these

166. Let $f \in R(a,b)$ then

$$\lim_{n \to \infty} \sum_{1}^{h} hf(a + rh) = \int_a^b f \text{ if } h =$$ **[Meerut 2015]**

a. $b - a$ b. $b + a$
c. $\frac{b-a}{n}$ d. $\frac{b+a}{n}$

167. If $f(x) = \begin{cases} x + x^2 & \text{if } x \text{ is irrational } x \in (0,2) \\ x^2 + x^3 & \text{if } x \text{ is rational } x \in (0,2) \end{cases}$

then the value of $\int_{a}^{-2} f$ and $\int_{-0}^{2} f$ are : **[Meerut 2015]**

a. $\frac{53}{12}, \frac{83}{12}$ b. $\frac{83}{12}, \frac{53}{12}$
c. $\frac{7}{12}, \frac{83}{12}$ d. None of these

168. If f is continuous on $[a,b]$, then : **[Meerut 2016]**

a. $f \in R[a,b]$ b. $f \notin R[a,b]$
c. $f \notin R[a,b]$ d. None of these

169. If f be continuous function on $[a,b]$ and ϕ be a differentiable function on $[a,b]$, such that

$\phi'(x) = f(x)\ \forall x \in [a,b]$ then : **[Meerut 2016]**

a. $\int_a^b f(x)\, dx = \phi(a) + \phi(b)$
b. $\int_a^b f(x)\, dx = \phi(b) - \phi(a)$
c. $\int_a^b f(x)\, dx = \phi(a) - \phi(b)$
d. None of these

170. Let g be a bounded function defined on $[a,b]$ and let R be a partition of $[a,b]$. If R^* is a refinement of R, then : **[Meerut 2016]**

a. $L(R^*, g) \leq L(R, g)$ b. $U(R^*, g) \leq U(R, g)$
c. $U(R^*, g) \geq U(R, g)$ d. None of these

171. If f is Riemann integrable on $[a,b]$ then : **[Meerut 2016]**

a. $\left|\int_a^b f(x)\, dx\right| = \int_a^b |f(x)|\, dx$
b. $\int_a^b |f(x)|\, dx \leq \left|\int_a^b f(x)\, dx\right|$
c. $\left|\int_a^b f(x)\, dx\right| \leq \int_a^b |f(x)|\, dx$
d. None of these

172. Let f be a bounded function defined on the bounded interval $[a,b]$ then f is called Riemann integral if : **[Meerut 2016]**

a. $\int_{-a}^{b} f\,dx = \int_{a}^{-b} f\,dx$ b. $\int_{-a}^{b} f\,dx = \int_{a}^{-b} f\,dx$

c. $\int_{-a}^{b} f\,dx = \int_{a}^{b} f\,dx$ d. None of these

173. A bounded function f is Riemann integrable in $[a,b]$. Then the set of its points of discontinuity is : **[Meerut 2016]**

a. Finite b. Infinite

c. Oscillatory d. None of these

174. If the partition $P_1, P_2 \in [a,b]$, then $P_1 \cup P_2$ is : **[Meerut 2016]**

a. Common refinement

b. Norm

c. Segment

d. None of these

175. Let f be a bounded function defined on $[a,b]$ then for each partition P of $[a,b]$: **[Meerut 2016]**

a. $U(P,-f) = -U(P,f)$

b. $U(P,-f) = -U(P,f)$

c. $U(P,-f) = -L(P,f)$

d. None of these

176. If f is bounded on $[a,b]$ and P is a partition of $[a,b]$ then : **[Meerut 2016]**

a. $\int_{-a}^{b} f = \lim_{\|P\|\to 0} \cup(P,f)$

b. $\int_{-a}^{b} f = \lim_{\|P\|\to 0} L(P,f)$

c. $\int_{-a}^{b} f = \lim_{\|P\|\to \infty} \cup(P,f)$

d. None of these

177. Let f be a continuous function on $[a,b]$ and Let

$F(x) = \int_{a}^{b} f(t)\,dt \;\forall x \in [a,b]$ then : **[Meerut 2016]**

a. $F'(x) = f(x) \;\forall x \in [a,b]$

b. $F'(x) = f'(x) \;\forall x \in [a,b]$

c. $F'(x) = f'(x) \;\forall x \in [a,b]$

d. None of these

178. Let $f(x)$ be a function bounded on $[a,b]$ and let P_1 and P_2 be two partitions of $[a,b]$ such that $P_1 < P_2$ then : **[Meerut 2017]**

a. $U(P_1,f) - L(P_1,f) \ge U(P_2,f) - L(P_2,f)$

b. $U(P_1,f) - L(P_1,f) \le U(P_2,f) - L(P_2,f)$

c. Both (a) and (b)

d. None of these

179. Let f be a bounded function defined on $[a,b]$ and let P_1 be the partition of $[a,b]$. If P_2 is a refinement of P_1 then :

a. $L(P_2,f) \le L(P_1,f)$ b. $U(P_2,f) \le U(P_1,f)$

c. $U(P_2,f) \ge U(P_1,f)$ d. $L(P_2,f) = U(P_1,f)$

180. If $f:[a,b] \to R$, P and Q are partitions of $[a,b]$ such that $P \subset Q$ then : **[Meerut 2017]**

a. $L(P,f) \le L(Q,f)$

b. $L(P,f) \ge L(Q,f)$

c. $L(P,f) \ne L(Q,f)$

d. (a) is true, (b) is false

181. Let $f(x) = \begin{cases} 0 & \text{when } x \notin Q \\ 1 & \text{when } x \in Q \end{cases}$ be a function defined on [0, 1] then the value of $\int_{-0}^{1} f\,dx$ and $\int_{0}^{-1} f\,dx$ is : **[Meerut 2017]**

a. 1 and 1 b. 0 and 0

c. 1 and 0 d. 0 and 1

182. If $f:[0,1] \to R$ such that

$$f(x) = \begin{cases} 0 & x \text{ is irrational} \\ 1 & x \text{ is rational} \end{cases} \text{ then :}$$

[Meerut 2015,17]

a. Upper and lower integral of f not exist

b. f is R-integrable

c. f is not R-integrable

d. None of these

183. Let $f(x) = x\,(0 \le x \le 1)$. Let P be the partition $\left(0, \frac{1}{3}, \frac{2}{3}\right)$ of [0, 1] then value of $U(P,f)$ is : **[Meerut 2017]**

a. $\frac{2}{3}$ b. $\frac{1}{3}$

c. $\frac{2}{3}$ d. 0

184. If $f(x) = x^2, \forall x \in [0, 1]$ then $U\,L(P, f)$ is equal to : **[Meerut 2018]**

a. $\frac{n(n+1)(2n+1)}{6}$ b. $\frac{n(n-1)(2n+1)}{6}$

c. $\frac{n(n-1)(2n-1)}{6}$ d. $\frac{n(n+1)(2n-1)}{6}$

185. Which is not true, if $f\,;[a, b] \to R$ and M supremum of f is : **[Meerut 2018]**

a. $L(P, f) \le U(P, f)$

b. $M(b-a) \ge L(P, f)$

c. $U(P, f) \ge M(b-a)$

d. Both (b) and (c)

186. Which is true :

a. $L(P, f) = -U(P, -f)$

b. $L(P, -f) = -U(P, f)$

c. $U(P, -f) = -L(P, f)$

d. All the above

187. If $f(x) = \begin{cases} 1 & \text{when } x \in Q \\ -1 & \text{when } x \in R - Q \end{cases}$ $f : [0, 1] \to R$ then $L(P, f)$ is equal to : **[Meerut 2018]**

a. 1 b. –1

c. 0 d. 2

188. Let f is R-Integrable on $[a, b]$ and P_2 is refinement of P_1 then : **[Meerut 2019]**

a. $U(P_2, f) > U(P_1, f)$

b. $U(P_2, f) \le U(P_1, f)$

c. $U(P_2, f) \ge U(P_2, f)$

d. $U(P_2, f) = U(P_1, f)$

189. If $f(x) = \begin{cases} x^2 + x^3, & x \in Q \\ x + x^2, & x \in R - Q \end{cases}$ then $\int_0^{-2} f$ and $\int_{-0}^{2} f$ are : **[Meerut 2019]**

a. $\frac{12}{53}, \frac{12}{83}$ b. $\frac{53}{12}, \frac{83}{12}$

c. $\frac{83}{12}, \frac{53}{12}$ d. $\frac{12}{83}, \frac{12}{53}$

190. If $f(x) = x\ \forall x \in [0, 1]$ then $L(P, f)$ equal to ______ for the partition $\left\{0, \frac{1}{2}, \frac{2}{3}, 1\right\}$:

a. $\frac{2}{3}$ b. $-\frac{2}{3}$

c. $\frac{1}{3}$ d. $-\frac{1}{3}$

191. If $f(x) = \begin{cases} x^2, & x \in Q \\ x^3, & x \in R - Q \end{cases}$ then $\int_{-0}^{\pi/2} f(x)\,dx$ is equal to :

a. $\frac{12}{31}$ b. $-\frac{12}{31}$

c. $-\frac{31}{12}$ d. $\frac{31}{12}$

192. Let $f(x)$ be a function on [0,1] defined by $f(x) = \frac{1}{2}$ and $f\left(\frac{1}{2}\right) = 0$ then :

a. $\int_{-0}^{1} f(x) = \int_{0}^{-1} f(x)$ b. $\int_{-0}^{1} f(x) \ne \int_{0}^{1} f(x)$

c. $\int_{-0}^{1} f(x) < \int_{0}^{-1} f(x)$ d. $f \notin R[0, 1]$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(b)	**2.**	(c)	**3.**	(d)	**4.**	(a)	**5.**	(a)	**6.**	(c)	**7.**	(a)	**8.**	(c)	**9.**	(b)	**10.**	(d)
11.	(b)	**12.**	(a)	**13.**	(d)	**14.**	(c)	**15.**	(a)	**16.**	(b)	**17.**	(c)	**18.**	(b)	**19.**	(c)	**20.**	(a)
21.	(b)	**22.**	(c)	**23.**	(c)	**24.**	(c)	**25.**	(c)	**26.**	(a)	**27.**	(c)	**28.**	(b)	**29.**	(c)	**30.**	(c)
31.	(c)	**32.**	(b)	**33.**	(c)	**34.**	(b)	**35.**	(d)	**36.**	(b)	**37.**	(a)	**38.**	(a)	**39.**	(c)	**40.**	(c)
41.	(d)	**42.**	(b)	**43.**	(c)	**44.**	(b)	**45.**	(b)	**46.**	(a)	**47.**	(b)	**48.**	(c)	**49.**	(d)	**50.**	(a)
51.	(c)	**52.**	(a)	**53.**	(d)	**54.**	(a)	**55.**	(b)	**56.**	(d)	**57.**	(a)	**58.**	(c)	**59.**	(a)	**60.**	(b)
61.	(c)	**62.**	(d)	**63.**	(b)	**64.**	(a)	**65.**	(c)	**66.**	(a)	**67.**	(c)	**68.**	(c)	**69.**	(c)	**70.**	(b)
71.	(b)	**72.**	(d)	**73.**	(a)	**74.**	(c)	**75.**	(d)	**76.**	(c)	**77.**	(d)	**78.**	(b)	**79.**	(a)	**80.**	(a)
81.	(c)	**82.**	(c)	**83.**	(c)	**84.**	(b)	**85.**	(c)	**86.**	(a)	**87.**	(b)	**88.**	(c)	**89.**	(d)	**90.**	(c)
91.	(d)	**92.**	(b)	**93.**	(b)	**94.**	(c)	**95.**	(c)	**96.**	(b)	**97.**	(a)	**98.**	(c)	**99.**	(b)	**100.**	(c)
101.	(a)	**102.**	(b)	**103.**	(a)	**104.**	(c)	**105.**	(b)	**106.**	(a)	**107.**	(b)	**108.**	(a)	**109.**	(a)	**110.**	(c)
111.	(a)	**112.**	(c)	**113.**	(a)	**114.**	(c)	**115.**	(c)	**116.**	(d)	**117.**	(a)	**118.**	(a)	**119.**	(c)	**120.**	(b)
121.	(b)	**122.**	(d)	**123.**	(c)	**124.**	(c)	**125.**	(a)	**126.**	(c)	**127.**	(c)	**128.**	(c)	**129.**	(d)	**130.**	(c)
131.	(b)	**132.**	(a)	**133.**	(b)	**134.**	(c)	**135.**	(d)	**136.**	(d)	**137.**	(d)	**138.**	(b)	**139.**	(d)	**140.**	(b)
141.	(c)	**142.**	(c)	**143.**	(a)	**144.**	(b)	**145.**	(d)	**146.**	(c)	**147.**	(c)	**148.**	(d)	**149.**	(c)	**150.**	(c)
151.	(a)	**152.**	(a)	**153.**	(a)	**154.**	(b)	**155.**	(c)	**156.**	(b)	**157.**	(c)	**158.**	(b)	**159.**	(c)	**160.**	(a)
161.	(a)	**162.**	(b)	**163.**	(d)	**164.**	(a)	**165.**	(c)	**166.**	(c)	**167.**	(b)	**168.**	(a)	**169.**	(b)	**170.**	(b)
171.	(c)	**172.**	(a)	**173.**	(b)	**174.**	(a)	**175.**	(c)	**176.**	(b)	**177.**	(a)	**178.**	(a)	**179.**	(b)	**180.**	(d)
181.	(d)	**182.**	(c)	**183.**	(a)	**184.**	(a)	**185.**	(d)	**186.**	(d)	**187.**	(b)	**188.**	(b)	**189.**	(c)	**190.**	(c)
191.	(d)	**192.**	(a)																

HINTS AND SOLUTIONS

3. Let $p = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a, b]$. Let M_r and m_r be the supremum and infimum of f in I_r.

Again f is bounded on $[a, b] \Rightarrow -f$ is bounded on $[a, b]$.

Again M_r, m_r are supremum and infimum of f in I_r then $-m_r, -\mu_r$ are supremum and infimum of $-f$ in I_r.

$$U(P, -f) = \sum_{r=1}^{n} (-m_r)\,\Delta x_r$$

by definition of upper R-sum

$$= -\sum_{r=1}^{n} m_r\,\Delta x_r = -L(P, f)$$

6. If P_1 and P_2 are two partitions of $[a, b]$ then we know

$$L(P_1, f) \le U(P_2, f)$$

Keeping P_2 fixed and taking the supremum over all partitions P_1 it gives

$$\int_{-a}^{b} f \le U(P_2, f)$$

Now taking infimum over all partition P_2 it gives

$$\int_{-a}^{b} f \le \int_{a}^{-b} f$$

11. Let $P = \{a = x_0, x_1, ..., x_n = b\}$ be any partition of $[a, b]$. Then $I_r = [x_{r-1}, x_r]$, $r = 1, 2, ..., n$ are the subintervals of $[a, b]$. Let m_r and M_r be the infimum and supremum of $f(x)$ in $[x_{r-1}, x_r]$ then

$$m \le m_r \le M_r \le M$$

or $\quad m\,\Delta x_r \le m_r\,\Delta x_r \le \mu_r\,\Delta x_r \le \mu\,\Delta x_r$

$\because \quad \Delta x_r > 0$

or $\quad \sum_{r=1}^{n} m\,\Delta x_r \le \sum_{r=1}^{n} \mu\,\Delta x_r$

Now $\quad \sum_{r=1}^{n} m\,\Delta x_r = m \sum_{r=1}^{n} \Delta x_r$

$$= m(x_n - x_0) = m(b-a)$$

Similarly, $\quad \sum_{r=1}^{n} \mu\,\Delta x_r = M(b-a)$

$\therefore \quad m(b-a) \le M(b-a)$

12. $f(x) = k \;\forall x \in [a, b]$ is bounded over $[a, b]$

Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a, b]$. Then for any subinterval $[x_{r-1}, x_r]$.

We have $m_r = k$ and $M_r = k$

Now, $\quad U(P, f) = \sum_{r=1}^{n} m_r\,\Delta x_r = \sum_{r=1}^{n} k\,\Delta x_r$

$$= k \sum_{r=1}^{n} \Delta x_r$$

$$= k(x_n - x_0) = k(b-a)$$

and $\quad L(P, f) = \sum_{r=1}^{n} m_r\,\Delta x_r$

$$= \sum_{r=1}^{n} k\,\Delta x_r = k(b-a)$$

Hence, $\quad \int_{a}^{-b} f = \inf U(P, f)$

$$= \inf\{k(b-a)\} = k(b-a)$$

and $\quad \int_{-a}^{b} f = \sup L(P, f)$

$$= \sup\{k(b-a)\} = k(b-a)$$

Since, $\quad \int_{-a}^{b} f = \int_{a}^{-b} f = k(b-a)$ so $f \in R[a, b]$

16. Let $P = \{a = x_0, x_1, ..., x_n = b\}$ be any partition of $[a, b]$. Let M_r and m_r be the supremum and infimum of f in I_r.

Now f is bounded on $[a, b] \Rightarrow -f$ is bounded on $[a, b]$

If M_r, m_r are supremum and infimum of f in I_r then $-m_r, -M_r$ are supremum and infimum of $-f$ in I_r.

Now $\quad L(p, -f) = \sum_{r=1}^{n} (-M_r)\,\Delta x_r$

$$= -\sum_{r=1}^{n} M_r\,\Delta x_r = -U(P, f)$$

19. Let $\varepsilon > 0$ be given and $\inf U(P, f) = \int_{a}^{-b} f$

with $\sup L(p, f) = \int_{-a}^{b} f$ for all partitions P.

So, for $\varepsilon > 0$ there exists partitions P_1 and P_2 such that

$$U(P_1, f) < \int_{a}^{-b} f + \varepsilon$$

and $\quad L(P_2, f) > \int_{-a}^{b} f - \varepsilon \quad ...(1)$

If P_3 be the common refinement of P_1 and P_2 then

$$U(P_3, f) \le U(P, f)$$

and $\quad L(P_3, f) \ge L(P_2, f) \quad ...(2)$

Thus by (1) and (2) we get

$$U(P, f) < \int_{a}^{-b} f + \varepsilon$$

and $\quad L(P, f) > \int_{-a}^{b} f - \varepsilon$

23. Given that

$$f(x) = \begin{cases} x + x^2, & \text{when } x \text{ is rational} \\ x^2 + x^3, & \text{when } x \text{ is irrational} \end{cases}$$

Here, $\quad (x + x^2) - (x^2 + x^3) = x(1 - x^2)$

But $\quad x(1 - x^2) \ge 0$

If $x \in [0, 1]$ and $x(1 - x^2) \le 0$ if $x \in [1, 2]$, So that $x + x^2 \geq x^2 + x^3$ if $x \in [0, 1]$

and $x + x^2 \leq x^2 + x^3$ if $x \in [1, 2]$

Let $$m_r = \begin{cases} x^2 + x^3 & \text{when } x \in [0,1] \\ x + x^2 & \text{when } x \in [1,2] \end{cases}$$

and $$M_r = \begin{cases} x + x^2 & \text{when } x \in [0,1] \\ x^2 + x^3 & \text{when } x \in [1,2] \end{cases}$$

where M_r and m_r are the supremum and infimum of $f(x)$ in $[x_{r-1}, x_r]$.

Hence, $\int_{-0}^{2} f(x)\,dx = \int_0^1 (x^2 + x^3)\,dx + \int_1^2 (x + x^2)\,dx$

$$= \left[\frac{x^3}{3} + \frac{x^4}{4}\right]_0^1 + \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_1^2 - \frac{53}{12}$$

Also $\int_9^{-2} f(x)\,dx = \int_0^1 (x + x^2)\,dx + \int_1^2 (x^2 + x^3)\,dx$

$$= \frac{83}{12}$$

31. Let $$P = \left\{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{r-1}{n}, \frac{r}{n} .... \frac{n}{n} = 1\right\}$$

Here $$m_r = \frac{r-1}{n}, M_r = \frac{r}{4}$$

and $$\Delta x_r = \frac{1}{4} \text{ for } r = 1, 2, ..., n$$

Then $$L(P, f) = \sum_{r=1}^{n} \mu_r \Delta x_r$$

$$= \sum_{r=1}^{n} \frac{r-1}{n} \cdot \frac{1}{n^2} \sum_{r=1}^{n} r - 1$$

$$= \frac{n(n-1)}{2n^2} = \frac{n-1}{2n}$$

So, $$\int_{-0}^{1} x\,dx = \lim_{\|p\| \to 0} L(p, f)$$

$$= \lim_{n \to \infty} \frac{n-1}{2n} = \frac{1}{2}$$

36. See the solution of question (2,3).

38. Given that

$$f(x) = \begin{cases} 1, & x \text{ is rational} \\ -1, & x \text{ is irrational} \end{cases}$$

Here $f(x)$ is bounded on $[0,1]$. If P is any partition of $[0,1)$ then for any subinterval $[x_{r_{-1}}, x_r]$ of P, we have $m_r = -1$ and $M_r = 1, r = 1, 2, ..., n.$

So $$U(P, f) = \sum_{r=1}^{n} m_r \Delta x_r$$

$$= 1 \sum_{r=1}^{n} \Delta x_r = 1(1 - 0) = 1$$

and then $$\int_0^{-1} f = \lim_{n \to \infty} U(P, f) = 1$$

39. Let $$P = \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, ..., \frac{(r-1)\pi}{2n}, \frac{r\pi}{2n} .... \frac{n\pi}{2n} = \frac{\pi}{2}\right\}$$

be the partition of $\left[0, \frac{\pi}{2}\right]$ in which rth subinterval

$$I_r = \left[\frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}\right] \text{with length} = \frac{\pi}{2n}$$

Here, $$m_r = \sin\frac{(r-1)\pi}{2}$$

and $$M_r = \sin\frac{r\pi}{2n}, r = 1, 2, ..., n$$

So, $$U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r$$

$$= \sum_{r=1}^{n} \left(\sin\frac{r\pi}{2n}\right)\frac{\pi}{2n}$$

$$= \frac{\pi}{2n}\left[\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + ... + \sin\frac{n\pi}{2n}\right]$$

$$= \frac{\pi}{4n}\left(\cot\frac{\pi}{4n} + 1\right)$$

Since, $$\sin a + \sin(a + d) + ... + \sin(a + (n-1)d)$$

$$= \frac{\sin\left[a + \frac{n-1}{2}d\right] \sin n\frac{d}{2}}{\sin\frac{d}{2}}$$

Similarly, $$L(P, f) = \frac{\pi}{4n}\left(\cot\frac{\pi}{4n} - 1\right)$$

$\therefore$ $$\int_{-0}^{\pi/2} f = \lim_{n \to \infty} L(P, f)$$

$$= \lim_{n \to \infty} \frac{\pi}{4n}\left(\cot\frac{\pi}{4n} + 1\right) = 1$$

and $$\int_0^{-\pi/2} f = \lim_{n \to \infty} U(P, f)$$

$$= \lim_{n \to \infty} \frac{\pi}{4n}\left(\cot\frac{\pi}{4n} + 1\right) = 1$$

$\therefore$ $$\int_{-0}^{\pi/2} f = \int_0^{-\pi/2} f = 1 \text{ so } f \in R\left[0, \frac{\pi}{2}\right]$$

43. Given that $f(x) = x \ \forall x \in [0, 1]$

Let $P = \left\{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{r-1}{n}, \frac{r}{n} .... \frac{n}{n} = 1\right\}$

then $m_r = \frac{r-1}{n}$

and $M_r = \frac{r}{n}$ with $\Delta x_r = \frac{1}{n} \ \forall r = 1, 2, ..., n$

So, $L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r$

$= \sum_{r=1}^{n} \frac{r-1}{n} \frac{1}{n} = \frac{1}{n^2} \sum_{r=1}^{n} (r-1)$

$= \frac{1}{n^2}[1 + 2 + ... + (n-1)] = \frac{n-1}{2n}$

Similarly, $U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r = \frac{n+1}{2n}$

So, $\int_{-0}^{1} x\,dx = \lim_{\|P\| \to 0} L(P, f)$

$= \lim_{n \to \infty} \frac{n-1}{2n} = \frac{1}{2}$

and $\int_{0}^{-1} x\,dx \ \lim_{\|P\| \to 0} U(P, f)$

$= \lim_{n \to \infty} \frac{n+1}{2n} = \frac{1}{2}$

So, $\int_{-0}^{1} f = \int_{0}^{-1} f = \frac{1}{2} = \int_{0}^{1} f$

i.e., $f \in R[0, 1]$

44. Given that

$$f(x) = \begin{cases} 0, & \text{when } x \text{ is irrational} \\ 1, & \text{when } x \text{ is rational} \end{cases}$$

Obviously $f(x)$ is bounded.

i.e. $0 \le f(x) \le 1 \ \forall x \in [0, 1]$

Let P be any partition of [0,1] then for any subinterval $[x_{r-1}, x_r]$ of P, $m_r = 0, M_r = 1$

So, $U(P, f) = \sum_{r=1}^{n} \mu_r \Delta x_r = \sum_{r=1}^{n} 1.\Delta x_r = 1$

Also $\int_{0}^{-1} f = \lim_{n \to \infty} U(P, f) = 1$

45. $f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ -1, & \text{when } x \text{ is irrational} \end{cases}$

Obviously $f(x)$ is bounded on [0,1]. If P is any partition of [0,1] then for $[x_{r-1}, x_r]$ of P $m_r = -1$ and $M_r = 1 \ \forall r = 1, 2, ..., n$

So, $L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r$

$= \sum_{r=1}^{n} -1.\Delta x_r = -\sum_{r=1}^{n} \Delta x_r$

$= -[(x_1 - x_0) + (x_2 - x_1) + ...$

$+ (x_n - x_{n-1})]$

$= (x_n - x_0) = -1$

50. $f(x) = \begin{cases} 0 & \text{when } x \text{ is irrational} \\ 1 & \text{when } x \text{ is rational} \end{cases}$

Here, $0 \le f(x) \le 1 \ \forall x \in [0, 1]$

Let $[x_{r-1}, x_r]$ be any subinterval of a partition P then $m_r = 0, M_r = 1$ then

$$L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum 0.\Delta x_r = 0$$

$\therefore \int_{-0}^{1} f = \lim_{n \to \infty} L(P, f) = 0$

54. $f(x) = x$ over [0,3] and $P = \{0, 1, 2, 3\}$

Consider $I_1 = [0, 1], I_2 = [1, 2], I_3 = [2, 3]$

then length of these subintervals are

$\Delta_1 = 1 - 1 = 0, \Delta_2 = 2 - 1 = 1, \Delta_3 = 3 - 2 = 1$

Let M_r and m_r be the supremum and infimum of $f(x) = x$ in I_r, $r = 1, 2, 3$ then

$$M_1 = 1, m_1 = 0; M_2 = 2, m_2 = 1; M_3 = 3, m_3 = 2$$

then $U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r$

$= M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3$

$= 1 \times 1 + 2 \times 1 + 3 \times 1 = 6$

56. $f(x) = x^3$ over $[0, a]$

Let $P = \left\{0, \frac{a}{n}, \frac{2a}{n}, ..., \frac{(n-1)a}{n}, \frac{na}{n} = a\right\}$

be the partition of $[0, a]$ then

$$I_r = \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right] \text{ and } \Delta x_r = \frac{a}{n}$$

Let M_r and m_r be the supremum and infimum of f in I_r then

$$M_r = \frac{r^3a^3}{n^3}, m_r = \frac{(r-1)^3a^3}{n^3}, r = 1, 2,..,n$$

So, $\quad L(P,f) = \sum_{r=1}^{n} m_r \Delta x_r$

$$\sum_{r=1}^{n} \frac{(r-1)^3a^3}{n^3}\frac{a}{n} = \frac{a^4}{n^4}\sum_{r=1}^{n}(r-1)^3$$

$$= \frac{a^4}{n^4}[1^3 + 2^3 + ... + (n-1)^3]$$

$$= \frac{a^4}{n^4}\left[\frac{(n-1)n}{2}\right]^2$$

$$L(P,f) = \frac{a^4}{n}\left(1 - \frac{1}{n}\right)^2$$

Similarly, $U(P,f) = \sum_{r=1}^{n} \mu_r \Delta x_r = \frac{a^4}{4}\left(1+\frac{1}{n}\right)^2$

So, $\quad \int_0^{-a} f = \lim_{n\to\infty} U(P,f)$

$$= \lim_{n\to\infty} \frac{a^4}{4}\left(1+\frac{1}{n}\right)^2 = \frac{a^4}{4}$$

and $\quad \int_{-0}^{a} f = \lim_{n\to\infty} L(P,f)$

$$= \lim_{n\to\infty} \frac{a^4}{n}\left(1+\frac{1}{n}\right)^2 = \frac{a^4}{n}$$

So, $\quad \int_{-0}^{a} f = \int_0^{-a} f = \frac{a^4}{n}$ so $f \in R[0,a]$

64. Since, $f \in R[a,b]$, f is bounded on $[a,b]$.

So, $\quad |f(t)| \le \mu \ \forall t \in [a,b]$

Let $\quad a \le x < y \le b$ then

$$|F(y) - F(x)| = \left|\int_a^y f(t)\,dt - \int_a^x f(t)\,dt\right|$$

$$= \left|\int_a^y f(t)\,dt + \int_x^a f(t)\,dt\right|$$

$$= \left|\int_x^y f(t)\,dt\right| \le M|y-x| = M(y-x)$$

Let $\varepsilon > 0$ be given then if $|y-x| < \frac{\varepsilon}{\mu}$ then $|F(y) - F(x)| < \varepsilon$.

Thus given $\varepsilon > 0$ there exists $\delta = \frac{\varepsilon}{\mu} > 0$ such that

$$|F(y) - F(x)| < \in \text{ whenever } |y-x| \ \forall x, y \in [a,b]$$

Thus, F is uniformly continuous on $[a,b)$ and hence continuous on $[a,b]$.

69. $f(x) = \begin{cases} \cos x, & \text{if } x \text{ is rational} \\ \sin x, & \text{if } x \text{ is irrational} \end{cases}$ over $\left[0, \frac{\pi}{4}\right]$

Let $P = \left\{0, \frac{\pi}{4n}, \frac{2\pi}{4n}, ..., \frac{(r-1)\pi}{4n}, \frac{r\pi}{4n}, ..., \frac{n\pi}{4n} = \frac{\pi}{4}\right\}$

be a partition then

$$I_r = \left[\frac{(r-1)\pi}{4n}, \frac{r\pi}{4n}\right]$$

and $\quad \Delta r = \frac{\pi}{4n} \ \forall r = 1, 2, ...n$

$$M_r = \cos\frac{(r-1)\pi}{4n}$$

and $\quad m_r = \sin\frac{(r-1)\pi}{4n}$

So, $\quad L(P,f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n} \sin\frac{(r-1)\pi}{4n} \cdot \frac{\pi}{4n}$

$$= \frac{\pi}{4n}\left[\sin\frac{\pi}{4n} + ... + \sin\frac{(n-1)\pi}{4n}\right]$$

$$= \frac{\pi}{4n}\frac{\sin\left(\frac{\pi}{4n} + \frac{n-2}{2}\frac{\pi}{4n}\right)\cdot\sin\frac{n\pi}{8n}}{\sin\frac{\pi}{84}}$$

$$L(P,f) = \frac{\left(\frac{\pi}{8n}\right)}{\sin\left(\frac{\pi}{8n}\right)} \cdot 2\sin^2\frac{\pi}{8}$$

Similarly, $U(P,f) = \sum_{r=1}^{n} M_r \Delta x_r$

$$= \frac{\frac{\pi}{84}}{\sin\frac{\pi}{84}} \cdot 2\cos\frac{(n-1)\pi}{8n}\sin\frac{\pi}{8}$$

$\therefore \quad \int_{-0}^{\pi/4} f = \lim_{n\to\infty} L(P,f) = 2\sin^2\frac{\pi}{8}$

$$= 1 - \cos\frac{\pi}{4} = 1 - \frac{1}{\sqrt{a}}$$

and $\quad \int_0^{-\pi/4} f = \lim_{n\to\infty} U(p,f)$

$$= \lim_{n\to\infty} \frac{\frac{\pi}{84}}{\sin\frac{\pi}{84}} \cdot 2\cos\frac{(n-1)\pi}{8n} \cdot \sin\frac{\pi}{8}$$

$$= 2\cos\frac{\pi}{8}\sin\frac{\pi}{8} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$\because \quad \int_0^{-\pi/4} f \neq \int_{-0}^{\pi/4} f$

So, f is not R-integrable over $\left[0, \frac{\pi}{4}\right]$.

70. $f(x) = 2rx$

when $\frac{1}{r+1} < x \le \frac{1}{r}$, $r = 1, 2, 3, ...$ over $[0,1]$

then $$f(x) = \begin{cases} 2x & \frac{1}{2} < x \le 1 \\ 4x & \frac{1}{3} < x \le \frac{1}{2} \\ 2(r-1)x & \frac{1}{r} < x \le \frac{1}{r-1} \\ 2rx & \frac{1}{r+1} < x \le \frac{1}{r} \end{cases}$$

So, $f\left(\frac{1}{r}+0\right) = \lim_{h\to 0} f\left(\frac{1}{r}+h\right)$

$$= \lim_{h\to 0} 2(r-1)\left(\frac{1}{r}+h\right) = 2 - \frac{2}{r}$$

$f\left(\frac{1}{r}-0\right) = \lim_{h\to 0} f\left(\frac{1}{r}-h\right)$

$$= \lim_{h\to 0} 2r\left(\frac{1}{r}-h\right) = 2$$

So, $f\left(\frac{1}{r}+0\right) \neq f\left(\frac{1}{r}-0\right)$

i.e. f is not continuous at $x = \frac{1}{r}$.

Also $f(1) = 2$ and $f(1-0)$

$$= \lim_{h\to\infty} 2(1-h) = 2$$

So, f is continuous at $x = 1$. Thus f is not continuous at $x = \frac{1}{r}$, $r = 2, 3, 4, ...$ and the set of point of discontinuity $\frac{1}{2}, \frac{1}{3}, ...$ of f has only one limit point at $x = 0$, So f is R-integrable.

72. $$f(x) = \begin{cases} \sqrt{1-x^2} & \text{if } x \text{ is rational} \\ 1-x & \text{if } x \text{ is irrational} \end{cases}$$

$(1-x^2)-(1-x)^2 = 2x(1-x) > 0 \ \forall x \in \,]0,1[$

So, $M_r = \sqrt{1-x^2}$ and $m_r = 1-x$

for the subinterval $[x_{r-1}, x_r]$.

$\therefore \quad \int_0^{-1} f = \int_0^1 \sqrt{1-x^2}.dx$

$$= \left[\frac{x}{2}\sqrt{1-x^2} + \frac{\sin^{-1}x}{2}\right]_0^1 = \frac{\pi}{4}$$

75. Let $p = \left\{0, \frac{\pi}{4n}, \frac{2\pi}{4n}, ..., \frac{(r-1)\pi}{4n}, \frac{r\pi}{4n} .... \frac{n\pi}{4n} = \frac{\pi}{4}\right\}$

be the partition and $I_r = \left[\frac{(r-1)\pi}{4n}, \frac{r\pi}{4n}\right]$ with

$\Delta x_r = \frac{\pi}{4n} \ \forall r = 1, 2, 3, ... n.$

So, $M_r = \cos\frac{(r-1)\pi}{4n}$

Thus, $U(P,f) = \sum_{r=1}^{n} M_r \Delta x_r$

$$= \sum_{r=1}^{n} \cos\frac{(r-1)\pi}{4n} \cdot \frac{\pi}{4n}$$

$$= \frac{\frac{\pi}{84}}{\sin\left(\frac{\pi}{84}\right)} \cdot 2\cos\frac{(n-1)\pi}{8n} \cdot \sin\frac{\pi}{8}$$

(after solving)

So, $\int_0^{-\pi/4} f = \lim_{n\to\infty} U(P,f)$

$$= \lim_{n\to\infty} \frac{\frac{\pi}{84}}{\sin\left(\frac{\pi}{84}\right)} \cdot 2\cos\frac{(n-1)\pi}{8n} \cdot \sin\frac{\pi}{8}$$

$$= 2\cos\frac{\pi}{8}\sin\frac{\pi}{8} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

78. $f(x) = \cos x \ \forall x \in \left[0, \frac{\pi}{2}\right]$

then $0 \le f(x) \le 1 \ \forall x \in \left[0, \frac{\pi}{2}\right]$

Let $P = \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, ... \frac{r\pi}{2n}, ... \frac{n\pi}{2n} = \frac{\pi}{2}\right\}$

be the partition of $\left[0, \frac{\pi}{2}\right]$ then

$$I_r = \left[\frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}\right]$$

and $\Delta x_r = \frac{\pi}{2a} \ \forall r = 1, 2, ...n$

So, $M_r = \cos\frac{(r-1)\pi}{2n}$

and $m_r = \cos\frac{r\pi}{2n}$

Then, $U(P,f) = \sum_{r=1}^{n} M_r \Delta x_r$

$$= \sum_{r=1}^{n} \cos\frac{(r-1)\pi}{2n} \cdot \frac{\pi}{2n}$$

$$= \frac{\pi}{2n} \cdot \left[\cos 0 + \cos\frac{\pi}{2n} + ... + \cos\frac{(n-1)\pi}{2n}\right]$$

$$= \frac{\pi}{2n} \frac{\cos\left[0 + \left(\frac{n-1}{2}\right) \cdot \frac{\pi}{2n}\right] \cdot \sin\frac{n\pi}{4n}}{\sin\frac{\pi}{4n}}$$

$$= \frac{\frac{\pi}{4n}}{\sin\left(\frac{\pi}{4n}\right)} \cdot 2\cos\left[\frac{\pi}{4}\left(1 - \frac{1}{n}\right)\right] \sin\frac{\pi}{4}$$

$$\therefore \quad \int_0^{-\pi/2} f(x)\,dx = \lim_{n\to\infty} U(P,f)$$

$$= 1.2\cos\frac{\pi}{4}\frac{\pi}{4} = 1$$

81. $f(x)$ is bounded in [0,1] since $0 \le f(x) \le 1 \ \forall x \in [0, 1]$

Let P be a partition of [0,1] such that $\frac{1}{2}$ belongs to the $]x_{s-1}, x_s[$. Then $m_r = M_r = 1$ for $r = 1, 2, ...n$ and $r \neq s, m_s = 0, M_s = 1$.

So, $U(P,f) - L(P,f) = \sum_{\substack{r=1 \\ r\neq s}}^{n} (M_r - m_r)\Delta x_r$

$$+ (M_s - m_s)(x_s - x_{s-1})$$

$$= \sum_{r=1}^{n} (1-1)(x_r - x_{r-1}) + (1-0)(x_s - x_{s-1})$$

$$= x_s - x_{s-1} \qquad ...(1)$$

Let $\varepsilon > 0$, choose a partition p such that $\frac{1}{2}$ is an interior point of one of the subinterval whose length is less then ε then by (1) $U(P,f) - L(P,f) < \varepsilon$ so $f \in R[0, 1]$.

84. $f(x) = \frac{1}{2^n}$ for $\frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$, $n = 0, 1, 2,...$

we have $f\left(\frac{1}{2^n} + 0\right) = \frac{1}{2^{n-1}}$

and $f\left(\frac{1}{2^n} - 0\right) = \frac{1}{2^n}$

which shows that the function $f(x)$ is discontinuous at $x = \frac{1}{2^n}$, $n = 1,2,3,.....$ Also for $n = 0$,

$$f\left(\frac{1}{2^n}\right) = f(1) = 1$$

and $f\left(\frac{1}{2^n} - 0\right) = 1$

so that $f(x)$ is continuous at $x = \frac{1}{2^0} = 1$

Thus the points of discontinuous of f are $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, ... \frac{1}{2^n}, ...$

Since, the set of these points of discontinuity of f has only one limiting point at $x = 0$ so $f \in R[0, 1]$.

87. $f(x) = \begin{cases} \sqrt{1-x^2} & \text{if } x \text{ is rational} \\ 1-x & \text{if } x \text{ is irrational} \end{cases}$

Hence, $(1-x^2) - (1-x)^2 = 2x(1-x) > 0 \ \forall x \in]0, 1[$

So, $\sqrt{1-x^2} > (1-x) \ \forall x \in]0, 1[$

Thus, $m_r = (1-x), M_r = \sqrt{1-x^2}$

So, $\int_{-0}^{1} f = \int_0^1 (1-x)\,dx = \left(x - \frac{x^2}{2}\right)_0^1$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

89. $f(x) = x^4$ defined on [0,1]

Let $P = \left\{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, \frac{n}{n} = 1\right\}$

be the partition of [0,1] then

$$I_r = \left[\frac{r-1}{n}, \frac{r}{n}\right]$$

and $\Delta_r = \frac{1}{n} \ \forall r = 1, 2 - n$

Then supremum $M_r = \frac{r^4}{n^4}$

and infimum $\mu_r = \frac{(r-1)^4}{n^4}, r = 1, 2, 3, \dots n$

$$\therefore \quad U(P, f) = \sum_{r=1}^{n} \mu_r \Delta x_r = \sum_{r=1}^{n} \frac{r^4}{n^4} \cdot \frac{1}{n}$$

$$= \frac{1}{n^5} \sum_{r=1}^{n} r^4$$

$$= \frac{1}{n^5} \cdot \frac{n(n+1)(2n+1)}{30}(3n^2 + 3n - 1)$$

$$= \frac{1}{30}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\left(3 + \frac{3}{n} - \frac{1}{n^2}\right)$$

$$\therefore \quad \int_0^{-1} f(x)\,dx = \lim_{n\to\infty} U(P, f)$$

$$= \frac{1}{30} \cdot 1 \cdot 2 \cdot 3 = \frac{1}{5}$$

91. $f(x) = 3x + 1$ over [1,2] then $f(x)$ is bounded.

Since, $u \le f(x) \le 7 \ \forall x \in [1, 2]$

Let $P = \left\{1, 1 + \frac{1}{n}, 1 + \frac{2}{n}, \dots, 1 + \frac{n}{4} = 2\right\}$

be the partition of [1,2) then

$$I_r = \left[1 + \frac{r-1}{n}, 1 + \frac{r}{n}\right]$$

and $\Delta x_r = \frac{1}{n} \ \forall r = 1, 2, \dots n$

Supremum of $f = M_r = 3\left(1 + \frac{r}{n}\right) + 1$

$$= 4 + \frac{3r}{n}$$

and infimum of $f = m_r = 3\left(1 + \frac{r-1}{n}\right) + 1$

$$= 4 + \frac{3(r-1)}{n}$$

So, $U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r = \sum_{r=1}^{n}\left(n + \frac{3r}{n}\right) \cdot \frac{1}{n}$

$$= \frac{1}{n}\sum_{r=1}^{n}\left(4 + \frac{3r}{n}\right) = \frac{1}{4}\left[4n + \frac{3}{n}\sum_{r=1}^{n} r\right]$$

$$U(p, f) = 4 + \frac{3}{n}(1 + 2 + \dots + n)$$

$$= 4 + \frac{3}{2}\left(1 + \frac{1}{n}\right)$$

and $L(P, f) = \sum_{r=1}^{n} m_r \Delta x_r$

$$= \sum\left[\left\{4 + \frac{3(r-1)}{n}\right\} \cdot \frac{1}{n}\right]$$

$$= 4 + \frac{3}{2}\left(1 - \frac{1}{n}\right)$$

So, $\int_1^{-2} f(x)\,dx = \lim_{n\to\infty} U(P, f)$

$$= 4 + \frac{3}{2} = \frac{11}{2}$$

and $\int_{-1}^{2} f(x)\,dx = \lim_{n\to\infty} L(P, f)$

$$= 4 + \frac{3}{2} = \frac{11}{2}$$

So, $\int_{-1}^{2} f(x)\,dx = \int_1^{-2} f(x)\,dx$

$$= \int_1^2 f(x)\,dx = \frac{11}{2} \quad i.e. \ f \in R[1, 2]$$

92. $f(x) = \frac{n}{n+1}$ over [0,1] when $\frac{1}{n+1} < x \le \frac{1}{n}$, $n = 1, 2, 3, \dots$ and $f(x) = 1, x = 0$

$$f(x) = \begin{cases} \frac{n-1}{n} & \text{when } \frac{1}{n} < x \le \frac{1}{n-1} \\ \frac{n}{n+1} & \text{when } \frac{1}{n+1} < x \le \frac{1}{n} \end{cases}$$

So, $f\left(\frac{1}{n} + 0\right) = \lim_{h\to 0} f\left(\frac{1}{n} + h\right)$

$$= \lim_{h\to 0}\left(\frac{h-1}{n} + h\right)$$

$$f\left(\frac{1}{n} + 0\right) = 1 - \frac{1}{n}$$

and $f\left(\frac{1}{n} - 0\right) = \lim_{h\to 0}\left(\frac{n}{n+1} - h\right) = \frac{n}{n+1}$

Since, $f\left(\frac{1}{n} + 0\right) \ne f\left(\frac{1}{n} - 0\right)$

f is not continuous at $x = \frac{1}{4}$. The set of points of discontinuity $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$ of f has only one limit point at $x = 0$, so $f \in R[0, 1]$.

96. $f(x) = \frac{1}{a^{r-1}}$

when $\frac{1}{a^r} < x < \frac{1}{a^{r-1}}$, $r = 1, 2, 3, ...$

$$f\left(\frac{1}{a^r} + 0\right) = \lim_{h \to 0} f\left(\frac{1}{a^r} + h\right) = \frac{1}{a^{r-1}}$$

$$f\left(\frac{1}{a^r} - 0\right) = \lim_{h \to 0} f\left(\frac{1}{a^r} - h\right) = \frac{1}{a^r}$$

Thus, $f\left(\frac{1}{a^r} + 0\right) \neq f\left(\frac{1}{a^r} - 0\right)$

i.e. f is not continuous at $x = \frac{1}{a^r}$, $r = 1,2,3,....$ and the set of points of discontinuity $\frac{1}{a}, \frac{1}{a^2}, \frac{1}{a^3}....$ of f has only one limit point at $x = 0$ so $f \in R[0, 1]$.

98. $f(x) = [x]$ on $[0, 4]$

Consider $I_1 = [0, 1], I_2 = [1, 2],$
$I_3 = [2, 3], I_4 = [3, 4]$

then $$\int_0^4 [x]\,dx = \int_0^1 [x]\,dx + \int_1^2 [x]\,dx + \int_2^3 [x]\,dx + \int_3^4 [x]\,dx$$

$$= \int_0^1 0dx + \int_1^2 1.dx + \int_2^3 2dx + \int_3^4 3dx$$

$$= 1 + 2 + 3 = 6$$

99. $$\int_0^1 f(x)\,dx = \sum_{n=1}^{\infty} \int_{1/n+1}^{1/n} \frac{n}{n+1}\,dx$$

$$= \sum_{n=1}^{\infty} \frac{n}{n+1}\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$$

100. $$\int_0^1 f(x)\,dx = \sum_{r=1}^{\infty} \int_{1/a^r}^{1/a^{r-1}} \frac{1}{a^{r-1}}\,dx$$

$$= \sum_{r=1}^{\infty} \frac{1}{a^{r-1}}\left[\frac{1}{a^{r-1}} - \frac{1}{a^r}\right]$$

$$= 1\left(1 - \frac{1}{a}\right) + \frac{1}{a}\left(\frac{1}{a} - \frac{1}{a^2}\right) + \frac{1}{a^2}\left(\frac{1}{a^2} - \frac{1}{a^3}\right) + ...$$

$$= 1 - \frac{1}{a} + \frac{1}{a^2} - \frac{1}{a^3} + ...$$

$$= \frac{1}{1 + \frac{1}{a}} = \frac{a}{a+1}$$

104. $f(x) = \begin{cases} x^2 & x \text{ is rational} \\ x^3 & x \text{ is irrational} \end{cases}$ over $[0,2]$

Obviously f is bounded since

$$0 \le f(x) < 8 \ \forall x \in [0, 2]$$

Now $x^2 - x^3 = x^2(1 - x)$

So, $x^2 > x^3$ if $0 < x < 1$

and $x^2 < x^3$ if $1 < x \le 2$

If P be any partition of $[0,2]$ then for rth subinterval

$$M_r = \begin{cases} x^2 & 0 < x < 1 \\ x^3 & 1 < x < 2 \end{cases}$$

and $$m_r = \begin{cases} x^3 & 0 < x < 1 \\ x^2 & 1 < x < 2 \end{cases}$$

So, $$\int_0^{-2} f(x)\,dx = \int_0^1 x^2dx + \int_1^2 x^3dx$$

$$= \frac{1}{3}[x^3]_0^1 + \frac{1}{4}[x^4]_1^2$$

$$= \frac{1}{3} + \frac{15}{4} = \frac{49}{12}$$

107. $f(x) = \begin{cases} 0 & \text{when } 0 < x \le \frac{1}{2} \\ 1 & \text{when } \frac{1}{2} < x \le 1 \end{cases}$ over $[0,1]$

Let $$p = \left\{0, \frac{1}{2n}, \frac{2}{2n}, ... \frac{2n-1}{2n}, \frac{2n}{2n} = 1\right\}$$

be the partitions of $I = [0, 1]$ which divides I into $2n$ sub-intervals $\left[\frac{r-1}{2n}, \frac{r}{2n}\right]$ where $x = 1, 2, ...2n$ and $\Delta x_r = \frac{1}{2n}$.

So, $$M_r = \sup\left\{f(x) : x \in \left[\frac{r-1}{2n}, \frac{r}{2n}\right]\right\}$$

$$= \begin{cases} 0 & \text{when } r = 1, 2, ...n \\ 1 & \text{when } r = n+1, n+2,....2n \end{cases}$$

and $m_r = \begin{cases} 0 & \text{when } r = 1, 2, \ldots.n+1 \\ 1 & \text{when } r = n+2, n+3, \ldots.\ 2n \end{cases}$

Now $L(P,f) = \sum_{r=1}^{2n} m_r \Delta x_r$

$$= \frac{1}{2n}\sum_{r=1}^{2n} m_r = \frac{n-1}{2n}$$

and $U(P,f) = \sum_{r=1}^{2n} \mu_r \Delta x_r$

$$= \frac{1}{2n}\sum_{r=1}^{2n} M_r = \frac{n}{2n} = \frac{1}{2}$$

So, $\lim_{n\to\infty} \{U(P,f) - L(P,f)\}$

$$= \lim_{n\to\infty}\left[\frac{1}{2} - \frac{n-1}{2n}\right] = 0$$

i.e., $f \in R[0, 1]$ and

$$\int_0^1 f(x) = \lim_{n\to\infty} L(P,f)$$

$$= \lim_{n\to\infty} \frac{n-1}{2n} = \frac{1}{2}$$

168. Let $F(x) = \int_a^x f(t)\,dt$

then $F'(x) = f(x)\ \forall x \in [a,b]$ ...(1)

By hypothesis

$\varphi'(x) = f(x)\ \forall x \in [a,b]$ ...(2)

By (1) and (2) we have

$$F'(x) = \varphi'(x)$$

or $F'(x) - \varphi'(x) = 0\ \forall x \in [a,b]$

$\Rightarrow\ (F - \varphi)'(x) = 0$

$\Rightarrow\ f(f - \varphi)(x) = c$ for some $c \in R$

$\Rightarrow\ F(x) - \varphi(x) = c$

Thus, $F(x) = \varphi(x) + c$

So, $F(b) - F(a) = \varphi(b) - \varphi(a)$

Also $F(a) = \int_a^a f(t)\,dt = 0$

and $F(b) = \int_a^b f(t)\,dt$

Thus, we get

$$\int_a^b f(t)\,dt = \varphi(b) - \varphi(a)$$

or $\int_a^b f(x)\,dx = \varphi(b) - \varphi(a)$

191. $f(x) = x\ \forall x \in [0, 1]$ and $P = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$

Let $I_1 = \left[0, \frac{1}{3}\right], I_2 = \left[\frac{1}{3}, \frac{2}{3}\right], I_3 = \left[\frac{2}{3}, 1\right]$

Then, $m_1 = 0, m_2 = \frac{1}{3}, m_3 = \frac{2}{3}$

$$\Delta_1 = \frac{1}{3}, \Delta_2 = \frac{1}{3}, \Delta_3 = \frac{1}{3}$$

So, $L(P,f) = \sum_{r=1}^{3} m_r \Delta_r = m_1\Delta_1 + m_2\Delta_2 + m_3\Delta_3$

$$= 0\cdot\frac{1}{3} + \frac{1}{3} - \frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3}$$

$$= \frac{1}{a} + \frac{2}{a} = \frac{1}{3}$$

❍❍❍

CHAPTER

10 Convergence of Improper Integral

SOME DEFINITIONS

1. **Finite interval :** The interval whose length is finite is called a finite interval. Thus, the interval $[a, b]$ is finite or its length is $b - a$ which is finite.
2. **Infinite interval :** The interval whose length is infinite is called a infinite interval. Thus, the intervals (a, ∞), $(-\infty, b)$ are infinite intervals.
3. **Bounded functions :** A function $f(x)$ is said to be bounded over the interval I if there exists two real numbers a and b such that

$$a \le f(x) \le b \ \forall \, x \in I$$

or $$|f(x)| \le k \ \forall \, x \in I$$

For example, $f(x) = \cos x$ is bounded over the interval $[-\pi, \pi]$ since

$$|\cos x| \le \ \forall \, x \in [-\pi, \pi]$$

4. **Unbounded functions :** A function $f(x)$ is said to be unbounded over the interval I if it becomes infinite at a point belongs to I.

For example, $f(x) = \dfrac{x}{(x-a)(x-b)}$ is bounded at $x = a$ and $x = b$.

5. **Monotonic functions :** A real valued function f defined on an interval I is said to be :

(i) increasing (non-decreasing) if

$$x > y \Rightarrow f(x) \ge f(y) \ \forall \, x, y \in I$$

(ii) strictly increasing if

$$x > y \Rightarrow f(x) > f(y) \ \forall \, x, y \in I$$

(iii) decreasing (non-increasing) if

$$x > y \Rightarrow f(x) \le f(y) \ \forall \, x, y \in I$$

(iv) strictly decreasing if

$$x > y \Rightarrow f(x) < f(y) \ \forall \, x, y \in I$$

For example, $f(x) = \sin x$ is monotonic increasing in $\left[0, \dfrac{\pi}{2}\right]$ and monotonic decreasing in $\left[\dfrac{\pi}{2}, \pi\right]$.

6. **Proper Integrals**

The definite integral $\int_a^b f(x)\,dx$ is said to be a proper integral if the range of integration is finite and the integrand $f(x)$ is bounded. For examples, the integrals $\int_0^{\pi/2} \sin x\,dx$ and $\int_0^{\pi/2} \cos x dx$ are proper integrals.

7. **Improper Integrals**

The definite integral $\int_a^b f(x)\,dx$ is said to be an improper integral if

(i) The interval (a, b) is infinite and $f(x)$ is bounded over this interval.

(ii) The interval (a, b) is finite and $f(x)$ is not bounded over this interval.

(iii) Neither the interval (a, b) is finite nor $f(x)$ is bounded over this interval.

For examples,

$$\int_a^\infty \frac{dx}{x}, \int_{-\infty}^\infty \frac{dx}{1+x}, \int_0^2 \frac{dx}{(x-1)(x-2)}$$

are the improper integrals.

8. **Improper Integral of the First Kind (Infinite Integrals)**

A definite integrals $\int_a^b f(x)\,dx$ in which the range of integration is infinite (either $a = -\infty$ or $b = \infty$ or both) and the integrand $f(x)$ is bounded, is called an improper integral of the first kind or an infinite integral.

For example, $\int_0^\infty \frac{dx}{1+x^4}$ is an improper integral of the first kind.

For improper integrals of the first kind, we define

(i) $\int_a^\infty f(x)\,dx = \lim_{x\to\infty}\int_a^x f(x)\,dx$, provided the limit exist finitely

(ii) $\int_{-\infty}^b f(x)\,dx = \lim_{x\to\infty}\int_{-x}^b f(x)\,dx$, provided the limit exists finitely

(iii) $\int_{-\infty}^\infty f(x)\,dx = \lim_{x_1\to\infty}\int_{-x_1}^c f(x)\,dx + \lim_{x_2\to\infty}\int_c^{x_2} f(x)\,dx$, provided both the limits exist finitely.

9. **Improper Integrals of the Second Kind**

A definite integral $\int_a^b f(x)\,dx$ in which the range of integration is finite but the integrand $f(x)$ is unbounded at one or more points of $[a, b]$, is called an improper integral of the second kind.

For example, $\int_0^3 \frac{x\,dx}{(x-1)(x-2)}$ is improper integral of the second kind.

If $\int_a^b f(x)\,dx$ is an improper integral of the second kind then the value is defined as follows :

(i) If $f(x)\to\infty$ or $x\to a$ only then

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0}\int_{a+\varepsilon}^b f(x)\,dx,$$

provided the limit exists finitely.

(ii) If $f(x)\to\infty$ as $x\to b$ only then

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0}\int_a^{b-\varepsilon} f(x)\,dx,$$

provided the limit exists finitely.

(iii) If $f(x)\to\infty$ at $x\to c$ only where $a<c<b$ then

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0}\int_a^{c-\varepsilon} f(x)\,dx + \lim_{\varepsilon'\to 0}\int_{c+\varepsilon'}^b f(x)\,dx$$

provided that both these limits exist finitely.

(iv) If $f(x)$ is unbounded at both the points a and b of the interval (a, b) and is bounded at each other point of this interval then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

where $a<c<b$ and the value of the integral exists only if each of the integrals on the right hand side exists.

CONVERGENCE OF IMPROPER INTEGRAL

1. If the limit of an improper integral or defined above, is a definite finite number then the given definite integral is convergent and the value of the integral is equal to the value of that limit.
2. If the limit is ∞ or $-\infty$, the integral is said to be divergent and the value of the integral does not exist.
3. If the limit is neither a definite number nor ∞ or $-\infty$, the integral is said to be oscillatory and the value of the integral does not exist.
4. The integral $\int_a^\infty f(x)\,dx$ is said to converge to the value I, if for any arbitrary chosen number $\varepsilon > 0$, however small but not zero, there exists a corresponding positive number (integer) n_0 such that

$$\left|\int_a^b f(x)\,dx - I\right| < \varepsilon \;\forall\; b \ge n_0$$

Similarly, we can define the convergence of an integral, when the lower limit is infinite or when the integrand becomes infinite at the upper or lower limit.

TEST FOR CONVERGENCE OF IMPROPER INTEGRALS OF THE FIRST KIND

If an integral of the form $\int_a^\infty f(x)\,dx$ or $\int_{-\infty}^b f(x)\,dx$ cannot be actually integrated, its convergence is determined by the following tests.

1. **Comparisons test :**

Let $f(x)$ and $g(x)$ be two functions which are bounded and integrable in the interval (a, ∞). Also, let $g(x)$ is positive.

(i) If $|f(x)| \le g(x)$ for $x \ge a$ and $\int_a^x g(x)\,dx$ is convergent, then $\int_a^\infty f(x)\,dx$ is also convergent.

(ii) If $|f(x)| \ge g(x)$ for all values of x greater than some number $x_0 > a$ and $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is also divergent.

(iii) **Alternative form :** If $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)}$ is a definite number, other than zero, the integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both converge or both diverge.

Results :

1. The comparison integral $\int_a^\infty \frac{dx}{x^n}$, where $a > 0$, is convergent when $n > 1$ and divergent when $n \le 1$.

2. **The μ-test** : Let $f(x)$ be bounded and integrable in the interval (a, ∞) where $a > 0$.

(i) If there is a number $\mu > 1$, such that $\lim\limits_{x\to\infty} x^\mu f(x)$ exists, then $\int_a^\infty f(x)\,dx$ is convergent.

(ii) If there is a number $\mu \le 1$, such that $\lim\limits_{x\to\infty} x^\mu f(x)$ exists and is non-zero, then $\int_a^\infty f(x)\,dx$ is divergent and the same is true if $\lim\limits_{x\to\infty} x^\mu f(x)$ is $+\infty$ or $-\infty$.

Note : While applying the μ-test, the value of μ is usually taken to be equal to the highest power of x in the denominator of the integrand minus the highest power of x in the numerator of the integrand.

3. **Abel's test for the convergence of integral of a product**

If $\int_a^\infty f(x)\,dx$ converges and $\phi(x)$ is bounded and monotonic for $x > a$, then $\int_a^\infty f(x)\,\phi x\,dx$ is convergent.

4. **Dirichlet's test for the convergence of integral of a product**

If $f(x)$ be bounded and monotonic in the interval $[a, \infty]$ and if $\lim\limits_{x\to\infty} f(x) = 0$, then the integral $\int_a^\infty f(x)\,\phi x\,dx$ converges provided $\left|\int_a^x \phi(x)\,dx\right|$ is bounded as x takes all finite values.

5. **Absolute convergence :** The infinite integral $\int_a^\infty f(x)\,dx$ is said to be absolutely convergent if the integral $\int_a^\infty |f(x)|\,dx$ is convergent.

If the integral $\int_a^\infty f(x)\,dx$ is absolutely convergent, it is necessarily convergent but not conversely.

TEST FOR CONVERGENCE OF IMPROPER INTEGRALS OF THE SECOND KIND

Now we shall discuss the convergence of a definite integral of the type $\int_a^b f(x)\,dx$ in which the range of integration is finite and the integrand $f(x)$ is unbounded at one or more points of the given interval $[a, b]$.

1. **Comparison test**

Consider the improper integral $\int_a^b f(x)\,dx$, where the interval of integration (a, b) is finite and $f(x)$ is unbounded at $x = a$. Let $g(x)$ is positive in $(a + \varepsilon, b)$ for $\varepsilon \to 0$ then

(i) If $|f(x)| \le g(x)$ for all $x\, \varepsilon\, [a + \varepsilon, b]$ and $\int_a^b g(x)\,dx$ is convergent, then $\int_a^b f(x)\,dx$ is also convergent.

(ii) If $|f(x)| \geq g(x)$ for all $x \, \varepsilon \, (a+\varepsilon, b)$ and $\int_a^b g(x)\,dx$ is divergent, then $\int_a^b f(x)\,dx$ is also divergent.

2. **Alternative form of the above comparison test**

If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is a non-zero definite finite quantity, the integrals $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ are either both convergent or both divergent.

Results :

(i) The comparison integral $\int_a^b \frac{dx}{(x-a)^n}$ is convergent when $n<1$ and divergent when $n \geq 1$.

(ii) If $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$ and $\int_a^b g(x)\,dx$ is converges then $\int_a^b f(x)\,dx$ is also converges.

(iii) If $\lim_{x\to a} \frac{f(x)}{g(x)} = \infty$ and $\int_a^b g(x)\,dx$ is diverges then $\int_a^b f(x)\,dx$ is also diverges.

(iv) If $f(x)$ is unbounded at $x=b$ then find $\lim_{x\to b} \frac{f(x)}{g(x)}$ and the result will be same.

3. **The μ-Test**

Let $f(x)$ be unbounded at $x=a$ and be bounded and integrable in $(a+\varepsilon, b)$ where $0<\varepsilon<b-a$.

(i) If there exist a number μ lying between 0 and 1 such that $\lim_{x\to a+0} (x-a)^{\mu} f(x)$ exist then $\int_a^b f(x)\,dx$ is convergent.

(ii) If there exist a number $\mu \geq 1$ such that $\lim_{x\to a+0} (x-a)^{\mu} f(x)$ exists, and is non-zero then $\int_a^b f(x)\,dx$ is divergent, and the same is true if $\lim_{x\to a+0} (x-a)^{\mu} f(x) = \infty$ or $-\infty$.

(iii) If $f(x)$ is unbounded at $x=b$ then we find $\lim_{x\to b-0} (b-x)^{\mu} . f(x)$, but the other condition of the test remaining the same.

4. **Abel's test for the convergence of integral of a product**

If $\int_a^b f(x)\,dx$ converges and $\phi(x)$ is bounded an monotonic in the interval (a, b) then $\int_a^b f(x) . \phi(x)\,dx$ is convergent.

5. **Dirichlet's test for the convergence of integral of a product**

If $\int_{a+\varepsilon}^b f(x)\,dx$ is bounded and $\phi(x)$ is bounded and monotonic in (a, b) such that

$$\lim_{x\to a} \phi(x) = 0, \text{ then } \int_a^b f(x)\,\phi x\,dx$$

is convergent.

6. **Absolute convergence of $\int_a^b f(x)\,dx$**

An improper integral $\int_a^b f(x)\,dx$ is said to be absolutely convergence if the integral $\int_a^b |f(x)|\,dx$ is convergent.

IMPROPER INTEGRAL OF THE THIRD KIND

The definite integral with infinite limits and infinite integrand is called improper integral of third kind. Thus, if is a combinations of both first kind and second kind of unproper integral.

For example, $\int_0^\infty \frac{dx}{x(x-1)}$ is an improper integral of third kind.

EXERCISE

MULTIPLE CHOICE QUESTIONS

Direction : Each of the following questions has four alternative answers. One of them is correct. Choose the correct answer.

1. The integral $\int_1^\infty \frac{dx}{\sqrt{x}}$ is :
   a. Convergent b. Divergent
   c. Oscillatory d. None of these

2. The integral $\int_0^3 \frac{dx}{(x-1)(x-2)}$ is an improper integral of :
   a. First kind b. Second kind
   c. Third kind d. None of these

3. If one or both limits of the integral $\int_a^b f(x)\,dx$ is/are infinite then it is an improper integral of :
   a. First kind b. Second kind
   c. Third kind d. None of these

4. The integral $\int_1^\infty \frac{dx}{x}$ is :
   a. Converges to ∞ b. Diverges to ∞
   c. Converges to $-\infty$ d. Diverges to $-\infty$

5. Consider the integrals :
   (i) $\int_{-\infty}^0 e^x\,dx$ and (ii) $\int_{-\infty}^0 e^{-x}\,dx$ then
   a. (i) is convergent (ii) is divergent
   b. Both are convergent
   c. Both are divergent
   d. (i) is divergent and (ii) is convergent

6. The integral $\int_1^\infty \frac{dx}{x^3+a^2}$ is :
   a. Convergent b. Divergent
   c. Oscillatory d. None of these

7. The integral $\int_3^\infty \frac{dx}{(x-2)}$ is : **[Meerut 2017]**
   a. Convergent b. Divergent
   c. Oscillatory d. None of these

8. The integral $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$ is :
   a. Convergent b. Divergent
   c. Oscillatory d. None of these

9. The integral $\int_0^1 e^{-x^2}\,dx$ is :
   a. Convergent b. Divergent
   c. Oscillatory d. None of these

10. The integral $\int_a^b f(x)\,dx$ in which (a,b) is finite but integrand is unbounded at one or more points of $[a,b]$ then it is improper integral of :
    a. First kind b. Second kind
    c. Third kind d. Zero kind

11. The integral $\int_0^\infty e^{-mx}\,dx$ $(m>0)$ if :
    a. Convergent b. Divergent
    c. Oscillatory d. None of these

12. The integral $\int_0^1 \frac{dx}{\sqrt{x}}$ is :
    a. Converges to 0 b. Diverges to ∞
    c. Converges to 2 d. Diverges to $-\infty$

13. Every proper integral is :
    a. Convergent
    b. Divergent
    c. May be convergent or divergent
    d. Oscillatory

14. The integral $\int_a^b \frac{dx}{(x-a)^n}$ is convergent for :
    a. $n>1$ b. $n\geq 1$
    c. $n<1$ d. $n\leq 1$

15. If limit of the integral $\int_a^b f(x)\,dx$ are finite and integrand is infinite then it is a proper integral of :
    a. Zero kind b. First kind
    c. Second kind d. Third kind

16. The integral $\int_0^\infty e^{2x}\,dx$ is :
    a. Convergent b. Divergent
    c. Oscillatory d. None of these

17. $\int_a^{\infty} \frac{dx}{x\sqrt{(1+x^2)}}$ where $a > 0$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

18. The integral $\int_1^{\infty} e^{-x} \frac{\sin x}{x} dx$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

19. The integral $\int_{-\infty}^{0} \sinh x \, dx$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

20. The integral $\int_a^{\infty} \frac{dx}{x^n}$ where $a > 0$ is divergent when :

a. $n > 1$ only b. $n \geq 1$ only
c. $n < 1$ only d. $n \leq 1$ only

21. If sum of two improper integrals is convergent then both of them are :

a. Convergent
b. Divergent
c. One is convergent and other is divergent
d. None of these

22. The integral $\int_a^{b} \frac{dx}{(x-a)^n}$ is divergent for :

a. $n > 1$ b. $n \geq 1$
c. $n < 1$ d. $n \leq 1$

23. The definite integral with infinite limits and unbounded integrand is improper integral of :

a. Zero kind b. First kind
c. Second kind d. Third kind

24. The integral $\int_0^{\infty} e^{-x} \frac{\sin x}{x} dx$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

25. The integral $\int_0^{a} \frac{x\,dx}{(1+x)^3}$ where $a > 0$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

26. If $f(x)$ is bounded and integrable over (a, ∞), $a > 0$ such that $\lim_{n\to\infty} x^{\mu} f(x)$ exists its then $\int_a^{\infty} f(x)\,dx$ is convergent when :

a. $\mu = 1$ b. $\mu > 1$
c. $\mu < 1$ d. $\mu \geq 1$

27. The integral $\int_0^{\infty} \frac{dx}{(1+x)^{2/3}}$ is :

a. Converge to 0 b. Converge to 1
c. Diverge to ∞ d. Diverge to $-\infty$

28. The integral $\int_0^{1} \frac{dx}{\sqrt{1-x}}$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

29. The integral $\int_1^{\infty} \frac{dx}{x^{1/3}(1+x^{1/2})}$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

30. The integral $\int_1^{\infty} x^{n-1} e^{-x}\, dx$ is convergent for :

a. $n \geq 1$ only b. $n \leq 1$ only
c. $n = 0$ only d. All vale of n

31. If $\int_a^{\infty} f(x)\,dx$ converges and $\phi(x)$ is bounded and monotonic for $x > a$ then $\int_a^{\infty} f(x)\,\phi(x)\,dx$ is convergent then it is called :

a. Abel's test b. Dirichelet's test
c. μ-test d. Comparison test

32. The integral $\int_{-\infty}^{0} \cos hx\, dx$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

33. The integral $\int_0^{\infty} \cos x\, dx$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

34. The integral $\int_0^{a} \frac{\sin^2 x}{x^2} dx$, where $a > 0$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

35. If the sum of two improper integrals is divergent then :

a. Both are convergent

b. Both are divergent only

c. One is convergent and one is divergent only

d. (b) and (c) are true

36. The integral $\int_a^b \frac{dx}{(b-x)^n}$ is convergent for :

[Kanpur 2018]

a. $n > 1$ b. $n \geq 1$

c. $n < 1$ d. $n \leq 1$

37. The integral $\int_0^1 \frac{dx}{x^3}$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

38. The integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

39. If $f(x)$ is bounded and integrable on (a,∞) where $a > 0$ such that $\lim_{n\to\infty} x^{\mu} f(x)$ exists and non-zero then $\int_a^{\infty} f(x)\,dx$ is divergent when :

a. $\mu \geq 1$ b. $\mu > 1$

c. $\mu \leq 1$ d. $\mu = 1$

40. The integral $\int_a^{\infty} f(x)\,dx$ is said to be absolutely convergent if $\int_a^{\infty} |f(x)|\,dx$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

41. The integral $\int_{-\infty}^{\infty} e^{-x}\,dx$ is :

a. Convergent to 0 b. Divergent to ∞

c. Divergent to $-\infty$ d. Convergent to 2

42. The integral $\int_a^{\infty} \frac{x\,dx}{(1+x)^3}$ where $a > 0$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

43. The integral $\int_a^{\infty} \frac{\cos x}{x^2}\,dx$ where $a > 0$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

44. The integral $\int_0^{\infty} \frac{\sin mx}{a^2+x^2}\,dx$ is :

a. Absolutely convergent

b. Not absolutely convergent

c. Not convergent

d. Oscillatory

45. The integral $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$ is :

a. Convergent to 0 b. Divergent to ∞

c. Divergent to $-\infty$ d. Convergent to π

46. The integral $\int_a^b \frac{dx}{(b-x)^n}$ is divergent for :

a. $n \geq 1$ b. $n \leq 1$

c. $n \geq 0$ d. $n \leq 0$

47. The integral $\int_0^a \frac{x^{2m}}{1+x^{2n}}\,dx$ where $a > 0$ and m and n are positive integer is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

48. The integral $\int_0^{\infty} \frac{x^{3/2}}{(b^2x^2+c^2)}\,dx$ is :

a. Convergent b. Absolutely convergent

c. Divergent d. Oscillatory

49. The integral $\int_0^1 \frac{dx}{1-x}$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

50. The integral $\int_{-1}^1 \frac{dx}{x^2}$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

51. The integral $\int_a^{\infty} \frac{dx}{x(\sin x)^{n+1}}$, $a > 1$ is divergent when :

a. $n \geq 1$ b. $n \geq 0$

c. $n \leq 1$ d. $n \leq 0$

52. The integral $\int_0^1 x^{n-1} e^{-x}\,dx$ is divergent for :

a. $p < 0$ b. $p > 0$

c. $p \geq 0$ d. $p \leq 0$

53. The integral $\int_{-1}^{1} \frac{dx}{x^{2/3}}$ is :

a. Convergent to 2/3 b. Convergent to 0
c. Convergent to 3 d. Convergent to 6

54. $\int_a^\infty \frac{\sin^2 x}{x^2} dx$ where $a > 0$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

55. The integral $\int_a^\infty \frac{x^{2m}}{1+x^{2n}} dx$ is divergent when (where m in are positive integers) :

a. $m \le n$ b. $m < n$
c. $m \ge n$ d. None of these

56. $\int_0^1 \sin x^2 \, dx$ is :

a. Convergent only b. Proper integral only
c. Bounded only d. All the above

57. The integral $\int_0^{\pi/2} \frac{\cos x}{x^n} dx$ is convergent for :

a. $n < 1$ b. $n \ge 1$
c. $0 < n < 2$ d. $n > -1$

58. The integral $\int_0^1 e^{-x} \frac{\sin x}{x} dx$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

59. If $f(x)$ is bounded and integrable over (a, ∞) where $a > 0$ such that $\lim_{x \to \infty} x^\mu f(x)$ is $+\infty$ or $-\infty$ then $\int_a^\infty f(x)\, dx$ is divergent when :

a. $\mu > 1$ only b. $\mu < 1$ only
c. $\mu \ge 1$ d. $\mu \le 1$

60. The integral $\int_0^\infty \frac{x\, dx}{(1+x)^3}$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

61. If $f(x)$ is bounded and monotonic in (a, ∞) and $\lim_{x \to \infty} f(x) = 0$, then the integral $\int_a^\infty f(x)\,\phi(x)\, dx$ is convergent, provided $\left|\int_a^\infty \phi(x)\, dx\right|$ is bounded as x takes all finite values then it is called :

a. Comparison test b. Abel's test
c. Dirichlet's test d. μ-test

62. The integral $\int_a^\infty \frac{\sin x}{x^2} dx$ where $a > 0$ is :

a. Convergent b. Divergent
c. Oscillatory d. Unbounded

63. The integral $\int_b^\infty \frac{x^{3/2}\, dx}{\sqrt{(x^4 - a^4)}}$, where $b > a$ is :

a. Convergent b. Divergent
c. Oscillatory d. Absolutely convergent

64. The integral $\int_0^\infty \frac{dx}{x^{1/3}(1+\sqrt{x})}$ is :

a. Convergent b. Absolutely convergent
c. Oscillatory d. Divergent

65. The integral $\int_1^\infty e^{-x^2} dx$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

66. The integral $\int_a^\infty \frac{dx}{x(\log x)^{n+1}}$, where $a > 1$ is convergent when :

a. $n > -1$ b. $n > 0$
c. $0 < n < 1$ d. $n < 1$

67. The integral $\int_a^\infty \frac{dx}{(x-2)^2}$ is :

a. Convergent
b. Divergent
c. Both convergent and divergent
d. None of these

68. If $\int_a^b f(x)\, dx$ is convergent and $\phi(x)$ is bounded and monotonic in (a, b) such that $\int_a^b f(x)\,\phi(x)$ is convergent then it is called : **[Kanpur 2018]**

a. μ-test b. Comparison test
c. Abel's test d. Dirichlet's test

69. The integral $\int_1^2 \frac{dx}{\sqrt{x^2 - 1}}$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

70. The integral $\int_0^1 x^{n-1} \log x \, dx$ is divergent for :

a. $n \geq 0$ b. $n \leq 0$

c. $n \geq 1$ d. $0 \leq n \leq 1$

71. The integral $\int_0^\infty \frac{1-\cos x}{x^2} dx$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

72. The integral $\int_0^1 \frac{dx}{(x+1)\sqrt{1-x^2}}$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

73. The integral $\int_a^\infty (1-e^{-x}) \frac{\cos x}{x^2} dx$ where $a > 0$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

74. The integral $\int_a^\infty \frac{\sin x}{\sqrt{x}} dx$ where $a > 0$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

75. The integral $\int_1^\infty \sin x^2 \, dx$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

76. The integral $\int_a^\infty \frac{\cos \alpha x}{x} dx$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

77. If $g(x)$ is positive when $|f(x)| \leq g(x) \ \forall x \geq a$ and $\int_0^\infty g(x) \, dx$ is convergent then $\int_a^\infty f(x) \, dx$ is :

a. Convergent b. Divergent

c. Oscillatory d. (a) or (b)

78. The integral $\int_0^\infty e^{-x} \cos mx \, dx$ is :

a. Oscillatory

b. Divergent

c. Absolutely convergent

d. None of these

79. The integral $\int_0^1 \frac{\sec x}{x} dx$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

80. The integral $\int_2^\infty \frac{dx}{\sqrt{x^2-x-1}}$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

81. The integral $\int_0^{\pi/2} \cdot \frac{\sin x}{x^{n+1}} dx$ is convergent for :

a. $n > 1$ b. $n \geq 1$

c. $n < 1$ d. $n \leq 1$

82. The integral $\int_0^\infty \frac{\sin x}{x^{3/2}} dx$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

83. The integral $\int_a^\infty e^{-x} \frac{\sin x}{x^2} dx$ where $a > 0$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

84. If $\int_{a+\varepsilon}^b f(x) \, dx$ be bounded and $\phi(x)$ is bounded and monotonic in the interval (a, b) such that $\lim_{x \to a} \phi(x) = 0$, then $\int_a^b f(x) \phi(x)$ is convergent is known as :

a. Abel's test b. Comparison test

c. µ-test d. Dirichlet's test

85. The integral $\int_0^\infty \frac{\sin x}{x} dx$ is : **[Kanpur 2018]**

a. Convergent b. Divergent

c. Oscillatory d. None of these

86. The integral $\int_1^2 \frac{dx}{\sqrt{x^4-1}}$ is :

a. Oscillatory b. Divergent

c. Convergent d. Bounded

87. The integral $\int_0^\infty x^{n-1} e^{-x} \, dx$ is convergent for :

a. $n < 0$ b. $n \leq 0$

c. $n > 0$ d. $n \geq 0$

88. The integral $\int_0^1 x^{n-1} \log x \, dx$ is convergent for :

a. $n < 0$ b. $n \leq 0$

c. $n > 0$ d. $n \geq 0$

89. The integral $\int_0^{\pi/2} \log \sin x \, dx$ is : **[Meerut 2017]**

a. Convergent b. Divergent
c. Oscillatory d. None of these

90. The integral $\int_\pi^\infty \frac{\sin^2 x}{x^2} dx$ is :

a. Divergent b. Convergent
c. Oscillatory d. None of these

91. $\int_q^\infty \frac{dx}{\sqrt{(x+1)(x-1)}}$ is : **[Meerut 2017]**

a. Divergent
b. Convergent
c. Oscillatory
d. Absolutely convergent

92. The integral $\int_0^\infty e^{-ax} \frac{\sin x}{x} dx$, where $a \geq 0$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

93. The integral $\int_1^\infty \frac{\sin x}{x^n} dx$ is :

a. Bounded only
b. Convergent only
c. Absolutely convergent only
d. All the above are true

94. If $f(x)$ is unbounded at $x = a$ and $\lim_{x \to a} \frac{f(x)}{g(x)}$ is a non-zero definite number such that $\int_a^b g(x)\, dx$ is convergent then $\int_a^b f(x)\, dx$ is :

a. Absolutely convergent
b. Convergent
c. Divergent
d. Oscillatory

95. The integral $\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

96. $\int_0^\infty \frac{\sin mx}{x^2 + a^2}$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

97. The integral $\int_0^\infty \frac{x \sin x}{1+x^2} dx$ is :

a. Oscillatory b. Convergent
c. Divergent d. None of these

98. The integral $\int_0^{\pi/2} \frac{\cos x}{x^2} dx$ is :

a. Divergent
b. Convergent
c. Absolutely convergent
d. Oscillatory

99. The integral $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

100. The integral $\int_0^{\pi/2} \frac{\cos x}{x^n} dx$ is divergent for :

a. $n \geq 1$ b. $n \leq 1$
c. $0 \leq n \leq 1$ d. $n \geq 0$

101. The integral $\int_0^\infty \frac{\cos x}{1+x^2} dx$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

102. The integral $\int_0^\infty e^{-a^2x^2} \cos bx \, dx$ is :

a. Convergent only
b. Absolutely convergent
c. Divergent
d. Oscillatory

103. The integral $\int_0^\infty \frac{x^2\, dx}{(1+x)^3}$ is :

a. Oscillatory b. Divergent
c. Convergent d. Absolutely convergent

104. $\int_a^\infty x^{n-1} e^{-x} \, dx$, where $a > 0$ is :

a. Convergent b. Divergent
c. Oscillatory d. None of these

105. The integral $\int_0^{\pi/4} \sqrt{\cot x} \, dx$ is :

a. Oscillatory b. Divergent
c. Convergent d. None of these

106. The integral $\int_0^\infty \frac{dx}{x\sqrt{1+x^2}}$ is :

a. Divergent

b. Convergent

c. Absolutely convergent

d. None of these

107. The integral $\int_0^\infty \frac{\sqrt{x}}{x^2+4}dx$ is :

a. Convergent b. Absolutely convergent

c. Divergent d. Oscillatory

108. Which of the following intervals are infinite intervals: **[Kanpur 2018]**

a. (a,∞) b. $(-\infty, b)$

c. $(-\infty, \infty)$ d. All of these

109. The definite integral $\int_a^b f(x)\,dx$ is called proper integral if :

(I) The range of integration is finite

(II) The integrand $f(x)$ is bounded

a. I is true only

b. II is true only

c. I and II both are true

d. None of these

110. If the limit of improper integral is ∞ or $-\infty$ then the integral is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

111. If Σu_n is convergent then $\sum \frac{u_n}{n}$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

112. If $\int_a^\infty |f(x)|\,dx < \infty$ then $\int_a^\infty f(x)\,dx$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

113. The improper integral $\int_1^\infty \frac{dx}{x^{3/2}}$ is : **[Kanpur 2018]**

a. Convergent and its value is 2

b. Convergent and its value is $\frac{1}{2}$

c. Convergent and its value is 0

d. Divergent

114. The integral $\int_0^\infty \frac{x\tan^{-1}x}{(1+x^4)^{1/3}}dx$ is :

a. Convergent b. Divergent

c. Oscillatory d. None of these

115. Which of the following is convergent ?

a. $\int_{e^2}^\infty \frac{dx}{x\log x}$ b. $\int_0^\infty x\sin x\,dx$

c. $\int_{-\infty}^\infty \frac{1+x}{1+x^2}dx$ d. $\int_a^\infty (1-e^{-x})\frac{\cos x}{x^2}dx$

116. The integral $\int_1^4 \frac{dx}{(x-1)(x+1)}$ is : **[Kanpur 2018]**

a. Proper integral

b. Improper integral of the first kind

c. Improper integral of the second kind

d. Improper integral of the third kind

117. The integral $\int_0^1 \frac{\sec x}{x}dx$ is :

a. Convergent b. Divergent

c. Ooscillatory d. None of these

118. The integral $\int_0^{\pi/2} \frac{x^m}{\sin^n x}dx$ exist when :

a. $n<m+1$ b. $n>m+1$

c. $n=m+1$ d. $n=m$

119. Which of the following is divergent ?

a. $\int_{-1}^1 \frac{dx}{(2-x)\sqrt{1-x^2}}$ b. $\int_0^1 \frac{\log x}{\sqrt{2-x}}dx$

c. $\int_0^1 \frac{dx}{\sqrt{1-x}}$ d. None of these

120. The integral $\int_0^{\pi/2} \frac{\log(\sin x)}{(\sin x)^n}dx$ is convergent when :

a. $0<n<1$ b. $n>1$

c. $n<1$ d. $n=0$

121. The integral $\int_0^\infty \frac{dx}{1+x^2}$ is :

a. Proper integral of first kind

b. Proper integral of second kind

c. Improper integral of first kind

d. None of these

122. If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is a non-zero definite finite quantity then $\int_a^b f(x)\,dx$ and $\int_a^b g(x)\,dx$ are either both convergent or both divergent, this is called :

a. Abel's test b. Dirichelt's test

c. Comparison test d. None of these

123. The sum of finite number of improper integrals diverges iff one or more of these integrals : **[Meerut 2017]**

a. Convergent b. Divergent

c. Oscillatory d. None of these

124. The definite integral $\int_a^b f(x)\,dx$ is said to be a proper integral if : **[Meerut 2017]**

a. The range of integration is finite

b. The integrand of $f(x)$ is bounded

c. Both (a) and (b)

d. None of these

125. The integral $\int_0^4 \frac{dx}{(x-2)(x-3)}$ is an improper integral of the : **[Meerut 2017]**

a. First kind b. Second kind

c. Neither (a) nor (b) d. None of these

126. The integral $\int_a^\infty \frac{dx}{x^p}$, where $a > 0$ is divergent when : **(Meerut 2017,18)**

a. $p = 1$ b. $p < 1$

c. $p > 1$ d. $p \le 1$

127. The integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is convergent when : **[Meerut 2016]**

a. $m > 0$ b. $n > 0$

c. $m > 0, n > 0$ d. None of these

128. The integral $\int_a^\infty \frac{1}{x^2} \cdot \sin^2 x\,dx,\ a > 0$ is : **[Meerut 2016]**

a. Convergent

b. Divergent

c. Uniformly convergent

d. None of these

129. The integral $\int_0^\infty \frac{dx}{1+x^2}$ is an improper integral of the : **[Meerut 2016]**

a. First kind b. Second kind

c. Neither (a) or (b) d. None of these

130. The integral $\int_0^\infty e^{-x} x^{n-1}\,dx$ is convergent if :

a. $n = 0$ b. $n > 0$

c. $n < 0$ d. None of these

131. The integral $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ is : **[Meerut 2016]**

a. Convergent

b. Divergent

c. May or may not be convergent

d. None of these

132. By comparison test integral $\int_0^1 \frac{\sec x}{x}\,dx$ is divergent because : **[Meerut 2016]**

a. $\int_0^1 \frac{1}{x}\,dx$ is convergent

b. $\int_0^1 \sec x\,dx$ is convergent

c. $\int_0^1 \sec x\,dx$ divergent

d. $\int_0^1 \frac{1}{x}\,dx$ is divergent

133. The value of integral $\int_1^\infty \frac{dx}{\sqrt{x}}$ is : **[Meerut 2016]**

a. 2 b. –2

c. 1 d. ∞

134. For the integral $\int_0^\infty v^{-mx}\,dx$ where $m > 0$, which is true : **[Meerut 2016]**

a. The value of integral is $\frac{1}{m}$

b. Integral convergent

c. Both (a) and (b)

d. None of these

135. The sum of finite number of improper integrals diverges iff one or more of these integrals : **[Meerut 2017]**

a. Converges b. Diverges

c. Oscillates d. None of these

136. Which of the following integrals is divergent ? **[Meerut 2017]**

a. $\int_3^\infty \frac{dx}{(x-2)^2}$ b. $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$

c. $\int_1^\infty \frac{dx}{(x^3+1)^{1/2}}$ d. $\int_2^\infty \frac{dx}{(x^2-1)^{1/2}}$

137. The integral $\int_0^{\pi/2} \log \sin x \, dx$ is : **[Meerut 2017]**

a. Converges

b. Diverges

c. Both (a) and (b) false

d. None of these

138. The definite integral $\int_a^b f(x)\,dx$ is said to be a proper integral if : **[Meerut 2017]**

a. The range of integration is finite

b. The integrand of $f(x)$ is bounded

c. Both (a) and (b)

d. None of these

139. The integral $\int_3^\infty \frac{dx}{(x-2)^2}$ is : **[Meerut 2017]**

a. Convergent   b. Divergent

c. (a) false   d. None of these

140. The integral $\int_0^1 \frac{dx}{x^3(1+x^2)}$ is :

[Meerut 2017; Kanpur 2017]

a. Convergent

b. Divergent

c. Both (a) and (b) false

d. None of these

141. If $\int_a^b f(x)\,dx$ converges and $\phi(x)$ is bounded and monotonic for $a \le x \le b$ then $\int_a^b f(x)\,\phi(x)\,dx$ is :

[Meerut 2017]

a. Converges   b. Diverges

c. Neither (a) nor (b)   d. None of these

142. The integral $\int_0^\infty \frac{x^{a-1}}{1+x} dx$ is : **[Meerut 2017]**

a. Convergent if $0 < a < 1$

b. Divergent if $a \ge 1$

c. Divergent if $a \le 0$

d. All above

143. The function $f(x) = \frac{x}{(x-1)(x-2)}$ is unbounded at the points : **[Meerut 2017]**

a. $x = 0, x = 1$ b. $x = 1, x = 2$

c. $x = -1, x = 2$ d. $x = 1, x = -2$

144. Which is true : **[Meerut 2018]**

a. $\int_0^\infty \frac{\cos mx}{x^2 + a^2} dx$ is convergent

b. $\int_0^\infty \frac{\sin^2 x}{x}$ is convergent

c. $\int_0^\infty \frac{\sin x}{x}$ is divergent

d. Both (a) and (b) true

145. Which integral is divergent : **[Meerut 2018]**

a. $\int_1^\infty \frac{1}{\sqrt{x}} dx$ b. $\int_0^\infty \frac{1}{x} dx$

c. $\int_1^\infty \frac{dx}{x^{3/2}}$ d. None of these

146. The integral $\int_0^\infty \frac{4a}{x^2 + 4a^2} dx$ is : **[Meerut 2018]**

a. Convergent   b. Divergent

c. Converges to π d. Equal to zero

147. The value of the integral $\int_0^1 \frac{1}{\sqrt{x}} dx$: **[Meerut 2018]**

a. 2   b. –2

c. 1   d. –1

148. The integral $\int_{-\infty}^\infty \frac{1}{1+x^2} dx$ is : **[Meerut 2018]**

a. Equal to zero   b. Convergent

c. Divergent   d. Converges to π

149. The integral $\int_{-\infty}^\infty e^{-x}\,dx$ is : **[Meerut 2018]**

a. Convergent   b. Convergent to zero

c. Divergent   d. Equal to zero

150. The value of integral $\int_0^\infty e^{-mx}.dx$ is : **[Meerut 2018]**

a. m b. $\frac{1}{m}$

c. $-m$ d. $-\frac{1}{m}$

151. The integral $\int_{-\infty}^{\infty} \frac{1+x}{1+x^2} dx$ is : **[Meerut 2014]**
a. Convergent
b. Divergent
c. May or may not be convergent
d. None of these

152. Integral $\int_{a}^{\infty} \frac{\cos mx}{a^2+x^2} dx$ is : **[Meerut 2014]**
a. Neither convergent nor divergent
b. Divergent
c. Convergent
d. Oscillatory

153. Find value of μ for $\int_{a}^{\infty} \frac{dx}{x^{1/3}(1+x^{1/2})}$ and test it convergence : **[Meerut 2014]**
a. $\mu = \frac{5}{3}$, convergent
b. $\mu = \frac{5}{6}$, convergent
c. $\mu = \frac{5}{3}$, divergent
d. $\mu = \frac{5}{6}$, divergent

154. Test the convergence $\int_{0}^{\infty} \frac{e^{-x}\cos x}{x^2} dx$ and name the test also : **[Meerut 2014]**
a. Convergent by comparison test
b. Convergent by Abel's test
c. Divergent by comparison test
d. Divergent by Abel's test

155. For the integral $\int_{-1}^{1} \frac{dx}{x^3}$ which is/are true : **[Meerut 2014]**
a. It is divergent
b. Its principal value is zero
c. Both (a) and (b) are true
d. None of these

156. The integral $\int_{a}^{b} \frac{dx}{(x-a)}$ is convergent when : **[Meerut 2014]**
a. $n<1$ b. $n>1$
c. $n=1$ d. $n \geq 1$

157. The integral $\int_{0}^{\infty} \frac{e^{-x}\sin x}{x^2} dx$ is : **[Meerut 2014]**
a. Convergent
b. Divergent
c. May or may not be convergent
d. None of these

158. What is the value of μ in the μ-test of convergency for the integral $\int_{0}^{\infty} \frac{x}{(1+x)^3} dx$: **[Meerut 2014]**
a. 1 b. –1
c. 2 d. 3

159. Find $\lim_{n\to\infty} \int_{1}^{x} \frac{dx}{x^{5/2}}$ and test the convergence of $\int_{1}^{\infty} \frac{dx}{x^{5/2}}$: **[Meerut 2014]**
a. $\frac{3}{2}$, convergent b. $\frac{3}{2}$, divergent
c. $\frac{2}{3}$, convergent d. $\frac{2}{3}$, divergent

160. The β function $\int_{0}^{1} x^{m-1}(1-x)^{n-1} dx$ is convergent if : **[Meerut 2014]**
a. $m>0 \quad n>0$ b. $m<0 \quad n>0$
c. $m \leq 0 \quad n \leq 0$ d. $m \geq 0 \quad n \leq 0$

161. The integral $\int_{0}^{\infty} x^{n-1} e^{-x} dx$ is divergent when : **[Meerut 2014]**
a. $n>0$ b. $n>1$
c. $n \leq 0$ d. None of these

162. Let $f(x)$ be bounded and integrable in the interval (a, ∞) where $a>0$, such that $\lim_{x\to\infty} x^{\mu} f(x)$ exist then $\int_{a}^{\infty} f(x)\,dx$ is convergent if : **[Meerut 2015]**
a. $\mu \geq 1$ b. $\mu < 1$
c. $\mu \leq 1$ d. None of these

163. For the integral $\int_{0}^{\infty} e^{-mx} dx\,(m>0)$ which is/are correct : **[Meerut 2015]**
a. The value at integral is $\frac{1}{m}$
b. The integral is convergent
c. Both (a) and (b)
d. None of these

164. If the limit of an improper integral is neither a definite number nor ∞ or $-\infty$ then integral is said to be : **[Meerut 2016]**

a. Convergent b. Divergent
c. Oscillatory d. None of these

165. The integral $\int_a^{\infty}\frac{1}{x^n}dx$, where $a>0$ is convergent when : **[Meerut 2014, 16]**

a. $n=1$ b. $n<1$
c. $n\le 1$ d. $n>1$

166. $\int_0^{\infty}\frac{\sin mx}{c^2+x^2}dx$ is : **[Meerut 2019]**

a. Convergent
b. Divergent
c. Absolutely convergent
d. May be convergent

167. Integral $\int_0^1 x^{n-1}.\log x\,dx$ is : **[Meerut 2019]**

a. Proper integral, when $x>1$
b. Proper integral when $x<1$
c. Proper integral when $n>1$
d. Proper integral when $n<1$

168. $\int_0^{\infty}\frac{\sin x}{x^2}dx$ is : **[Meerut 2019]**

a. Convergent b. Divergent
c. Proper d. Always divergent

169. $\int_0^1\frac{dx}{(1+x^2)}$ is : **[Meerut 2019]**

a. Divergent
b. Convergent
c. May be convergent
d. Finite

170. $\int_0^4\frac{1}{x-1}dx$ is an integral of : **[Meerut 2014]**

a. Improper integral
b. Improper integral of first kind
c. Improper integral of second kind
d. Proper integral

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(b)	**2.**	(b)	**3.**	(a)	**4.**	(b)	**5.**	(a)	**6.**	(a)	**7.**	(a)	**8.**	(a)	**9.**	(a)	**10.**	(b)
11.	(a)	**12.**	(c)	**13.**	(a)	**14.**	(c)	**15.**	(c)	**16.**	(b)	**17.**	(a)	**18.**	(a)	**19.**	(b)	**20.**	(d)
21.	(a)	**22.**	(b)	**23.**	(d)	**24.**	(a)	**25.**	(a)	**26.**	(b)	**27.**	(c)	**28.**	(a)	**29.**	(b)	**30.**	(d)
31.	(a)	**32.**	(b)	**33.**	(c)	**34.**	(a)	**35.**	(d)	**36.**	(c)	**37.**	(b)	**38.**	(a)	**39.**	(c)	**40.**	(a)
41.	(b)	**42.**	(a)	**43.**	(a)	**44.**	(a)	**45.**	(d)	**46.**	(a)	**47.**	(a)	**48.**	(c)	**49.**	(b)	**50.**	(b)
51.	(a)	**52.**	(d)	**53.**	(d)	**54.**	(a)	**55.**	(c)	**56.**	(b)	**57.**	(a)	**58.**	(b)	**59.**	(d)	**60.**	(a)
61.	(c)	**62.**	(a)	**63.**	(b)	**64.**	(d)	**65.**	(a)	**66.**	(b)	**67.**	(b)	**68.**	(c)	**69.**	(a)	**70.**	(b)
71.	(a)	**72.**	(a)	**73.**	(a)	**74.**	(a)	**75.**	(c)	**76.**	(a)	**77.**	(a)	**78.**	(c)	**79.**	(b)	**80.**	(b)
81.	(c)	**82.**	(a)	**83.**	(a)	**84.**	(d)	**85.**	(a)	**86.**	(c)	**87.**	(c)	**88.**	(c)	**89.**	(a)	**90.**	(b)
91.	(a)	**92.**	(a)	**93.**	(d)	**94.**	(b)	**95.**	(a)	**96.**	(a)	**97.**	(b)	**98.**	(a)	**99.**	(a)	**100.**	(a)
101.	(a)	**102.**	(b)	**103.**	(b)	**104.**	(a)	**105.**	(c)	**106.**	(a)	**107.**	(a)	**108.**	(d)	**109.**	(c)	**110.**	(b)
111.	(a)	**112.**	(a)	**113.**	(a)	**114.**	(b)	**115.**	(d)	**116.**	(c)	**117.**	(b)	**118.**	(a)	**119.**	(d)	**120.**	(a)
121.	(c)	**122.**	(c)	**123.**	(b)	**124.**	(c)	**125.**	(b)	**126.**	(d)	**127.**	(c)	**128.**	(a)	**129.**	(a)	**130.**	(b)

131.	(a)	**132.**	(d)	**133.**	(d)	**134.**	(c)	**135.**	(b)	**136.**	(d)	**137.**	(a)	**138.**	(c)	**139.**	(a)	**140.**	(b)
141.	(a)	**142.**	(d)	**143.**	(b)	**144.**	(d)	**145.**	(a)	**146.**	(a)	**147.**	(a)	**148.**	(d)	**149.**	(c)	**150.**	(b)
151.	(b)	**152.**	(c)	**153.**	(d)	**154.**	(b)	**155.**	(c)	**156.**	(a)	**157.**	(a)	**158.**	(c)	**159.**	(c)	**160.**	(a)
161.	(c)	**162.**	(d)	**163.**	(c)	**164.**	(c)	**165.**	(d)	**166.**	(a)	**167.**	(c)	**168.**	(a)	**169.**	(b)	**170.**	(c)

HINTS AND SOLUTIONS

1. $\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{x\to\infty} \int_1^x \frac{dx}{\sqrt{x}}$

$= \lim_{x\to\infty} \int_1^x x^{-1/2}\,dx$

$= \lim_{x\to\infty} [2x^{1/2}]_1^x$

$= \lim_{x\to\infty} [2\sqrt{x} - 2] = \infty$

Thus, the limit does not exist finitely and so the given integral is divergent.

5. $\int_{-\infty}^0 e^x\,dx = \lim_{x\to\infty} \int_{-x}^0 e^x\,dx$

$= \lim_{x\to\infty} [e^x]_{-x}^0$

$= \lim_{x\to\infty} [1 - e^{-x}] = [1 - 0] = 1$

i.e. limit exist and is unique and finite so convergent.

$\int_{-\infty}^0 e^{-x}\,dx = \lim_{x\to\infty} \int_{-x}^0 e^{-x}\,dx$

$= \lim_{x\to\infty} [-e^{-x}]_{-x}^0$

$= -\lim_{x\to\infty} [e^0 - e^x] = \infty$

i.e. limit does not exist so integral is divergent.

7. $I = \int_3^\infty \frac{dx}{(x-2)^2}$

$= \lim_{x\to\infty} \int_3^x \frac{dx}{(x-2)^2}$

$= \lim_{x\to\infty} \int_3^x (x-2)^{-2}\,dx$

$= \lim_{x\to\infty} \left[\frac{(x-2)^{-1}}{-1}\right]_3^x$

$= \lim_{x\to\infty} \left[-\frac{1}{x-2} + 1\right] = 1$

a definite number so I is convergent.

8. $I = \int_0^\infty \frac{\sin^2 x}{x^2}\,dx$

$= \int_0^a \frac{\sin^2 x}{x^2}\,dx + \int_a^\infty \frac{\sin^2 x}{x^2}\,dx,\ a > 0$

Since, $\lim_{x\to\infty} \frac{\sin^2 x}{x^2} = 1$

So, $\int_0^a \frac{\sin^2 x}{x^2}\,dx$ is a proper integral so it is convergent.

Now consider $\int_a^\infty \frac{\sin^2 x\,dx}{x^2}$,

Let $f(x) = \frac{\sin^2 x}{x^2},\ g(x) = \frac{1}{x^2}$

obviously $g(x)$ is positive in (a, ∞).

Now, $|f(x)| = \left|\frac{\sin^2 x}{x^2}\right| \le \frac{1}{x^2}$

and $\int_a^\infty \frac{dx}{x^2}$ is convergent so by comparison test $\int_a^R \frac{\sin^2 x\,dx}{x^2}$ is also convergent. Thus I is convergent.

11. $I = \int_0^\infty e^{-mx}\,dx\ (m > 0)$

$= \lim_{x\to\infty} \int_0^\infty e^{-mx}\,dx$

$= \lim_{x\to\infty} \left(\frac{e^{-mx}}{-m}\right)_0^x$

$= \lim_{x\to\infty} \left[-\frac{1}{m}(e^{-mx} - 1)\right] = \frac{1}{m}$

Thus, the limit exists and finitely unique so I is convergent.

14. $\int_a^b \frac{dx}{(x-a)^n}$

$$= \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b \frac{dx}{(x-a)^n}$$

$$= \lim_{\varepsilon \to 0} \left[\frac{(x-a)^{-n+1}}{-n+1} \right]_{a+\varepsilon}^b$$

$$= \lim_{\varepsilon \to 0} \left[\frac{(b-a)^{1-n}}{1-n} - \frac{\varepsilon^{1-n}}{1-n} \right]$$

If $n < 1$ the $\lim_{\varepsilon \to 0} \varepsilon^{1-n} = 0$ so

$$\int_a^b \frac{dx}{(x-a)^n} = \frac{(b-a)^{1-n}}{1-n} \text{ if } n < 1,$$

so this integral is convergent when $n < 1$.

17. Let $I = \int_a^\infty \frac{dx}{x\sqrt{1+x^2}}, a > 0$

and $f(x) = \frac{1}{x\sqrt{1+x^2}} = \frac{1}{x^2\sqrt{1+\frac{1}{x^2}}}$

put $g(x) = \frac{1}{x^2}$ we get $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$ which is finite and non-zero. So, $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both convergent or both diverge. But $\int_a^\infty \frac{dx}{x^2}$ is divergent so $\int_a^\infty \frac{dx}{x\sqrt{1+x^2}}$ is also convergent.

19. $\int_{-\infty}^0 \sin hx\,dx = \lim_{x \to \infty} \int_{-x}^0 \sin hx$

$$= \lim_{x \to \infty} \int_{-x}^0 \frac{e^x - e^{-x}}{2} dx$$

$$= \frac{1}{2}[\lim_{x \to \infty} [e^x]_{-x}^0 - \lim_{x \to \infty} [-e^{-x}]_x^0]^{-x}$$

$$= \frac{1}{2}[1 - \infty] = -\infty$$

So, the given integral diverges to $-\infty$.

24. Let $I = \int_0^\infty e^{-x} \frac{\sin x}{x} dx$

$$= \int_0^1 e^{-x} \frac{\sin x}{x} dx + \int_1^\infty e^{-x} \frac{\sin x}{x} dx$$

Since, $\lim_{x \to \infty} e^{-x} \frac{\sin x}{x} dx = 1$, so it is bounded.

i.e., $\int_0^1 e^{-x} \frac{\sin x}{x} dx$ is proper integral and so convergent.

Let $f(x) = e^{-x} \frac{\sin x}{x}$

then $f(x)$ is bounded in $(1, \infty)$.

Put $g(x) = e^{-x}$ then it is positive in $(1, \infty)$.

$$|f(x)| = \left| e^{-x} \frac{\sin x}{x} \right|$$

$$= e^{-x} |\sin x| \cdot \frac{1}{x} \leq e^{-x}$$

Thus, $|f(x)| \leq g(x)$ through out $(1, \infty)$.

Now $\int_1^\infty g(x)\,dx = \int_1^\infty e^{-x}\,dx$

$$= \lim_{x \to \infty} \int_1^\infty e^x\,dx$$

$$= \lim_{x \to \infty} [-e^{-x}]_1^x = \frac{1}{e}$$

which a definite finite number. So $\int_1^\infty g(x)\,dx$ is convergent. So by comparison test $\int_0^\infty e^{-x} \frac{\sin x}{x} dx$ is convergent.

27. $I = \int_0^\infty \frac{dx}{(1+x)^{2/3}}$

$$= \lim_{x \to \infty} \int_0^x (1+x)^{-2/3}\,dx$$

$$= \lim_{x \to \infty} \left[\frac{(1+x)^{1/3}}{1/3} \right]_0^x$$

$$= \lim_{x \to \infty} 3\{(1+x)^{1/3} - 1\} = \infty$$

Thus, the limit does not exist finitely and so the given integral is convergent.

29. Given integral is

$$I = \int_1^\infty \frac{dx}{x^{1/3}(1+x^{1/2})}$$

Let $f(x) = \frac{1}{x^{1/3}(1+x^{1/2})} = \frac{1}{x^{5/6}\left(1+\frac{1}{x^{1/2}}\right)}$

so $f(x)$ is bounded in $(1, \infty)$. Choose $\mu = \frac{5}{6}$ we have

$$\lim_{x \to \infty} x^\mu f(x) = \lim_{x \to \infty} \frac{1}{1+\frac{1}{x^{1/2}}} = 1$$

which is finite and non-zero. Since, $\mu = \frac{5}{6} < 1$ so by μ test I is divergent.

30. Given that $I = \int_1^\infty x^{n-1} e^{-x}\, dx$

Let $f(x) = x^{n-1} e^{-x}$

then $f(x)$ is bounded in $(1, \infty)$.

Now $\lim\limits_{x\to\infty} x^\mu f(x) = \lim\limits_{x\to\infty} \dfrac{x^\mu x^{n-1}}{e^x}$

$$= \lim_{x\to\infty} \frac{x^{\mu+n-1}}{1 + x + \frac{x^2}{\lfloor 2} + \ldots} = 0$$

$= 0$ for all μ and n

Taking $\mu > 1$, by μ-test $\int_1^\infty x^{n-1} e^{-x}\, dx$ is convergent for all values of n.

33. $\int_0^x \cos x\, dx = \lim\limits_{x\to\infty} \int_0^x \cos x\, dx = \lim\limits_{x\to\infty} \sin x$

This limit does not exist finitely. Hence, this integral oscillatore and so not convergent *i.e.* oscillatory between –1 and +1.

38. $I = \int_{-\infty}^\infty \dfrac{dx}{1+x^2}$

$$= \lim_{x\to\infty} \int_{-x}^x \frac{dx}{1+x^2}$$

$$= \lim_{x\to\infty} (\tan^{-1} x)_{-x}^x$$

$$= \lim_{x\to\infty} [\tan^{-1} x - \tan^{-1}(-x)]$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

a finite and unique number so I is convergent.

42. $I + \int_a^\infty \dfrac{x\, dx}{(1+x)^3} = \int_a^\infty f(x)\, dx,$

then $f(x)$ is bounded in (a, ∞).

Take $\mu = 3 - 1 = 2$ then,

$$\lim_{x\to\infty} x^\mu f(x) = \lim_{x\to\infty} x^2 \frac{x}{(1+x)^3} = 1$$

which exists *i.e.* is equal to a definite real number.

Since $\mu > 1$ so by μ-test $\int_a^\infty \dfrac{x\, dx}{(1+x)^3}$ is convergent.

43. Let $I = \int_a^\infty \dfrac{\cos x}{x^2} dx$

Here, $\left|\dfrac{\cos x}{x^2}\right| \le \dfrac{1}{x^2}$ as $|\cos x| \le 1$

Since, $\int_a^\infty \dfrac{dx}{x^2}$ is convergent so by comparison test $\int_a^\infty \dfrac{\cos x}{x^2} dx$ is also convergent.

44. Given that $I = \int_0^\infty \dfrac{\sin mx}{a^2 + x^2} = \int_0^\infty f(x)\, dx$

Now $\left|\dfrac{\sin mx}{a^2+x^2}\right| \le \dfrac{|\sin mx|}{a^2+x^2} \le \dfrac{1}{a^2+x^2}$

Also $\int_0^\infty \dfrac{dx}{a^2+x^2} = \lim\limits_{x\to\infty} \int_0^x \dfrac{dx}{a^2+x^2}$

$$\lim_{x\to\infty} \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^x = \lim_{x\to\infty} \left[\frac{1}{a}\tan^{-1}\frac{x}{a} - 0\right]$$

$$= \frac{\pi}{2a}$$

which is definite real number

So $\int_0^\infty \dfrac{dx}{a^2+x^2}$ is convergent.

Hence, $\int_0^\infty \left(\dfrac{\sin mx}{a^2+x^2}\right) dx$ is convergent *i.e.* I is also solutely convergent.

47. Let $I = \int_0^a \dfrac{x^{2m}}{1+x^{2n}} dx$

where m, n are positive integers. Since, I is a proper integral so it is convergent.

51. $I = \int_a^\infty \dfrac{dx}{x(\log x)^{n+1}}$ where $a > 1$

Let $\log x = t$ so $\dfrac{1}{x} dx = dt$

$$I = \int_{\log a}^\infty \frac{dt}{t^{n+1}} = \int_{\log a}^\infty f(t)\, dt$$

Then $f(t)$ is bounded in $(\log a, \infty)$.

Put $\mu = n + 1$ then

$$\lim_{t\to\infty} t^\mu f(t) = \lim_{t\to\infty} \frac{t^{n+1}}{t^{n+1}} = 1$$

which is finite and non-zero.

So by μ-test I is divergent if

$$\mu \le 1 \;\; i.e. \;\; n+1 \le 1 \;\; i.e. \;\; n \le 0$$

54. $I = \int_a^\infty \dfrac{\sin^2 x}{x^2} dx$

Let $f(x) = \dfrac{\sin^2 x}{x^2}$ and $g(x) = \dfrac{1}{x^2}$

then $g(x)$ is positive in (a, ∞).

$$|f(x)| = \left|\frac{\sin^2 x}{x^2}\right| \le \frac{1}{x^2}$$

so by comparison test, $\int_a^\infty \frac{\sin^2 x}{x^2} dx$ is convergent if $\int_a^\infty \frac{dx}{x^2}$ is convergent. But $\int_a^\infty \frac{dx}{x^2}$ is convergent so I is also convergent.

55. $I = \int_a^\infty \frac{x^{2m}}{1+x^{2n}} dx$

put $\mu = 2n - 2m$

we have

$$\lim_{x\to\infty} x^\mu \frac{x^{2m}}{1+x^{2n}}$$

$$= \lim_{x\to\infty} x^{2n-2m} \cdot \frac{x^{2m}}{x^{2n} \cdot \left[1 + \frac{1}{x^{2n}}\right]}$$

$$= \lim_{x\to\infty} \frac{1}{1 + \frac{1}{x^{2n}}} = 1$$

57. $I = \int_0^{\pi/2} \frac{\cos x}{x^n} dx$

when $n \le 0, I$ is a proper integral and hence convergent. when $n > 0$ the integrand becomes unbounded at $x = 0$.

Let $\qquad f(x) = \frac{\cos x}{x^n}$

then $\lim_{x\to 0} x^\mu f(x) = \lim_{x\to 0} x^{\mu-n} \cos x = 1$

if $\mu = n$

Hence, by μ-test it follows that the given integral I is convergent when $r < n < 1$.

62. Since $\left|\frac{\sin x}{x^2}\right| \le \frac{1}{x^2}$ and $\int_a^\infty \frac{dx}{x^2}$ is convergent, therefore by comparison test $\int_a^\infty \frac{\sin x}{x^2} dx$ is also convergent.

65. Let $\qquad I = \int_1^\infty e^{-x^2} dx$

and $\qquad f(x) = e^{-x^2}$

Put $g(x) = xe^{-x^2}$ so $g(x)$ is positive throughout the interval $(1, \infty)$ so $|f(x)| = e^{-x^2} \le xe^{-x^2}$ for $x \ge 1$ so $|f(x)| \le g(x)$ throughout $(1, \infty)$.

So by comparison test $\int_1^\infty e^{-x^2} dx$ is convergent if $\int_1^\infty xe^{-x^2} dx$ is convergent.

Now $\qquad \int_1^\infty x e^{-x^2} dx = \lim_{x\to\infty} \int_1^x x e^{-x^2} dx$

$$= \lim_{x\to\infty} \left(-\frac{1}{2}e^{-x^2}\right)_1^x$$

$$= \lim_{x\to\infty} \left(-\frac{1}{2}e^{-x^2} + \frac{1}{2}e^{-1}\right)$$

$$= \frac{1}{2}e^{-1}$$

which is a definite number

$\therefore \qquad \int_1^\infty x e^{-x^2} dx$ is convergent and so $\int_1^\infty e^{-x^2} dx$ is also convergent.

70. Let $\qquad I = \int_0^1 x^{n-1} \log x \, dx$

and $\qquad f(x) = x^{n-1} \log x$

then $\qquad \lim_{x\to 0} x^\mu f(x) = \lim_{x\to 0} x^{\mu+n-1} \log x$

$$= \begin{cases} 0 & \text{if} \quad \mu > 1-n \\ -\infty & \text{if} \quad \mu \le 1-n \end{cases}$$

when $n \le 0$ we can take $\mu = 1$ so by μ-test the integral is divergent when $n \le 0$.

73. Let $\qquad I = \int_a^\infty (1 - e^{-x}) \frac{\cos x}{x^2} dx$, when $a > 0$

Let $\qquad f(x) = \frac{\cos x}{x^2}$

and $\qquad g(x) = 1 - e^{-x}$

we have $\left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2}$ and $\int_a^x \frac{1}{x^2} dx$ is convergent

so by comparison test $\int_a^\infty \frac{\cos x}{x^2} dx$ is also convergent.

Again $g(x) = 1 - e^{-x}$ is monotonic increasing and bounded function for $x > a$. Hence, by Abel's test $\int_a^\infty (1 - e^{-x}) \frac{\cos x}{x^2} dx$ is convergent.

78. $\int_0^\infty e^{-x} \cos mx \, dx$ will be absolutely convergent if $\int_0^\infty |e^{-x} \cos mx| dx$ is convergent.

Let $\qquad f(x) = |e^{-x} \cos mx|$

Then $f(x)$ is bounded in $(0, \infty)$.

Now $f(x) = |e^{-x} \cos mx| = e^{x} |\cos mx|$

$\leq e^{-x}$, since $|\cos mx| \leq 1$

So by comparison test $\int_0^\infty f(x)\,dx$ is convergent if $\int_0^\infty e^{-x}\,dx$ is convergent.

But $\int_0^\infty e^{-x}\,dx = \lim_{x\to\infty} \int_0^\infty e^{-x}\,dx$

$= \lim_{x\to\infty} [-e^{-x}]_0^\infty$

$= \lim_{x\to\infty} [-e^{-x} + 1] = 1$

So, $\int_0^\infty e^{-x}\,dx$ is convergent.

Hence, $\int_0^\infty f(x)\,dx$ is convergent and so the given integral is absolutely convergent.

81. $I = \int_0^{\pi/2} \frac{\sin x}{x^{n+1}}\,dx$

$$= \int_0^{\pi/2} \left(\frac{\sin x}{x}\right)\frac{1}{x^n}\,dx$$

Now $\lim_{x\to 0} \frac{\sin x}{x} = 1$ for $n \leq 0$

So integrand is bounded in $\left(0, \frac{\pi}{2}\right)$ *i.e.* I is a proper integral and hence it is convergent if $n \leq 0$.

If $n > 0$, the integrand is unbounded at $x = 0$

So, $\lim_{x\to 0} x^{\mu} \frac{\sin x}{x^{n+1}} = \lim_{y\to 0} \left\{x^{\mu-n}\left(\frac{\sin x}{x}\right)\right\}$

$= 1$ if $\mu - x = 0$ *i.e.* $\mu = n$

∴ by μ-test if $0 \leq \mu < 1$ *i.e.* $0 < n < 1$, I is convergent.

83. Given that

$$I = \int_a^\infty e^{-x} \frac{\sin x}{x^2}\,dx,\ a > 0$$

Let $f(x) = \frac{\sin x}{x^2}$ and $\phi(x) = e^{-x}$

∵ $\left|\frac{\sin x}{x^2}\right| \leq \frac{1}{x^2}$

and $\int_a^\infty \frac{1}{x^2}\,dx$ is convergent so by comparison test $\int_a^\infty \frac{\sin x}{x^2}\,dx$ is also convergent. Again e^{-x} is monotonic decreasing and bounded function for $x > a$. Hence by Abel's test $\int_a^\infty e^{-x} \cdot \frac{\sin x}{x^2}\,dx$ is convergent.

87. If $n \geq 1$ then $I = \int_0^1 x^{n-1} e^{-x}\,dx$ is a proper integral so I is convergent.

If $0 < n < 1$, then integrand $f(x) = x^{n-1} e^{-x}$ is unbounded at $x = 0$. If $g(x) = x^{n-1}$ then

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = x \to 0\ e^{-x} = 1$$

a finite and non-zero number. So by comparison test $\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ either both convergent or both divergent.

But $\int_0^1 g(x)\,dx = \int_0^1 x^{n-1}\,dx$

$= \lim_{\varepsilon\to 0} \int_\varepsilon^1 x^{n-1}\,dx$

$= \lim_{\varepsilon\to 0} \left[\frac{x^n}{x}\right]_\varepsilon^1 = \frac{1}{n}$

which is definite real number.

∴ $\int_0^1 g(x)\,dx$ is convergent. Hence, $\int_0^1 x^{n-1} e^{-x}\,dx$ also convergent.

89. Let $I = \int_0^{\pi/2} \log \sin x\,dx$

The only point of infinite discontinuity of the integrand is $x = 0$.

Now $\lim_{x\to 0} x^{\mu} \log \sin x$ when $\mu > 0$

$= \lim_{x\to 0} \frac{\log \sin x}{x^{-\mu}}$

$= \lim_{x\to 0} \frac{\cot x}{-\mu x^{-\mu-1}}$

$= \lim_{x\to 0} -\frac{1}{\mu} \frac{x^{\mu+1}}{\tan x}$

$= \lim_{x\to 0} -\frac{1}{\mu} \frac{(\mu+1)x^{\mu}}{\sec^2 x}$

$= 0$ if $\mu > 0$

Taking $0 \leq \mu < 1$, by μ-test I is convergent.

91. $I = \int_2^\infty \frac{dx}{(x+1)(x-1)}$

Let $f(x) = \frac{1}{\sqrt{x^2 - 1}}$

Take $g(x) = \frac{1}{x}$

then $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{1}{\sqrt{1-\frac{1}{x^2}}} = 1$

which is finite and non-zero so $\int_2^{\infty} f(x)\,dx$ and $\int_2^{\infty} g(x)\,dx$ either both converge or both diverge. By comparison test $\int_2^{\infty}\frac{1}{x^2}dx$ is divergent so $I = \int_2^{\infty}\frac{dx}{\sqrt{x^2-1}}$ is also divergent.

96. Given integral is $\int_0^{\infty}\frac{\sin mx}{x^2+a^2}dx$

Let $f(x) = \frac{\sin mx}{x^2+a^2}$

and $g(x) = \frac{1}{x^2+a^2}$

Here $g(x)$ is positive in $(0, \infty)$.

Also $|f(x)| = \left|\frac{\sin mx}{x^2+a^2}\right| \le \frac{1}{x^2+a^2}$

Then $|f(x)| \le g(x)$ when $x \ge 0$

By comparison test $\int_0^{\infty}\frac{\sin mx}{x^2+a^2}dx$ is convergent if $\int_0^{\infty}\frac{dx}{x^2+a^2}$ is convergent.

But $\int_0^{\infty}\frac{dx}{x^2+a^2} = \lim_{x\to\infty}\int_0^{x}\frac{dx}{x^2+a^2}$

$$= \lim_{x\to\infty}\left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^x = \frac{\pi}{2a}$$

a definite real number so $\int_0^{\infty}\frac{dx}{x^2+a^2}$ is convergent

i.e. $\int_0^{\infty}\frac{\sin mx}{x^2+a^2}dx$ is also convergent.

98. $I = \int_0^{\pi/2}\frac{\cos x}{x^2}dx$

Let $f(x) = \frac{\cos x}{x^2}$ and $g(x) = \frac{1}{x^2}$

Here $f(x)$ is unbounded at $x = 0$.

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\left[\frac{\cos x}{x^2}\cdot x^2\right] = 1$$

So by comparison test $\int_0^{\pi/2} f(x)\,dx$ and $\int_0^{\pi/2} g(x)\,dx$ either both converge or both diverge.

But $\int_0^{\pi/2} g(x)\,dx = \int_0^{\pi/2}\frac{1}{x^2}dx$

$$= \lim_{\varepsilon\to 0}\frac{dx}{x^2}$$

$$= \lim_{\varepsilon\to 0}\left[-\frac{1}{x}\right]_{\varepsilon}^{\pi/2}$$

$$= \lim_{\varepsilon\to 0}\left[-\frac{2}{\pi}+\frac{1}{\varepsilon}\right] = \infty$$

$\therefore$ $\int_0^{\pi/2} g(x)\,dx$ is diverge. Hence, I is also diverge.

100. Let $I = \int_0^{\pi/2}\frac{\cos x}{x^n}dx$

when $n \le 0$, I is proper so convergent. When $n > 0$, the integrand becomes unbounded at $x = 0$.

Let $f(x) = \frac{\cos x}{x^n}$

then $\lim_{x\to 0} x^{\mu} f(x) = \lim_{x\to 0} x^{\mu-n}\cos x$

$= 1$ if $\mu = n$

Hence by μ-test I is divergent when $n \ge 1$.

102. $\int_0^{\infty}|e^{-a^2x^2}\cos bx|\,dx \le \int_0^{\infty}|e^{-a^2x^2}|\,dx$

$$\int_0^{\infty} e^{-a^2x^2}\,dx = \int_0^{b} e^{-a^2x^2}\,dx + \int_b^{\infty} e^{-a^2x^2}\,dx$$

where $b > 0$

But $\int_0^{\infty} e^{-a^2x^2}\,dx$ is proper integral so convergent.

Also $\int_0^{\infty} e^{-a^2x^2}\,dx$ is convergent by μ-test. for

$$\lim_{x\to\infty} x^{\mu} e^{-a^2x^2} = \lim_{x\to\infty}\frac{x^{\mu}}{1+a^2x^2+\frac{a^4x^4}{\angle 2}+\ldots.}$$

$= 0$ for all values of μ

Taking $\mu > 1$, $\int_b^{\infty} e^{-a^2x^2}\,dx$ is convergent.

So $\int_0^{\infty} e^{-a^2x^2}\,dx$ is convergent.

Hence, $\int_0^{\infty}|e^{-a^2x^2}\cos bx|\,dx$ is convergent *i.e.* given integral is absolutely convergent.

105. Given that $I = \int_0^{\pi/4}\sqrt{\cos x}\,dx$

Let $f(x) = \sqrt{\cot x}$

which is unbounded at $x = 0$.

Take $\mu = \frac{1}{2}$ we have

$$\lim_{x\to 0} x^{\mu}\sqrt{\cot x} = \lim_{x\to 0}\sqrt{\frac{x}{\sin}}\sqrt{\cos x} = 1$$

Since $0 \le \mu < 1$ so by μ-test the given integral is convergent.

107. $I = \int_0^\infty \frac{\sqrt{x}}{x^2+4} dx$

$$= \int_0^a \frac{\sqrt{x}}{x^2+4} dx + \int_a^\infty \frac{\sqrt{x}}{x^2+4} dx$$

The first integral is proper integral so it is convergent.

Let $f(x) = \frac{\sqrt{x}}{x^2+4}$

and $g(x) = \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}$

then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1}{1+\frac{4}{x^2}} = 1$

which is finite and non-zero.

Therefore, $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ either both converge or both diverge. But by comparison test $\int_a^\infty \frac{dx}{x^{3/2}}$ is convergent so $\int_a^\infty \frac{\sqrt{x}}{x^2+4} dx$ is also convergent. Hence, I is convergent.

117. Let $I = \int_0^1 \frac{\sec x}{x} dx$

and $f(x) = \frac{\sec x}{x}$

which is bounded at lower limit *i.e.* at $x = 0$.

Take $g(x) = \frac{1}{x}$

Then $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \left[\frac{\sec x}{x} \cdot x\right] = 1$

which is finite and non-zero.

So by comparison test $\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ either both converge or both diverge.

Now, $\int_0^1 g(x)\,dx = \int_0^1 \frac{1}{x} dx$

divergent. Hence, $\int_0^1 \frac{\sec x}{x} dx$ is also divergent.

120. Let $I = \int_0^{\pi/2} \frac{\log \sin x}{(\sin x)^n} dx$

The integrand has an infinity at $x = 0$. Also

$$\lim_{x\to 0} x^\mu \frac{\log \sin x}{(\sin x)^n} = \lim_{x\to 0} x^{\mu-n} \log \sin x \left(\frac{x}{\sin x}\right)^n$$

$$= \lim_{x\to 0} x^{\mu-n} \log \sin x$$

$$= \lim_{x\to 0} \frac{\log \sin x}{x^{x-\mu}}$$

$$= \lim_{x\to 0} \frac{\cot x}{(n-\mu)x^{n-\mu-1}}$$

$$= \frac{1}{n-\mu} \lim_{x\to 0} \frac{x^{\mu+1-n}}{\tan x}$$

$$= \frac{1}{n-\mu} \lim_{x\to 0} \frac{(\mu+1-n)x^{\mu-n}}{\sec^2 x}$$

$$= \frac{\mu+1-n}{n-\mu} \lim_{x\to 0} \frac{x^{\mu-n}}{\sec^2 x} = 0$$

μ can be taken between 0 and 1 if $0 < n < 1$ so by μ-test I is convergent.

146. $I = \int_0^\infty \frac{4a\,dx}{x^2+4a^2}$

$$= \lim_{x\to\infty} \int_0^x \frac{4a\,dx}{x^2+4a^2}$$

$$= \lim_{x\to\infty} \frac{4a}{2a}\left(\tan^{-1}\frac{x}{2a}\right]_0^x$$

$$= 2\cdot\frac{\pi}{2} = \pi$$

So, a finite and unique limit *i.e.* I is convergent.

152. Let $f(x) = \frac{\cos mx}{x^2+a^2}$, $g(x) = \frac{1}{x^2+a^2}$

Here, $g(x)$ is positive in the interval $(0, \infty)$.

Also $|f(x)| = \left|\frac{\cos mx}{x^2+a^2}\right| \le \frac{1}{x^2+a^2}$

Thus $|f(x)| \le g(x)$ when $x \ge 0$

$$\int_0^\infty \frac{dx}{x^2+a^2} = \lim_{x\to\infty} \int_0^x \frac{dx}{x^2+a^2}$$

$$= \lim_{x\to\infty} \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^x$$

$$= \lim_{x\to\infty} \frac{1}{a}\tan^{-1}\frac{x}{a} = \frac{\pi}{2a}$$

So, $\int_0^\infty \frac{dx}{x^2+a^2}$ is convergent so by comparison test $\int_0^\infty \frac{\cos mx\,dx}{x^2+a^2}$ is also convergent.

OOO

CHAPTER

11 Integral as a Function of a Parameter

DEFINITE INTEGRAL AS A FUNCTION OF A PARAMETER

Let f be a continuous function of two variables x and t. For a fixed $t \in [t_1, t_2]$, the function of x with domain $[a, b]$ is continuous so $\int_a^b f(x, t)\, dx$ exists and is a function of a single variable t.

If ϕ be a function of t such that

$$\phi(t) = \int_a^b f(x, t)\, dx$$

then ϕ is called in integral function of $f(x, t)$.

Results :

1. The function $\phi(t) = \int_a^b f(x, t)\, dx$, where $f(x, t)$ is continuous in $a \le x \le b$, $t_1 \le t \le t_2$, is a continuous function of t on $[t_1, t_2]$.

2. **Derivability of proper integrals**

   If $f(x, t)$ and f_t be continuous in $[a, b] \times [t_1, t_2]$ then $\phi(t)$ is differentiable and

$$\phi'(t) = \int_a^b f_t(x, t)\, dx \;\forall\; t \in [t_1, t_2]$$

or

$$\frac{d}{dt}\left\{\int_a^b f(x, t)\, dx\right\} = \int_a^b f_t(x, t)\, dx$$

   where f_t denotes the partial derivative of f with respect to t.

3. If $f(x, t)$ and f_t be continuous in $[a, b] \times [t_1, t_2]$ and $h(t)$, $g(t)$ be two functions differentiable in $[t_1, t_2]$ such that for all $t \in [t_1, t_2]$ the points $(h(t), t)$ and $(g(t), t) \in [a, b] \times [t_1, t_2]$ then the function

$$(t) = \int_{h(t)}^{g(t)} f(x, f)\, dx$$

   is differentiable in $[t_1, t_2]$

   and
$$\phi'(t) = \left[\int_{h(t)}^{g(t)} f_t(x, t)\, dx\right] - h'(t) . f(h(t), t) + g'(t) . f(g(t), t)$$

4. **Integrability of proper integrals**

   If $f(x, t)$ be continuous in $[a, b] \times [t_1, t_2]$, then

$$\int_{t_1}^{t_2}\left\{\int_a^b f(x, t)\, dx\right\} dt = \int_a^b \left\{\int_{t_1}^{t_2} f(x, t)\, dt\right\} dx$$

DIFFERENTIATION UNDER THE SIGN OF INTEGRATION (LEIBNITZ RULE)

Let $f(x, t)$ and $\frac{\partial f}{\partial t}$ be continuous on $[a, b] \times [c, d]$ in a xt-plane. Let u and v be functions of t, which are differentiable in $[c, d]$ such that for all t in $[c, d]$ the points $(u(t), t)$ and $(v(t), t)$ belong to R. Then the function ϕ defined by

$$\phi(t) = \int_u^v f(x, t)\, dx$$

is differentiable in $[c, d]$ and

$$\frac{d\phi}{dt} = \int_u^v \frac{\partial f}{\partial t}\, dx + f(v, t)\frac{dv}{dt} - f(u, t)\frac{du}{dt}$$

If u and v are constants then

$$\frac{d\phi}{dt} = \int_u^v \frac{\partial f}{\partial t}\, dx$$

IMPORTANT RESULTS

1. $\int_0^\infty \frac{x^{n-1}}{1+x}\, dx = \frac{\pi}{\sin n\pi}$ if $0 < n < 1$

2 $\int_0^\infty \frac{\sin nx}{x}\, dx = \frac{\pi}{2}$ if $n > 0$

3. $\int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2 + b^2}$

4. $\int_0^\infty e^{-ax} \cos bx\, dx = \frac{a}{a^2 + b^2}$

5. $\int_0^\infty \frac{x^{n-1}}{(1-x)^{m+n}}\, dx = \frac{t(m) . t(n)}{t(m+n)} = \beta(m, n)$

6. $\int_0^\infty e^{-cx} x^n \, dx = \frac{\lfloor n}{c^{n+1}}$

CONTINUITY OF IMPROPER INTEGRALS AS A FUNCTION OF A PARAMETER

The improper integral

$$\phi(t) = \int_a^\infty f(x, t) \, dx$$

is continuous in $[t_1, t_2]$ if

(i) $f(x, t)$ is continuous for $x \geq t$ and $t \in [t_1, t_2]$.

(ii) $\int_a^\infty f(x, t) \, dx$ is uniformly convergent for $t \in [t_1, t_2]$.

INTEGRABILITY OF IMPROPER INTEGRALS AS A FUNCTION OF A PARAMETER

If (i) $f(x, t)$ is continuous for $x \geq t$ and $t \in [t_1, t_2]$ and (ii) $\int_a^\infty f(x, t) \, dx$ is uniformly convergent for $t \in [t_1, t_2]$ then the improper integral

$$\phi(t) = \int_a^\infty f(x, t) \, dx$$

can be integrated under the integral sign and

$$\int_{t_1}^{t_2} \left\{ \int_a^\infty f(x, t) \, dx \right\} dt = \int_{t_1}^{t_2} \phi(t) \, dt$$

$$= \int_a^\infty \left\{ \int_{t_1}^{t_2} f(x, t) \, dt \right\} dx$$

DERIVABILITY OF IMPROPER INTEGRALS AS A FUNCTION OF A PARAMETER

If (i) $f(x, t)$ is continuous and has a continuous partial derivative with respect to t for $x \geq t$ and $t \in [t_1, t_2]$ and (ii) $\int_a^\infty f_t(x, t) \, dx$ converges uniformly in $t \in [t_1, t_2]$, then $\phi(t)$ is differentiable and

$$\phi'(t) = \int_a^\infty f_t(x, t) \, dx$$

EXERCISE

MULTIPLE CHOICE QUESTIONS

Direction : Each of the following questions has four alternative answers. One of them is correct. Choose the correct answer.

1. The value of $\int_0^\infty e^{-x^2} dx$ is :

   a. $\sqrt{\frac{\pi}{2}}$ b. $\sqrt{\pi}$

   c. $\sqrt{\frac{\pi}{2}}$ d. $\frac{2}{\sqrt{\pi}}$

2. If $I = \int_0^\pi \frac{\log(1 + y\cos x)}{\cos x} dx$ then $\frac{dI}{dy}$ is equal to :

   a. $\frac{y^2}{\sqrt{1-\pi}}$ b. $\frac{\pi}{\sqrt{1-y^2}}$

   c. $\pi\sqrt{1-y^2}$ d. None of these

3. The definite integral as a function of a parameter is defined as :

   a. $\phi(t) = \int_a^b f(x) \, dx$ b. $\phi(x) = \int_a^b f(x,t) \, dt$

   c. $\phi(t) = \int_a^b f(x,t) \, dx$ d. None of these

4. The value of $\int_0^\infty x^3 e^{-cx^2} dx$ is :

   a. c^2 b. $\frac{1}{2c^2}$

   c. $\frac{1}{c^2}$ d. None of these

5. The value of $\int_0^\pi \frac{\log(1 + y\cos x)}{\cos x} dx$ when $-1 < y < 1$ is:

   a. $\pi \tan^{-1} y$ b. $\pi \cos^{-1} y$

   c. $\pi \sin^{-1} y$ d. None of these

6. If $\int_0^\infty e^{-cx} dx = \frac{1}{c}, c > 0$ then $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} dx$ is :

   a. $\log \frac{b}{a}$ b. $\log \frac{a}{b}$

   c. $\log a + \log b$ d. None of these

7. The order of integration in a definite integral changed when :

   a. Integrand is constant

   b. Limits depend on variables

   c. Limits are not depend on variables

   d. None of these

8. If $I = \int_0^{\pi/2} \log(a^2\cos^2\theta + b^2\sin^2\theta)\,d\theta$ and $a \neq b$ then :

a. $I = \pi\log\frac{a}{b} + c$ b. $I = \pi\log ab + c$

c. $I = \pi\log(a-b) + c$ d. $I = \pi\log\frac{(a+b)}{2} + c$

9. If $f(x,t)$ is continuous in $[a,b]\times[t_1,t_2]$ and $\phi(x,t) = \int_t^{t_2}\left[\int_a^x f(x,t)\,dx\right]dt$ then :

a. $\frac{\partial^2\phi}{\partial x\partial t} + \frac{\partial^2\phi}{\partial t\partial n} = 0$ b. $\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial u^2} = 0$

c. $\frac{\partial^2\phi}{\partial x\partial t} - \frac{\partial^2\phi}{\partial t\partial x} = 0$ d. None of these

10. The value of $\int_0^{\pi}\frac{\log(1+y\cos x)}{\cos x}dx$ when $|y|<1$ is :

a. $\pi\cos^{-1}\alpha$ b. $\pi\sin^{-1}\alpha$

c. π d. None of these

11. The value of $\int_0^{\infty} e^{-cx^2}dx$ is equal to :

a. $\sqrt{\frac{\pi}{c}}$ b. $\sqrt{\frac{\pi}{2c}}$

c. $\sqrt{\frac{\pi}{4c}}$ d. None of these

12. The value of $\int_0^{\infty}\frac{\cos mx}{a^2+x^2}dx$ is equal to :

a. $\frac{\pi}{2a}e^{ma}$ b. $\frac{\pi}{2a}e^{-ma}$

c. $\frac{\pi}{a}e^{-ma}$ d. None of these

13. $\int_0^{\infty}\frac{\sin mx}{x}dx$ if $m>0$ is equal to :

a. $\frac{\pi}{2}$ b. π

c. 0 d. $-\frac{\pi}{2}$

14. If the function $\phi(t) = \int_a^b f(x,t)\,dx$, where $f(x,t)$ is continuous in $a \le x \le b, t_1 \le t \le t_2$ then in the integral $[t_1,t_2]$ for the function $\phi(t)$:

a. Limit exist only b. Continuous

c. Differentiable d. None of these

15. The value of $\int_0^1\frac{\tan^{-1}yx}{x\sqrt{1-x^2}}dx$ is :

a. $\frac{\pi}{2}\log(y-\sqrt{1+y^2})$

b. $\frac{\pi}{2}\log\sqrt{1+y^2}$

c. $\frac{\pi}{2}\log(y+\sqrt{1+y^2})$

d. None of these

16. If $\int_0^c\frac{\log(1+cx)}{1+x^2}dx = \frac{1}{2}\log(1+c^2)\tan^{-1}c$, then the value of $\int_0^1\frac{\log(1+x)}{1+x^2}dx$ is :

a. $\frac{\pi}{8}\log^2$ b. $\frac{\pi}{4}\log^2$

c. $\frac{\pi}{2}\log 2$ d. None of these

17. The value of $\int_0^{\pi}\frac{\log(1+\sin y\cos x)}{\cos x}dx$ is :

a. πy b. $\frac{y}{\pi}$

c. $\frac{\pi y}{2}$ d. $\frac{\pi}{y}$

18. If f is continuous in a rectangle $[a,b]\times[t_1,t_2]$ then :

a. $\int_{t_1}^{t_2}\left[\int_a^b f(x,t)\,dt\right]dx = \int_a^b\left[\int_{t_1}^{t_2} f(x,t)\,dt\right]dx$

b. $\int_{t_1}^{t_2}\left[\int_a^b f(x,t)\,dt\right]dx = \int_a^b\left[\int_{t_1}^{t_2} f(x,t)\,dx\right]dt$

c. $\int_{t_1}^{t_2}\left[\int_a^b f(x,t)\,dx\right]dt = \int_a^b\left[\int_{t_1}^{t_2} f(x,t)\,dt\right]dx$

d. None of these

19. The value of integral $\int_0^{\infty} e^{-x^2}\cos yx\,dx$ is :

a. $e^{-y^2/4}$ b. $\frac{\sqrt{\pi}}{2}e^{-y^2/2}$

c. $\frac{\pi}{2}e^{-y^2/4}$ d. None of these

20. If $f(x,t)$ is continuous function in $[a,b]\times[c,d]$ then the function $\phi(t) = \int_a^b f(x,t)\,dx$ is : **[Kanpur 2018]**

a. Continuous function of t on $[a,b]$

b. Continuous function of t on $[c,d]$

c. Continuous function of t on $[a,b]\times[c,d]$

d. None of these

21. If $a, b > 0$ then the integral

$$\int_0^{\infty}(e^{-ax} - e^{-bx})\frac{\cos mx}{x}dx$$ is equal to :

a. $\frac{1}{2}\log\left(\frac{m^2+b^2}{m^2+a^2}\right)$ b. $\frac{1}{2}\log\left(\frac{m^2-a^2}{m^2-b^2}\right)$

c. $2\log\left(\frac{m^2+b^2}{m^2+a^2}\right)$ d. $2\log\left(\frac{m^2-a^2}{m^2-b^2}\right)$

22. The value of $\int_0^{\infty}\frac{\sin mx}{x}dx$ if $m = 0$ is :

a. $\frac{\pi}{2}$ b. $\frac{-\pi}{2}$

c. 0 d. None of these

23. The value of the integral $\int_0^{\infty}\frac{\log(1+a^2x^2)}{1+b^2x^2}$ is :

a. $\frac{\pi}{a}\log\left(\frac{a+b}{b}\right)$

b. $\frac{\pi}{b}\log\left(\frac{a+b}{b}\right)$

c. $\frac{\pi}{a}\log\left(\frac{a-b}{a}\right)$

d. $\frac{\pi}{a}\log\left(\frac{a-b}{b}\right)$

24. If F is bounded and integrable in $[a,b]$ and $f(x,t)$ is continuous in $[a,b]\times[t_1,t_2]$ then $\int_a^b f(x,t), F(x)\,dx$ is :

a. Continuous in (t_1, t_2)

b. Differentiable in (t_1, t_2)

c. Continuous in $[t_1, t_2]$

d. Differentiable in $[t_1, t_2]$

25. If $\int_0^{\infty} e^{-ax}\,dx = \frac{1}{a}$ then the value of $\int_0^{\infty} e^{-ax}.x^n dx$ is:

a. $\frac{\lfloor n}{a^{n+1}}$ b. $\frac{\lfloor n}{a^n}$

c. $\frac{\lfloor n-1}{a^{n+1}}$ d. None of these

26. The value of $\int_0^{\infty} e^{-c^2x^2}dx$ is :

a. $\frac{\pi}{2\sqrt{a}}$ b. $\frac{\sqrt{\pi}}{2a}$

c. $\sqrt{\frac{\pi}{a}}$ d. None of these

27. If there exists $\mu(x) > \forall x \geq 0$ such that $\int_a^{\infty}\mu(x)\,dx$ converges and $|f(x,t)| \leq \mu(x)$ for all $x \geq \alpha$ and $t \in [c,d]$ then $\phi(t) = \int_a^{\infty} f(x,t)\,dx$ is : **[Kanpur 2018]**

a. Discontinuous

b. Divergent

c. Uniformly convergent

d. None of these

28. The value of $\int_0^{\infty}\frac{e^{-ax}\sin bx}{x}dx,\ a > 0$ is :

a. $\tan^{-1}\left(\frac{b}{a}\right)$ b. $\tan^{-1}\left(\frac{a}{b}\right)$

c. $\cot^{-1}\left(\frac{b}{a}\right)$ d. None of these

29. The value of $\int_0^1\frac{x^y-1}{\log x}dx$, when $y > -1$ is :

a. $\log y$ b. $\log(1+y)$

c. $\log\left(1+\frac{1}{y}\right)$ d. None of these

30. $\int_0^{\pi/2}\frac{1}{\sin x}\log\left(\frac{1+y\sin x}{1-y\sin x}\right)dx$ when $y^2 < 1$ is equal to:

a. $\pi\cos^{-1}y$ b. $\pi\tan^{-1}y$

c. $\pi\sin^{-1}y$ d. None of these

31. If $\int_0^{\infty}\frac{\cos mx}{a^2+x^2}dx = \frac{\pi}{2a}e^{-ma}$ then $\int_0^{\infty}\frac{x\sin mx}{a^2+x^2}dx$ is equal to :

a. $\frac{\pi}{2}e^{-ma}$ b. $\frac{\pi}{2}e^{ma}$

c. πe^{-ma} d. None of these

32. The value of $\int_0^{\infty}\frac{e^{-ax}\sin mx}{x}dx, c > 0$ is :

a. $\tan^{-1}\frac{a}{m}$ b. $\cot^{-1}\frac{m}{a}$

c. $\tan^{-1}\frac{m}{a}$ d. $\cot^{-1}\frac{a}{m}$

33. If $f(x,t) = \frac{\partial^2\phi}{\partial x\partial t}$ then which one is true :

a. $\int_{t_1}^{t_2} f(x,t)\,dt = \phi_x(x_1,t_1) + \phi_x(x_1,t_2)$

b. $\int_{t_1}^{t_2} f(x,t)\,dt = \phi_x(x_1,\alpha_1) + \phi_x(x_1,t_2)$

c. $\int_{t_1}^{t_2} f(x,t)\,dt = \phi_x(x_1,t_2) + \phi_x(x_1,t_1)$

d. None of these

34. If $\phi(t) = \int_u^v f(x,t)\,dx$ is differentiable then by Leibnitz rule $\frac{d\phi}{dt}$ is equal to :

a. $\int_u^v \frac{\partial}{\partial t} f(x,t)\,dx + f(v,t)\frac{dv}{dt} - f(u,t)\frac{du}{dt}$

b. $\int_u^v \frac{\partial}{\partial t} f(x,t)\,dx - f(v,t)\frac{dv}{dt} - f(u,t)\frac{du}{dt}$

c. $\int_u^v \frac{\partial}{\partial t} f(x,t)\,dx + f(v,t)\frac{dv}{dt} + f(u,t)\frac{du}{dt}$

d. None of these

35. If $\phi(t) = \int_a^\infty f(x,t)$ is continuous in $[t_1,t_2]$ then :

a. $f(x,t)$ is continuous for $x \geq t_1,\ t \in [t_1,t_2]$

b. $f(x,t)$ is differentiable in $[t_1,t_2]$

c. $\int_a^\infty f(x,t)\,dx$ is uniformly continuous in $[t_1,t_2]$

d. None of these

36. If $\int_0^{\pi/2} \sec x \log\left(\frac{1+a\cos x}{1+b\cos x}\right)dx$

$$= \frac{1}{2}[(\cos^{-1} - b) - (\cos^{-1} a)^2]$$

for all $0 \leq a < 1,\ 0 \leq b < 1$ then

$\int_0^{\pi/2} \sec x.\log\left(1 + \frac{1}{2}\cos x\right)dx$ is equal to :

a. $\frac{3\pi^2}{58}$ b. $\frac{5\pi^2}{72}$

c. $\frac{7\pi^2}{61}$ d. None of these

37. If $f(x,t)$ and f_t be continuous in $[a,b] \times [t_1,t_2]$ then :

a. f_t is differentiable

b. $\phi(t)$ is continuous

c. $\phi(f)$ is differentiable

d. None of these

38. The value of integral $\int_0^1 \frac{\log(1+x)}{(1+x^2)}dx$ is :

[Meerut 2015]

a. $\frac{\pi}{2}$ b. $\frac{\pi}{8}\log 2$

c. $\frac{\pi}{8}\log 3$ d. None of these

39. The value of $\int_0^\infty \frac{\sin mx}{x}dx$ if $m < 0$ is :

a. $\frac{\pi}{2}$ b. $\frac{-\pi}{2}$

c. 0 d. None of these

40. If $f(x,t)$ is continuous for $x \geq t$ and $t \in [t_1,t_2]$ and $\int_a^\infty f(x,t)\,dx$ is uniformly convergent for $t \in [t_1,t_2]$ then $\int_a^\infty f(x,t)\,dx$ is :

a. Continuous in $[t_1,t_2]$

b. Differentiable in $[t_1,t_2]$

c. Differentiable in (t_1,t_2)

d. None of these

41. The value of integral $\int_0^{\pi/2} \frac{\log(1+\cos y\cos x)}{\cos x}dx$ is :

a. $\frac{\pi^2}{4} + y^2$ b. $\frac{1}{2}\left(\frac{\pi^2}{4} - y^2\right)$

c. $\frac{\pi^2}{2} - y^2$ d. $2\left(\frac{\pi^2}{2} - y^2\right)$

42. In the Leibnitz rule if $\phi(t) = \int_u^v f(x,t)\,dx$ is differentiable and u is constant then $\frac{d\phi}{dt}$ is equal to :

a. $\int_u^v \frac{\partial f}{\partial t}du + f(v,t)\frac{dv}{dt}$

b. $\int_u^v \frac{\partial f}{\partial t}du + f(v,t)\frac{du}{dt} - f(u,t)\frac{du}{dt}$

c. $\int_u^v \frac{\partial f}{\partial t}du$

d. None of these

43. If $f(x,t)$ is continuous for $x \geq t$ and $t \in [t_1,t_2]$ and $\int_a^\infty f(x,t)\,dx$ is uniformly convergent for $t \in [t_1,t_2]$ then $\int_a^\infty f(x,t)\,dx$ is :

a. Differentiable in $[t_1,t_2]$

b. Integrated under the integral sign

c. Cannot be integrated under the integral sign

d. None of these

44. The value of $\int_0^\infty e^{-ax}\frac{\sin x}{x}dx,\ a > 0$ is :

a. $\tan^{-1}\left(\frac{1}{a}\right)$ b. $\tan^{-1}(a)$

c. $-\tan^{-1}\left(\frac{1}{a}\right)$ d. $-\tan^{-1} a$

45. The value of $\int_0^\infty \frac{x \sin mx}{a^2+x^2} dx$ is :

a. $\frac{\pi}{3} e^{-ma}$ b. $\frac{\pi}{2} e^{-ma}$

c. $\frac{\pi}{2a} e^{-ma}$ d. $\frac{\pi}{3a} e^{-ma}$

46. The value of $\int_0^\infty \frac{1-e^{-cx}}{xe^x} dx$ when $c > -1$ is :

a. $\log(1+c)$ b. $\frac{1}{2}\log(1+c)$

c. $\frac{\pi}{2}\log(1+c)$ d. None of these

47. If $f(x,t)$ is continuous and has a continuous partial derivative w.r.t. t for $x \geq t$ and $t \in [t_1, t_2]$ and $\int_a^\infty f_t(x,t)\, dx$ converges uniformly in $t \in [t_1, t_2]$ then $\phi(t)$ is :

a. Continuous only

b. Differentiable

c. Uniformly convergent

d. None of these

48. The value of $\int_0^{\pi/2} \frac{\log(1+y\sin^2 x)}{\sin^2 x} dx$ is :

a. $\pi\sqrt{1+y^2}$ b. $\pi[\sqrt{1+y^2}-1]$

c. $\pi[\sqrt{1+y^2}+1]$ d. None of these

49. The value of $\int_0^{\pi/2} \frac{\log(1+\cos m \cos x)}{\cos x} dx$ is :

a. $\frac{\pi^2}{8} - \frac{m^2}{2}$ b. $\frac{\pi^2}{8} + m^2$

c. $\frac{\pi^2}{4} - \frac{m^2}{2}$ d. None of these

50. Differentiable under the sign of integration is studied under the rule :

a. Lagranges b. Maclawn

c. Dirichlet's d. Leibnitz

51. If $f(x,t)$ is continuous and has a continuous partial derivative w.r.t. t for $x \geq t$ and $t \in [t_1, t_2]$ and $\int_a^\infty f_t(x,t)\, dx$ converges uniformly in $t \in [t_1, t_2]$ then $\phi'(t)$ is equal to :

a. $\int_a^\infty f_t(x,t)\, dx$ b. $\int_a^x f_t(x,t)\, dx$

c. $\int_a^\infty f_t(x,t)\, dx$ d. None of these

52. If f be a continuous functions of two variables x and t in $[a,b]\times[c,d]$ then $\int_a^b f(x,t)\, dx$ defines a function of :

a. A single variable x with domain $[a,b]$

b. A single variable t with domain $[c,d]$

c. Two variable x and y with domain $[a,b]\times[c,d]$

d. None of these

53. The value of $\int_0^\infty e^{-\left(x^2+\frac{\alpha^2}{x^2}\right)\beta^2} dx$ is :

a. $\frac{\pi}{2\alpha} e^{-\beta\alpha^2}$ b. $\frac{\pi}{2\alpha} e^{-2\beta\alpha^2}$

c. $\frac{\sqrt{\pi}}{2\beta} e^{-2\alpha\beta^2}$ d. None of these

54. The value of $\int_0^\infty \left(\frac{e^{-\alpha x}-e^{\beta x}}{x}\right)\sin px\, dx$ when $\beta > \alpha > 0$ and $p > 0$ is :

a. $\tan^{-1}\frac{\beta}{p} - \tan^{-1}\frac{\alpha}{p}$ b. $\tan^{-1}\frac{\beta}{p} + \tan^{-1}\frac{\alpha}{p}$

c. $\tan^{-1}\frac{p}{\beta} - \tan^{-1}\frac{p}{\alpha}$ d. $\tan^{-1}\frac{p}{\beta} + \tan^{-1}\frac{p}{\alpha}$

55. If $\int_0^\pi \frac{dx}{a+b\cos x} = \frac{\pi}{\sqrt{a^2-b^2}}$, $a > 0$, $|b| < a$ then the value of $\int_0^\pi \frac{dx}{(a+b\cos x)^2}$ is :

a. $\frac{\pi}{\sqrt{a^2-b^2}}$ b. $\frac{\pi a}{(a^2-b^2)^{3/2}}$

c. $\frac{\pi b}{\sqrt{a^2-b^2}}$ d. $\frac{\pi b}{(a^2-b^2)^{3/2}}$

56. The value of $\int_0^1 \frac{x^2-1}{\log x} dx$ is equal to :

a. log (3) b. log (2)

c. $-\log 3$ d. None of these

57. If $\int_0^\infty e^{-\alpha x}\frac{\sin \beta x}{x} dx = \tan^{-1}\frac{\beta}{\alpha}$, $\alpha \geq 0$ then $\int_0^\infty \frac{\sin \beta x}{x} dx$ for $\beta > 0$ is :

a. 0 b. $\frac{-\pi}{2}$

c. $\frac{\pi}{2}$ d. π

58. If $\beta > \alpha > 0$ and $p > 0$ such that

$$\int_0^\infty \left(\frac{e^{-\alpha x} - e^{-\beta x}}{x}\right) \sin px \, dx = \tan^{-1}\frac{\beta}{p} - \tan^{-1}\frac{\alpha}{p}$$

then for $p > 0$ the value of $\int_0^\infty \frac{\sin px}{x} dx$ is :

a. $\frac{\pi}{2}$ b. $\frac{\pi}{3}$

c. $\frac{\pi}{4}$ d. 0

59. If $\int_0^\infty e^{-\alpha x} \frac{\sin \beta x}{x} dx = \tan^{-1}\frac{\beta}{\alpha}$ for $\alpha \geq 0$ then $\int_0^\infty \frac{\sin \beta x}{x} dx$ when $\beta = 0$ is :

a. $\frac{\pi}{2}$ b. 0

c. $\frac{-\pi}{2}$ d. None of these

60. The function $y = \int_0^\infty \frac{e^{-x^2}}{1+z^2} dz$ satisfy the following differential equation :

a. $y'' - y = x$ b. $y'' + y = x$

c. $y'' + y = \frac{1}{x}$ d. $y'' - y = \frac{1}{x}$

61. If $I = \int_{\frac{\pi}{2}-\alpha}^{\pi/2} \sin\theta \cos^{-1}(\cos\alpha \text{ cosec}\,\theta)\, d\theta$ then $\frac{dI}{d\alpha}$ is equal to :

a. $\pi \sin\alpha$ b. $\pi \cos\alpha$

c. $\frac{\pi}{2}\sin\alpha$ d. $\frac{\pi}{2}\cos\alpha$

62. If $I = \int_0^\pi \frac{\log(1 + a\cos x)}{\cos x} dx, |a| < 1$ then $\frac{dI}{d\alpha}$ is equal to:

a. $\pi\sqrt{1-a^2}$ b. $\pi\sqrt{1+a^2}$

c. $\frac{\pi}{\sqrt{1+a^2}}$ d. $\frac{\pi}{\sqrt{1-a^2}}$

63. If $I = \int_0^{\pi/2} \log(a^2\cos^2\theta + b^2\sin^2\theta)\, d\theta$ where $a > 0$, $b > 0$ then $\frac{dI}{da}$ is equal to :

a. $\frac{\pi}{\sqrt{a^2 - b^2}}$ b. $\frac{\pi}{\sqrt{a^2 + b^2}}$

c. $\frac{\pi}{a+b}$ d. $\pi\sqrt{a^2 - b^2}$

64. If $I = \int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)} dx = \frac{\pi}{2}\log(1+a)$ for $a \geq 0$ then the value of I for $a < 0$ is :

a. $\frac{\pi}{2}\sin(1-a)$ b. $\frac{-\pi}{2}\sin(1-a)$

c. $\frac{-\pi}{2}\sin(1-a)$ d. None of these

65. If $\int_0^\infty e^{-ax} dx = \frac{1}{a}$ then the value of $\int_0^\infty e^{-ax} x^4 dx$ is :

a. $\frac{\lfloor 4}{a^4}$ b. $\frac{\lfloor 4}{a^5}$

c. $\frac{\lfloor 4}{a^3}$ d. $\frac{\lfloor 4}{a^6}$

66. If $\int_0^1 \frac{x^y - 1}{\sin x} dx = \sin(1+y)$ then $\int_0^1 \frac{x - a - \sin x}{(\sin x)^2} dx$ is equal to :

a. $\log 2 - 1$ b. $\log 3 - 1$

c. $\log 4 - 1$ d. $2\log 2 + 1$

67. If f is continuous on [0, 1] then **[Meerut 2014]**

$$\lim_{n\to\infty} \int_0^1 \frac{nf(x)}{1+n^2x^2} dx =$$

a. $\frac{\pi}{4}f(0)$ b. $\frac{\pi}{2}f(0)$

c. $\frac{\pi}{2}f(\pi)$ d. $\frac{\pi}{4}f(\pi/2)$

68. An improper integral is called convergent if : **[Meerut 2015]**

a. Limit is a finite number

b. Value of integral and limit same

c. Both (a) and (b)

d. None of these

69. Let $f(x,\alpha)$ be continuous in $[a,b] \times (\alpha_1, \alpha_2)$ and $\phi(x,\alpha) = \int_{\alpha_1}^{\alpha_2}\left[\int_a^x f(x,\alpha)\, dx\right] d\alpha$ then : **[Meerut 2016]**

a. $\frac{\partial^2\phi}{\partial x\,\partial\alpha} = \frac{\partial^2\phi}{\partial\alpha\,\partial x}$

b. $\frac{\partial^2\phi}{\partial x\,\partial\alpha} = -\frac{\partial^2\phi}{\partial\alpha\,\partial x}$

c. Both (a) and (b)

d. None of these

70. The integral $\int_0^{\infty} x^{n-1} e^{-x}\, dx$ is divergent if : **[Meerut 2015]**

a. $n > 0$ b. $n > 1$

c. $n \leq 0$ d. $n = \frac{1}{2}$

71. The integral $\int_0^{\infty} \frac{\cos x}{1+x^2}\, dx$ is : **[Meerut 2015]**

a. Divergent b. Convergent

c. Both (a) and (b) d. None of these

72. The value of integral $\int_1^{\infty} \frac{dx}{x^{3/2}}$ is : **[Meerut 2015]**

a. 0 b. ∞

c. 2 d. -2

73. The integral $\int_0^1 \frac{dx}{x^{1/3}(1+x^5)}$ is : **[Meerut 2015]**

a. Convergent b. Divergent

c. Both (a) and (b) d. None of these

74. Which of the following integral is divergent : **[Meerut 2015]**

a. $\int_0^1 \frac{dx}{x^3(1+x^2)}$ b. $\int_0^{\infty} e^{-x^2} dx$

c. $\int_0^{\infty} \frac{\cos x}{1+x^2} dx$ d. $\int_0^{\infty} \frac{\sin^2 x}{x^2} dx$

75. If the limit of an improper integral is ∞ or $-\infty$ the integral is said to be : **[Meerut 2015]**

a. Convergent b. Divergent

c. Oscillatory d. None of these

76. The integral $\int_0^{\infty} \frac{dx}{1+x^2}$ is an improper integral of the : **[Meerut 2015]**

a. First kind

b. Second kind

c. Neither (a) nor (b)

d. None of these

77. The integral $\int_a^b \frac{dx}{(x-a)^n}$ is convergent if : **[Meerut 2015]**

a. $n = 0$ b. $n > 0$ but $n < 1$

c. $n < 0$ d. None of these

78. For integral $\int_0^{\infty} \frac{4a}{x^2+4a^2} dx$ which is/are correct : **[Meerut 2015]**

a. The value at integral is π

b. The integral is divergent

c. Both (a) and (b)

d. None of these

79. If $f(x,\alpha) = \frac{\partial^2 \theta}{\partial x\, \partial \alpha}$ then we have : **[Meerut 2015]**

a. $\int_{\alpha_1}^{\alpha_2} f(x,\alpha)\, d\alpha = \phi_x(x,\alpha_1) - \phi_x(x,\alpha_2)$

b. $\int_{\alpha_1}^{\alpha_2} f(x,\alpha)\, d\alpha = \phi_x(x,\alpha_2) - \phi_x(x,\alpha_1)$

c. $\int_{\alpha_1}^{\alpha_2} f(x,\alpha)\, dx = \phi_x(x,\alpha_1) - \phi_x(x,\alpha_2)$

d. None of these

80. If $\alpha > -1$, then $\int_0^1 \frac{x^{\alpha}-1}{\log x} dx$ is equal to : **[Meerut 2015]**

a. $\log(1+\alpha)^{-1}$ b. $\log(1+\alpha)^2$

c. $\log(1+\alpha)$ d. None of these

81. The value of integral P is where

$$P = \int_0^1 \frac{\tan^{-1} ax}{\sqrt{1-x^2}} dx :$$ **[Meerut 2015]**

a. $\frac{\pi}{2} \log\,[\alpha + \sqrt{1+\alpha^2}]$

b. $\frac{\pi}{2} \log\,[a - \sqrt{1+\alpha^2}]$

c. $\pi \log\,[\alpha + \sqrt{1+\alpha^2}]$

d. None of these

82. If f is continuous in $[a,b] \times [\alpha_1, \alpha_2]$ then we have

$$\int_{\alpha_1}^{\alpha_2} \left\{ \int_a^b f(x,\alpha)\, dx \right\} d\alpha =$$ **[Meerut 2015]**

a. $\int_a^b \left| \int_{\alpha_1}^{\alpha_2} f(x,\alpha)\, d\alpha \right| dx$

b. $\int_a^b \left\{ \int_{\alpha_1}^{\alpha_2} f(x,\alpha)\, d\alpha \right\} dx$

c. Both (a) and (b)

d. None of these

83. The value of $\int_0^\infty x^3 e^{-\alpha x^2}\,dx$ is : **[Meerut 2015]**

a. $\frac{1}{2\alpha^2}$ b. $\frac{1}{\alpha^2}$

c. $\frac{2}{\alpha^2}$ d. None of these

84. If $|\alpha|< 1$ then value of integral

$$\int_0^\pi \frac{\log(1+\alpha\cos x)}{\cos x}\,dx \text{ is :}$$ **[Meerut 2016]**

a. π

b. $\sin^{-1}\alpha$

c. $\pi\sin^{-1}\alpha$

d. None of these

85. The improper integral

$$\phi(\alpha) = \int_a^\infty f(x,\alpha)\,dx$$

is continuous in $[\alpha_1,\alpha_2]$ if : **[Meerut 2016]**

a. $f(x,\alpha)\,dx$ is uniformly continuous for $x \ge a$, $\alpha \in [\alpha_1,\alpha_2]$

b. $\int_a^\infty f(x,\alpha)\,dx$ is uniformly continuous for $\alpha \in [\alpha_1,\alpha_2]$

c. Both (a) and (b)

d. None of these

86. If f is continuous in $[a,b]\times[\alpha_1,\alpha_2]$ then : **[Meerut 2016]**

a. $\int_{\alpha_1}^{\alpha_2}\left[\int_a^b f(x,\alpha)\,d\alpha\right]dx = \int_a^b\left|\int_{\alpha_1}^{\alpha_2} f(x,\alpha)\,d\alpha\right|dx$

b. $\int_{\alpha_1}^{\alpha_2}\left[\int_a^b f(x,\alpha)\,d\alpha\right]d\alpha = \int_a^b\left|\int_{\alpha_1}^{\alpha_2} f(x,\alpha)\,d\alpha\right|dx$

c. Both (a) and (b)

d. None of these

87. If F is a function of which is bounded and Integrable in $[a,b]$ and $f(x,\alpha)$ is continuous in $[a,b]\times[\alpha_1,\alpha_2]$ then : **[Meerut 2017]**

a. $\int_a^b f(x,\alpha)\,dx$ is continuous in $[\alpha_1,\alpha_2]$

b. $\int_a^b f(x,\alpha)\,dx$ is continuous in (α_1,α_2)

c. $\int_a^b f(x,\alpha)F(x)\,dx$ continuous in $[\alpha_1,\alpha_2]$

d. $\int_a^b f(x,a)F(x)\,dx$ is continuous in (α_1,α_2)

88. If f is continuous in $[a,b]\times[\alpha_1,\alpha_2]$ then $\int_{\alpha_1}^{\alpha_2}\{\int_a^b f(x,\alpha)\,dx\}\,d\alpha =$ **[Meerut 2017]**

a. $\int_a^b\left|\int_{\alpha_1}^{\alpha_2} f(x,\alpha)\,d\alpha\right|dx$

b. $\int_a^b\{\int_{\alpha_1}^{\alpha_2} f(x,\alpha)\,d\alpha\}\,dx$

c. $\int_{\alpha_1}^{\alpha_2}\{\int_a^b f(x,\alpha)\,d\alpha\}\,dx$

d. None of these

89. The integral $\int_a^\infty (1-e^x)\frac{\cos x}{x^2}\,dx$ where $a > 0$ is : **[Meerut 2017]**

a. Diverges

b. Converges

c. Both (a) and (b) true

d. None of these

90. $\int_0^1 \frac{\sin x}{x}\,dx$ is : **[Meerut 2019]**

a. Proper integral

b. Improper integral

c. Improper integral of first kind

d. Improper integral of second kind

91. $\int_0^\infty e^{-rt}\,dt$ is equal to :

a. r b. $\frac{1}{r}$

c. $-r$ d. $-\frac{1}{r}$

92. $\int_{-\infty}^\infty e^x\,dx$ is :

a. Convergent

b. Divergent

c. Conditional convergent

d. Zero

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(c)	**2.**	(b)	**3.**	(c)	**4.**	(b)	**5.**	(c)	**6.**	(b)	**7.**	(c)	**8.**	(d)	**9.**	(c)	**10.**	(b)
11.	(c)	**12.**	(b)	**13.**	(a)	**14.**	(b)	**15.**	(c)	**16.**	(a)	**17.**	(a)	**18.**	(c)	**19.**	(b)	**20.**	(b)
21.	(a)	**22.**	(c)	**23.**	(b)	**24.**	(c)	**25.**	(a)	**26.**	(b)	**27.**	(c)	**28.**	(a)	**29.**	(b)	**30.**	(c)
31.	(a)	**32.**	(c)	**33.**	(c)	**34.**	(a)	**35.**	(a)	**36.**	(b)	**37.**	(c)	**38.**	(b)	**39.**	(b)	**40.**	(a)
41.	(b)	**42.**	(a)	**43.**	(b)	**44.**	(a)	**45.**	(b)	**46.**	(a)	**47.**	(b)	**48.**	(b)	**49.**	(a)	**50.**	(d)
51.	(a)	**52.**	(b)	**53.**	(c)	**54.**	(a)	**55.**	(b)	**56.**	(a)	**57.**	(c)	**58.**	(a)	**59.**	(b)	**60.**	(c)
61.	(c)	**62.**	(c)	**63.**	(c)	**64.**	(b)	**65.**	(b)	**66.**	(c)	**67.**	(b)	**68.**	(c)	**69.**	(a)	**70.**	(a)
71.	(b)	**72.**	(c)	**73.**	(b)	**74.**	(a)	**75.**	(b)	**76.**	(a)	**77.**	(b)	**78.**	(a)	**79.**	(a)	**80.**	(c)
81.	(a)	**82.**	(b)	**83.**	(a)	**84.**	(c)	**85.**	(c)	**86.**	(b)	**87.**	(c)	**88.**	(b)	**89.**	(b)	**90.**	(a)
91.	(b)	**92.**	(b)																

HINTS AND SOLUTIONS

2. Let $f(x,y) = \dfrac{\log(1+b\cos x)}{\cos x}$

$\because \quad \lim\limits_{x\to\pi/2} \dfrac{\log(1+y\cos x)}{\cos x} = y$

and $f_y(x,y) = \dfrac{1}{1+y\cos x}$

Here f and f_y are continuous in $[0, \pi]$ and $|y| < 1$.

Let $\phi(y) = \int_0^\pi f(x,y)\,dx$

So, $\phi'(y) = \int_0^\pi f_y(x,y)\,dx$

$$= \int_0^\pi \frac{dx}{1+y\cos x}$$

$$= \int_0^\pi \frac{dx}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + y\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}$$

$$= \int_0^\pi \frac{\sec^2 x/2\,dx}{(1+y)+(1-y)\tan^2 x/2}$$

put $\tan\dfrac{x}{2} = t, \dfrac{1}{2}\sec^2\dfrac{x}{2}dx = dt$

So, $\phi'(y) = \int_0^\infty \dfrac{2dt}{(1+y)+(1-y)t^2}$

$$= \frac{2}{(1-y)}\frac{1}{\sqrt{\frac{1+y}{1-y}}} - \frac{\pi}{2} = \frac{\pi}{\sqrt{1-y^2}}$$

4. Let $I = \int_0^\infty x^3 e^{-cx^2}\,dx$

Put $cx^2 = t$

then $2cx\,dx = dt$

or $dx = \dfrac{dt}{2cx}$

$\therefore \quad I = \int_0^\infty \dfrac{t^{3/2}}{c^{3/2}} e^{-t} \dfrac{dt\,c^{1/2}}{2c.t^{1/2}}$

$$= \frac{1}{2c^2}\int_0^\infty e^{-t}\,t^{2-1}\,dt$$

$\because \quad \int_0^\infty e^{-1} t^{n-1}\,dt = \tau(t)$

So, $I = \dfrac{1}{2c^2}\tau(1) = \dfrac{1}{2c^2}$

6. Given that

$$\int_0^\infty e^{-cx}\,dx = \frac{1}{c} \text{ for } c > 0$$

Let $$\phi(b) = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx$$

$$= \int_0^\infty f(x, b)\,dx \qquad ...(1)$$

So, $f_b(x, b) = e^{-bx}$

Since, $\int_0^\infty f(x, b)\,dx$ and $\int_0^\infty f_b(x, b)\,dx$ are uniformly convergent so $\phi'(b)$ exists.

i.e., $$\phi'(b) = \int_0^\infty f(x, b)\,dx$$

$$= \int_0^\infty e^{-bx}\,dx = \frac{1}{b}$$

Integrating $\phi(b) = \log b + c \qquad ...(2)$

Put $b = a$ in equation (1) and (2) we get,

$$\phi(a) = 0 \text{ for } a > 0$$

and $$\phi(a) = \log a + c$$

$\therefore$ $$c = -\log a, \text{ for } a > 0$$

Put it in (2), we get

$$\phi(b) = \log b - \log a = \log\frac{b}{a}$$

So, $$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx = \log\frac{b}{a}$$

8. Let $$f(\theta, a) = \log(a^2\cos^2\theta + b^2\sin^2\theta)$$

then $$f_a(\theta, a) = \frac{2a\cos^2\theta}{a^2\cos^2\theta + b^2\sin^2\theta} \qquad ...(1)$$

Let $$\phi(a) = \int_0^{\pi/2} f(\theta, a)\,d\theta$$

$$= \int_0^{\pi/2} \log(a^2\cos^2\theta + b^2\sin^2\theta)\,d\theta \qquad ...(2)$$

then $$\phi'(a) = \int_0^{\pi/2} f_a(\theta, a)\,d\theta$$

$$= \int_0^{\pi/2} \frac{2a\cos^2\theta\,d\theta}{a^2\cos^2\theta + b^2\sin^2\theta}$$

$$= \int_0^{\pi/2} \frac{2a\,d\theta}{a^2 + b^2\tan^2\theta}$$

$$= \frac{2ab}{b^2 - a^2}\int_0^\infty \left[\frac{1}{a^2 + t^2} - \frac{1}{b^2 + t^2}\right]dt$$

By putting $t = b\tan\theta$ or $dt = b\sec^2\theta\,d\theta$

So, $$\phi'(a) = \frac{\pi}{a + b}$$ Integrating w.r.t. a

$$\phi(a) = \int \frac{\pi da}{(a + b)} + c$$

$$= \pi\log(a + b) + c \qquad ...(3)$$

or $$\phi(0) = \pi\log b + c \qquad ...(4)$$

put $a = 0$ in equal (2), we get

$$\phi(0) = \int_0^{\pi/2} \log(b^2\sin^2\theta)\,d\theta$$

$$= \pi\log b + \pi\log\frac{1}{2}$$

So, we get $$c = \pi\log\frac{1}{2}$$

Put it in equation (3), we get

$$\phi(a) = \pi\log(a + b) + \pi\log\frac{1}{2}$$

$$= \pi\log\left(\frac{a + b}{2}\right)$$

11. Let $$I = \int_0^\infty e^{-cx^2}\,dx$$

Put $$cx^2 = t \text{ or } x = \sqrt{\frac{t}{c}}$$

or $$dx = \frac{t^{-1/2}}{2\sqrt{c}}\,dt$$

$$I = \int_0^\infty \frac{e^{-t}\,t^{-1/2}}{2\sqrt{c}}\,dt$$

$$= \frac{1}{2\sqrt{c}}\int_0^\infty e^{-t}\,t^{1/2-1}\,dt$$

$$= \frac{1}{2\sqrt{c}}\,t\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2\sqrt{c}} = \sqrt{\frac{\pi}{4c}}$$

15. Let $$\phi(y) = \int_0^1 f(x, y)\,dx$$

$$= \int_0^1 \frac{\tan^{-1} yx}{x\sqrt{1 - x^2}}\,dx$$

then $$f_y(x, y) = \frac{1}{(1 + y^2 - x^2)\sqrt{1 - x^2}}$$

So, $$\phi'(y) = \int_0^1 f_y(x, y)\,dx$$

$$= \int_0^1 \frac{dx}{(1 + y^2x^2)\sqrt{1 - x^2}}$$

Put $x = \sin\theta$, $dx = \cos\theta\, d\theta$, we get

$$\phi'(y) = \int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{(1+y^2)\tan^2\theta + 1}$$

$$= \frac{\pi}{2\sqrt{1+y^2}}$$

Integrating w.r.t. y,

$$\phi(y) = \frac{\pi}{2}\log(y + \sqrt{1+y^2}) + c$$

put $y = 0$ we get

$$\phi(0) = 0 = \frac{\pi}{2}\log 1 + c \Rightarrow c = 0$$

$$\therefore \quad \phi(y) = \int_0^1 \frac{\tan^{-1} yx}{x\sqrt{1-x^2}}\,dx$$

$$= \frac{\pi}{2}\log(y + \sqrt{1+y^2})$$

16. Given that

$$\int_0^c \frac{\log(1+cx)}{1+x^2}\,dx = \frac{1}{2}\log(1+c^2)\tan^{-1} c$$

put $c = 1$ we get

$$\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{1}{2}\log(1+1)\tan^{-1} 1$$

$$= \frac{1}{2}\cdot\frac{\pi}{4}\log 2 = \frac{\pi}{8}\log 2$$

19. Let $\phi(y) = \int_0^\infty f(x,y)\,dx$

$$= \int_0^\infty e^{-x^2}\cos yx\,dx \qquad \ldots(1)$$

then $fy(x,y) = -xe^{-x^2}\sin yx$. Obviously $\int_0^\infty f(x,y)\,dx$ and $\int_0^\infty f_y(x,y)\,dx$ are uniformly convergent *i.e.* $\phi'(y)$ exists.

So, $$\phi'(y) = \int_0^\infty -xe^{-x^2}\sin yx\,dx$$

$$= \left[\frac{1}{2}e^{-x^2}\sin x\right]_0^\infty - \frac{y}{2}\int_0^\infty e^{-x^2}\cos yx\,dx$$

$$= \frac{-y}{2}\phi(y)$$

or $$\frac{\phi'(y)}{\phi(y)} = -\frac{y}{2},$$

Integrating it, we get

$$\phi(y) = -\frac{y^2}{4} + c$$

or $$\phi(y) = ce^{-y^2/4} \qquad \ldots(2)$$

put $y = 0$ in (1) and (2)

$$\phi(0) = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

and $$\phi(0) = ce^0 = c$$

So, $$c = \frac{\sqrt{\pi}}{2}$$

i.e. $$\phi(y) = \frac{\sqrt{\pi}}{2}e - \frac{y^2}{4}$$

21. Let $f(x,t) = e^{-xt}\cos mx \ \forall x \geq 0, t \geq \alpha \geq 0$

Let $$\phi(m) = \int_0^\infty e^{-xt}\sin mx\,dx$$

and $$\psi(m) = \int_0^\infty e^{-xt}\cos mx\,dx$$

$$\phi(x) = \lim_{c\to\infty}\int_0^c e^{-xt}\sin mx\,dx$$

$$= \lim_{c\to\infty}\left[\left(\frac{e^{-xt}\sin mx}{-t}\right)_0^c + \frac{m}{t}\int_0^c e^{-xt}\cos mx\,dx\right]$$

$$= \frac{m}{t}\psi(m) = \frac{m}{t}\lim_{c\to\infty}\int_0^c e^{-xt}\cos mx\,dx$$

$$= \frac{m}{t^2} - \frac{m^2}{t^2}\phi(m)$$

So, we get

$$\psi(m) = \int_0^\infty e^{-xt}\cos mx\,dx$$

$$= \frac{t}{m^2+t^2}, t > 0$$

or $$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\cos mx\,dx = \int_a^b \frac{t\,dt}{m^2+t^2}$$

$$= \frac{1}{2}[\log(t^2+m^2)]_a^b$$

$$= \frac{1}{2}\log\frac{b^2+m^2}{a^2+m^2},\ a, b > 0$$

25. Let $I = \int_0^\infty e^{-ax}\, dx = \frac{1}{a}$

Differentiating w.r.t. a, we get

$$\frac{dI}{da} = \int_0^\infty e^{-ax}.(-x)\, dx$$

$$= -\frac{1}{a^2} = (-1)^1 \frac{\lfloor 1}{a^2}$$

Differentiating again

$$\frac{d^2I}{d_{a^2}} = \int_0^\infty e^{-ax}(-x)^2\, dx = \frac{\lfloor 2}{a^2}$$

continuity it, we get

$$\int_0^\infty e^{-ax} x^n\, dx = \frac{\lfloor n}{a^{n+1}}$$

28. Let $f(a,b) = \int_0^\infty e^{-ax} \frac{\sin bx}{x} dx,\ a \ge 0$

$\because \quad \left| e^{-ax} \frac{\sin bx}{x} \right| \le \frac{e^{-ax}}{x}$

for $x > 0$ and $\int_0^\infty \frac{e^{-ax}}{x} dx$ is convergent at ∞ if $a > 0$

so by weierstrass μ-test $\int_0^\infty e^{-ax} \frac{\sin bx}{x} dx$ is uniformly convergent.

Similarly, $\int_0^\infty e^{-ax} \cos x\, bx\, dx$ is also uniformly convergent so $f_b(a,b)$ exists.

$$f_b(a,b) = \int_0^\infty e^{-ax} \cos x\, bx\, dx$$

$$= [e^{-ax}\{b\sin bx - a\cos bx\}]_0^\infty$$

$$f_b(a,b) = \frac{a}{a^2+b^2}$$

Integrating it w.r.t. b we get

$$f(a,b) = \tan^{-1}\frac{b}{a} + c$$

Put $b = 0$ we get,

$$0 = f(a, 0) = 0 + c \quad \Rightarrow \quad c = 0$$

$\therefore \quad f(a,b) = \int_0^\infty e^{-ax} \frac{\sin bx}{x} dx$

$$= \tan^{-1}\frac{b}{a}$$

29. Let $f(x,y) = \int_0^1 \frac{x^y - 1}{\log x}$ for $y > -1$

then $f_y(x,y) = \frac{x^y \log x}{\log x} = x^y$

Let $\phi(y) = \int_0^1 \frac{x^y - 1}{\sin x} dx$

or $\phi'(y) = \int_0^1 x^y dx = \frac{1}{y+1}$ when $y > -1$

Integrating it

$$\phi(y) = \log(1+y) + c \qquad \ldots(1)$$

Put $y = 0$ we get

$$\phi(x) = \int_0^1 0\, dx = 0 \text{ and } \phi(0) = c$$

So, $c = 0$ i.e. by (1)

$$\phi(y) = \log(1+y) = \int_0^1 \frac{x^y - 1}{\log x} dx$$

31. Given that

$$\int_0^\infty \frac{\cos mx}{a^2+x^2} dx = \frac{\pi}{2a} e^{-ma}$$

Differentiating both sides w.r.t. m we have

$$\int_0^\infty \frac{-x\sin mx}{a^2+x^2} dx = \frac{\pi}{2a} e^{-ma}(-a)$$

or $\int_0^\infty \frac{x\sin mx}{a^2+x^2} dx = \frac{\pi}{2} e^{-ma}$

36. Given that

$$\int_0^{\pi/2} \sec x \log\left(\frac{1+b\cos x}{1+a\cos x}\right) dx$$

$$= \frac{1}{2}\left[(\cos^{-1} a)^2 - (\cos^{-1} b)^2\right]$$

put $b = \frac{1}{2}, a = 0$ we get

$$\int_0^{\pi/2} \sec x \log\left(1 + \frac{1}{2}\cos x\right) dx$$

$$= \frac{1}{2}\left[(\cos^{-1} 0)^2 - \left(\cos^{-1}\frac{1}{2}\right)^2\right]$$

$$= \frac{1}{2}\left[\frac{\pi^2}{4} - \frac{\pi^2}{9}\right] = \frac{5\pi^2}{72}$$

38. Consider that

$$I = \int_0^a \frac{\log(1+ax)}{1+x^2} dx$$

$$\frac{dI}{da} = \int_0^a \frac{x dx}{(1+x^2)(1+ax)} + \frac{\log(1+a^2)}{1+a^2}$$

$$= \frac{1}{1+a^2}\int_0^a \left[\frac{x}{1+x^2} + \frac{a}{1+x^2} - \frac{a}{1+ax}\right]dx + \frac{\log(1+a^2)}{1+a^2}$$

$$= \frac{1}{1+a^2}\left[\frac{1}{2}\log(1+a^2) + a\tan^{-1}a - \log(1+a^2)\right] + \frac{\log(1+a^2)}{1+a^2}$$

$$\frac{dI}{da} = \frac{1}{2(1+a^2)}\log(1+a^2) + \frac{a\tan^{-1}a}{1+a^2}$$

Integration w.r.t. a, we get

$$I = \frac{1}{2}\log(1+a^2).\tan^{-1}a + c$$

But $I = 0$ when $a = 0$, so $c = 0$

$$\therefore \quad I = \frac{1}{2}\log(1+a^2).\tan^{-1}a$$

$$= \int_0^a \frac{\log(1+ax)}{1+x^2} dx$$

put $a = 1$ we get

$$\int_0^1 \frac{\log(1+x)}{1+x^2} dx = \frac{\pi}{8}\log 2$$

45. Consider that

$$I = \int_0^\infty \frac{\cos mx}{a^2+m^2} dx$$

we have

$$\int 2z\, e^{-(a^2+x^2)z^2} dz = \frac{1}{a^2+x^2}$$

Multiplying both sides by $\cos mx$ and integrating w.r.t. x from 0 to ∞, we have,

$$\int_0^\infty \int_0^\infty \cos mz.2z\, e^{-(a^2+x^2)z^2} dz\, dx = \int_0^\infty \frac{\cos mx}{a^2+x^2} dx$$

$$= \int_0^\infty 2z\, e^{-a^2z^2} \frac{\sqrt{\pi}}{2z} e^{\left(\frac{-m^2}{4z^2}\right)} dz$$

$$= \sqrt{\pi}\int_0^\infty e^{(-a^2z^2 - m^2/4z^2)} dz$$

$$= \sqrt{\pi}.\frac{\sqrt{\pi}}{2a} e^{\left(\frac{-2ma^2}{2a}\right)}$$

or $$\int_0^\infty \frac{\cos mx}{a^2+x^2} dx = \frac{\pi}{2a} e^{-ma}$$

Differentiating both sides w.r.t. m we get

$$\int_0^\infty \frac{-x\sin mx}{a^2+x^2} dx = \frac{\pi}{2a} e^{-ma}(-a)$$

or $$\int_0^\infty \frac{x\sin mx}{a^2+x^2} dx = \frac{\pi}{2} e^{-ma}$$

46. Let $$\phi(c) = \int_0^\infty \frac{1-e^{-cx}}{xe^x} dx$$

$$= \int_0^\infty f(x,c)\, dx \quad \text{...(1)}$$

Then $$f_c(x,c) = \frac{e^{-cx}}{e^x} = e^{-(c+1)x} \quad \text{...(2)}$$

Since, $\int_0^\infty f(x,c)\,dx$ and $\int_0^\infty f_c(x,c)\,dx$ we uniformly convergent so $\phi'(c)$ exists.

$$\phi'(c) = \int_0^\infty f_c(x,c)\, dx$$

$$= \int_0^\infty e^{-(c+1)x} dx = \frac{1}{c+1}$$

if integrating w.r.t. c we get

$$\phi(c) = \log(1+c) + d$$

then $\phi(0) = d$

put $c = 0$ in (1) we get

$$\phi(0) = 0$$

So, $c = 0$

i.e., $$\phi(c) = \int_0^\infty \frac{\phi - e^{-cx}}{xe^x} dx = \log(1+c)$$

53. Let $$I = \int_0^\infty \exp\left\{-\left(x^2 + \frac{\alpha^2}{x^2}\right)\beta^2\right\} dx \quad \text{...(1)}$$

Differentiating w.r.t. α we get

$$\frac{dI}{d\alpha} = -2\int_0^\infty \exp\left\{-\left(x^2 + \frac{\alpha^2}{x^2}\right)\beta^2\right\} \frac{\alpha\beta^2}{x^2} dx$$

put $\frac{\alpha}{x} = z$ or $\frac{\alpha}{-x^2}dx = dz$ we have

$$\frac{dI}{d\alpha} = -2\beta^2 \int_0^\infty \exp\left\{-\left(\frac{\alpha^2}{z^2}+z^2\right)\beta^2\right\}dx$$

$$= -2\pi\beta^2$$

or $$\int \frac{dI}{I} = -2\beta^2 \int d\alpha$$

Integrating we get,

$$\log I = -2\alpha\beta^2 + \log c$$

or $$I = ce^{-2\alpha\beta^2} \quad ...(2)$$

put $\alpha = 0$ in (1) we get

$$I = \int_0^\infty e^{-\beta^2 x^2} dx = \frac{\sqrt{\pi}}{2\beta}$$

Thus, $$c = \frac{\sqrt{\pi}}{2\beta}$$

By (2) we get,

$$\int_0^\infty \exp\left\{-\left(x^2+\frac{\alpha^2}{x^2}\right)\beta^2\right\}dx = \frac{\sqrt{\pi}}{2\beta}e^{-2\alpha\beta^2}$$

54. Let $f(x,t) = e^{-xt}\sin px$

then it continue for all $x \geq 0$ and $t \geq \alpha > 0$ so

$$\int_0^\infty \left(\int_\alpha^\beta e^{-xt}\sin px\, dx\right)dx = \int_\alpha^\beta \left(\int_0^\infty e^{-xt}\sin px\, dx\right)dt$$

or $$\int_0^\infty \frac{e^{-\alpha x}-e^{-\beta x}}{x}\sin px\, dx = \int_\alpha^\beta \frac{pdt}{p^2+x^2}$$

$$= \left[\tan^{-1}\frac{t}{p}\right]_\alpha^\beta = \tan^{-1}\frac{\beta}{p} - \tan^{-1}\frac{\alpha}{p}$$

where $\alpha,\beta > 0$

55. Given that

$$\int_0^\pi \frac{dx}{(a+b\cos x)} = \frac{\pi}{a^2-b^2}$$

for $a > 0$ and $|b| < a$

Differentiating it w.r.t. a, we get

$$-\int_0^\pi \frac{dx}{(a+b\cos x)^2} = \frac{-a\pi}{(a^2-b^2)^{3/2}}$$

or $$\int_0^\pi \frac{dx}{(a+b\cos x)^2} = \frac{a\pi}{(a^2-b^2)^{3/2}}$$

57. Given that

$$\int_0^\infty e^{-\alpha x}\frac{\sin\beta x}{x}dx = \tan^{-1}\frac{\beta}{\alpha}, \alpha \geq 0$$

put $\alpha = 0$ we get

$$\int_0^\infty \frac{\sin\beta x}{x}dx = \tan^{-1}\frac{\beta}{0}$$

$$= \tan^{-1}\alpha = \frac{\pi}{2}$$

59. Given that

$$\int_0^\infty \left(\frac{e^{-\alpha x}-e^{-\beta x}}{x}\right)\sin px\, dx = \tan^{-1}\frac{\beta}{p} - \tan^{-1}\frac{\alpha}{p}$$

put $\alpha \to 0$ and $\beta \to \infty$ we get

$$\int_0^\infty \frac{\sin px}{x}(1-0)\,dx = \tan^{-1}\alpha - \tan^{-1}0$$

or $$\int_0^\infty \frac{\sin px}{x}dx = \frac{\pi}{2}$$

60. Given that

$$y = \int_0^\infty \frac{e^{-x^2}}{1+z^2}dz \quad ...(1)$$

Differentiating w.r.t. x we get

$$y' = \int_0^\infty \frac{-ze^{-xz}}{1+z^2}dz \quad ...(2)$$

Again differentiating w.r.t. x

$$y'' = \int_0^\infty \frac{-z^2e^{-x^2}}{1+z^2}dz \quad ...(3)$$

Adding (1) and (3), we get

$$y''+y = \int_0^\infty \frac{1+z^2}{1+z^2}e^{-xz}\,dz$$

$$= \int_0^\infty e^{-xz}\,dz = \left[\frac{e^{-xz}}{-x}\right]_0^\infty = \frac{1}{x}$$

61. Let $$I = \int_{\frac{\pi}{2}-\alpha}^{\pi/2} \sin\theta\cos^{-1}(\cos\alpha\ \text{cosec}\ \theta)\,d\theta$$

$$\frac{dI}{d\alpha} = \int_{\frac{\pi}{2}-\alpha}^{\pi/2} \frac{-\sin\theta}{\sqrt{1-\cos^2\alpha\ \text{cosec}^2\ \theta}}$$

$$(-\sin\theta\ \text{cosec}\ \theta)\,d\theta - \sin\left(\frac{\pi}{2}-\alpha\right)\cos^{-1}$$

$$\left\{\cos\alpha\ \text{cosec}\left(\frac{\pi}{2}-\alpha\right)\right\}\cdot\frac{d}{d\alpha}\left(\frac{\pi}{2}-\alpha\right)$$

or $$\frac{dI}{d\alpha} = \int_{\frac{\pi}{2}-\alpha}^{\pi/2} \frac{\sin\alpha\sin\theta\, d\theta}{\sqrt{\sin^2\theta-\cos^2\alpha}}$$

$$= \int_{\frac{\pi}{2}-\alpha}^{\pi/2} \frac{\sin\alpha \sin\theta \, d\theta}{\sqrt{(1-\cos^2\theta)-(1-\sin^2\alpha)}}$$

$$= \int_{\frac{\pi}{2}-\alpha}^{\pi/2} \frac{\sin\alpha \sin\theta \, d\theta}{\sqrt{\sin^2\alpha - \cos^2\theta}}$$

$$= \sin\alpha \int_0^1 \frac{dt}{\sqrt{1-t^2}}$$

putting $\cos\theta = t\sin\alpha$

or $\frac{dI}{d\alpha} = \sin\alpha[\sin^{-1} t]_0^1 = \frac{\pi}{2}\sin\alpha$

64. Given that $\phi(a) = \int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)} dx$

$$= \frac{\pi}{2}\log(1+a) \text{ if } a \geq 0$$

then $\frac{d\phi}{da} = \int_0^\infty \frac{1}{x(1+x^2)} \cdot \frac{x}{(1+a^2x^2)} dx$

$$= \int_0^\infty \frac{1}{1-a^2}\left(\frac{1}{1+x^2} - \frac{az}{1+a^2x^2}\right) dx$$

provided $a \neq \pm 1$

$$\frac{d\phi}{da} = \frac{1}{1-a^2}[\tan^{-1} x - a\tan^{-1} ax]_0^\infty$$

when $a < 0$, $\tan^{-1} ax \to -\frac{\pi}{2}$ as $x \to \infty$ so

$$\frac{d\phi}{da} = \frac{1}{1-a^2}\left[\frac{\pi}{2} - a\left(\frac{-\pi}{2}\right)\right]$$

$$= \frac{\pi}{2}\left(\frac{1}{1-a}\right)$$

Integrating w.r.t. a we get

$$\phi = \frac{-\pi}{2}\log(1-a) + c$$

But $\phi \to 0$ as $a \to 0$ so $c = 0$

Hence, $\phi(a) = \frac{-\pi}{2}\log(1-a)$

65. By solution of question (25), we have

$$\int_0^\infty e^{-ax} x^n dx = \frac{\lfloor n}{a^{n+1}}$$

put $n = 4$, we get

$$\int_0^\infty e^{-ax} x^4 \, dx = \frac{\lfloor 4}{a^5}$$

❍❍❍

CHAPTER

12 Limits and Continuity of Functions of Two Variables

FUNCTIONS OF TWO VARIABLES

The function $z = f(x, y)$ is a real valued function of two independent real variables x and y, if for each pair of values of x and y *i.e.* (x, y) of $A \subseteq R^2$, there corresponds a unique value of z in R.

Examples $z = xy + x \sin y$ is a real valued function of two real independent variables x and y.

Here x and y are independent variable and z the dependent variable.

NEIGHBOURHOOD OF A POINT

1. **Circular neighbourhood**

   The set $N(a, b) = \{(x, y) : x \in R, y \in R$ and $\sqrt{(x-a)^2 + (y-b)^2} < \delta\ \}$ is called a circular neighbourhood of the point (a, b), where δ is an arbitrarily small positive real number.

2. **Rectangular neighbourhood**

   The set $N(a, b) = \{(x, y) : x \in R, y \in R, a - h < x < a + h, b - k < y < b + k\}$ is called a rectangular neighbourhood of (a, b), where h and k are arbitrarily small positive real numbers.

   It should be noted that energy circular neighbourhood of a point contains a rectangular neighbourhood and vice versa.

3. **Deleted neighbourhood**

   The set obtained by deleting the point (a, b) from its neighbourhood $N(a, b)$ is called deleted neighbourhood of (a, b).

LIMIT OF FUNCTIONS OF TWO VARIABLES

1. **Simultaneous limits (Double limits)**

   Let f be a function defined on some neighbourhood of a point (a, b) except possibly at (a, b), then a real number l is said to be the simultaneous limit of f at (a, b) if for each $\varepsilon > 0$, however small, there exists a $\delta > 0$ such that

   $$|f(x, y) - l| < \varepsilon \text{ whenever } (x, y) \text{ satisfies}$$

   $$|x - a| < \delta \text{ and } |y - b| < \delta$$

   It is denoted by $\lim\limits_{(x,y)\to(a,b)} f(x, y)$

   or $\lim\limits_{\substack{x\to a\\ y\to b}} f(x, y)$

   Non-existence criterion for simultaneous limit. If $f(x, y)$ tends to two different real numbers as $(x, y) \to (a, b)$ through two different points, then the simultaneous limit does not exists.

2. **Iterated limits (Repeated limits)**

   If function f is defined in some deleted neighbourhood of (a, b), then the limit,

   $\lim\limits_{y\to b} f(x, y)$ if it exists is a function of x say $g(x)$. If then the limit, $\lim\limits_{x\to a} \phi(x)$ exists and is equal to λ, we write

   $$\lim_{x\to a}\lim_{y\to b} f(x, y) = \lambda$$

   and then λ is called the repeated limit of f as $y \to b, x \to a$.

   If the order of taking the limit is changed, we obtained the other iterated limit

   $$\lim_{y\to b}\lim_{x\to a} (f(x, y)) = \mu$$

   or $\quad x \to$ and $y \to b$

   These two limits may or may not be equal.

   Note : If the simultaneous limit exists, these two repeated limits if they exist are necessarily equal but the converse is not true. Also, if the repeated limits are not equal, the simultaneous limit cannot exist.

Results :

1. Let the simultaneous limit $\lim_{(x,y)\to(a,b)} f(x, y)$ exist and be equal to A and let the limit. $\lim_{x\to a} f(x, y)$ exist for each constant value of y in the nbd of b and likewise set the limit $\lim_{y\to b} f(x, y)$ exist for each constant value of x in the nbd of a then

$$\lim_{x\to a}\lim_{y\to b} f(x, y) = \lim_{y\to b}\lim_{x\to a} f(x, y) = A$$

CONTINUITY OF FUNCTIONS OF TWO VARIABLES

The function $f(x, y)$ is said to be continuous at (a, b) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - a| < \delta, |y - b| < \delta \Rightarrow |f(x, y) - f(a, b)| < \varepsilon$$

Thus $f(x, y)$ is said to be continuous at (a, b) if the simultaneous limit $\lim_{(x,y)\to(a,b)} f(x, y)$ exists and is equal to its functional value $f(a, b)$ at (a, b). The function f is said to be continuous on the domain $D \subseteq R^2$ if f is continuous at each point of domain D.

If f is not continuous at $(a, b) \in D$ then f is said to be discontinuous at (a, b).

Removalbe discontinuity : A function $f(x, y)$ is said to be removable discontinuity at (a, b) if both $\lim_{(x,y)\to(a,b)} f(x, y)$ and $f(a, b)$ exist but are not equal.

Results :

1. If $A \times B = D \subseteq R^2$ and $f : D \to R$ be a function continuous at (a, b) then
2. The function $f(x) = f(x, b)$ is continuous at $x = a \in A$.
3. The function $f_2(y) = f(a, y)$ is continuous at $y = b \in B$.

Differentiability : A function $f(x, y)$ of two variables is said to be differentiable at (x, y) if there exists two numbers A and B such that we have the simultaneous double limit.

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) - A\Delta x - B\Delta y}{\Delta(x, y)}$$

where $\Delta(x, y) = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

$$A = fx(x_0, y_0)$$

$$= \lim_{\Delta x\to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$B = f_y(x_0, y_0)$$

$$= \lim_{\Delta y\to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

TAYLOR'S THEOREM FOR FUNCTIONS OF TWO VARIABLES

If $f(x, y)$ and all its partial derivatives are continuous in a certain domain of the point (x, y) then

$$f(x + h, y + k) = f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{\lfloor 2}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots.$$

In particular

$$f(x, y) = f(0,0) + x\left(\frac{\partial f}{\partial x}\right)_{(0,0)} + y\left(\frac{\partial f}{\partial y}\right)_{(0,0)} + \frac{x^2}{\lfloor 2}\left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} + xy\left(\frac{\partial^2 f}{\partial x_0 y}\right)_{(0,0)} + \frac{y^2}{\lfloor 2}\left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)} + \ldots\ldots$$

This is called Maclaurin's theorem for two variables.

MAXIMA AND MINIMA OF TWO VARIABLES

Let $f(x, y)$ be a continuous function and finite for all values of x and y in the neighbourhood of $x = a$ and $y = b$. Then $f(x, y)$ is maximum at (a, b) if

$$f(a + h, b + k) < f(a, b)$$

and minimum if $f(a - h, b - k) > f(a, b)$ for the small values of h and k.

Necessary and sufficient conditions for the existence of maxima and minima.

Necessary condition : The function $f(x, y)$ should maximum or minimum at $x = a$ and $y = b$ if

$$\left(\frac{\partial f}{\partial x}\right)_{\substack{x=a\\ y=b}} = 0 \text{ and } \left(\frac{\partial f}{\partial y}\right)_{\substack{x=a\\ y=b}} = 0$$

Sufficient condition : Consider $r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{\substack{x=a\\y=b}}$

$$s = \left(\frac{\partial^2 f}{\partial x.\partial y}\right)_{\substack{x=a\\y=b}}, t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a\\y=b}}$$

1. $f(x, y)$ is maximum if $rt - s^2 > 0$ for $r < 0$.
2. $f(x, y)$ is minimum if $rt - s^2 > 0$ for $r > 0$.
3. $f(x, y)$ is neither maximum nor minimum if $rt - s^2 < 0$.
4. $f(x, y)$ is doubtful when $rt - s^2 = 0$.

EXERCISE

MULTIPLE CHOICE QUESTIONS

Direction : Each of the following questions has four alternative answers. One of them is correct. Choose the correct answer.

1. The simultaneous limit of $f(x,y)$ exist and equal to A as $(x,y) \to (a,b)$ if for every $\varepsilon > 0, \exists\, \delta > 0$ such that

$$|f(x,y) - A| < \varepsilon \;\forall x, y$$

in the neighbourhood of (a,b) defined by :

a. $|x-a| > \delta, |y-b| > \delta$
b. $|x-a| < \delta, |y-b| > \delta$
c. $|x-a| < \delta, |y-b| > \delta$
d. $|x-a| < \delta, |y-b| < \delta$

2. The simultaneous limit $\lim\limits_{(x,y)\to(0,0)} y \sin\frac{1}{x}$ exists and is equal to :

a. 1 b. 0
c. –1 d. None of these

3. $\lim\limits_{x\to 0}\lim\limits_{y\to 0} \frac{y-x}{y+x}\cdot\frac{1+x}{1+y}$ is equal to :

a. 1 b. –1
c. 0 d. Does not exist

4. $\lim\limits_{(x,y)\to(0,0)} \frac{3x-2y}{2x-3y}$ is equal to :

a. $\frac{3}{2}$ b. $\frac{-2}{3}$
c. 0 d. Does not exist

5. The simultaneous limit of

$$f(x) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x} & , \quad xy \neq 0 \\ 0 & , \quad xy = 0 \end{cases}$$

at origin :

a. Exist
b. Does not exist
c. Cannot be determined
d. None of these

6. If $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists then it has :

a. Unique value b. Finite values
c. Infinite values d. None of these

7. $\lim\limits_{\substack{x\to a\\y\to b}} f(x,y)$ is called :

a. Iterated limits
b. Repeated limits
c. Simultaneous limits
d. None of these

8. $\lim\limits_{y\to 0}\lim\limits_{x\to 0} \frac{1+x}{1+y}\cdot\frac{y-x}{y+x}$ is equal to :

a. 0 b. –1
c. 1 d. Not exists

9. If simultaneous limit $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does not exist then iterated limits are :

a. Exist
b. Not exist
c. May or may not exist
d. None of these

10. $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$ is equal to :

a. 1 b. –1
c. $\frac{1}{2}$ d. 0

11. The function $f(x,y)$ is continuous at (a,b) if :
a. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists
b. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and equal to $f(a,b)$
c. $f(x,y)$ at (a,b) exists
d. None of these

12. The simultaneous limt $\lim_{(x,y)\to(0,0)} (x+y)$ is equals to :
a. 0 b. 1
c. 2 d. Does not exist

13. The function

$$f(x,y) = \begin{cases} x^2+2y & (x,y)\neq(1,2) \\ 0 & (x,y)=(1,2) \end{cases}$$

at (1,2) is : **[Kanpur 2018]**
a. Continuous b. Discontinuous
c. Differentiable d. None of these

14. If $\lim_{(x,y)\to(a,b)} f(x,y)$ exists then its value is :
a. Dependent of the path
b. Independent of the path
c. May or may not depend on the path
d. None of these

15. If $(a,b)\in R^2$ then

$$N(a,b) = \{(x,y) : x,y\in R,\ a-\delta < x < a+\delta, b-\delta < y < b+\delta\}$$

is called :
a. Circular neighbourhood
b. Rectangular neighbourhood
c. Square neighbourhood
d. None of these

16. Which of the following limits does not exist : **[Kanpur 2018]**
a. $\lim_{(x,y)\to(0,0)} y\sin\frac{1}{x}$ b. $\lim_{(x,y)\to(0,0)} (x+y)$
c. $\lim_{(x,y)\to(0,0)}$ d. None of these

17. If $\lim_{(x,y)\to(a,b)} f(x,y)$ exists then two iterated limits if exist are :
a. Equal
b. Not equal
c. May or may not be equal
d. None of these

18. $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x-y}$ is equal to :
a. 1 b. 0
c. $\frac{1}{2}$ d. Does not exist

19. The function $f(x,y)$ is continuous at (a,b) if for all $\in > 0, \delta > 0$ we have :
a. $|f(x,y)-f(a,b)| < \varepsilon$ when $|x-a|>\delta, |y-b|>\delta$
b. $|f(x,y)-f(a,b)| > \varepsilon$ when $|x-a|>\delta, |y-b|>\delta$
c. $|f(x,y)-f(a,b)| < \varepsilon$ when $|x-a|<\delta, |y-b|<\delta$
d. None of these

20. $\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$ is equal to :
a. 0 b. $\frac{1}{2}$
c. 1 d. Does not exist

21. $\lim_{(x,y)\to(0,0)} \frac{\sqrt{(x^2y^2+1)}-1}{x^2+y^2}$ is equal to :
a. 0 b. 1
c. $\frac{1}{2}$ d. Does not exist

22. If $(a,b)\in R^2$ then $N(a,b) = \{(x,y) : x,y\in R, \sqrt{(x-a)^2+(y-b)^2} < \delta\}$ is called :
a. Square nbd b. Rectangular nbd
c. Circular nbd d. None of these

23. If $\lim_{x\to a}\lim_{y\to b} f(x,y)$ and $\lim_{y\to b}\lim_{x\to 2} f(x,y)$ exists then they are :
a. Equal
b. Not equal
c. May or may not be equal
d. None of these

24. The domain of $z = e^{-(x^2+y^2)}$ is :
a. Whole xy-plane b. Whole yz-plane
c. Whole zx-plane d. None of these

25. The function $f(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x} & xy\neq 0 \\ 0 & xy = 0 \end{cases}$

the repeated limit :
a. Exist
b. Does not exist
c. May or may not exist
d. None of these

26. $\lim_{x\to a}\lim_{y\to b} f(x,y)$ is called :

a. Iterated limit b. Simultaneous limit

c. Double limit d. None of these

27. If $\lim_{(x,y)\to(a,b)} f(x,y)$ and $f(a,b)$ exist at (a,b) but are not equal then f is :

a. Continuous

b. Discontinuous of first kind

c. Discontinuous of second kind

d. Removable discontinuous

28. If $(a,b)\in R^2$ then the set $N(a,b)=\{(x,y): x,y\in R, a-h<x<a+h, b-k<y<b+k\}$ is called :

a. Circular nbd b. Square nbd

c. Rectangular nbd d. None of these

29. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ is equal to :

a. $\frac{1}{2}$ b. 0

c. 1 d. Does not exist

30. If simultaneous limit $\lim_{(x,y)\to(a,b)} f(x,y)$ exists then iterated limits are :

a. Exist

b. Not exist

c. May or may not exist

d. None of these

31. On expending $f(x+h, y+k)$ by Taylor's theorem the second term is :

a. $f(x,y)$ b. $hf_x + kf_y$

c. $kf_x + hf_y$ d. $h^2 f_x + k^2 f_y$

32. $\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^2+y^2}$ is equal to :

a. 0 b. 1

c. $\frac{1}{2}$ d. Does not exist

33. If $\lim_{(x,y)\to(a,b)} f(x,y)$ exists then which one of the following exist :

a. $\lim_{x\to a} f(x,y)$ b. $\lim_{y\to b} f(x,y)$

c. $\lim_{x\to a} f(x,b)$ d. Both (a) and (b)

34. $\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2}$ is equal to :

a. 0 b. 1

c. -1 d. Does not exist

35. If $f: A\times B\subseteq R^2 \to R$ be a continuous function at $(a,b)\in D$ such that $f_1(x)=f(x,b)$ then f_1 is :

a. Continuous at $x=b$

b. Continuous at $x=a$

c. Discontinuous at $x=a$

d. Discontinuous at $x=b$

36. If $\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{y\to b}\lim_{x\to a} f(x,y)$ then $\lim_{\substack{x\to a\\ y\to b}} f(x,y)$:

a. Exist

b. Not exist

c. May or may not be exist

d. None of these

37. The expansion of $e^x\cos(1+y)$ in Taylor series in the nbd of (0,0) is :

a. $x+xy+y^2+.....$ b. $x-xy+\frac{y^2}{2}+.....$

c. $y+xy-\frac{y^2}{2}+.....$ d. $y-xy+\frac{y^2}{2}+......$

38. Consider the following statements :

(i) $\lim_{(x,y)\to(0,0)} y\sin\frac{1}{x}$ exists

(ii) $\lim_{(x,y)\to(0,0)} y\sin\frac{1}{x}=0$

(iii) $\lim_{x\to 0} y\sin\frac{1}{x}$ exist

a. I and III are true b. II and III are true

c. I and II are true d. I,II,III are true

39. The function

$$f(x,y)=\begin{cases} 3xy & , (x,y)\neq(2,3) \\ 6 & , (x,y)=(2,3) \end{cases} \text{ is :}$$

a. Continuous at (2,3)

b. Discontinuity of first kind at (2,3)

c. Removable discontinuity at (2,3)

d. None of these

40. If repeated limits do not exist the simultaneous limit:

a. Exist

b. Does not exists

c. May or may not exists

d. None of these

41. The value of simultaneous limit $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$ is: **[Kanpur 2018]**

a. 0 b. 1

c. $\frac{1}{2}$ d. Does not exist

42. The second degree term in Taylor's expansion of $\cos x \cos y$ is :

a. $xy - \frac{x^2}{2}$ b. $xy - \frac{y^2}{2}$

c. $\frac{x^2+y^2}{2}$ d. $-\left(\frac{x^2+y^2}{2}\right)$

43. $f(x,y)$ has a removable discontinuity at (a,b) if :

a. $\lim_{(x,y)\to(a,b)} f(x,y)$ exist

b. $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist

c. $\lim_{(x,y)\to(a,b)} f(x,y)$ exist and equal to $f(a,b)$

d. $\lim_{(x,y)\to(a,b)} f(x,y)$ exist but not equal to $f(a,b)$

44. The second degree term in Taylor's series for the function $f(x,y) = e^x \log(1+y)$ in the nbd of (0,0) is:

a. $xy - \frac{y^2}{2}$ b. $-xy + \frac{y^2}{2}$

c. $\frac{x^2-y^2}{2}$ d. None of these

45. Which of the following statement is true for the function $f(x,y) = \frac{xy}{x^2+y^2}$, where $(x,y) \neq (0,0)$:

(I) order of iterated limit can be interchanged

(II) simultaneous limit exist

a. I is true

b. II is true

c. Both I and II are true

d. None of these

46. Which of the following statement is true for the function

$$f(x,y) = \frac{y-x}{y+x} \cdot \frac{1+x}{1+y}, (x,y) \neq (0,0)$$

(I) repeated limit exist at (0,0)

(II) repeated limit are unequal

a. I is true only

b. II is true only

c. I and II both are true

d. None of these

47. $\lim_{\substack{x\to0\\y\to0}} \frac{2xy}{x^2+y^2}$ is equal to :

a. 1 b. 2

c. 0 d. Does not exist

48. The second degree term of $e^x \sin y$ in Taylor series expansion is :

a. $xy + \frac{x^2}{2}$ b. $xy - \frac{y^2}{2}$

c. xy d. None of these

49. For the function $f(x,y) = \begin{cases} 1 \text{ if } xy \neq 0 \\ 0 \text{ if } xy = 0 \end{cases}$ at (0,0) :

a. Simultaneous limit exist

b. Repeated limit exist

c. Both simultaneous and repeated limit exist

d. None of these

50. For the function

$$f(x,y) = \begin{cases} x\sin\frac{1}{y} + y & \text{if } x \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

which of the following statements are true :

I $\lim_{(x,y)\to(0,0)} f(x,y)$ exist

II $\lim_{y\to0}\lim_{x\to0} f(x,y)$ does not exist

III $\lim_{x\to0}\lim_{y\to0} f(x,y)$ exist

a. I and II are true b. II and III are true

c. I,II and II are true d. I is true

51. If $f : A\times B \subseteq R^2 \to R$ be a continuous function at (a,b) such that

$$f_1(y) = f(a,y) \text{ then } f_1 \text{ is :}$$

a. Discontinuous at a

b. Discontinuous at b

c. Continuous at b

d. Continuous at a

52. For the function

$$f(x,y) = \frac{2xy^2}{x^3+3y^3}, (x,y) \neq (0,0) \text{ and } f(0,0) = 0$$

consider the following statements :

I $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist

II f is not continuous at (0,0)

a. II is true only

b. I is true only

c. I and II both are true

d. I and II both are false

53. The function $f(x,y) = \begin{cases} 3xy & , (x,y) \neq (2,3) \\ 6 & , (x,y) = (2,3) \end{cases}$ at (2,3) is :

a. Continuous b. Discontinuous

c. Differentiable d. None of these

54. The function $f(x,y) = |x| + |y|$ at (0,0) is :

a. Differentiable

b. Not differentiable

c. Cannot be determined

d. None of these

55. The function

$$f(x,y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y} & , xy \neq 0 \\ 0 & , x = 0, y = 0 \end{cases}$$

at (0,0) is :

a. Discontinuous

b. Not differentiable

c. Continuous but not differentiable

d. None of these

56. The function $f(x,y) = \frac{1}{x^2+y^2}, (x,y) \neq (0,0)$ and $f(0,0) = 0$ at (0, 0) is : **(Kanpur 2018)**

a. Continuous

b. Discontinuous

c. Uniformly continuous

d. None of these

57. The function $f(x,y) = \frac{x^3y^3}{x^2+y^2}, (x,y) \neq (0,0)$ and $f(0,0) = 0$ at (0,0) is :

a. Continuous only

b. Differentiable only

c. Continuous but not differentiable

d. Continuous and differentiable both

58. If $f(x,y)$ is differentiable at (a,b) then at (a,b) it is :

a. Continuous

b. Discontinuous

c. May be continuous or discontinuous

d. None of these

59. The function $f(x,y) = \begin{cases} \frac{x^4-y^4}{x^4+y^4} & , (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ at (0,0) is :

a. Continuous

b. Differentiable

c. Continuous but not differentiable

d. None of these

60. The second degree term in taylors expansion of the function $e^x \cos y$ is :

a. $\frac{x-y}{2}$ b. $\frac{x+y}{2}$

c. $\frac{x^2-y^2}{\lfloor 2}$ d. $\frac{x^2+y^2}{\lfloor 2}$

61. The function $f(x,y) = \frac{xy}{\sqrt{x^2-y^2}}, (x,y) \neq (0,0)$ and $f(0,0) = 0$ at (0,0) is :

a. Continuous only

b. Differentiable only

c. Continuous and differentiable both

d. None of these

62. The function $f(x,y) = \frac{x^2y^2}{x^4+y^4}, (x,y) \neq (0,0)$ and $f(0,0) = 0$ at (0,0) is :

a. Continuous

b. Differentiable

c. Continuous but not differentiable

d. None of these

63. The function

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

at origin is :

a. Continuous
b. Discontinuous
c. Cannot be determined
d. None of these

64. The function $f(x,y) = \dfrac{xy^2}{x^2+y^2}$, $(x,y) \neq (0,0)$ and $f(0,0) = 0$ then at $(0,0)$ $f(x,y)$ is :

a. Continuous and differentiable both
b. Continuous but not differentiable
c. Differentiable but not continuous
d. None of these

65. The third degree terms in the Taylor's expansion of the function $e^x \sin y$ is :

a. $\dfrac{x^2y}{2} - \dfrac{y^3}{6}$ b. $\dfrac{xy^2}{2} + \dfrac{x^3}{6}$
c. $\dfrac{xy^2}{2} - \dfrac{y^3}{6}$ d. None of these

66. If $f(x,y) = \dfrac{x-y}{x+y}$, $(x,y) \neq (0,0)$ then iterated limit are :

a. Exist and equal
b. Not exist but equal
c. Exist but not equal
d. None of these

67. The function $f(x,y) = \dfrac{x^3-y^3}{x^2+y^2}$, $(x,y) \neq (0,0)$ and $f(x,y) = 0$ then at (0,0), f is :

a. Continuous
b. Differentiable
c. Continuous but not differentiable
d. None of these

68. If $f(x,y) = \dfrac{xy^2}{x^2+y^2}$, $(x,y) \neq (0,0)$ and $f(0,0) = 0$ then at (0,0), $f(x,y)$ is :

a. Continuous b. Discontinuous
c. Differentiable d. None of these

69. The function $f(x,y) = (|x,y|)^{1/2}$ when $x,y \neq 0$ and $f(0,0) = 0$ then :

a. f_x and f_y not exist at origin
b. f is differentiable at origin
c. f_x and f_y both exist but is not differentiable
d. None of these

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(d)	**2.**	(b)	**3.**	(b)	**4.**	(d)	**5.**	(a)	**6.**	(a)	**7.**	(c)	**8.**	(c)	**9.**	(c)	**10.**	(d)
11.	(b)	**12.**	(a)	**13.**	(b)	**14.**	(b)	**15.**	(c)	**16.**	(c)	**17.**	(a)	**18.**	(d)	**19.**	(c)	**20.**	(d)
21.	(a)	**22.**	(c)	**23.**	(c)	**24.**	(a)	**25.**	(b)	**26.**	(a)	**27.**	(b)	**28.**	(c)	**29.**	(d)	**30.**	(c)
31.	(b)	**32.**	(a)	**33.**	(c)	**34.**	(a)	**35.**	(b)	**36.**	(c)	**37.**	(c)	**38.**	(c)	**39.**	(c)	**40.**	(c)
41.	(d)	**42.**	(d)	**43.**	(d)	**44.**	(a)	**45.**	(a)	**46.**	(c)	**47.**	(d)	**48.**	(c)	**49.**	(b)	**50.**	(d)
51.	(c)	**52.**	(c)	**53.**	(b)	**54.**	(b)	**55.**	(d)	**56.**	(b)	**57.**	(d)	**58.**	(a)	**59.**	(d)	**60.**	(c)
61.	(c)	**62.**	(d)	**63.**	(a)	**64.**	(b)	**65.**	(a)	**66.**	(c)	**67.**	(a)	**68.**	(a)	**69.**	(b,c)		

HINTS AND SOLUTIONS

2. Let $\varepsilon > 0$ be given and $\delta = \varepsilon$ then for all x, y satisfying $0 < |x| < \delta$ and $0 < |y| < \delta$, we have

$$\left| y\sin\frac{1}{x} - 0 \right| = \left| y\sin\frac{1}{x} \right| = |y| \left| \sin\frac{1}{x} \right|$$

$$\leq |y|$$

$$< \delta = \varepsilon$$

So, $\lim\limits_{(x,y)\to(0,0)} y\sin\frac{1}{x} = 0$

i.e., the simultaneous limit exists.

3. $\lim\limits_{x\to 0} \lim\limits_{y\to 0} \frac{y-x}{y+x} \frac{1+x}{1+y}$

$$= \lim_{x\to 0} -\left(\frac{1+x}{1}\right) = -1$$

4. When $(x, y) \to (0,0)$ along the line $y = x$ then

$$\lim_{(x,y)\to(0,0)} \frac{3x-2y}{2x-3y} = \lim_{x\to 0} \frac{3x-2x}{2x-3x}$$

$$= \lim_{x\to 0} \frac{x}{-x} = -1$$

Again when $(x, y) \to (0,0)$ along the line $y = 0$ then

$$\lim_{(x,y)\to(0,0)} \frac{3x-2y}{2x-3y} = \lim_{x\to 0} \frac{3x-0}{2x-0} = \frac{3}{2}$$

Since here different limiting values exist so the simultaneous limit does not exist.

5. Let $\varepsilon > 0$ and for all $(x, y) \neq (0,0)$ such that $xy = 0$ we have

$$|f(x, y) - 0| = |0 - 0| = 0 < \varepsilon$$

Again for all $(x, y) \neq (0,0)$ such that $xy \neq 0$

We have,

$$|f(x,y) - 0| = |f(x,y)| = \left| x\sin\frac{1}{y} + y\sin\frac{1}{x} \right|$$

$$\leq \left| x\sin\frac{1}{y} \right| + \left| y\sin\frac{1}{x} \right|$$

$$= |x| \left| \sin\frac{1}{y} \right| + |y| \left| \sin\frac{1}{x} \right|$$

$$\leq |x| + |y|$$

$$\leq 2\sqrt{x^2 + y^2}$$

$\because$ $|x| \leq \sqrt{x^2+y^2}$ and $|y| \leq \sqrt{x^2+y^2}$

$$< \varepsilon \text{ if } x^2 < \frac{\varepsilon^2}{\delta} \text{ and } y^2 < \frac{\varepsilon^2}{8}$$

i.e. if $x < \frac{\varepsilon}{2\sqrt{2}}$ and $|y| < \frac{\varepsilon}{2\sqrt{2}}$

Choose $\delta = \frac{\varepsilon}{2\sqrt{2}}$ then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x, y) - 0| < \varepsilon$ whenever $|x| < \delta$ and $|y| < \delta$.

Hence, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$

8. $\lim_{y\to 0}\lim_{x\to 0}\frac{1+x}{1+y}, \frac{y-x}{y+1}$

$$= \lim_{y\to 0}\left(\frac{1}{1+y}\right) = 1$$

10. Take $\varepsilon > 0$ then for all $(x, y) \neq (0,0)$ we have

$$\left|\frac{xy^2}{x^2+y^2} - 0\right| = \left|\frac{xy^2}{x^2+y^2}\right| = \left|\frac{r^3\cos\theta\sin^2\theta}{r^2}\right|$$

by taking $x = r\cos\theta, y = r\sin\theta$

$$= r\,|\cos\theta|\,|\sin\theta|^2 \leq r$$

$$= \sqrt{x^2+y^2}$$

$$< \varepsilon \text{ if } x^2 < \frac{\varepsilon^2}{2} \text{ and } y^2 < \frac{\varepsilon^2}{2}$$

i.e. if $|x| < \frac{\varepsilon}{\sqrt{2}}$ and $|y| < \frac{\varepsilon}{\sqrt{2}}$

Choose $\delta = \frac{\varepsilon}{\sqrt{2}}$ we have for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|\frac{xy^2}{x^2+y^2} - 0\right| < \varepsilon \text{ whenever } |x| < \delta \text{ and } |y| < \delta.$$

Hence, $\lim_{(x,y)\to(0,0)}\frac{xy^2}{x^2+y^2} = 0$

13. We have

$$\lim_{(x,y)\to(1,2)} x^2 + 2y = 1 + 4 = 5$$

So the limit exists and equal to 5

Since, $f(1, 2) = 0$ and $\lim_{(x,y)\to(1,2)} f(x, y) = 5$

i.e., $\lim_{(x,y)\to(1,2)} f(x, y) \neq f(1, 2)$

So, the function is not continuous at (1,2).

18. Let $(x, y) \to (0,0)$ along the curve $y = x - mx^3$ then

$$\lim_{(x,y)\to(0,0)}\frac{x^3+y^3}{x-y}$$

$$= \lim_{x\to 0}\frac{x^3+(x-mx^3)^3}{x-(x-mx^3)}$$

$$= \lim_{x\to 0}\frac{x^3[1+(1-mx^3)^3]}{mx^3}$$

$$= \lim_{x\to 0}\frac{2-3mx^2+3m^2x^4-m^3x^6}{m}$$

$$= \frac{2}{m}$$

which is different for different values of m.

So, $\lim_{(x,y)\to(0,0)}\frac{x^3+y^3}{x-y}$ does not exist.

20. Let $(x, y) \to (0,0)$ along the line $y = x$ then

$$\lim_{(x,y)\to(0,0)}\frac{xy^3}{x^2+y^6} = \lim_{x\to 0}\frac{x^4}{x^2+x^6} = 0$$

Again let $(x, y) \to (0,0)$ along $x = y^3$ then

$$\lim_{(x,y)\to(0,0)}\frac{xy^3}{x^2+y^6} = \lim_{y\to 0}\frac{y^6}{y^6+y^6} = \frac{1}{2}$$

Two limits are different so the limit does not exist.

21. Since, x and y are small in absolute values so

$$\frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} = \frac{(1+x^2y^2)^{1/2}-1}{x^2+y^2}$$

$$\approx \frac{\frac{1}{2}x^2y^2}{x^2+y^2}$$

Choose $\varepsilon > 0$ we have

$$\left|\frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} - 0\right| \approx \frac{\frac{1}{2}x^2y^2}{x^2+y^2}$$

$$= \frac{\frac{1}{2}r^4\cos^2\theta\sin^2\theta}{r^2}$$

putting $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{1}{2}r^2\cos^2\theta\sin^2\theta \leq \frac{r^2}{2} = \frac{1}{2}(x^2+y^2)$$

$$< \varepsilon \text{ if } x^2 < \varepsilon \text{ or } x < \sqrt{\varepsilon}$$

and $y^2 < \varepsilon$ or $y < \sqrt{\varepsilon}$

Choose $\delta = \sqrt{\varepsilon}$ we get for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|\frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} - 0\right| < \varepsilon \text{ whenever } |x| < \delta \text{ and } |y| < \delta$$

So $\lim_{(x,y)\to(0,0)}\frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} = 0$

25. Let $f(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x} & xy \neq 0 \\ 0 & xy = 0 \end{cases}$

Since, $\lim_{y\to 0} f(x,y)$ and $\lim_{x\to 0} f(x,y)$ do not exist and so both the repeated limits $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ and $\lim_{y\to 0}\lim_{x\to 0} f(x,y)$ do not exist.

29. Given that $f(x,y) = \dfrac{xy}{x^2+y^2}$

Let $(x,y) \to (0,0)$ along the line $y = x$ we have

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

Again let $(x,y) \to (0,0)$ along y axis *i.e.* $x = 1$.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{y\to 0} \frac{0}{0+y^2} = 0$$

Since two methods give different results the simultaneous limits does not exist.

32. Choose $\varepsilon > 0$ then for all $(x,y) \neq (0,0)$ we have

$$\left|\frac{x^3y^3}{x^2+y^2} - 0\right| = \left|\frac{x^3y^3}{x^2+y^2}\right|$$

$$= |r^4\cos^3\theta\sin^3\theta|,$$

putting $x = r\cos\theta, y = r\sin\theta$

$$= r^4|\cos\theta|^3|\sin\theta|^3$$

$$\leq r^4 = (x^2+y^2)^2 < \varepsilon$$

if $\quad x^2 < \dfrac{\sqrt{\varepsilon}}{2}$ or $|x| < \left(\dfrac{\sqrt{\varepsilon}}{2}\right)^{1/2}$

and $\quad y^2 < \dfrac{\sqrt{\varepsilon}}{2}$ or $(y) < \left(\dfrac{\sqrt{\varepsilon}}{2}\right)^{1/2}$

Choose $\delta = \sqrt{\dfrac{\sqrt{\varepsilon}}{2}}$ we have for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|\frac{x^3y^3}{x^2+y^2} - 0\right| < \varepsilon \text{ whenever } |x| < \delta \text{ and } |y| < \delta$$

Hence, $\quad \displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^2+y^2} = 0$

34. Take $\varepsilon > 0$ and putting $x = r\cos\theta$, $y = r\sin\theta$

$$\left|xy\frac{x^2-y^2}{x^2+y^2} - 0\right| = \left|xy\frac{x^2-y^2}{x^2+y^2}\right|$$

$$= |r^2\sin\theta\cos\theta.\cos 2\theta|$$

$$= \left|\frac{r^2}{4}\sin 4\theta\right| \leq \frac{r^2}{4} = \frac{x^2+y^2}{4}$$

$$< \varepsilon \text{ if } \frac{x^2}{4} < \frac{\varepsilon}{2} \text{ and } \frac{y^2}{4} < \frac{\varepsilon}{2}$$

i.e., if $\quad |x| < \sqrt{2\varepsilon}$ and $|y| < 2\varepsilon$

Take $\delta = \sqrt{2\varepsilon}$ then for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|xy\frac{x^2-y^2}{x^2+y^2} - 0\right| < \varepsilon \text{ whenever } |x| < \delta \text{ and } |y| < \delta$$

Hence, $\quad \displaystyle\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2} = 0$

37. Let $\quad F(x,y) = e^x\log(1+y)$ then $f(0,0) = 0$

$F_x(x,y) = e^x\log(1+y)$

so $\quad F_x(0,0) = 0$

$F_{xx}(x,y) = e^x\log(1+y)$

so $\quad F_{xx}(0,0) = 0$

$F_y(x,y) = \dfrac{e^x}{1+y}$

so $\quad F_y(0,0) = 1$

$F_{yy}(x,y) = -\dfrac{e^x}{(1+y)^2}$

so $\quad F_{yy}(0,0) = -1$

$F_{xy}(x,y) = \dfrac{e^x}{1+y}$

so $\quad F_{xy}(0,0) = 1$

Taylor's theorem states that

$$F(x,y) = F(0,0) + xF_x(0,0) + yF_y(0,0) + \frac{x^2}{\lfloor 2}$$

$$F_{xx}(0,0) + xyF_{xy}(0,0) + \frac{y^2}{\lfloor 2}F_{yy}(0,0) + \ldots$$

or $e^x \log(1+y) = 0 + x\cdot 0 + y\cdot 1 + \frac{x^2}{\lfloor 2}\cdot 0 + xy\cdot 1$

$$+\frac{y^2}{\lfloor 2}(-1) + \ldots.$$

41. $\lim_{(x,y)\to(2,3)} f(x,y) = \lim_{(x,y)\to(2,3)} 3xy = 18$

Since, $f(2, 3) = 6$

So, $\lim_{(x,y)\to(2,3)} f(x,y) \neq f(2,3)$

Hence, $f(x,y)$ is discontinuous at (2, 3). Since, $\lim_{(x,y)\to(2,3)} f(x,y)$ exists but not equal to $f(2,3)$ so the discontinuous is removable.

42. Let $f(x,y) = \cos x \cos y$

then $f(0, 0) = 1$

$f_x(x,y) = -\sin x \cos y$

then $f_x(0,0) = 0$

$f_y(x,y) = -\cos x \sin y$

then $f_y(0,0) = 0$

$f_{xx}(x,y) = -\cos x \cos y$

so $f_{xx}(0,0) = -1$

$f_{xy}(x,y) = \sin x \sin y$

so $f_{xy}(0,0) = 0$

$f_{yy}(x,y) = -\cos x \cos y$

so $f_{yy}(0,0) = -1$

Taylor's theorem states that

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)f(0,0)$$

$$+\frac{1}{\lfloor 2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) + \frac{1}{\lfloor 3}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(0,0) + \ldots$$

$$= f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{x^2}{2}$$

$$f_{xx}(0,0) + xy f_{xy}(0,0) + \frac{y^2}{2} f_{yy}(0,0) + \ldots.$$

$$\cos x.\cos y = 1 + 0 + 0 + \frac{x^2}{2}(-1)$$

$$+xy(0) + \frac{y^2}{2}(-1) + \frac{x^3}{6}(0) + \ldots..$$

$$= 1 - \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^4}{24} + \ldots..$$

So, second degree ferm is $\frac{-x^2}{2} - \frac{y^2}{2}$

43. Let $(x,y) \to (0,0)$ along the curve $x = my^2$

We have

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{2my^4}{(m^2+1)y^4}$$

$$= \lim_{y\to 0} \frac{2m}{1+m^2} = \frac{2m}{1+m^2}$$

which is different for different values of m since it gives different limiting values so the simultaneous limit does not exist.

48. Let $f(x,y) = e^x \sin y$

then $f(0,0) = 0$

$f_x(x,y) = e^x \sin y$ so $f_x(0,0) = 0$

$f_y(x,y) = e^x \cos y$ so $f_y(0,0) = 1$

$f_{xx}(x,y) = e^x \sin y$ so $f_{xx}(0,0) = 0$

$f_{xy}(x,y) = e^x \cos y$ so $f_{xy}(0,0) = 1$

$f_{yy}(x,y) = -e^x \sin y$ so $f_{yy}(0,0) = 0$

By taylor's theorem,

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)f(0,0)$$

$$+\frac{1}{\lfloor 2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) + \ldots..$$

$$= f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{x^2}{2}$$

$$f_{xx}(0,0) + xy f_{xy}(0,0) + \frac{y^2}{2} f_{yy}(0,0) + \ldots..$$

or $e^x \sin y = 0 + x(0) + y(1) + \frac{x^2}{2}(0)$

$$+\frac{y^2}{2}(0) + xy(1) + \ldots..$$

or $e^x \sin y = y + xy + \ldots.$

49. We have

$$\lim_{y\to 0} f(x,y) = \begin{cases} 1 & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad y = 0 \end{cases}$$

$\therefore$ $$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = 1$$

Similarly, $$\lim_{x\to 0} f(x,y) = \begin{cases} 1 & \text{if} \quad y \neq 0 \\ 0 & \text{if} \quad y = 0 \end{cases}$$

$\therefore$ $$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = 1$$

Hence, repeated limits exist and are equal.

Let $(x,y) \to (0,0)$ along the line $y = 0$, so $f(x,y) = 0$

i.e., $$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x,y) = \lim_{x\to 0} 0 = 0$$

Again let $(x,y) \to (0,0)$ along $y = x$ then $f(x,y) = 1$

So $$\lim_{\substack{(x,y)\to(0,0)\\ y=r}} f(x,y) = \lim_{x\to 0} 1 = 1$$

Since, these gives different limiting values so the simultaneous limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

52. Given that

$$f(x,y) = \frac{2xy^2}{x^3 + 3y^3}, (x,y) \neq (0,0)$$

and $f(0,0) = 0$

when $(x,y) \to (0,0)$ along the line $y = x$, we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{2x.x^2}{x^3 + 3x^3}$$

$$= \lim_{x\to 0} \frac{2x^3}{4x^3} = \frac{1}{2}$$

Again when $(x,y) \to (0,0)$ along the line $y = 0$, we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{2x.0^2}{x^3 + 3.0^2}$$

$$= \lim_{x\to 0} 0 = 0$$

Since, two different limiting values exists so the simultaneous limit does not exist. Hence, the function is not continues at (0,0).

53. Given that

$$f(x,y) = \begin{cases} 3xy & , \quad (x,y) \neq (2,3) \\ 6 & , \quad (x,y) = (2,3) \end{cases}$$

$$\lim_{(x,y)\to(2,3)} f(x,y) = \lim_{(x,y)\to(2,3)} 3xy = 18$$

Since, $f(2,3) = 6$ and $\lim_{(x,y)\to(2,3)} f(x,y) = 18$

So, the function $f(x,y)$ is discontinuous at (2,3).

Again $\lim_{(x,y)\to(2,3)} f(x,y)$ exists but is not equal to $f(2,3)$ so the discontinuity is removable.

56. Given that

$$f(x,y) = \frac{1}{x^2 + y^2}, (x,y) \neq (0,0), f(0,0) = 0$$

Let $(x,y) \to (0,0)$ along the line $y = 0$ then

$$\lim_{\substack{(x,y)\to(0,0)\\ (y\to 0)}} f(x,y) \lim_{\substack{(x,y)\to(0,0)\\ (y\to 0)}} \frac{1}{x^2 + y^2}$$

$$= \lim_{x\to 0} \frac{1}{x^2} = \infty$$

$$= \lim_{(x,y)\to(0,0)} f(x,y)$$

does not exist *i.e.* $f(x,y)$ is discontinuous at (0,0).

Let $\varepsilon > 0$ be given then for all $(x,y) \neq (0,0)$ we have

$$|f(x,y) - 0| = \left|\frac{xy}{\sqrt{x^2 + y^2}} - 0\right| = \left|\frac{xy}{\sqrt{x^2 + y^2}}\right|$$

$$= \left|\frac{r\cos\theta . r\sin\theta}{r}\right|$$

putting $x = r\cos\theta, y = r\sin\theta$

$$= r|\cos\theta||\sin\theta| = r$$

$$\leq \sqrt{x^2 + y^2} < \varepsilon$$

provided $x^2 < \frac{1}{2}\varepsilon^2$ and $y^2 < \frac{\varepsilon^2}{2}$

i.e. $$|x| < \frac{\varepsilon}{2} \text{ and } |y| < \frac{\varepsilon}{\sqrt{2}}$$

Choose $\delta = \frac{\varepsilon}{\sqrt{2}}$ we get

$$|f(x,y) - 0| < \varepsilon \text{ whenever } |x| < \delta \text{ and } |y| < \delta$$

So, $$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Also $$f(0,0) = 0$$

So, $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

Hence, f is continuous at (0,0).

Now $f(x)(0,0) = \lim_{\Delta x\to 0} \frac{f(\Delta x,0) \to f(0,0)}{\Delta x}$

$$= \lim_{\Delta x\to 0} \frac{\Delta x.0}{\Delta x\sqrt{\Delta x^2+0}} = 0$$

Similarly, $f_y(0,0) = 0$

Hence,

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(\Delta x,\Delta y) - f(0,0) - \Delta x\, f_x(0,0) - \Delta y\, f_y(0,0)}{\Delta(x,y)}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\frac{\Delta x.\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}} - 0 - \Delta x.0 - \Delta y.0}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\Delta x.\Delta y}{(\Delta x)^2+(\Delta y)^2}$$

Let $\Delta y = m\Delta x$ we get

$$\lim_{\Delta x\to 0} \frac{\Delta x.m\Delta r}{(\Delta x)^2 + m^2(\Delta x)^2} = \frac{m}{1+m^2}$$

which depends on m so limit does not exist. Hence, $f(x,y)$ is not totally differentiable at (0,0).

62. Let $(x,y) \to (0,0)$ along the line $y = mx$ then

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{x^2y^2}{x^4+y^4} = \lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{x^2 - m^2x^2}{x^4+m^4x^4}$$

$$= \lim_{x\to 0} \frac{x^4m^2}{x^4(1+m^4)} = \frac{m^2}{1+m^4}$$

Which is different for different values of m so $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Hence, $f(x,y)$ is discontinuous at (0,0). Since, $f(x,y)$ is not continuous it is not differentiable.

63. Given that

$$f(x,y) = \frac{x^2y^2}{x^2+y^2}, (x,y) \neq (0,0) \text{ and } f(0,0) = 0$$

Let $\varepsilon > 0$ then for $(x,y) \neq (0,0)$ we have

$$|f(x,y) - 0| = \left|\frac{x^2y^2}{x^2+y^2}\right| = \frac{x^2y^2}{x^2+y^2}$$

$$= r^2\cos^2\theta\sin^2\theta \leq r^2$$

putting $x = r\cos\theta$, $y = r\sin\theta$

$$= x^2 + y^2 < \varepsilon \text{ is } x^2 < \frac{\varepsilon^2}{2} \text{ and } y^2 < \frac{\varepsilon^2}{2}$$

if $|x| < \sqrt{\frac{\varepsilon}{2}}$ and $|y| < \sqrt{\frac{\varepsilon}{2}}$

Choose $\delta = \sqrt{\frac{\varepsilon}{2}}$ then for $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x,y) - 0| < \varepsilon$ whenever $|x| < \delta$ and so

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

Hence, $f(x,y)$ is continuous at (0,0).

64. Given that

$$f(x,y) = \frac{xy^2}{x^2+y^2}$$

when $(x,y) \neq (0,0)$ and $f(0,0) = 0$

Let $\varepsilon > 0$ and $x = r\cos\theta$, $y = r\sin\theta$ then

$$f(x,y) = \frac{r\cos\theta r^2.\sin^2\theta}{r^2} = r\cos\theta.\sin^2\theta$$

So, $|f(x,y) - f(0,0)| = |r\cos\theta.\sin^2\theta|$

$$= r|\cos\theta||\sin\theta|^2$$

$$\leq r \text{ for all values of } \theta$$

Choose $r_0 = \varepsilon$ then for all values of θ and $r < r_0$ we have

$$|f(r\cos\theta, r\sin\theta) - f(0,0)| < \varepsilon$$

So, $f(x,y)$ is uniformly continuous in r for all θ *i.e.* $f(x,y)$ is continuous in (x,y) together at (0,0)

Now, $f_x(0,0) = \lim_{\Delta x\to 0} \frac{f(\Delta x,0) - f(0,0)}{\Delta x}$

$$= \lim_{\Delta x\to 0} \frac{1}{\Delta x}\left[\frac{\Delta x.0}{\Delta x' + 0} - 0\right] = 0$$

Similarly $f_y(0,0) = 0$

So, $\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \dfrac{f(\Delta x,\Delta y)-f(0,0)-\Delta x\, f_x(0,0) -\Delta y\, f_y(0,0)}{\Delta(x,y)}$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{1}{\sqrt{(\Delta x)^2+(\Delta y)^2}} \left[\frac{\Delta x.(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2} - \Delta x.0 - \Delta y.0\right]$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\Delta x(\Delta y)^2}{[(\Delta x)^2+(\Delta y)^2]^{3/2}}$$

putting $\Delta y = m\,\Delta x$ we get

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\Delta x(\Delta y)^2}{[(\Delta x)^2+(\Delta y)^2]^{3/2}}$$

$$= \lim_{\Delta x\to 0} \frac{\Delta x\, m^2(\Delta x)^2}{(\Delta x)^2(1+m^2)^{3/2}}$$

$$= \frac{m^2}{(1+m^2)^{3/2}}$$

Hence, this limit does not exist since it depends upon m. So $f(x,y)$ is not differentiable at (0,0).

67. Given that

$$f(x,y) = \frac{x^3-y^3}{x^2+y^2}, (x,y) \neq (0,0)$$

and $f(0,0) = 0$

Put $x = r\cos\theta$, $y = r\sin\theta$ we have

$$|f(x,y)-f(0,0)| = \left|\frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2(\cos^2\theta+\sin^2\theta)} - 0\right|$$

$$= r|\cos^3\theta - \sin^3\theta| \leq r[\,|\cos^3\theta| + |\sin^3\theta|)$$

$$\leq 2r \text{ for all values of } \theta$$

Choose $r_0 = \dfrac{\varepsilon}{2}$ then for all values of θ and $r < r_0$ we have

$|f(x,y) - f(0,y)| < \varepsilon$ so $f(x,y)$ is continuous at (0,0)

Now, $f_x(0,0) = \lim_{\Delta x\to 0} \dfrac{f(\Delta x,0)-f(0,0)}{\Delta x}$

$$= \lim_{\Delta x\to 0} \frac{1}{\Delta x}\left[\frac{(\Delta x)^2-0}{(\Delta x)^2+0} - 0\right] = 1$$

and $f_y(0,0) = \lim_{\Delta y\to 0} \dfrac{f(0,\Delta y)-f(0,0)}{\Delta y}$

$$= \lim_{\Delta y\to 0} \frac{1}{\Delta y}\left[\frac{0-(\Delta y)^3}{0+(\Delta y)^3} - 0\right] = -1$$

So,

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(\Delta x,\Delta y)-f(0,0)-\Delta x\, f_x(0,0)-\Delta y\, f_y(0,0)}{\Delta(x,y)}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{1}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\left[\frac{(\Delta x)^3-(\Delta y)^3}{(\Delta x)^2+(\Delta y)^2} - \Delta x + \Delta y\right]$$

$$[(\Delta x)^2 + \Delta x.\Delta y + (\Delta y)^2$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} (\Delta x - \Delta y)\frac{-(\Delta x)^2-(\Delta y)^2]}{[(\Delta x)^2+(\Delta y)^2]^{3/2}}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{(\Delta x-\Delta y)\,\Delta x.\Delta y}{[(\Delta x)^2+(\Delta y)^2]^{3/2}}$$

If we put $\Delta y = m\,\Delta x$ then,

$$\lim_{\Delta x\to 0} \frac{(\Delta x - m\,\Delta x)\,\Delta x\, m\Delta x}{(\Delta x)^3(1+m^2)^{3/2}} = \frac{(1-m)m}{(1+m^2)^{3/2}}$$

Hence, this limit does not exist since it depends upon m. So $f(x,y)$ is not differentiable at (0,0).

68. Given that

$$f(x,y) = \frac{xy^2}{x^2+y^2}, (x,y) \neq (0,0) \text{ at } f(0,0) = 0$$

Let $\varepsilon > 0$ and $x = r\cos\theta, y = r\sin\theta$ then

$$f(r\cos\theta, r\sin\theta) = \frac{r\cos\theta\, r^2\sin^2\theta}{r^2(\cos^2\theta+\sin^2\theta)} = r\cos\theta\sin^2\theta$$

So, $|f(r\cos\theta, r\sin\theta) - f(0,0)| = |r\cos\theta\sin^2\theta - 0|$

$$= r|\cos\theta|\sin^2\theta \leq r$$

for all values of θ put $r_0 = \varepsilon$ then for all values of θ and $r < r_0$ we have

$|f(r\cos\theta, r\sin\theta) - f(0,0)| < \varepsilon$

So, f is uniformly continuous in r for all values of θ *i.e.* $f(x,y)$ is continuous in (x,y) together at (0,0).

Now $f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x,0) - f(0,0)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{\Delta x.0}{(\Delta x)^2 + 0}\right]$$

Similarly $f_y(0,0) = 0$

and
$$f(\Delta x, \Delta y) - f(0,0)$$
$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{-\Delta x\, f_x(0,0) - \Delta y\, f_y(0,00}{\Delta(x,y)}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{1}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$
$$\left[\frac{\Delta x\,(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2} - 0 - \Delta x.0 - \Delta y.0\right]$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x\,(\Delta y)^2}{[(\Delta x)^2 + (\Delta y)^2]^{3/2}}$$

Put $\Delta y = m\Delta x$ we get

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x\,(\Delta y)^2}{[(\Delta x)^2 + (\Delta y)^2]^{3/2}}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x m^2 (\Delta x)^2}{(\Delta x)^2 (1 + m^2)^{3/2}}$$

$$= \frac{m^2}{(1 + m^2)^{3/2}}$$

which depends upon in so f is not differentiable at (0,0).

❍❍❍

CHAPTER

13 Maxima and Minima of Three Variables

MAXIMA AND MINIMA OF SEVERAL VARIABLES

Let $f(x, y, z, \ldots)$ be a function of several independent variables $x, y, z, \ldots$. Further let f be continuous and finite for all values of $x, y, z, \ldots$ in the nbd of their values $a, b, c, \ldots$ respectively. Then $f(a, b, c, \ldots)$ is said to be a maximum or a minimum values of $f(x, y, z)$ according as $f(a+h, b+k, c+l, \ldots.)$ is less or greater than $f(a, b, c, \ldots)$ for all sufficiently small independent values of $h, k, l, \ldots$ positive or negative, provided they are not all zero.

NECESSARY CONDITIONS FOR THE EXISTENCE OF MAXIMA OR MINIMA

The necessary conditions for the existence of maxima or minima is

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0, \ldots.$$

It should be noted that the conditions,

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0 \ldots.$$

are necessary but not sufficient for the existence of maxima and minima.

STATIONARY AND EXTREME POINTS

Stationary Point

A point $(a_1, a_2, \ldots., a_n)$ is called a stationary point, if all the first order partial derivatives of the function $f(x_1, x_2, \ldots, x_n)$ vanish at that point. Also then the value of $f(x_1, x_2, \ldots x_n)$ is said to be stationary at the point

Extreme Points

A stationary point which is either a maxima or a minima is called an extreme point and the value of the function at that point is called an extreme value.

Note : Stationary point is not necessarily an extreme point. Thus a stationary value may be a maximum or a minimum or neither of these two.

MAXIMA AND MINIMA FOR A FUNCTION OF THREE VARIABLES LAGRANGE'S CONDITIONS

To discuss the maximum or minimum of $f(x, y, z)$ at any point (a, b, c) obtained on solving the equations.

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0$$

We find the values at (a, b, c) of six partial derivatives of second order of $f(x, y, z)$ defined as

$$A = \frac{\partial^2 f}{\partial x^2}, B = \frac{\partial^2 f}{\partial y^2}, C = \frac{\partial^2 f}{\partial z^2}$$

$$F = \frac{\partial^2 f}{\partial y \partial z}, G = \frac{\partial^2 f}{\partial z \partial x}, H = \frac{\partial^2 f}{\partial x \partial y}$$

If the expressions $A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$

(i) All positive, we shall have a minimum of $f(x, y, z)$ at (a, b, c).

(ii) Alternately negative and positive, we shall have a maximum of $f(x, y, z)$ at (a, b, c).

LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS

Consider the function $f(x, y, z)$ subject to the conditions

$$\phi_1(x, y, z) = 0 \quad \text{and} \quad \phi_2(x, y, z) = 0$$

To find the maximum and minimum of f.

(i) Put $F = f + \lambda_1 \phi_1 + \lambda_2 \phi_2$

(ii) Find the equations

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda_1 \frac{\partial \phi_1}{\partial x} + \lambda_2 \frac{\partial \phi_2}{\partial x} = 0$$

$$\frac{\partial F}{\partial y}=\frac{\partial f}{\partial y}+\lambda_1\frac{\partial \phi_1}{\partial y}+\lambda_2\frac{\partial \phi_2}{\partial y}=0$$

$$\frac{\partial F}{\partial z}=\frac{\partial f}{\partial z}+\lambda_1\frac{\partial \phi_1}{\partial z}+\lambda_2\frac{\partial \phi_2}{\partial z}=0$$

(iii) If $x=a, y=b, z=c$ be the solution of these equation then find d^2F at (a, b, c).

If $d^2F<0$ then f is maximum

If $d^2F>0$ then f is minimum

where $d^2F=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}+dz\frac{\partial}{\partial z}\right)^2F$

$$=\frac{\partial^2F}{\partial x^2}dx^2+\frac{\partial^2F}{\partial y^2}dy^2+\frac{\partial^2F}{\partial z^2}dz^2$$

$$+2\left(\frac{\partial^2F}{\partial x\partial y}dxdy+\frac{\partial^2F}{\partial y\partial z}dydz+\frac{\partial^2F}{\partial z\partial x}dzdx\right)$$

$$=\sum\frac{\partial^2F}{\partial x^2}dx^2+2\sum\frac{\partial^2F}{\partial x\partial y}dx\,dy$$

EXERCISE

MULTIPLE CHOICE QUESTIONS

Direction : Each of the following questions has four alternative answers. One of them is correct. Choose the correct answer.

1. Two conditions $\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0, \frac{\partial f}{\partial z}=0$ for the existence of maxima and minima are :
   a. Necessary
   b. Sufficient
   c. Necessary and sufficient
   d. None of these

2. The points at which all first order partial derivatives of $f(x,y,z)$ vanish are called :
   a. Boundary points b. Extreme points
   c. Stationary points d. None of these

3. The function $\mu=x^2+y^2+z^2+x-2z-xy$ has its minimum value at : **[Kanpur 2018]**
   a. $\left(\frac{-2}{3},\frac{-1}{3},1\right)$ b. $\left(\frac{-2}{3},\frac{1}{3},1\right)$
   c. $\left(\frac{2}{3},\frac{-1}{3},-1\right)$ d. $\left(\frac{2}{3},\frac{1}{3},1\right)$

4. If the expressions $A, \begin{vmatrix}A & H\\ H & B\end{vmatrix}$ and $\begin{vmatrix}A & H & G\\ H & B & F\\ G & F & C\end{vmatrix}$ be all positive for a function $f(x,y,z)$ at (a,b,c) then f is :
   a. Minimum
   b. Maximum
   c. Neither maximum nor minimum
   d. None of these

5. The maximum of the function $u=(x+y+z)^3-3(x+y+z)-24xyz+a^3$ exist at :
   a. (1,1,1) b. (1,1,0)
   c. (–1,–1,–1) d. (–1,0,–1)

6. If $u=\sum(x-a_1)^2+\sum(y-b_1)^2+\sum(z-c_1)^2$ then the stationary points are :
   a. $(n\Sigma a_1, n\Sigma b_1, n\Sigma c_1)$
   b. $\left(\frac{n}{\Sigma a_1},\frac{n}{\Sigma b_1},\frac{n}{\Sigma c_1}\right)$
   c. $\left(\frac{\Sigma a_1}{n},\frac{\Sigma b_1}{n},\frac{\Sigma c_1}{n}\right)$
   d. $(-n\Sigma c_1, -n\Sigma b_1, -n\Sigma c_1)$

7. A stationary point which is either maximum or a minimum is called :
   a. Supremum b. Infimum
   c. Boundary point d. Extreme point

8. If for a function $f(x,y,z)$ at (a,b,c) the expression $A, \begin{vmatrix}A & H\\ H & B\end{vmatrix}\begin{vmatrix}A & H & G\\ H & B & F\\ G & F & C\end{vmatrix}$ are all negatives then $f(x,y,z)$ at (a,b,c) is :
   a. Maximum
   b. Neither maximum nor minimum
   c. Minimum
   d. None of these

9. Consider the following statements :

   (I) Every stationary point is extreme point

   (II) The point at which the function is neither maximum nor minimum is called saddle point.

   a. I is true
   b. II is true
   c. I and II both are true
   d. I and II are not true

10. The maximum of $u = f(x_1, x_2, ... x_n)$ exist only when :

   a. $du = 0$ b. $d^2u = 0$
   c. $du < 0$ d. $d^2u > 0$

11. For a rectangular parallelopiped the cube has :

   a. Maximum surface
   b. Minimum surface
   c. Neither maximum nor minimum
   d. None of these

12. The stationary points for the function

$$u = \sin x \sin y \sin z$$

where x, y, z are the angles of a triangle is :

   a. $\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$ b. $\left(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}\right)$
   c. $\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right)$ d. None of these

13. If the expressions $A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$ be alternatively negative and positive for $f(x, y, z)$ at (a, b, c) then $f(x, y, z)$ is :

   a. Maximum
   b. Minimum
   c. Neither maximum nor minimum
   d. None of these

14. The minimum value of

$$x = x^2 + y^2 + z^2 + x - 2z - xy \text{ is :}$$

   a. $\frac{-1}{3}$ b. $\frac{-2}{3}$
   c. $\frac{-4}{3}$ d. $\frac{2}{3}$

15. The stationary point of the function

$$u = axy^2z^3 - x^2y^2z^3 - xy^3z^3 - xy^2z^4 \text{ is :}$$

[Kanpur 2018]

   a. $\left(\frac{a}{7}, \frac{2a}{7}, \frac{3a}{7}\right)$ b. $\left(\frac{3a}{7}, \frac{2a}{7}, \frac{a}{7}\right)$
   c. $(3a, 2a, 7a)$ d. $(2a, 3a, 7a)$

16. The maximum value of

$$(ax + by + cz).e^{-(\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2)} \text{ is :}$$

   a. $\sqrt{\frac{1}{2e}\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}\right)}$
   b. $\sqrt{\frac{1}{2e}\left(\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}\right)}$
   c. $\frac{1}{2e}\sqrt{\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}}$
   d. none of these

17. The value of the greatest rectangular parallelopiped that can be inscribed in the ellepsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is :

   a. $\frac{2abc}{\sqrt{3}}$ b. $\frac{8abc}{\sqrt{3}}$
   c. $\frac{2abc}{3\sqrt{3}}$ d. $\frac{8abc}{3\sqrt{3}}$

18. In a plane triangle, the maximum value of $u = \cos x \cos y \cos z$ is :

   a. $\frac{1}{8}$ b. $\frac{1}{16}$
   c. $\frac{1}{4}$ d. $\frac{1}{2}$

19. The maximum and minimum values of $u = a^2x^2 + b^2y^2 + c^2z^2$ where $x^2 + y^2 + z^2 = 1$ and $lx + my + nz = 0$ are the roots of the equation :

   a. $\frac{l}{4 - a^2} + \frac{m}{4 - b^2} + \frac{n}{4 - c^2} = 0$
   b. $\frac{l^2}{4 - a^2} + \frac{m^2}{4 - b^2} + \frac{n^2}{4 - c^2} = 0$
   c. $\frac{l^2}{4 - a} + \frac{m^2}{4 - b} + \frac{n^2}{4 - c} = 0$
   d. None of these

20. Which one of the following is not a stationary point for the function

$$u = (x+y+z)^2 - 3(x+y+z) - 24xyz + a^3$$

a. (1,1,1) b. (–1,–1,–1)
c. (1,–1,1) d. None of these

21. The maximum value of

$$u = \frac{xyz}{(x+a)(x+y)(y+z)(z+b)} \text{ is :}$$

a. $\frac{1}{\sqrt{a^{1/2}+b^{1/2}}}$ b. $\frac{1}{(a^{1/4}+b^{1/4})^4}$
c. $\frac{1}{(a^{1/4}+b^{1/4})^2}$ d. None of these

22. Consider the following statements :
(I) Every extreme point is stationary point
(II) The value of function at extreme point is called supremum
a. I is true
b. II is true
c. I and II both are true
d. None of these

23. For all rectangular parallelopiped of same volume the surface of cube is :
a. Greatest
b. Least
c. Cannot be determined
d. None of these

24. The stationary points for the function minimum value of $x^2+y^2+z^2$ subject to the condition $ax+by+cz=p$ is :
a. $\left(\frac{p}{a^2+b^2+c^2}, \frac{p}{a^2+b^2+c^2}, \frac{p}{a^2+b^2+c^2}\right)$
b. $\left(\frac{p}{\sqrt{a^2+b^2+c^2}}, \frac{p}{\sqrt{a^2+b^2+c^2}}, \frac{p}{\sqrt{a^2+b^2+c^2}}\right)$
c. $\left(\frac{ap}{\sqrt{a^2+b^2+c^2}}, \frac{bp}{\sqrt{a^2+b^2+c^2}}, \frac{cp}{\sqrt{a^2+b^2+c^2}}\right)$
d. $\left(\frac{ap}{a^2+b^2+c^2}, \frac{bp}{a^2+b^2+c^2}, \frac{cp}{a^2+b^2+c^2}\right)$

25. The maximum value of the function

$$u = \sin x \sin y \sin z$$

where x, y, z are the angles of a triangle is :
a. $\frac{1}{8}$ b. $\frac{3}{8}$
c. $\frac{\sqrt{3}}{8}$ d. $\frac{3\sqrt{3}}{8}$

26. Maximum of $x^m y^n z^p$ subject to $x+y+z=a$ is stationary at :
a. $\left(\frac{am}{m+n+p}, \frac{bn}{m+n+p}, \frac{cn}{m+n+p}\right)$
b. $\left(\frac{am}{m+n+p}, \frac{an}{m+n+p}, \frac{ap}{m+n+p}\right)$
c. $\left(\frac{a}{m+n+p}, \frac{b}{m+n+p}, \frac{c}{m+n+p}\right)$
d. None of these

27. The stationary values of $x^2+y^2+z^2$ subject to the conditions $ax^2+by^2+cz^2=1$ and $lx+my+nz=0$ is given by :
a. $\frac{a^2}{lu-1}+\frac{b^2}{mu-1}+\frac{c^2}{nu-1}=0$
b. $\frac{al}{u^2-1}+\frac{bm}{u^2-1}+\frac{cn}{u^2-1}=0$
c. $\frac{l^2}{au-1}+\frac{m^2}{bu-1}+\frac{n^2}{cu-1}=0$
d. None of these

28. The stationary points of x^p, y^q, z^r subject to the conditions $ax+by+cz=p+q+r$ is :
a. $\left(\frac{a}{p}, \frac{b}{q}, \frac{c}{r}\right)$ b. $\left(\frac{a}{2p}, \frac{b}{2q}, \frac{c}{2r}\right)$
c. $\left(\frac{p}{a}, \frac{q}{b}, \frac{r}{c}\right)$ d. $\left(\frac{2p}{a}, \frac{2q}{b}, \frac{2r}{c}\right)$

29. The minimum value of $x^2+y^2+z^2$ subject to the condition $ax+by+cz=p$ is :
a. $\frac{p}{a^2+b^2+c^2}$ b. $\frac{p^2}{\sqrt{a^2+b^2+c^2}}$
c. $\frac{p}{\sqrt{a^2+b^2+c^2}}$ d. $\frac{p^2}{a^2+b^2+c^2}$

30. The stationary point for the function $u = \cos x \cos y \cos z$ is :
a. $\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$ b. $\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right)$
c. $\left(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}\right)$ d. None of these

31. The minimum of $x^4 + y^4 + z^4$ subject to $xyz = c^3$ is :

a. $3c^4$ b. $3c^3$

c. $2c^4$ d. c^4

32. The stationary points of maximum of $\frac{5xyz}{x + 2y + 4z}$ subject to the condition $xyz = 8$ is :

a. (1,2,4) b. (4,2,1)

c. (2,1,4) d. (4,1,2)

33. The stationary points at which the value of the greatest rectangular parallelopiped that can be inscribed in the ellepsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is :

a. (a,b,c) b. $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}, \frac{c}{\sqrt{2}}\right)$

c. $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ d. None of these

34. The maximum value of xyz subject to the condition $x + y + z = a$ is :

a. $\frac{a}{3}$ b. $\frac{a^3}{27}$

c. $\frac{a^3}{9}$ d. $\frac{a}{9}$

35. The maximum value of $x^p.y^q.z^r$ subject to the condition $ax + by + cz = p + q + r$ is :

a. $\left(\frac{p}{a}\right)^p\left(\frac{q}{b}\right)^q\left(\frac{r}{c}\right)^r$ b. $\left(\frac{a}{p}\right)^p\left(\frac{b}{q}\right)^q\left(\frac{c}{r}\right)^r$

c. $\left(\frac{p}{a}\right)^a\left(\frac{q}{b}\right)^b\left(\frac{r}{c}\right)^c$ d. $\left(\frac{a}{p}\right)^a\left(\frac{b}{q}\right)^b\left(\frac{c}{r}\right)^c$

36. If the area of a triangle is maximum with constant perimeter then the triangle is :

a. Isoscele b. Right angle

c. Equilateral d. None of these

37. The minimum of $x + y + z$ subject to the condition $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$ is :

a. $(a + b + c)^2$ b. $(\sqrt{a} + \sqrt{b} + c)^2$

c. $a^2 + b^2 + c^2$ d. $a + b + c$

38. The maximum of $\sin^m A \sin^n B \sin^n C$ subject to $A + B + C = \pi$ is given by the equation :

a. $m \sin A = n \sin B = p \sin C$

b. $m \cos A = n \cos B = p \cos C$

c. $m \tan A = n \tan B = p \tan C$

d. $m \cot A = n \cot B = p \cot C$

39. The stationary points of maximum of $a^3x^2 + b^3y^2 + c^3z^2$ subject to $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ are :

a. $(a + b + c, a + b + c, a + b + c)$

b. (a,b,c)

c. $\left(\frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}\right)$

d. $\left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right)$

40. The area of the quadrilateral is greatest when it can be inscribed in a :

a. Triangle b. Ellipse

c. Parabola d. Circle

41. The maximum and minimum values of $u = x^2 + y^2 + z^2$ subject to the conditions $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$ and $z = x + y$ is given by :

a. $\frac{1}{u-4} + \frac{1}{u-5} + \frac{1}{u-25} = 0$

b. $\frac{4}{u-4} + \frac{5}{u-5} + \frac{25}{u-25} = 0$

c. $\frac{2}{u-4} + \frac{3}{u-5} + \frac{4}{u-25} = 0$

d. None of these

42. The extreme value of $a^3x^2 + b^3y^2 + c^3z^2$ subject to $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ is given by :

a. $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$ b. $\frac{x}{a^2} = \frac{y}{b^2} = \frac{z}{c^2}$

c. $\frac{x}{a^3} = \frac{y}{b^3} = \frac{z}{c^3}$ d. $ax = by = cz$

43. If $u = xy + yz + zx$ is a function of three independent variables x,y,z then u has :

a. Maximum value

b. Minimum value

c. Neither maximum nor minimum

d. None of these

44. The extreme value of $u = x^2 + y^2 + xy$ subject to $ax^2 + by^2 = ab$ is given by :

a. $(u+a)(u+b) = ab$

b. $(u-a)(u-b) = ab$

c. $u(u+a)(u+b) = ab$

d. $u(u-a)(u-b) = ab$

45. The extreme value of $u = x^2 + y^2 + z^2$ subject to $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $px + qy + rz = 0$ is given by the equation :

a. $\frac{a^2}{u-a^2} + \frac{b^2}{u-b^2} + \frac{c^2}{u-c^2} = 0$

b. $\frac{p^2}{u-a^2} + \frac{q^2}{u-b^2} + \frac{r^2}{u-c^2} = 0$

c. $\frac{p^2a^2}{u-a^2} + \frac{q^2b^2}{u-b^2} + \frac{r^2c^2}{u-c^2} = 0$

d. None of these

46. The maximum value of $x^2y^3z^4$ subject to $2x + 3y + 4z = a$ is given by :

a. $\left(\frac{a}{9}\right)^9$ b. $\left(\frac{a}{2}\right)^9$

c. $\left(\frac{a}{3}\right)^9$ d. $\left(\frac{a}{4}\right)^9$

47. The highest distance from origin to the curve

$x^2 + y^2 + 2z^2 = 5, x + 2y + z = 5$ is :

a. 5 b. $\sqrt{5}$

c. 10 d. $\sqrt{10}$

48. The extremum of $u = x^2 + y^2$ subject to $ax^2 + by^2 + 2hxy = 1$ is given by the equation :

a. $(u-a)(u-b) = h^2$

b. $\left(a - \frac{1}{u}\right)\left(b - \frac{1}{u}\right) = h^2$

c. $\left(u - \frac{1}{a}\right)\left(u - \frac{1}{b}\right) = h^2$

d. $\left(\frac{1}{u} - \frac{1}{a}\right)\left(\frac{1}{u} - \frac{1}{b}\right)\left(\frac{1}{u} - \frac{1}{c}\right) = h^2$

49. The stationary points of maximum of $x^2y^3z^4$ subject to the condition $2x + 3y + 4z = a$ are :

a. $\left(\frac{a}{2}, \frac{a}{3}, \frac{a}{4}\right)$ b. $\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$

c. $\left(\frac{a}{4}, \frac{a}{4}, \frac{a}{4}\right)$ d. $\left(\frac{a}{9}, \frac{a}{9}, \frac{a}{9}\right)$

50. The maximum or minimum value of $u = x^2 + xy + y^2$ subject to $x^2 + y^2 = 1$ is given by the equation :

a. $u(u+1)^2 = 1$ b. $(u+1)^2 = 1$

c. $u(u-1)^2 = 1$ d. $(u-1)^2 = 1$

51. The shortest distance between origin (0,0,0) and the curve, $x^2 + 7y^2 + 8xy = 225, z = 0$ is :

a. 5 b. 10

c. 15 d. 25

52. The minimum value of $x + y + z$ subject to $\frac{1}{x} + \frac{2}{y} + \frac{3}{z} = 1$ is :

a. 6 b. $(1 + \sqrt{2} + \sqrt{3})$

c. 14 d. $(1 + \sqrt{2} + \sqrt{3})^2$

53. If $u = xy^2z^3$ subject to $x + 2y + 3z = 6$ then x is stationary at :

a. (1,1,1) b. (1,2,3)

c. (1,3,2) d. None of these

54. The minimum value of $x^2 + y^2 + z^2$ subject to $2x + 3y + z = p$ is :

a. $\frac{p^2}{6}$ b. $\frac{p^2}{14}$

c. $\frac{p}{14}$ d. $\frac{p}{6}$

55. The maximum value of xyz subject to the condition $x + y + z = 2$ is :

a. $\frac{2}{3}$ b. $\frac{4}{9}$

c. $\frac{8}{27}$ d. $\frac{16}{81}$

56. The minimum values of $x + y + z$ subject to $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$ is :

a. 1 b. 3

c. $\sqrt{3}$ d. 9

57. The maximum value of xyz subject to $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ is :

a. 9 b. 1

c. 3 d. 6

58. The minimum value of $x^4+y^4+z^4$ with $xyz=1$ is :

a. 1 b. 3

c. 4 d. 6

59. If a number 9 be divided into there parts such that their product will be maximum then the division is :

a. (1,3,5) b. (2,3,4)

c. (3,3,3) d. (1,4,4)

60. The maximum value of $u=x^2y^3z^4$ subject to $2x+3y+4z=9$ is :

a. 1 b. 9

c. 9^2 d. 9^9

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(a)	**2.**	(c)	**3.**	(a)	**4.**	(a)	**5.**	(c)	**6.**	(c)	**7.**	(d)	**8.**	(b)	**9.**	(b)	**10.**	(a)
11.	(b)	**12.**	(c)	**13.**	(a)	**14.**	(c)	**15.**	(a)	**16.**	(b)	**17.**	(d)	**18.**	(a)	**19.**	(b)	**20.**	(c)
21.	(b)	**22.**	(a)	**23.**	(b)	**24.**	(d)	**25.**	(d)	**26.**	(b)	**27.**	(c)	**28.**	(c)	**29.**	(d)	**30.**	(b)
31.	(a)	**32.**	(b)	**33.**	(c)	**34.**	(b)	**35.**	(a)	**36.**	(c)	**37.**	(b)	**38.**	(d)	**39.**	(c)	**40.**	(d)
41.	(b)	**42.**	(d)	**43.**	(c)	**44.**	(d)	**45.**	(c)	**46.**	(a)	**47.**	(b)	**48.**	(b)	**49.**	(d)	**50.**	(c)
51.	(a)	**52.**	(d)	**53.**	(a)	**54.**	(b)	**55.**	(c)	**56.**	(d)	**57.**	(a)	**58.**	(b)	**59.**	(c)	**60.**	(a)

HINTS AND SOLUTIONS

3. Given $u = x^2 + y^2 + z^2 + x - 2z - xy$

For a maximum or a minimum of u, we have

$$\frac{\partial u}{\partial x} = 2x - y + 1 = 0$$

$$\frac{\partial u}{\partial y} = -x + 2y = 0$$

$$\frac{\partial u}{\partial z} = 2z - 2 = 0$$

The solution is

$$x = \frac{-2}{3}, y = \frac{-1}{3}, z = 1$$

Now $$\frac{\partial^2 u}{\partial x^2} = 2, \frac{\partial^2 u}{\partial y^2} = 2, \frac{\partial^2 u}{\partial z^2} = 2$$

$$\frac{\partial^2 u}{\partial y \partial z} = 0, \frac{\partial^2 u}{\partial z \partial x} = 0, \frac{\partial^2 u}{\partial x \partial y} = 1$$

So at $\left(\frac{-2}{3}, \frac{-1}{3}, 1\right)$ we have

$$A = \frac{\partial^2 u}{\partial x^2} = 2, B = \frac{\partial^2 u}{\partial y^2} = 2, C = \frac{\partial^2 u}{\partial z^2} = 2$$

$$F = \frac{\partial^2 u}{\partial y \partial z} = 0, G = \frac{\partial^2 u}{\partial z \partial x} = 0, H = \frac{\partial^2 u}{\partial x \partial y} = -1$$

So, $$A = 2, \begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6$$

Since these three expressions are all positive, we have minimum of u at $\left(\frac{-2}{3}, \frac{-1}{3}, 1\right)$

5. $u = (x + y + z)^3 - 3(x + y + z) - 2u\,xyz + a^3$

For maximum or minimum

$$\frac{\partial u}{\partial x} = 3(x + y + z)^2 - 3 - 2uyz = 0$$

$$\frac{\partial u}{\partial y} = 3(x + y + z)^2 - 3 - 2uzx = 0$$

$$\frac{\partial u}{\partial z} = 3(x + y + z)^2 - 3 - 2uxy = 0$$

After solving we get (1,1,1) and (–1,–1,–1) are the stationary points.

Now at (–1,–1,–1)

$$A = \frac{\partial^2 u}{\partial x^2} = 6(x + y + z) = -18$$

$$B = \frac{\partial^2 u}{\partial y^2} = 6(x + y + z) = -18$$

$$C = \frac{\partial^2 u}{\partial z^2} = 6(x + y + z) = -18$$

$$F = \frac{\partial^2 u}{\partial y \partial z} = 6(x + y + z) - 24x = 6$$

$$G = \frac{\partial^2 u}{\partial z \partial x} = 6(x + y + z) - 24y = 6$$

$$H = \frac{\partial^2 u}{\partial x \partial y} = 6(x + y + z) - 24z = 6$$

So, $$A = -18, \begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} -18 & 6 \\ 6 & -18 \end{vmatrix} = 288$$

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} -18 & 6 & 6 \\ 6 & -18 & 6 \\ 6 & 6 & -18 \end{vmatrix} = -6^3.16$$

These expressions are alternately negative and positive so u is maximum at (–1,–1,–1).

6. Given that

$$u = \sum[(x - a_1)^2 + (y - b_1)^2 + (z - c_1)^2]$$

or $$u = \sum(x - a_1)^2 + \sum(y - b_1)^2 + \sum(z - c_1)^2$$

For maximum or a minimum of u, we have

$$\frac{\partial u}{\partial x} = 2\sum(x - a_1) = 2nx - 2\sum a_1 = 0$$

$$\frac{\partial u}{\partial y} = 2\sum(y - b_1) = 2ny - 2\sum b_1 = 0$$

$$\frac{\partial u}{\partial z} = 2\sum(z - c_1) = 2nz - 2\sum c_1 = 0$$

Solving these equations, stationary points are

$$x = \frac{\sum a_1}{n},\ y = \frac{\sum b_1}{n},\ z = \frac{\sum c_1}{n}$$

11. If x, y, z be the dimensions of the rectangular parallelopiped then

Surface $S = 2xy + 2yz + 2zx$ ...(1)

and volume $V = xyz = \text{constant}$ ...(2)

For maximum or a minimum of $s, ds = 0$

eg. $(y+z)\,dx + (z+x)\,dy + (x+y)\,dz = 0$ ...(3)

Differentiating equation (2)

$$yz\,dx + zx\,dy + xy\,dz = 0 \quad ...(4)$$

Multiplying (3) by 1 and (4) by λ and adding and then equating to zero the coefficients of dx, dy and dz, we get

$$(y+z) + \lambda yz = 0, (z+x) + \lambda zx = 0, (x+y) + \lambda xy = 0$$

These give $-\lambda = \frac{1}{y} + \frac{1}{z} = \frac{1}{z} + \frac{1}{x} = \frac{1}{x} + \frac{1}{y}$

So, $x = y = z = v^{1/3}$

Thus, S is stationary when the rectangular parallelopiped is a cube.

By (1) $\frac{\partial S}{\partial x} = 2y + 2y\frac{\partial z}{\partial x} + 2z + 2x\frac{\partial z}{\partial x}$

By (2) $yz + xy\frac{\partial z}{\partial x} = 0$

or $\frac{\partial z}{\partial x} = \frac{-z}{x}$

So, $\frac{\partial S}{\partial x} = 2y - \frac{2yz}{x}$

and $\frac{\partial^2 S}{\partial x^2} = \frac{4yz}{x^2} = 4 \text{ at } x = y = z$

Similarly by symmetry $\frac{\partial^2 S}{\partial y^2} = 4 \text{ at } x = y = z$

and $\frac{\partial^2 S}{\partial x \partial y} = 2 - \frac{2z}{x} - \frac{2y}{x}\frac{\partial z}{\partial y}$

Differentiating partials eq. (2) w.r.t. y, we get

$$xz + xy\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad \frac{\partial z}{\partial y} = -\frac{z}{y}$$

So, $\frac{\partial^2 S}{\partial x \partial y} = 2 - \frac{2z}{x} + \frac{2z}{x} = 2$

Thus at $(v^{1/3}, v^{1/3}, v^{1/3})$, we have

$$r = \frac{\partial^2 S}{\partial x^2} = 4,\ s = \frac{\partial^2 S}{\partial x \partial y} = 2,\ t = \frac{\partial^2 S}{\partial y^2} = 4$$

$\because$ $r + -s^2 = 12 > 0$ and $r = 4$ (positive)

Hence, s is minimum *i.e.* all rectangular parallelopipeds of the same value, the cube has the least surface.

12. Given that

$$u = \sin x.\sin y.\sin z \quad ...(1)$$

with $x + y + z = \pi$ ...(2)

For a maximum or a minimum of u, we have

$$du = \cos x.\sin y.\sin z\,dx + \sin x.\cos y.\sin z\,dy + \sin x.\sin y.\cos z\,dz = 0 \quad ...(3)$$

By (2) $dx + dy + dz = 0$ ...(4)

Multiplying (3) by (1) and (4) by λ and adding and then equating to zero the coefficients of dx, dy and dz.

$$\cos x \sin y.\sin z + \lambda = 0$$

$$\sin x \cos y \sin z + \lambda = 0$$

and $\sin x \sin y \cos z + \lambda = 0$

These gives $-\lambda = \cos x \sin y \sin z$

$$= \sin x \cos y \sin z = \sin x \sin y \cos z$$

or $\cot x = \cot y = \cot z$

$\Rightarrow$ $x = y = z = \frac{\pi}{3}$

Thus, u is stationary at $\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right)$.

15. Given that

$$u = axy^2z^3 - x^2y^2z^3 - xy^3z^3 - xy^2z^4$$

$$\frac{\partial u}{\partial x} = y^2z^3(a - 2x - y - z) = 0$$

$$\frac{\partial u}{\partial y} = xyz^3(2a - 2x - 3y - 2z) = 0$$

$$\frac{\partial u}{\partial z} = xy^2z^2(3a - 3x - 3y - 4z) = 0$$

Solving these equations the stationary points are

$$x = \frac{a}{7},\ y = \frac{2a}{7},\ z = \frac{3a}{7}$$

16. Let $U(ax+by+cz)e^{-(\alpha^2x^2+\beta^2y^2+\gamma^2z^2)}$

$$\log u = \log(ax+by+cz) - (\alpha^2x^2+\beta^2y^2+\gamma^2z^2)$$

For maximum or minimum

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{a}{ax+by+cz} - 2\alpha^2 x = 0$$

$$\frac{1}{u}\frac{\partial u}{\partial y} = \frac{b}{ax+by+cz} - 2\beta^2 z = 0$$

$$\frac{1}{u}\frac{\partial u}{\partial z} = \frac{c}{ax+by+cz} - 2\gamma^2 z = 0$$

These gives $x(ax+by+cz) = \frac{a}{2\alpha^2}$

$$y(ax+by+cz) = \frac{b}{2\beta^2}$$

and $z(ax+by+cz) = \frac{c}{2\gamma^2}$

Multiplying by a, b, c respectively and adding

$$(ax+by+cz)^2 = \frac{1}{2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)$$

or $ax+by+cz = \sqrt{\frac{1}{2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)} = d$ say

Then $x = \frac{a}{2\alpha^2 d}, y = \frac{b}{2\beta^2 d}, z = \frac{c}{2\gamma^2 d}$

are the stationary points.

Now $\frac{1}{u}\frac{\partial^2 u}{\partial x^2} - \frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2 = \frac{-a^2}{(ax+by+cz)^2} - 2\alpha^2$

At the stationary points $\frac{\partial u}{\partial x} = 0$

So, $\frac{\partial^2 u}{\partial x^2} = -u\left[\frac{a^2}{(ax+by+cz)} + 2\alpha^2\right]$

which is negative so u is maximum at these points.

Also, $u_{max.} = de^{-\frac{1}{4d^2}\frac{1}{2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)}$

$$= de^{-\frac{1}{4d^2}2R^2} = de^{-\frac{1}{2}} = \frac{d}{\sqrt{e}}$$

$$u_{max.} = \sqrt{\frac{1}{2e}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)}$$

17. Let x, y, z be the dimensions of the rectangular parallelopiped then

maximum of volume

$$V = 8xyz \quad ...(1)$$

subject to the condition

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1 \quad ...(2)$$

For a maximum or a minimum of V, we have

$$dV = 8yz\,dx + 8zx\,dy + 8xy\,dz = 0$$

or $yz\,dx + zx\,dy + xy\,dz = 0$ ...(3)

Differentiating (2),

$$\frac{x}{a^2}dx + \frac{y}{b^2}dy + \frac{z}{c^2}dz = 0 \quad ...(4)$$

Multiplying (3) by (1) and (4) λ adding and then equating the coefficients of dx, dy, dz to zero.

$$yz + \frac{\partial x}{a^2} = 0, zx + \frac{\lambda y}{b^2} = 0 \text{ and } xy + \frac{\lambda z}{c^2} = 0$$

After solving we get

$$x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$$

By (1), $\frac{\partial V}{\partial x} = 8yz + 8xy\frac{\partial z}{\partial x}$

Differentiating (2) partially w.r.t. x, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2} - \frac{\partial z}{\partial x} = 0$$

or $\frac{\partial z}{\partial x} = -\frac{c^2 x}{a^2 z}$

$\therefore$ $\frac{\partial V}{\partial x} = 8yz - \frac{8c^2x^2y}{a^2z}$

and $\frac{\partial^2 v}{\partial x^2} = 8y\frac{\partial z}{\partial x} - \frac{16c^2xy}{a^2z} + \frac{8c^2x^2y}{a^2z^2}\cdot\frac{\partial z}{\partial x}$

$$= 8y\left(\frac{-c^2x}{a^2z}\right) - \frac{16c^2xy}{a^2z} - \frac{8c^2x^2y}{a^2z}\cdot\frac{c^2x}{a^2z}$$

which is the when $x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$

So, maximum of $V = \frac{8abc}{3\sqrt{3}}$

19. $u = a^2x^2 + b^2y^2 + c^2z^2$ ...(1)

with $x^2 + y^2 + z^2 = 1$ ...(2)

and $lx + my + nz = 0$ ...(3)

For maximum or minimum of u, $du = 0$

or $d^2x\,dx + b^2y\,dy + c^2z\,dz = 0$ ...(4)

Differentiating (2) and (3) we have

$$x\,dx + y\,dy + z\,dz = 0 \quad ...(5)$$

and $l\,dx + m\,dy + n\,dz = 0$ ...(6)

Multiplying (4),(5) and (6) by (1) λ and μ respectively and adding and the equating to zero the coefficients of dx, dy and dz

$$c_1^2x + \lambda x + \mu l = 0 \quad ...(7)$$

$$b^2y + \lambda y + \mu m = 0 \quad ...(8)$$

$$c^2z + \lambda z + \mu n = 0 \quad ...(9)$$

Multiplying (7), (8) and (9) by x, y and z respectively and adding,

$$u + \lambda = 0 \quad \lambda = -u$$

putting $\lambda = -u$ in (7),

$$a^2x - ux + \mu l = 0$$

or $$x = \frac{\mu l}{u - a^2}$$

Similarly, $$y = \frac{\mu m}{u - b^2}, z = \frac{\mu n}{u - c^2}$$

putting these in $lx + my + nz = 0$, we get

$$\frac{l^2}{u - a^2} + \frac{m^2}{u - b^2} + \frac{n^2}{u - c^2} = 0$$

it gives the maximum or minimum values of u.

21. Given that

$$u = \frac{xyz}{(a + x)(x + y)(y + z)(z + b)}$$

Taking logarithms,

$$\log u = \log x + \log y + \log z - \log(a + x)$$
$$- \log(x + y) - \log(y + z) - \log(z + b)$$

Differentiating partially w.r.t. x, we get

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{x} - \frac{1}{a + x} - \frac{1}{x + y}$$

$$= \frac{ay - x^2}{x(a + x)(x + y)}$$

So, for maximum or minimum

$$\frac{\partial u}{\partial x} = \frac{(ay - x^2)u}{x(a + x)(x + y)} = 0$$

Similarly, $$\frac{\partial u}{\partial y} = \frac{(zx - y^2)u}{y(x + y)(y + z)} = 0$$

$$\frac{\partial u}{\partial z} = \frac{(by - z^2)u}{z(y + z)(z + b)} = 0$$

or $ay - x^2 = 0, zx - y^2 = 0, by - z^2 = 0$

or $$\frac{x}{a} = \frac{y}{x} = \frac{z}{y} = \frac{b}{z}$$

so each of these function is $\left(\frac{x}{a}\cdot\frac{y}{x}\cdot\frac{z}{y}\cdot\frac{b}{z}\right)^{1/4}$

$$= \left(\frac{b}{a}\right)^{1/4} = d$$

Thus, $x = ad, y = xd = ad^2, z = yd = ad^3$

and $b = zd = ad^4$

putting these values in given equation we get,

$$u = \frac{ad.ad^2.ad^3}{a(1 + d)\,ad(1 + d)\,ad^2(1 + d)\,ad^3(1 + d)}$$

$$u = \frac{1}{a(1 + d)^4} = \frac{1}{a\left[1 + \left(\frac{b}{a}\right)^{1/4}\right]^4}$$

$$= \frac{1}{(a^{1/4} + b^{1/4})^4}$$

It can be easily shown that it is maximum value.

24. Let $F = (x^2 + y^2 + z^2) + \lambda(ax + by + cz - p)$

For maxima and minima of F, we have

$$\frac{\partial F}{\partial x} = 2x + a\lambda = 0 \quad ...(1)$$

$$\frac{\partial F}{\partial y} = 2y + b\lambda = 0 \quad ...(2)$$

$$\frac{\partial F}{\partial z} = 2z + c\lambda = 0 \quad ...(3)$$

Multiplying (1) by a, (2) by b, (3) by c and adding the resulting equations, we have

$$2(ax+by+cz)+\lambda(a^2+b^2+c^2)=0$$

i.e., $2p+\lambda(a^2+b^2+c^2)=0$

or $\lambda=\dfrac{-2p}{a^2+b^2+c^2}$

Using it, by equation (1),(2) and (3) we get

$$x=\frac{ap}{a^2+b^2+c^2},\ y=\frac{bp}{a^2+b^2+c^2},$$

$$z=\frac{cp}{a^2+b^2+c^2}$$

26. Let $u=x^m y^n z^p$ ...(1)

with $x+y+z=a$ ...(2)

By (1) $\log u=m\log u+x\log y+p\log z$

Differentiating $\dfrac{1}{u}du=\dfrac{m}{x}dx+\dfrac{n}{y}dy+\dfrac{p}{z}dz$

For maximum or minimum of x, $du=0$

or $\dfrac{m}{x}dx+\dfrac{n}{y}dy+\dfrac{p}{z}dz=0$ ...(3)

Differentiating equation (2), we have

$$dx+dy+dz=0 \quad ...(4)$$

Multiplying (3) by (1) and (4) by λ and adding and then equating the coefficients of dx, dy, dz to zero.

$$\frac{m}{x}+\lambda=0,\ \frac{n}{y}+\lambda=0,\ \frac{p}{z}+\lambda=0$$

or $x=\dfrac{-m}{\lambda},\ y=\dfrac{-n}{\lambda},\ z=\dfrac{-p}{\lambda}$

putting these values in equation (2), we get

$$-\left(\frac{m}{\lambda}+\frac{n}{\lambda}+\frac{p}{\lambda}\right)=a$$

or $-\dfrac{1}{\lambda}=\dfrac{a}{m+n+p}$

So, u is stationary when

$$x=\frac{am}{m+n+p},\ y=\frac{an}{m+n+p},\ z=\frac{ap}{m+n+p}$$

28. Let $u=x^p y^q z^r$ ...(1)

with $ax+by+cz=p+q+r$ ...(2)

By (1), $\log x=p\log x+q\log y+r\log z$

Differentiating $\dfrac{1}{u}du=\dfrac{p}{x}dx+\dfrac{q}{y}dy+\dfrac{r}{z}dz$

For maximum or minimum, $du=0$

or $\dfrac{p}{x}dx+\dfrac{q}{y}dy+\dfrac{r}{z}dz=0$ ...(3)

Differentiating equation (2),

$$adx+bdy+cdz=0 \quad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero,

$$\frac{p}{x}+\lambda a=0,\ \frac{q}{y}+\lambda b=0,\ \frac{r}{z}+\lambda c=0$$

or $x=\dfrac{-p}{\lambda a},\ y=\dfrac{-q}{\lambda b},\ z=\dfrac{-r}{\lambda c}$

putting in equation (2), we get

$$-\left(\frac{p}{\lambda}+\frac{q}{\lambda}+\frac{r}{\lambda}\right)=p+q+r \quad\Rightarrow\quad \lambda=-1$$

So, u is stationary when

$$x=\frac{p}{a},\ y=\frac{q}{b},\ z=\frac{r}{c}$$

30. $u=\cos x\cos y\cos z$ ...(1)

with $x+y+z=\pi$ ...(2)

By (1), $\log u=\log\cos x+\log\cos y+\log\cos z$

or $\dfrac{1}{u}du=-\tan x\,dx-\tan y\,dy-\tan z\,dz$

For maximum or minimum of x, $du=0$

or $\tan x\,dx+\tan y\,dy+\tan z\,dz=0$ ...(3)

Differentiating equation (2), we get

$$dx+dy+dz=0 \quad ...(4)$$

Multiplying (3) by 1 and (4), by λ and adding and then equating to zero the coefficients of dx, dy, dz

$$\tan x+\lambda=0,\ \tan y+\lambda=0,\ \tan z+\lambda=0$$

$$-\lambda=\tan x=\tan y=\tan z$$

or $x=y=z=\dfrac{\pi}{3}$

Thus, u is maximum at $\left(\dfrac{\pi}{3},\dfrac{\pi}{3},\dfrac{\pi}{3}\right)$.

32. Given that $x=\dfrac{5xyz}{x+2y+4z}$ ...(1)

with $xyz=8$ ...(2)

By (1) $u=\dfrac{u_0}{x+2y+x^2}$

So, $du = \frac{-u_0}{(x+2y+uz)^2}(dx + 2dy + udz)$

For maximum or minimum of $u = du = 0$

or $dx + 2dy + udz = 0$ ...(3)

By (2), $\log x + \log y + \log z = \log 8$

Differentiating $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$ ...(4)

Multiplying (3) by 1 and (4) by λ and adding and then equating to zero the coefficients of dx, dy and dz

$$1 + \frac{\lambda}{x} = 0, 2 + \frac{\lambda}{y} = 0, 4 + \frac{\lambda}{2} = 0$$

or $x = -\lambda,\ y = \frac{-\lambda}{2},\ z = \frac{-\lambda}{4}$

putting in (2) we get

$$\frac{-\lambda^3}{8} = 8 \quad \text{or} \quad \lambda = -4$$

So, u is stationary at $x = 4, y = 2, z = 1$.

33. See the solution of question (17).

34. Let $u = xyz$, with $x + y + z = a$

$$\log u = \log x + \log y + \log z$$

Define $F = \log x + \log y + \log z + \lambda(x + y + z - a)$

For maxima or minima of F,

$$\frac{\partial F}{\partial x} = \frac{1}{x} + \lambda = 0$$

$$\frac{\partial F}{\partial y} = \frac{1}{y} + \lambda = 0$$

$$\frac{\partial F}{\partial z} = \frac{1}{z} + \lambda = 0$$

or $\frac{1}{x} = \frac{1}{y} = \frac{1}{z} = -\lambda$

or $x = y = z = \frac{x+y+z}{3} = \frac{a}{3}$

So, $x = \frac{a}{3}, y = \frac{a}{3}, z = \frac{a}{3}$ is stationary point

For maximum or minimum

$$d^2F = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + dz\frac{\partial}{\partial y}\right)^2 F$$

$$= \sum \frac{\partial^2 F}{\partial x^2}dx^2 + 2\sum \frac{\partial^2 F}{\partial x \partial y}dx\,dy$$

$$= -\left(\frac{1}{x^2}dx^2 + \frac{1}{y^2}dy^2 + \frac{1}{z^2}dz^2\right)$$

$\because$ $\frac{\partial^2 F}{\partial x \partial y} = 0$

$$= -9\left(\frac{dx^2}{a^2} + \frac{dy^2}{a^2} + \frac{dz^2}{a^2}\right)$$

$\because$ $d^2F < 0$, so u is maximum at $\left(\frac{a}{3}, \frac{a}{3}, \frac{a}{3}\right)$.

$$u_{\text{max.}} = \left(\frac{a}{3}\right)\left(\frac{a}{3}\right)\left(\frac{a}{3}\right) = \frac{a^3}{27}$$

36. Let a, b, c be the sides and $2s$ be its parameter then its area is :

$$u^2 = s(s-a)(s-b)(s-c) \qquad ...(1)$$

with $a + b + c = 2s$ ...(2)

By (1), $2\log u = \log s + \log(s-a) + \log(s-b) + \log(s-c)$

or $\frac{2}{u}du = -\frac{1}{s-a}da - \frac{1}{s-b}db - \frac{1}{s-c}dc$

For maximum or minimum of u, $du = 0$

$$\frac{da}{s-a} + \frac{db}{s-b} + \frac{dc}{s-c} = 0 \qquad ...(3)$$

Differentiating (2),

$$da + db + dc = 0 \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ and adding and the equating to zero the coefficients da, db and dc.

$$\frac{1}{s-a} + \lambda = 0, \frac{1}{s-b} + \lambda = 0, \frac{1}{s-c} + \lambda = 0$$

or $s - a = s - b = s - c$

or $a = b = c$

Thus, u is stationary at (a, a, a).

Differentiating (1) logarithmically with a and b as independent variables and c is a function of a and b

$$\frac{2}{u}\frac{\partial u}{\partial a} = -\frac{1}{s-a} - \frac{1}{s-c}\frac{\partial c}{\partial a}$$

By (2), $1 + \frac{\partial c}{\partial a} = 0 \Rightarrow \frac{\partial c}{\partial a} = -1$

$\therefore \quad \frac{2}{a}\frac{\partial u}{\partial a} = -\frac{1}{s-a} + \frac{1}{s-c}$

So, $\frac{2}{u}\frac{\partial^2 u}{\partial a^2} - \frac{2}{u^2}\left(\frac{\partial u}{\partial a}\right)^2 = -\frac{1}{(s-a)^2} + \frac{1}{(s-c)^2}\frac{\partial c}{\partial u}$

$$= -\frac{1}{(s-a)^2} - \frac{1}{(s-c)^2}$$

u is stationary at $\frac{\partial u}{\partial a} = 0$

$\therefore \quad u^2\frac{\partial^2 u}{\partial a^2} = -\frac{u}{2}\left[\frac{1}{(s-a)^2} + \frac{1}{(s-c)^2}\right]$

which is negative so area is maximum at $a = b = c$ *i.e.* when it is equilateral.

38. Given that

$$u = \sin^m A \sin^n B \sin^p C \quad ...(1)$$

and $\quad A + B + C = \pi \quad ...(2)$

From (1),

$$\log x = m \log \sin A + n \log \sin B + p \log \sin C$$

or $\quad \frac{1}{u}du = m \cot A dA + n \cot B dB + p \cot C dC$

For maximum or minimum, $du = 0$

i.e., $m \cot A . dA + n \cot B . dB + p \cot C dC = 0 \quad ...(3)$

Differentiating (2), we get

$$dA + dB + dC = 0 \quad ...(4)$$

Multiplying (3) by 1 and (4) by λ and adding and then equating to zero the coefficients of dA, dB and dC, we get

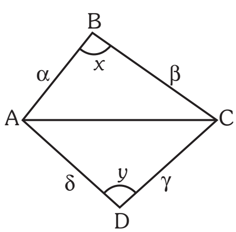
$$m \cot A + \lambda = 0,\ n \cot B + \lambda = 0,\ p \cot C + \lambda = 0$$

or $\quad -\lambda = m \cot A = n \cot B = p \cot C$

Hence, u is stationary when

$$m \cot A = n \cot B = p \cot C$$

40. The area of a quadrilateral $ABCD$ will be a function of x and y (see figure).



So, area A is given by

$$A = \frac{1}{2}(\alpha\beta . \sin x + \gamma\delta \sin y) \quad ...(1)$$

In ΔABC,

$$AC^2 = \alpha^2 + \beta^2 - 2\alpha\beta \cos x$$

and In ΔADC,

$$AC^2 = \gamma^2 + \delta^2 - 2\gamma\delta \cos y$$

Thus, we get

$$\alpha^2 + \beta^2 - 2\alpha\beta \cos x = \gamma^2 + \delta^2 - 2\gamma\delta \cos y$$

or $\quad \alpha\beta \cos x - \gamma\delta \cos y + c = 0 \quad ...(2)$

where $\quad C = \frac{1}{2}(\gamma^2 + \delta^2 - \alpha^2 - \beta^2)$

which is constant.

Define $\quad F = \frac{1}{2}(\alpha\beta \sin x + \gamma\delta \sin y) + \lambda(\alpha\beta \cos x - \gamma\delta \cos y + z)$

For maximum and minimum of F, we have

$$\frac{\partial F}{\partial x} = \frac{1}{2}\alpha\beta \cos x - \lambda\alpha\beta \sin x = 0 \quad ...(3)$$

and $\quad \frac{\partial F}{\partial y} = \frac{1}{2}\gamma\delta \cos y + \lambda\gamma\delta \sin y = 0 \quad ...(4)$

i.e., $\quad \lambda = \frac{1}{q}\cot x = -\frac{1}{2}\cot y$

or $\quad \cot x = -\cot y \Rightarrow \cot x = \cos(-y)$

or $\quad \cot x = \cos(\pi - y)$

i.e., $\quad x = -y$ or $x = \pi - y$

But $x = -y$ is not possible so $x = \pi - y$ *i.e.* $x + y = \pi$

So, $\quad d^2F = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)^2 F$

$$= \frac{\partial^2 F}{\partial x}dx^2 + \frac{\partial^2 F}{\partial y^2}dy^2 + \frac{2\partial^2 F}{\partial x \partial y}dx\,dy$$

$$= -\frac{1}{2}\alpha\beta(\sin x + 2\lambda \cos x)\,dx^2$$

$$-\frac{1}{2}\gamma\delta(\sin y - 2\lambda \cos y)$$

$$= -\frac{1}{2}(\alpha\beta \text{cosec } cx\ dx^2 + \gamma\delta \text{ cosec } y\ dy^2)$$

$= $ A negative quantity

So F and hence u is maximum when $x + y = \pi$

Thus, area of the quadrilateral is greatest when $x + y = \pi$ *i.e.*, when it can be inscribed in a circle.

41. Define $F = (x^2 + y^2 + z^2) + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1\right)$

$$+\lambda_2(x + y - z)$$

For maximum and minimum of F,

$$\frac{\partial F}{\partial x} = 2x + 2x\frac{\lambda_1}{4} + \lambda_2 = 0 \quad ...(1)$$

$$\frac{\partial F}{\partial y} = 2y + 2y\frac{\lambda_1}{5} + \lambda_2 = 0 \quad ...(2)$$

$$\frac{\partial F}{\partial z} = 2z + 2z\frac{\lambda_1}{5} - \lambda_2 = 0 \quad ...(3)$$

Multiplying (1) by x, (2) by y and (3) by z and then adding, we get

$$2(x^2 + y^2 + z^2) + 2\lambda_1\left(\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25}\right)$$

$$+\lambda_2(z + y - z) = 0$$

or $\quad 2u + 2\lambda_1 = 0 \Rightarrow \lambda_1 = -u$

put it in (1) we get

$$2x - 2x\frac{u}{n} + \lambda_2 = 0$$

$$\Rightarrow \quad x = \frac{u\lambda_2}{2(u-4)}$$

Similarly, $\quad y = \frac{5\lambda_2}{2(u-5)}, z = \frac{25\lambda_2}{2(u-25)}$

putting these values in $x + y - z = 0$ we get

$$\frac{4\lambda_2}{2(u-4)} + \frac{5\lambda_2}{2(u-5)} + \frac{25\lambda_2}{2(u-25)}$$

$$\text{or} \quad \frac{4}{u-4} + \frac{5}{u-5} + \frac{25}{u-25} = 0$$

43. We know that u will be minimum if

$$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

be all positive and a maximum if they be alternately negative and positive.

Now $\quad A = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(z + y) = 0$

Since A is neither negative nor positive, so u has neither a maximum nor a minimum, *i.e.* no extreme point exist.

45. Define $\quad F = (x^2 + y^2 + z^2) + \lambda_1(px + qy + rz)$

$$+\lambda_2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

For maximum and minimum of F,

$$\frac{\partial F}{\partial x} = 2x + p\lambda_1 + 2x\frac{\lambda_2}{a^2} = 0 \quad ...(1)$$

$$\frac{\partial F}{\partial y} = 2y + q\lambda_1 + 2y\frac{\lambda_2}{b^2} = 0 \quad ...(2)$$

$$\frac{\partial F}{\partial z} = 2z + r\lambda_1 + 2z\frac{\lambda_2}{c^2} = 0 \quad ...(3)$$

Multiplying (1) by x, (2) by y, (3) by z and adding

$$2(x^2 + y^2 + z^2) + \lambda_1(px + qy + rz)$$

$$+2\lambda_2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0$$

or $\quad 2u + 2\lambda_2 = 0 \Rightarrow \lambda_2 = -u$

put it in equation (1),

$$2x + p\lambda_1 - 2x\frac{u}{a^2} = 0$$

$$\text{or} \quad x = \frac{\lambda_1 pa^2}{2(u - a^2)}$$

Similarly, $\quad y = \frac{\lambda_1 qb^2}{2(u - b^2)}, z = \frac{\lambda_1 \pi c^2}{2(u - c^2)}$

putting these values in $px + qy + rz = 0$

we get $\quad \frac{p^2a^2}{u - a^2} + \frac{q^2b^2}{u - b^2} + \frac{r^2c^2}{u - c^2} = 0$

46. Let $\quad u = x^2y^3z^4 \quad ...(1)$

with $\quad 2x + 3y + 4z = a \quad ...(2)$

By (1), $\quad \log x = 2\log x + 3\log y + 4\log z$

or $\quad \frac{1}{u}du = \frac{2}{x}dx + \frac{3}{y}dy + \frac{4}{z}dz$

For maximum or a minimum of $u, du = 0$

i.e., $\quad \frac{2}{x}dx + \frac{3}{y}dy + \frac{4}{z}dz = 0 \quad ...(3)$

and by (2), $2dx + 3dy + 4dz = 0$ ...(4)

Multiplying (3) by 1 and (4) by λ and adding and then equating the coefficients of du, dy, dz we get

$$\frac{2}{x} + 2\lambda = 0, \frac{3}{y} + 3\lambda = 0, \frac{4}{z} + 4\lambda = 0$$

or $$x = y = z = -\frac{1}{\lambda}$$

So by equation (2)

$$\lambda = \frac{-9}{a}$$

Thus, u is stationary at $x = y = z = \frac{a}{9}$

By (1), $$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{2}{x} + \frac{4}{z}\frac{\partial z}{\partial x}$$

when z is a function of x and y.

Differentiating (2) partially w.r.t. x

$$2 + 4\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-1}{2}$$

$\therefore$ $$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{2}{x} - \frac{2}{z}$$

So, $$\frac{1}{u}\frac{\partial^2 u}{\partial x^2} - \frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2 = \frac{-2}{x^2} + \frac{2}{z^2} - \frac{\partial z}{\partial x}$$

$$= \frac{-2}{x^2} - \frac{1}{z^2}$$

when u is stationary $\frac{\partial u}{\partial x} = 0$

So, $$\frac{\partial^2 u}{\partial x^2} = -u\left(\frac{2}{x^2} + \frac{1}{z^2}\right)$$

$$= -x^2y^3z^4\left(\frac{z}{x^2} + \frac{1}{z^2}\right)$$

which is negative, hence u is maximum at

$$x = y = z = \frac{a}{9}$$

and $$u_{max.} = \left(\frac{a}{9}\right)^2\left(\frac{a}{9}\right)^3\left(\frac{a}{9}\right)^4 = \left(\frac{a}{9}\right)^9$$

48. Let $u = x^2 + y^2$ ...(1)

with $ax^2 + 2hxy + by^2 = 1$ ...(2)

For maximum or a minimum of u, $du = 0$

i.e., $x\,dx + y\,dy = 0$ ...(3)

Differentiating (2),

$$2ax\,dx + 2hx\,dy + 2hy\,du + 2by\,dy = 0$$

or $(ax + hy)\,dx + (hx + by)\,dy = 0$ ...(4)

Multiplying (3) by 1, (4) by λ and adding and then equating the coefficients of dx, dy to zero.

$x + \lambda(ax + hy) = 0$ ...(5)

and $y + \lambda(hx + by) = 0$ ...(6)

Multiplying (5) by x, (6) by and adding

$$x^2 + y^2 + \lambda(ax^2 + by^2 + 2hxy) = 0$$

or $u + \lambda = 0 \Rightarrow \lambda = -u$

So, by (5), $x - u(ax + hy) = 0$

or $x(1 - au) - hxy = 0$

or $\left(a - \frac{1}{u}\right)x + hy = 0$ ...(7)

Similarly by (6),

$hx + \left(b - \frac{1}{u}\right)y = 0$ ...(8)

By (7) and (8) eliminate x and y

$$\begin{vmatrix} a - \frac{1}{u} & h \\ h & b - \frac{1}{u} \end{vmatrix} = 0$$

or $$\left(a - \frac{1}{u}\right)\left(b - \frac{1}{u}\right) = h^2$$

50. Given that

$u = x^2 + y^2 + xy$ ...(1)

and $x^2 + y^2 = 1$ ...(2)

For maximum or minimum of u, $du = 0$

i.e., $2x\,dx + 2y\,dy + y\,dx + x\,dy = 0$

or $(2x + y)\,dx + (2y + x)\,dy = 0$ ...(3)

By (2), $x\,dx + y\,dy = 0$ ...(4)

Multiplying (3) by 1, (4) by λ and adding and then equating to zero the coefficients of dx and dy,

$(2x + y) + \lambda x = 0$ ...(5)

and $(2y + x) + \lambda y = 0$ ...(6)

Multiplying (5) by x and (6) by y and adding

$$2(x^2 + y^2 + xy) + \lambda(x^2 + y^2) = 0$$

or $\quad 2u + \lambda = 0 \quad$ or $\quad \lambda = -2u$

but it in equation (5), we get

$$(2x + y) - 2u\,x = 0$$

$\Rightarrow \quad 2(1-u)x + y = 0 \quad \ldots(7)$

Similarly by (6),

$$(2y + x) - 2u\,y = 0$$

or $\quad x + 2(1-u)y = 0 \quad \ldots(8)$

Eliminating x and y from (7) and (8),

$$\begin{vmatrix} 2(1-u) & 1 \\ 1 & 2(1-u) \end{vmatrix} = 0$$

$\Rightarrow \quad 4(1-u)^2 = 1 \quad$ or $\quad 4(u-1)^2 = 1$

52. Let $\quad u = x + y + z \quad \ldots(1)$

with $\quad \dfrac{1}{x} + \dfrac{2}{y} + \dfrac{3}{2} = 1 \quad \ldots(2)$

For maximum or a minimum of x, $du = 0$

i.e., $\quad dx + dy + dz = 0 \quad \ldots(3)$

By (2), $\quad -\dfrac{1}{x^2}dx - \dfrac{2}{y^2}dy - \dfrac{3}{z^2}dz = 0 \quad \ldots(4)$

Multiplying (3) by 1, (4) by λ and adding and then equating the coefficients of dx, dy and dz to zero.

$$1 - \frac{\lambda}{x^2} = 0, 1 - \frac{2\lambda}{y^2} = 0, 1 - \frac{3\lambda}{z^2} = 0$$

or $\quad x = \sqrt{\lambda}, y = \sqrt{2\lambda}, z = \sqrt{3\lambda}$

putting these values in equation (2)

$$\frac{1}{\sqrt{\lambda}}(1 + \sqrt{2} + \sqrt{3}) = 1$$

$\Rightarrow \quad \sqrt{\lambda} = 1 + \sqrt{2} + \sqrt{3}$

so u is stationary when

$$x = 1 + \sqrt{2} + \sqrt{3}$$

$$y = \sqrt{2}(1 + \sqrt{2} + \sqrt{3})$$

$$z = \sqrt{3}(1 + \sqrt{2} + \sqrt{3})$$

By (2), $\quad \dfrac{\partial u}{\partial x} = 1 + \dfrac{\partial z}{\partial u}$

if z is a function of x and y.

Differentiation (2) partially w.r.t. x taking y as constants

$$-\frac{1}{x^2} - \frac{3}{z^2}\frac{\partial z}{\partial x} = 0$$

or $\quad \dfrac{\partial z}{\partial x} = \dfrac{-z^2}{3x^2}$

$\therefore \quad \dfrac{\partial u}{\partial x} = 1 - \dfrac{z^2}{3x^2}$

So, $\quad \dfrac{\partial^2 u}{\partial x^2} = \dfrac{2z^2}{3x^3} - \dfrac{2z}{3x^2}\dfrac{\partial z}{\partial x}$

$$= \frac{2z^2}{3x^3} + \frac{2z}{3x^2}\frac{z^2}{3x^2}$$

which is positive so u is minimum at these values and

$$u_{\text{min.}} = (1 + \sqrt{2} + \sqrt{3}) + \sqrt{2}(1 + \sqrt{2} + \sqrt{3}) + \sqrt{3}(1 + \sqrt{2} + \sqrt{3})$$

$$= (1 + \sqrt{2} + \sqrt{3})^2$$

58. $u = x^4 + y^4 + z^4 \quad \ldots(1)$

with $\quad xyz = 1 \quad \ldots(2)$

By (1), $\quad du = ux^3dx + uy^3dy + uz^3dz = 0$

For maximum or minimum of u, $du = 0$

So, $\quad x^3dx + y^3dy + z^3dz = 0 \quad \ldots(3)$

By (2), $\quad \log x + \log y + \log z = 0$

or $\quad \dfrac{1}{x}du + \dfrac{1}{y}dy + \dfrac{1}{z}dz = 0 \quad \ldots(4)$

Multiplying (3) by 1, (4) by λ, adding and equating the coefficients of dx, dy, dz to zero.

$$x^3 + \frac{\lambda}{x} = 0, y^3 + \frac{\lambda}{y} = 0, z^3 + \frac{\lambda}{2} = 0$$

or $\quad x^4 = y^4 = z^4 = -\lambda$

so by (2), $\quad -\lambda^3 = 1 \quad$ or $\quad \lambda = -1$

so u is stationary at (1,1,1)

By (1), $\quad \dfrac{\partial u}{\partial x} = 4x^3 + 4z^3\dfrac{\partial z}{\partial x}$

and by (2), $\log x + \log y + \log z = 0$

or $\quad \dfrac{1}{x} + \dfrac{1}{z}\dfrac{\partial z}{\partial u} = 0$

$\Rightarrow \quad \dfrac{\partial z}{\partial u} = \dfrac{-z}{x}$

$\therefore \quad \dfrac{\partial u}{\partial x} = 4x^3 - \dfrac{4z^4}{x}$

and $\quad \dfrac{\partial^2 u}{\partial u^2} = 12x^2 + \dfrac{4z^4}{x^2} + \dfrac{16z^4}{x^2}$

At (1,1,1), $\dfrac{\partial^2 u}{\partial x^2} = 32$ which is positive.

So, u is minimum at (1,1,1) and $u_{min.} = 1+1+1 = 3$

60. $u = x^2 y^3 z^4$ ...(1)

$$2x + 3y + 4z = 9 \qquad ...(2)$$

By (1), $\quad \log 4 = 2\log x + 3\log y + 4\log z$

or $\quad \dfrac{1}{u}du = \dfrac{2}{x}dx + \dfrac{3}{y}dy + \dfrac{4}{z}dz$

For maximum or minimum of 4, $du = 0$

So, $\quad \dfrac{2}{x}dx + \dfrac{3}{y}dy + \dfrac{4}{2}dz = 0$ ...(3)

Differentiating (2),

$$2du + 3dy + 4dz = 0 \qquad ...(4)$$

Multiplying (3) by 1, (4) by λ and adding and then equating the coefficients of dx, dy, dz to zero.

$$\frac{2}{x} + 2\lambda = 0, \frac{3}{y} + 3\lambda = 0, \frac{4}{2} + 4\lambda = 0$$

or $\quad x = y = z = -\dfrac{1}{\lambda}$

put these values in (2)

$$-\frac{2}{\lambda} - \frac{3}{\lambda} - \frac{4}{\lambda} = 9 \Rightarrow \lambda = -1$$

So u is stationary at (1,1,1)

$\because \quad \log x = 2\log x + 3\log y + 4\log z$

So, $\quad \dfrac{1}{u}\dfrac{\partial u}{\partial x} = \dfrac{2}{x} + \dfrac{4}{z}\dfrac{\partial z}{\partial u}$

By (2), $\quad 2 + 4\left(\dfrac{\partial z}{\partial x}\right) = 0$

$\Rightarrow \quad \dfrac{\partial z}{\partial x} = -\dfrac{1}{2}$

$\therefore \quad \dfrac{1}{u}\dfrac{\partial u}{\partial x} = \dfrac{2}{x} - \dfrac{2}{z}$

and $\quad \dfrac{1}{u}\dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{u^2}\left(\dfrac{\partial u}{\partial x}\right)^2 = -\dfrac{2}{x^2} - \dfrac{1}{z^2}$

But u is stationary so $\dfrac{\partial u}{\partial x} = 0$

i.e., $\quad \dfrac{\partial^2 u}{\partial x^2} = -u\left(\dfrac{2}{x^2} + \dfrac{1}{z^2}\right)$

$$= -x^2 y^3 z^4\left(\frac{2}{x^2} + \frac{1}{z^2}\right)$$

which is negative for $x = y = z = 1$

So, maximum of u is

$$u_{max} = (1)^2(1)^3(1)^4 = 1$$

❍❍❍

CHAPTER 14

Metric Spaces

METRIC SPACE

Let X be a non-empty set. A mapping d of $X \times X$ into R *i.e.* $d : X \times X \to R$ is said to be a metric or distance function on X iff d satisfies the following axioms :

$[m1] : d(x, y) \geq 0 \;\forall\, x, y \in X$ (non-negativity)

$[m2] : d(x, y) = 0$ iff $x = y$

$[m3] : d(x, y) = d(x, y) \;\forall\, x, y \in X$ (symmetry)

$[m4] : d(x, y) \leq d(x, z) + d(z, y) \;\forall\, x, y, z \in X$ (triangle inequality)

If d is metric for X, then the ordered pair (X, d) is called a metric space and $d(x, y)$ is called the distance between x and y.

Pseudo-metric : A mapping $d : X \times X \to R$ is called a pseudo-metric or semi-metric for X iff d satisfies the axioms $[m_1], [m_3], [m_4]$ of the metric d and the axioms.

$[m'2]$ $d(x, x) = 0 \;\forall\, x \in X$

Thus, every metric is a pseudo-metric but a pseudo-metric is not necessarily a metric.

Results :

1. Let (X, d) be a metric space and $x, y, z \in X$ then
$$d(x, y) \geq |d(x, z) - d(z, y)|$$
2. Let (X, d) be a metric space and $x, y, x', y' \in X$ then
$$|d(x, y) - d(x', y')| \leq d(x, x') + d(y, y')$$
3. Let X be a non-empty set. Then $d : X \times X \to R$ is metric on X iff the following condition are satisfied.

   $[m^*1] : d(x, y) = 0$ iff $x = y \;\forall\, x, y \in X$

   $[m^*2] : d(x, y) \leq d(x, z) + d(y, z) \;\forall\, x, y, z \in X$

Examples :

1. Let $d : R \times R \to R$ defined by
$$d(x, y) = |x - y| \;\forall\, x, y \in R$$
is a metric on R called usual metric on R.

Solution :

$[m1]$: since $|x - y| \geq 0 \;\forall\, x, y \in R$ so $d(x, y) \geq 0$

$[m2]$: we have $|x - y| = 0 \Rightarrow x - y = 0 \Leftrightarrow x = y$

$\therefore \quad d(x, y) = 0$ iff $x = y$

$[m3]$: we have $|x - y| = |y - x| \;\forall\, x, y \in R$

$\therefore \quad d(x, y) = d(y, x) \;\forall\, x \in R$

$$[m4] : |x - y| = |(x - z) + (z - y)|$$
$$\leq |x - z| + |z - y| \;\forall\, x, y, z \in R$$

So, $\quad d(x, y) \leq d(x, z) + d(z, y) \;\forall\, x, y, z \in R$

Hence, d is a metric space.

2. Let X be a non-empty set. The mapping $d : X \times X \to R$ defined by
$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$
is a metric on X called the discrete metric space.

APPLICATION OF METRIC OR A DISTANCE FUNCTION

1. **Distance of a point from a set**

   Let (X, d) be a metric space and $A \subseteq X$. Then the distance between a point $x \in X$ and the set A is defined by
$$d(x, A) = \inf\{d(x, a) : a \in A\}$$
   If $x \in A$ then
$$d(x, A) = 0$$
   Result : If A is non-empty subset of a metric space (X, d), then

$$|d(x, A) - d(y, A)| \le d(x, y)$$

for any points $x, y \in X$.

2. **Distance between two subsets of a metric space**

Let (X, d) be a metric space and A and B be any two non-empty subsets of X. The distance between the sets A and B is defined by

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in A\}$$

Evidently $d(A, B) \ge 0$

If $A \cap B \ne 0$ then $d(A, B) = 0$ but converge is not necessarily true.

3. **Diameter of subset of a metric space**

Let (X, d) be a metric space and set A be any non-empty subset of X. Then the diameter of A, is defined by

$$\delta(A) = d(A) = \sup\{d(x, y) : x, y \in A\}$$

Evidently $d(A) \ge 0$

If $d(A) < \infty$ then A is bounded and the diameter of A is said to be finite otherwise infinite.

By convention $d(\phi) = -\infty$

Result : Let A and B be non-empty subsets of a metric space (X, d), then

$$\delta(A \cup B) \le \delta(A) + \delta(B) + d(A, B)$$

BOUNDED AND UNBOUNDED METRIC SPACE

Let (X, d) be a metric space. Then X is bounded if there exists a positive number M such that $d(x, y) \le M$ for every pair of points $x, y \in X$. A metric space which is not bounded is said to be unbounded.

Thus, a metric space (X, d) is bounded if its diameter is finite or $\delta(A)$ is finite.

Example :

1. Let $X = R$ and $d(x, y) = |x - y|\ \forall x, y \in R$. Then X is unbounded since the diameter of R is infinite.
2. The discrete metric space (X, d) where

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \ne y \end{cases}$$

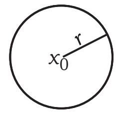
is bounded since $\delta(X) = 1$.

Result : Let (X, d) be any metric space and let M be a positive number, then there exists a metric d^* on X such that the metric (X, d^*) is bounded with $\delta(X) \le M$.

OPEN AND CLOSED SETS IN A METRIC SPACE

1. **Spheres (or balls) :** Let (X, d) be a metric space and $x_0 \in X$. If $r \in R^+$ then the set

$$\{x \in X : d(x, x_0) < r\}$$



is called an open sphere (or open ball). The point x_0 is called the centre and r the radius of the sphere. It is denoted by $S(x_0, r)$ or $B(x_0, r)$ or $S_r(x_0)$ or $B_r(x_0)$.

So, $S(x_0, r) = \{x \in X : d(x, x_0) < r\}$

Similarly a closed sphere (or closed ball) is defined by

$$S[x_0, r] = \{x \in X : d(x, x_0) \le r\}$$

2. **Open sets :** Let (X, d) be a metric space. A subset A of X is said to be d-open iff to each $x \in A$, there exists $r > 0$ such that $S(x, r) \subseteq A$.

Example : On the real line every open interval is an open set.

3. **Properties :**

(i) In a metric space (X, d) the empty set ϕ and the whole space X are open sets.

(ii) In a metric space (X, d) every open sphere is an open set.

(iii) In a metric space, the union of an arbitrary collection of open sets is open.

(iv) In a metric space, the intersection of a finite number of open sets is open.

(v) A subset of a metric space is open iff it is the union of a family of open spheres.

(vi) Every non-empty open set on the real line is the union of a countable collection of pairwise disjoint open intervals.

(vii) In a discrete metric space every set is open.

CLOSED SETS

Let (X, d) be a metric space. A subset A of X is said to be d-closed if the complement of A is open.

Example : In a usual metric for R every closed interval is a closed set for,

let $a, b \in R$ with $a < b$ then

$$\begin{aligned} R - [a, b] &= \{x \in R : x < a \text{ or } x > b\} \\ &= \{x \in R : x < a\} \cup \{x \in R : x > b\} \\ &=]-\infty, a[\cup]\, b, \infty[\end{aligned}$$

which is open, being a union of two open sets. Hence, $[a, b]$ is closed.

Properties :

(i) In a metric space every closed sphere is a closed set.

(ii) The intersection of an arbitrary collection of closed sets is closed.

(iii) The union of a finite number of closed sets is closed.

(iv) In a metric space every finite subset is closed.

NEIGHBOURHOODS

Let (X, d) be a metric space and $x \in X$. A subset N of X is said to be a neighbourhood of x if there exists an open set G such that

$$x \in G \subseteq N$$

If N is an open then it is called an open nbd of x. In a similar way N is said to be neighbourhood of a subset A of X if there exists an open set G such that

$$A \subseteq G \subseteq N$$

Properties :

(i) Every superset of a nbd of $\frac{A}{x}$ is also a nbd of $\frac{A}{x}$.

(ii) The intersection of a finite number of nbds of $\frac{M}{x}$ is also a nbds of $\frac{A}{x}$.

(iii) A set is open iff it contains a nbd of each of its points.

(iv) Let (X, d) be a metric space and let $A \subseteq X$. Then the set $N_r(A) = \{x \in X : d(x, A) < r\}, r > 0$ is an open neighbourhood of A.

LIMIT POINTS AND ADHERENT POINTS

1. **Limit point of a set :** Let (X, d) be a metric space and let A be a subset of X. A point $x \in X$ is called a limit point or accumulation point of A if every neighbourhood of x contains a point of A other then x.

   The set of all limit points of A is called the derived set of A and is denoted by $D(A)$.

2. **Adherent point of a set :** Let (X, d) be a metric space and $A \subseteq X$ and let $x \in X$. Then x is called an adherent point of A if every neighbourhood of x contains a point of A not necessarily distinct from x.

   The set of all adherent points of A is called the adherence of A and is denoted by $Adh(A)$.

3. **Point of condensation :** Let (X, d) be a metric space. A point $x \in X$ is said to be a point of condensation of A if every nbd of x contains uncountably many points of A.

4. **Isolated point of a set :** A point x of a metric space X is said to be an isolated point of a subset A of X if $x \in A$ but x is not limit point of A.

5. **Perfect set :** A closed set having x_0 isolated points is said to be a perfect set.

Results :

1. Let (X, d) be a metric space and $A \subseteq X$. A point $x \in X$ is limit point of A iff every open sphere centred at x contains infinitely may points of A.

2. Let (X, d) be metric space and let $A \subseteq X$. A point $x \in X$ is an adherent point of A iff $d(x, A) = 0$.

3. A subset A of a metric space X is closed iff $D(A) \subseteq A$ *i.e.* A contains all its limit points.

4. Let A be any subset of a metric space (X, d). Then the derived set of A *i.e.* $D(A)$ is a closed set.

5. Let A and B be two subsets of a metric space X then,

   (i) $D(\phi) = (\phi)$

   (ii) $A \subseteq B = D(A) \subseteq D(B)$

(iii) $D(A \cap B) \subseteq D(A) \cap D(B)$

(iv) $D(A \cup B) = D(A) \cup D(B)$

CLOSURE OF A SET

Let A be a subset of a metric space X. The closure of A denoted by $\overline{A}$ or $cl\ A$ is the intersection of all closed sets containing A *i.e.*,

$$\overline{A} = \cap\{F : F \text{ is closed and } F \supseteq A\}$$

If $x \in \overline{A}$ then x is called the point of closure.

Properties :

(i) If A is any closed subset of a metric space X, then $\overline{A}$ is the smallest closed set containing A.

(ii) A subset A of a metric space is closed iff $\overline{A} = A$.

(iii) Let A be a subset of a metric space. Then

$$\overline{A} = A \cup D(A)$$

(iv) Let A and B two subsets of a metric space (X, d) then

(a) $\overline{\phi} = \phi$ (b) $A \subseteq \overline{A}$

(c) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$ (d) $\overline{A \subseteq B} = \overline{A} \cup \overline{B}$

(e) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ (f) $\overline{\overline{A}} = \overline{A}$

(v) Let (X, d) be a metric space and $A \subseteq X$, then the following are equivalent.

(a) A is closed

(b) A contains all its limiting points

(c) $\overline{A} = A$

INTERIOR, EXTERIOR, FRONTIER AND BOUNDARY OF A SET

1. **Interior points :** Let (X, d) be a metric space and $A \subseteq X$. The point $x \in X$ is said to be an interior point of A if A is a neighbourhood of x. The set of all interior points of A is called the interior of A and is denoted by A° or int (A).

2. **Exterior points :** Let (X, d) be a metric space and $A \subseteq X$. The point $x \in X$ is said to be an exterior point of A if it is an interior point of A^e.

   The set of all exterior points of A is called the exterior of A and is denoted by ext (A) or A^e.

3. **Frontier points :** A point $x \in X$ is said to be frontier point of a subset A of X iff it is neither an interior nor an exterior point of A. The set of all frontier points of A is called the frontier of A and is denoted by $Fr(A)$.

4. **Boundary points :** A point $x \in X$ is said to be a boundary point of a subset A of X if It is a frontier point of A and belong to A. The set of all boundary points of A is called the boundary of A and is denoted by $b(A)$.

5. **Dense sets :** Let X be a metric space and let A, B be two subsets of X then

   (i) A is said to be dense in B if $B \subseteq \overline{A}$

   (ii) A is said to be dense in X or everywhere dense if $\overline{A} = X$.

   (iii) A is said to be non-dense in X if $(\overline{A})^\circ = \phi$

   (iv) A is said to be dense-in itself if $A \subseteq D(A)$.

6. **Separable spaces :** A metric space X is said to be separable if X contains a countable dense subset *i.e.* there exists a countable subset A of X such that $\overline{A} = X$.

Properties :

(i) Let (X, d) be a metric space and $A \subseteq X$, then

(a) A° is an open set

(b) A° is the largest open set contained in A

(c) A is open iff $A^\circ = A$

(ii) Let (X, d) be a metric space and $A \subseteq X$, then

(a) $A^\circ = U\{G : G \text{ is open, } G \subseteq A\}$

(b) $\text{ext } A = U\{G : G \text{ is open } G \subseteq A'\}$

(iii) A point $x \in X$ is an exterior point of A iff x is not an adherent point of A *i.e.* $x \in (\overline{A})'$.

(iv) A point $x \in X$ is a frontier point of $A \subseteq X$ iff every neighbourhood of x intersects both A and A'.

(v) Let A be any subset of a metric space X then A°, ext (A) and $Fr(A)$ are disjoint and $X = A^\circ \cup \text{ext}\,(A) \cup Fr(A)$.

(vi) Let A and B are two subsets of a metric space (X, d) then

(a) $X^\circ = X, \phi^\circ = \phi$

(b) $A^\circ \subseteq A$

(c) $A \subseteq B \Rightarrow A^\circ \subseteq B^\circ$

(d) $(A \cap B)^\circ = A^\circ \cap B^\circ$

(e) $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$

(f) $A^{\circ\circ} = A^\circ$

(vii) **Bolzano-weierstrass property :** Every infinite subset of a compact metric space X has a limit point in X.

(viii) **Bolzano-weierstrass theorem :** Every infinite bounded set of real numbers has a limit point.

(ix) Let A and B be two subsets of a metric space (X, d) then

(a) ext $(X) = \phi$, ext $(\phi) = X$

(b) ext $(A) \subseteq A'$

(c) $A \subseteq B \Rightarrow$ ext $(B) \subseteq$ ext (A)

(d) $A^\circ \subseteq$ ext (ext (A))

(e) ext $(A \cup B) =$ ext $(A) \cap$ ext (B)

(x) Let A and B be two subsets of a metric space (X, d) then :

(a) $Fr(A) = \overline{A} - A^\circ$

(b) $Fr(A^\circ) \subseteq Fr(A)$

(c) $Fr(\overline{A}) \subseteq Fr(A)$

(d) $Fr(A \cup B) \subseteq Fr(A) \cup Fr(B)$

(e) $Fr(A \cap B) \subseteq Fr(A) \cup Fr(B)$

(f) A is open iff $A \cap Fr(A) = \phi$ *i.e.*, $Fr(A) \subseteq A'$

(g) A is closed iff $Fr(A) \subseteq A$.

BASES

1. **Base of the neighbourhood system of a point :** Let $N(x)$ be the family of all neighbourhoods of point x in a metric space X. Then a subfamily $B(x)$ of $N(x)$ is said to be a base for $N(x)$ if to each member N of $N(x)$ there exists $B \in B(x)$ such that $B \subseteq N$. $B(x)$ is also called a local base at x or a fundamental system of neighbourhoods of x.

   Example : For the usual metric $d(x, y) = |x - y|$ for R and any $x \in R$ the collection.

   $B(x) = \{]x - \varepsilon, x + \varepsilon[\ \because \varepsilon > 0\}$ of all open intervals with x as mid-point constitutes a, b etc. for the neighbourhood system of x.

2. **Base for the open sets of a metric space :** Let h be the family of all open subsets of a metric space (X, d). A subfamily β of h is said to be a base for h if for each points $x \in X$ and each nbd N of x, there exists some $B \in \beta$ such that

   $$x \in \beta \subseteq \text{N}$$

3. **First countable space :** A metric space (X, d) is said to satisfy the first axiom of countability if each point of X possesses a countable local base. (X, d) is then called first countable or first axiom space.

4. **Second countable space :** A metric space (X, d) is said to satisfy the second axiom of countability if there exists a countable base for h. The space (X, d) then called second countable or second axiom space.

Properties :

1. Let (X, d) be a metric space and h be the family of all subsets of X. A subfamily β of h is a base for h iff every member of h can be expressed as a union of member of β.
2. Every metric space is first countable.
3. A metric space is separable iff it is second countable.

SUB-SPACES OF A METRIC SPACES

Let (X, d) be a metric space and y be a proper subset of X. Let d^* be the restriction of d to $Y \times Y$ *i.e.* $d^*(x, y) = d(x, y)$ whenever $x, y \in Y$. Then the metric space (Y, d^*) is called a subspace of metric space (X, d).

Properties :

1. Let (X, d) be a metric space and (Y, d^*) be its subspace. Then a subset A of Y is d^* open iff there exists a d-open subset G of X such that
$$A = h \cap Y$$
2. Let (Y, d^*) be a sub-space of a metric space (X, d). Then for $B \subseteq Y$, which is open in Y be open in X. It is necessary and sufficient that Y be open in$X$.
3. Let (Y, d^*) be a subspace a metric space (X, d) then
   (i) A subset A of Y is closed in Y iff there exists a closed set F in X such that $A = F \cap Y$.
   (ii) $\overline{A}^* = \overline{A} \cap Y$ where A^* is a subset of Y.
   (iii) A point $y \in Y$ is d^*-limit point of a subset A & Y iff it is a d-limit point of A and
   $$D^*(A) = D(A) \cap Y$$
   where $D^*(A)$ and $D(A)$ are d^*-derived set and d-derived set of A respectively.
   (iv) A subset N^* of Y is a d^*-nbd of a point $y \in Y$ iff $N^* = N \cap Y$ for some d-nbd N of y.
   (v) If $A \subseteq Y$ then
   $$A^\circ \subseteq A^{\circ *} \text{ and } Fr^*(A) \subseteq Fr(A)$$
5. Let (Y, d^*) be a subspace of a metric space (X, d) and β be a base for the family h of all open subsets of X then the base for family G^* of all open subsets of Y relative to d^* is :
$$B^* = \{B \cap Y : B \in \beta\}$$
6. Every subspace of a second countable space is second countable.
7. Every subspace of a separable metric space is separable.

SEQUENCE AND SUBSEQUENCE IN A METRIC SPACE

1. **Sequence in a metric space :** Let X be a metric space. Then a function $f : N \to X$ is called a sequence in X, where N is the set of natural numbers. The value of function f at $n \in N$ is denoted by $f(x) = x_n$. The sequence of f be denoted by $< x_n >$.
2. **Subsequence in a metric space :** If $< x_n >$ be a sequence in a metric space (X, d) and $< i_n >$ is a strictly increasing sequence in N such that $i_1 < i_2 < \ldots i_n < \ldots..$ then $< x_{in} >$ is called a subsequence of $< x_n >$.

   Example : $\left\langle \frac{1}{2n} \right\rangle$ is a subsequence of $\left\langle \frac{1}{n} \right\rangle$.
3. **Convergent sequence in a metric space :** Let (X, d) be a metric space then a sequence $< x_n >$ in X is said to convergent to $x_0 \in X$ if for $\varepsilon > 0$ there exists a positive integer n_0 such that $\forall\, n \geq n_0 \; d(x_n, x_0) < \epsilon$.

   The point $x_0 \in X$ is called the limit of the sequence $< x_n >$ and denoted by
   $$\lim_{n\to\infty} x_n = x_0 \text{ or } \lim x_n = x_0$$
4. **Cluster points of sequence :** Let (X, d) be a metric space. A point $x_0 \in X$ is said to be a cluster point of $< x_n >$ iff for any $\varepsilon > 0$ and positive integer m, $\exists$ an integer $n \geq m$ such that $d(x_0, x_n) < \varepsilon$.

   Thus, x_0 will be a cluster point of $< x_n >$ iff every open sphere centred at x_0 contains infinitely many terms of the sequence.

Properties :

(i) The limit of a convergent sequence is unique.

(ii) Let (X, d) be a metric space. If x_0 is a limit point of a subset A of X, then there exists a sequence $< x_n >$ of A, all distinct from x_0, which converges to x_0.

(iii) If the range set of a convergent sequence in a metric space consists of infinitely many distinct points, then the limit of the sequence is a limit point of the range set of the sequence.

(iv) Let (X, d) be a metric space and x_0, y_0 be two points of X. If $< y_n >$ in X converges to y_0 then $< d(x_0, y_n) >$ of real number converges to $d(x_0, y_0)$.

(v) Let (x, d) be a metric space. If $< x_n >$ and $< y_n >$ in X converge respectively to x_0 and y_0 in X, then the sequence $< d(x_n, y_n) >$ converges to $d(x_0, y_0)$.

CAUCHY SEQUENCE IN A METRIC SPACE

Let (X, d) be metric space. A sequence $< x_n >$ in X is said to be a cauchy sequence in X if for given $\varepsilon > 0$, there exists a positive integer m such that

$$d(x_n, x_m) < \varepsilon \text{ for all } n \geq m$$

or $$d(x_p, x_q) < \varepsilon \text{ for all } p, q \geq m(\varepsilon)$$

Complete metric space : A metric space (X, d) is said to be complete iff every cauchy sequence $< x_n >$ in X converges to a point in X.

Properties :

1. Every convergent sequence in a metric space is a cauchy sequence but not conversely.
2. If a cauchy sequence in a metric space has a convergent subsequence then the sequence is convergent.
3. If $< x_n >$ be a sequence in a metric space (X, d) and E_n be defined by

   $$E_1 = \{x_1, x_2, \ldots\}, E_2 = \{x_2, x_3, \ldots\}$$
   $$\ldots. E_n = \{x_n, x_{n+1}, \ldots.\}$$

   then $< -x_n >$ is a cauchy sequence iff $\delta(E_n) \to 0$ as $n \to \infty$.
4. Let $< x_n >$ be a cauchy sequence in a metric space (X, d) and let $< x_{i_n} >$ be a subsequence of $< x_n >$ then $\lim_{n \to \infty} d(x_n, x_{i_n}) = 0$.
5. Let $< x_n >$ be a cauchy sequence in a metric space (X, d) and $< x_{i_n} >$ be a subsequence of $< x_n >$ converging to $x_0 \in X$. Then $< x_n >$ also converges to x_0.
6. Let (X, d) be a metric space and $< x_n >$ be a cauchy sequence in X. If $< y_n >$ be sequence in X such that $d(x_n, y_n) < \frac{1}{n}$ for every $n \in I^+$ then
   (i) $< y_n >$ is also a cauchy sequence in X.
   (ii) $< y_n >$ converges to $y_0 \in X$ iff $< x_n >$ converges to y_0.
7. Let (X, d) be a complete metric space and let Y be a subspace of X. Then Y is complete iff Y is closed.

Nested sequence

Let (X, d) be a metric space. A sequence A_n of subsets of X is said to be monotonic decreasing iff $A_1 \supset A_2 \supset A_3 \supset \ldots.$

Such a sequence is also called a nested sequence.

Cantor's Intersection theorem

Let (x, d) be a metric space and let $< F_n >$ be a nested sequence of non-empty closed subsets of X such that $\delta(F_n) \to 0$ as $n \to \infty$. Then X is complete iff $\bigcap_{n=1}^{\infty} F_n$ consists of exactly one point.

Properties :

1. The real line *i.e.* usual metric for R is complete metric space.
2. The set c of complex numbers with usual metric is a complete metric space.
3. The set R^n of all n-tuples $x = (x_1, x_2, \ldots. x_n)$ of real numbers is a complete metric space with respect to the usual metric d-defined by

   $$d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$
4. The metric space of rational number with the usual metric is incomplete.

PRODUCT OF COMPLETE METRIC SPACES

Let (X, d) and (Y, e) be two complete metric spaces. Then the product space $z = X \times Y$ with metric

$$\rho(z_1, z_2) = \sqrt{[d^2(x_1, x_2) + e^2(y_1, y_2)]}$$

is complete where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$

CANTOR'S TERNARY SET

The cantor set is the set of all numbers in the interval [0,1] which have a ternary expansion without the digit 1.

EXERCISE

MULTIPLE CHOICE QUESTIONS

Direction : Each of the following questions has four alternative answers. One of them is correct. Choose the correct answer.

1. In a metric space (X,d) the metric d is a function from $X \times X$ to :
   a. R^2 b. N
   c. N^2 d. R

2. The function $d : R \times R \to R$ defined by
   $$d(x,y) = |x - y| \ \forall x,y \in R$$
   is called : **[Kanpur 2018]**
   a. Discrete metric b. Indiscrete metric
   c. Usual metric d. Euclidean metric

3. If (X,d) is a metric space then $\forall x,y \in X$:
   a. $d(x,y) \le 0$ b. $d(x,y) \ge 0$
   c. $d(x,y) = 0$ d. None of these

4. If (X,d) is a metric space then $\forall x,y,z \in X$:
   a. $d(x,y) \ge d(x,z) + d(z,y)$
   b. $d(x,y) \le d(x,z) + d(z,y)$
   c. $d(x,y) = d(x,z) + d(z,y)$
   d. None of these

5. The metric defined by
   $$d(x,y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \ \ \forall x = (x_1, y_1)$$
   and $y = (x_2, y_2) \in R \times R$ is :
   a. Usual metric on R
   b. Usual metric on R^2
   c. Discrete metric on R
   d. Discrete metric on R^2

6. In a metric space (X,d) if $x,y,z \in X$ then :
   a. $|d(x,z) - d(z,y)| \le d(x,y)$
   b. $|d(x,z) + d(z,y)| \le d(x,y)$
   c. $|d(x,z) - d(z,y)| \ge d(x,y)$
   d. None of these

7. Which of the following is not a metric :
   a. $d(x,y) = |x - y| \ \forall x,y \in R$
   b. $d(x,y) = 0$ iff $x = y$ and 1 iff $x \ne y \ \forall x,y \in R$
   c. $d(x,y) = \min\{1, d(x,y)\} \ \forall x,y \in R$
   d. $d(x,y) = |x^2 - y^2| \ \forall x,y \in R$

8. If (X,d) be a metric space then $d(x,y) = 0 \ \forall x,y \in X$ if and only if :
   a. $x = y$ b. $x > y$
   c. $x < y$ d. $(x,y) = 0$

9. If (X,d) be a metric space then symmetric property is defined by :
   a. $d(x,y) \le d(y,x)$ b. $d(x,y) = 0$ iff $x = y$
   c. $d(x,y) = 0$ d. $d(x,y) = d(y,x)$

10. The mapping $d : c \times c \to R$ defined by
    $d(z_1,z_2) = |z_1 - z_2|, \ \forall z_1,z_2 \in c$ is :
    a. A metric space
    b. Not a metric space
    c. Usual metric space over R
    d. Discrete metric space

11. Which of the following is true :
    a. Every metric is a pseudo-metric
    b. Every pseudo-metric is a metric
    c. Metric and pseudo-metric are independent
    d. None of these

12. If $d : X \times X \to R$ be defined by
    (i) $d(x,y) = 0$ iff $x = y \ \forall x,y \in X$
    (ii) $d(x,y) \le d(x,z) + d(y,z) \ \forall x,y,z \in X$, then d is:
    a. Pseudo-metric only
    b. Metric only
    c. Both pseudo and metric
    d. None of these

13. If x,y are two real numbers then :
    a. $|x + y| = |x| + |y|$ b. $|x + y| \ge |x| + |y|$
    c. $|x - y| \le |x| - |y|$ d. $|x + y| \le |x| + |y|$

14. If $d(x,y) = \begin{cases} 0 & \text{iff} \quad x = y \\ 1 & \text{iff} \quad x \ne y \end{cases} \ \forall x,y \in R$ then the metric d is called :
    a. Usual metric on R
    b. Indiscrete metric on R
    c. Discrete metric on R
    d. Euclidean metric on R

15. A sequence in a set X is a mapping whose domain is the set of :

a. I b. N
c. R d. C

16. In a metric space (X,d) the distance between a point $p \in X$ and set A *i.e.* $d(p,A)$ is defined by :

a. $\sup\{d(p,x) : x \in A\}$
b. $\sup\{d(p,x) : x \in X\}$
c. $\inf\{d(p,x) : x \in A\}$
d. $\inf\{d(p,x) : x \in X\}$

17. If A and B are non-empty subsets of a metric space (X,d) with diameter δ then :

a. $\delta(A \cup B) = \delta(A) + \delta(B)$
b. $\delta(A \cup B) \le \delta(A) + \delta(B)$
c. $\delta(A \cup B) \le \delta(A) + \delta(B) + d(A,B)$
d. $\delta(A \cup B) \le \delta(A) + \delta(B) - d(A,B)$

18. The metric space over R defined by $d(x,y) = |x-y|$ is :

a. Finite b. Bounded
c. Unbounded d. Diameter is finite

19. If $d(x,y) = |x-y|$ be a usual metric on $X = [0, 1]$ then $S\left(\frac{1}{2}, 1\right) =$

a. $\left[\frac{1}{2}, 1\right]$ b. $\left[\frac{-1}{2}, \frac{3}{2}\right]$
c. $[0,1]$ d. None of these

20. For the usual metric $d(x,y) = |x-y|$, which of the following set is open set :

a. $\{1\}$ b. $\{1,2,3\}$
c. $]0,1]$ d. $]0,1[\cup]2,3[$

21. If (X_1,d_1) and (X_2,d_2) be metric space and $X = X_1 \times X_2$ then $d(x,y) = d_1(x_1,y_1) + d_2(x_2,y_2)$ $\forall x = (x_1,x_2)$ and $y = (y_1,y_2) \in X$ is :

a. Metric
b. Not a metric
c. May or may not be metric
d. None of these

22. The mapping $d(x,y) = |x^2 - y^2|, \ \forall x,y \in R$ is :

a. Metric space
b. Pseudo-metric space
c. Usual metric space
d. Discrete metric space

23. Which of the following shows that d is not a bounded metric :

a. $d(x,y) \le k \ \ \forall x,y \in X$
b. $d(x,y) \ge k \ \ \forall x,y \in X$
c. $d(X) \le k$
d. None of these

24. If (X,d) is any metric space then d^* defined by

$$d^*(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

is a metric for X with :

a. $\delta(X,d^*) = 1$ b. $\delta(X,d^*) \ge 1$
c. $\delta(X,d^*) \le 1$ d. None of these

25. For the usual metric $d(x,y) = |x-y|$ over the set $X = [0, 1]$, $S\left(\frac{1}{16}, \frac{1}{16}\right)$ is equal to :

a. $\left[\frac{1}{16}, \frac{1}{16}\right]$ b. $\left[0, \frac{1}{8}\right]$
c. $0, \frac{1}{8}$ d. none of these

26. If $d(x,y) = \begin{cases} 0 & \text{when } x = y \\ 1 & \text{when } x \ne y \end{cases}$ and $x_0 \in X$ then $S(x_0, 1)$ is equal to :

a. X b. $\{x_0\}$
c. 1 d. 0

27. In a metric space (X,d) which of the following statements are true :

(I) union of open sets is open
(II) finite intersection of open sets is open

a. I is true only b. II is true only
c. I and II are true d. None of these

28. In a metric space (X,d) if $p \in A \subseteq X$ then $d(p,A)$:

a. p b. A
c. 0 d. Not exist

29. If A be any non-empty subset of a metric space (X,d) then for any points $x,y \in X : |d(x,A) - d(y,A)|$ is :

a. $\ge d(x,y)$ b. $\le d(x,y)$
c. $= d(x,y)$ d. $d(x-y,A)$

30. If $A=[0,1)$ and $B=(1,2]$ in the usual metric space (R,d) then $d(A,B)$ is equal to :

a. 2 b. 1
c. 0 d. 3

31. If (x,d) be a discrete metric space and $x_0 \in X$ then for a positive real number $r>1$, $s(x_0,r)$ is equal to :

a. ϕ b. 1
c. X d. Not exist

32. Which of the following is not true in a metric space (X,d) :

a. ϕ and X are open sets
b. Every open sphere is open set
c. Union of arbitrary collection of open set is open
d. Intersection of arbitrary collection of open set is open

33. If the usual metric space $d(x,y)=|x-y|\ \forall x,y \in R$ $\cap\left\{\left]\frac{-1}{n},\frac{1}{n}\right[: n\in N\right\}$ is :

a. Closed
b. Open
c. Both open and closed
d. Not necessarily open

34. Which of the following is not a nbd of 1 for the usual metric $d(x,y)=|x-y|$ for R :

a. $]0,2[-1\frac{1}{2}$ b. $[0,2]-\frac{1}{2}$
c. R d. $[1,2[$

35. Let (X,d) be a metric space and $A\subseteq X$ such that $x\in X$. If every nbd of x contains a point of A other than x then x is called :

a. Limit point b. Point of condensation
c. Isolated point d. None of these

36. If $d : X\times X \to R$ be defined as

(i) $d(x,y)\geq 0$ (ii) $d(x,x)=0$
(iii) $d(x,y)=d(y,x)$ (iv) $d(x,y)\leq d(x,z)+d(z,y)$

Then d is a :

a. Metric space
b. Pseudo-metric space
c. Both metric and pseudo-metric space
d. None of these

37. If $A=\left\{1,\frac{1}{3},\frac{1}{5},\ldots.\frac{1}{2n-1}\right\}$ and $B=\left\{\frac{1}{2},\frac{1}{4},\frac{1}{6},\ldots.\frac{1}{2n}\ldots\right\}$ are two subsets in a metric space (X,d) then $d(A,B)$ is equal to :

a. 0 b. $\frac{1}{2}$
c. $A=B$ d. ∞

38. For the usual metric $d(x,y)=|x-y|\ \forall x,y\in[0,1]$ $S\left(0,\frac{1}{8}\right)$ is equal to :

a. $S\left(2,\frac{17}{8}\right)$ b. $S\left(1,\frac{9}{8}\right)$
c. $S\left(\frac{1}{32},\frac{3}{32}\right)$ d. $S\left(\frac{1}{32},\frac{2}{32}\right)$

39. In a discrete metric space every set is :

a. Closed
b. Open
c. Either open or closed
d. None of these

40. For the usual metric d for R the singleton set in R is :

a. Closed
b. Open
c. May be open or closed
d. None of these

41. In a metric space (X,d), if $x\in A\subseteq X$ but x is not a limit point of A then x is called :

a. Limit point b. Adherent point
c. Isolated point d. Exterior point

42. If A° be the interior of A in a metric space (x,d) then which one is not true :

a. A° is the largest open set contained in A
b. A° is an open set
c. A is open iff $A^\circ = A$
d. A is open iff $A^\circ = X$

43. If A be a subset of a metric space (X,d) then the diameter of $A,\delta(A)$ is defined by : **[Kanpur 2018]**

a. $\sup\{d(x,y) : x,y\in X\}$
b. $\sup\{d(x,y) : x,y\in A\}$
c. $\inf\{d(x,y) : x,y\in X\}$
d. $\inf\{d(x,y) : x,y\in A\}$

44. If in a metric space $(X,d), d(p,A)=0\ \forall p \in A \subseteq X$ then :

a. $p \in X$ b. $p \in A$

c. $p \notin A$ d. $p \notin X$

45. If A and B are two non-empty subsets of a non space (X,d) then diameter of A is :

a. $d(x,y)\ \forall x,y \in A$

b. $\inf\{d(x,y): x,y \in A\}$

c. $\sup\{d(x,y): x,y \in A\}$

d. $\sup\{d(x,y): x,y \in X\}$

46. If $\delta(A \cup B) \le \delta(A)+\delta(B)$ where δ be the diameter exist only when :

a. $A \cup B = \phi$ b. $A \cap B = \phi$

c. $A \cap B \ne \phi$ d. None of these

47. If d be a usual metric for R defined by

$$d(x,y)=|x-y|\ \forall x,y \in R$$

then $S(-1, 1)$ is equal to :

a. [0,1] b. [–2,0]

c.]–2, 0[d.]–1, 1[

48. If a metric space every singleton set is :

a. Open

b. Closed

c. May be open or closed

d. None of these

49. Which of the following is true in a metric space :

a. Every closed sphere is a closed set

b. Intersection of an arbitrary collection of closed sets is closed

c. Union of finite number of closed sets is closed

d. All of the above

50. If in a metric space every nbd of $x \in X$ contains a point of $A \subseteq X$ not necessarily distinct from x then x is called :

a. A limit point

b. Adherent point

c. Isolated point

d. Interior point

51. If $d(A,B)=0$ for A and B are non-empty subsets of X then :

a. $A \cap B = \phi$ only

b. $A \cap B \ne 0$ only

c. Non-necessarily $A \cap B \ne 0$

d. None of these

52. The supremum of the set of all distances between the points of A is called :

a. Displacement b. Circumference

c. Radius d. Diameter

53. Consider the usual metric $d(x,y)=|x-y|$ and $A=[1, 2]$, $B=[2, 4]$ then $d(5, B)$ is equal to :

a. 1 b. 2

c. 4 d. 5

54. If $d(z_1,z_2)=|x_1-x_2|+|y_1-y_2|$ where $z_1=(x_1,y_1)$ and $z_2=(x_2,y_2)\in R^2$ then the open sphere of limit radius about (0,0) is :

a. $|x+y|<1$ b. $|x|+|y|<1$

c. $|x|+|y|>1$ d. $|x+y|>1$

55. If $d(x,y)=|x-y|\ \forall x,y \in X$ where $X=[0, 1]$ then the set $A=[0, 1[$ is :

a. Closed in X b. Open in X

c. Semi-open in X d. Semi closed in X

56. In a metric space (X,d), consider the following statements

(I) Every singleton set is open

(II) Complement of a finite set is open

a. I is true only

b. II is true only

c. I and II both are true

d. None of these

57. For the usual metric $d(x,y)=|x-y|$ consider $I_n=\left\{\left]\frac{-1}{n},\frac{1}{n}\right[: n \in N\right\}$ of open interval in R then $\cap I_n$ is equal to :

a. $]-\infty, \infty[$ b.]–1,1[

c. {0} d. 0

58. In a metric space (X,d), $d(\phi)$ is equal to :

a. 0 b. 1

c. ∞ d. $-\infty$

59. If $d(x,y)=|x-y|$ is a usual metric for R and $A=[1, 2[$, then $d\left(\frac{5}{4}, A\right)$ is :

a. $\frac{1}{2}$ b. 1

c. 2 d. 0

60. If $d(z_1,z_2)=\max.\{|x_1-x_2|, |y_1-y_2|\}$ above $z_1=(x_1,y_1)$, $z_2=(x_2,y_2)\in R^2$ then the open sphere of unit radius about (0,0) is :

a. $\{(x,y)\in R^2 : \max.\{|x|,|y|\}<1\}$

b. $\{(x,y)\in R^2 : \max.\{|x-y|<1\}$

c. $\{(x,y)\in R^2 : \max.\{|x+y|<1\}$

d. $\{(x,y)\in R^2 : \max.\{|x|,|y|\}>1\}$

61. If $d(x,y)=\begin{cases}0 & \text{when } x=y\\ 1 & \text{when } x\neq y\end{cases}$ and $x_0\in X$ then $S\left(x_0,\frac{3}{2}\right)$ is equal to :

a. ϕ b. $\{x_0\}$

c. X d. None of these

62. If every open sphere centred at x contains infinite many points of $A\subseteq X$ in a metric space (X,d) then x is called :

a. Limit point b. Isolated point

c. Interior point d. None of these

63. Let R be the set of real numbers. The metric space (R^n,d) with the usual metric d on R^n is called :

[Kanpur 2018]

a. Usual n-space

b. Real n-space

c. Real euclidean n-space

d. Frechet space

64. If A is subset of metric space X then true statement is :

a. Int (A) equals the union of all open subsets of A

b. Int (A) equals the intersection of all closed subsets of A

c. Int (A) equals the intersection of all open subsets of A

d. Int (A) equals the union of all closed subsets of A

65. If $d(z_1,z_2)=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ then open sphere of limit radius about (0,0) is :

a. $x^2+y^2=1$ b. $x^2+y^2\geq 1$

c. $x^2+y^2\leq 1$ d. None of these

66. For the usual metric on R the internal $[a,b[$ is :

a. Open

b. Closed

c. Both open and closed

d. Neither open nor closed

67. Which of the following is not true in a metric space :

a. Every open sphere containing x is an open nbd. of x

b. Every superset of nbd of x is again a nbd of x

c. A is open iff it contains a nbd of each of its Points

d. every subset of nbd. of x is again a nbd of x

68. The intersection of an infinite number of neighbourhoods of a set is :

a. Neighbourhood

b. Not a neighbourhood

c. May or may not be neighbourhood

d. None of these

69. A subset A of a metric space (X,d) is closed if and only if :

a. A is open b. A' is open

c. A' is closed d. None of these

70. A subset G in a metric space (X,d) is said to be open if to each $x\in G$, $\exists\, r>0$ such that :

a. $\delta(x,r)\subseteq G$ b. $\delta(x,r)=G$

c. $\delta(x,r)\geq G$ d. None of these

71. If (X,d) is a pseudo-metric space and if $d(x,y)=0$ then :

a. $x\neq y$

b. $x=y$

c. Either $x=y$ or $x\neq y$

d. None of these

72. If $d(x,y) = x^2 - y^2 \ \forall x,y \in R$ then d is :

a. Pseudo-metic

b. Metric

c. Both metric and pseudo-metric

d. None of these

73. If (X,d) be a discrete metric space and $x \in X$, then every subset of X containing x is :

a. Open set b. Open sphere

c. Neighbourhood d. All of these

74. A point $x \in X$ is a limit point of $A \subseteq X$ in a metric space (X,d) if $d(x,[A \sim \{x\}])$ is equal to :

a. 0 b. $\{0\}$

c. $\{x\}$ d. A

75. A point $x \in X$ is an adherent point of $A \subseteq X$ iff :

a. $d(x,A) = A$ b. $d(x,A) = \{x\}$

c. $d(x,A) = X$ d. $d(x,A) = 0$

76. If A is a finite set in a metric space (X,d) then $D(A)$ is equal to :

a. X b. A

c. $X - A$ d. ϕ

77. The theorem 'Every infinite subset of a compact metric space X has a limit point in X' is stated by :

a. Bounded theorem

b. Abel's theorem

c. Bolzano-weierstrass theorem

d. Dirichlet's theorem

78. Which of the following is not a base for x in metric space $d(x,y) = |x-y| \ \forall x,y \in R$:

a. $]x-\varepsilon, x+\varepsilon[: 0 < \varepsilon \in R$

b. $]x - \frac{1}{n}, x + \frac{1}{n}[: n \in N$

c. $[x-\varepsilon, x+\varepsilon] : 0 < \varepsilon \in R$

d. None of these

79. If A be any subset of a metric space (X,d) then :

a. ext A = ext $(X - A)$

b. ext A = int(X)

c. ext A = int$(X - A)$

d. ext $(X - A)$ = int X

80. The frontier of a subset A of X in a metric spaces :

a. Open

b. Closed

c. Either open or close

d. None of these

81. A subset A of a metric space (X,d) is closed iff :

a. $\overline{A} = \phi$ b. $\overline{A} = X$

c. $\overline{A} = A$ d. None of these

82. A subset A of a metric space X is closed if and only if:

a. $D(A) = \phi$ b. $A \subset D(A)$

c. $D(A) \subseteq A$ d. None of these

83. The derived set of A *i.e.* $D(A)$ in a metric space (X,d) is :

a. Open set

b. Closed set

c. Both open and closed set

d. None of these

84. Which of the following is not true :

a. $A \subseteq \overline{A}$ b. $\overline{(A \cap B)} \subseteq \overline{A} \cap \overline{B}$

c. $\overline{\overline{A}} = A$ d. $\overline{A \cup B} = \overline{A} \cap \overline{B}$

85. If A be a subset of a metric space (X,d) then :

a. $A^\circ = \cap\{G : G \text{ is open and } G \supseteq A\}$

b. $A^\circ = \cup\{G : G \text{ is open and } G \supseteq A\}$

c. $A^\circ = \cup\{G : G \text{ is open and } G \subseteq A\}$

d. $A^\circ = \cup\{G : G \text{ is open and } G \subset A'\}$

86. In the usual metric space (R,d) if $A =]a,b[$ then $\overline{A}$ is equal to :

a. $]a,b[$ b. $]a,b]$

c. $[a,b[$ d. $[a,b]$

87. In a metric space (X,d) a subset N is called neighbourhood of x if there exists $r > 0$ such that :

a. $s(x,r) = N$ b. $s(x,r) \subseteq N$

c. $s(x,r) \supseteq N$ d. None of these

88. A subset A of a metric space (X,d) is closed if and only if :

a. $\overline{A} = A \cup D(A)$ b. $\overline{A} = A$

c. $\overline{A} = D(A)$ d. $\overline{A} = A \cap D(A)$

89. If A be a subset of a metric space (X,d) then :

a. $\overline{A} = A$ b. $\overline{A} = D(A)$

c. $\overline{A} = A \cap D(A)$ d. $\overline{A} = A \cup D(A)$

90. Which one of the following is not true in a metric space (X, d) :

a. $D(\phi) = \phi$

b. $A \subseteq B \Rightarrow D(A) \subseteq D(B)$

c. $D(A \cup B) = D(A) \cup D(B)$

d. $D(A \cap B) = D(A) \cap D(B)$

91. If $x \in X$ is frontier point of A and belongs to A then x is called :

a. Limit point b. Isolated point

c. Boundary point d. Interior point

92. In a metric space (X, d), A is said to be dense in B if :

a. $\overline{A} = B$ b. $A = \overline{B}$

c. $A \subseteq \overline{B}$ d. $B \subseteq \overline{A}$

93. If $d(x, y) = |x - y| \ \forall x, y \in R$ then int $]0,1[$ is :

a. $[0,1]$ b. $]0,1]$

c. $[0,1[$ d. $]0,1[$

94. In the usual metric space (R, d) the derived set of the set ϕ of all rational numbers *i.e.* $D(\phi)$ is equal to :

a. Q b. R

c. $R - Q$ d. ϕ

95. In a usual metric space (R, d), the following is true :

a. $\overline{Q} = \phi$ b. $\overline{Q} = \theta$

c. $\overline{Q} = R$ d. None of these

96. Let R^n be the set of all ordered n-tubes of real numbers and $x = (x_1, x_2, ... x_n)$, $y = (y_1, y_2 ... y_n)$ then $d(x, y)$ usual metric d on R^n is defined by :

a. $\sum (x_i - y_i)^2$ b. $\left(\sum (x_i - y_i)^2\right)^{1/2}$

c. $\left(\sum (x_i - y_i)\right)^{1/2}$ d. None of these

97. In a usual metric space (R, d), $A = \left\{\frac{1}{n} : n \in N\right\}$ then $\overline{A}$ is equal to :

a. $\{0\}$ b. $\left\{\frac{1}{n} : n \in N\right\} \cup \{0\}$

c. $\left\{\frac{1}{n} : n \in N\right\}$ d. ϕ

98. In the metric space (X, d) a subset A of X is open iff :

a. $A^\circ \subset A$ b. $A \subset A^\circ$

c. $A^\circ = X$ d. $A^\circ = A$

99. If every neighbourhood of x intersects both A and A' in a metric space (X, d) then x is called :

a. Limit point b. Isolated point

c. Adherent point d. Frontier point

100. Which of the following is correct in a metric space (X, d) :

a. A subset A of X is open iff $\overline{A} = A$

b. A subset A of X is closed iff $\text{int}(A) = A$

c. If a subset A of X is not open then it is closed

d. A subset A is open iff A is a nbd of each of its points

101. In a discrete metric space every set is :

[Kanpur 2018]

a. Open

b. Closed

c. Either open or closed

d. None of these

102. If x is the limit point of A in a metric space (X, d) then:

a. $x \in X$ b. $x \in A$

c. $x \notin X$ d. $x \notin A$

103. In a metric space (X, d) $\overline{A}$ is defined as :

a. Union of all closed subsets of A

b. Intersection of all closed subsets of A

c. Union of all closed supersets of A

d. Intersection of all closed supersets of A

104. Which of the following is true in a metric space (X, d):

a. Union of arbitrary collection of closed subsets of X is closed

b. $A \subset X$ is open iff A constant all its limit points

c. If $A \subset X$ then $\text{int}(X - A)$ is closed

d. Every convergent sequence in X is a Cauchy sequence

105. The diameter of a finite subset in a metric space (X, d) is :

a. Finite

b. Infinite

c. May be finite or infinite

d. None of these

106. The mapping $d : R \times R \to R$ defined by

$d(x, y) = |x^2 - y^2|, \ \forall x, y \in R$ is : **[Kanpur 2018]**

a. Discrete metric on R

b. Usual metric on R

c. Indiscrete metric on R

d. Pseudo metric on R

107. Consider the following statements in a metric space R.

(I) $d(x, y) = \dfrac{|x - y|}{1 + |x - y|} \ \forall x, y \in R$

(II) $d(x, y) = \dfrac{|x - y|}{1 - |x - y|} \ \forall x, y \in R$

a. I is metric

b. II is metric

c. Both I and II are metric

d. None of these

108. In a metric space (X, d) :

a. $\phi^\circ = \phi$

b. $X^\circ = X$

c. $\phi^\circ = \phi$ and $X^\circ = X$ both

d. None of these

109. If in a metric space (X, d) each point of X has a countable local base then X is called :

a. Countable space

b. First countable space

c. Second countable space

d. None of these

110. Which one of the following statements is a metric for metric space (X, d) :

(I) $d^*(x, y) = \max. \{1, d(x, y)\}$

(II) $d^*(x, y) = \min \{1, d(x, y)\}$

a. I   b. II

c. Both I and II   d. Neither I nor II

111. If A and B are disjoint subsets of a metric space (X, d) then $d(A, B)$ is equal to :

a. 0   b. ∞

c. Finite distance   d. None of these

112. Let (R, d) be a metric space, where d is the usual metric on R and $A = \{x \in R : 0 < x \le 1\}$ then $d(0, A)$ is equal to : **[Kanpur 2018]**

a. 0   b. 1

c. –1   d. None of the above

113. The usual metric space (R, d) is :

a. First countable space

b. Second countable space

c. Both (a) and (b)

d. None of these

114. Consider the following statements :

(I) Every separable metric space is second countable.

(II) Every second countable space is separable.

a. I is true only

b. II is true only

c. Both I and II are true

d. I and II are not true

115. Every metric space is :

a. First countable

b. Second countable

c. First and second countable both

d. None of these

116. If X is a metric space and N is the set of natural numbers then function f is sequence in X when :

a. $f : X \to N$ b. $f : N \to X$

c. Both (a) and (b)   d. None of these

117. Consider the following statements in a metric space (X, d) :

(I) $d^*(x, y) = \max. \left\{\frac{1}{2}, d(x, y)\right\}$

(II) $d^*(x, y) = \min. \left\{\frac{1}{2}, d(x, y)\right\}$

a. I is metric

b. II is metric

c. Both I and II are metric

d. None of these

118. Let (y, d^*) be a subspace of (X, d) then a subset A of Y is d^* open iff there exist a d-open subset G of X such that :

a. $G = A \cap Y$ b. $G = A \cup Y$
c. $A = Y \cap G$ d. $X = Y \cap G$

119. If A be a subset of a metric space (X, d) then $d(A, \phi)$ is equal to :

a. 0 b. ∞
c. $-\infty$ d. Cannot defined

120. Consider the metric space (R, d), where d is the usual metric on R and $A = \left\{1, \frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{2n-1}\right\}$ and $B = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6} \ldots \frac{1}{2n} \ldots\right\}$ then $d(A, B) =$ **[Kanpur 2018]**

a. ∞ b. $\frac{1}{3}$
c. $\frac{1}{2}$ d. 0

121. Which of the following function is a metric on R :

a. $d(x, y) = \frac{1}{3}|x - y|\ \forall x, y \in R$

b. $\begin{cases} 0 & \text{if} \quad x \neq y \\ 1 & \text{if} \quad x = y \end{cases}$

c. $d(x, y) = 3(x - y)\ \forall x, y \in R$

d. None of these

122. Which of the following is not a subspace of the usual metric space C of complex numbers :

a. Unit circle b. Open disc
c. Closed unit disc d. Open sphere

123. A sequence $< x_n >$ in a metric space (X, d) is said to be convergent sequence if for each $\varepsilon > 0$ there exists a positive integer x_0 such that for $n \geq n_0$:

a. $d(x_n, x) > \varepsilon$ b. $d(x_n, x) = \varepsilon$
c. $d(x_n, x) < \varepsilon$ d. None of these

124. If (X, d) be a metric space then $d'(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$ is a :

a. Metric on X
b. Pseudometric on X
c. Neither metric nor pseudometric on X
d. None of these

125. Which of the following is a subspace of the usual metric space R :

a. [0,1] b. Q
c. R d. All the above

126. Every finite subset of R with respect to usual metric for R is :

a. Open b. Closed
c. Both (a) and (b) d. None of these

127. If sequence $< x_n >$ is convergent in a metric space X then its limit is/are :

a. Finite b. Unique
c. Infinite d. None of these

128. If A and B are non-empty subsets of a metric space (X, d) such that $d(A, B) = 0$ then :

a. $A \cap B = \phi$ b. $A \cap B \neq \phi$
c. Either (a) or (b) d. None of these

129. Metrics $d(x, y)$ and $\dfrac{d(x, y)}{1 + d(x, y)}$ defined on a non-empty set X are : **[Kanpur 2018]**

a. Equivalent b. Reciprocal
c. Complementary d. None of these

130. To each $x \in R$ the family $\left\{]x - \frac{1}{n}, x + \frac{1}{n}[: n \in N \right\}$ is:

a. Local base at x only
b. Countable only
c. Countable local base at x
d. None of these

131. A sequence $< x_n >$ in a metric space (X, d) is called Cauchy sequence if for $\varepsilon > 0$ there exists a positive integer n_0 such that for $m, n \geq n_0$:

a. $d(x_n, x_m) \leq \varepsilon$ b. $d(x_n, x_m) \geq \varepsilon$
c. $d(x_n, x_m) < \varepsilon$ d. $d(x_n, x_m) > \varepsilon$

132. If A be a subset of a metric space (X, d) such that $d(x, A) = 0$ then :

a. $x \in A$
b. $x \notin A$
c. Either $x \in A$ or $x \notin A$
d. None of these

133. Consider the following statements in a metric space R (I) $d(x,y) = \dfrac{1+|x-y|}{1-|x-y|}$ $\forall x,y \in R$

(II) $d(x,y) = |x+y|$ $\forall x,y \in R$

a. I is metric

b. II is metric

c. I and II both are metric

d. Neither I nor II are metric

134. If $< x_n >$ and $< y_n >$ are sequences in a metric space (X,d) such that $x_{n \to x}$ and $y_n \to y$ then the sequence $< d(x_n, y_n) >$ of real numbers converges to :

a. (x,y) b. $d(x,y)$

c. $(0,0)$ d. None of these

135. Which of the following is a metric on R :

a. $d(x,y) = |x+y|$ $\forall x,y \in R$

b. $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \infty & \text{if } x \neq y \end{cases}$

c. $d(x,y) = \dfrac{1-|x-y|}{1+|x+y|}$ $\forall x,y \in R$

d. $d(x,y) = |x-y|$ $\forall x,y \in R$

136. Cantor's ternary set is a :

a. Closed set

b. Open set

c. Both open and closed set

d. None of these

137. Consider the following statements in a metric space (x,d)

(I) Every convergent sequence is a Cauchy sequence

(II) Every Cauchy sequence is a convergent sequence

a. I is true

b. II is true

c. I and II both are true

d. None of these

138. If $d(x,y) = |x-y|$ for R and $Y = [0,1]$ then $\left]\dfrac{1}{2}, 1\right]$ is :

a. Open in R b. Closed in R

c. Open relative to Y d. Semi closed relative to Y

139. Which of the following is a countable base :

a. $]a,b[$ where a,b are rationals

b. $]a,b[$ where a,b are reals

c. $]a,b[$ where a,b are irrationals

d. $]a,b[$ where a,b are integers

140. In a discrete metric space (x,d) the open sphere $s(x_0, r)$ is defined as :

a. $\begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

b. $\begin{cases} X & \text{if } r > 1 \\ \{x_0\} & \text{if } 0 < r \leq 1 \end{cases}$

c. $\begin{cases} X & \text{if } 0 < r \leq 1 \\ \{x_0\} & \text{if } r > 1 \end{cases}$

d. $\begin{cases} X & \text{if } r > 1 \\ 0 & \text{if } 0 < r \leq 1 \end{cases}$

141. If $(0,1]$ be ths subspace of usual metric space R then the sequence $< \dfrac{1}{n} >$ is :

a. Cauchy sequence

b. Convergent sequence

c. Both Cauchy and convergent

d. None of these

142. The subset $[0,3]$ in $X = [0,3]$ under the metric $d(x,y) = |x-y|$ $\forall x,y \in X$ is :

a. Closed

b. Open

c. Both open and closed

d. None of these

143. If $F_n = \left[0, \dfrac{n}{n+1}\right]$ $\forall n \in N$ in the usual metric (R,d) then $\bigcup_{n=1}^{\infty} F_n$ is :

a. $[0,1]$ b. $\{0, n]$

c. $\{0\}$ d. $[0,1[$

144. The set $A = \{1, 2, 3, 4, ..., n...\}$ in the usual metric on R is :

a. Open

b. Closed

c. Both open and closed

d. None of these

145. If every Cauchy sequence in X is convergent in a metric space then it is called :

a. Pseudo-metric space
b. Compact metric space
c. Cauchy metric space
d. Complete metric space

146. If $d(x,y) = |x-y|$ for R and $Y = [0,1]$ then $\left]0, \frac{1}{2}\right]$ is :

a. Open relative to Y b. Open in R
c. Closed in R d. None of these

147. If $A = \{1, 2, 3, ...\}$, $B = \left\{n + \frac{1}{n} : n \in N\right\}$ in a usual metric R then $d(A,B) =$

a. 0 b. ∞
c. 1 d. $1-n$

148. If $A = [a,b]$ in the usual metric space (R,d) then boundary (A) is :

a. $[a,b]$ b. $]a,b[$
c. $\{a,b\}$ d. None of these

149. In the discrete metric space (X,d) the closed sphere $s(x_0,r)$ is defined by :

a. $\begin{cases} X & \text{if} \quad 0 < r \leq 1 \\ \{x_0\} & \text{if} \quad r \geq 1 \end{cases}$

b. $\begin{cases} X & \text{if} \quad 0 < r \leq 1 \\ 0 & \text{if} \quad r > 1 \end{cases}$

c. $\begin{cases} X & \text{if} \quad r \geq 1 \\ \{x_0\} & \text{if} \quad 0 < r < 1 \end{cases}$

d. $\begin{cases} X & \text{if} \quad r > 1 \\ \{x_0\} & \text{if} \quad 0 < r \leq 1 \end{cases}$

150. The set $A = \left\{n + \frac{1}{n} : n \in N\right\}$ in usual metric on R is :

a. Closed
b. Open
c. Both closed and open
d. None of these

151. If A be any subset of a metric space (X,d) then $\overline{A}$ is defined by :

a. $\{x \in X : d(x,x) = 0\}$ b. $\{x \in X : d(x,A) = 0\}$
c. $\{x \in A : d(x,A) = 0\}$ d. $\{x \in X : d(x,A) \neq 0\}$

152. If $A = [a,b]$ is closed interval in the usual metric space (R,d) then $Fr(A)$ is :

a. $[a,b]$ b. $]a,b[$
c. $\{a,b\}$ d. ϕ

153. Consider the following statements :

(I) R with usual metric is complete
(II) C with usual metric is complete

a. I is true only b. II is true only
c. I and II are true d. None of these

154. The theorem "Let X be metric space and $< F_n >$ be a decreasing sequence of non-empty closed subsets of X such that $d(F_n) \to 0$, then $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point" is called :

a. Abel's theorem
b. Cantro's intersection theorem
c. Weierstrass theorem
d. None of these

155. In the metric space (R,d) where d is the usual metric on R, boundary of set of integers z is :

a. ϕ b. Q
c. R d. Z

156. Consider the following statements :

(I) R^n under usual metric is complete
(II) Every subspace of a complete metric space is complete

a. I is true
b. II is true
c. I and II both are true
d. None of these

157. The diameter of a finite subset of a metric space is :

a. Finite b. Infinite
c. Does not exist d. None of these

158. The set Q of all rational number with usual metric is:

a. Complete
b. Not complete
c. May or may not be complete
d. None of these

159. Consider the following statements :

(I) The set $\left\{\frac{1}{2},\frac{-1}{2},\frac{2}{3},\frac{-2}{3}.....\frac{n}{n+1},\frac{-n}{n+1}\right\}$ has two limit points.

(II) The set]1,2[has no limit points.

a. I is true
b. II is true
c. I and II both are true
d. None of these

160. The metric space X is called complete if every Cauchy sequence in X is :

a. Convergent
b. Not convergent
c. May or may not be convergent
d. None of these

161. A metric space d on a non-empty set X is said to be bounded if there exists a real number $x > 0$ such that: **[Kanpur 2018]**

a. $d(x,y) \le k \;\forall x,y \in X$
b. $d(x,y) > k \;\forall x,y \in X$
c. $d(x,y) = \infty \;\forall x,y \in X$
d. None of these

162. Consider the following statements :

(I) The set of positive integers z^+ has no limit points

(II) Every point of $A = [2, 3]$ is limit points of A

a. I is true only
b. II is true
c. I and II both are true
d. None of these

163. The set of all real valued bounded continuous functions on [0,1] is :

a. Complete
b. Not complete
c. May or may not be complete
d. None of these

164. In a metric space (X,d) which one of the following is true : **[Kanpur 2018]**

a. Every singleton set is open set
b. ϕ and X are closed
c. Every subset is neither open nor closed
d. None of these

165. In the usual metric space (R,d) the interior of the set $A=\left\{\frac{1}{n} : n\in N\right\}$ is : **(Kanpur 2018)**

a. A
b. ϕ
c. $\{1\}$
d. $A\cup\{0\}$

166. Consider the statements :

(I) (R,U) is second countable

(II) (R^2,U) is second countable

a. I is true only
b. II is true
c. I and II both are true
d. None of these

167. In a usual metric space R, if $A=\left\{0,\frac{1}{2},\left(\frac{1}{2}\right)^2,....\right\}$ then :

a. $A\subset D(A)$
b. $A\supset D(A)$
c. $A=D(A)$
d. $D(A)=\phi$

168. If $A = [0, 1[$, $B =]1, 2]$ in a usual metric space on R then $D(A)\cap D(B)$ is equal to :

a. $\{1\}$
b. $[0,2]$
c. $]0,2[$
d. $\{0,1,2\}$

169. Consider the statements :

(I) Q is dense in R

(II) Z is no where dense in R

a. I is true only
b. II is true only
c. I and II both are true
d. None of these

170. If $d(x,y) = |x-y| \;\forall x,y\in R$ then $[a,b[$ is :

a. Open only
b. Closed only
c. Both open and closed
d. None of these

171. If A be a subset of a metric space (X,d) then :

a. $A\cup ext(A)=\phi$
b. $A\cap ext(A)=X$
c. $A\cap ext(A)=\phi$
d. $A\cup ext\ A=A$

ANSWERS

MULTIPLE CHOICE QUESTIONS

1.	(d)	**2.**	(c)	**3.**	(b)	**4.**	(b)	**5.**	(b)	**6.**	(a)	**7.**	(d)	**8.**	(a)	**9.**	(d)	**10.**	(a)
11.	(a)	**12.**	(c)	**13.**	(d)	**14.**	(c)	**15.**	(b)	**16.**	(c)	**17.**	(c)	**18.**	(c)	**19.**	(c)	**20.**	(d)
21.	(a)	**22.**	(b)	**23.**	(b)	**24.**	(c)	**25.**	(c)	**26.**	(b)	**27.**	(c)	**28.**	(c)	**29.**	(b)	**30.**	(c)
31.	(c)	**32.**	(d)	**33.**	(d)	**34.**	(d)	**35.**	(a)	**36.**	(b)	**37.**	(a)	**38.**	(c)	**39.**	(b)	**40.**	(a)
41.	(c)	**42.**	(d)	**43.**	(b)	**44.**	(b)	**45.**	(c)	**46.**	(c)	**47.**	(c)	**48.**	(b)	**49.**	(d)	**50.**	(b)
51.	(c)	**52.**	(d)	**53.**	(a)	**54.**	(b)	**55.**	(b)	**56.**	(c)	**57.**	(c)	**58.**	(d)	**59.**	(d)	**60.**	(a)
61.	(c)	**62.**	(a)	**63.**	(c)	**64.**	(a)	**65.**	(a)	**66.**	(d)	**67.**	(c)	**68.**	(c)	**69.**	(b)	**70.**	(a)
71.	(c)	**72.**	(a)	**73.**	(d)	**74.**	(a)	**75.**	(d)	**76.**	(d)	**77.**	(c)	**78.**	(c)	**79.**	(c)	**80.**	(b)
81.	(c)	**82.**	(c)	**83.**	(b)	**84.**	(d)	**85.**	(c)	**86.**	(d)	**87.**	(b)	**88.**	(b)	**89.**	(d)	**90.**	(d)
91.	(c)	**92.**	(d)	**93.**	(d)	**94.**	(b)	**95.**	(c)	**96.**	(b)	**97.**	(b)	**98.**	(d)	**99.**	(d)	**100.**	(d)
101.	(a)	**102.**	(a)	**103.**	(d)	**104.**	(d)	**105.**	(a)	**106.**	(d)	**107.**	(a)	**108.**	(c)	**109.**	(b)	**110.**	(b)
111.	(a)	**112.**	(a)	**113.**	(c)	**114.**	(c)	**115.**	(a)	**116.**	(b)	**117.**	(d)	**118.**	(c)	**119.**	(b)	**120.**	(d)
121.	(c)	**122.**	(d)	**123.**	(c)	**124.**	(a)	**125.**	(d)	**126.**	(b)	**127.**	(b)	**128.**	(c)	**129.**	(a)	**130.**	(c)
131.	(c)	**132.**	(c)	**133.**	(d)	**134.**	(b)	**135.**	(d)	**136.**	(a)	**137.**	(a)	**138.**	(c)	**139.**	(a)	**140.**	(b)
141.	(a)	**142.**	(b)	**143.**	(d)	**144.**	(b)	**145.**	(d)	**146.**	(d)	**147.**	(a)	**148.**	(c)	**149.**	(c)	**150.**	(a)
151.	(b)	**152.**	(c)	**153.**	(c)	**154.**	(b)	**155.**	(d)	**156.**	(a)	**157.**	(a)	**158.**	(b)	**159.**	(a)	**160.**	(a)
161.	(a)	**162.**	(c)	**163.**	(a)	**164.**	(b)	**165.**	(b)	**166.**	(c)	**167.**	(b)	**168.**	(a)	**169.**	(c)	**170.**	(d)
171.	(c)																		

HINTS AND SOLUTIONS

6. We have

$$d(x,z) \le d(x,y) + d(y,z) \text{ by } [M4]$$

$$= d(x,y) + d(z,y) \text{ by } [M3]$$

So, $d(x,y) \ge d(x,z) - d(z,y)$ ...(1)

Also, $d(z,y) \le d(z,x) + d(x,y)$ by $[M4]$

So, $d(x,y) \ge d(z,y) - d(x,z)$

$$= d(x,z) + d(x,y) \text{ by } [M3]$$

or $d(x,y) \ge -[d(x,z) - d(z,y)]$ ...(2)

By (1) and (2), we get

$$d(x,y) \ge |d(x,z) - d(z,y)|$$

10. Given that

$$d(z_1,z_2) = |z_1 - z_2|,\ \forall z_1,z_2 \in C$$

$[M1]$: we have $|z_1 - z_2| \ge 0\ \forall z_1,z_2 \in C$

$\therefore$ $d(z_1,z_2) \ge 0\ \forall z_1,z_2 \in C$

$[M2]$: since $|z_1 - z_2| = 0 \Leftrightarrow z_1 - z_2 = 0 \Leftrightarrow z_1 = z_2$

So, $d(z_1,z_2) = 0$ iff $z_1 = z_2$

$[M3]$ since $|z_1 - z_2| = |z_2 - z_1|\ \forall z_1,z_2 \in C$

So, $d(z_1,z_2) = d(z_2,z_1)\ \forall z_1,z_2 \in C$

$[M4]$: If $z_1,z_2,z_3 \in C$ then

$$|z_1 - z_2| = |z_1 - z_3 + z_3 - z_2|$$

$$\le |z_1 - z_3| + |z_3 - z_2|$$

So, $d(z_1, z_2) \le d(z_1, z_3) + d(z_3, z_2)$

So, d is a metric on C.

19. Given that

$$d(x,y) = |x - y| \text{ over the set } X = [0, 1]$$

then $S\left(\frac{1}{2}, 1\right) = \left\{x \in [0,1] : \left|x - \frac{1}{2}\right| < 1\right\}$

$$= \left\{x \in [0, 1] : \frac{1}{2} - 1 < x < \frac{1}{2} + 1\right\}$$

$$= \left\{x \in [0, 1] : -\frac{1}{2} < x < \frac{3}{2}\right\}$$

$$= [0,1]$$

22. Given that

$$d(x,y) = |x^2 - y^2| \ \forall x, y \in R$$

[$m1$] $\because$ $|x^2 - y^2| \ge 0 \ \forall x, y \in R$

So, $d(x,y) \ge 0 \ \forall x, y \in R$

[$m2$] $d(x,x) = |x^2 - x^2| = 0 \ \forall x \in R$

[$m3$] since $|x^2 - y^2| = |y^2 - x^2| \ \forall x, y \in R$

So, $d(x,y) = d(y,x) \ \forall x, y \in R$

[$m4$] Let $x, y, z \in R$ then

$$|x^2 - y^2| = |(x^2 - z^2) + (z^2 - y^2)|$$

$$\le |x^2 - z^2| + |z^2 - y^2|$$

$\therefore$ $d(x,y) \le d(x,z) + d(z,y) \ \forall x, y, z \in R$

Hence, d is a pseudo-metric on R. However d is not a metric space on R for,

$$d(x,y) = |x^2 - y^2| = 0$$

$\Rightarrow$ $x^2 - y^2 = 0 \ \Rightarrow \ x = \pm\, y$

Thus, $d(x,y) = 0$ does not necessarily imply that $x = y$.

24. $\because$ $d^X(x,y) = \dfrac{d(x,y)}{1 + d(x,y)}$

$\therefore$ $d^*(x,y) \le 1$ for every pair of points x, y of X.

$\because$ $0 \le \dfrac{d(x,y)}{1 + d(x,y)} < 1$

Hence d^* is a bounded metric for X with $\delta(X) \le 1$.

25. $d(x,y) = |x - y|$ over the set $X = \{0, 1]$

$\therefore$ $S\left(\frac{1}{16}, \frac{1}{16}\right) = \left\{x \in [0,1] : \frac{1}{16} - \frac{1}{16} < x < \frac{1}{16} + \frac{1}{16}\right\}$

$$= \left\{x \in [0, 1] : 0 < x < \frac{1}{8}\right\}$$

$$= \left]0, \frac{1}{8}\right[$$

26. Given that

$$d(x,y) = \begin{cases} 0 & \text{when } x = y \\ 1 & \text{when } x \ne y \end{cases}$$

Let $x_0 \in X$ and r is any positive real number greater than 1, then

$$S(x_0, r) = \{x \in X : d(x, y < 0) < r\}$$

$$= X$$

Since, $d(x, x_0) = 0$ or 1 each of which is less than r so that $x \in X \Rightarrow x \in S(x_0, r)$

If $0 < r \le 1$ then

$$S(x_0, r) = \{x \in X : d(x, x_0) < r\}$$

$$= \{x_0\}$$

Since, $d(x_0, x_0) = 0 < r$

and $d(x, x_0) = 1 \not< r$ if $x \ne x_0$

put $r = 1$ we get

$$S(x_0, 1) = \{x_0\}$$

29. Since, $d(x,y) = d(y,x)$, we may assume that

$$d(x, A) \ge d(y, A)$$

Let $\varepsilon > 0$ be given

Since, $d(y, A) = \inf\{d(y,z) : z \in A\}$

we can choose $a \in A$ such that

$$d(y,a) < d(y,A) + \varepsilon$$

then $|d(x,A) - d(y,A)| = d(x,A) - d(y,A)$

$$\le d(x,a) - d(y,A)$$

$$\le d(x,a) - d(y,a) + \varepsilon$$

$$\le d(x,y) + \varepsilon$$

Since, ε is arbitrary, we have

$$|d(x,A) - d(y,A)| \le d(x,y).$$

33. $\because$ $d(x,y) = |x-y|$ then

$$\cap\left\{\left]\frac{-1}{n},\frac{1}{n}\right[: n\in N\right\} = \{0\}$$

which is not open since there exists $n_0 r > 0$ such that $]-r, r[\subseteq \{0\}$.

38. $d(x,y) = |x-y|$ over $X = [0,1]$ then

$$S\left(0,\frac{1}{8}\right) = \left\{x\in[0,1] : 0-\frac{1}{8} < x < 0+\frac{1}{8}\right\}$$

$$= \left\{x\in[0,1] : \frac{-1}{8} < x < \frac{1}{8}\right\}$$

$$= \left[0,\frac{1}{8}\right[$$

Again

$$S\left(\frac{1}{32},\frac{3}{32}\right) = \left\{x\in[0,1] : \frac{1}{32}-\frac{3}{32} < x < \frac{1}{32}+\frac{3}{32}\right\}$$

$$= \left\{x\in\{0,1\} : \frac{-1}{16} < x < \frac{1}{8}\right\}$$

$$= \left[0,\frac{1}{8}\right[$$

$\therefore$ $S\left(0,\frac{1}{8}\right) = S\left(\frac{1}{32},\frac{3}{32}\right)$

47. If $d(x,y) = |x-y|\ \forall x,y\in R$ then

$$S(-1,1) =]-1-1, -1+1[=]-2,0[$$

53. Given that

$$d(x,y) = |x-y|$$

and $A = \{1,2\}, B = [2,4]$

Since, $d(A,B) = \inf\{d(x,y) : x\in A, y\in B\}$

then $d(5,B) = \inf\{d(5,y) : y\in B\}$

$= d(5,4) = 1$

54. $S\{(0,0),1\} = \{(x,y)\in R^2 : |x-0| + |y-0| < 1\}$

$= \{(x,y)\in R^2 : |x| + |y| < 1\}$

59. Given that

$$d(x,y) = |x-y|\ \forall x,y\in R^2$$

and $A = [1,2[$ then

$$d\left(\frac{5}{4},A\right) = \inf\left\{\left(\frac{5}{4},x\right) : x\in[1,2[\right\}$$

$$= 0, \text{ since } \frac{5}{4}\in[1,2[$$

60. Given that

$$d(z_1,z_2) = \max.\{|x_1-x_2|, |y_1-y_2|\}$$

So, $S\{(0,0),1\} = \{(x,y)\in R^2 : \max.\{|x-0|, |y-0|\}$

$= \{(x,y)\in R^2 : \max.\{|x|,|y|\} < 1\}$

61. Given that

$$d(x,y) = \begin{cases} 0 & \text{when } x = y \\ 1 & \text{when } x \neq y \end{cases}$$

Let $x_0\in X$, then

$$S\left(x_0,\frac{3}{2}\right) = \{x\in X : d(x,x_0) < 3/2\}$$

$$= X[\because d(x,x_0) = 0$$

or each of which is less than r so that $x\in X \Rightarrow x\in S(x_0,r)]$

65. $S\{(0,0),1\} = \{(x,y)\in R^2 : (x-0)^2 + (y-0)^2 = 1\}$

$\because$ $d(z_1,z_2) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

$\therefore$ $S\{(0,0),1\} = \{(x,y)\in R^2 : x^2+y^2 = 1\}$

76. Let S be any finite subset of X. Since S is finite so if $r > 0$, then $S(p,r)$ contains only finitely many points of the set S. Thus, $S(p,r)$ is a nbd of p which does not contain infinitely many points of S and so p is not a limit point of S. Thus, every $p\in X$ is not a limit point of S and so a finite set S has no limit points *i.e.* the derived set of a finite set is empty.

83. Let p be any limit point of the derived set $D(A)$. Then for every $r > 0$, the open sphere $S(p,r)$ contains infinitely many points of $D(A)$ and since each point of $D(A)$ is a limit point of A, every open sphere $S(p,r)$ must contain infinitely many points of A. Thus p is also a limit point of A and so $p\in D(A)$. Therefore $D(A)$ contains all its limit points and so $D(A)$ is closed.

93. $d(x,y) = |x-y|\ \forall x,y\in R$ and $A =]0,1[$

Since, A is an open set, it is a nbd of each of its points and so every point of A is its interior point.

Hence, $A^\circ = A =]0,1[$.

94. Let $p\in R$ and $\varepsilon > 0$ be given, Then $p-\varepsilon$ and $p+\varepsilon$ are two distinct real numbers and we know that between two distinct real numbers there lie infinitely

many rational numbers. Therefore for every $\in > 0$, the open interval $]p - \varepsilon, p + \varepsilon[$ contains at least over point of Q other than p. Hence, p is a limit point of Q. Since, p is an arbitrary real number so every real number is limit point of Q. Hence,

$$D(Q) = R$$

97. Given that $A = \left\{\frac{1}{n} : n \in N\right\}$ is a usual metric space (R, d)

Here, $D(A) = \{0\}$

So $\bar{A} = A \cup D(A) = \left\{\frac{1}{n} : n \in N\right\} \cup \{0\}$

101. Let A be any subset of a discrete metric space (X, d). If $A = \phi$, then A is open. If $A \neq \phi$, let x be an arbitrary point of A. Since, $s\left(x, \frac{1}{2}\right) = \{x\}$, we have $s\left(x, \frac{1}{2}\right) \leq A$. Hence, A is open.

106. Given that

$$d(x, y) = |x^2 - y^2| \ \forall x, y \in R$$

[$m1$] we have

$$|x^2 - y^2| \geq 0 \ \forall x, y \in R$$

So, $d(x, y) \geq 0 \ \forall x, y \in R$

$[M_2]$ $d(x, x) = |x^2 - x^2| = 0 \ \forall x \in R$

$[M_3]$ $|x^2 - y^2| = |y^2 - x^2| \ \forall x, y \in R$

So, $d(x, y) = d(y, x) \ \forall x, y \in R$

$[M_4]$ Let $x, y, z \in R$ then

$$|x^2 - y^2| = |(x^2 - z^2) + (z^2 - y^2)|$$
$$\leq |x^2 - z^2| + |z^2 - y^2|$$

So, $d(x, y) \leq d(x, z) + d(z, y) \ \forall x, y, z \in R$

Hence, d is a pseudo-metric on R. However d is not a metric on R. For if d is metric on R then $d(x, y) = 0$ iff $x = y$.

But $d(x, y) = |x^2 - y^2| = 0$

$\Rightarrow \quad x^2 - y^2 = 0 \ \Rightarrow \ x = \pm y$

Thus, $d(x, y) = 0$ does not necessarily imply that $x = y$. For example,

$$d(2, -2) = |2^2 - (-2)^2| = 0 \text{ while } 2 \neq -2$$

Hence, d is not a metric on R.

107. Given that

$$d(x, y) = \frac{|x - y|}{1 + |x - y|} \ \forall x, y \in R$$

[$m1$] $|x - y| \geq 0 \ \forall x, y \in R$

so $\frac{|x - y|}{1 + |x - y|} \geq 0 \ \forall x, y \in R$

$\Rightarrow \quad d(x, y) \geq 0 \ \forall x, y \in R$

[$m2$] $d(x, y) = 0 \ \Leftrightarrow \ \frac{|x - y|}{1 + |x - y|} = 0$

$\Rightarrow \quad |x - y| = 0 \ \Leftrightarrow \ x = y$

[$m3$] $d(x, y) = \frac{|x - y|}{1 + |x - y|} = \frac{|y - x|}{1 + |y - x|}$

$$= d(x, y) \ \forall x, y \in R$$

[$m4$] $|x - y| = |x - z + z - y| \leq |x - z| + |z - y|$

$\therefore \quad 1 + |x - y| \leq 1 + |x - z| + |z - y|$

or $\frac{1}{1 + |x - y|} \geq \frac{1}{1 + |x - z| + |z - y|}$

or $-\frac{1}{1 + |x - y|} \leq -\frac{1}{1 + |x - z| + |z - y|}$

$$1 - \frac{1}{1 + |x - y|} \leq 1 - \frac{1}{1 + |x - z| + |z - y|}$$

or $\frac{|x - y|}{1 + |x - y|} \leq \frac{|x - z| + |z - y|}{1 + |x - z| + |z - y|}$

or $\frac{|x - y|}{1 + |x - y|} \leq \frac{|x - z|}{1 + |x - z| + |z - y|} + \frac{|z - y|}{1 + |x - z| + |z - y|}$

or $\frac{|x - y|}{1 + |x - y|} \leq \frac{|x - z|}{1 + |x - z|} + \frac{|z - y|}{1 + |z - y|}$

or $d(x, y) \leq d(x, z) + d(z, y) \ \forall x, y, z \in R$

Thus d is a metric space.

Now it can be easily shown that for $d(x, y) = \frac{|x - y|}{1 - |x - y|} \ \forall x, y \in R$, the fourty property does not hold *i.e.* it is not a metric space.

110. Given that $d^*(x,y) = \min\{1, d(x,y)\}$ $\forall x,y \in X$ and d is a metric on X.

$[m_1]$: since d is a metric on X, so

$$d(x,y) \geq 0 \ \forall x,y \in X$$

Now, $d^*(x,y) = 1$ or $d^*(x,y) = d(x,y)$

So, $d^*(x,y) \geq 0 \ \forall x,y \in X$

$[m2]$: If $d^*(x,y) = 0$, then

$$d^*(x,y) = d(x,y) = 0$$

Since, d is a metric, so

$$d(x,y) = 0 \ \Rightarrow \ x = y$$

Again if $x = y$ then $d(x,y) = 0$ and so

$$d^*(x,y) = d(x,y) = 0$$

Hence, $d^*(x,y) = 0$ iff $x = y$

$[m3]$: we have either

$$d^*(x,y) = d(x,y) \text{ or } d^*(x,y) = 1$$

If $d^*(x,y) = d(x,y)$ then $d(x,y) < 1$

Hence, $d(y,x) = d(x,y) < 1$

But $d(y,x) < 1$

$\Rightarrow$ $d^*(y,x) = d(y,x) = d(x,y) = d^*(x,y)$

A and if $d^*(x,y) = 1$ then

$$d(x,y) \geq 1 \text{ and so } d(y,x) \geq 1$$

But $d(y,x) \geq 1 \ \Rightarrow \ d^*(y,x) = 1$

Hence, $d^*(x,y) = d^*(y,x)$

Thus in either case,

$$d^*(x,y) = d^*(y,x)$$

$[m4]$: we show that

$$d^*(x,y) \leq d^*(x,z) + d^*(z,y) \quad ...(1)$$

Now, $d^*(x,y) \leq 1$

Hence, if either

$$d^*(x,z) = 1 \text{ or } d^*(z,y) \neq 1$$

then $d^*(x,z) = d(x,z)$ and $d^*(z,y) = d(z,y)$

So we have, $d(x,y) \leq d(x,z) + d(z,y)$

$$= d^*(x,z) + d^*(z,y) \quad ...(2)$$

But $d^*(x,y) = \min\{1, d(x,y)\} \leq d(x,y)$ ...(3)

By (2) and (3), we get

$$d^*(x,y) \leq d^*(x,z) + d^*(z,y)$$

Hence, d^* is a metric for X.

Since, $d^*(x,y) \leq 1$ for every pair of points $x,y \in X$ therefore d^* is a bounded metric for X with $\delta(X) \leq 1$.

It can be easily shown that for

$$d^*(x,y) = \max.\{1, d(x,y)\}$$

fourth property does not hold hence it is not a metric.

113. Consider the usual metric space (R,d) *i.e.* $d: R \times R \rightarrow R$ defined by

$$d(x,y) = |x - y| \ \forall x,y \in R$$

then to each $x \in R$ the family.

$\left\{\left]x - \frac{1}{n}, x + \frac{1}{n}\right[: n \in N\right\}$ is a countable local base at x r_0 (R,d) is first countable. Again the collection of all open intervals $]a,b[$ where a,b are rational numbers forms a countable base for G_1 where G is the family of all open subsets of X. So, (R,d) is second countable also.

115. Let (X,d) be a metric space and $x \in X$. Consider the collections of open spheres $\beta(x) = \left\{S\left(x, \frac{1}{n}\right) : n \in N\right\}$.

We claim that this collection form a countable local base at x. Let N be a nbd of x, then there exists an open set G such that

$$x \in G \subseteq N$$

By definition of open sets, there exists $\varepsilon > 0$ such that $S(n, \varepsilon) \subseteq G \subseteq N$

Choose n so large that $\frac{1}{n} < \varepsilon$ then

$$S\left(x, \frac{1}{n}\right) \subseteq S(x, \varepsilon) \subseteq G \subseteq N$$

Thus every nbd of x contains a member of $\beta(x)$ and so $\beta(x)$ forms a countable local base at x. Thus (X,d) is first countable.

124. Given that

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

where d is a metric

[m1] $\because$ $d(x,y) \geq 0$ so $\frac{d(x,y)}{1+d(x,y)} \geq 0$

i.e., $d'(x,y) \geq 0 \ \forall x,y \in X$

[m2] $d'(x,y) = 0 \Leftrightarrow \frac{d(x,y)}{1+d(x,y)} = 0 \Leftrightarrow d'(y,x) = 0$

$$\Leftrightarrow x = y$$

[m3] $d'(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = d'(y,x)$

[m4] Let $x,y,z \in X$ then,

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)} = 1 - \frac{1}{1+d(x,y)}$$

$$\leq 1 - \frac{1}{1+d(x,z)+d(z,y)} = \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)}$$

$$= \frac{d(x,z)}{1+d(x,z)+d(z,y)} + \frac{d(z,y)}{1+d(x,z)+d(z,y)}$$

$$\leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)}$$

$$\leq d(x,z) + d'(z,y)$$

So, d' is a metric for X.

137. Let $< x_n >$ be a convergent sequence in a metric space (X,d). Take $\varepsilon > 0$ and $< x_n >$ converges to $x_0 \in X$ so there exists $m \in N$ such that

$$d(x_n, x_0) < \frac{\varepsilon}{2} \ \forall n \geq m$$

In particular $d(x_m, x_0) < \frac{\varepsilon}{2}$

Now for $n \geq m$ we have

$$d(x_n, x_m) \leq d(x_n, x_0) + d(x_0, x_m)$$

$$= d(x_n, x_0) + d(x_m, x_0)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, for $\in > 0$, there exists $m \in N$ such that

$$d(x_m, x_n) < \varepsilon \ \forall n \geq m$$

Hence, $< x_n >$ is a Cauchy sequence in X.

Converge is not necessarily true for

$$d(x,y) = |x-y| \ \forall x,y \in X$$

and $X =]0,1]$

Consider $< x_n >$ in X such that

$$x_n = \frac{1}{n} \ \forall n \in N$$

Obviously $0 < x_n = \frac{1}{n} \leq 1 \ \forall n \in N$ so $x_n \in X$

Also $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} = 0$, but $0 \notin X$

so $< x_n >$ is not converge to a point of X.

$< x_n >$ is a Cauchy sequence in X for,

Choose $m > \frac{1}{\varepsilon}$ or $\frac{1}{m} < \varepsilon$.

Now for all $n \geq m$, we have

$$|x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| = \left|\frac{m-n}{mn}\right| = \frac{n-m}{nm} < \frac{1}{m}$$

or $|x_n - x_m| < \varepsilon$

Thus, for $\varepsilon > 0$ there exists $m \in N$ such that

$$d(x_n, x_m) = |x_n - x_m| < \varepsilon \ \forall n \geq m$$

Thus, $< x_n >$ is a Cauchy sequence in X.

141. See the solution of Questions (137).

143. Given that $F_n = \left[0, \frac{n}{n+1}\right] \ \forall n \in N$, then

$$\bigcup_{n=1}^{\infty} F_n = \left[0, \frac{1}{2}\right] \cup \left[0, \frac{2}{3}\right] \cup \ldots \{0\} = [0,1[$$

152. $A = [a,b]$

Here every point of A is its interior point except at a and b, so

$$A^\circ = A - \{a,b\} =]a,b[$$

Also $A' =]-\infty, a[\cup]b, \infty[$

Hence, ext $A = (A')^\circ =]-\infty, a[\cup]b, \infty[$

$\therefore$ $Fr(A) = [A^\circ \cup \text{ext}(A)]'$

$$= [\,]a,b[\cup]-\infty, a[\cup]b,\infty[\,]'$$

$$= \{a,b\}$$

155. $\bar{z} = z \cup D(z) = z \cup \phi$ $\quad \because D(z) = \phi$

So, $\bar{z} = z$

Also z is not a neighbourhood of any of its points. So no point of z is an interior point of z.

$\therefore$ $Z^\circ = \phi$

Now, $Fr(z) = \bar{z} - z^\circ = z - \phi = z$

So boundary of $z = \{x : x \in Fr(z) \text{ and } x \in z\}$

$= z$

159. Let $A = \left\{\frac{1}{2}, \frac{-1}{2}, \frac{2}{3}, \frac{-2}{3} \ldots \frac{n}{n+1}, \frac{-n}{n+1}\right\}$

then A has exactly two limit points and they are $+1$ and -1. Again let $B =]1,2[$ then B has infinite number of limit points. Each point of the closed interval $[1,2]$ is its limit point.

165. Given that $A = \left\{\frac{1}{n} : n \in N\right\}$

then A cannot be a nbd of any of its points $\frac{1}{n}$, $n = 1, 2, 3\ldots$, since there exists x_0 $\varepsilon > 0$ such that

$\left]\frac{1}{n} - \varepsilon, \frac{1}{n} + \varepsilon\right[\leq A.$

Hence, no point of A can be its interior point so that $A^\circ = \phi$.

167. $A = \left\{0, \frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \ldots\right\}$

The only limit point of A is 0 and r_0 $D(A) = \{0\}$. Thus,

$D(A) \subseteq A$

168. $d(x, y) = |x - y| \; \forall x, y \in R$

and $A = [0, 1[, B =]1, 2]$

So, $D(A) = [0, 1]$

and $D(B) = [1, 2]$

$\therefore$ $D(A) \cap D(B) = \{0, 1\} \cap [1, 2] = \{1\}$

169. $\because$ $Q \subseteq R$

So, $\bar{Q} = Q \cup D(Q)$

$= Q \cup R = R$

So, Q is dense in R.

Again $Z \subseteq R$ so $\bar{Z} = Z \cup D(Z) = Z \cup \phi = Z$

$\therefore$ $(\bar{Z})^\circ = Z^\circ = \phi$

Hence, Z is nowhere dense in R.

OOO