THIRD EDITION

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Introduction to PARTIAL DIFFERENTIAL EQUATIONS

K. Sankara Rao

PHI

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K. SANKARA RAO
Formerly Professor
Department of Mathematics
Anna University, Chennai

PHI Learning Private Limited
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INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS, Third Edition
K. Sankara Rao

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This book is dedicated with affection and gratitude to
the memory of my respected Father

(Late) KOMMURI VENKATESWARLU

and

to my respected Mother

SHRIMATI VENKATARATNAMMA

Contents

Preface

The objective of this third edition is the same as in previous two editions: to provide a broad coverage of various mathematical techniques that are widely used for solving and to get analytical solutions to Partial Differential Equations of first and second order, which occur in science and engineering. In fact, while writing this book, I have been guided by a simple teaching philosophy: An ideal textbook should teach the students to solve problems. This book contains hundreds of carefully chosen worked-out examples, which introduce and clarify every new concept. The core material presented in the second edition remains unchanged.

I have updated the previous edition by adding new material as suggested by my active colleagues, friends and students.

Chapter 1 has been updated by adding new sections on both homogeneous and non-homogeneous linear PDEs, with constant coefficients, while Chapter 2 has been repeated as such with the only addition that a solution to Helmholtz equation using variables separable method is discussed in detail.

In Chapter 3, few models of non-linear PDEs have been introduced. In particular, the exact solution of the IVP for non-linear Burger's equation is obtained using Cole–Hopf function.

Chapter 4 has been updated with additional comments and explanations, for better understanding of normal modes of vibrations of a stretched string.

Chapters 5–7 remain unchanged.

I wish to express my gratitude to various authors, whose works are referred to while writing this book, as listed in the Bibliography. Finally, I would like to thank all my old colleagues, friends and students, whose feedback has helped me to improve over previous two editions.

It is also a pleasure to thank the publisher, PHI Learning, for their careful processing of the manuscript both at the editorial and production stages.

Any suggestions, remarks and constructive comments for the improvement of text are always welcome.

K. SANKARA RAO

Preface to the First and Second Edition

With the remarkable advances made in various branches of science, engineering and technology, today, more than ever before, the study of partial differential equations has become essential. For, to have an in-depth understanding of subjects like fluid dynamics and heat transfer, aerodynamics, elasticity, waves, and electromagnetics, the knowledge of finding solutions to partial differential equations is absolutely necessary.

This book on Partial Differential Equations is the outcome of a series of lectures delivered by me, over several years, to the postgraduate students of Applied Mathematics at Anna University, Chennai. It is written mainly to acquaint the reader with various well-known mathematical techniques, namely, the variables separable method, integral transform techniques, and Green's function approach, so as to solve various boundary value problems involving parabolic, elliptic and hyperbolic partial differential equations, which arise in many physical situations. In fact, the Laplace equation, the heat conduction equation and the wave equation have been derived by taking into account certain physical problems.

The book has been organized in a logical order and the topics are discussed in a systematic manner. In Chapter 0, partial differential equations of first order are dealt with. In Chapter 1, the classification of second order partial differential equations, and their canonical forms are given. The concept of adjoint operators is introduced and illustrated through examples, and Riemann's method of solving Cauchy's problem described. Chapter 2 deals with elliptic differential equations. Also, basic mathematical tools as well as various properties of harmonic functions are discussed. Further, the Dirichlet and Neumann boundary value problems are solved using variables separable method in cartesian, cylindrical and spherical coordinate systems. Chapter 3 is devoted to a discussion on the solution of boundary value problems describing the parabolic or diffusion equation in various coordinate systems using the variables separable method. Elementary solutions are also given. Besides, the maximum-minimum principle is discussed, and the concept of Dirac delta function is introduced along with a few properties. Chapter 4 provides a detailed study of the wave equation representing the hyperbolic partial differential equation, and gives D'Alembert's solution.

In addition, the chapter presents problems like vibrating string, vibration of a circular membrane, and periodic solutions of wave equation, shows the uniqueness of the solutions, and illustrates Duhamel's principle. Chapter 5 introduces the basic concepts in the construction of

Green's function for various boundary value problems using the eigenfunction method and the method of images. Chapter 6 on Laplace transform method is self-contained since the subject matter has been developed from the basic definition. Various properties of the transform and inverse transform are described and detailed proofs are given, besides presenting the convolution theorem and complex inversion formula. Further, the Laplace transform methods are applied to solve several initial value, boundary value and initial boundary value problems. Finally in Chapter 7, the theory of Fourier transform is discussed in detail. Finite Fourier transforms are also introduced, and their applications to diffusion, wave and Laplace equations have been analyzed.

The text is interspersed with solved examples; also, miscellaneous examples are given in most of the chapters. Exercises along with hints are provided at the end of each chapter so as to drill the student in problem-solving. The preprequisites for the book include a knowledge of advanced calculus, Fourier series, and some understanding about ordinary differential equations and special functions.

The book is designed as a textbook for a first course on partial differential equations for the senior undergraduate engineering students and postgraduate students of applied mathematics, physics and engineering. The various topics covered in the book can be taught either in one semester or in two semesters depending on the syllabi. The book would also be of interest to scientists and engineers engaged in research.

In the second edition, I have added a new chapter (Partial Differential Equations of First Order). Also, some additional examples are included, which are taken from question papers for GATE in the last 10 years. This, I believe, would surely benefit students intending to appear for the GATE examination.

I am indebted to many of my colleagues in the Department of Mathematics, particularly to Prof. N. Muthiyalu, Prof. Prabhamani, R. Patil, Dr. J. Pandurangan, Prof. K. Manivachakan, for their many useful comments and suggestions. I am also grateful to the authorities of Anna University, for the encouragement and inspiration provided by them.

I wish to thank Mr. M.M. Thomas for the excellent typing of the manuscript. Besides, my gratitude and appreciation are due to the Publishers, PHI Learning, for the very careful and meticulous processing of the manuscript, both during the editorial and production stages.

Finally, I sincerely thank my wife, Leela, daughter Aruna and son-in-law R. Parthasarathi, for their patience and encouragement while writing this book. I also appreciate the understanding shown by my granddaughter Sangeetha who had to forego my attention and care during the course of my book writing.

Any constructive comments for improving the contents of this volume will be warmly appreciated.

K. SANKARA RAO

CHAPTER 0

Partial Differential Equations of First Order

0.1 INTRODUCTION

Partial differential equations of first order occur in many practical situations such as Brownian motion, the theory of stochastic processes, radioactive disintegration, noise in communication systems, population growth and in many problems dealing with telephone traffic, traffic flow along a highway and gas dynamics and so on. In fact, their study is essential to understand the nature of solutions and forms a guide to find the solutions of higher order partial differential equations.

A first order partial differential equation (usually denoted by PDE) in two independent variables x, y and one unknown z, also called dependent variable, is an equation of the form

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0. \tag{0.1}$$

Introducing the notation

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y} \tag{0.2}$$

Equation (0.1) can be written in symbolic form as

$$F(x, y, z, p, q) = 0. \tag{0.3}$$

A solution of Eq. (0.1) in some domain Ω of $\mathbb{R}^2$ is a function $z = f(x, y)$ defined and is of C' in Ω should satisfy the following two conditions:

(i) For every $(x, y) \in \Omega$, the point (x, y, z, p, q) is in the domain of the function F.

(ii) When $z = f(x, y)$ is substituted into Eq. (0.1), it should reduce to an identity in x, y for all $(x, y) \in \Omega$.

We classify the PDE of first order depending upon the form of the function F. An equation of the form

$$P(x, y, z)\frac{\partial z}{\partial x} + Q(x, y, z)\frac{\partial z}{\partial y} = R(x, y, z) \tag{0.4}$$

is a quasi-linear PDE of first order, if the derivatives $\partial z/\partial x$ and $\partial z/\partial y$ that appear in the function F are linear, while the coefficients P, Q and R depend on the independent variables x, y and also on the dependent variable z. Similarly, an equation of the form

$$P(x, y)\frac{\partial z}{\partial x} + Q(x, y)\frac{\partial z}{\partial y} = R(x, y, z) \tag{0.5}$$

is called almost linear PDE of first order, if the coefficients P and Q are functions of the independent variables only. An equation of the form

$$a(x, y)\frac{\partial z}{\partial x} + b(x, y)\frac{\partial z}{\partial y} + c(x, y)z = d(x, y) \tag{0.6}$$

is called a linear PDE of first order, if the function F is linear in $\partial z/\partial x$, $\partial z/\partial y$ and z, while the coefficients a, b, c and d depend only on the independent variables x and y. An equation which does not fit into any of the above categories is called non-linear. For example,

(i) $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = nz$

is a linear PDE of first order.

(ii) $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z^2$

is an almost linear PDE of first order.

(iii) $P(z)\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$

is a quasi-linear PDE of first order.

(iv) $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 = 1$

is a non-linear PDE of first order.

Before discussing various methods for finding the solutions of the first order PDEs, we shall review some of the basic definitions and concepts needed from calculus.

0.2 SURFACES AND NORMALS

Let Ω be a domain in three-dimensional space $\mathbb{R}^3$ and suppose $F(x, y, z)$ is a function in the class $C'(\Omega)$, then the vector valued function grad F can be written as

$$\text{grad } F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) \tag{0.7}$$

If we assume that the partial derivatives of F do not vanish simultaneously at any point then the set of points (x, y, z) in Ω, satisfying the equation

$$F(x, y, z) = C \tag{0.8}$$

is a surface in Ω for some constant C. This surface denoted by S_C is called a level surface of F. If (x_0, y_0, z_0) is a given point in Ω, then by taking $F(x_0, y_0, z_0) = C$, we get an equation of the form

$$F(x, y, z) = F(x_0, y_0, z_0), \tag{0.9}$$

which represents a surface in Ω, passing through the point (x_0, y_0, z_0). Here, Eq. (0.8) represents a one-parameter family of surface in Ω. The value of grad F is a vector, normal to the level surface. Now, one may ask, if it is possible to solve Eq. (0.8) for z in terms of x and y. To answer this question, let us consider a set of relations of the form

$$x = f_1(u, v), \qquad y = f_2(u, v), \qquad z = f_3(u, v) \tag{0.10}$$

Here for every pair of values of u and v, we will have three numbers x, y and z, which represents a point in space. However, it may be noted that, every point in space need not correspond to a pair u and v. But, if the Jacobian

$$\frac{\partial(f_1, f_2)}{\partial(u, v)} \neq 0 \tag{0.11}$$

then, the first two equations of (0.10) can be solved and u and v, can be expressed as functions of x and y like

$$u = \lambda(x, y), \qquad v = \mu(x, y).$$

Thus, u and v are obtained once x and y are known, and the third relation of Eq. (0.10) gives the value of z in the form

$$z = f_3[\lambda(x, y), \mu(x, y)] \tag{0.12}$$

This is, of course, a functional relation between the coordinates x, y and z as in Eq. (0.8). Hence, any point (x, y, z) obtained from Eq. (0.10) always lie on a fixed surface. Equations (0.10) are also called parametric equations of a surface. It may be noted that the parametric equation of a surface need not be unique, which can be seen in the following example:

The parametric equations

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta$$

and

$$x = r\frac{(1-\phi^2)}{(1+\phi^2)}\cos\theta, \qquad y = r\frac{(1-\phi^2)}{(1+\phi^2)}\sin\theta, \qquad z = \frac{2r\phi}{1+\phi^2}$$

both represent the same surface $x^2 + y^2 + z^2 = r^2$ which is a sphere, where r is a constant.

If the equation of the surface is of the form

$$z = f(x, y) \tag{0.13}$$

Then

$$F = f(x, y) - z = 0. \tag{0.14}$$

Differentiating partially with respect to x and y, we obtain

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0, \qquad \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0$$

from which we get

$$\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} = \frac{\partial F}{\partial x} \text{ (using 0.14)}$$

or

$$\frac{\partial F}{\partial x} = p.$$

Similarly, we obtain

$$\frac{\partial F}{\partial y} = q \quad \text{and} \quad \frac{\partial F}{\partial z} = -1.$$

Hence, the direction cosines of the normal to the surface at a point (x, y, z) are given as

$$\left(\frac{p}{\sqrt{p^2+q^2+1}}, \frac{q}{\sqrt{p^2+q^2+1}}, \frac{-1}{\sqrt{p^2+q^2+1}}\right) \tag{0.15}$$

Now, returning to the level surface given by Eq. (0.8), it is easy to write the equation of the tangent plane to the surface S_c at a point (x_0, y_0, z_0) as

$$(x - x_0)\left[\frac{\partial F}{\partial x}(x_0, y_0, z_0)\right] + (y - y_0)\left[\frac{\partial F}{\partial y}(x_0, y_0, z_0)\right] + (z - z_0)\left[\frac{\partial F}{\partial z}(x_0, y_0, z_0)\right] = 0. \tag{0.16}$$

0.3 CURVES AND THEIR TANGENTS

A curve in three-dimensional space $\mathbb{R}^3$ can be described in terms of parametric equations. Suppose $\vec{r}$ denotes the position vector of a point on a curve C, then the vector equation of C may be written as

$$\vec{r} = \vec{F}(t) \qquad \text{for } t \in I, \tag{0.17}$$

where I is some interval on the real axis. In component form, Eq. (0.17) can be written as

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t) \tag{0.18}$$

where $\vec{r} = (x, y, z)$ and $\vec{F}(t) = [f_1(t), f_2(t), f_3(t)]$ and the functions f_1, f_2 and f_3 belongs to $C'(I)$.

Further, we assume that

$$\left(\frac{df_1(t)}{dt}, \frac{df_2(t)}{dt}, \frac{df_3(t)}{dt}\right) \neq (0, 0, 0) \tag{0.19}$$

This non-vanishing vector is tangent to the curve C at the point (x, y, z) or at $[f_1(t), f_2(t), f_3(t)]$ of the curve C.

Another way of describing a curve in three-dimensional space $\mathbb{R}^3$ is by using the fact that the intersection of two surfaces gives rise to a curve.

Let

$$\left.\begin{aligned} F_1(x, y, z) &= C_1 \\ F_2(x, y, z) &= C_2 \end{aligned}\right\} \tag{0.20}$$

and

are two surfaces. Their intersection, if not empty, is always a curve, provided grad F_1 and grad F_2 are not collinear at any point of Ω in $\mathbb{R}^3$. In other words, the intersection of surfaces given by Eq. (0.20) is a curve if

$$\text{grad } F_1(x, y, z) \times \text{grad } F_2(x, y, z) \neq (0, 0, 0) \tag{0.21}$$

for every $(x, y, z) \in \Omega$. For various values of C_1 and C_2, Eq. (0.20) describes different curves. The totality of these curves is called a two parameter family of curves. Here, C_1 and C_2 are referred as parameters of this family. Thus, if we have two surfaces denoted by S_1 and S_2 whose equations are in the form

$$\left.\begin{aligned} F(x, y, z) &= 0 \\ G(x, y, z) &= 0 \end{aligned}\right\} \tag{0.22}$$

and

Then, the equation of the tangent plane to S_1 at a point $P(x_0, y_0, z_0)$ is

$$(x - x_0)\frac{\partial F}{\partial x} + (y - y_0)\frac{\partial F}{\partial y} + (z - z_0)\frac{\partial F}{\partial z} = 0 \tag{0.23}$$

Similarly, the equation of the tangent plane to S_2 at the point $P(x_0, y_0, z_0)$ is

$$(x - x_0)\frac{\partial G}{\partial x} + (y - y_0)\frac{\partial G}{\partial y} + (z - z_0)\frac{\partial G}{\partial z} = 0. \tag{0.24}$$

Here, the partial derivatives $\partial F/\partial x$, $\partial G/\partial x$, etc. are evaluated at $P(x_0, y_0, z_0)$. The intersection of these two tangent planes is the tangent line L at P to the curve C, which is the intersection of the surfaces S_1 and S_2. The equation of the tangent line L to the curve C at (x_0, y_0, z_0) is obtained from Eqs. (0.23) and (0.24) as

$$\frac{(x - x_0)}{\dfrac{\partial F}{\partial y}\dfrac{\partial G}{\partial z} - \dfrac{\partial F}{\partial z}\dfrac{\partial G}{\partial y}} = \frac{(y - y_0)}{\dfrac{\partial F}{\partial z}\dfrac{\partial G}{\partial x} - \dfrac{\partial F}{\partial x}\dfrac{\partial G}{\partial z}} = \frac{(z - z_0)}{\dfrac{\partial F}{\partial x}\dfrac{\partial G}{\partial y} - \dfrac{\partial F}{\partial y}\dfrac{\partial G}{\partial x}} \tag{0.25}$$

or

$$\frac{(x-x_0)}{\dfrac{\partial(F,G)}{\partial(y,z)}} = \frac{(y-y_0)}{\dfrac{\partial(F,G)}{\partial(z,x)}} = \frac{(z-z_0)}{\dfrac{\partial(F,G)}{\partial(x,y)}} \tag{0.26}$$

Therefore, the direction cosines of L are proportional to

$$\left[\frac{\partial(F,G)}{\partial(y,z)}, \frac{\partial(F,G)}{\partial(z,x)}, \frac{\partial(F,G)}{\partial(x,y)}\right]. \tag{0.27}$$

For illustration, let us consider the following examples:

EXAMPLE 0.1 Find the tangent vector at $(0, 1, \pi/2)$ to the helix described by the equation

$$x = \cos t, \quad y = \sin t, \quad z = t, \quad t \in I \text{ in } \mathbb{R}'.$$

Solution The tangent vector to the helix at (x, y, z) is

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (-\sin t, \cos t, 1).$$

We observe that the point $(0, 1, \pi/2)$ corresponds to $t = \pi/2$. At this point $(0, 1, \pi/2)$, the tangent vector to the given helix is $(-1, 0, 1)$.

EXAMPLE 0.2 Find the equation of the tangent line to the space circle

$$x^2 + y^2 + z^2 = 1, \quad x + y + z = 0$$

at the point $(1/\sqrt{14},\ 2/\sqrt{14},\ -3/\sqrt{14})$.

Solution The space circle is described as

$$F(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$$

$$G(x, y, z) = x + y + z = 0$$

Recalling Eq. (0.25), the equation of the tangent plane at $(1/\sqrt{14},\ 2/\sqrt{14},\ -3/\sqrt{14})$ can be written as

$$\frac{x - 1/\sqrt{14}}{2 \times \dfrac{2}{\sqrt{14}} - 2\left(\dfrac{-3}{\sqrt{14}}\right)} = \frac{y - 2/\sqrt{14}}{2\left(\dfrac{-3}{\sqrt{14}}\right) - 2\left(\dfrac{1}{\sqrt{14}}\right)}$$

$$= \frac{z + 3/\sqrt{14}}{2\left(\dfrac{1}{\sqrt{14}}\right) - 2\left(\dfrac{2}{\sqrt{14}}\right)}$$

or

$$\frac{x - 1/\sqrt{14}}{10/\sqrt{14}} = \frac{y - 2/\sqrt{14}}{-8/\sqrt{14}} = \frac{z + 3/\sqrt{14}}{-2/\sqrt{14}}.$$

0.4 FORMATION OF PARTIAL DIFFERENTIAL EQUATION

Suppose u and v are any two given functions of x, y and z. Let F be an arbitrary function of u and v of the form

$$F(u, v) = 0 \tag{0.28}$$

We can form a differential equation by eliminating the arbitrary function F. For, we differentiate Eq. (0.28) partially with respect to x and y to get

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p\right] + \frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p\right] = 0 \tag{0.29}$$

and

$$\frac{\partial F}{\partial u}\left[\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q\right] + \frac{\partial F}{\partial v}\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q\right] = 0 \tag{0.30}$$

Now, eliminatin $\partial F/\partial u$ and $\partial F/\partial v$ from Eqs. (0.29) and (0.30), we obtain

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial z} p & \dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial z} p \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} q & \dfrac{\partial v}{\partial y} + \dfrac{\partial v}{\partial z} q \end{vmatrix} = 0$$

which simplifies to

$$p\frac{\partial (u, v)}{\partial (y, z)} + q\frac{\partial (u, v)}{\partial (z, x)} = \frac{\partial (u, v)}{\partial (x, y)}. \tag{0.31}$$

This is a linear PDE of the type

$$Pp + Qq = R, \tag{0.32}$$

where

$$P = \frac{\partial (u, v)}{\partial (y, z)}, \quad Q = \frac{\partial (u, v)}{\partial (z, x)}, \quad R = \frac{\partial (u, v)}{\partial (x, y)}. \tag{0.33}$$

Equation (0.32) is called Lagrange's PDE of first order. The following examples illustrate the idea of formation of PDE.

EXAMPLE 0.3 Form the PDE by eliminating the arbitrary function from

(i) $z = f(x + it) + g(x - it)$, where $i = \sqrt{-1}$

(ii) $f(x + y + z,\ x^2 + y^2 + z^2) = 0.$

Solution

(i) Given $z = f(x + it) + g(x - it)$ (1)

Differentiating Eq. (1) twice partially with respect to x and t, we get

$$\frac{\partial z}{\partial x} = f'(x+it) + g'(x-it)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+it) + g''(x-it). \tag{2}$$

Here, f' indicates derivative of f with respect to $(x+it)$ and g' indicates derivative of g with respect to $(x-it)$. Also, we have

$$\frac{\partial z}{\partial t} = if'(x+it) - ig'(x-it)$$

$$\frac{\partial^2 z}{\partial t^2} = -f''(x+it) - g''(x-it). \tag{3}$$

From Eqs. (2) and (3), we at once, find that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0, \tag{4}$$

which is the required PDE.

(ii) The given relation is of the form

$$\phi(u, v) = 0,$$

where $u = x + y + z$, $v = x^2 + y^2 + z^2$

Hence, the required PDE is of the form

$$Pp + Qq = R, \text{ (Lagrange equation)} \tag{1}$$

where

$$P = \frac{\partial(u, v)}{\partial(y, z)} = \begin{vmatrix} \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial z} & \dfrac{\partial v}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 2y \\ 1 & 2z \end{vmatrix} = 2(z - y)$$

$$Q = \frac{\partial(u, v)}{\partial(z, x)} = \begin{vmatrix} \dfrac{\partial u}{\partial z} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} 1 & 2z \\ 1 & 2x \end{vmatrix} = 2(x - z)$$

and

$$R = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 2x \\ 1 & 2y \end{vmatrix} = 2(y - x)$$

Hence, the required PDE is

$$2(z-y)p + 2(x-z)q = 2(y-x)$$

or

$$(z-y)p + (x-z)q = y - x.$$

EXAMPLE 0.4 Eliminate the arbitrary function from the following and hence, obtain the corresponding partial differential equation:

(i) $z = xy + f(x^2 + y^2)$

(ii) $z = f(xy/z)$.

Solution

(i) Given $z = xy + f(x^2 + y^2)$ (1)

Differentiating Eq. (1) partially with respect to x and y, we obtain

$$\frac{\partial z}{\partial x} = y + 2xf'(x^2 + y^2) = p \tag{2}$$

$$\frac{\partial z}{\partial y} = x + 2yf'(x^2 + y^2) = q \tag{3}$$

Eliminating f' from Eqs. (2) and (3) we get

$$yp - xq = y^2 - x^2, \tag{4}$$

which is the required PDE.

(ii) Given $z = f(xy/z)$ (1)

Differentiating partially Eq. (1) with respect to x and y, we get

$$\frac{\partial z}{\partial x} = \frac{y}{z} f'(xy/z) = p \tag{2}$$

$$\frac{\partial z}{\partial y} = \frac{x}{z} f'(xy/z) = q \tag{3}$$

Eliminating f' from Eqs. (2) and (3), we find

$$xp - yq = 0$$

or

$$px = qy \tag{4}$$

which is the required PDE.

EXAMPLE 0.5 Form the partial differential equation by eliminating the constants from

$$z = ax + by + ab.$$

Solution Given $z = ax + by + ab$ (1)

Differentiating Eq. (1) partially with respect to x and y we obtain

$$\frac{\partial z}{\partial x} = a = p \tag{2}$$

$$\frac{\partial z}{\partial y} = b = q \tag{3}$$

Substituting p and q for a and b in Eq. (1), we get the required PDE as

$$z = px + qy + pq$$

EXAMPLE 0.6 Find the partial differential equation of the family of planes, the sum of whose x, y, z intercepts is equal to unity.

Solution Let $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ be the equation of the plane in intercept form, so that $a + b + c = 1$. Thus, we have

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{1 - a - b} = 1 \tag{1}$$

Differentiating Eq. (1) with respect to x and y, we have

$$\frac{1}{a} + \frac{p}{1 - a - b} = 0 \quad \text{or} \quad \frac{p}{1 - a - b} = -\frac{1}{a} \tag{2}$$

and

$$\frac{1}{b} + \frac{q}{1 - a - b} = 0 \quad \text{or} \quad \frac{q}{1 - a - b} = -\frac{1}{b} \tag{3}$$

From Eqs. (2) and (3), we get

$$\frac{p}{q} = \frac{b}{a} \tag{4}$$

Also, from Eqs. (2) and (4), we get

$$pa = a + b - 1 = a + \frac{p}{q}a - 1$$

or

$$a\left(1 + \frac{p}{q} - p\right) = 1.$$

Therefore,

$$a = \frac{q}{(p+q-pq)} \tag{5}$$

Similarly, from Eqs. (3) and (4), we find

$$b = \frac{p}{(p+q-pq)} \tag{6}$$

Substituting the values of a and b from Eqs. (5) and (6) respectively to Eq. (1), we have

$$\frac{p+q-pq}{q}x + \frac{p+q-pq}{p}y + \frac{p+q-pq}{-pq}z = 1$$

or

$$\frac{x}{q} + \frac{y}{p} - \frac{z}{pq} = \frac{1}{p+q-pq}.$$

That is,

$$px + qy - z = \frac{pq}{p+q-pq}, \tag{7}$$

which is the required PDE.

0.5 SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER

In Section 0.4, we have observed that relations of the form

$$F(x, y, z, a, b) = 0 \tag{0.34}$$

give rise to PDE of first order of the form

$$f(x, y, z, p, q) = 0 \tag{0.35}$$

Thus, any relation of the form (0.34) containing two arbitrary constants a and b is a solution of the PDE of the form (0.35) and is called a complete solution or complete integral.

Consider a first order PDE of the form

$$P(x, y, z)\frac{\partial z}{\partial x} + Q(x, y, z)\frac{\partial z}{\partial y} = R(x, y, z) \tag{0.36}$$

or simply

$$Pp + Qq = R \tag{0.37}$$

where x and y are independent variables. The solution of Eq. (0.37) is a surface S lying in the (x, y, z)-space, called an integral surface. If we are given that $z = f(x, y)$ is an integral surface of the PDE (0.37). Then, the normal to this surface will have direction cosines proportional to $(\partial z/\partial x,\ \partial z/\partial y,\ -1)$ or $(p,\ q,\ -1)$. Therefore, the direction of the normal is given by $\vec{n} = \{p,\ q,\ -1\}$. From the PDE (0.37), we observe that the normal $\vec{n}$ is perpendicular to the direction defined by the vector $\vec{t} = \{P,\ Q,\ R\}$ (see Fig. 0.1).

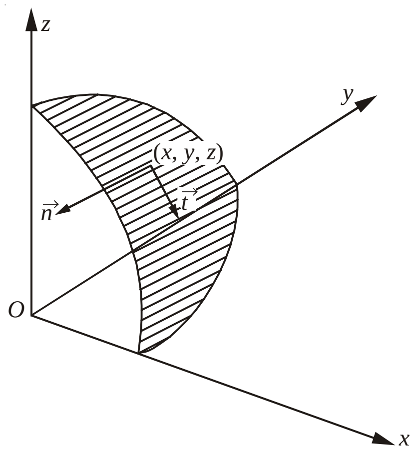

Fig. 0.1 Integral surface $z = f(x, y)$.

Therefore, any integral surface must be tangential to a vector with components $\{P,\ Q,\ R\}$, and hence, we will never leave the integral surface or solutions surface. Also, the total differential dz is given by

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \tag{0.38}$$

From Eqs. (0.37) and (0.38), we find

$$\{P,\ Q,\ R\} = \{dx,\ dy,\ dz\} \tag{0.39}$$

Now, the solution to Eq. (0.37) can be obtained using the following theorem:

Theorem 0.1 The general solution of the linear PDE

$$Pp + Qq = R$$

can be written in the form $F(u, v) = 0$, where F is an arbitrary function, and $u(x, y, z) = C_1$ and $v(x, y, z) = C_2$ form a solution of the equation

$$\frac{dx}{P(x, y, z)} = \frac{dy}{Q(x, y, z)} = \frac{dz}{R(x, y, z)} \tag{0.40}$$

Proof We observe that Eq. (0.40) consists of a set of two independent ordinary differential equations, that is, a two parameter family of curves in space, one such set can be written as

$$\frac{dy}{dx} = \frac{Q(x, y, z)}{P(x, y, z)} \tag{0.41}$$

which is referred to as "characteristic curve". In quasi-linear case, Eq. (0.41) cannot be evaluated until $z(x, y)$ is known. Recalling Eqs. (0.37) and (0.38), we may recast them using matrix notation as

$$\begin{bmatrix} P & Q \\ dx & dy \end{bmatrix} \begin{pmatrix} \partial z/\partial x \\ \partial z/\partial y \end{pmatrix} = \begin{pmatrix} R \\ dz \end{pmatrix} \tag{0.42}$$

Both the equations must hold on the integral surface. For the existence of finite solutions of Eq. (0.42), we must have

$$\begin{vmatrix} P & Q \\ dx & dy \end{vmatrix} = \begin{vmatrix} P & R \\ dx & dz \end{vmatrix} = \begin{vmatrix} R & Q \\ dz & dy \end{vmatrix} = 0 \tag{0.43}$$

on expanding the determinants, we have

$$\frac{dx}{P(x, y, z)} = \frac{dy}{Q(x, y, z)} = \frac{dz}{R(x, y, z)} \tag{0.44}$$

which are called auxiliary equations for a given PDE.

In order to complete the proof of the theorem, we have yet to show that any surface generated by the integral curves of Eq. (0.44) has an equation of the form $F(u, v) = 0$.

Let

$$u(x, y, z) = C_1 \quad \text{and} \quad v(x, y, z) = C_2 \tag{0.45}$$

be two independent integrals of the ordinary differential equations (0.44). If Eqs. (0.45) satisfy Eq. (0.44), then, we have

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = du = 0$$

and

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = dv = 0.$$

Solving these equations, we find

$$\frac{dx}{\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z} - \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}} = \frac{dy}{\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}}$$

$$= \frac{dz}{\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}}$$

which can be rewritten as

$$\frac{dx}{\frac{\partial(u,v)}{\partial(y,z)}} = \frac{dy}{\frac{\partial(u,v)}{\partial(z,x)}} = \frac{dz}{\frac{\partial(u,v)}{\partial(x,y)}} \tag{0.46}$$

Now, we may recall from Section 0.4 that the relation $F(u, v) = 0$, where F is an arbitrary function, leads to the partial differential equation

$$p\frac{\partial(u,v)}{\partial(y,z)} + q\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)} \tag{0.47}$$

By virtue of Eqs. (0.37) and (0.47), Eq. (0.46) can be written as

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

The solution of these equations are known to be $u(x, y, z) = C_1$ and $v(x, y, z) = C_2$. Hence, $F(u, v) = 0$ is the required solution of Eq. (0.37), if u and v are given by Eq. (0.45),

We shall illustrate this method through following examples:

EXAMPLE 0.7 Find the general integral of the following linear partial differential equations:

(i) $y^2 p - xy\, q = x(z - 2y)$

(ii) $(y + zx)p - (x + yz)q = x^2 - y^2$.

Solution

(i) The integral surface of the given PDE is generated by the integral curves of the auxiliary equation

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} \tag{1}$$

The first two members of the above equation give us

$$\frac{dx}{y} = \frac{dy}{-x} \quad \text{or} \quad x\,dx = -y\,dy,$$

which on integration results in

$$\frac{x^2}{2} = -\frac{y^2}{2} + C \quad \text{or} \quad x^2 + y^2 = C_1 \tag{2}$$

The last two members of Eq. (1) give

$$\frac{dy}{-y} = \frac{dz}{z-2y} \quad \text{or} \quad z\,dy - 2y\,dy = -y\,dz$$

That is,

$$2y\,dy = y\,dz + z\,dy,$$

which on integration yields

$$y^2 = yz + C_2 \quad \text{or} \quad y^2 - yz = C_2 \tag{3}$$

Hence, the curves given by Eqs. (2) and (3) generate the required integral surface as

$$F(x^2 + y^2,\ y^2 - yz) = 0.$$

(ii) The integral surface of the given PDE is generated by the integral curves of the auxiliary equation

$$\frac{dx}{y + zx} = \frac{dy}{-(x + yz)} = \frac{dz}{x^2 - y^2} \tag{1}$$

To get the first integral curve, let us consider the first combination as

$$\frac{x\,dx + y\,dy}{xy + zx^2 - xy - y^2 z} = \frac{dz}{x^2 - y^2}$$

or

$$\frac{x\,dx + y\,dy}{z(x^2 - y^2)} = \frac{dz}{x^2 - y^2}.$$

That is,

$$x\,dx + y\,dy = z\,dz.$$

On integration, we get

$$\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = C \quad \text{or} \quad x^2 + y^2 - z^2 = C_1 \tag{2}$$

Similarly, for getting the second integral curve, let us consider the combination such as

$$\frac{y\,dx + x\,dy}{y^2 + xyz - x^2 - xyz} = \frac{dz}{x^2 - y^2}$$

or

$$y\,dx + x\,dy + dz = 0,$$

which on integration results in

$$xy + z = C_2 \tag{3}$$

Thus, the curves given by Eqs. (2) and (3) generate the required integral surface as

$$F(x^2 + y^2 - z^2, xy + z) = 0.$$

EXAMPLE 0.8 Use Lagrange's method to solve the equation

$$\begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} & -1 \end{vmatrix} = 0$$

where $z = z(x, y)$.

Solution The given PDE can be written as

$$x\left[-\beta - \gamma\frac{\partial z}{\partial y}\right] - y\left[-\alpha - \gamma\frac{\partial z}{\partial x}\right] + z\left[\alpha\frac{\partial z}{\partial y} - \beta\frac{\partial z}{\partial x}\right] = 0$$

or

$$(\gamma y - \beta z)\frac{\partial z}{\partial x} + (\alpha z - \gamma x)\frac{\partial z}{\partial y} = \beta x - \alpha y \tag{1}$$

The corresponding auxiliary equations are

$$\frac{dx}{(\gamma y - \beta z)} = \frac{dy}{(\alpha z - \gamma x)} = \frac{dz}{(\beta x - \alpha y)} \tag{2}$$

Using multipliers x, y, and z we find that each fraction is

$$= \frac{x\,dx + y\,dy + z\,dz}{0}.$$

Therefore,

$$x\,dx + y\,dy + z\,dz = 0,$$

which on integration yields

$$x^2 + y^2 + z^2 = C_1 \tag{3}$$

Similarly, using multipliers α, β, and γ, we find from Eq. (2) that each fraction is equal to

$$\alpha\,dx + \beta\,dy + \gamma\,dz = 0,$$

which on integration gives

$$\alpha x + \beta y + \gamma z = C_2 \tag{4}$$

Thus, the general solution of the given equation is found to be

$$F(x^2 + y^2 + z^2,\ \alpha x + \beta y + \gamma z) = 0$$

EXAMPLE 0.9 Find the general integrals of the following linear PDEs:

(i) $pz - qz = z^2 + (x + y)^2$

(ii) $(x^2 - yz)\, p + (y^2 - zx)\, q = z^2 - xy.$

Solution

(i) The integral surface of the given PDE is generated by the integral curves of the auxiliary equation

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x + y)^2} \tag{1}$$

The first two members of Eq. (1) give

$$dx + dy = 0,$$

which on integration yields

$$x + y = C_1 \tag{2}$$

Now, considering Eq. (2) and the first and last members of Eq. (1), we obtain

$$dx = \frac{z\, dz}{z^2 + C_1^2}$$

or

$$\frac{2z\, dz}{z^2 + C_1^2} = 2\, dx,$$

which on integration yields

$$\ln\,(z^2 + C_1^2) = 2x + C_2$$

or

$$\ln\,[z^2 + (x + y)^2] - 2x = C_2 \tag{3}$$

Thus, the curves given by Eqs. (2) and (3) generates the integral surface for the given PDE as

$$F(x + y,\ \log\,\{x^2 + y^2 + z^2 + 2xy\} - 2x) = 0$$

(ii) The integral surface of the given PDE is given by the integral curves of the auxiliary equation

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \tag{1}$$

Equation (1) can be rewritten as

$$\frac{dx - dy}{(x - y)\,(x + y + z)} = \frac{dy - dz}{(y - z)\,(x + y + z)} = \frac{dz - dx}{(z - x)\,(x + y + z)} \tag{2}$$

Considering the first two terms of Eq. (2) and integrating, we get

$$\ln(x-y) = \ln(y-z) + \ln C_1$$

or
$$\frac{x-y}{y-z} = C_1 \tag{3}$$

Similarly, considering the last two terms of Eq. (2) and integrating, we obtain

$$\frac{y-z}{z-x} = C_2 \tag{4}$$

Thus, the integral curves given by Eqs. (3) and (4) generate the integral surface

$$F\left(\frac{x-y}{y-z}, \frac{y-z}{z-x}\right) = 0.$$

0.6 INTEGRAL SURFACES PASSING THROUGH A GIVEN CURVE

In the previous section, we have seen how a general solution for a given linear PDE can be obtained. Now, we shall make use of this general solution to find an integral surface containing a given curve as explained below:

Suppose, we have obtained two integral curves described by

$$\left.\begin{aligned} u(x, y, z) &= C_1 \\ v(x, y, z) &= C_2 \end{aligned}\right\} \tag{0.48}$$

from the auxiliary equations of a given PDE. Then, the solution of the given PDE can be written in the form

$$F(u, v) = 0 \tag{0.49}$$

Suppose, we wish to determine an integral surface, containing a given curve C described by the parametric equations of the form

$$x = x(t), \quad y = y(t), \quad z = z(t), \tag{0.50}$$

where t is a parameter. Then, the particular solution (0.48) must be like

$$\left.\begin{aligned} u\{x(t), y(t), z(t)\} &= C_1 \\ v\{x(t), y(t), z(t)\} &= C_2 \end{aligned}\right\} \tag{0.51}$$

Thus, we have two relations, from which we can eliminate the parameter t to obtain a relation of the type

$$F(C_1, C_2) = 0, \tag{0.52}$$

which leads to the solution given by Eq. (0.49). For illustration, let us consider the following couple of examples.

EXAMPLE 0.10 Find the integral surface of the linear PDE

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$$

containing the straight line $x+y=0,\ z=1$.

Solution The auxiliary equations for the given PDE are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z} \tag{1}$$

Using the multiplier xyz, we have

$$yz\,dx + zx\,dy + xy\,dz = 0.$$

On integration, we get

$$xyz = C_1 \tag{2}$$

Suppose, we use the multipliers x, y and z. Then find that each fraction in Eq. (1) is equal to

$$x\,dx + y\,dy + z\,dz = 0,$$

which on integration yields

$$x^2+y^2+z^2 = C_2 \tag{3}$$

For the curve in question, we have the equations in parametric form as

$$x=t, \quad y=-t, \quad z=1$$

Substituting these values in Eqs. (2) and (3), we obtain

$$\left.\begin{aligned} -t^2 &= C_1 \\ 2t^2+1 &= C_2 \end{aligned}\right\} \tag{4}$$

Eliminating the parameter t, we find

$$1-2C_1 = C_2$$

or

$$2C_1 + C_2 - 1 = 0$$

Hence, the required integral surface is

$$x^2+y^2+z^2+2xyz-1=0.$$

EXAMPLE 0.11 Find the integral surface of the linear PDE

$$xp + yq = z$$

which contains the circle defined by

$$x^2+y^2+z^2=4, \quad x+y+z=2.$$

Solution The integral surface of the given PDE is generated by the integral curves of the auxiliary equation

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} \tag{1}$$

Integration of the first two members of Eq. (1) gives

$$\ln x = \ln y + \ln C$$

or

$$\frac{x}{y} = C_1 \tag{2}$$

Similarly, integration of the last two members of Eq. (1) yields

$$\frac{y}{z} = C_2 \tag{3}$$

Hence, the integral surface of the given PDE is

$$F\left(\frac{x}{y}, \frac{y}{z}\right) = 0 \tag{4}$$

If this integral surface also contains the given circle, then we have to find a relation between x/y and y/z.

The equation of the circle is

$$x^2 + y^2 + z^2 = 4 \tag{5}$$

$$x + y + z = 2 \tag{6}$$

From Eqs. (2) and (3), we have

$$y = x/C_1, \quad z = y/C_2 = x/C_1C_2$$

Substituting these values of y and z in Eqs. (5) and (6), we find

$$x^2 + \frac{x^2}{C_1^2} + \frac{x^2}{C_1^2C_2^2} = 4, \quad \text{or} \quad x^2\left(1 + \frac{1}{C_1^2} + \frac{1}{C_1^2C_2^2}\right) = 4 \tag{7}$$

and

$$x + \frac{x}{C_1} + \frac{x}{C_1C_2} = 2, \quad \text{or} \quad x\left(1 + \frac{1}{C_1} + \frac{1}{C_1C_2}\right) = 2 \tag{8}$$

From Eqs. (7) and (8) we observe

$$1+\frac{1}{C_1^2}+\frac{1}{C_1^2C_2^2}=\left(1+\frac{1}{C_1}+\frac{1}{C_1C_2}\right)^2,$$

which on simplification gives us

$$\frac{2}{C_1}+\frac{2}{C_1C_2}+\frac{2}{C_1^2C_2}=0$$

That is,

$$C_1C_2+C_1+1=0. \tag{9}$$

Now, replacing C_1 by x/y and C_2 by y/z, we get the required integral surface as

$$\frac{x}{y}\frac{y}{z}+\frac{x}{y}+1=0,$$

or $$\frac{x}{z}+\frac{x}{y}+1=0,$$

or $$xy+xz+yz=0.$$

0.7 THE CAUCHY PROBLEM FOR FIRST ORDER EQUATIONS

Consider an interval I on the real line. If $x_0(s)$, $y_0(s)$ and $z_0(s)$ are three arbitrary functions of a single variable $s \in I$ such that they are continuous in the interval I with their first derivatives. Then, the Cauchy problem for a first order PDE of the form

$$F(x, y, z, p, q)=0 \tag{0.53}$$

is to find a region $\mathbb{R}$ in (x, y), i.e. the space containing $(x_0(s), y_0(s))$ for all $s \in I$, and a solution $z=\phi(x, y)$ of the PDE (0.53) such that

$$Z[x_0(s), y_0(s)]=Z_0(s)$$

and $\phi(x, y)$ together with its partial derivatives with respect to x and y are continuous functions of x and y in the region $\mathbb{R}$.

Geometrically, there exists a surface $z=\phi(x, y)$ which passes through the curve Γ, called datum curve, whose parametric equations are

$$x=x_0(s), \qquad y=y_0(s), \qquad z=z_0(s)$$

and at every point of which the direction $(p, q, -1)$ of the normal is such that

$$F(x, y, z, p, q)=0$$

This is only one form of the problem of Cauchy.

In order to prove the existence of a solution of Eq. (0.53) containing the curve Γ, we have to make further assumptions about the form of the function F and the nature of Γ. Based on these assumptions, we have a whole class of existence theorems which is beyond the scope of this book. However, we shall quote one form of the existence theorem without proof, which is due to Kowalewski (see Senddon, 1986).

Theorem 0.2 If

(i) $g(y)$ and all of its derivatives are continuous for $|y - y_0| < \delta$,

(ii) x_0 is a given number and $z_0 = g(y_0)$, $q_0 = g'(y_0)$ and $f(x, y, z, q)$ and all of its partial derivatives are continuous in a region S defined by

$$|x - x_0| < \delta,\ |y - y_0| < \delta,\ |q - q_0| < \delta,$$

then, there exists a unique function $\phi(x, y)$ such that

(a) $\phi(x, y)$ and all of its partial derivatives are continuous in a region $\mathbb{R}$ defined by

$$|x - x_0| < \delta_1, \quad |y - y_0| < \delta_2,$$

(b) For all (x, y) in $\mathbb{R}$, $z = \phi(x, y)$ is a solution of the equation

$$\frac{\partial z}{\partial x} = f\left(x, y, z, \frac{\partial z}{\partial y}\right) \quad \text{and}$$

(c) For all values of y in the interval $|y - y_0| < \delta_1$, $\phi(x_0, y) = g(y)$.

0.8 SURFACES ORTHOGONAL TO A GIVEN SYSTEM OF SURFACES

One of the useful applications of the theory of linear first order PDE is to find the system of surfaces orthogonal to a given system of surfaces. Let a one-parameter family of surfaces is described by the equation

$$F(x, y, z) = C \tag{0.54}$$

Then, the task is to determine the system of surfaces which cut each of the given surfaces orthogonally. Let (x, y, z) be a point on the surface given by Eq. (0.54), where the normal to the surface will have direction ratios $(\partial F/\partial x, \partial F/\partial y, \partial F/\partial z)$ which may be denoted by P, Q, R.

Let

$$Z = \phi(x, y) \tag{0.55}$$

be the surface which cuts each of the given system orthogonally (see Fig. 0.2).

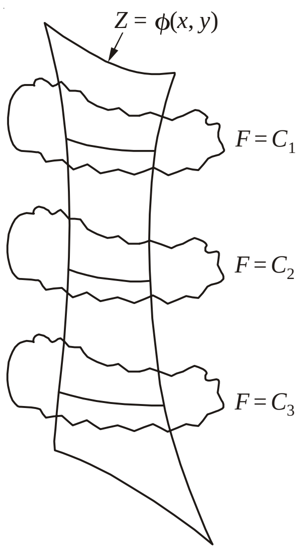

Fig. 0.2 Orthogonal surface to a given system of surfaces.

Then, its normal at the point (x, y, z) will have direction ratios $(\partial z/\partial x, \partial z/\partial y, -1)$ which, of course, will be perpendicular to the normal to the surfaces characterized by Eq. (0.54). As a consequence we have a relation

$$P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} - R = 0 \tag{0.56}$$

or

$$Pp + Qq = R \tag{0.57}$$

which is a linear PDE of Lagranges type, and can be recast into

$$\frac{\partial F}{\partial x}\frac{\partial z}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial z}{\partial y} = \frac{\partial F}{\partial z} \tag{0.58}$$

Thus, any solution of the linear first order PDE of the type given by either Eq. (0.57) or (0.58) is orthogonal to every surface of the system described by Eq. (0.54). In other words, the surfaces orthogonal to the system (0.54) are the surfaces generated by the integral curves of the auxiliary equations

$$\frac{dx}{\partial F/\partial x} = \frac{dy}{\partial F/\partial y} = \frac{dz}{\partial F/\partial z} \tag{0.59}$$

0.9 FIRST ORDER NON-LINEAR EQUATIONS

In this section, we will discuss the problem of finding the solution of first order non-linear partial differential equations (PDEs) in three variables of the form

$$F(x, y, z, p, q) = 0, \tag{0.60}$$

where

$$p = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y}.$$

We also assume that the function possesses continuous second order derivatives with respect to its arguments over a domain Ω of (x, y, z, p, q)-space, and either F_p or F_q is not zero at every point such that

$$F_p^2 + F_q^2 \neq 0.$$

The PDE (0.60) establishes the fact that at every point (x, y, z) of the region, there exists a relation between the numbers p and q such that $\phi(p, q) = 0$, which defines the direction of the normal $\vec{n} = \{p, q, -1\}$ to the desired integral surface $z = z(x, y)$ of Eq. (0.60). Thus, the direction of the normal to the desired integral surface at certain point (x, y, z) is not defined uniquely. However, a certain cone of admissable directions of the normals exist satisfying the relation $\phi(p, q) = 0$ (see Fig. 0.3).

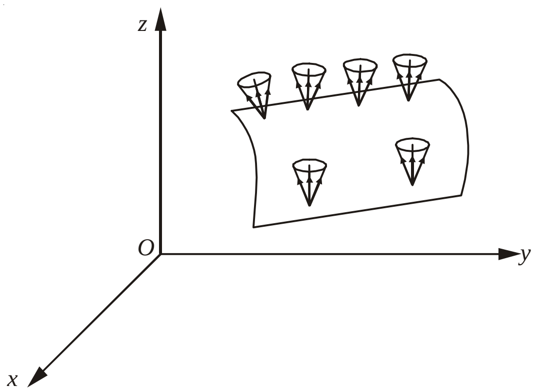

Fig. 0.3 Cone of normals to the integral surface.

Therefore, the problem of finding the solution of Eq. (0.60) reduces to finding an integral surface $z = z(x, y)$, the normals at every point of which are directed along one of the permissible directions of the cone of normals at that point.

Thus, the integral or the solution of Eq. (0.60) essentially depends on two arbitrary constants in the form

$$f(x, y, z, a, b) = 0, \tag{0.61}$$

which is called a complete integral. Hence, we get a two-parameter family of integral surfaces through the same point.

0.9.1 Cauchy's Method of Characteristics

The integral surface $z = z(x, y)$ of Eq. (0.60) that passes through a given curve $x_0 = x_0(s)$, $y_0 = y_0(s)$, $z_0 = z_0(s)$ may be visualized as consisting of points lying on a certain one-parameter family of curves $x = x(t, s)$, $y = y(t, s)$, $z = z(t, s)$, where s is a parameter of the family called characteristics.

Here, we shall discuss the Cauchy's method for solving Eq. (0.60), which is based on geometrical considerations. Let $z = z(x, y)$ represents an integral surface S of Eq. (0.60) in (x, y, z)-space. Then, $\{p, q, -1\}$ are the direction ratios of the normal to S. Now, the differential equation (0.60) states that at a given point $P(x_0, y_0, z_0)$ on S, the relationship between p_0 and q_0, that is $F(x_0, y_0, z_0, p_0, q_0)$, need not be necessarily linear. Hence, all the tangent planes to possible integral surfaces through P form a family of planes enveloping a conical surface called Monge Cone with P as its vertex. In other words, the problem of solving the PDE (0.60) is to find surfaces which touch the Monge cone at each point along a generator. For example, let us consider the non-linear PDE

$$p^2 - q^2 = 1. \tag{0.62}$$

At every point of the xyz-space, the relation (0.62) can be expressed parametrically as

$$p = \cosh \mu, \qquad q = \sinh \mu, \qquad -\infty < \mu < \infty \tag{0.63}$$

Then, the equation of the tangent planes at (x_0, y_0, z_0) can be written as

$$(x - x_0)\cosh \mu + (y - y_0)\sinh \mu - (z - z_0) = 0 \tag{0.64}$$

Let $P(x_0, y_0, z_0)$ be the vertex and $Q(x, y, z)$ be any point on the generator. Then, the direction ratios of the generator are $(x - x_0), (y - y_0), (z - z_0)$. Now, the direction ratios of the axis of the cone which is parallel to x-axis are (1, 0, 0) (see Fig. 0.4). Let the semi-verticle angle of the cone be $\pi/4$. Then,

$$\cos\frac{\pi}{4} = \frac{(x - x_0)1 + (y - y_0)0 + (z - z_0)0}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} = \frac{1}{\sqrt{2}}$$

or

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = 2(x - x_0)^2$$

or

$$(x - x_0)^2 - (y - y_0)^2 - (z - z_0)^2 = 0 \tag{0.65}$$

Thus, we see that the Monge cone of the PDE (0.62) is given by Eq. (0.65). This is a right circular cone with semi-vertical angel $\pi/4$ whose axis is the straight line passing through (x_0, y_0, z_0) and parallel to z-axis.

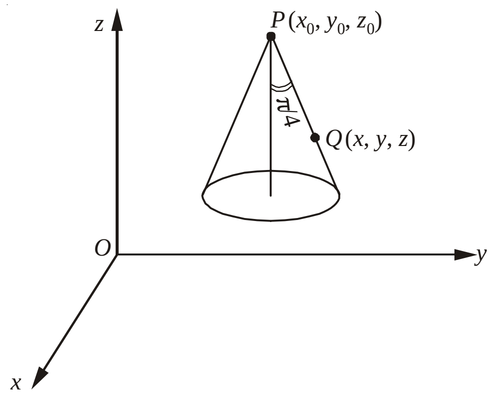

Fig. 0.4 Monge cone.

Since an integral surface is touched by a Monge cone along its generator, we must have a method to determine the generator of the Monge cone of the PDE (0.60) which is explained below:

It may be noted that the equation of the tangent plane to the integral surface $z = z(x, y)$ at the point (x_0, y_0, z_0) is given by

$$p(x - x_0) + q(y - y_0) = (z - z_0). \tag{0.66}$$

Now, the given non-linear PDE (0.60) can be recasted into an equivalent form as

$$q = q(x_0, y_0, z_0, p) \tag{0.67}$$

indicating that p and q are not independent at (x_0, y_0, z_0). At each point of the surface S, there exists a Monge cone which touches the surface along the generator of the cone. The lines of contact between the tangent planes of the integral surface and the corresponding cones, that is the generators along which the surface is touched, define a direction field on the surface S. These directions are called the characteristic directions, also called Monge directions on S and lie along the generators of the Monge cone. The integral curves of this field of directions on the integral surface S define a family of curves called characteristic curves as shown in Fig. 0.5. The Monge cone can be obtained by eliminating p from the following equations:

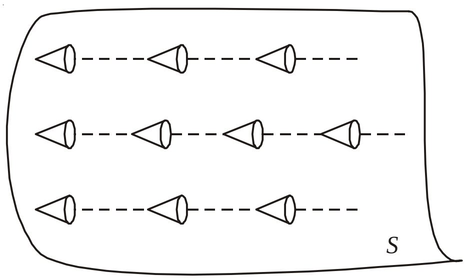

Fig. 0.5 Characteristic directions on an integral surface.

$$p(x-x_0)+q(x_0, y_0, z_0, p)(y-y_0)=(z-z_0) \tag{0.68}$$

and

$$(x-x_0)+(y-y_0)\frac{dq}{dp}=0. \tag{0.69}$$

observing that q is a function of p and differentiating Eq. (0.60) with respect to p, we get

$$\frac{dF}{dp}=\frac{\partial F}{\partial p}+\frac{\partial F}{\partial q}\frac{dq}{dp}=0. \tag{0.70}$$

Now, eliminating (dq/dp) from Eqs. (0.69) and (0.70), we obtain

$$\frac{\partial F}{\partial p}-\frac{\partial F}{\partial q}\frac{(x-x_0)}{(y-y_0)}=0$$

or

$$\frac{x-x_0}{F_p}=\frac{y-y_0}{F_q} \tag{0.71}$$

Therefore, the equations describing the Monge cone are given by

$$\left.\begin{aligned} q &= q(x_0, y_0, z_0, p), \\ (x-x_0)p+(y-y_0)q &= (z-z_0) \\ \text{and} \quad \frac{x-x_0}{F_p} &= \frac{y-y_0}{F_q}. \end{aligned}\right\} \tag{0.72}$$

The second and third of Eqs. (0.72) define the generator of the Monge cone. Solving them for $(x-x_0), (y-y_0)$ and $(z-z_0)$, we get

$$\frac{x-x_0}{F_p}=\frac{y-y_0}{F_q}=\frac{z-z_0}{pF_p+qF_q} \tag{0.73}$$

Finally, replacing $(x-x_0), (y-y_0)$ and $(z-z_0)$ by dx, dy and dz respectively, which corresponds to infinitesimal movement from (x_0, y_0, z_0) along the generator, Eq. (0.73) becomes

$$\frac{dx}{F_p}=\frac{dy}{F_q}=\frac{dz}{pF_p+qF_q}. \tag{0.74}$$

Denoting the ratios in Eq. (0.74) by dt, we observe that the characteristic curves on S can be obtained by solving the ordinary differential equations

$$\frac{dx}{dt} = F_p\{x, y, z(x, y), p(x, y), q(x, y)\} \tag{0.75}$$

and

$$\frac{dy}{dt} = F_q\{x, y, z(x, y), p(x, y), q(x, y)\}. \tag{0.76}$$

Also, we note that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = p\frac{dx}{dt} + q\frac{dy}{dt}$$

Therefore,

$$\frac{dz}{dt} = pF_p + qF_q \tag{0.77}$$

Along the characteristic curve, p is a function of t, so that

$$\frac{dp}{dt} = \frac{\partial p}{\partial x}\frac{dx}{dt} + \frac{\partial p}{\partial y}\frac{dy}{dt}$$

Now, using Eqs. (0.75) and (0.76), the above equation becomes

$$\frac{dp}{dt} = \frac{\partial p}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial p}{\partial y}\frac{\partial F}{\partial q}.$$

Since $z_{xy} = z_{yx}$ or $p_y = q_x$, we have

$$\frac{dp}{dt} = \frac{\partial p}{\partial x}\frac{\partial F}{\partial p} + \frac{\partial q}{\partial x}\frac{\partial F}{\partial q} \tag{0.78}$$

Also, differentiating Eq. (0.60) with respect to x, we find

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0 \tag{0.79}$$

Using Eq. (0.79), Eq. (0.78) becomes

$$\frac{dp}{dt} = -(F_x + pF_z) \tag{0.80}$$

Similarly, we can show that

$$\frac{dq}{dt} = -(F_y + qF_z) \tag{0.81}$$

Thus, given an integral surface, we have shown that there exists a family of characteristic curves along which x, y, z, p and q vary according to Eqs. (0.75), (0.76), (0.77), (0.80) and (0.81). Collecting these results together, we may write

$$\left.\begin{aligned} \frac{dx}{dt} &= F_p, \\ \frac{dy}{dt} &= F_q, \\ \frac{dz}{dt} &= pF_p + qF_q, \\ \frac{dp}{dt} &= -(F_x + pF_z) \text{ and} \\ \frac{dq}{dt} &= -(F_y + qF_z). \end{aligned}\right\} \tag{0.82}$$

These equations are known as characteristic equations of the given PDE (0.60). The last three equations of (0.82) are also called compatibility conditions. Without knowing the solution $z = z(x, y)$ of the PDE (0.60), it is possible to find the functions $x(t), y(t), z(t), p(t), q(t)$ from Eqs. (0.82). That is, we can find the curves $x = x(t), y = y(t), z = z(t)$ called characteristics and at each point of a characteristic, we can find the numbers $p = p(t)$ and $q = q(t)$ that determine the direction of the plane

$$p(X - x) + q(Y - y) = (Z - z). \tag{0.83}$$

The characteristics, together with the plane (0.83) referred to each of its points is called a characteristic strip. The solution $x = x(t), y = y(t), z = z(t), p = p(t), q = q(t)$ of the characteristic equations (0.82) satisfy the strip condition

$$\frac{dz}{dt} = p(t)\frac{dx}{dt} + q(t)\frac{dy}{dt} \tag{0.84}$$

It may be noted that not every set of five functions can be interpreted as a strip. A strip should satisfy that the planes with normals $(p, q, -1)$ be tangential to the characteristic curve. That is, they must satisfy the strip condition (0.84) and the normals should vary continuously along the curve.

An important consequence of the Cauchy's method of characteristic is stated in the following theorem.

Theorem 0.3 Along every strip (characteristic strip) of the PDE: $F(x, y, z, p, q) = 0$, the function $F(x, y, z, p, q)$ is constant.

Proof Along the characteristic strip, we have

$$\frac{d}{dt}F\{x(t), y(t), z(t), p(t), q(t)\} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} + \frac{\partial F}{\partial p}\frac{dp}{dt} + \frac{\partial F}{\partial q}\frac{dq}{dt}$$

Now, using the results listed in Eq. (0.82), the right-hand side of the above equation becomes

$$F_x F_p + F_y F_q + F_z(pF_p + qF_q) - F_p(F_x + pF_z) - F_q(F_y + qF_z) = 0.$$

Hence, the function $F(x, y, z, p, q)$ is constant along the strip of the characteristic equations of the PDE defined by Eq. (0.60).

For illustration, we consider the following examples:

EXAMPLE 0.12 Find the characteristics of the equation $pq = z$ and determine the integral surface which passes through the straight line $x = 1, z = y$.

Solution If the initial data curve is given in parametric form as

$$x_0(s) = 1, \quad y_0(s) = s, \quad z_0(s) = s,$$

then ordinarily the solution is sought in parametric form as

$$x = x(t, s), \quad y = y(t, s), \quad z = z(t, s).$$

Thus, using the given data, the differential equation becomes

$$p_0(s)q_0(s) - s = 0 = F \tag{1}$$

and the strip condition gives

$$1 = p_0(0) + q_0(1) \quad \text{or} \quad q_0 = 1. \tag{2}$$

Therefore,

$$q_0 = 1, \quad p_0 = s \quad \text{(unique initial strip)} \tag{3}$$

Now, the characteristic equations for the given PDE are

$$\frac{dx}{dt} = q, \quad \frac{dy}{dt} = p, \quad \frac{dz}{dt} = 2pq, \quad \frac{dp}{dt} = p, \quad \frac{dq}{dt} = q \tag{4}$$

On integration, we get

$$\left.\begin{aligned} &p = c_1 \exp(t),\ q = c_2 \exp(t),\ x = c_2 \exp(t) + c_3 \\ &y = c_1 \exp(t) + c_4,\ z = 2c_1c_2 \exp(2t) + c_5 \end{aligned}\right\} \tag{5}$$

Now, taking into account the initial conditions

$$x_0 = 1, \ y_0 = s, \ z_0 = s, \ p_0 = s, \ q_0 = 1 \tag{6}$$

we can determine the constants of integration and obtain (since $c_2 = 1,\ c_3 = 0$)

$$\left.\begin{aligned} &p = s\exp(t),\ \ q = \exp(t),\ \ x = \exp(t) \\ &y = s\exp(t),\ \ z = s\exp(2t) \end{aligned}\right\} \tag{7}$$

Consequently, the required integral surface is obtained from Eq. (7) as

$$z = xy.$$

EXAMPLE 0.13 Find the characteristics of the equation $pq = z$ and hence, determine the integral surface which passes through the parabola $x = 0,\ y^2 = z.$

Solution The initial data curve is

$$x_0(s) = 0,\ \ y_0(s) = s,\ \ z_0(s) = s^2.$$

Using this data, the given PDE becomes

$$p_0(s)\,q_0(s) - s^2 = 0 = F \tag{1}$$

The strip condition gives

$$2s = p_0(0) + q_0(1) \quad \text{or} \quad q_0 - 2s = 0 \tag{2}$$

Therefore,

$$q_0 = 2s \quad \text{and} \quad p_0 = z_0/q_0 = s^2/2s = \frac{s}{2} \tag{3}$$

Now, the characteristic equations of the given PDE are given by

$$\frac{dx}{dt} = q,\ \ \frac{dy}{dt} = p,\ \ \frac{dz}{dt} = 2pq,\ \ \frac{dp}{dt} = p,\ \ \frac{dq}{dt} = q \tag{4}$$

On integration, we obtain

$$\left.\begin{aligned} &p = c_1\exp(t),\ \ q = c_2\exp(t),\ \ x = c_2\exp(t) + c_3 \\ &y = c_1\exp(t) + c_4,\ \ z = c_1c_2\exp(2t) + c_5 \end{aligned}\right\} \tag{5}$$

Taking into account the initial conditions

$$x_0 = 0,\ \ y_0 = s,\ \ z_0 = s^2,\ \ p_0 = s/2,\ \ q_0 = 2s,$$

we find

$$c_1 = s/2,\ \ c_2 = 2s,\ \ c_3 = -2s,\ \ c_4 = s/2,\ \ c_5 = 0$$

Therefore, we have

$$\left.\begin{aligned} & p=\frac{s}{2}\exp(t),\ \ q=2s\exp(t), \\ & x=2s[\exp(t)-1],\ \ y=\frac{s}{2}[\exp(t)+1] \\ & z=s^2\exp(2t) \end{aligned}\right\} \tag{6}$$

Eliminating s and t from the last three equations of (6), we get

$$16z=(4y+x)^2.$$

This is the required integral surface.

EXAMPLE 0.14 Find the characteristics of the PDE

$$p^2+q^2=2$$

and determine the integral surface which passes through $x=0, z=y$.

Solution The initial data curve is

$$x_0(s)=0,\ \ y_0(s)=s,\ \ z_0(s)=s.$$

Using this data, the given PDE becomes

$$p_0^2+q_0^2-2=0=F \tag{1}$$

and the strip condition gives

$$1=p_0(0)+q_0(1) \quad \text{or} \quad q_0-1=0 \tag{2}$$

Hence,

$$q_0=1,\ \ p_0=\pm 1 \tag{3}$$

Now, the characteristic equations for the given PDE are given by

$$\left.\begin{aligned} & \frac{dx}{dt}=2p,\ \ \frac{dy}{dt}=2q,\ \ \frac{dz}{dt}=2p^2+2q^2=4 \\ & \frac{dp}{dt}=0,\ \ \frac{dq}{dt}=0 \end{aligned}\right\} \tag{4}$$

On integration, we get

$$\left.\begin{aligned} & p=c_1,\ \ q=c_2,\ \ x=2c_1t+c_3 \\ & y=2c_2t+c_4,\ \ z=4t+c_5 \end{aligned}\right\} \tag{5}$$

Taking into account the initial conditions

$$x_0 = 0, \; y_0 = s, \; z_0 = s, \; p_0 = \pm 1, \; q_0 = 1,$$

we find

$$\left.\begin{aligned} &p = \pm 1, \; q = 1, \; x = \pm 2t, \\ &y = 2t + s, \; z = 4t + s. \end{aligned}\right\} \tag{6}$$

The last-three equations of (6) are parametric equations of the desired integral surface. Eliminating the parameters s and t, we get

$$z = y \pm x.$$

0.10 COMPATIBLE SYSTEMS OF FIRST ORDER EQUATIONS

Two first order PDEs are said to be compatible, if they have a common solution. We shall now derive the necessary and sufficient conditions for the two partial differential equations

$$f(x, y, z, p, q) = 0 \tag{0.85}$$

and

$$g(x, y, z, p, q) = 0 \tag{0.86}$$

to be compatible.

Let

$$J = \frac{\partial (f, g)}{\partial (p, q)} \neq 0 \tag{0.87}$$

Since Eqs. (0.85) and (0.86) have common solution, we can solve them and obtain explicit expressions for p and q in the form

$$p = \phi(x, y, z), \; q = \psi(x, y, z) \tag{0.88}$$

and then, the differential relation

$$p\, dx + q\, dy = dz$$

or

$$\phi(x, y, z)\, dx + \psi(x, y, z)\, dy = dz \tag{0.89}$$

should be integrable, for which the necessary condition is

$$\vec{X} \cdot \operatorname{curl} \vec{X} = 0$$

where $X = \{\phi, \psi, -1\}$. That is,

$$(\phi \hat{i} + \psi \hat{j} - \hat{k}) \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \phi & \psi & -1 \end{vmatrix} = 0$$

or

$$\phi(-\psi_z) + \psi(\phi_z) = \psi_x - \phi_y$$

which can be rewritten as

$$\psi_x + \phi\psi_z = \phi_y + \psi\phi_z \tag{0.90}$$

Now, differentiating Eq. (0.85) with respect to x and z, we get

$$f_x + f_p \frac{\partial p}{\partial x} + f_q \frac{\partial q}{\partial x} = 0$$

and

$$f_z + f_p \frac{\partial p}{\partial z} + f_q \frac{\partial q}{\partial z} = 0$$

But, from Eq. (0.89), we have

$$\frac{\partial p}{\partial x} = \frac{\partial \phi}{\partial x}, \quad \frac{\partial q}{\partial x} = \frac{\partial \psi}{\partial x} \text{ and so on.}$$

Using these results, the above equations can be recast into

$$f_x + f_p\phi_x + f_q\psi_x = 0$$

and

$$f_z + f_p\phi_z + f_q\psi_z = 0.$$

Multiplying the second one of the above pair by ϕ and adding to the first one, we readily obtain

$$(f_x + \phi f_z) + f_p(\phi_x + \phi\phi_z) + f_q(\psi_x + \phi\psi_z) = 0$$

Similarly, from Eq. (0.86) we can deduce that

$$(g_x + \phi g_z) + g_p(\phi_x + \phi\phi_z) + g_q(\psi_x + \phi\psi_z) = 0$$

Solving the above pair of equations for $(\psi_x + \phi\psi_z)$, we have

$$\frac{(\psi_x + \phi\psi_z)}{f_p(g_x + \phi g_z) - g_p(f_x + \phi f_z)} = \frac{1}{f_q g_p - g_q f_p} = \frac{1}{J}$$

or

$$\begin{aligned}\psi_x + \phi\psi_z &= \frac{1}{J}[(f_p g_x - g_p f_x) + \phi(f_p g_z - g_p f_z)] \\ &= \frac{1}{J}\left[\frac{\partial(f, g)}{\partial(x, p)} + \phi\frac{\partial(f, g)}{\partial(z, p)}\right]\end{aligned} \tag{0.91}$$

where J is defined in Eq. (0.87). Similarly, differentiating Eq. (0.85) with respect to y and z and using Eq. (0.88), we can show that

$$\phi_y + \psi\phi_z = -\frac{1}{J}\left[\frac{\partial(f, g)}{\partial(y, q)} + \psi\frac{\partial(f, g)}{\partial(z, q)}\right] \tag{0.92}$$

Finally, substituting the values of $\psi_x + \phi\psi_z$ and $\phi_y + \psi\phi_z$ from Eqs. (0.91) and (0.92) into Eq. (0.90), we obtain

$$\frac{\partial(f, g)}{\partial(x, p)} + \phi\frac{\partial(f, g)}{\partial(z, p)} = -\left[\frac{\partial(f, g)}{\partial(y, q)} + \psi\frac{\partial(f, g)}{\partial(z, q)}\right]$$

In view of Eqs. (0.88), we can replace ϕ and ψ by p and q, respectively to get

$$\frac{\partial(f, g)}{\partial(x, p)} + p\frac{\partial(f, g)}{\partial(z, p)} + \frac{\partial(f, g)}{\partial(y, q)} + q\frac{\partial(f, g)}{\partial(z, q)} = 0 \tag{0.93}$$

This is the desired compatibility condition. For illustration, let us consider the following examples:

EXAMPLE 0.15 Show that the following PDEs

$$xp - yq = x \quad \text{and} \quad x^2p + q = xz$$

are compatible and hence, find their solution.

Solution Suppose, we have

$$f = xp - yq - x = 0, \tag{1}$$

and

$$g = x^2 p + q - xz = 0. \tag{2}$$

Then,

$$\frac{\partial(f, g)}{\partial(x, p)} = \begin{vmatrix} (p-1) & x \\ (2xp - z) & x^2 \end{vmatrix} = px^2 - x^2 - 2x^2p + xz = xz - x^2p - x^2,$$

$$\frac{\partial(f, g)}{\partial(z, p)} = \begin{vmatrix} 0 & x \\ -x & x^2 \end{vmatrix} = x^2,$$

$$\frac{\partial(f, g)}{\partial(y, q)} = \begin{vmatrix} -q & -y \\ 0 & 1 \end{vmatrix} = -q,$$

$$\frac{\partial(f, g)}{\partial(z, q)} = \begin{vmatrix} 0 & -y \\ -x & 1 \end{vmatrix} = -xy.$$

and we find

$$\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = xz - x^2p - x^2 + px^2 - q - qxy$$

$$= xz - q - qxy - x^2$$

$$= xz - q - x(qy + x)$$

$$= xz - q - x^2 p$$

$$= 0$$

Hence, the given PDEs are compatible.

Now, solving Eqs. (1) and (2) for p and q, we obtain

$$\frac{p}{xyz + x} = \frac{q}{x^3 + x^2 z} = \frac{1}{x + x^2 y}$$

from which we get

$$p = \frac{x(1+yz)}{x(1+xy)} = \frac{1+yz}{1+xy}$$

and

$$q = \frac{x^2(z-x)}{x(1+xy)} = \frac{x(z-x)}{1+xy}$$

In order to get the solution of the given system, we have to integrate Eq. (0.89), that is

$$dz = \frac{(1+yz)}{1+xy}dx + \frac{x(z-x)}{1+xy}dy \tag{3}$$

or

$$dz - dx = \frac{y(z-x)}{1+xy}dx + \frac{x(z-x)}{1+xy}dy$$

or

$$\frac{dz - dx}{z - x} = \frac{y\,dx + x\,dy}{1+xy}$$

On integration, we get

$$\ln(z - x) = \ln(1+xy) + \ln c.$$

That is,

$$z - x = c(1+xy)$$

Hence, the solution of the given system is found to be

$$z = x + c(1+xy), \tag{4}$$

which is of one-parameter family.

0.11 CHARPIT'S METHOD

In this section, we will discuss a general method for finding the complete integral or complete solution of a nonlinear PDE of first order of the form

$$f(x, y, z, p, q) = 0. \tag{0.94}$$

This method is known as Charpit's method. The basic idea in Charpit's method is the introduction of another PDE of first order of the form

$$g(x, y, z, p, q) = 0 \tag{0.95}$$

and then, solve Eqs. (0.94) and (0.95) for p and q and substitute in

$$dz = p(x, y, z, a)\,dx + q(x, y, z, a)\,dy. \tag{0.96}$$

Now, the solution of Eq. (0.96) if it exists is the complete integral of Eq. (0.94).

The main task is the determination of the second equation (0.95) which is already discussed in the previous section. Now, what is required, is to seek an equation of the form

$$g(x, y, z, p, q) = 0$$

compatible with the given equation

$$f(x, y, z, p, q) = 0$$

for which the necessary and sufficient condition is

$$\frac{\partial(f, g)}{\partial(x, p)} + p\frac{\partial(f, g)}{\partial(z, p)} + \frac{\partial(f, g)}{\partial(y, q)} + q\frac{\partial(f, g)}{\partial(z, q)} = 0. \tag{0.97}$$

On expansion, we have

$$\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}\right) + p\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial z}\right)$$
$$+\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial y}\right) + q\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial z}\right) = 0$$

which can be recast into

$$f_p\frac{\partial g}{\partial x} + f_q\frac{\partial g}{\partial y} + (pf_p + qf_q)\frac{\partial g}{\partial z} - (f_x + pf_z)\frac{\partial g}{\partial p} - (f_y + qf_z)\frac{\partial g}{\partial q} = 0. \tag{0.98}$$

This is a linear PDE, from which we can determine g. The auxiliary equations of (0.98) are

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)}$$
$$= \frac{dq}{-(f_y + qf_z)} \tag{0.99}$$

These equations are called Charpit's equations. Any integral of Eq. (0.99) involving p or q or both can be taken as the second relation (0.95). Then, the integration of Eq. (0.96) gives the complete integral as desired. It may be noted that all charpits equations need not be used, but it is enough to choose the simplest of them. This method is illustrated through the following examples:

EXAMPLE 0.16 Find the complete integral of

$$(p^2 + q^2)y = qz$$

Solution Suppose

$$f = (p^2 + q^2)y - qz = 0 \tag{1}$$

then, we have

$$f_x = 0, \quad f_y = p^2 + q^2, \quad f_z = -q$$

$$f_p = 2py, \quad f_q = 2qy - z,$$

Now, the charpits auxiliary equations are given by

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

That is,

$$\frac{dx}{2py} = \frac{dy}{2qy - z} = \frac{dz}{2p^2y + 2q^2y - qz}$$

$$= \frac{dp}{pq} = \frac{dq}{-[(p^2 + q^2) - q^2]} \tag{2}$$

From the last two members of Eq. (2), we have

$$\frac{dp}{pq} = \frac{dq}{-p^2}$$

or

$$p\,dp + q\,dq = 0$$

On integration, we get

$$p^2 + q^2 = a \text{ (constant)} \tag{3}$$

From Eqs. (1) and (3), we obtain

$$\left.\begin{aligned} ay - qz = 0 \quad &\text{or} \quad q = ay/z \\ \text{and} \quad p = \sqrt{a - \left(\frac{ay}{z}\right)^2} &= \sqrt{(az^2 - a^2y^2)/z^2} \end{aligned}\right\} \tag{4}$$

Substituting these values of p and q in

$$dz = p\,dx + q\,dy,$$

we get

$$dz = \frac{\sqrt{az^2 - a^2y^2}}{z}dx + \frac{ay}{z}dy$$

or

$$z\,dz - ay\,dy = \sqrt{az^2 - a^2y^2}\;dx$$

which can be rewritten as

$$\frac{d\,(az^2 - a^2y^2)^{1/2}}{a} = dx$$

On integration, we find

$$\frac{\sqrt{az^2 - a^2y^2}}{a} = x + b$$

or

$$(x+b)^2 = (z^2/a) - y^2$$

Hence, the complete integral is

$$(x+b)^2 + y^2 = z^2/a.$$

EXAMPLE 0.17 Find the complete integral of the PDE:

$$z^2 = pq\,xy.$$

Solution In this example, given

$$f = z^2 - pq\,xy. \tag{1}$$

Then, we have

$$f_x = -pqy, \qquad f_y = -pqx, \qquad f_z = 2z$$

$$f_p = -qxy, \qquad f_q = -pxy.$$

Now, the Charpit's auxiliary equations are given by

$$\frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{-(f_x+pf_z)}=\frac{dq}{-(f_y+qf_z)}.$$

That is,

$$\frac{dx}{-qxy}=\frac{dy}{-pxy}=\frac{dz}{-2pq\,xy}=\frac{dp}{pqy-2pz}=\frac{dq}{pqx-2qz} \tag{2}$$

From Eq. (2), it follows that

$$\frac{dp/p}{qy-2z}=\frac{dq/q}{px-2z}=\frac{dx/x}{-qy}=\frac{dy/y}{-px}$$

which can be rewritten as

$$\frac{dp/p-dq/q}{qy-px}=\frac{-dx/x+dy/y}{qy-px}$$

or

$$\frac{dp}{p}-\frac{dq}{q}=\frac{dy}{y}-\frac{dx}{x}$$

On integration, we find

$$\frac{p}{q}\,\frac{x}{y}=c\,(\text{constant})$$

or

$$p=cqy/x$$

From the given PDE, we have

$$z^2=pq\,xy=cq^2y^2$$

which gives

$$q^2=z^2/cy^2 \quad \text{or} \quad q=z/\sqrt{c}\,y=az/y,$$

where $a=1/\sqrt{c}$.

Hence,

$$p=z/ax.$$

Substituting these values of p and q in

$$dz=p\,dx+q\,dy,$$

we get

$$dz=\frac{z}{ax}dx+\frac{az}{y}dy$$

or

$$\frac{dz}{z} = \frac{1}{a}\frac{dx}{x} + a\frac{dy}{y}$$

On integration, we obtain

$$\ln z = \frac{1}{a}\ln x + a\ln y + \ln b$$

or

$$z = bx^{1/a}y^{a}$$

which is the complete integral of the given PDE.

EXAMPLE 0.18 Find the complete integral of

$$x^2p^2 + y^2q^2 - 4 = 0$$

using Charpit's method.

Solution The Charpit's equations for the given PDE can be written as

$$\frac{dx}{2x^2p} = \frac{dy}{2y^2q} = \frac{dz}{2(x^2p^2 + y^2q^2)} = \frac{dp}{-2xp^2}$$

$$= \frac{dq}{-2yq^2} \tag{1}$$

Considering the first and last but one of Eq. (1), we have

$$\frac{dx}{2x^2p} = \frac{dp}{-2xp^2} \quad \text{or} \quad \frac{dx}{x} + \frac{dp}{p} = 0$$

On integration, we get

$$\ln (xp) = \ln a \quad \text{or} \quad xp = a \tag{2}$$

From the given PDE and using the result (2), we get

$$y^2q^2 = 4 - a^2 \tag{3}$$

Substituting one set of p and q values from Eqs. (2) and (3) in

$$dz = p\,dx + q\,dy,$$

we find that

$$dz = a\frac{dx}{x} + \sqrt{4 - a^2}\,\frac{dy}{y}.$$

On integration, the complete integral of the given PDE is found to be

$$z = a\ln x + \sqrt{4 - a^2}\,\ln y + b.$$

0.11.1 Special Types of First Order Equations

Type I Equations Involving p and q only.
That is, equations of the type

$$f(p, q) = 0. \tag{0.100}$$

Let $z = ax + by + c = 0$ is a solution of the given PDE, described by $f(p, q) = 0$, then

$$p = \frac{\partial z}{\partial x} = a, \qquad q = \frac{\partial z}{\partial y} = b.$$

Substituting these values of p and q in the given PDE, we get

$$f(a, b) = 0 \tag{0.101}$$

Solving for b, we get, $b = \phi(a)$, say. Then,

$$z = ax + \phi(a)\, y + c \tag{0.102}$$

is the complete integral of the given PDE.

EXAMPLE 0.19 Find a complete integral of the equation

$$\sqrt{p} + \sqrt{q} = 1.$$

Solution The given PDE is of the form $f(p, q) = 0$. Therefore, let us assume the solution in the form

$$z = ax + by + c$$

where

$$\sqrt{a} + \sqrt{b} = 1 \quad \text{or} \quad b = (1 - \sqrt{a})^2$$

Hence, the complete integral is found to be

$$z = ax + (1 - \sqrt{a})^2\, y + c.$$

EXAMPLE 0.20 Find the complete integral of the PDE

$$pq = 1.$$

Solution Since the given PDE is of the form $f(p, q) = 0$, we assume the solution in the form $z = ax + by + c$, where $ab = 1$ or $b = 1/a$. Hence, the complete integral is

$$z = ax + \frac{1}{a}\, y + c.$$

Type II Equations Not Involving the Independent Variables.
That is, equations of the type

$$f(z, p, q) = 0 \tag{0.103}$$

As a trial solution, let us assume that z is a function of $u = x + ay$, where a is an arbitrary constant. Then,

$$z = f(u) = f(x + ay) \tag{0.104}$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = a\frac{dz}{du}$$

Substituting these values of p and q in the given PDE, we get

$$f\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0 \tag{0.105}$$

which is an ordinary differential equation of first order.

Solving Eq. (0.105) for dz/du, we obtain

$$\frac{dz}{du} = \phi(z, a) \quad \text{(say)}$$

or

$$\frac{dz}{\phi(z, a)} = du.$$

On integration, we find

$$\int \frac{dz}{\phi(z, a)} = u + c$$

That is,

$$F(z, a) = u + c = x + ay + c \tag{0.106}$$

which is the complete integral of the given PDE.

EXAMPLE 0.21 Find the complete integral of

$$p(1 + q) = qz$$

Solution Let us assume the solution in the form

$$z = f(u) = x + ay$$

Then,

$$p = \frac{dz}{du}, \qquad q = a\frac{dz}{du}$$

Substituting these values in the given PDE, we get

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = az\frac{dz}{du}.$$

That is,

$$a\frac{dz}{du} = az - 1 \quad \text{or} \quad a\frac{dz}{az-1} = du$$

On integration, we find

$$\ln (az - 1) = u + c = x + ay + c$$

which is the required complete integral.

EXAMPLE 0.22 Find the complete integral of the PDE:

$$p^2z^2 + q^2 = 1.$$

Solution Let us assume that $z = f(u) = x + ay$ is a solution of the given PDE. Then,

$$p = \frac{dz}{du}, \quad q = a\frac{dz}{du}$$

Substituting these values of p and q in the given PDE, we obtain

$$\left(\frac{dz}{du}\right)^2 z^2 + a^2\left(\frac{dz}{du}\right)^2 = 1$$

That is,

$$\left(\frac{dz}{du}\right)^2 (z^2 + a^2) = 1 \quad \text{or} \quad \frac{dz}{du} = \frac{1}{\sqrt{z^2 + a^2}}$$

or

$$\sqrt{z^2 + a^2}\; dz = du$$

On integration, we get

$$\frac{z\sqrt{z^2+a^2}}{2} + \frac{a^2}{2}\ln\left[\frac{z + \sqrt{z^2+a^2}}{a}\right] = x + ay + b$$

which is the required complete integral of the given PDE.

Type III Separable Equations.

An equation in which z is absent and the terms containing x and p can be separated from those containing y and q is called a separable equation.

That is, equations of the type

$$f(x, p) = F(y, q) \tag{0.107}$$

As a trial solution, let us assume that

$$f(x, p) = F(y, q) = a \text{ (say)} \tag{0.108}$$

Now, solving them for p and q, we obtain

$$p = \phi(x), \quad q = \psi(y)$$

Since

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = p\,dx + q\,dy$$

or

$$dz = \phi(x)\,dx + \psi(y)\,dy$$

On integration, we get the complete integral in the form

$$z = \int \phi(x)\,dx + \int \psi(y)\,dy + b \tag{0.109}$$

EXAMPLE 0.23 Find the complete integral of the PDE:

$$p^2 y(1+x^2) = qx^2$$

Solution The given PDE is of separable type and can be rewritten as

$$\frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = a \quad \text{(say), an arbitrary constant.}$$

Then,

$$p = \frac{\sqrt{a}x}{\sqrt{1+x^2}}, \quad q = ay$$

Substituting these values of p and q in

$$dz = p\,dx + q\,dy,$$

we get

$$dz = \frac{\sqrt{a}x}{\sqrt{1+x^2}} dx + ay\,dy$$

On integration, we obtain

$$z = \sqrt{a}\sqrt{1+x^2} + \frac{a}{2}y^2 + b$$

which is the complete integral of the given PDE.

EXAMPLE 0.24 Find the complete integral of

$$p^2 + q^2 = x + y$$

Solution The given PDE is of separable type and can be rewritten as

$$p^2 - x = y - q^2 = a \quad \text{(say)}$$

Then,

$$p=\sqrt{x+a}, \qquad q=\sqrt{y+a}.$$

Now, substituting these values of p and q in

$$dz = p\,dx + q\,dy$$

we find

$$dz=\sqrt{x+a}\;dx+\sqrt{y+a}\;dy$$

On integration, the complete integral is found to be

$$z=\frac{2}{3}(x+a)^{3/2}+\frac{2}{3}(y+a)^{3/2}+b.$$

Type IV Clairaut's Form

A first order PDE is said to be of Clairaut's form if it can be written as

$$z = px+qy+f(p,q) \tag{0.110}$$

The corresponding Charpit's equations are

$$\frac{dx}{x+f_p}=\frac{dy}{y+f_q}=\frac{dz}{px+qy+pf_p+qf_q}$$

$$=\frac{dp}{p-p}=\frac{dq}{q-q} \tag{0.111}$$

The integration of the last two equations of (0.111) gives us

$$p=a, \qquad q=b$$

Substituting these values of p and q in the given PDE, we get the required complete integral in the form

$$z=ax+by+f(a,b) \tag{0.112}$$

EXAMPLE 0.25 Find the complete integral of the equation

$$z = px+qy+\sqrt{1+p^2+q^2}$$

Solution The given PDE is in the Clairaut's form. Hence, its complete integral is

$$z=ax+by+\sqrt{1+a^2+b^2}.$$

EXAMPLE 0.26 Find the complete integral of

$$(p+q)(z-xp-yq)=1$$

Solution The given PDE can be rewritten as

$$z = xp + yq + \frac{1}{p+q}$$

which is in the Clairaut's form,

$$z = px + qy + f(p, q)$$

Hence, the complete integral of the given PDE is

$$z = ax + by + \frac{1}{a+b}.$$

EXAMPLE 0.27 Solve the following Cauchy IVP:

PDE: $u_t + cu_x = 0,\ x \in \mathbf{R},\ t > 0$

IC: $u(x, 0) = f(x),\ x \in \mathbf{R}$.

Solution The characteristic equations for the given problem is given by $\dfrac{dt}{1} = \dfrac{dx}{c} = \dfrac{du}{0}$.

On integration, we get: u = constant and $x - ct = \xi$ (constant). This linear problem has a unique solution, given by $u(x, t) = f(x - ct)$.

This is a right travelling wave with speed c, of course with no change in shape. The characteristic line $x = \xi + ct$, gives rise to a system of parallel, straight lines in the (x, t)-plane as shown in Fig. 0.6.

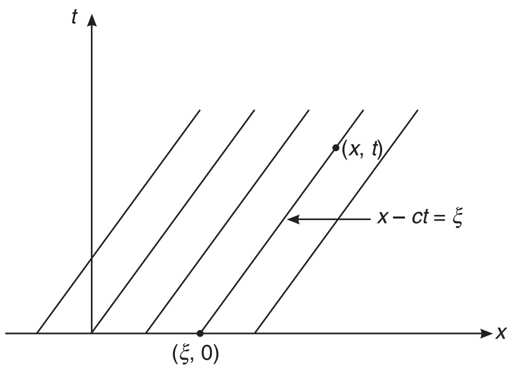

Fig. 0.6 The family of characteristic lines: $x - ct = \xi$.

We observe that one of these lines that passes through the point (x, t) intersects the x-axis at $(\xi, 0)$ moving with speed c, having a slope $-1/c$.

EXAMPLE 0.28 In classical mechanics, the Hamilton–Jacobi equation for the problem of one-dimensional, Harmonic oscillator is given by the differential equation as (see–Sankara Rao, 2005).

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}Kq^2 + \frac{\partial S}{\partial t} = 0,$$

where $S = S(p, q, t)$, $p = \dfrac{\partial S}{\partial q}$ and K is a constant. Using Charpits method, find S.

Solution Following the notation of Eq. (0.94) we rewrite

$$f(t, q, S, S_t, S_q) = \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}Kq^2 + \frac{\partial S}{\partial t} \tag{1}$$

which gives us

$$f_t = 0,\ f_q = Kq,\ f_S = 0,\ f_{S_t} = 1,\ f_{S_q} = \frac{S_q}{m}.$$

Then, the Charpits auxiliary equations (0.99) assumes the following form:

$$\frac{dt}{1} = \frac{dq}{S_q/m} = \frac{dS}{S_t + S_q^2/m} = \frac{dS_t}{0} = \frac{dS_q}{-Kq} \tag{2}$$

Considering the second and last members, we have $\dfrac{dq}{S_q/m} = \dfrac{dS_q}{-Kq}$.

On integration, we get

$$\frac{S_q^2}{2m} + \frac{1}{2}Kq^2 = a \quad \text{(constant of integration).}$$

Equation (1) then becomes

$$S_t = -a,$$

and

$$S_q^2 = Km\left(\frac{2a}{K} - q^2\right).$$

Substituting S_t and S_q into

$$dS = S_t\,dt + S_q\,dq$$

and integrating, we arrive at

$$S = -at + \sqrt{Km}\int\left(\frac{2a}{K} - q^2\right)^{1/2} dq + C$$

or

$$S = -at + \sqrt{Km}\int(\alpha^2 - q^2)^{1/2}\,dq + C$$

where $\alpha^2 = \dfrac{2a}{K}$ and C is another constant of integration.

EXERCISES

1. Eliminate the arbitrary function in the following and hence obtain the corresponding PDE

$$z = x + y + f(xy).$$

2. Form the PDE from the following by eliminating the constants

$$z = (x^2 + a)(y^2 + b).$$

3. Find the integral surface (general solution) of the differential equation

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x + y) z.$$

4. Find the general integrals of the following linear PDEs:

   (i) $\dfrac{y^2 z}{x} p + xzq = y^2$

   (ii) $(y+1) p + (x+1) q = z.$

5. Find the integral surface of the linear PDE

$$xp - yq = z$$

which contains the circle $x^2 + y^2 = 1,\ z = 1.$

6. Find the equation of the integral surface of the PDE

$$2y(z-3)p + (2x - z)q = y(2x-3)$$

which contains the circle $x^2 + y^2 = 2x,\ z = 0.$

7. Find the general integral of the PDE

$$(x - y) p + (y - x - z) q = z$$

which contains the circle $x^2 + y^2 = 1,\ z = 1.$

8. Find the solution of the equation

$$z = \frac{1}{2}(p^2 + q^2) + (p - x)(q - y)$$

which passes through the x-axis.

9. Find the characteristics of the equation

$$pq = xy$$

and determine the integral surface which passes through the curve $z = x,\ y = 0.$

10. Determine the characteristics of the equation

$$z = p^2 - q^2$$

and find the integral surface which passes through the parabola $4z + x^2 = 0,\ y = 0.$

11. Show that the PDEs

$$xp = yq \quad \text{and} \quad z(xp + yq) = 2xy$$

are compatible and hence find its solution.

12. Show that the equations

$$p^2 + q^2 = 1 \quad \text{and} \quad (p^2 + q^2)x = pz$$

are compatible and hence find its solution.

13. Find the complete integral of the equation

$$(p^2 + q^2)x = pz$$

where $p = \partial z/\partial x$, $q = \partial z/\partial y$.

14. Find the complete integrals of the equations

(i) $px^5 - 4q^3x^2 + 6x^2z - 2 = 0$

(ii) $2(z + xp + yq) = yp^2$.

15. Find the complete integral of the equation

$$p + q = pq.$$

16. Find the complete integrals of the following equations:

(i) $z\,pq = p + q$

(ii) $p^2q^2 + x^2y^2 = x^2q^2(x^2 + y^2)$.

17. Find the complete integral of the PDE

$$z = px + qy - \sin(pq)$$

18. Find the complete integrals of the following PDEs:

(i) $xp^3q^2 + yp^2q^3 + (p^3 + q^3) - zp^2q^2 = 0$

(ii) $p\,qz = p^2(xq + p^2) + q^2(yp + q^2)$.

19. Find the surface which intersects the surfaces of the system

$$z(x + y) = c(3z + 1)$$

orthogonally and passes through the circle

$$x^2 + y^2 = 1, \; z = 1.$$

20. Find the complete integral of the equation

$$(p^2 + q^2)x = pz$$

where $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$. (GATE-Maths, 1996)

21. Find the integral surface of the linear PDE

$$(x-y)\frac{\partial z}{\partial x}-(x-y+z)\frac{\partial z}{\partial y}=z$$

which contains $z=1$ and $x^2+y^2=1$. (GATE-Maths, 1999)

Choose the correct answer in the following questions:

22. Using the transformation $u=W/y$ in the PDE $xu_x=u+yu_y$, the transformed equation has a solution of the form $W=$

(A) $f(x/y)$ (B) $f(x+y)$

(C) $f(x-y)$ (D) $f(xy)$. (GATE-Maths, 1997)

23. The complete integral of the partial differential equation

$$xp^3q^2+yp^2q^3+(p^3+q^3)-zp^2q^2=0$$

is $z=$

(A) $ax+by+(ab^{-2}+ba^{-2})$ (B) $ax-by+(ab^{-2}+ba^{-2})$

(C) $-ax+by+(ba^{-2}-ab^{-2})$ (D) $ax+by-(ab^{-2}+ba^{-2})$.

(GATE-Maths, 1997)

24. The partial differential equation of the family of surfaces $z=(x+y)+A(xy)$ is

(A) $xp-yq=0$ (B) $xp-yq=x-y$

(C) $xp+yq=x+y$ (D) $xp+yq=0$.

(GATE-Maths, 1998)

25. The complete integral of the PDE $z=px+qy-\sin(pq)$ is

(A) $z=ax+by+\sin(ab)$ (B) $z=ax+by-\sin(ab)$

(C) $z=ax+y+\sin(b)$ (D) $z=x+by-\sin(a)$.

(GATE-Maths, 2003)

Chapter 1

Fundamental Concepts

1.1 INTRODUCTION

Many practical problems in science and engineering, when formulated mathematically, give rise to partial differential equations (often referred to as PDE). In order to understand the physical behaviour of the mathematical model, it is necessary to have some knowledge about the mathematical character, properties, and the solution of the governing PDE. An equation which involves several independent variables (usually denoted by x, y, z, t, ...), a dependent function u of these variables, and the partial derivatives of the dependent function u with respect to the independent variables such as

$$F(x, y, z, t, \ldots, u_x, u_y, u_z, u_t, \ldots, u_{xx}, u_{yy}, \ldots, u_{xy}, \ldots) = 0 \tag{1.1}$$

is called a partial differential equation. A few well-known examples are:

(i) $u_t = k(u_{xx} + u_{yy} + u_{zz})$ [linear three-dimensional heat equation]

(ii) $u_{xx} + u_{yy} + u_{zz} = 0$ [Laplace equation in three dimensions]

(iii) $u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz})$ [linear three-dimensional wave equation]

(iv) $u_t + uu_x = \mu u_{xx}$ [nonlinear one-dimensional Burger equation].

In all these examples, u is the dependent function and the subscripts denote partial differentiation with respect to these variables.

Definition 1.1 The order of the partial differential equation is the order of the highest derivative occurring in the equation. Thus the above examples are partial differential equations of second order, whereas

$$u_t = uu_{xxx} + \sin x$$

is an example for third order partial differential equation.

1.2 CLASSIFICATION OF SECOND ORDER PDE

The most general linear second order PDE, with one dependent function u on a domain Ω of points $X = (x_1, x_2, \ldots, x_n), n > 1,$ is

$$\sum_{i,\, j=1}^{n} a_{ij} u_{xi} x_j + \sum_{i=1}^{n} b_i u_{xi} + F(u) = G \tag{1.2}$$

The classification of a PDE depends only on the highest order derivatives present.

The classification of PDE is motivated by the classification of the quadratic equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{1.3}$$

which is elliptic, parabolic, or hyperbolic according as the discriminant $B^2 - 4AC$ is negative, zero or positive. Thus, we have the following second order linear PDE in two variables x and y:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{1.4}$$

where the coefficients A, B, C, … may be functions of x and y, however, for the sake of simplicity we assume them to be constants. Equation (1.4) is elliptic, parabolic or hyperbolic at a point (x_0, y_0) according as the discriminant

$$B^2(x_0, y_0) - 4A(x_0, y_0)\, C(x_0, y_0)$$

is negative, zero or positive. If this is true at all points in a domain Ω, then Eq. (1.4) is said to be elliptic, parabolic or hyperbolic in that domain. If the number of independent variables is two or three, a transformation can always be found to reduce the given PDE to a canonical form (also called normal form). In general, when the number of independent variables is greater than 3, it is not always possible to find such a transformation except in certain special cases. The idea of reducing the given PDE to a canonical form is that the transformed equation assumes a simple form so that the subsequent analysis of solving the equation is made easy.

1.3 CANONICAL FORMS

Consider the most general transformation of the independent variables x and y of Eq. (1.4) to new variables $\xi, \eta,$ where

$$\xi = \xi(x, y), \qquad \eta = \eta(x, y) \tag{1.5}$$

such that the functions ξ and η are continuously differentiable and the Jacobian

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = (\xi_x \eta_y - \xi_y \eta_x) \neq 0 \tag{1.6}$$

in the domain Ω where Eq. (1.4) holds. Using the chain rule of partial differentiation, the partial derivatives become

$$
\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x \\
u_y &= u_\xi \xi_y + u_\eta \eta_y \\
u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx} \\
u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy} \\
u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}
\end{aligned} \tag{1.7}
$$

Substituting these expressions into the original differential equation (1.4), we get

$$\bar{A}u_{\xi\xi} + \bar{B}u_{\xi\eta} + \bar{C}u_{\eta\eta} + \bar{D}u_\xi + \bar{E}u_\eta + \bar{F}u = \bar{G} \tag{1.8}$$

where

$$
\begin{aligned}
\bar{A} &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
\bar{B} &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
\bar{C} &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
\bar{D} &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
\bar{E} &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
\bar{F} &= F, \qquad \bar{G} = G
\end{aligned} \tag{1.9}
$$

It may be noted that the transformed equation (1.8) has the same form as that of the original equation (1.4) under the general transformation (1.5).

Since the classification of Eq. (1.4) depends on the coefficients *A*, *B* and *C*, we can also rewrite the equation in the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} = H(x, y, u, u_x, u_y) \tag{1.10}$$

It can be shown easily that under the transformation (1.5), Eq. (1.10) takes one of the following three canonical forms:

(i) $u_{\xi\xi} - u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$ (1.11a)

or

$u_{\xi\eta} = \phi, (\xi, \eta, u, u_\xi, u_\eta)$ in the hyperbolic case

(ii) $u_{\xi\xi} + u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta)$ in the elliptic case (1.11b)

(iii) $u_{\xi\xi} = \phi(\xi, \eta, u, u_\xi, u_\eta)$ (1.11c)

or

$$u_{\eta\eta} = \phi(\xi, \eta, u, u_\xi, u_\eta) \text{ in the parabolic case}$$

We shall discuss in detail each of these cases separately.

Using Eq. (1.9) it can also be verified that

$$\overline{B}^2 - 4\overline{A}\overline{C} = (\xi_x\eta_y - \xi_y\eta_x)^2 (B^2 - 4AC)$$

and therefore we conclude that the transformation of the independent variables does not modify the type of PDE.

1.3.1 Canonical Form for Hyperbolic Equation

Since the discriminant $\overline{B}^2 - 4\overline{A}\overline{C} > 0$ for hyperbolic case, we set $\overline{A} = 0$ and $\overline{C} = 0$ in Eq. (1.9), which will give us the coordinates ξ and η that reduce the given PDE to a canonical form in which the coefficients of $u_{\xi\xi}, u_{\eta\eta}$ are zero. Thus we have

$$\overline{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$

$$\overline{C} = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0$$

which, on rewriting, become

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0$$

$$A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0$$

Solving these equations for (ξ_x/ξ_y) and (η_x/η_y), we get

$$\frac{\xi_x}{\xi_y} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

$$\frac{\eta_x}{\eta_y} = \frac{-B - \sqrt{B^2 - 4AC}}{2A} \tag{1.12}$$

The condition $B^2 > 4AC$ implies that the slopes of the curves $\xi(x, y) = C_1, \eta(x, y) = C_2$ are real. Thus, if $B^2 > 4AC$, then at any point (x, y), there exists two real directions given by the two roots (1.12) along which the PDE (1.4) reduces to the canonical form. These are called *characteristic equations*. Though there are two solutions for each quadratic, we have considered only one solution for each. Otherwise we will end up with the same two coordinates.

Along the curve $\xi(x, y) = c_1$, we have

$$d\xi = \xi_x\, dx + \xi_y\, dy = 0$$

Hence,

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) \tag{1.13}$$

Similarly, along the curve $\eta(x, y) = c_2$, we have

$$\frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) \tag{1.14}$$

Integrating Eqs. (1.13) and (1.14), we obtain the equations of family of characteristics $\xi(x, y) = c_1$ and $\eta(x, y) = c_2$, which are called the characteristics of the PDE (1.4). Now to obtain the canonical form for the given PDE, we substitute the expressions of ξ and η into Eq. (1.8) which reduces to Eq. (1.11a).

To make the ideas clearer, let us consider the following example:

$$3u_{xx} + 10u_{xy} + 3u_{yy} = 0$$

Comparing with the standard PDE (1.4), we have $A = 3, B = 10, C = 3, B^2 - 4AC = 64 > 0$. Hence the given equation is a hyperbolic PDE. The corresponding characteristics are:

$$\frac{dy}{dx} = -\left(\frac{\xi_x}{\xi_y}\right) = -\left(\frac{-B + \sqrt{B^2 - 4AC}}{2A}\right) = \frac{1}{3}$$

$$\frac{dy}{dx} = -\left(\frac{\eta_x}{\eta_y}\right) = -\left(\frac{-B - \sqrt{B^2 - 4AC}}{2A}\right) = 3$$

To find ξ and η, we first solve for y by integrating the above equations. Thus, we get

$$y = 3x + c_1, \qquad y = \frac{1}{3}x + c_2$$

which give the constants as

$$c_1 = y - 3x, \qquad c_2 = y - x/3$$

Therefore,

$$\xi = y - 3x = c_1, \qquad \eta = y - \frac{1}{3}x = c_2$$

These are the characteristic lines for the given hyperbolic equation. In this example, the characteristics are found to be straight lines in the (x, y)-plane along which the initial data, impulses will propagate.

To find the canonical equation, we substitute the expressions for ξ and η into Eq. (1.9) to get

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 3(-3)^2 + 10(-3)(1) + 3 = 0$$

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

$$= 2(3)(-3)\left(-\frac{1}{3}\right) + 10\left[(-3)(1) + 1\left(-\frac{1}{3}\right)\right] + 2(3)(1)(1)$$

$$= 6 + 10\left(-\frac{10}{3}\right) + 6 = 12 - \frac{100}{3} = -\frac{64}{3}$$

$$\bar{C} = 0, \qquad \bar{D} = 0, \qquad \bar{E} = 0, \qquad \bar{F} = 0$$

Hence, the required canonical form is

$$\frac{64}{3}u_{\xi\eta} = 0 \quad \text{or} \quad u_{\xi\eta} = 0$$

On integration, we obtain

$$u(\xi, \eta) = f(\xi) + g(\eta)$$

where f and g are arbitrary. Going back to the original variables, the general solution is

$$u(x, y) = f(y - 3x) + g(y - x/3)$$

1.3.2 Canonical Form for Parabolic Equation

For the parabolic equation, the discriminant $\bar{B}^2 - 4\bar{A}\bar{C} = 0$, which can be true if $\bar{B} = 0$ and $\bar{A}$ or $\bar{C}$ is equal to zero. Suppose we set first $\bar{A} = 0$ in Eq. (1.9). Then we obtain

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$

or

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0$$

which gives

$$\frac{\xi_x}{\xi_y} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Using the condition for parabolic case, we get

$$\frac{\xi_x}{\xi_y} = -\frac{B}{2A} \tag{1.15}$$

Hence, to find the function $\xi = \xi(x, y)$ which satisfies Eq. (1.15), we set

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B}{2A}$$

and get the implicit solution

$$\xi(x, y) = C_1$$

In fact, one can verify that $\bar{A} = 0$ implies $\bar{B} = 0$ as follows:

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

Since $B^2 - 4AC = 0$, the above relation reduces to

$$\begin{aligned}\bar{B} &= 2A\xi_x\eta_x + 2\sqrt{AC}\,(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\ &= 2(\sqrt{A}\xi_x + \sqrt{C}\xi_y)(\sqrt{A}\eta_x + \sqrt{C}\eta_y)\end{aligned}$$

However,

$$\frac{\xi_x}{\xi_y} = -\frac{B}{2A} = -\frac{2\sqrt{AC}}{2A} = -\sqrt{\frac{C}{A}}$$

Hence,

$$\bar{B} = 2(\sqrt{A}\xi_x - \sqrt{A}\xi_x)(\sqrt{A}\eta_x + \sqrt{C}\eta_y) = 0$$

We therefore choose ξ in such a way that both $\bar{A}$ and $\bar{B}$ are zero. Then η can be chosen in any way we like as long as it is not parallel to the ξ-coordinate. In other words, we choose η such that the Jacobian of the transformation is not zero. Thus we can write the canonical equation for parabolic case by simply substituting ξ and η into Eq. (1.8) which reduces to either of the forms (1.11c).

To illustrate the procedure, we consider the following example:

$$x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} = e^x$$

The discriminant $B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$, and hence the given PDE is parabolic everywhere. The characteristic equation is

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B}{2A} = -\frac{2xy}{2x^2} = -\frac{y}{x}$$

On integration, we have

$$xy = c$$

and hence $\xi = xy$ will satisfy the characteristic equation and we can choose $\eta = y$. To find the canonical equation, we substitute the expressions for ξ and η into Eq. (1.9) to get

$$\bar{A} = Ay^2 + Bxy + cx^2 = x^2y^2 - 2x^2y^2 + y^2x^2 = 0$$
$$\bar{B} = 0, \qquad \bar{C} = y^2, \qquad \bar{D} = -2xy$$
$$\bar{E} = 0, \qquad \bar{F} = 0, \qquad \bar{G} = e^x$$

Hence, the transformed equation is

$$y^2 u_{\eta\eta} - 2xyu_\xi = e^x$$

or

$$\eta^2 u_{\eta\eta} = 2\xi u_\xi + e^{\xi/\eta}$$

The canonical form is, therefore,

$$u_{\eta\eta} = \frac{2\xi}{\eta^2}u_\xi + \frac{1}{\eta^2}e^{\xi/\eta}$$

1.3.3 Canonical Form for Elliptic Equation

Since the discriminant $B^2 - 4AC < 0$, for elliptic case, the characteristic equations

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A}$$

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A}$$

give us complex conjugate coordinates, say ξ and η. Now, we make another transformation from (ξ, η) to (α, β) so that

$$\alpha = \frac{\xi + \eta}{2}, \qquad \beta = \frac{\xi - \eta}{2i}$$

which give us the required canonical equation in the form (1.11b).

To illustrate the procedure, we consider the following example:

$$u_{xx} + x^2 u_{yy} = 0$$

The discriminant $B^2 - 4AC = -4x^2 < 0$. Hence, the given PDE is elliptic. The characteristic equations are

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -\frac{\sqrt{-4x^2}}{2} = -ix$$

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = ix$$

Integration of these equations yields

$$iy + \frac{x^2}{2} = c_1, \qquad -iy + \frac{x^2}{2} = c_2$$

Hence, we may assume that

$$\xi = \frac{1}{2}x^2 + iy, \qquad \eta = \frac{1}{2}x^2 - iy$$

Now, introducing the second transformation

$$\alpha = \frac{\xi + \eta}{2}, \qquad \beta = \frac{\xi - \eta}{2i}$$

we obtain

$$\alpha = \frac{x^2}{2}, \qquad \beta = y$$

The canonical form can now be obtained by computing

$$\bar{A} = A\alpha_x^2 + \beta\alpha_x\alpha_y + c\alpha_y^2 = x^2$$

$$\bar{B} = 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2c(\alpha_y\beta_y) = 0$$

$$\bar{C} = A\beta_x^2 + B\beta_x\beta_y + c\beta_y^2 = x^2$$

$$\bar{D} = A\alpha_{xx} + B\alpha_{xy} + c\alpha_{yy} + D\alpha_x + E\alpha_y = 1$$

$$\bar{E} = A\beta_{xx} + B\beta_{xy} + c\beta_{yy} + D\beta_x + E\beta_y = 0$$

$$\bar{F} = 0, \qquad \bar{G} = 0$$

Thus the required canonical equation is

$$x^2u_{\alpha\alpha} + x^2u_{\beta\beta} + u_\alpha = 0$$

or

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{u_\alpha}{2\alpha}$$

EXAMPLE 1.1 Classify and reduce the relation

$$y^2 u_{xx} - 2xyu_{xy} + x^2 u_{yy} = \frac{y^2}{x} u_x + \frac{x^2}{y} u_y$$

to a canonical form and solve it.

Solution The discriminant of the given PDE is

$$B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$$

Hence the given equation is of a parabolic type. The characteristic equation is

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B}{2A} = \frac{-2xy}{2y^2} = -\frac{x}{y}$$

Integration gives $x^2 + y^2 = c_1$. Therefore, $\xi = x^2 + y^2$ satisfies the characteristic equation. The η-coordinate can be chosen arbitrarily so that it is not parallel to ξ, i.e. the Jacobian of the transformation is not zero. Thus we choose

$$\xi = x^2 + y^2, \qquad \eta = y^2$$

To find the canonical equation, we compute

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 4x^2y^2 - 8x^2y^2 + 4x^2y^2 = 0$$

$$\bar{B} = 0, \qquad \bar{C} = 4x^2y^2, \qquad \bar{D} = \bar{E} = \bar{F} = \bar{G} = 0$$

Hence, the required canonical equation is

$$4x^2y^2 u_{\eta\eta} = 0 \quad \text{or} \quad u_{\eta\eta} = 0$$

To solve this equation, we integrate it twice with respect to η to get

$$u_\eta = f(\xi), \qquad u = f(\xi)\eta + g(\xi)$$

where $f(\xi)$ and $g(\xi)$ are arbitrary functions of ξ. Now, going back to the original independent variables, the required solution is

$$u = y^2 f(x^2 + y^2) + g(x^2 + y^2)$$

EXAMPLE 1.2 Reduce the following equation to a canonical form:

$$(1 + x^2)u_{xx} + (1 + y^2)u_{yy} + xu_x + yu_y = 0$$

Solution The discriminant of the given PDE is

$$B^2 - 4AC = -4(1 + x^2)(1 + y^2) < 0$$

Hence the given PDE is an elliptic type. The characteristic equations are

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -\frac{\sqrt{-4(1+x^2)(1+y^2)}}{2(1+x^2)} = -i\sqrt{\frac{1+y^2}{1+x^2}}$$

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = i\sqrt{\frac{1+y^2}{1+x^2}}$$

On integration, we get

$$\xi = \ln(x + \sqrt{x^2+1}) - i\ln(y + \sqrt{y^2+1}) = c_1$$

$$\eta = \ln(x + \sqrt{x^2+1}) + i\ln(y + \sqrt{y^2+1}) = c_2$$

Introducing the second transformation

$$\alpha = \frac{\xi + \eta}{2}, \qquad \beta = \frac{\eta - \xi}{2i}$$

we obtain

$$\alpha = \ln(x + \sqrt{x^2+1})$$

$$\beta = \ln(y + \sqrt{y^2+1})$$

Then the canonical form can be obtained by computing

$$\bar{A} = A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 = 1, \quad \bar{B} = 0, \quad \bar{C} = 1, \quad \bar{D} = \bar{E} = \bar{F} = \bar{G} = 0$$

Thus the canonical equation for the given PDE is

$$u_{\alpha\alpha} + u_{\beta\beta} = 0$$

EXAMPLE 1.3 Reduce the following equation to a canonical form and hence solve it:

$$u_{xx} - 2\sin x u_{xy} - \cos^2 x u_{yy} - \cos x u_y = 0$$

Solution Comparing with the general second order PDE (1.4), we have

$$A = 1, \quad B = -2\sin x, \quad C = -\cos^2 x,$$

$$D = 0, \quad E = -\cos x, \quad F = 0, \quad G = 0$$

The discriminate $B^2 - 4AC = 4(\sin^2 x + \cos^2 x) = 4 > 0$. Hence the given PDE is hyperbolic. The relevant characteristic equations are

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = -\sin x - 1$$

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = 1 - \sin x$$

On integration, we get

$$y = \cos x - x + c_1, \qquad y = \cos x + x + c_2$$

Thus, we choose the characteristic lines as

$$\xi = x + y - \cos x = c_1, \qquad \eta = -x + y - \cos x = c_2$$

In order to find the canonical equation, we compute

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$

$$= 2(\sin x + 1)(\sin x - 1) - 4\sin^2 x - 2\cos^2 x = -4$$

$$\bar{C} = 0, \qquad \bar{D} = 0, \qquad \bar{E} = 0, \qquad \bar{F} = 0, \qquad \bar{G} = 0$$

Thus, the required canonical equation is

$$u_{\xi\eta} = 0$$

Integrating with respect to ξ, we obtain

$$u_\eta = f(\eta)$$

where f is arbitrary. Integrating once again with respect to η, we have

$$u = \int f(\eta)\, d\eta + g(\xi)$$

or

$$u = \psi(\eta) + g(\xi)$$

where $g(\xi)$ is another arbitrary function. Returning to the old variables x, y, the solution of the given PDE is

$$u(x, y) = \psi(y - x - \cos x) + g(y + x - \cos x)$$

EXAMPLE 1.4 Reduce the Tricomi equation

$$u_{xx} + xu_{yy} = 0, \qquad x \neq 0$$

for all x, y to canonical form.

Solution The discriminant $B^2 - 4AC = -4x$. Hence the given PDE is of mixed type: hyperbolic for $x < 0$ and elliptic for $x > 0$.

Case I In the half-plane $x < 0$, the characteristic equations are

$$\frac{dy}{dx} = -\frac{\xi_x}{\xi_y} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{-2\sqrt{-x}}{2} = -\sqrt{-x}$$

$$\frac{dy}{dx} = -\frac{\eta_x}{\eta_y} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \sqrt{-x}$$

Integration yields

$$y = \frac{2}{3}(-x)^{3/2} + c_1$$

$$y = -\frac{2}{3}(-x)^{3/2} + c_2$$

Therefore, the new coordinates are

$$\xi(x, y) = \frac{3}{2}y - (\sqrt{-x})^3 = c_1$$

$$\eta(x, y) = \frac{3}{2}y + (\sqrt{-x})^3 = c_2$$

which are cubic parabolas.

In order to find the canonical equation, we compute

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = -\frac{9}{4}x + 0 + \frac{9}{4}x = 0$$

$$\bar{B} = 9x, \qquad \bar{C} = 0, \qquad \bar{D} = -\frac{3}{4}(-x)^{-1/2} = -\bar{E}, \qquad \bar{F} = \bar{G} = 0$$

Thus, the required canonical equation is

$$9xu_{\xi\eta} - \frac{3}{4}(-x)^{-1/2}u_\xi + \frac{3}{4}(-x)^{-1/2}u_\eta = 0$$

or

$$u_{\xi\eta} = \frac{1}{6(\xi - \eta)}(u_\xi - u_\eta)$$

Case II In the half-plane $x > 0$, the characteristic equations are given by

$$\frac{dy}{dx} = i\sqrt{x}, \qquad \frac{dy}{dx} = -i\sqrt{x}$$

On integration, we have

$$\xi(x, y) = \frac{3}{2}y - i(\sqrt{x})^3, \qquad \eta(x, y) = \frac{3}{2}y + i(\sqrt{x})^3$$

Introducing the second transformation

$$\alpha = \frac{\xi + \eta}{2}, \qquad \beta = \frac{\xi - \eta}{2i}$$

we obtain

$$\alpha = \frac{3}{2}y, \qquad \beta = -(\sqrt{x})^3$$

The corresponding normal or canonical form is

$$u_{\alpha\alpha} + u_{\beta\beta} + \frac{1}{3\beta} u_\beta = 0$$

EXAMPLE 1.5 Find the characteristics of the equation

$$u_{xx} + 2u_{xy} + \sin^2(x) u_{yy} + u_y = 0$$

when it is of hyperbolic type.

Solution The discriminant $B^2 - 4AC = 4 - 4\sin^2 x = 4\cos^2 x$. Hence for all $x \neq (2n - 1)\pi/2$, the given PDE is of hyperbolic type. The characteristic equations are

$$\frac{dy}{dx} = \frac{B \mp \sqrt{B^2 - 4AC}}{2A} = 1 \mp \cos x$$

On integration, we get

$$y = x - \sin x + c_1, \qquad y = x + \sin x + c_2$$

Thus, the characteristic equations are

$$\xi = y - x + \sin x, \qquad \eta = y - x - \sin x$$

EXAMPLE 1.6 Reduce the following equation to a canonical form and hence solve it:

$$yu_{xx} + (x + y) u_{xy} + xu_{yy} = 0$$

Solution The discriminant

$$B^2 - 4AC = (x + y)^2 - 4xy = (x - y)^2 > 0$$

Hence the given PDE is hyperbolic everywhere except along the line $y = x$; whereas on the line $y = x$, it is parabolic. When $y \neq x$, the characteristic equations are

$$\frac{dy}{dx} = \frac{B \mp \sqrt{B^2 - 4AC}}{2A} = \frac{(x + y) \mp (x - y)}{2y}$$

Therefore,

$$\frac{dy}{dx} = 1, \qquad \frac{dy}{dx} = \frac{x}{y}$$

On integration, we obtain

$$y = x + c_1, \qquad y^2 = x^2 + c_2$$

Hence, the characteristic equations are

$$\xi = y - x, \qquad \eta = y^2 - x^2$$

These are straight lines and rectangular hyperbolas. The canonical form can be obtained by computing

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = y - x - y + x = 0, \qquad \bar{B} = -2(x-y)^2,$$

$$\bar{C} = 0, \qquad \bar{D} = 0, \qquad \bar{E} = 2(x-y), \qquad \bar{F} = \bar{G} = 0$$

Thus, the canonical equation for the given PDE is

$$-2(x-y)^2 u_{\xi\eta} + 2(x-y)u_\eta = 0$$

or

$$-2\xi^2 u_{\xi\eta} + 2(-\xi)\,u_\eta = 0$$

or

$$\xi u_{\xi\eta} + u_\eta = \frac{\partial}{\partial\xi}\left(\xi\frac{\partial u}{\partial\eta}\right) = 0$$

Integration yields

$$\xi\frac{\partial u}{\partial\eta} = f(\eta)$$

Again integrating with respect to η, we obtain

$$u = \frac{1}{\xi}\int f(\eta)\,d\eta + g(\xi)$$

Hence,

$$u = \frac{1}{y-x}\int f(y^2 - x^2)\,d(y^2 - x^2) + g(y-x)$$

is the general solution.

EXAMPLE 1.7 Classify and transform the following equation to a canonical form:

$$\sin^2(x)u_{xx} + \sin(2x)u_{xy} + \cos^2(x)u_{yy} = x$$

Solution The discriminant of the given PDE is

$$B^2 - 4AC = \sin^2 2x - 4\sin^2 x\cos^2 x = 0$$

Hence, the given equation is of parabolic type. The characteristic equation is

$$\frac{dy}{dx} = \frac{B}{2A} = \cot x$$

Integration gives

$$y = \ln\sin x + c_1$$

Hence, the characteristic equations are:

$$\xi = y - \ln \sin x, \qquad \eta = y$$

η is chosen in such a way that the Jacobian of the transformation is nonzero. Now the canonical form can be obtained by computing

$$\bar{A} = 0, \qquad \bar{B} = 0, \qquad \bar{C} = \cos^2 x, \qquad \bar{D} = 1,$$
$$\bar{E} = 0, \qquad \bar{F} = 0, \qquad \bar{G} = x$$

Hence, the canonical equation is

$$\cos^2(x)\, u_{\eta\eta} + u_{\xi} = x$$

or

$$[1 - e^{2(\eta-\xi)}]u_{\eta\eta} = \sin^{-1}(e^{\eta-\xi}) - u_{\xi}$$

EXAMPLE 1.8 Show that the equation

$$u_{xx} + \frac{2N}{x}u_x = \frac{1}{a^2}u_{tt}$$

where N and a are constants, is hyperbolic and obtain its canonical form.

Solution Comparing with the general PDE (1.4) and replacing y by t, we have $A = 1$, $B = 0$, $C = -1/a^2$, $D = 2N/x$, and $E = F = G = 0$. The discriminant $B^2 - 4AC = 4/a^2 > 0$. Hence, the given PDE is hyperbolic. The characteristic equations are

$$\frac{dt}{dx} = \frac{B \mp \sqrt{B^2 - 4AC}}{2A} = \mp\frac{\sqrt{4/a^2}}{2} = \mp\frac{1}{a}$$

Therefore,

$$\frac{dt}{dx} = -\frac{1}{a}, \qquad \frac{dt}{dx} = \frac{1}{a}$$

On integration, we get

$$t = -\frac{x}{a} + c_1, \qquad t = \frac{x}{a} + c_2$$

Hence, the characteristic equations are

$$\xi = x + at, \qquad \eta = x - at$$

The canonical form can be obtained by computing

$$\bar{A} = A\xi_x^2 + B\xi_x\xi_t + C\xi_t^2 = 0,$$

$$\bar{B} = 2A\xi_x\eta_x + B(\xi_x\eta_t + \xi_t\eta_x) + 2C\xi_t\eta_t = 4,$$

$$\bar{C} = 0, \qquad \bar{D} = D\xi_x + E\xi_t = \frac{2N}{x}, \qquad \bar{E} = D\eta_x + E\eta_t = \frac{2N}{x}$$

Thus, the canonical equation for the given PDE is

$$4u_{\xi\eta} + \frac{2N}{x}(u_\xi + u_\eta) = 0$$

Expressing x in terms of ξ and η, the required canonical equation is

$$u_{\xi\eta} + \frac{N}{\xi + \eta}(u_\xi + u_\eta) = 0$$

EXAMPLE 1.9 Transform the following differential equation to a canonical form:

$$u_{xx} + 2u_{xy} + 4u_{yy} + 2u_x + 3u_y = 0$$

Solution The discriminant $B^2 - 4AC = -12 < 0$. Hence, the given PDE is elliptic. The characteristic equations are

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = +1 - i\sqrt{3}$$

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = +1 + i\sqrt{3}$$

Integration of these equations yields

$$y = +(1 - i\sqrt{3})x + c_1, \qquad y = +(1 + i\sqrt{3})x + c_2$$

Hence, we may take the characteristic equations in the form

$$\xi = y - (1 - i\sqrt{3})x, \qquad \eta = y - (1 + i\sqrt{3})x$$

In order to avoid calculations with complex variables, we introduce the second transformation

$$\alpha = \frac{\xi + \eta}{2}, \qquad \beta = \frac{\xi - \eta}{2i}$$

Therefore,

$$\alpha = y - x, \qquad \beta = \sqrt{3x}$$

The canonical form can now be obtained by computing

$$\bar{A} = A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 = 3$$

$$\bar{B} = 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y = 0$$

$$\bar{C} = A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2 = 3$$

$$\bar{D} = A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y = 1$$

$$\bar{E} = A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y = 2\sqrt{3}$$

$$\bar{F} = 0, \qquad \bar{G} = 0$$

Thus the required canonical form is

$$3u_{\alpha\alpha} + 3u_{\beta\beta} + u_\alpha + 2\sqrt{3}u_\beta = 0$$

or

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{3}(u_\alpha + 2\sqrt{3}u_\beta)$$

1.4 ADJOINT OPERATORS

Let

$$Lu = \phi \tag{1.16}$$

where L is a differential operator given by

$$L = a_0(x)\frac{d^n}{dx^n} + a_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_n(x)$$

One way of introducing the adjoint differential operator L^* associated with L is to form the product vLu and integrate it over the interval of interest. Let

$$\int_A^B vLu\,dx = [\quad]_A^B + \int_A^B uL^*v\,dx \tag{1.17}$$

which is obtained after repeated integration by parts. Here, L^* is the operator adjoint to L, where the functions u and v are completely arbitrary except that Lu and L^*v should exist.

EXAMPLE 1.10 Let $Lu = a(x)(d^2u/dx^2) + b(x)(du/dx) + c(x)u$; construct its adjoint L^*.

Solution Consider the equation

$$\int_A^B vLu\,dx = \int_A^B v\left[a(x)\frac{d^2u}{dx^2} + b(x)\frac{du}{dx} + c(x)u\right]dx$$

$$= \int_A^B (av)\frac{d^2u}{dx^2}dx + \int_A^B (bv)\frac{du}{dx}dx \int_A^B (cv)\,u\,dx$$

However,

$$\int_A^B (av)\frac{d^2u}{dx^2}dx = \int_A^B (av)\frac{d}{dx}(u')dx$$

$$= [u'va]_A^B - \int_A^B (av)'u'dx$$

$$= [u'av]_A^B - [u(av)']_A^B + \int_A^B u(av)''dx$$

$$\int_A^B (bv)\frac{du}{dx}dx = [u(bv)]_A^B - \int_A^B u(bv)'dx$$

$$\int_A^B (cv)u\,dx = \int_A^B u(cv)\,dx$$

Therefore,

$$\int_A^B vLu\, dx = [u'(av) - u(av)' + u(bv)]_A^B + \int_A^B u[(av)'' - (bv)' + (cv)]\, dx$$

Comparing this equation with Eq. (1.17), we get

$$L^* v = (av)'' - (bv)' + (cv) = av'' + (2a' - b)v' + (a'' - b' + c)v$$

Therefore,

$$L^* = a\frac{d^2}{dx^2} + (2a' - b)\frac{d}{dx} + (a'' - b' + c)$$

Consider the partial differential equation

$$L(u) = Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = \phi \tag{1.18}$$

which is valid in a region S of the xy-plane, where A, B, C, …, ϕ are functions of x and y. In addition, linear boundary conditions of the general form

$$\alpha u + \beta u_x = f$$

are prescribed over the boundary curve ∂S of the region S. Let

$$\iint_S vLu\, d\sigma = [\quad] + \iint_S uL^* v\, d\sigma$$

where the integrated part [ ] is a line integral evaluated over ∂S, the boundary of S, then L^* is called the adjoint operator. In general, a second order linear partial differential operator L is denoted by

$$L(u) = \sum_{i,\,j=1}^{n} A_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i \frac{\partial u}{\partial x_i} + Cu \tag{1.19}$$

Its adjoint operator is defined by

$$L^*(v) = \sum_{i,\,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(A_{ij}v) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(B_i v) + Cv \tag{1.20}$$

Here it is assumed that $A_{ij} \in C^{(2)}$ and $B_i \in C^{(1)}$. For any pair of functions $u, v \in C^{(2)}$, it can be shown that

$$vL(u) - uL^*(v) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[\sum_{j=1}^{n} A_{ij}\left(v\frac{\partial u}{\partial x_j} - u\frac{\partial v}{\partial x_j}\right) + uv\left(B_i - \sum_{j=1}^{n}\frac{\partial A_{ij}}{\partial x_j}\right)\right] \tag{1.21}$$

This is known as Lagrange's identity.

EXAMPLE 1.11 Construct an adjoint to the Laplace operator given by

$$L(u) = u_{xx} + u_{yy} \tag{1.22}$$

Solution Comparing Eq. (1.22) with the general linear PDE (1.19), we have $A_{11} = 1, A_{22} = 1.$ From Eq. (1.20), the adjoint of (1.22) is given by

$$L^*(v) = \frac{\partial^2}{\partial x^2}(v) + \frac{\partial^2}{\partial y^2}(v) = v_{xx} + v_{yy}$$

Therefore,

$$L^*(u) = u_{xx} + u_{yy}$$

Hence, the Laplace operator is a self-adjoint operator.

EXAMPLE 1.12 Find the adjoint of the differential operator

$$L(u) = u_{xx} - u_t \tag{1.23}$$

Solution Comparing Eq. (1.23) with the general second order PDE (1.19), we have $A_{11} = 1, \ B_1 = -1.$ From Eq. (1.20), the adjoint of (1.23) is given by

$$L^*(v) = \frac{\partial^2}{\partial x^2}(v) - \frac{\partial}{\partial t}(-v) = v_{xx} + v_t$$

Therefore,

$$L^*(u) = u_{xx} + u_t$$

It may be noted that the diffusion operator is not a self-adjoint operator.

1.5 RIEMANN'S METHOD

In Section 1.2, we have noted with interest that a linear second order PDE

$$L(u) = G(x, y)$$

is classified as hyperbolic if $B^2 > 4AC$, and it has two families of real characteristic curves in the xy-plane whose equations are

$$\xi = f_1(x, y) = c_1, \qquad \eta = f_2(x, y) = c_2$$

Here, (ξ, η) are the natural coordinates for the hyperbolic system. In the xy-plane, the curves $\xi(x, y) = c_1$ and $\eta(x, y) = c_2$ are the characteristics of the given PDE as shown in Fig. 1.1(a), while in the $\xi\eta$-plane, the curves $\xi = c_1$ and $\eta = c_2$ are families of straight lines parallel to the axes as shown in Fig. 1.1(b).

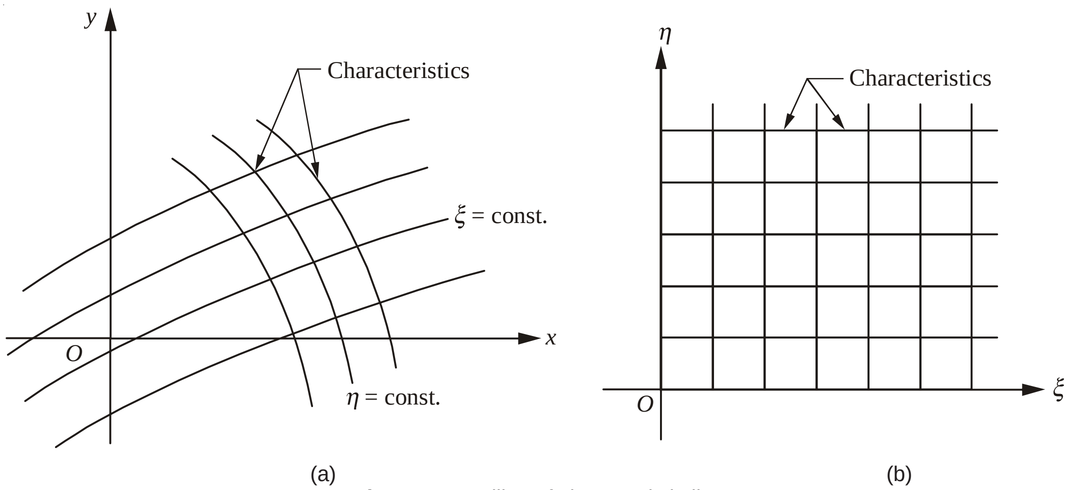

Fig. 1.1 Families of characteristic lines.

A linear second order partial differential equation in two variables, once classified as a hyperbolic equation, can always be reduced to the canonical form

$$\frac{\partial^2 u}{\partial x\, \partial y} = F(x, y, u, u_x, u_y)$$

In particular, consider an equation which is already reduced to its canonical form in the variables x, y:

$$L(u) = \frac{\partial^2 u}{\partial x\, \partial y} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu = F(x, y) \tag{1.24}$$

where L is a linear differential operator and a, b, c, F are functions of x and y only and are differentiable in some domain $\mathbb{R}$.

Let $v(x, y)$ be an arbitrary function having continuous second order partial derivatives. Let us consider the adjoint operator L^* of L defined by

$$L^*(v) = \frac{\partial^2 v}{\partial x\, \partial y} - \frac{\partial}{\partial x}(av) - \frac{\partial}{\partial y}(bv) + cv \tag{1.25}$$

Now we introduce

$$M = auv - u\frac{\partial v}{\partial y}, \qquad N = buv + v\frac{\partial u}{\partial x} \tag{1.26}$$

then

$$M_x + N_y = u_x(av) + u(av)_x - u_x v_y - uv_{xy} + u_y(bv) + u(bv)_y + v_y u_x + vu_{xy}$$

Adding and subtracting cuv, we get

$$M_x + N_y = -u\left[\frac{\partial^2 v}{\partial x\,\partial y} - \frac{\partial}{\partial x}(av) - \frac{\partial}{\partial y}(bv) + cv\right] + v\left[\frac{\partial^2 u}{\partial x\,\partial y} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu\right]$$

i.e.

$$vLu - uL^*v = M_x + N_y \tag{1.27}$$

This is known as *Lagrange identity* which will be used in the subsequent discussion. The operator L is a self-adjoint if and only if $L = L^*$. Now we shall attempt to solve Cauchy's problem which is described as follows: Let

$$L(u) = F(x, y) \tag{1.28}$$

with the condition (Cauchy data)

(i) $u = f(x)$ on Γ, a curve in the xy-plane;

(ii) $\dfrac{\partial u}{\partial n} = g(x)$ on Γ.

That is u, and its normal derivatives are prescribed on a curve Γ which is not a characteristic line.

Let Γ be a smooth initial curve which is also continuous as shown in Fig. 1.2. Since Eq. (1.24) is in canonical form, x and y are the characteristic coordinates. We also assume that the tangent to Γ is nowhere parallel to the coordinate axes.

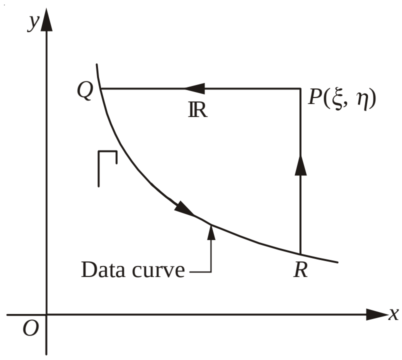

Fig. 1.2 Cauchy data.

Let $P(\xi, \eta)$ be a point at which the solution to the Cauchy problem is sought. Let us draw the characteristics PQ and PR through P to meet the curve Γ at Q and R. We assume that u, u_x, u_y are prescribed along Γ. Let $\partial\mathbb{R}$ be a closed contour $PQRP$ bounding $\mathbb{R}$. Since Eq. (1.28) is already in canonical form, the characteristics are lines parallel to x and y axes. Using Green's theorem, we have

$$\iint_R (M_x + N_y)\,dxdy = \oint_{\partial\mathbb{R}} (M\,dy - N\,dx) \tag{1.29}$$

where $\partial\mathbb{R}$ is the boundary of $\mathbb{R}$. Applying this theorem to the surface integral of Eq. (1.27), we obtain

$$\int_{\partial\mathbb{R}} (M\,dy - N\,dx) = \iint_{\mathbb{R}} [vL(u) - uL^*(v)]\,dx\,dy \tag{1.30}$$

In other words,

$$\int_{\Gamma} (M\,dy - N\,dx) + \int_{RP} (M\,dy - N\,dx) + \int_{PQ} (M\,dy - N\,dx) = \iint_{R} [vL(u) - uL^*(v)]\,dx\,dy$$

Now using the fact that $dy = 0$ on PQ and $dx = 0$ on PR, we have

$$\int_{\Gamma} (M\,dy - N\,dx) + \int_{RP} M\,dy - \int_{PQ} N\,dx = \iint_{\mathbb{R}} [vL(u) - uL^*(v)]\,dx\,dy \tag{1.31}$$

From Eq. (1.26), we find that

$$\int_{PQ} N\,dx = \int_P^Q buv\,dx + \int_P^Q vu_x\,dx$$

Integrating by parts the second term on the right-hand side and grouping, the above equation becomes

$$\int_{PQ} N\,dx = [uv]_p^Q + \int_P^Q u(bv - v_x)\,dx$$

Substituting this result into Eq. (1.31), we obtain

$$[uv]_p = [uv]_Q + \int_{PQ} u(bv - v_x)\,dx - \int_{RP} u(av - v_y)\,dy$$

$$- \int_{\Gamma} (M\,dy - N\,dx) + \iint_{\mathbb{R}} [vL(u) - uL^*(v)]\,dx\,dy \tag{1.32}$$

Let us choose $v(x, y; \xi, \eta)$ to be a solution of the adjoint equation

$$L^*(v) = 0 \tag{1.33}$$

and at the same time satisfy the following conditions:

(i) $v_x = bv$ when $y = \eta$, i.e., on PQ (1.34a)

(ii) $v_y = av$ when $x = \xi$, i.e., on PR (1.34b)

(iii) $v = 1$ when $x = \xi$, $y = \eta$ (1.34c)

We call this function $v(x, y; \xi, \eta)$ as the *Riemann function* or the *Riemann-Green function*. Since $L(u) = F$, Eq. (1.32) reduces to

$$[u]_P = [uv]_Q - \int_\Gamma [u(av - v_y)\,dy - v(bu + u_x)\,dx] + \iint_{\mathbb{R}} (vF)\,dx\,dy \qquad (1.35)$$

This is called the Riemann-Green solution for the Cauchy problem described by Eq. (1.28) when u and u_x are prescribed on Γ. Equation (1.35) can also be written as

$$[u]_P = [uv]_Q - \int_\Gamma uv(a\,dy - b\,dx) + \int_\Gamma (uv_y\,dy - vu_x\,dx) + \iint_{\mathbb{R}} (vF)\,dx\,dy \qquad (1.36)$$

This relation gives us the value of u at a point P when u and u_x are prescribed on Γ. But when u and u_y are prescribed on Γ, we obtain

$$[u]_P = [uv]_R - \int_\Gamma uv(a\,dy - b\,dx) - \int_\Gamma (uv_x\,dx + vu_y\,dy) + \iint_{\mathbb{R}} (vF)\,dx\,dy \qquad (1.37)$$

By adding Eqs. (1.36) and (1.37), the value of u at P is given by

$$[u]_P = \frac{1}{2}\{[uv]_Q + [uv]_R\} - \int_\Gamma uv(a\,dy - b\,dx) - \frac{1}{2}\int_\Gamma u(v_x\,dx - v_y\,dy)$$

$$+ \frac{1}{2}\int_\Gamma v(u_x\,dx - u_y\,dy) + \iint_{\mathbb{R}} (vF)\,dx\,dy \qquad (1.38)$$

Thus, we can see that the solution to the Cauchy problem at a point (ξ, η) depends only on the Cauchy data on Γ. The knowledge of the Riemann-Green function therefore enables us to solve Eq. (1.28) with the Cauchy data prescribed on a noncharacteristic curve.

EXAMPLE 1.13 Obtain the Riemann solution for the equation

$$\frac{\partial^2 u}{\partial x\,\partial y} = F(x, y)$$

given

(i) $u = f(x)$ on Γ

(ii) $\dfrac{\partial u}{\partial n} = g(x)$ on Γ

where Γ is the curve $y = x$.

Solution Here, the given PDE is

$$L(u) = \frac{\partial^2 u}{\partial x\,\partial y} = F(x, y) \qquad (1.39)$$

We construct the adjoint L^* of L as follows: setting

$$M = auv - uv_y, \qquad N = buv + vu_x$$

and comparing the given equation (1.39) with the standard canonical form of hyperbolic equation (1.24), we have

$$a = b = c = 0$$

Therefore,

$$M = -uv_y, \qquad N = vu_x \tag{1.40}$$

and

$$M_x + N_y = vu_{xy} - uv_{xy} = vL(u) - uL^*(v)$$

Thus,

$$L^*(v) = \frac{\partial^2 v}{\partial x\, \partial y} \tag{1.41}$$

Here, $L = L^*$ and is a self-adjoint operator. Using Green's theorem

$$\iint_{\mathbb{R}} (M_x + N_y)\, dx\, dy = \int_{\partial \mathbb{R}} (M\, dy - N\, dx)$$

we have

$$\iint_{\mathbb{R}} [vL(u) - uL^*(v)]\, dx\, dy = \int_{\partial \mathbb{R}} (M\, dy - N\, dx)$$

or

$$\iint_{\mathbb{R}} [vF - uL^*(v)]\, dx\, dy = \int_{\partial \mathbb{R}} (M\, dy - N\, dx) \tag{1.42}$$

But

$$\int_{\partial \mathbb{R}} (M\, dy - N\, dx) = \int_{\Gamma} (M\, dy - N\, dx) + \int_{QP} (M\, dy - N\, dx) + \int_{PR} (M\, dy - N\, dx) \tag{1.43}$$

where

$$\int_{\Gamma} (M\, dy - N\, dx) = \int_{\Gamma} \left(-u \frac{\partial v}{\partial y} dy - v \frac{\partial u}{\partial x} dx \right)$$

From Fig. 1.3, we have on Γ, $x = y$. Therefore, $dx = dy$. Hence

$$\int_{\Gamma} (M\, dy - N\, dx) = \int_{\Gamma} \left(-u \frac{\partial v}{\partial y} - v \frac{\partial u}{\partial x} \right) dx \tag{1.44}$$

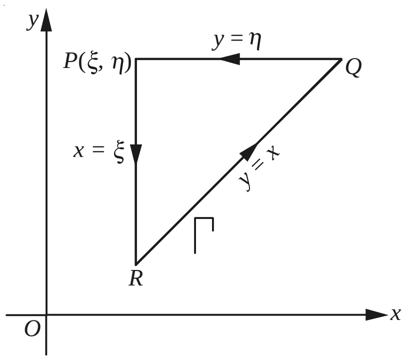

Fig. 1.3 An illustration of Example 1.13.

since on QP, y = constant. Therefore, $dy = 0$. Thus,

$$\int_{QP}(M\,dy - N\,dx) = \int_{QP} -N\,dx = \int_{QP} -v\frac{\partial u}{\partial x}dx \tag{1.45}$$

Similarly, on PR, x = constant. Hence, $dx = 0$. Thus,

$$\int_{PR}(M\,dy - N\,dx) = \int_{PR} M\,dy = \int_{PR} -u\frac{\partial v}{\partial y}dy \tag{1.46}$$

Substituting Eqs. (1.44)–(1.46) into Eq. (1.43), we obtain from Eq. (1.42), the relation

$$\iint_{\mathbb{R}} [vF - uL^*(v)]\,dx\,dy = \int_{\Gamma}\left(-u\frac{\partial v}{\partial y}dx - v\frac{\partial u}{\partial x}dx\right) + \int_{QP} -v\frac{\partial u}{\partial x}dx + \int_{PR} -u\frac{\partial v}{\partial y}dy$$

But

$$\int_{QP} -v\frac{\partial u}{\partial x}dx = [-vu]_Q^P + \int_{QP} u\frac{\partial v}{\partial x}dx$$

Therefore,

$$\iint_{\mathbb{R}} [vF - uL^*(v)]dx\,dy = \int_{\Gamma}\left(-u\frac{\partial v}{\partial y}dx - v\frac{\partial u}{\partial x}dx\right)$$

$$+[-vu]_Q^P + \int_{QP} u\frac{\partial v}{\partial x}dx + \int_{PR} -u\frac{\partial v}{\partial y}dy \tag{1.47}$$

Now choosing $v(x, y; \xi, \eta)$ as the solution of the adjoint equation such that

(i) $L^*v = 0$ throughout the xy-plane

(ii) $\dfrac{\partial v}{\partial x} = 0$ when $y = \eta$, i.e., on QP

(iii) $\dfrac{\partial v}{\partial y}=0$ when $x=\xi$, i.e., on PR

(iv) $v=1$ at $P(\xi,\eta)$.

Equation (1.47) becomes

$$\iint\limits_{R}(vF)\,dx\,dy=\int_{\Gamma}\left(-u\frac{\partial v}{\partial y}dx-v\frac{\partial u}{\partial x}dx\right)+(uv)_Q-(u)_P$$

or

$$(u)_P=(uv)_Q+\int_{\Gamma}\left(-u\frac{\partial v}{\partial y}dx-v\frac{\partial u}{\partial x}dx\right)-\iint\limits_{R}(vF)\,dx\,dy \tag{1.48}$$

However,

$$(uv)_Q-(uv)_R=\int_{\Gamma}d\,(uv)=\int_{\Gamma}\left[\frac{\partial}{\partial x}(uv)\,dx+\frac{\partial}{\partial y}(uv)\,dy\right]$$

$$=\int_{\Gamma}(u_x v\,dx+uv_x\,dx+u_y v\,dy+uv_y\,dy)$$

Now Eq. (1.48) can be rewritten as

$$(u)_P=(uv)_R=\int_{\Gamma}(u_y v\,dy+uv_y\,dy)-\iint\limits_{R}(vF)\,dx\,dy \tag{1.49}$$

Finally, adding Eqs. (1.48) and (1.49), we get

$$(u)_P=\frac{1}{2}[(uv)_Q+(uv)_R]+\frac{1}{2}\int_{\Gamma}(-uv_x\,dx-vu_x\,dx)$$

$$+\frac{1}{2}\int_{\Gamma}(u_y v\,dy+uv_y\,dy)-\iint\limits_{R}(vF)\,dx\,dy$$

EXAMPLE 1.14 Verify that the Green function for the equation

$$\frac{\partial^2 u}{\partial x\,\partial y}+\frac{2}{x+y}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)=0$$

subject to $u=0$, $\partial u/\partial x=3x^2$ on $y=x$, is given by

$$v(x,y;\xi,\eta)=\frac{(x+y)\{2xy+(\xi-\eta)(x-y)+2\xi\eta\}}{(\xi+\eta)^3}$$

and obtain the solution of the equation in the form

$$u=(x-y)(2x^2-xy+2y^2)$$

Solution In the given problem,

$$L(u) = \frac{\partial^2 u}{\partial x\, \partial y} + \frac{2}{x+y}\frac{\partial u}{\partial x} + \frac{2}{x+y}\frac{\partial u}{\partial y} = 0 \tag{1.50}$$

Comparing this equation with the standard canonical hyperbolic equation (1.24), we have

$$a = b = \frac{2}{x+y}, \quad C = 0, \quad F = 0$$

Its adjoint equation is $L^*(v) = 0$, where

$$L^*(v) = \frac{\partial^2 v}{\partial x\, \partial y} - \frac{\partial}{\partial x}\left(\frac{2v}{x+y}\right) - \frac{\partial}{\partial y}\left(\frac{2v}{x+y}\right). \tag{1.51}$$

such that

(i) $L^* v = 0$ throughout the xy-plane

(ii) $\dfrac{\partial v}{\partial x} = \dfrac{2}{x+y} v$ on PQ, i.e., on $y = \eta$

(iii) $\dfrac{\partial v}{\partial y} = \dfrac{2}{x+y} v$ on PR, i.e., on $x = \xi$
(1.52)

(iv) $v = 1$ at $P(\xi, \eta)$.

If v is defined by

$$v(x, y; \xi, \eta) = \frac{(x+y)}{(\xi+\eta)^3}[2xy + (\xi - \eta)(x - y) + 2\xi\eta] \tag{1.53}$$

Then

$$\frac{\partial v}{\partial x} = \frac{x+y}{(\xi+\eta)^3}[2y + (\xi - \eta)] + \left[\frac{2xy + (\xi - \eta)(x-y) + 2\xi\eta}{(\xi+\eta)^3}\right]$$

or

$$\frac{\partial v}{\partial x} = \frac{1}{(\xi+\eta)^3}[4xy + 2y^2 + 2x(\xi - \eta) + 2\xi\eta] \tag{1.54}$$

and

$$\frac{\partial^2 v}{\partial x\, \partial y} = \frac{4(x+y)}{(\xi+\eta)^3} \tag{1.55}$$

$$\frac{\partial v}{\partial y} = \frac{1}{(\xi+\eta)^3}[4xy + 2x^2 - 2y(\xi - \eta) + 2\xi\eta] \tag{1.56}$$

Using the results described by Eqs. (1.53)–(1.56), Eq. (1.51) becomes

$$L^*(v) = \frac{\partial^2 v}{\partial x\,\partial y} - \frac{2}{x+y}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{4v}{(x+y)^2}$$

$$= \frac{4(x+y)}{(\xi+\eta)^3} - \frac{2}{(x+y)(\xi+\eta)^3}[4xy + 2(x^2+y^2)]$$

or

$$L^*(v) = \frac{4(x+y)}{(\xi+\eta)^3} - \frac{4(x+y)}{(\xi+\eta)^3} = 0$$

Hence condition (i) of Eq. (1.52) is satisfied. Also, on $y = \eta$.

$$\frac{\partial v}{\partial x} = \frac{1}{(\xi+\eta)^3}[4x\eta + 2\eta^2 + 2x(\xi-\eta) + 2\xi\eta]$$

or

$$\left.\frac{\partial v}{\partial x}\right|_{y=\eta} = \frac{1}{(\xi+\eta)^3}[2\eta^2 + 2x(\xi+\eta) + 2\xi\eta] \tag{1.57}$$

Also, $2v/(x+y)$ at $y = \eta$ is given by

$$\frac{2v}{x+y} = \frac{2}{x+\eta}\,\frac{x+\eta}{(\xi+\eta)^3}[2x\eta + (\xi-\eta)(x-\eta) + 2\xi\eta]$$

$$= \frac{1}{(\xi+\eta)^3}[2\eta^2 + 2x(\xi+\eta) + 2\xi\eta] \tag{1.58}$$

From Eqs. (1.57) and (1.58), we get

$$\frac{\partial v}{\partial x} = \frac{2}{x+y}v \quad \text{at } y = \eta$$

Thus, property (ii) in Eq. (1.52) has been verified. Similarly, property (iii) can also be verified. Also, at $x = \xi$, $y = \eta$,

$$v = \frac{\xi+\eta}{(\xi+\eta)^3}[2\xi\eta + (\xi-\eta)^2 + 2\xi\eta] = \frac{(\xi+\eta)(\xi+\eta)^2}{(\xi+\eta)^3} = 1$$

Thus property (iv) in Eq. (1.52) has also been verified.

From Eqs. (1.50) and (1.51), we have

$$vL(u) - uL^*(v) = v\frac{\partial^2 u}{\partial x\,\partial y} - u\frac{\partial^2 v}{\partial x\,\partial y} + \frac{\partial}{\partial x}\left(\frac{2vu}{x+y}\right) + \frac{\partial}{\partial y}\left(\frac{2vu}{x+y}\right)$$

$$= \frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{2vu}{x+y}\right) + \frac{\partial}{\partial y}\left(\frac{2vu}{x+y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{2vu}{x+y} - u\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{2vu}{x+y} + v\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

where

$$M = \frac{2uv}{x+y} - u\frac{\partial v}{\partial y}, \qquad N = \frac{2vu}{x+y} + v\frac{\partial u}{\partial x}$$

Now using Green's theorem, we have

$$\iint\limits_{\mathbb{R}} [vL(u) - uL^*(v)]\,dx\,dy = \int_{\partial\mathbb{R}} (M\;dy - N\;dx) = \int_R^Q (M\;dy - N\;dx)$$

$$+ \int_Q^P (M\;dy - N\;dx) + \int_P^R (M\;dy - N\;dx) \tag{1.59}$$

(see Fig. 1.3) on QP, $y = C$. Hence, $dy = 0$; on PR, $x = C$. Therefore, $dx = 0$

$$= \int_R^Q \left[\left\{\frac{2uv}{x+y} - u\frac{\partial v}{\partial y}\right\}dy - \left\{\frac{2uv}{x+y} + v\frac{\partial u}{\partial x}\right\}dx\right]$$

$$-\int_Q^P \left[\left\{\frac{2uv}{x+y} + v\frac{\partial u}{\partial x}\right\}\right]dx + \int_P^R \left[\left\{\frac{2uv}{x+y} - u\frac{\partial v}{\partial y}\right\}\right]dy$$

However,

$$\int_Q^P \left(\frac{2uv}{x+y} + v\frac{\partial u}{\partial x}\right)dx = \int_Q^P \frac{2uv}{x+y}dx + (uv)_Q^P - \int_Q^P u\frac{\partial v}{\partial x}dx$$

Now, using the condition $u = 0$ on $y = x$, Eq. (1.59) becomes

$$\iint\limits_{\mathbb{R}} [vL(u) - uL^*(v)]\,dx\,dy = \int_R^Q \left(\frac{2uv}{x+y} - u\frac{\partial v}{\partial y}\right)dy - \int_R^Q \left(\frac{2uv}{x+y} + v\frac{\partial u}{\partial x}\right)dx$$

$$-\int_Q^P \frac{2uv}{x+y}dx - (uv)_P + (uv)_Q + \int_Q^P u\frac{\partial v}{\partial x}dx$$

$$+\int_Q^P \frac{2uv}{x+y}dy - \int_P^R \left(u\frac{\partial v}{\partial y}\right)dy$$

Also, using conditions (ii)–(iv) of Eq. (1.52), the above equation simplifies to

$$(u)_P = (uv)_Q - \int_R^Q v \frac{\partial u}{\partial x} dx$$

Now using the given condition, viz.

$$\frac{\partial u}{\partial x} = 3x^2 \text{ on } RQ$$

we obtain

$$(u)_P = (uv)_Q - 3\int_R^Q x^2 \left[\frac{2x[2x^2 + 2\xi\eta]}{(\xi+\eta)^3}\right] dx$$

$$= -\frac{12}{(\xi+\eta)^3} \int_\xi^\eta (x^5 + x^3 \xi\eta)\, dx$$

$$= -\frac{12}{(\xi+\eta)^3} \left[\frac{1}{6}(\eta^6 - \xi^6) + \frac{1}{4}\xi\eta(\eta^4 - \xi^4)\right]$$

$$= \frac{\xi^2 - \eta^2}{(\xi+\eta)^3} [2(\xi^4 + \xi^2\eta^2 + \eta^4) + 3\xi\eta(\xi^2 + \eta^2)]$$

$$= (\xi - \eta)(2\xi^2 - \xi\eta + 2\eta^2)$$

Therefore,

$$u(x, y) = (x - y)(2x^2 - xy + 2y^2)$$

Hence the result.

EXAMPLE 1.15 Show that the Green's function for the equation

$$\frac{\partial^2 u}{\partial x\, \partial y} + u = 0$$

is

$$v(x, y; \xi, \eta) = J_0 \sqrt{2(x-\xi)(y-\eta)}$$

where J_0 denotes Bessel's function of the first kind of order zero.

Solution Comparing with the standard canonical hyperbolic equation (1.24), we have

$$a = b = 0, \qquad c = 1$$

It is a self-adjoint equation and, therefore, the Green's function v can be obtained from

$$\frac{\partial^2 v}{\partial x\, \partial y} + v = 0$$

subject to

$$\frac{\partial v}{\partial x} = 0 \qquad \text{on } \; y = \eta$$

$$\frac{\partial v}{\partial y} = 0 \qquad \text{on } \; x = \xi$$

$$v = 1 \qquad \text{at } \; x = \xi, \; y = \eta$$

Let

$$\phi^k = a(x-\xi)(y-\eta)$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \phi}\frac{\partial \phi}{\partial x}$$

But

$$k\phi^{k-1}\frac{\partial \phi}{\partial x} = a(y-\eta)$$

Therefore,

$$\frac{\partial \phi}{\partial x} = \frac{a}{k}\phi^{1-k}(y-\eta)$$

Thus,

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial \phi}\frac{a}{k}\phi^{1-k}(y-\eta)$$

$$\frac{\partial^2 v}{\partial x\,\partial y} = \frac{\partial}{\partial y}\left[\frac{\partial v}{\partial \phi}\frac{a}{k}\phi^{1-k}(y-\eta)\right]$$

$$= \frac{a}{k}\left[\phi^{1-k}\frac{\partial v}{\partial \phi} + (1-k)\phi^{-k}\frac{\partial v}{\partial \phi}\frac{\partial \phi}{\partial y}(y-\eta) + \phi^{1-k}(y-\eta)\frac{\partial^2 v}{\partial \phi^2}\frac{\partial \phi}{\partial y}\right]$$

However,

$$\frac{\partial \phi}{\partial y} = \frac{a}{k}\phi^{1-k}(x-\xi)$$

Therefore,

$$\frac{\partial^2 v}{\partial x\,\partial y} = \frac{a}{k}\left[\phi^{1-k}\frac{\partial v}{\partial \phi} + (1-k)\phi^{-k}(x-\xi)(y-\eta)\frac{a}{k}\phi^{1-k}\frac{\partial v}{\partial \phi} + \phi^{1-k}(y-\eta)\frac{\partial^2 v}{\partial \phi^2}\frac{a}{k}\phi^{1-k}(x-\xi)\right]$$

Hence,

$$\frac{\partial^2 v}{\partial x\,\partial y} + v = 0$$

gives

$$\frac{a}{k}\left[\frac{\phi^k}{k}\phi^{2(1-k)}\frac{d^2 v}{d\phi^2} + \frac{\phi^{1-k}}{k}(1-k)\frac{dv}{d\phi} + \phi^{1-k}\frac{dv}{d\phi}\right] + v = 0$$

or

$$\frac{a}{k^2}\left(\phi^{2-k}\frac{d^2 v}{d\phi^2} + \phi^{1-k}\frac{dv}{d\phi}\right) + v = 0$$

or

$$\phi^2 v'' + \phi v' + \frac{k^2}{a}\phi^k v = 0$$

Let $k = 2,\ a = 4.$ Then the above equation reduces to

$$\phi^2 v'' + \phi v' + \phi^2 v = 0 = v'' + \frac{1}{\phi}v' + v \quad \text{(Bessel's equation)}$$

Its solution is known to be of the form

$$v = J_0(\phi) = J_0\sqrt{2(x-\xi)(y-\eta)}$$

which is the desired Green's function.

1.6 LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

An nth order linear PDE with constant coefficients can be written in the form

$$a_0\frac{\partial^n u}{\partial x^n} + a_1\frac{\partial^n u}{\partial x^{n-1}\partial y} + a_2\frac{\partial^n u}{\partial x^{n-2}\partial^2 y} + \cdots + a_n\frac{\partial^n u}{\partial y^n}$$
$$= f(x,\ y) \tag{1.60}$$

where $a_0,\ a_1,\ \ldots,\ a_n$ are constants; u is the dependent variable; x and y are independent variables. Introducing the standard differential operator notation, such as $D = \dfrac{\partial}{\partial x}$, $D' = \dfrac{\partial}{\partial y}$, the above equation can be rewritten as

$$F(D,\ D')u = (a_0 D^n + a_1 D^{n-1}D' + a_2 D^{n-2}D'^2 + \cdots + a_n D'^n)u = f(x,\ y) \tag{1.61}$$

It can also be written in more compact form as

$$F(D, D')u = \sum_i \sum_j C_{ij} D^i D'^j \tag{1.62}$$

where C_{ij} are constants, $D = \dfrac{\partial}{\partial x}, D' = \dfrac{\partial}{\partial y}, D^n = \dfrac{\partial^n}{\partial x^n}, D'^n = \dfrac{\partial^n}{\partial y^n}$, etc.

As in the case of linear ODE with constant coefficients, the complete solution of Eq. (1.61) consists of two parts:

(i) the complementary function (CF), which is the most general solution of the equation $F(D, D')u = 0$, the one containing, n arbitrary functions, where n is the order of the DE.

(ii) the particular integral (PI), is a particular solution, which is free from arbitrary constants or functions of the equation $F(D, D')u = f(x, y)$.

The complete solution of Eq. (1.61) is then

$$u = \text{CF} + \text{PI} \tag{1.63}$$

It may be noted that, if all the terms on the left-hand side of Eq. (1.61) are of the same order, it is said to be a homogeneous equation otherwise, it is a non-homogeneous equation. Now, we shall study few basic theorems as is the case in ODEs.

Theorem 1.1 If u_{CF} and u_{PI} are respectively the complementary function and particular integral of a linear PDE, then their sum $(u_{\text{CF}} + u_{\text{PI}})$ is a general solution of the given PDE.

Proof Since

$$F(D, D')u_{\text{CF}} = 0,$$

and

$$F(D, D')u_{\text{PI}} = f(x, y),$$

we arrive at

$$F(D, D')u_{\text{CF}} + F(D, D')u_{\text{PI}} = f(x, y).$$

showing that $(u_{\text{CF}} + u_{\text{PI}})$ is in fact a general solution of Eq. (1.61). Hence proved.

Theorem 1.2 If $u_1, u_2, \ldots, u_n$ are the solutions of the homogeneous PDE: $F(D, D')u = 0$, then $\sum_{i=1}^{n} C_i u_i$, where C_i are arbitrary constants, is also a solution.

Proof Since we observe that

$$F(D, D')(C_i u_i) = C_i F(D, D')u_i$$

and

$$F(D, D')\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} F(D, D')v_i$$

For any set of functions v_i, we find at once,

$$F(D, D')\sum_{i=1}^{n} C_i u_i = \sum_{i=1}^{n} F(D, D')(C_i u_i)$$

$$= \sum_{i=1}^{n} C_i F(D, D') u_i = 0$$

Hence proved.

We shall now classify linear differential operator $F(D, D')$ into reducible and irreducible types, in the sense that $F(D, D')$ is reducible if it can be expressed as the product of linear factors of the form $(aD + bD' + c)$, where a, b and c are constants, otherwise $F(D, D')$ is irreducible. For example, the operator

$$F(D, D')u = (D^2 - D'^2 + 3D + 2D' + 2)u$$

$$= (D + D' + 1)(D - D' + 2)u$$

is reducible. While the operator $F(D, D')u = (D^2 - D')$, is irreducible, due to the fact that it cannot be factored into linear factors.

1.6.1 General Method for Finding CF of Reducible Non-homogeneous Linear PDE

The general strategy adopted for finding the CF of reducible equations is stated in the following theorems:

Theorem 1.3 If the operator $F(D, D')$ is reducible that is, if $(a_i D + b_i D' + c_i)$ is a factor of $F(D, D')$ and $\phi_i(\xi)$ is an arbitrary function of a single variable ξ, then, if $a_i \neq 0$,

$$u_i = \exp\left(-\frac{c_i}{a_i}x\right) \phi_i(b_i x - a_i y) \tag{1.64}$$

is a solution of the equation $F(D, D')u = 0$ (Sneddon, 1986).

Proof Using product rule of differentiation, Eq. (1.64) gives

$$Du_i = \left(-\frac{c_i}{a_i}\right)\exp\left(-\frac{c_i}{a_i}x\right)\phi_i(b_i x - a_i y) + b_i \exp\left(-\frac{c_i}{a_i}x\right)\phi_i'(b_i x - a_i y)$$

$$= -\frac{c_i}{a_i}u_i + b_i \exp\left(-\frac{c_i}{a_i}x\right)\phi'(b_i x - a_i y).$$

Similarly, we get

$$D'u_i = -a_i \exp\left(-\frac{c_i}{a_i}x\right)\phi'(b_i x - a_i y).$$

Thus, we observe that

$$(a_iD + b_iD' + c_i)u_i = 0 \tag{1.65}$$

That is, if the operator $F(D, D')$ is reducible, the order in which the linear factors appear is immaterial. Thus, if

$$F(D, D')u_i = \left\{\prod_{j=1}^{n}{}'(a_jD + b_jD' + c_j)\right\}(a_iD + b_iD' + c_i)u_i \tag{1.66}$$

where, the prime on the product indicates that the factor corresponding to $i = j$ is omitted. Combining Eqs. (1.65) and (1.66), we arrive at the result $F(D, D')u_i = 0$. Hence proved.

It may be noted that if no two factors of Eq. (1.65) are linearly independent, then the general solution of Eq. (1.66) is the sum of the general solutions of the equations of the form (1.65). For illustration, we consider the following examples:

EXAMPLE 1.16 Solve the following equation $(D^2 + 2DD' + D'^2 - 2D - 2D')u = 0$.

Solution Observe that the given PDE is non-homogeneous and can be factored as

$$(D^2 + 2DD' + D'^2 - 2D - 2D')u = (D + D')(D + D' - 2)u.$$

Using the result of Theorem 1.3, we get the general solution or the CF as

$$u_{CF} = \phi_1(x - y) + e^{2x}\phi_2(x - y).$$

On similar lines, we can also establish the following result:

Theorem 1.4 (Sneddon, 1986) Let $(b_i\ D' + c_i)$ is a factor of $F(D, D')u$, and $\phi_i(\xi)$ is an arbitrary function of a single variable ξ, then, if $b_i \neq 0$, we have

$$u_i = \exp\left(-\frac{c_i}{b_i}y\right)\phi_i(b_ix) \tag{1.67}$$

as a solution of the equation $F(D, D')u = 0$.

Proof Suppose, the factorisation of $F(D, D') = 0$ gives rise to a multiple factors of the form $(a_iD + b_iD' + c_i)^n$, the solution of $F(D, D')u = 0$ can be obtained by the application of Theorems 1.3 and 1.4. For example, let us find the solution of

$$(a_iD + b_iD' + c_i)^2u = 0 \tag{1.68}$$

We set,

$$(a_iD + b_iD' + c_i)u = U,$$

then, Eq. (1.68) becomes

$$(a_iD + b_iD' + c_i)U = 0.$$

Using Theorem 1.3, its solution is found to be

$$U = \exp\left(-\frac{c_i}{a_i}x\right)\phi_i(b_ix - a_iy).$$

Further, assume that $a_i \neq 0$, now, in order to find u, we have to solve

$$(a_iD + b_iD' + c_i)u = \exp\left(-\frac{c_i}{a_i}x\right)\phi_i(b_ix - a_iy).$$

This is a first order PDE. Using the Lagrange's method (Section 0.8), its auxiliary equations are

$$\frac{dx}{a_i} = \frac{dy}{b_i} = \frac{du}{-c_iu + \exp\left(-\frac{c_i}{a_i}x\right)\phi_i(b_ix - a_iy)} \tag{1.69}$$

One solution of which is given by

$$b_ix - a_iy = \lambda \text{ (a constant)} \tag{1.70}$$

Substituting this solution into the first and third of the auxiliary equations, we obtain

$$\frac{dx}{a_i} = \frac{du}{-c_iu + \exp\left(-\frac{c_i}{a_i}x\right)\phi_i(\lambda)}$$

or

$$\frac{du}{dx} + \frac{c_i}{a_i}u = \frac{1}{a_i}\exp\left(-\frac{c_i}{a_i}x\right)\phi_i(\lambda).$$

This being an ODE, its solution can be readily written as

$$u \exp\left(\frac{c_i}{a_i}x\right) = \frac{1}{a_i}x\phi_i(\lambda) + \mu \text{ (constant)}$$

or

$$u = \frac{1}{a_i}[x\phi_i(\lambda) + \mu] \exp\left(-\frac{c_i}{a_i}x\right) \tag{1.71}$$

Thus, the solution of Eq. (1.68) is given by

$$u = [x\phi_i(b_ix - a_iy) + \psi_i(b_ix - a_iy]e^{-c_i/a_ix} \tag{1.72}$$

where ϕ_i and ψ_i are arbitrary functions.

In general, if there are n, multiple factors of the form $(a_iD + b_iD' + c_i)$, then the solution of $(a_iD + b_iD' + c_i)^n\ u = 0$ can be written as

$$u = \exp\left(-\frac{c_i}{a_i}x\right)\left[\sum_{j=1}^{n} x^{j-1}\phi_{ij}(b_ix - a_iy)\right] \tag{1.73}$$

Here follows an example for illustration.

EXAMPLE 1.17 Find the solution of the equation $(2D - D' + 4)(D + 2D' + 1)^2 u = 0$.

Solution The complementary function (CF) corresponding to the factor $(2D - D' + 4)$ is $e^{-2x}\,\phi(-x - 2y)$. Similarly, CF corresponding to

$$(D + 2D' + 1)^2 \text{ is } e^{-x}[\psi_1(2x - y) + x\psi_2(2x - y)].$$

Thus, the CF for the given PDE is given by

$$u = e^{-2x}\phi(x + 2y) + e^{-x}[\psi_1(2x - y) + x\psi_2(2x - y)],$$

where, ϕ, ψ_1, ψ_2 are arbitrary functions.

1.6.2 General Method to Find CF of Irreducible Non-homogeneous Linear PDE

If the operator $F(D, D')$ is irreducible, we can find the complementary function, containing as many arbitrary functions as we wish by a method which is stated in the following theorem:

Theorem 1.5 The solution of irreducible PDE $F(D, D')u = 0$ is

$$u = \sum_{i=1}^{\infty} c_i \exp(a_i x + b_i y) \tag{1.74}$$

Proof Let us assume the solution of $F(D, D')u = 0$ in the form, $u = ce^{ax+by}$, where a, b and c are constants to be determined. Then, we have

$$D^i u = ca^i e^{ax+by},\ D'^j u = cb^j\, e^{ax+by},$$

$$D^i D'^j u = ca^i b^j e^{ax+by}$$

Thus, $$F(D, D')u = 0 \text{ yields}$$

$$c[F(a,\ b)]e^{ax+by} = 0$$

where c is an arbitrary constant, not zero, holds true iff

$$F(a,\ b) = 0, \tag{1.75}$$

indicating that there exists infinite pair of values (a_i, b_i) satisfying Eq. (1.75). Hence,

$$u = \sum_{i=1}^{\infty} c_i\, e^{a_i x + b_i y} \tag{1.76}$$

is a solution of irreducible equation

$$F(D, D')u = 0, \tag{1.77}$$

provided $$F(a_i, b_i) = 0 \tag{1.78}$$

It may be noted that this method is applicable even for reducible equations. Here follows an example for illustration:

EXAMPLE 1.18 Solve the following equation $(2D^2 - D'^2 + D)u = 0$.

Solution The given equation is an irreducible non-homogeneous PDE. Using the result of Theorem 1.5, it follows immediately that

$$u = u_{CF} = \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y}$$

where a_i, b_i are related by $F(a_i, b_i) = 0$.

That is,

$$2a_i^2 - b_i^2 + a_i = 0$$

which gives $b_i^2 = 2a_i^2 + a_i$.

1.6.3 Methods for Finding the Particular Integral (PI)

To find the PI of Eq. (1.61), we rewrite the same in the form

$$u = \frac{1}{F(D, D')} f(x, y) \tag{1.79}$$

Very often, the operator $F^{-1}(D, D')$ can be expanded, using binomial theorem and interpret the operators D^{-1}, $(D')^{-1}$ as integrations. That is,

$$D^{-1} f(x, y) = \frac{1}{D} f(x, y) = \int_{y \text{ constant}} f(x, y)dx,$$

and

$$\frac{1}{D'} f(x, y) = \int_{x \text{ constant}} f(x, y)dy.$$

We present below different cases for finding the PI, depending on the nature of $f(x, y)$.

Case I Let $f(x, y) = \exp(ax + by)$, then

$$\frac{1}{F(D, D')} e^{ax+by} = \frac{1}{F(a, b)} e^{ax+by} \tag{1.80}$$

By direct differentiation, we find $D^i D'^j e^{ax+by} = a^i b^j e^{ax+by}$.

In other words,

$$F(D, D')e^{ax+by} = F(a, b)e^{ax+by},$$

that is,

$$e^{ax+by} = F(a, b) \frac{1}{F(D, D')} e^{ax+by}.$$

Dividing both sides by $F(a, b)$, we get

$$\frac{1}{F(D, D')} e^{ax+by} = \frac{1}{F(a, b)} e^{ax+by},$$

provided $F(a, b) \neq 0$.

Case II Let $f(x, y) = \sin(ax + by)$ or $\cos(ax + by)$, where a and b are constants, then, since

$$D^2 \sin(ax + by) = -a^2 \sin(ax + by)$$
$$DD' \sin(ax + by) = -ab \sin(ax + by)$$

$$D'^2 \sin(ax + by) = -b^2 \sin(ax + by)$$

We notice that

$$\frac{1}{F(D, D')} \sin(ax + by)$$

is obtained by setting, $D^2 = -a^2$, $DD' = -ab$, $D'^2 = -b^2$ provided $F(D, D') \neq 0$. Thus,

$$F(D^2, DD', D'^2) \sin(ax + by) = F(-a^2, -ab, -b^2) \sin(ax + by)$$

or

$$\frac{1}{F(D^2, DD', D'^2)} \sin(ax + by) = \frac{1}{F(-a^2, -ab, -b^2)} \sin(ax + by) \qquad (1.81)$$

Similarly,

$$\frac{1}{F(D^2, DD', D'^2)} \cos(ax + by) = \frac{1}{F(-a^2, -ab, -b^2)} \cos(ax + by) \qquad (1.82)$$

Case III Let $f(x, y)$ is of the form $x^p y^q$, where p and q are positive integers. Then, the PI can be obtained by expanding $F(D, D')$ in ascending powers of D or D'.

Case IV Let $f(x, y)$ is of the form $e^{ax+by}\, \phi(x, y)$.

Then,

$$F(D, D')[e^{ax+by}\, \phi(x, y)] = e^{ax+by}\, F(D + a, D' + b)\, \phi(x, y).$$

Let us recall Leibnitz's theorem for the nth derivative of a product of functions; thus, we have

$$D^n[e^{ax}\phi] = \sum_{r=0}^{n} nc_r (D^r e^{ax})(D^{n-r}\phi)$$

$$= e^{ax}\left(\sum_{r=0}^{n} nc_r a^r D^{n-r}\phi\right)$$

$$= e^{ax}(D + a)^n \phi$$

Applying this result, we arrive at

$$F(D, D')[e^{ax+by}\phi(x, y)] = e^{ax+by}\, F(D + a, D' + b)\, \phi(x, y) \qquad (1.83)$$

Hence, it follows that

$$\frac{1}{F(D, D')}[e^{ax+by}\phi(x, y)] = e^{ax+by} \frac{1}{F(D + a, D' + b)} \phi(x, y)$$

$$= e^{ax} \cdot \frac{1}{F(D + a, D')} e^{by} \phi(x, y)$$

$$= e^{by} \cdot \frac{1}{F(D, D' + b)} e^{ax} \phi(x, y) \qquad (1.84)$$

For illustration of various cases to find PI, here follows several examples:

EXAMPLE 1.19 Solve the equation $(D^2 + 3DD' + 2D'^2)u = x + y$.

Solution The given PDE is reducible, since it can be factored as

$$(D + D')(D + 2D')u = x + y \tag{1}$$

Therefore,

$$\text{CF} = \phi_1(x - y) + \phi_2(2x - y) \tag{2}$$

where ϕ_1 and ϕ_2 are arbitrary functions.

The PI of the given PDE is obtained as follows:

$$\text{PI} = \frac{1}{(D^2 + 3DD' + 2D'^2)}(x + y)$$

$$= \frac{1}{D^2\left(1 + 3\dfrac{D'}{D} + 2\dfrac{D'^2}{D^2}\right)}(x + y)$$

$$= \frac{1}{D^2}\left[1 + \left(3\frac{D'}{D} + 2\frac{D'^2}{D^2}\right)\right]^{-1}(x + y)$$

$$= \frac{1}{D^2}\left[1 - 3\frac{D'}{D} - \cdots\right](x + y)$$

$$= \frac{1}{D^2}\left[x + y - \frac{3}{D}(1)\right]$$

$$= \frac{1}{D^2}[y - 2x] = y\frac{x^2}{2} - \frac{x^3}{3} \tag{3}$$

Adding Eqs. (2) and (3), we have the complete solution of the given PDE as

$$u = \phi_1(y - x) + \phi_2(y - 2x) + y\frac{x^2}{2} - \frac{x^3}{3}.$$

EXAMPLE 1.20 Solve the following equation $(D - D' - 1)(D - D' - 2)u = e^{2x-y} + x$.

Solution The CF of the given PDE is

$$\text{CF} = e^x\phi_1(x + y) + e^{2x}\phi_2(x + y) \tag{1}$$

The PI corresponding to the term e^{2x-y} is

$$= \frac{1}{(2 + 1 - 1)(2 + 1 - 2)}e^{2x-y} = \frac{1}{2}e^{2x-y} \tag{2}$$

while the PI corresponding to the term x is

$$= \frac{1}{2}(1 - D + D')^{-1}\left(1 - \frac{D}{2} + \frac{D'}{2}\right)^{-1} x$$

$$= \frac{1}{2}\left[(1 + D - D' + \cdots)\left(1 + \frac{D}{2} - \frac{D'}{2} + \cdots\right)\right] x$$

$$= \frac{1}{2}(1 + D - D')\left(x + \frac{1}{2}\right) = \frac{1}{2}\left(x + \frac{3}{2}\right) \tag{3}$$

Combining Eqs. (1), (2) and (3), the complete solution of the given PDE is found to be

$$u = e^{x}\phi_1(x + y) + e^{2x}\phi_2(x + y) + \frac{1}{2}e^{2x-y} + \frac{x}{2} + \frac{3}{4}.$$

EXAMPLE 1.21 Solve the following equation

$$(D^2 + 2DD' + D'^2 - 2D - 2D')u = \sin(x + 2y).$$

Solution The given PDE can be factored and rewritten as

$$(D + D')(D + D' - 2)u = \sin(x + 2y) \tag{1}$$

for which the CF is given by

$$\text{CF} = \phi_1(x - y) + e^{2x}\phi_2(x - y) \tag{2}$$

while

$$\text{PI} = \frac{1}{(D^2 + 2DD' + D'^2 - 2D - 2D')}\sin(x + 2y).$$

Setting $D^2 = -1$, $DD' = -2$, $D'^2 = -4$, we get

$$\text{PI} = -\frac{1}{(2D + 2D' + 9)}\sin(x + 2y)$$

$$= -\frac{[2(D + D') - 9]}{[4(D^2 + 2DD' + D'^2) - 81]}\sin(x + 2y)$$

$$= \frac{2(D + D') - 9}{117}\sin(x + 2y)$$

$$= \frac{1}{117}\,[2\cos(x + 2y) + 4\cos(x + 2y) - 9\sin(x + 2y)]$$

$$= \frac{1}{39}\,[2\cos(x + 2y) - 3\sin(x + 2y)] \tag{3}$$

Adding Eqs. (1), (2) and (3), we find that the complete solution of the given PDE as

$$u = \phi_1(x - y) + e^{2x}\phi_2(x - y) + \frac{1}{39}[2 \cos(x + 2y) - 3 \sin(x + 2y)].$$

EXAMPLE 1.22 Solve the following equation

$$(D^2 - DD')u = \cos x \cos 2y.$$

Solution The given PDE can be rewritten as

$$D(D - D')u = \cos x \cos 2y \tag{1}$$

Its CF is given by

$$\text{CF} = \phi_1(y) + \phi_2(y + x), \tag{2}$$

while its PI is given by

$$\text{PI} = \frac{1}{(D^2 - DD')} \cdot \frac{1}{2}[\cos(x + 2y) + \cos(x - 2y)]$$

$$= \frac{1}{2}\left[\frac{1}{(-1+2)}\cos(x+2y) + \frac{1}{(-1-2)}\cos(x-2y)\right]$$

$$= \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y) \tag{3}$$

Hence, the complete solution of the given PDE is given by

$$u = \phi_1(y) + \phi_2(y + x) + \frac{1}{2}\cos(x + 2y) - \frac{1}{6}\cos(x - 2y).$$

EXAMPLE 1.23 Find the solution of

$$(D^2 + DD' - 6D'^2)u = y \cos x.$$

Solution The given PDE can be rewritten as

$$(D + 3D')(D - 2D')u = y \cos x \tag{1}$$

Its CF is given by

$$\text{CF} = \phi_1(3x - y) + \phi_2(2x + y) \tag{2}$$

The PI of the given PDE is

$$\text{PI} = \frac{1}{(D+3D')} \cdot \frac{1}{(D-2D')} y \cos x \tag{3}$$

Applying the operator $\dfrac{1}{(D-2D')}$ first on $y \cos x$

$$(D - 2D')u = y \cos x$$

Its auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-2} = \frac{du}{y\cos x}\cdot$$

The first two members give

$$y + 2x = \lambda \text{ (constant)}.$$

From the first and the third members, we have

$$du = y \cos x\, dx = (\lambda - 2x) \cos x\, dx,$$

on integration, we get

$$u = \int (\lambda - 2x) \cos x\, dx$$

where λ is to be replaced by $(y + 2x)$ after integration. Now, Eq. (3) gives

$$\text{PI} = \frac{1}{(D + 3D')}\left[(\lambda - 2x)\sin x - \int \sin x(-2)dx\right]$$

$$= \frac{1}{(D + 3D')}[(\lambda - 2x)\sin x - 2\cos x]$$

$$= \frac{1}{(D + 3D')}[y \sin x - 2\cos x]$$

$$= \int [(\lambda + 3x)\sin x - 2\cos x]dx$$

$$= (\lambda + 3x)(-\cos x) + 3\int \cos x\, dx - 2\sin x.$$

Now, replacing λ by $(y - 3x)$, we get

$$\text{PI} = -y \cos x + \sin x \tag{4}$$

Hence, the solution of the given PDE is found to be

$$u = \text{CI} + \text{PI}$$

$$= \phi_1(3x - y) + \phi_2(2x + y) - y \cos x + \sin x.$$

EXAMPLE 1.24 Show that a linear PDE of the type

$$\sum_i \sum_j a_{ij}\, x^i y^j \frac{\partial^{i+j} u}{\partial x^i \partial y^j} = f(x, y)$$

can be reduced to a one with constant coefficients by the substitution

$$\xi = \log x,\ \eta = \log y.$$

Solution

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{1}{x}\frac{\partial u}{\partial \xi}$$

or
$$x\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}$$

That is,

$$x\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} = D \text{ (say)} \tag{1}$$

Therefore,

$$x\frac{\partial}{\partial x}\left(x^{n-1}\frac{\partial^{n-1}u}{\partial x^{n-1}}\right) = x^n\frac{\partial^n u}{\partial x^n} + (n-1)x^{n-1}\frac{\partial^{n-1}u}{\partial x^{n-1}}$$

or
$$x^n\frac{\partial^n u}{\partial x^n} = \left(x\frac{\partial}{\partial x} - n + 1\right)x^{n-1}\frac{\partial^{n-1}u}{\partial x^{n-1}} \tag{2}$$

By setting $n = 2, 3, 4, \ldots$ in Eq. (2), we obtain

$$x^2\frac{\partial^2 u}{\partial x^2} = (D-1)x\frac{\partial u}{\partial x} = D(D-1)u,$$

$$x^3\frac{\partial^3 u}{\partial x^3} = D(D-1)(D-2)u,$$

and so on. Similarly, we can show that

$$y\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta} = D'u,$$

$$y^2\frac{\partial^2 u}{\partial y^2} = D'(D'-1)u,$$

and
$$xy\frac{\partial^2 u}{\partial x \partial y} = DD'u$$

and so on. Substituting these results into the given PDE, it becomes

$$F(D, D')u = f(e^{\xi}, e^{\eta}) = f(\xi, \eta) \tag{3}$$

where,

$$D = \frac{\partial}{\partial \xi}, D' = \frac{\partial}{\partial \eta}$$

Thus, Eq. (3) can be seen as a PDE with constant coefficients.

For illustration, here follows an example.

EXAMPLE 1.25 Solve the following PDE

$$(x^2D^2 + 2xy\,DD' + y^2D'^2)u = x^2y^2 \tag{1}$$

Solution Using the substitution

$\xi = \log x$, $\eta = \log y$ and using the notation

$$\frac{\partial}{\partial \xi} = D, \frac{\partial}{\partial \eta} = D'$$

respectively, the given PDE reduces to a PDE with constant coefficients, in the form

$$[D(D - 1) + 2DD' + D'(D' - 1)]u = e^{2\xi+2\eta}.$$

On rewriting, we have

$$(D + D')(D + D' - 1)u = e^{2\xi+2\eta} \tag{2}$$

The CF of Eq. (2) is given by

$$\text{CF} = \phi_1(\xi - \eta) + e^{\xi}\,\phi_2\,(\xi - \eta), \tag{3}$$

while, its PI is obtained as

$$\text{PI} = \frac{1}{(2+2)(2+2-1)} e^{2\xi+2\eta}$$

$$= \frac{1}{12} e^{2\xi+2\eta}. \tag{4}$$

Transforming back from (ξ, η) to (x, y), we find the complete solution of the given PDE as

$$u = \phi_1(\log x - \log y) + x\,\phi_2(\log x - \log y) + \frac{x^2y^2}{12}$$

or

$$u = \psi_1\left(\frac{x}{y}\right) + x\,\psi_2\left(\frac{x}{y}\right) + \frac{1}{12}x^2y^2.$$

1.7 HOMOGENEOUS LINEAR PDE WITH CONSTANT COEFFICIENTS

Equation (1.61) is said to be linear PDE of nth order with constant coefficients. It is also called *homogeneous*, because all the terms containing derivatives are of the same order. Now, Eq. (1.61) can be rewritten in operator notation as

$$[a_0D^n + a_1D^{n-1}\,D' + a_2D^{n-2}D'^2 + \cdots + a_nD'^n]u = F(D, D')u = f(x, y) \tag{1.85}$$

As in the case of ODE, the complete solution of Eq. (1.85) consists of the sum of CF and PI.

1.7.1 Finding the Complementary Function

Let us assume that the solution of the equation $F(D, D')u = 0$ in the form

$$u = \phi(y + mx) \tag{1.86}$$

Then,

$$D^i u = m^i \phi^i(y + mx),\ D'^j u = \phi^j(y + mx),$$

and $$D^i D'^j u = m^i \phi^{i+j}(y + mx).$$

Substituting these results into $F(D, D')u = 0$, we obtain

$$(a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n)\phi^n(y + mx) = 0,$$

which will be true, iff

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n = 0 \tag{1.87}$$

This equation is called an *auxiliary equation* for $F(D, D')u = 0$.

Let $m_1, m_2, \ldots, m_n$ are the roots of Eq. (1.87). Depending on the nature of these roots, several cases arise:

Case I When the roots $m_1, m_2, \ldots, m_n$ are distinct. Corresponding to $m = m_1$, the CF is $u = \phi_1(y + m_1 x)$. Similarly, $u = \phi_2(y + m_2 x)$, $u = \phi_3(y + m_3 x)$, etc. are all complementary functions. Since, the given PDE is linear, using the principle of superposition, the CF of Eq. (1.85) can be written as

$$\text{CF} = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \cdots + \phi_n(y + m_n x)$$

where, $\phi_1, \phi_2, \ldots, \phi_n$ are arbitrary.

Case II When some of the roots are repeated. Let two roots say m_1 and m_2 are repeated, and each equal to m. Consider the equation

$$(D - mD')(D - mD')u = 0.$$

Setting $(D - mD')u = z$, the above equation becomes

$$(D - mD')z = 0,$$

or $$\frac{\partial z}{\partial x} - m\frac{\partial z}{\partial y} = 0$$

which is of Lagrange's form. Writing down its auxiliary equations, we have

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{0}.$$

The first two members gives $y + mx$ = constant = a(say).

The third member yields z = constant.

Therefore, $z = \phi(y + mx)$ is a solution.

Substituting for z, we get

$$(D - mD')u = \phi(y + mx)$$

which is again in the Lagranges form, whose auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{du}{\phi(y + mx)}$$

which gives,

$$y + mx = \text{constant}$$

and $$u = x\phi(y + mx) + \text{constant}$$

Hence, the CF is

$$u = x\phi(y + mx) + \psi(y + mx).$$

In general, if the root m is repeated r times, then the CF is given by

$$u = x^{r-1}\phi_1(y + mx) + x^{r-2}\phi_2(y + mx) + \cdots + \phi_r(y + mx)$$

where, ϕ_1, ϕ_2, …, ϕ_r are arbitrary.

For illustration, we consider the following couple of examples.

EXAMPLE 1.26 Solve the following PDE $(D^3 - 3D^2D' + 4D'^3)u = 0$.

Solution Observe that the given PDE is a linear homogeneous PDE. Dividing throughout by D' and denoting (D/D') by m, its auxiliary equation can be written as

$$m^3 - 3m^2 + 4 = (m + 1)(m - 2)^2 = 0.$$

Therefore, the roots of the auxiliary equation are –1, 2, 2. Thus,

$$\text{CF} = \phi_1(y - x) + \phi_2(y + 2x) + x\ \phi_3(y + 2x).$$

EXAMPLE 1.27 Solve the following PDE

$$(D^3 + DD'^2 - 10D'^3)u = 0.$$

Solution Observe that the given equation is a linear homogeneous PDE. Denoting (D/D') by m. The auxiliary equation for the given PDE is given by

$$m^3 + m - 10 = (m - 2)(m^2 + 2m + 5) = 0.$$

Its roots are: 2, $(-1 + 2i)$, $(-1 - 2i)$.

Hence, the required CF is found to be

$$u = \phi_1(y + 2x) + \phi_2(y - x + 2ix) + \phi_3(y - x - 2ix).$$

1.7.2 Methods for Finding the PI

Methods for finding the PI, in the case of linear homogeneous PDE's are, similar to the one's developed in the Section 1.6 for linear non-homogeneous PDEs. That is, the PI for the equation

$$F(D, D')u = f(x, y)$$

is obtained from

$$u_{\text{PI}} = \text{PI} = \frac{1}{F(D, D')} f(x, y).$$

However, when the above stated methods fail we have a general method, which is applicable whatever may be the form of $f(x, y)$, and is presented below:

We have already assumed that $F(D, D')$ can be factorised, in general, say into n linear factors. Thus,

$$\text{PI} = \frac{1}{F(D, D')} f(x, y)$$

$$\text{PI} = \frac{1}{(D - m_1 D')(D - m_2 D')\ldots(D - m_n D')} f(x, y)$$

$$= \frac{1}{(D - m_1 D')} \cdot \frac{1}{(D - m_2 D')} \cdots \frac{1}{(D - m_n D')} f(x, y).$$

In general, to evaluate

$$\frac{1}{(D - mD')} f(x, y),$$

we consider the equation

$$(D - mD')u = f(x, y)$$

or

$$p - mq = f(x, y) \qquad \text{(Lagrange's form)}$$

for which the auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{du}{f(x, y)}.$$

Its first two members, yield

$$y + mx = c \text{ (constant)}$$

The first and last members gives us

$$du = f(x, y)dx = f(x, c - mx).$$

On integration, we get

$$u = \int f(x, c - mx)dx$$

or

$$\frac{1}{(D - mD')} f(x, y) = \int f(x, c - mx)dx$$

After integration, we shall immediately replace c by $(y + mx)$. Applying this procedure repeatedly, we can find the PI for the given PDE. For illustration, we consider the following examples:

EXAMPLE 1.28 Solve the following PDE

$$(D^2 - 4DD' + 4D'^2)u = e^{2x+y}.$$

Solution The given equation is a linear homogeneous PDE. Its auxiliary equation can be written as

$$m^2 - 4m + 4 = (m - 2)^2 = 0.$$

In this example, the roots are repeated and they are 2, 2. The complementary function and particular integral are obtained as

$$\text{CF} = \phi_1(y + 2x) + x\phi_2(y + 2x) \tag{1}$$

and

$$\text{PI} = \frac{1}{(D - 2D')^2} e^{2x+y}$$

If we set $D = 2$, $D' = 1$, we observe that $F(D, D') = 0$, which is a failure case. Therefore, we shall adopt the general method to find PI. Now, noting that

$$\frac{1}{(D - mD')} f(x, y) = \int f(x, c - mx)dx.$$

$$\text{PI} = \frac{1}{(D - 2D')} \cdot \frac{1}{(D - 2D')} e^{2x+y}$$

$$= \frac{1}{(D - 2D')} \int e^{(2x+c-2x)}dx$$

$$= \frac{1}{(D - 2D')} xe^c = \frac{1}{(D - 2D')} xe^{y+2x}$$

$$= \int xe^{(c-2x+2x)}dx = e^c \int xdx$$

$$= e^c \frac{x^2}{2} = \frac{x^2}{2} e^{y+2x} \tag{2}$$

From Eqs. (1) and (2), the complete solution of the given PDE is found to be

$$u = \phi_1(y + 2x) + x\phi_2(y + 2x) + \frac{x^2}{2} e^{y+2x}.$$

EXAMPLE 1.29 Find a real function $u(x, y)$, which reduces to zero when $y = 0$ and satisfy the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\pi(x^2 + y^2).$$

Solution In symbolic form, the given PDE can be written as

$$(D^2 + D'^2)u = -\pi(x^2 + y^2)$$

Its auxiliary equation is given by

$$(m^2 + 1) = 0, \text{ which gives } m = \pm i.$$

Hence,

$$\text{CF} = \phi_1(y + ix) + \phi_2(y - ix) \tag{1}$$

and

$$\text{PI} = \frac{-1}{(D^2 + D'^2)}\pi(x^2 + y^2) = -\frac{\pi}{D'^2}\frac{(x^2 + y^2)}{(1 + D^2/D'^2)}$$

$$= -\frac{\pi}{D'^2}\left(1 + \frac{D^2}{D'^2}\right)^{-1}(x^2 + y^2)$$

$$= -\frac{\pi}{D'^2}\left[1 - \frac{D^2}{D'^2} + \cdots\right](x^2 + y^2)$$

$$= -\frac{\pi}{D'^2}(x^2 + y^2) + \frac{2\pi}{D'^4}$$

$$= -\frac{\pi}{D'}\left(x^2 y + \frac{y^3}{3}\right) + \frac{2\pi}{D'^3}y$$

or

$$\text{PI} = -\pi\left(\frac{x^2y^2}{2} + \frac{y^4}{12}\right) + \frac{2\pi}{D'^2}\frac{y^2}{2}$$

$$= -\pi\left(\frac{x^2y^2}{2} + \frac{y^4}{12}\right) + 2\pi\frac{y^4}{24}$$

$$= -\frac{\pi}{2}x^2y^2 \tag{2}$$

Hence, the complete solution of the given PDE is found to be

$$u = \phi_1(y + ix) + \phi_2(y - ix) - \frac{\pi}{2}x^2y^2 \tag{3}$$

Finally, the real function satisfying the given PDE is given by

$$u = -\frac{\pi}{2}x^2y^2 \tag{4}$$

which of course $\to 0$ as $y \to 0$.

EXERCISES

1. Find the region in the xy-plane in which the following equation is hyperbolic:

$$[(x - y)^2 - 1]u_{xx} + 2u_{xy} + [(x - y)^2 - 1]u_{yy} = 0$$

2. Find the families of characteristics of the PDE

$$(1-x^2)u_{xx} - u_{yy} = 0$$

in the elliptic and hyperbolic cases.

3. Reduce the following PDE to a canonical form

$$u_{xx} + xyu_{yy} = 0$$

4. Classify and reduce the following equations to a canonical form:

(a) $y^2u_{xx} - x^2u_{yy} = 0, \qquad x > 0, \qquad y > 0.$

(b) $u_{xx} + 2u_{xy} + u_{yy} = 0.$

(c) $e^x u_{xx} + e^y u_{yy} = u.$

(d) $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0.$

(e) $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2.$

5. Reduce the following equation to a canonical form and hence solve it:

$$3u_{xx} + 10u_{xy} + 3u_{yy} = 0$$

6. If $L(u) = c^2u_{xx} - u_{tt}$, then show that its adjoint operator is given by

$$L^* = c^2v_{xx} - v_{tt}$$

7. Determine the adjoint operator L^* corresponding to

$$L(u) = Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu$$

where A, B, C, D, E and F are functions of x and y only.

8. Find the solution of the following Cauchy problem

$$u_{xy} = F(x, y)$$

given

$$u = f(x), \qquad \frac{\partial u}{\partial n} = g(x) \quad \text{on the line } y = x$$

using Riemann's method which is of the form

$$u(x_0, y_0) = \frac{1}{2}[f(x_0) + f(y_0)] + \frac{1}{\sqrt{2}}\int_{x_0}^{y_0} g(x)\,dx - \iint_{\mathbb{R}} F(x, y)\,dx\,dy$$

where $\mathbb{R}$ is the triangular region in the xy-plane bounded by the line $y = x$ and the lines $x = x_0$, $y = y_0$ through (x_0, y_0).

9. The characteristics of the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x\,\partial y} + \cos^2 x \frac{\partial^2 z}{\partial y^2} + 2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = 0$$

when it is of hyperbolic type are… and… (GATE-Maths, 1997)

10. Using $\eta = x + y$ as one of the transformation variable, obtain the canonical form of

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$$

(GATE-Maths, 1998)

Choose the correct answer in the following questions (11–14):

11. The PDE

$$y^3 u_{xx} - (x^2 - 1) u_{yy} = 0 \quad \text{is}$$

(A) parabolic in $\{(x, y) : x < 0\}$

(B) hyperbolic in $\{(x, y) : y > 0\}$

(C) elliptic in $\mathbb{R}^2$

(D) parabolic in $\{(x, y) : x > 0\}$. (GATE-Maths, 1998)

12. The equation

$$x^2(y-1) z_{xx} - x(y^2 - 1) z_{xy} + y(y^2 - 1) z_{yy} + z_x = 0$$

is hyperbolic in the entire xy-plane except along

(A) x-axis (B) y-axis

(C) A line parallel to y-axis (D) A line parallel to x-axis.

(GATE-Maths, 2000)

13. The characteristic curves of the equation

$$x^2 u_{xx} - y^2 u_{yy} = x^2 y^2 + x, \quad x > 0 \text{ are}$$

(A) rectangular hyperbola (B) parabola

(C) circle (D) straight line.

(GATE-Maths, 2000)

14. Pick the region in which the following PDE is hyperbolic:

$$y u_{xx} + 2xy u_{xy} + x u_{yy} = u_x + u_y$$

(A) $xy \neq 1$ (B) $xy \neq 0$

(C) $xy > 1$ (D) $xy > 0$.

(GATE-Maths, 2003)

15. Solve the following PDEs:

(i) $(D - D' - 1)(D - D' - 2)u = 0$

(ii) $(D + D' - 1)(D + 2D' - 3)u = 0$

16. Solve the following PDE:

$$(D^2 + DD' + D + D' + 1)u = 0$$

17. Solve the following PDEs:

(i) $(D^2 - DD' + D' - 1)u = \cos(x + 2y)$

(ii) $D(D - 2D' - 3)u = e^{x+2y}$

(iii) $(2D + D' - 1)^2(D - 2D' + 2)^3 = 0$

18. Find the complete solution of the following PDEs:

(i) $(x^2D^2 - 2xy\, DD' + y^2D'^2 - xD + 3yD')u = 8(y/x)$

(ii) $D(D - 2D' - 3)u = e^{x+2y}$

19. Find the complete solution of the following PDEs:

(i) $(D^2 + 3DD' + 2D'^2)u = x + y$

(ii) $(D^2 + D'^2)u = \cos px \cos qy$

(iii) $(D^2 - DD' - 2D'^2)u = (y - 1)e^x$

(iv) $(4D^2 - 4DD' + D'^2)u = 16 \log (x + 2y)$

(v) $(D^2 - 3DD' + 2D'^2)u = e^{2x+3y} + \sin (x - 2y)$

CHAPTER 2

Elliptic Differential Equations

2.1 OCCURRENCE OF THE LAPLACE AND POISSON EQUATIONS

In Chapter 1, we have seen the classification of second order partial differential equation into elliptic, parabolic and hyperbolic types. In this chapter we shall consider various properties and techniques for solving Laplace and Poisson equations which are elliptic in nature.

Various physical phenomena are governed by the well known Laplace and Poisson equations. A few of them, frequently encountered in applications are: steady heat conduction, seepage through porous media, irrotational flow of an ideal fluid, distribution of electrical and magnetic potential, torsion of prismatic shaft, bending of prismatic beams, distribution of gravitational potential, etc. In the following two sub-sections, we shall give the derivation of Laplace and Poisson equations in relation to the most frequently occurring physical situation, namely, the gravitational potential.

2.1.1 Derivation of Laplace Equation

Consider two particles of masses m and m_1 situated at Q and P separated by a distance r as shown in Fig. 2.1. According to Newton's universal law of gravitation, the magnitude of the force, proportional to the product of their masses and inversely proportional to the square of the distance, between them is given by

$$F = G\frac{mm_1}{r^2} \tag{2.1}$$

where G is the gravitational constant. It $\mathbf{r}$ represents the vector $\overrightarrow{PQ}$, assuming unit mass at Q and $G = 1$, the force at Q due to the mass at P is given by

$$\mathbf{F} = -\frac{m_1\mathbf{r}}{r^3} = \nabla\left(\frac{m_1}{r}\right) \tag{2.2}$$

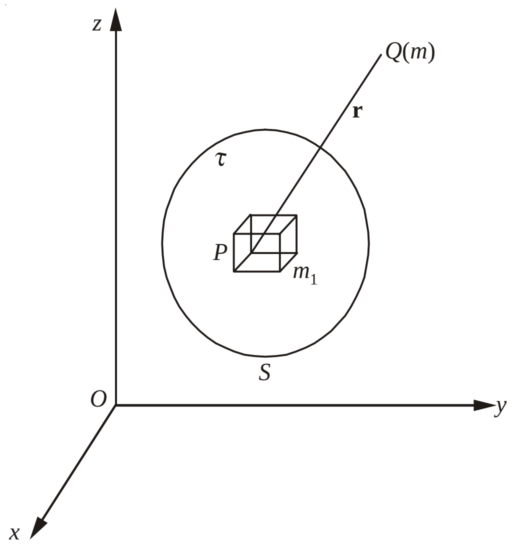

Fig. 2.1 Illustration of Newton's universal law of gravitation.

which is called the intensity of the gravitational force. Suppose a particle of unit mass moves under the attraction of a particle of mass m_1 at P from infinity up to Q; then the work done by the force $\mathbf{F}$ is

$$\int_{\infty}^{r} \mathbf{F} \cdot d\mathbf{r} = \int_{\infty}^{r} \nabla\left(\frac{m_1}{r}\right) \cdot d\mathbf{r} = \frac{m_1}{r} \tag{2.3}$$

This is defined as the potential V at Q due to a particle at P and is denoted by

$$V = -\frac{m_1}{r} \tag{2.4}$$

From Eq. (2.2), the intensity of the force at P is

$$\mathbf{F} = -\nabla V \tag{2.5}$$

Now, if we consider a system of particles of masses $m_1, m_2, \ldots, m_n$ which are at distances $r_1, r_2, \ldots, r_n$ respectively, then the force of attraction per unit mass at Q due to the system is

$$\mathbf{F} = \sum_{i=1}^{n} \nabla \frac{m_i}{r_i} = \nabla \sum_{i=1}^{n} \frac{m_i}{r_i} \tag{2.6}$$

The work done by the force acting on the particle is

$$\int_{\infty}^{r} \mathbf{F} \cdot d\mathbf{r} = \sum_{i=1}^{n} \frac{m_i}{r_i} = -V \tag{2.7}$$

Therefore,

$$\nabla^2 V = -\nabla^2 \sum_{i=1}^{n} \frac{m_i}{r_i} = \sum_{i=1}^{n} \nabla^2 \frac{m_i}{r_i} = 0, \qquad r_i \neq 0 \tag{2.8}$$

where

$$\nabla^2 = \text{div}\, \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is called the Laplace operator.

In the case of continuous distribution of matter of density ρ in a volume τ, we have

$$V(x, y, z) = \iiint_{\tau} \frac{\rho(\xi, \eta, \zeta)}{r} d\tau \tag{2.9}$$

where $r = \{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\}^{1/2}$ and Q is outside the body. It can be verified that

$$\nabla^2 V = 0 \tag{2.10}$$

which is called the Laplace equation.

2.1.2 Derivation of Poisson Equation

Consider a closed surface S consisting of particles of masses $m_1, m_2, \ldots, m_n$. Let Q be any point on S. Let $\sum_{i=1}^{n} m_i = M$ be the total mass inside S, and let $g_1, g_2, \ldots, g_n$ be the gravity field at Q due to the presence of $m_1, m_2, \ldots, m_n$ respectively within S. Also, let $\sum_{i=1}^{n} g_i = g$, the entire gravity field at Q. Then, according to Gauss law, we have

$$\iint_S \mathbf{g} \cdot d\mathbf{S} = -4\pi GM \tag{2.11}$$

where $M = \iiint_{\tau} \rho\, d\tau, \rho$ is the mass density function and τ is the volume in which the masses are distributed throughout. Since the gravity field is conservative, we have

$$\mathbf{g} = \nabla V \tag{2.12}$$

where V is a scalar potential. But the Gauss divergence theorem states that

$$\iint_S \mathbf{g} \cdot d\mathbf{S} = \iiint_{\tau} \nabla \cdot \mathbf{g}\, d\tau \tag{2.13}$$

Also, Eq. (2.11) gives

$$\iint_S \mathbf{g} \cdot d\mathbf{S} = -4\pi G \iiint_{\tau} \rho\, d\tau \tag{2.14}$$

Combining Eqs. (2.13) and (2.14), we have

$$\iiint_{\tau} (\nabla \cdot \mathbf{g} + 4\pi G\rho)\, d\tau = 0$$

implying

$$\nabla \cdot \mathbf{g} = -4\pi G\rho = \nabla \cdot \nabla V$$

Therefore,

$$\nabla^2 V = -4\pi G\rho \tag{2.15}$$

This equation is known as *Poisson's equation*.

2.2 BOUNDARY VALUE PROBLEMS (BVPs)

The function V, whose analytical form we seek for the problems stated in Section 2.1, in addition to satisfying the Laplace and Poisson equations in a bounded region $\mathbb{R}$ in R^3, should also satisfy certain boundary conditions on the boundary $\partial\mathbb{R}$ of this region. Such problems are referred to as boundary value problems (BVPs) for Laplace and Poisson equations. We shall denote the set of all boundary points of $\mathbb{R}$ by $\partial\mathbb{R}$. By the closure of $\mathbb{R}$, we mean the set of all interior points of $\mathbb{R}$ together with its boundary points and is denoted by $\overline{\mathbb{R}}$. Symbolically, $\overline{\mathbb{R}}. = \mathbb{R} U \partial\mathbb{R}$.

If a function $f \in c^{(n)}$ (f "belongs to" $c^{(n)}$), then all its derivatives of order n are continuous. If it belongs to $c^{(0)}$, then we mean f is continuous.

There are mainly three types of boundary value problems for Laplace equation. If $f \in c^{(0)}$ and is specified on the boundary $\partial\mathbb{R}$ of some finite region $\mathbb{R}$, the problem of determining a function $\psi(x, y, z)$ such that $\nabla^2\psi = 0$ within $\mathbb{R}$ and satisfying $\psi = f$ on $\partial\mathbb{R}$ is called the boundary value problem of first kind, or the Dirichlet problem. For example, finding the steady state temperature within the region $\mathbb{R}$ when no heat sources or sinks are present and when the temperature is prescribed on the boundary $\partial\mathbb{R}$, is a Dirichlet problem. Another example would be to find the potential inside the region $\mathbb{R}$ when the potential is specified on the boundary $\partial\mathbb{R}$. These two examples correspond to the interior Dirichlet problem.

Similarly, if $f \in c^{(0)}$ and is prescribed on the boundary $\partial\mathbb{R}$ of a finite simply connected region $\mathbb{R}$, determining a function $\psi(x, y, z)$ which satisfies $\nabla^2\psi = 0$ outside $\mathbb{R}$ and is such that $\psi = f$ on $\partial\mathbb{R}$, is called an exterior Dirichlet problem. For example, determination of the distribution of the potential outside a body whose surface potential is prescribed, is an exterior Dirichlet problem. The second type of BVP is associated with von Neumann. The problem is to determine the function $\psi(x, y, z)$ so that $\nabla^2\psi = 0$ within $\mathbb{R}$ while $\partial\psi/\partial n$ is specified at every point of $\partial\mathbb{R}$, where $\partial\psi/\partial n$ denotes the normal derivative of the field variable ψ. This problem is called the Neumann problem. If ψ is the temperature, $\partial\psi/\partial n$ is the heat flux representing the amount of heat crossing per unit volume per unit time along the normal direction, which is zero when insulated. The third type of BVP is concerned with the determination of the function $\psi(x, y, z)$ such that $\nabla^2\psi = 0$ within $\mathbb{R}$, while a boundary condition of the form $\partial\psi/\partial n + h\psi = f$, where $h \ge 0$ is specified at every point of $\partial\mathbb{R}$. This is called a mixed BVP or Churchill's problem. If we assume Newton's law of cooling, the heat lost is $h\psi$, where ψ is the temperature difference from the surrounding medium and $h > 0$ is a constant depending on the medium. The heat f supplied at a point of the boundary is partly conducted into the medium and partly lost by radiation to the surroundings. Equating these amounts, we get the third boundary condition.

2.3 SOME IMPORTANT MATHEMATICAL TOOLS

Among the mathematical tools we employ in deriving many important results, the Gauss divergence theorem plays a vital role, which can be stated as follows: Let $\partial\mathbb{R}$ be a closed surface in the xyz-space and $\mathbb{R}$ denote the bounded region enclosed by $\partial\mathbb{R}$ in which $\mathbf{F}$ is a vector belonging to $c^{(1)}$ in $\mathbb{R}$ and continuous on $\mathbb{R}$. Then

$$\iint_{\partial\mathbb{R}} \mathbf{F}\cdot\hat{n}\, dS = \iiint_{\mathbb{R}} \nabla\cdot\mathbf{F}\, dV \tag{2.16}$$

where dV is an element of volume, dS is an element of surface area, and $\hat{n}$ the outward drawn normal.

Green's identities which follow from divergence theorem are also useful and they can be derived as follows: Let $\mathbf{F} = \mathbf{f}\phi$, where $\mathbf{f}$ is a vector function of position and ϕ is a scalar function of position. Then,

$$\iiint_{\mathbb{R}} \nabla\cdot(\mathbf{f}\phi)\, dV = \iint_{\partial\mathbb{R}} \hat{n}\cdot\mathbf{f}\phi\, dS$$

Using the vector identity

$$\nabla\cdot(\mathbf{f}\phi) = \mathbf{f}\cdot\nabla\phi + \phi\nabla\cdot\mathbf{f}$$

we have

$$\iiint_{\mathbb{R}} \mathbf{f}\cdot\nabla\phi\, dV = \iint_{\partial\mathbb{R}} \hat{n}\cdot\mathbf{f}\phi\, dS - \iiint_{\mathbb{R}} \phi\nabla\cdot\mathbf{f}\, dV$$

If we choose $\mathbf{f} = \nabla\psi$, the above equation yields

$$\iint\limits_{\mathbb{R}} \nabla\phi\cdot\nabla\psi\, dV = \iint\limits_{\partial\mathbb{R}} \phi\hat{n}\cdot\nabla\psi\, dS - \iiint\limits_{\mathbb{R}} \phi\nabla^2\psi\, dV \tag{2.17}$$

Noting that $\hat{n}\cdot\nabla\psi$ is the derivative of ψ in the direction of $\hat{n}$, we introduce the notation

$$\hat{n}\cdot\nabla\psi = \partial\psi/\partial n$$

into Eq. (2.17) to get

$$\iiint\limits_{\mathbb{R}} \nabla\phi\cdot\nabla\psi\, dV = \iint\limits_{\partial\mathbb{R}} \phi\frac{\partial\psi}{\partial n} dS - \iiint\limits_{\mathbb{R}} \phi\nabla^2\psi\, dV \tag{2.18a}$$

This equation is known as *Green's first identity*. Of course, it is assumed that both ϕ and ψ possess continuous second derivatives.

Interchanging the role of ϕ and ψ, we obtain from relation (2.18a) the equation

$$\iiint\limits_{\mathbb{R}} \nabla\psi\cdot\nabla\phi\, dV = \iint\limits_{\partial\mathbb{R}} \psi\frac{\partial\phi}{\partial n} dS - \iiint\limits_{\mathbb{R}} \psi\nabla^2\phi\, dV \tag{2.18b}$$

Now, subtracting Eq. (2.18b) from Eq. (2.18a), we get

$$\iiint\limits_{\mathbb{R}} (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV = \iint\limits_{\partial\mathbb{R}} \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) dS \tag{2.19}$$

This is known as *Green's second identity*. If we set $\phi = \psi$ in Eq. (2.18a) we get

$$\iiint\limits_{\mathbb{R}} (\nabla\phi)^2\, dV = \iint\limits_{\partial\mathbb{R}} \phi\frac{\partial\phi}{\partial n} dS - \iiint\limits_{\mathbb{R}} \phi\nabla^2\phi\, dV \tag{2.20}$$

which is a special case of Green's first identity.

2.4 PROPERTIES OF HARMONIC FUNCTIONS

Solutions of Laplace equation are called harmonic functions which possess a number of interesting properties, and they are presented in the following theorems.

Theorem 2.1 If a harmonic function vanishes everywhere on the boundary, then it is identically zero everywhere.

Proof If ϕ is a harmonic function, then $\nabla^2\phi = 0$ in $\mathbb{R}$. Also, if $\phi = 0$ on $\partial\mathbb{R}$, we shall show that $\phi = 0$ in $\overline{\mathbb{R}} = \mathbb{R}\, U\, \partial\mathbb{R}$. Recalling Green's first identity, i.e., Eq. (2.20), we get

$$\iiint\limits_{\mathbb{R}} (\nabla\phi)^2\, dV = \iint\limits_{\partial\mathbb{R}} \phi\frac{\partial\phi}{\partial n} dS - \iiint\limits_{\mathbb{R}} \phi\nabla^2\phi\, dV$$

and using the above facts we have, at once, the relation

$$\iiint_{\mathbb{R}} (\nabla \phi)^2 \, dV = 0$$

Since $(\nabla \phi)^2$ is positive, it follows that the integral will be satisfied only if $\nabla \phi = 0$. This implies that ϕ is a constant in $\mathbb{R}$. Since ϕ is continuous in $\overline{\mathbb{R}}$ and ϕ is zero on $\partial \mathbb{R}$, it follows that $\phi = 0$ in $\mathbb{R}$.

Theorem 2.2 If ϕ is a harmonic function in $\mathbb{R}$ and $\partial \phi / \partial n = 0$ on $\partial \mathbb{R}$, then ϕ is a constant in $\overline{\mathbb{R}}$.

Proof Using Green's first identity and the data of the theorem, we arrive at

$$\iiint_{\mathbb{R}} (\nabla \phi)^2 \, dV = 0$$

implying $\nabla \phi = 0$, i.e., ϕ is a constant in $\mathbb{R}$. Since the value of ϕ is not known on the boundary $\partial \mathbb{R}$ while $\partial \phi / \partial n = 0$, it is implied that ϕ is a constant on $\partial \mathbb{R}$ and hence on $\overline{\mathbb{R}}$.

Theorem 2.3 If the Dirichlet problem for a bounded region has a solution, then it is unique.

Proof If ϕ_1 and ϕ_2 are two solutions of the interior Dirichlet problem, then

$$\nabla^2 \phi_1 = 0 \quad \text{in } \mathbb{R}; \qquad \phi_1 = f \quad \text{on } \partial \mathbb{R}$$

$$\nabla^2 \phi_2 = 0 \quad \text{in } \mathbb{R}; \qquad \phi_2 = f \quad \text{on } \partial \mathbb{R}$$

Let $\psi = \phi_1 - \phi_2$. Then

$$\nabla^2 \psi = \nabla^2 \phi_1 - \nabla^2 \phi_2 = 0 \qquad \text{in } \mathbb{R};$$

$$\psi = \phi_1 - \phi_2 = f - f = 0 \qquad \text{on } \partial \mathbb{R}$$

Therefore,

$$\nabla^2 \psi = 0 \text{ in } \mathbb{R}, \qquad \psi = 0 \quad \text{on } \partial \mathbb{R}$$

Now using Theorem 2.1, we obtain $\psi = 0$ on $\overline{\mathbb{R}}$, which implies that $\phi_1 = \phi_2$. Hence, the solution of the Dirichlet problem is unique.

Theorem 2.4 If the Neumann problem for a bounded region has a solution, then it is either unique or it differs from one another by a constant only.

Proof Let ϕ_1 and ϕ_2 be two distinct solutions of the Neumann problem. Then we have

$$\nabla^2\phi_1 = 0 \quad \text{in} \quad \mathbb{R}; \quad \frac{\partial \phi_1}{\partial n} = f \quad \text{on } \partial\mathbb{R},$$

$$\nabla^2\phi_2 = 0 \quad \text{in} \quad \mathbb{R}; \quad \frac{\partial \phi_2}{\partial n} = f \quad \text{on } \partial\mathbb{R}$$

Let $\psi = \phi_1 - \phi_2$. Then

$$\nabla^2\psi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0 \quad \text{in} \quad \mathbb{R}$$

$$\frac{\partial \psi}{\partial n} = \frac{\partial \phi_1}{\partial n} - \frac{\partial \phi_2}{\partial n} = 0 \qquad \text{on } \partial\mathbb{R}$$

Hence from Theorem 2.2, ψ is a constant on $\overline{\mathbb{R}}$, i.e., $\phi_1 - \phi_2$ = constant. Therefore, the solution of the Neumann problem is not unique. Thus, the solutions of a certain Neumann problem can differ from one another by a constant only.

2.4.1 The Spherical Mean

Let $\mathbb{R}$ be a region bounded by $\partial\mathbb{R}$ and let $P(x, y, z)$ be any point in $\mathbb{R}$. Also, let $S(P, r)$ represent a sphere with centre at P and radius r such that it lies entirely within the domain $\mathbb{R}$ as depicted in Fig. 2.2. Let u be a continuous function in $\mathbb{R}$. Then the spherical mean of u denoted by $\bar{u}$ is defined as

$$\bar{u}(r) = \frac{1}{4\pi r^2} \iint_{S(P,r)} u(Q)\, dS \tag{2.21}$$

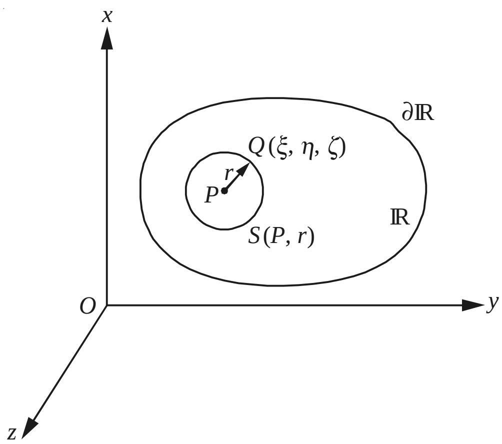

Fig. 2.2 Spherical mean.

where $Q(\xi,\eta,\zeta)$ is any variable point on the surface of the sphere $S(P,r)$ and dS is the surface element of integration. For a fixed radius r, the value $\bar{u}(r)$ is the average of the values of u taken over the sphere $S(P,r)$, and hence it is called the spherical mean. Taking the origin at P, in terms of spherical polar coordinates, we have

$$\xi = x + r\sin\theta\cos\phi$$
$$\eta = y + r\sin\theta\sin\phi$$
$$\zeta = z + r\cos\theta$$

Then, the spherical mean can be written as

$$\bar{u}(r) = \frac{1}{4\pi r^2}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} u(x + r\sin\theta\cos\phi,\ y + r\sin\theta\sin\phi,\ z + r\cos\theta)\,r^2\sin\theta\,d\theta\,d\phi$$

Also, since u is continuous on $S(P,r), \bar{u}$ too is a continuous function of r on some interval $0 < r \le R$, which can be verified as follows:

$$\bar{u}(r) = \frac{1}{4\pi}\iint u(Q)\sin\theta\,d\theta\,d\phi = \frac{u(Q)}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin\theta\,d\theta\,d\phi = u(Q)$$

Now, taking the limit as $r \to 0$, $Q \to P$, we have

$$\operatorname*{Lt}_{r\to 0}\bar{u}(r) = u(P) \tag{2.22}$$

Hence, $\bar{u}$ is continuous in $0 \le r \le R$.

2.4.2 Mean Value Theorem for Harmonic Functions

Theorem 2.5 Let u be harmonic in a region $\mathbb{R}$. Also, let $P(x, y, z)$ be a given point in $\mathbb{R}$ and $S(P, r)$ be a sphere with centre at P such that $S(P, r)$ is completely contained in the domain of harmonicity of u. Then

$$u(P) = \bar{u}(r) = \frac{1}{4\pi r^2}\iint_{S(P,r)} u(Q)\,dS$$

Proof Since u is harmonic in $\mathbb{R}$, its spherical mean $\bar{u}(r)$ is continuous in $\mathbb{R}$ and is given by

$$\bar{u}(r) = \frac{1}{4\pi r^2}\iint_{S(P,r)} u(Q)\,dS = \frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi} u(\xi,\eta,\zeta)\,r^2\sin\theta\,d\theta\,d\phi$$

Therefore,

$$\frac{d\bar{u}(r)}{dr} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}(u_\xi \xi_r + u_\eta \eta_r + u_\zeta \zeta_r)\sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}(u_\xi \sin\theta\cos\phi + u_\eta \sin\theta\sin\phi + u_\zeta \cos\theta)\sin\theta \, d\theta \, d\phi \quad (2.23)$$

Noting that $\sin\theta\cos\phi$, $\sin\theta\sin\phi$ and $\cos\theta$ are the direction cosines of the normal $\hat{n}$ on $S(P, r)$,

$$\nabla u = iu_\xi + ju_\eta + ku_\zeta, \qquad \hat{n} = (in_1, jn_2, kn_3),$$

the expression within the parentheses of the integrand of Eq. (2.23) can be written as $\nabla u \cdot \hat{n}$. Thus

$$\frac{d\bar{u}(r)}{dr} = \frac{1}{4\pi r^2}\iint_{S(P,r)} \nabla u \cdot \hat{n} r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{4\pi r^2}\iint_{S(P,r)} \nabla u \cdot \hat{n} \, dS$$

$$= \frac{1}{4\pi r^2}\iiint_{V(P,r)} \nabla \cdot \nabla u \, dV \text{ (by divergence theorem)}$$

$$= \frac{1}{4\pi r^2}\iiint_{V(P,r)} \nabla^2 u \, dV = 0 \text{ (since } u \text{ is harmonic)}$$

Therefore, $\dfrac{d\bar{u}}{dr} = 0$, implying $\bar{u}$ is constant.

Now the continuity of $\bar{u}$ at $r = 0$ gives, from Eq.(2.22), the relation

$$\bar{u}(r) = u(P) = \frac{1}{4\pi r^2}\iint_{S(P,r)} u(Q)\, dS \quad (2.24)$$

2.4.3 Maximum-Minimum Principle and Consequences

Theorem 2.6 Let $\mathbb{R}$ be a region bounded by $\partial\mathbb{R}$. Also, let u be a function which is continuous in a closed region $\overline{\mathbb{R}}$ and satisfies the Laplace equation $\nabla^2 u = 0$ in the interior of $\mathbb{R}$. Further, if u is not constant everywhere on $\overline{\mathbb{R}}$, then the maximum and minimum values of u must occur only on the boundary $\partial\mathbb{R}$.

Proof Suppose u is a harmonic function but not constant everywhere on $\overline{\mathbb{R}}$. If possible, let u attain its maximum value M at some interior point P in $\mathbb{R}$. Since M is the maximum of u which is not a constant, there should exist a sphere $S(P, r)$ about P such that some of the values of u on $S(P, r)$ must be less than M. But by the mean value property, the value of u at P is the average of the values of u on $S(P, r)$, and hence it is less than M. This contradicts the assumption that $u = M$ at P. Thus u must be constant over the entire sphere $S(P, r)$.

Let Q be any other point inside $\mathbb{R}$ which can be connected to P by an arc lying entirely within the domain $\mathbb{R}$. By covering this arc with spheres and using the Heine-Borel theorem to choose a finite number of covering spheres and repeating the argument given above, we can arrive at the conclusion that u will have the same constant value at Q as at P. Thus u cannot attain a maximum value at any point inside the region $\mathbb{R}$. Therefore, u can attain its maximum value only on the boundary $\partial\mathbb{R}$. A similar argument will lead to the conclusion that u can attain its minimum value only on the boundary $\partial\mathbb{R}$.

Some important consequences of the maximum-minimum principle are given in the following theorems.

Theorem 2.7 (Stability theorem). The solutions of the Dirichlet problem depend continuously on the boundary data.

Proof Let u_1 and u_2 be two solutions of the Dirichlet problem and let f_1 and f_2 be given continuous functions on the boundary $\partial\mathbb{R}$ such that

$$\nabla^2 u_1 = 0 \quad \text{in } \mathbb{R}; \qquad u_1 = f_1 \quad \text{on } \partial\mathbb{R},$$

$$\nabla^2 u_2 = 0 \quad \text{in } \mathbb{R}; \qquad u_2 = f_2 \quad \text{on } \partial\mathbb{R}$$

Let $u = u_1 - u_2$. Then,

$$\nabla^2 u = \nabla^2 u_1 - \nabla^2 u_2 = 0 \quad \text{in } \mathbb{R}; \qquad u = f_1 - f_2 \quad \text{on } \partial\mathbb{R}$$

Hence, u is a solution of the Dirichlet problem with boundary condition $u = f_1 - f_2$ on $\partial\mathbb{R}$. By the maximum-minimum principle, u attains the maximum and minimum values on $\partial\mathbb{R}$. Thus at any interior point in $\mathbb{R}$, we shall have, for a given $\varepsilon > 0$,

$$-\varepsilon < u_{\min} \le u \le u_{\max} < \varepsilon$$

Therefore,

$$|u| < \varepsilon \quad \text{in } \mathbb{R}, \text{ implying } |u_1 - u_2| < \varepsilon$$

Hence, if

$$|f_1 - f_2| < \varepsilon \quad \text{on } \partial\mathbb{R}, \quad \text{then } |u_1 - u_2| < \varepsilon \quad \text{on } \mathbb{R}$$

Thus, small changes in the initial data bring about an arbitrarily small change in the solution. This completes the proof of the theorem.

Theorem 2.8 Let $\{f_n\}$ be a sequence of functions, each of which is continuous on $\overline{\mathbb{R}}$ and harmonic on $\mathbb{R}$. If the sequence $\{f_n\}$ converges uniformly on $\partial\mathbb{R}$, then it converges uniformly on $\overline{\mathbb{R}}$.

Proof Since the sequence $\{f_n\}$ converges uniformly on $\partial\mathbb{R}$, for a given $\varepsilon > 0$, we can always find an integer N such that

$$|f_n - f_m| < \varepsilon \quad \text{for } n, m > N$$

Hence, from stability theorem, for all $n, m > N$, it follows immediately that

$$|f_n - f_m| < \varepsilon \quad \text{in } \mathbb{R}$$

Therefore, $\{f_n\}$ converges uniformly on $\overline{\mathbb{R}}$.

EXAMPLE 2.1 Show that if the two-dimensional Laplace equation $\nabla^2 u = 0$ is transformed by introducing plane polar coordinates r, θ defined by the relations $x = r\cos\theta, y = r\sin\theta$, it takes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Solution In many practical problems, it is necessary to write the Laplace equation in various coordinate systems. For instance, if the boundary of the region $\partial\mathbb{R}$ is a circle, then it is natural to use polar coordinates defined by $x = r\cos\theta$, $y = r\sin\theta$. Therefore,

$$r^2 = x^2 + y^2, \qquad \theta = \tan^{-1}(y/x)$$

$$r_x = \cos\theta, \qquad r_y = \sin\theta, \qquad \theta_x = -\frac{\sin\theta}{r}, \qquad \theta_y = \frac{\cos\theta}{r}$$

since

$$u = u(r, \theta) \qquad u_x = u_r r_x + u_\theta \theta_x = \left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}\right)$$

Similarly,

$$u_y = u_r r_y + u_\theta \theta_y = \left(u_r \sin\theta + u_\theta \frac{\cos\theta}{r}\right)$$

Now for the second order derivatives,

$$u_{xx} = (u_x)_x = (u_x)_r r_x + (u_x)_\theta \theta_x = \left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}\right)_r \cos\theta + \left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}\right)_\theta \left(-\frac{\sin\theta}{r}\right)$$

Therefore,

$$\begin{aligned} u_{xx} &= \left(u_{rr}\cos\theta - u_{\theta r}\frac{\sin\theta}{r} + u_\theta \frac{\sin\theta}{r^2}\right)\cos\theta \\ &\quad + \left(u_{r\theta}\cos\theta - u_r \sin\theta - u_{\theta\theta}\frac{\sin\theta}{r} - u_\theta \frac{\cos\theta}{r}\right)\left(-\frac{\sin\theta}{r}\right) \end{aligned} \tag{2.25}$$

Similarly, we can show that

$$\begin{aligned} u_{yy} &= \left(u_{rr}\sin\theta + u_{r\theta}\frac{\cos\theta}{r} - u_\theta \frac{\cos\theta}{r^2}\right)\sin\theta \\ &\quad + \left(u_{r\theta}\sin\theta + u_r \cos\theta + u_{\theta\theta}\frac{\cos\theta}{r} - u_\theta \frac{\sin\theta}{r}\right)\left(\frac{\cos\theta}{r}\right) \end{aligned} \tag{2.26}$$

By adding Eqs. (2.25) and (2.26) and equating to zero, we get

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \tag{2.27}$$

which is the Laplace equation in polar coordinates. One can observe that the Laplace equation in Cartesian coordinates has constant coefficients only, whereas in polar coordinates, it has variable coefficients.

EXAMPLE 2.2 Show that in cylindrical coordinates r, θ, z defined by the relations $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, the Laplace equation $\nabla^2 u = 0$ takes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Solution The Laplace equation in Cartesian coordinates is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

The relations between Cartesian and cylindrical coordinates give

$$r^2 = x^2 + y^2, \qquad \theta = \tan^{-1}(y/x), \qquad z = z$$

Since

$$u = u(r, \theta, z)$$

$$u_x = u_r r_x + u_\theta \theta_x + u_z z_x = u_r \cos\theta - u_\theta\left(\frac{\sin\theta}{r}\right)$$

$$u_y = u_r r_y + u_\theta \theta_y + u_z z_y = u_r \sin\theta + u_\theta\left(\frac{\cos\theta}{r}\right)$$

$$u_z = u_r r_z + u_\theta \theta_z + u_z = u_z$$

for the second order derivatives, we find

$$\begin{aligned} u_{xx} &= (u_x)_x = (u_x)_r r_x + (u_x)_\theta \theta_x + (u_x)_z z_x \\ &= \left[u_r \cos\theta - u_\theta\left(\frac{\sin\theta}{r}\right)\right]_r \cos\theta + \left[u_r \cos\theta - u_\theta\left(\frac{\sin\theta}{r}\right)\right]_\theta \left(-\frac{\sin\theta}{r}\right) \\ &= \left(u_{rr}\cos\theta - u_{r\theta}\frac{\sin\theta}{r} + u_\theta\frac{\sin\theta}{r^2}\right)\cos\theta \\ &\quad + \left(u_{r\theta}\cos\theta - u_r \sin\theta - u_{\theta\theta}\frac{\sin\theta}{r} - u_\theta\frac{\cos\theta}{r}\right)\left(-\frac{\sin\theta}{r}\right) \end{aligned} \tag{2.28}$$

Similarly

$$\begin{aligned} u_{yy} &= (u_y)_y = (u_y)_r r_y + (u_y)_\theta \theta_y + (u_y)_z z_y \\ &= \left[u_r \sin\theta + u_\theta \frac{\cos\theta}{r}\right]_r \sin\theta + \left[u_r \sin\theta + u_\theta\frac{\cos\theta}{r}\right]_\theta \left(\frac{\cos\theta}{r}\right) \\ &= \left(u_{rr}\sin\theta + u_{\theta r}\frac{\cos\theta}{r} - u_\theta\frac{\cos\theta}{r^2}\right)\sin\theta \\ &\quad + \left(u_{r\theta}\sin\theta + u_r \cos\theta + u_{\theta\theta}\frac{\cos\theta}{r} - u_\theta\frac{\sin\theta}{r}\right)\left(\frac{\cos\theta}{r}\right) \end{aligned} \tag{2.29}$$

$$u_{zz} = u_{zz} \tag{2.30}$$

Adding Eqs. (2.28)–(2.30), we obtain

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} \tag{2.31}$$

EXAMPLE 2.3 Show that in spherical polar coordinates r, θ, ϕ defined by the relations $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, the Laplace equations $\nabla^2 u = 0$ takes the form

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}=0$$

Solution In Cartesian coordinates, the Laplace equation is

$$\nabla^2 u = u_{xx}+u_{yy}+u_{zz}=0$$

In spherical coordinates, $u = u(r, \theta, \phi)$, $r^2 = x^2 + y^2 + z^2$, $\cos\theta = z/r$, $\tan\phi = y/x$.

It can be easily verified that

$$\theta_x = \frac{\cos\theta\cos\phi}{r}, \qquad \theta_y = \frac{\cos\theta\sin\phi}{r}, \qquad \theta_z = -\frac{\sin\theta}{r}$$

$$\phi_x = -\frac{\sin\phi}{r\sin\theta}, \qquad \phi_y = \frac{\cos\phi}{r\sin\theta}, \qquad \phi_z = 0$$

Now,

$$u_x = u_r r_x + u_\theta\theta_x + u_\phi\phi_x = u_r\sin\theta\cos\phi + u_\theta\frac{\cos\theta\cos\phi}{r} - u_\phi\frac{\sin\phi}{r\sin\theta}$$

$$u_y = u_r r_y + u_\theta\theta_y + u_\phi\phi_y = u_r\sin\theta\sin\phi + u_\theta\frac{\cos\theta\sin\phi}{r} + \frac{u_\phi\cos\phi}{r\sin\theta}$$

$$u_z = u_r r_z + u_\theta\theta_z + u_\phi\phi_z = u_r\cos\theta + u_\theta\left(-\frac{\sin\theta}{r}\right)$$

For the second order derivatives,

$$\begin{aligned}
u_{xx} &= (u_x)_r r_x + (u_x)_\theta\theta_x + (u_x)_\phi\phi_x \\
&= \left(u_r\sin\theta\cos\phi + u_\theta\frac{\cos\theta\cos\phi}{r} - u_\phi\frac{\sin\phi}{r\sin\theta}\right)_r \cdot(\sin\theta\cos\phi) \\
&\quad + \left(u_r\sin\theta\cos\phi + u_\theta\frac{\cos\theta\cos\phi}{r} - u_\phi\frac{\sin\phi}{r\sin\theta}\right)_\theta \left(\frac{\cos\theta\cos\phi}{r}\right) \\
&\quad + \left(u_r\sin\theta\cos\phi + u_\theta\frac{\cos\theta\cos\phi}{r} - u_\phi\frac{\sin\phi}{r\sin\theta}\right)_\phi \left(-\frac{\sin\phi}{r\sin\theta}\right)
\end{aligned}$$

$$
\begin{aligned}
&= (\sin^2\theta\cos^2\phi)u_{rr} + \frac{\cos^2\theta\cos^2\phi}{r^2}u_{\theta\theta} + \frac{\sin^2\phi}{r^2\sin^2\theta}u_{\phi\phi} \\
&\quad + u_{r\theta}\left(\frac{2\sin\theta\cos\theta\cos^2\phi}{r}\right) + u_{r\phi}\left(-\frac{2\sin\phi\cos\phi}{r}\right) \\
&\quad + u_{\theta\phi}\left(-\frac{2\cos\theta\cos\phi\sin\phi}{r^2\sin\theta}\right) + u_r\left(\frac{\cos^2\theta\cos^2\phi}{r} + \frac{\sin^2\phi}{r}\right) \\
&\quad + u_\phi\left(\frac{\sin\phi\cos\phi}{r^2} + \frac{\cos^2\theta\cos\phi\sin\phi}{r^2\sin^2\theta} + \frac{\sin\phi\cos\phi}{r^2\sin^2\theta}\right) \\
&\quad + u_\theta\left(\frac{\cos\theta\sin^2\phi}{r^2\sin\theta} - \frac{2\cos\theta\sin\theta\cos^2\phi}{r^2}\right)
\end{aligned}
\tag{2.32}
$$

$$
\begin{aligned}
u_{yy} &= (u_y)_r r_y + (u_y)_\theta\theta_y + (u_y)_\phi\phi_y = \left(u_r\sin\theta\sin\phi + u_\theta\frac{\cos\theta\sin\phi}{r} + u_\phi\frac{\cos\phi}{r\sin\theta}\right)_r(\sin\theta\sin\phi) \\
&\quad + \left(u_r\sin\theta\sin\phi + u_\theta\frac{\cos\theta\sin\phi}{r} + u_\phi\frac{\cos\phi}{r\sin\theta}\right)_\theta\frac{\cos\theta\sin\phi}{r} \\
&\quad + \left(u_r\sin\theta\sin\phi + u_\theta\frac{\cos\theta\sin\phi}{r} + u_\phi\frac{\cos\phi}{r\sin\theta}\right)_\phi\frac{\cos\phi}{r\sin\theta} \\
&= (\sin^2\theta\sin^2\phi)u_{rr} + \frac{\cos^2\theta\sin^2\phi}{r^2}u_{\theta\theta} + \frac{\cos^2\phi}{r^2\sin^2\theta}u_{\phi\phi} \\
&\quad + u_{r\theta}\left(\frac{2\sin\theta\cos\theta\sin^2\phi}{r}\right) + u_{r\phi}\left(\frac{2\cos\phi\sin\phi}{r}\right) \\
&\quad + u_{\theta\phi}\left(\frac{2\cos\theta\cos\phi\sin\phi}{r^2\sin\theta}\right) + u_r\left(\frac{\cos^2\theta\sin^2\phi}{r} + \frac{\cos^2\phi}{r}\right) \\
&\quad + u_\theta\left(-\frac{2\sin\theta\cos\theta\sin^2\phi}{r^2} + \frac{\cos\theta\cos^2\phi}{r^2\sin\theta}\right) \\
&\quad + u_\phi\left(-\frac{\sin\phi\cos\phi}{r^2} - \frac{\sin\phi\cos\phi}{r^2\sin^2\theta} - \frac{\cos^2\theta\sin\phi\cos\phi}{r^2\sin^2\theta}\right)
\end{aligned}
\tag{2.33}
$$

Similarly,

$$
\begin{aligned}
u_{zz} &= (u_z)_r r_z + (u_z)_\theta \theta_z + (u_z)_\phi \phi_z \\
&= \left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}\right)_r (\cos\theta) + \left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}\right)_\theta - \left(-\frac{\sin\theta}{r}\right) \\
&= u_{rr}\cos^2\theta - u_{r\theta}\frac{2\sin\theta\cos\theta}{r} + u_{\theta\theta}\frac{\sin^2\theta}{r^2} \\
&\quad + u_r \frac{\sin^2\theta}{r} + u_\theta \frac{\cos\theta\sin\theta}{r^2}
\end{aligned}
\tag{2.34}
$$

Adding Eqs. (2.32)–(2.34), we obtain

$$
\nabla^2 u = u_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r^2\sin^2\theta}u_{\phi\phi} + \frac{2}{r}u_r + \frac{\cos\theta}{r^2\sin\theta}u_\theta = 0
$$

which can be rewritten as

$$
\nabla^2 u = \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0 \tag{2.35}
$$

2.5 SEPARATION OF VARIABLES

The method of separation of variables is applicable to a large number of classical linear homogeneous equations. The choice of the coordinate system in general depends on the shape of the boundary. For example, consider a two-dimensional Laplace equation in Cartesian coordinates.

$$
\nabla^2 u = u_{xx} + u_{yy} = 0 \tag{2.36}
$$

We assume the solution in the form

$$
u(x, y) = X(x)\,Y(y) \tag{2.37}
$$

Substituting in Eq. (2.36), we get

$$
X''Y + Y''X = 0
$$

i.e.

$$
\frac{X''}{X} = -\frac{Y''}{Y} = k
$$

where k is a separation constant. Three cases arise.

Case I Let $k = p^2$, p is real. Then

$$\frac{d^2X}{dx^2} - p^2X = 0 \qquad \text{and} \qquad \frac{d^2Y}{dy^2} + p^2Y = 0$$

whose solution is given by

$$X = c_1e^{px} + c_2e^{-px}$$

and

$$Y = c_3 \cos py + c_4 \sin py$$

Thus, the solution is

$$u(x, y) = (c_1e^{px} + c_2e^{-px})(c_3 \cos py + c_4 \sin py) \tag{2.38}$$

Case II Let $k = 0$. Then

$$\frac{d^2X}{dx^2} = 0 \qquad \text{and} \qquad \frac{d^2Y}{dy^2} = 0$$

Integrating twice, we get

$$X = c_5x + c_6$$

and

$$Y = c_7y + c_8$$

The solution is therefore,

$$u(x, y) = (c_5x + c_6)(c_7y + c_8) \tag{2.39}$$

Case III Let $k = -p^2$. Proceeding as in Case I, we obtain

$$X = c_9 \cos px + c_{10} \sin px$$

$$Y = c_{11}e^{py} + c_{12}e^{-py}$$

Hence, the solution in this case is

$$u(x, y) = (c_9 \cos px + c_{10} \sin px)(c_{11}e^{py} + c_{12}e^{-py}) \tag{2.40}$$

In all these cases, $c_i (i = 1, 2, \ldots, 12)$ refer to integration constants, which are calculated by using the boundary conditions.

2.6 DIRICHLET PROBLEM FOR A RECTANGLE

The Dirichlet problem for a rectangle is defined as follows:

$$\text{PDE: } \nabla^2 u = 0, \qquad 0 \le x \le a, \; 0 \le y \le b$$

$$\text{BCs: } u(x, b) = u(a, y) = 0, \qquad u(0, y) = 0, \qquad u(x, 0) = f(x) \tag{2.41}$$

This is an interior Dirichlet problem. The general solution of the governing PDE, using the method of variables separable, is discussed in Section 2.5. The various possible solutions of the Laplace equation are given by Eqs. (2.38–2.40). Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem and the given boundary conditions as depicted in Fig. 2.3.

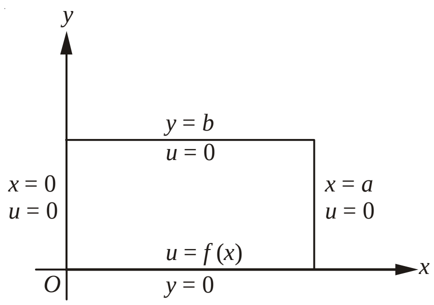

Fig. 2.3 Dirichlet boundary conditions.

Consider the solution given by Eq. (2.38):

$$u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

Using the boundary condition: $u(0, y) = 0$, we get

$$(c_1 + c_2)(c_3 \cos py + c_4 \sin py) = 0$$

which means that either $c_1 + c_2 = 0$ or $c_3 \cos py + c_4 \sin py = 0$. But $c_3 \cos py + c_4 \sin py \ne 0$; therefore,

$$c_1 + c_2 = 0 \tag{2.42}$$

Again, using the BC; $u(a, y) = 0$, Eq. (2.38) gives

$$(c_1 e^{ap} + c_2 e^{-ap})(c_3 \cos py + c_4 \sin py) = 0$$

implying thereby

$$c_1 e^{ap} + c_2 e^{-ap} = 0 \tag{2.43}$$

To determine the constants c_1, c_2, we have to solve Eqs. (2.42) and (2.43); being homogeneous, the determinant

$$\begin{vmatrix} 1 & 1 \\ e^{ap} & e^{-ap} \end{vmatrix} = 0$$

for the existence of non-trivial solution, which is not the case. Hence, only the trivial solution $u(x, y) = 0$ is possible.

If we consider the solution given by Eq. (2.39) $u(x, y) = (c_5 x + c_6)(c_7 y + c_8)$, the boundary conditions: $u(0, y) = u(a, y) = 0$ again yield a trivial solution. Hence, the possible solutions given by Eqs. (2.38) and (2.39) are ruled out. Therefore, the only possible solution obtained from Eq. (2.40) is

$$u(x, y) = (c_9 \cos px + c_{10} \sin px)(c_{11} e^{py} + c_{12} e^{-py})$$

Using the BC: $u(0, y) = 0$, we get $c_9 = 0$. Also, the other BC: $u(a, y) = 0$ yields

$$c_{10} \sin pa \, (c_{11} e^{py} + c_{12} e^{-py}) = 0$$

For non-trivial solution, c_{10} cannot be zero, implying $\sin pa = 0$, which is possible if $pa = n\pi$ or $p = n\pi/a$, $n = 1, 2, 3, \ldots$ Therefore, the possible non-trivial solution after using the superposition principle is

$$u(x, y) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} [a_n \exp(n\pi y/a) + b_n \exp(-n\pi y/a)] \tag{2.44}$$

Now, using the BC: $u(x, b) = 0$, we get

$$\sin \frac{n\pi x}{a} [a_n \exp(n\pi b/a) + b_n \exp(-n\pi b/a)] = 0$$

implying thereby

$$a_n \exp(n\pi b/a) + b_n \exp(-n\pi b/a) = 0$$

which gives

$$b_n = -a_n \frac{\exp(n\pi b/a)}{\exp(-n\pi b/a)}, \qquad n = 1, 2, \ldots, \infty$$

The solution (2.44) now becomes

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2a_n \sin(n\pi x/a)}{\exp(-n\pi b/a)} \left[\frac{\exp\{n\pi (y - b)/a\} - \exp\{-n\pi (y - b)/a\}}{2} \right]$$

$$= \sum_{n=1}^{\infty} \frac{2a_n}{\exp(-n\pi b/a)} \sin(n\pi x/a) \sin h \{n\pi (y - b)/a\}$$

Let $2a_n/[\exp(-n\pi b/a)] = A_n$. Then the solution can be written in the form

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/a) \sinh\{n\pi(y-b)/a\} \tag{2.45}$$

Finally, using the non-homogeneous boundary condition: $u(x, 0) = f(x)$, we get

$$\sum_{n=1}^{\infty} A_n \sin(n\pi x/a) \sinh(-n\pi b/a) = f(x)$$

which is a half-range Fourier series. Therefore,

$$A_n \sinh(-n\pi b/a) = \frac{2}{a}\int_0^a f(x)\sin(n\pi x/a)\,dx \tag{2.46}$$

Thus, the required solution for the given Dirichlet problem is

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/a) \sinh\{n\pi(y-b)/a\} \tag{2.47}$$

where

$$A_n = \frac{2}{a}\frac{1}{\sinh(-n\pi b/a)}\int_0^a f(x)\sin(n\pi x/a)\,dx$$

2.7 THE NEUMANN PROBLEM FOR A RECTANGLE

The Neumann problem for a rectangle is defined as follows:

$$\text{PDE: } \nabla^2 u = 0, \qquad 0 \le x \le a,\ 0 \le y \le b$$

$$\text{BCs: } u_x(0, y) = u_x(a, y) = 0, \qquad u_y(x, 0) = 0, \qquad u_y(x, b) = f(x) \tag{2.48}$$

The general solution of the Laplace equation using the method of variables separable is given in Section 2.5, and is found to be

$$u(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$$

The BC: $u_x(0, y) = 0$ gives

$$0 = c_2 p(c_3 e^{py} + c_4 e^{-py})$$

implying $c_2 = 0$. Therefore,

$$u(x, y) = c_1 \cos px(c_3 e^{py} + c_4 e^{-py}) \tag{2.49}$$

The BC: $u_x(a, y) = 0$ gives

$$0 = -c_1 p \sin pa\,(c_3 e^{py} + c_4 e^{-py})$$

For non-trivial solution, $c_1 \neq 0$, implying

$$\sin pa = 0, \qquad pa = n\pi, \qquad p = \frac{n\pi}{a} \qquad (n = 0, 1, 2, \ldots)$$

Thus the possible solution is

$$u(x, y) = \cos\frac{n\pi x}{a}\left(Ae^{n\pi y/a} + Be^{-n\pi y/a}\right) \tag{2.50}$$

Now, using the BC: $u_y(x, 0) = 0$, we get

$$0 = \cos\frac{n\pi x}{a}\left(A\frac{n\pi}{a} - B\frac{n\pi}{a}\right)$$

implying $B = A$. Thus, the solution is

$$u(x, y) = A\cos\frac{n\pi x}{a}\left[\exp(n\pi y/a) + \exp(-n\pi y/a)\right]$$

$$= 2A\cos\frac{n\pi x}{a}\cosh\frac{n\pi y}{a}$$

Using the superposition principle and defining $2A = A_n$, we get

$$u = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{a}\cosh\frac{n\pi y}{a} \tag{2.51}$$

Finally, using the BC: $u_y(x, b) = f(x)$, we get

$$f(x) = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{a}\frac{n\pi}{a}\sinh\frac{n\pi b}{a}$$

which is the half-range Fourier cosine series. Therefore,

$$A_n\frac{n\pi}{a}\sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx$$

Hence, the required solution is

$$u = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{a}\cosh\frac{n\pi y}{a} \tag{2.52}$$

where

$$A_n = \frac{2}{n\pi}\frac{1}{\sinh(n\pi b/a)}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx$$

2.8 INTERIOR DIRICHLET PROBLEM FOR A CIRCLE

The Dirichlet problem for the circle is defined as follows:

$$\begin{aligned} &\text{PDE}: \nabla^2 u = 0, && 0 \le r \le a,\ 0 \le \theta \le 2\pi \\ &\text{BC}:\ u(a,\theta) = f(\theta), && 0 \le \theta \le 2\pi \end{aligned} \tag{2.53}$$

where $f(\theta)$ is a continuous function on $\partial \mathbb{R}$. The task is to find the value of u at any point in the interior of the circle $\mathbb{R}$ in terms of its values on $\partial \mathbb{R}$ such that u is single valued and continuous on $\overline{\mathbb{R}}$.

In view of circular geometry, it is natural to choose polar coordinates to solve this problem and then use the variables separable method. The requirement of single-valuedness of u in $\overline{\mathbb{R}}$ implies the periodicity condition, i.e.,

$$u(r,\theta+2\pi) = u(r,\theta), \qquad 0 \le r \le a, \tag{2.54}$$

From Eq. (2.27), $\nabla^2 u = 0$ which in polar coordinates can be written as

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

If $u(r,\theta) = R(r)H(\theta)$, the above equation reduces to

$$R''H + \frac{1}{r}R'H + \frac{1}{r^2}RH'' = 0$$

This equation can be rewritten as

$$\frac{r^2R'' + rR'}{R} = -\frac{H''}{H} = k \tag{2.55}$$

which means that a function of r is equal to a function of θ and, therefore, each must be equal to a constant k (a separation constant).

Case I Let $k = \lambda^2$. Then

$$r^2R'' + rR' - \lambda^2 R = 0 \tag{2.56}$$

which is a Euler type of equation and can be solved by setting $r = e^z$. Its solution is

$$R = c_1e^{\lambda z} + c_2e^{-\lambda z} = c_1r^{\lambda} + c_2r^{-\lambda}$$

Also,

$$H'' + \lambda^2 H = 0$$

whose solution is

$$H = c_3 \cos \lambda\theta + c_4 \sin \lambda\theta$$

Therefore,

$$u(r,\theta) = (c_1 r^{\lambda} + c_2 r^{-\lambda})(c_3 \cos \lambda\theta + c_4 \sin \lambda\theta) \tag{2.57}$$

Case II Let $k = -\lambda^2$. Then

$$r^2 R'' + rR' + \lambda^2 R = 0, \qquad H'' - \lambda^2 H = 0$$

Their respective solutions are

$$R = c_1 \cos(\lambda \ln r) + c_2 \sin(\lambda \ln r)$$

$$H = c_3 e^{\lambda\theta} + c_4 e^{-\lambda\theta}$$

Thus

$$u(r,\theta) = [c_1 \cos(\lambda \ln r) + c_2 \sin(\lambda \ln r)](c_3 e^{\lambda\theta} + c_4 e^{-\lambda\theta}) \tag{2.58}$$

Case III Let $k = 0$. Then we have

$$rR'' + R' = 0$$

Setting $R'(r) = V(r)$, we obtain

$$r\frac{dV}{dr} + V = 0, \quad \text{i.e.,} \ \frac{dV}{V} + \frac{dr}{r} = 0$$

Integrating, we get $\ln Vr = \ln c_1$. Therefore,

$$V = \frac{c_1}{r} = \frac{dR}{dr}$$

On integration,

$$R = c_1 \ln r + c_2$$

Also,

$$H'' = 0$$

After integrating twice, we get

$$H = c_3\theta + c_4$$

Thus,

$$u(r,\theta) = (c_1 \ln r + c_2)(c_3\theta + c_4) \tag{2.59}$$

Now, for the interior problem, $r = 0$ is a point in the domain $\mathbb{R}$ and since ln r is not defined at $r = 0$, the solutions (2.58) and (2.59) are not acceptable. Thus the required solution is obtained from Eq. (2.57). The periodicity condition in θ implies

$$c_3 \cos \lambda\theta + c_4 \sin \lambda\theta = c_3 \cos (\lambda(\theta + 2\pi)) + c_4 \sin (\lambda(\theta + 2\pi))$$

i.e.

$$c_3[\cos \lambda\theta - \cos (\lambda\theta + 2\lambda\pi)] + c_4[\sin \lambda\theta - \sin (\lambda\theta + 2\lambda\pi)] = 0$$

or

$$2 \sin \lambda\pi [c_3 \sin (\lambda\theta + \lambda\pi) - c_4 \cos (\lambda\theta + \lambda\pi)] = 0$$

implying $\sin \lambda\pi = 0$, $\lambda\pi = n\pi$, $\lambda = n (n = 0, 1, 2, \ldots)$. Using the principle of superposition and renaming the constants, the acceptable general solution can be written as

$$u(r, \theta) = \sum_{n=0}^{\infty} (c_n r^n + d_n r^{-n}) (a_n \cos n\theta + b_n \sin n\theta) \tag{2.60}$$

At $r = 0$, the solution should be finite, which requires $d_n = 0$. Thus the appropriate solution assumes the form

$$u(r, \theta) = \sum_{n=0}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

For $n = 0$, let the constant A_0 be $A_0/2$. Then the solution is

$$u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \tag{2.61}$$

which is a full-range Fourier series. Now we have to determine A_n and B_n so that the BC: $u(a, \theta) = f(\theta)$ is satisfied, i.e.,

$$f(\theta) = \sum_{n=0}^{\infty} a^n (A_n \cos n\theta + B_n \sin n\theta)$$

Hence,

$$\begin{aligned} A_0 &= \frac{1}{\pi} \int_0^{2\pi} f(\theta)\, d\theta \\ a^n A_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta \\ a^n B_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta, \; n = 1, 2, 3, \ldots \end{aligned} \tag{2.62}$$

In Eqs. (2.62) we replace the dummy variable θ by ϕ to distinguish this variable from the current variable θ in Eq. (2.61). Substituting Eq. (2.62) into Eq. (2.61), we obtain the relation

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi + \sum_{n=1}^{\infty}\left[\frac{r^n}{a^n}\frac{\cos n\theta}{\pi}\int_0^{2\pi}\cos(n\phi)\,f(\phi)\,d\phi\right.$$
$$\left. + \frac{r^n}{a^n}\frac{\sin n\theta}{\pi}\int_0^{2\pi}\sin(n\phi)\,f(\phi)\,d\phi\right]$$

Interchanging the order of summation and integration, we get

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi)\,d\phi + \frac{1}{\pi}\int_0^{2\pi} f(\phi)\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\{\cos n\phi\cos n\theta + \sin n\phi \sin n\theta\}\,d\phi$$
$$= \frac{1}{\pi}\int_0^{2\pi} f(\phi)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\phi-\theta)\right]d\phi \qquad (2.63)$$

To obtain an alternative expression for $u(r,\theta)$ in closed integral form, we can proceed as follows:

Let

$$c = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\phi-\theta)$$
$$s = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\sin n(\phi-\theta)$$

so that

$$c + is = \sum_{n=1}^{\infty}\left[\frac{r}{a}e^{i(\phi-\theta)}\right]^n$$

Since $r < a, (r/a) < 1$ and $|e^{i(\phi-\theta)}| \le 1$,

$$c + is = \sum_{n=1}^{\infty}\left[\left(\frac{r}{a}\right)e^{i(\phi-\theta)}\right]^n = \frac{(r/a)e^{i(\phi-\theta)}}{[1-(r/a)e^{i(\phi-\theta)}]}$$
$$= \frac{(r/a)\{e^{i(\phi-\theta)} - (r/a)\}}{[1-(r/a)e^{i(\phi-\theta)}][1-(r/a)e^{-i(\phi-\theta)}]}$$

Equating the real part on both sides, we get

$$c = \frac{[r/a) \cos(\phi - \theta) - (r^2/a^2)]}{[1 - (2r/a) \cos(\phi - \theta) + (r^2/a^2)]}$$

Thus, the expression in the square brackets of Eq. (2.63) becomes

$$\frac{1}{2} + \frac{[(r/a) \cos(\phi - \theta) - (r^2/a^2)]}{[1 - (2r/a) \cos(\phi - \theta) + (r^2/a^2)]} = \frac{a^2 - r^2}{2[a^2 - 2ar \cos(\phi - \theta) + r^2]}$$

Thus, the required solution takes the form

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(a^2 - r^2) f(\phi)}{[a^2 - 2ar \cos(\phi - \theta) + r^2]} d\phi \tag{2.64}$$

This is known as Poisson's integral formula for a circle, which gives a unique solution for the Dirichlet problem. The solution (2.64) can be interpreted physically in many ways: It can be thought of as finding the potential $u(r, \theta)$ as a weighted average of the boundary potentials $f(\phi)$ weighted by the Poisson kernel P, given by

$$P = \frac{a^2 - r^2}{[a^2 - 2ar \cos(\phi - \theta) + r^2]}$$

It can also be thought of as a steady temperature distribution $u(r, \theta)$ in a circular disc, when the temperature u on its boundary $\partial\mathbb{R}$ is given by $u = f(\phi)$ which is independent of time.

2.9 EXTERIOR DIRICHLET PROBLEM FOR A CIRCLE

The exterior Dirichlet problem is described by

$$\text{PDE: } \nabla^2 u = 0$$
$$\text{BC: } u(a, \theta) = f(\theta) \tag{2.65}$$

u must be bounded as $r \to \infty$.

By the method of separation of variables, the general solution (2.60) of $\nabla^2 u = 0$ in polar coordinates can be written as

$$u(r, \theta) = \sum_{n=0}^{\infty} (c_n r^n + d_n r^{-n})(a_n \cos n\theta + b_n \sin n\theta)$$

Now as, $r \to \infty$, we require u to be bounded, and, therefore, $c_n = 0$.

After adjusting the constants, the general solution now reads

$$u(r, \theta) = \sum_{n=0}^{\infty} r^{-n} (A_n \cos n\theta + B_n \sin n\theta)$$

With no loss of generality, it can also be written as

$$u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^{-n} (A_n \cos n\theta + B_n \sin n\theta) \tag{2.66}$$

Using the BC: $u(a, \theta) = f(\theta)$, we obtain

$$f(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^{-n} (A_n \cos n\theta + B_n \sin n\theta)$$

This is a full-range Fourier series in $f(\theta)$, where

$$\begin{aligned} A_0 &= \frac{1}{\pi} \int_0^{2\pi} f(\theta)\, d\theta \\ a^{-n} A_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos n\theta\, d\theta \\ a^{-n} B_n &= \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin n\theta\, d\theta \end{aligned} \tag{2.67}$$

In Eq. (2.67) we replace the dummy variable θ by ϕ so as to distinguish it from the current variable θ. We then introduce the changed variable into solution (2.66) which becomes

$$\begin{aligned} u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(\phi)\, d\phi + \sum_{n=1}^{\infty} \Bigg[& \frac{r^{-n} a^n}{\pi} \cos n\theta \int_0^{2\pi} \cos (n\phi)\; f(\phi)\, d\phi \\ & + \frac{r^{-n} a^n}{\pi} \sin n\theta \int_0^{2\pi} \sin (n\phi)\; f(\phi)\, d\phi \Bigg] \end{aligned}$$

or

$$u(r, \theta) = \frac{1}{\pi} \int_0^{2\pi} f(\phi) \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \cos n(\phi - \theta) \right] d\phi \tag{2.68}$$

Let

$$C = \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \cos n(\phi - \theta)$$

$$S = \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \sin n(\phi - \theta)$$

Then,

$$C + iS = \sum_{n=1}^{\infty} \left[\left(\frac{a}{r}\right) e^{i(\phi-\theta)}\right]^n$$

Since $\dfrac{a}{r} < 1$, $|e^{i(\phi-\theta)}| \le 1$. We have

$$C + iS = \frac{a}{r}\frac{e^{i(\phi-\theta)}}{[1-(a/r)e^{i(\phi-\theta)}]} = \frac{(a/r)[e^{i(\phi-\theta)} - (a/r)]}{[1-(a/r)e^{i(\phi-\theta)}][1-(a/r)e^{-i(\phi-\theta)}]}$$

Hence,

$$C = \frac{[(a/r)\cos(\phi-\theta) - (a^2/r^2)]}{[1-(2a/r)\cos(\phi-\theta) + (a^2/r^2)]}$$

Thus the quantity in the square brackets on the right-hand side of Eq. (2.68) becomes

$$\frac{1}{2} + \frac{[(a/r)\cos(\phi-\theta) - (a^2/r^2)]}{[1-(2a/r)\cos(\phi-\theta) + (a^2/r^2)]} = \frac{r^2 - a^2}{2[r^2 - 2ar\cos(\phi-\theta) + a^2]}$$

Therefore, the solution of the exterior Dirichlet problem reduces to that of an integral equation of the form

$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2 - a^2) f(\phi)}{[r^2 - 2ar\cos(\phi-\theta) + a^2]} d\phi \tag{2.69}$$

EXAMPLE 2.4 Find the steady state temperature distribution in a semi-circular plate of radius a, insulated on both the faces with its curved boundary kept at a constant temperature U_0 and its bounding diameter kept at zero temperature as described in Fig. 2.4.

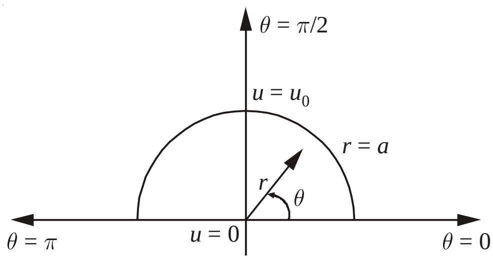

Fig. 2.4 Semi-circular plate.

Solution The governing heat flow equation is

$$u_t = \nabla^2 u$$

In the steady state, the temperature is independent of time; hence $u_t = 0$, and the temperature satisfies the Laplace equation. The problem can now be stated as follows: To solve

$$\text{PDE: } \nabla^2 u(r, \theta) = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

$$\text{BCs: } u(a, \theta) = U_0, \qquad u(r, 0) = 0, \qquad u(r, \pi) = 0$$

the acceptable general solution is

$$u(r, \theta) = (cr^{\lambda} + Dr^{-\lambda})(A\cos\lambda\theta + B\sin\lambda\theta) \tag{2.70}$$

From the BC: $u(r, 0) = 0$, we get $A = 0$; however, the BC: $u(r, \pi) = 0$ also gives

$$B\sin\lambda\pi(cr^{\lambda} + Dr^{-\lambda}) = 0$$

implying either $B = 0$ or $\sin\lambda\pi = 0$. $B = 0$ gives a trivial solution. For a non-trivial solution, we must have $\sin\lambda\pi = 0$, implying

$$\lambda\pi = n\pi, \qquad n = 1, 2, \ldots$$

meaning thereby $\lambda = n$. Hence, the possible solution is

$$u(r, \theta) = B\sin n\theta(Cr^{\lambda} + Dr^{-\lambda}) \tag{2.71}$$

In Eq. (2.71), we observe that as $r \to 0$, the term $r^{-\lambda} \to \infty$. But the solution should be finite at $r = 0$, and so $D = 0$. Then after adjusting the constants, it follows from the superposition principle that,

$$u(r, \theta) = \sum_{n=1}^{\infty} B_n r^n \sin n\theta$$

Finally, using the first BC: $u(a, \theta) = U_0$, we get

$$u(a, \theta) = U_0 = \sum_{n=1}^{\infty} B_n a^n \sin n\theta$$

which is a half-range Fourier sine series. Therefore,

$$B_n a^n = \frac{2}{\pi}\int_0^{\pi} U_0 \sin n\theta \, d\theta = \begin{cases} \dfrac{4U_0}{n\pi}, & \text{for } n = 1, 3, \ldots \\ 0, & \text{for } n = 2, 4, \ldots \end{cases}$$

Hence,

$$B_n = \frac{4U_0}{n\pi a^n}, \qquad n = 1, 3, \ldots$$

With these values of B_n, the required solution is

$$u(r, \theta) = \frac{4U_0}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{1}{n}\left(\frac{r}{a}\right)^n \sin n\theta$$

2.10 INTERIOR NEUMANN PROBLEM FOR A CIRCLE

The interior Neumann problem for a circle is described by

$$\text{PDE: } \nabla^2 u = 0, \qquad 0 \le r < a; \; 0 \le \theta \le 2\pi \tag{2.72}$$

$$\text{BC: } \frac{\partial u}{\partial n} = \frac{\partial u(a, \theta)}{\partial r} = g(\theta), \qquad r = a$$

Following the method of separation of variables, the general solution (2.60) of equation $\nabla^2 u = 0$ in polar coordinates is given by

$$u(r, \theta) = \sum_{n=0}^{\infty} (c_n r^n + d_n r^{-n})(a_n \cos n\theta + b_n \sin n\theta)$$

At $r = 0$, the solution should be finite and, therefore, $d_n = 0$. Hence, after adjusting the constants, the general solution becomes

$$u(r, \theta) = \sum_{n=0}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

With no loss of generality, this equation can be written as

$$u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \tag{2.73}$$

$$\frac{\partial u}{\partial r} = \sum_{n=1}^{\infty} n r^{n-1} (A_n \cos n\theta + B_n \sin n\theta)$$

Using the BC:

$$\frac{\partial u}{\partial r}(a, \theta) = g(\theta)$$

we get

$$g(\theta) = \sum_{n=1}^{\infty} na^{n-1}(A_n \cos n\theta + B_n \sin n\theta) \tag{2.74}$$

which is a full-range Fourier series in $g(\theta)$, where

$$na^{n-1}A_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos n\theta\, d\theta$$

$$na^{n-1}B_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin n\theta\, d\theta \tag{2.75}$$

Here, we replace the dummy variable θ by ϕ to distinguish from the current variable θ in Eq. (2.74). Now introducing Eq. (2.75) into Eq. (2.73), we obtain

$$u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \frac{r^n}{n\pi a^{n-1}} \int_0^{2\pi} g(\phi)\,(\cos n\phi \cos n\theta + \sin n\phi \sin n\theta)\, d\phi$$

or

$$u(r, \theta) = \frac{A_0}{2} + \int_0^{2\pi} g(\phi) \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \frac{a}{n\pi} \cos n(\phi - \theta)\, d\phi \tag{2.76}$$

This solution can also be expressed in an alternative integral form as follows: Let

$$C = \sum \left(\frac{r}{a}\right)^n \frac{a}{n\pi} \cos n(\phi - \theta)$$

$$S = \sum \left(\frac{r}{a}\right)^n \frac{a}{n\pi} \sin n(\phi - \theta)$$

Therefore,

$$C + iS = \sum \left(\frac{r}{a}\right)^n \frac{a}{n\pi} e^{in(\phi-\theta)} = \frac{a}{\pi} \sum_{n=1}^{\infty} \left[\frac{r}{a} e^{i(\phi-\theta)}\right]^n \frac{1}{n}$$

$$= \frac{a}{\pi}\left[\frac{\left\{\frac{r}{a}e^{i(\phi-\theta)}\right\}}{1} + \frac{\left\{\frac{r}{a}e^{i(\phi-\theta)}\right\}^2}{2} + \frac{\left\{\frac{r}{a}e^{i(\phi-\theta)}\right\}^3}{3} + \cdots\right]$$

or

$$C + iS = -\frac{a}{\pi}\ln\left[1-\frac{r}{a}e^{i(\phi-\theta)}\right] = -\frac{a}{\pi}\ln\left[1-\frac{r}{a}\cos(\phi-\theta)-i\frac{r}{a}\sin(\phi-\theta)\right] \quad (2.77)$$

To get the real part of $\ln z$, we may note that

$$W = \ln z \quad \text{or} \quad z = e^W$$

i.e., $x + iy = e^{u+iv} = e^u \cos v + ie^u \sin v$. Therefore,

$$x = e^u \cos v, \qquad y = e^u \sin v$$

$$e^{2u} = x^2 + y^2 = |z|^2$$

i.e., $u = \ln|z|$. Therefore,

$$C = -\frac{a}{\pi}\ln\sqrt{\left(1-\frac{r}{a}\cos(\phi-\theta)\right)^2 + \left(\frac{r}{a}\sin(\phi-\theta)\right)^2}$$

$$= -\frac{a}{\pi}\ln\sqrt{\frac{a^2-2ar\cos(\phi-\theta)+r^2}{a^2}}$$

Thus the required solution is

$$u(r,\theta) = \frac{A_0}{2} - \frac{a}{\pi}\int_0^{2\pi}\ln\sqrt{\frac{a^2-2ar\cos(\phi-\theta)+r^2}{a^2}}\,g(\phi)\,d\phi \quad (2.78)$$

which is again an integral equation.

2.11 SOLUTION OF LAPLACE EQUATION IN CYLINDRICAL COORDINATES

The Laplace equation in cylindrical coordinates assumes the following form:

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0 \quad (2.79)$$

We now seek a separable solution of the form

$$u(r,\theta,z) = F(r,\theta)\,Z(z) \quad (2.80)$$

Substituting Eq. (2.80) into Eq. (2.79), we get

$$\frac{\partial^2 F}{\partial r^2}Z + \frac{1}{r}\frac{\partial F}{\partial r}Z + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}Z + F\frac{d^2 Z}{dz^2} = 0$$

or

$$\left(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right)\frac{1}{F} = -\frac{d^2 Z}{dz^2}\frac{1}{Z} = k \ \text{(say)}$$

where k is a separation constant. Therefore, either

$$\frac{d^2Z}{dz^2}+kZ=0 \tag{2.81}$$

or

$$\frac{\partial^2 F}{\partial r^2}+\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}-KF=0 \tag{2.82}$$

If k is real and positive, the solution of Eq. (2.81) is

$$Z=c_1\cos\sqrt{k}z+c_2\sin\sqrt{k}z$$

If k is negative, the solution of Eq. (2.81) is

$$Z=c_1e^{\sqrt{k}z}+c_2e^{-\sqrt{k}z}$$

If k is equal to zero, the solution of Eq. (2.81) is

$$Z=c_1z+c_2$$

From physical considerations, one would expect a solution which decays with increasing z and, therefore, the solution corresponding to negative k is acceptable. Let $k=-\lambda^2$. Then

$$Z=c_1e^{\lambda z}+c_2e^{-\lambda z} \tag{2.83}$$

Equation (2.82) now becomes

$$\frac{\partial^2 F}{\partial r^2}+\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}+\lambda^2F=0$$

Let $F(r,\theta)=f(r)H(\theta)$. Substituting into the above equations, we get

$$f''H+\frac{1}{r}f'H+\frac{1}{r^2}fH''+\lambda^2 fH=0$$

or

$$(r^2f''+rf'+\lambda^2r^2f)\frac{1}{f}=-\frac{H''}{H}=k' \quad \text{(say)}$$

From physical consideration, we expect the solution to be periodic in θ, which can be obtained when k' is positive and $k'=n^2$. Therefore, the acceptable solution will be

$$H=c_3\cos n\theta+c_4\sin n\theta \tag{2.84}$$

When $k'=n^2$, we will also have

$$r^2\frac{d^2f}{dr^2}+r\frac{df}{dr}+(\lambda^2r^2-n^2)f=0 \tag{2.85}$$

which is a Bessel's equation whose general solution is given by

$$f = AJ_n(\lambda r) + BY_n(\lambda r) \tag{2.86}$$

Here, $J_n(\lambda r)$ and $Y_n(\lambda r)$ are the nth order Bessel functions of first and second kind, respectively. Since $Y_n(\lambda r) \to -\infty$ as $r \to 0$, $Y_n(\lambda r)$ becomes unbounded at $r = 0$. Continuity of the solution demands $B = 0$. Hence the most general and acceptable solution of $\nabla^2 u = 0$ is

$$u(r, \theta, z) = J_n(\lambda r)\,(c_1 e^{\lambda z} + c_2 e^{-\lambda z})\,(c_3 \cos n\theta + c_4 \sin n\theta) \tag{2.87}$$

EXAMPLE 2.5 A homogeneous thermally conducting cylinder occupies the region $0 \le r \le a$, $0 \le \theta \le 2\pi$, $0 \le z \le h$, where r, θ, z are cylindrical coordinates. The top $z = h$ and the lateral surface $r = a$ are held at 0°, while the base $z = 0$ is held at 100°. Assuming that there are no sources of heat generation within the cylinder, find the steady-temperature distribution within the cylinder.

Solution The temperature u must be a single valued continuous function. The steady state temperature satisfies the Laplace equation inside the cylinder. To compute the temperature distribution inside the cylinder, we have to solve the following BVP:

$$\begin{aligned}
&\text{PDE:} \quad \nabla^2 u = 0\\
&\text{BCs:} \quad u = 0^\circ \quad \text{on } z = h,\\
&\qquad\quad\; u = 0^\circ \quad \text{on } r = a,\\
&\qquad\quad\; u = 100^\circ \quad \text{on } z = 0
\end{aligned}$$

The general solution of the Laplace equation in cylindrical coordinates as given in Section 2.11 is

$$r(r, \theta, z) = J_n(\lambda r)\,(c_1 \cos n\theta + c_2 \sin n\theta)\,(c_3 e^{\lambda z} + c_4 e^{-\lambda z})$$

Since the face $z = 0$ is maintained at 100° and since the other face and lateral surface of the cylinder are maintained at 0°, the temperature at any point inside the cylinder is obviously independent of θ. This is possible only when $n = 0$ in the general solution. Thus,

$$u(r, z) = J_0(\lambda r)\,(Ae^{\lambda z} + Be^{-\lambda z})$$

Using the BC: $u = 0$ on $z = h$, we get

$$0 = J_0(\lambda r)\,(Ae^{\lambda h} + Be^{-\lambda h})$$

implying thereby $Ae^{\lambda h} + Be^{-\lambda h} = 0$, from which

$$B = -\frac{Ae^{\lambda h}}{e^{-\lambda h}}$$

Therefore, the solution is

$$u(r, z) = \frac{J_0(\lambda r) A}{e^{-\lambda h}} [e^{\lambda(z-h)} - e^{-\lambda(z-h)}]$$

or

$$u(r, z) = J_0(\lambda r) A_1 \sinh \lambda (z - h)$$

where $A_1 = 2A/e^{-\lambda h}$. Now using the BC: $u = 0$ on $r = a$, we have

$$0 = A_1 J_0(\lambda a) \sinh \lambda (z - h)$$

implying $J_0(\lambda a) = 0$, which has infinitely many positive roots. Denoting them by ξ_n, we have $\xi_n = \lambda a$, and therefore,

$$\lambda = \frac{\xi_n}{a}$$

Thus the solution is

$$u(r, z) = A_1 J_0\left(\frac{\xi_n r}{a}\right) \sinh\left[\frac{\xi_n}{a}(z - h)\right], \qquad n = 1, 2, \ldots$$

Using the principle of superposition, we have

$$u(r, z) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\xi_n r}{a}\right) \sinh\left[\frac{\xi_n}{a}(z - h)\right]$$

The BC: $u = 100°$ on $z = 0$ gives

$$100 = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{\xi_n h}{a}\right) J_0\left(\frac{\xi_n r}{a}\right)$$

which is a Fourier-Bessel series. Multiplying both sides with $rJ_0(\xi_m r/a)$ and integrating, we get

$$100 \int_0^a rJ_0\left(\frac{\xi_m r}{a}\right) dr = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{\xi_n h}{a}\right) \int_0^a rJ_0\left(\frac{\xi_n r}{a}\right) J_0\left(\frac{\xi_m r}{a}\right) dr$$

Using the orthogonality property of Bessel's function, namely,

$$\int_0^a xJ_n(\alpha_i x) J_n(\alpha_j x)\, dx = \begin{cases} 0, & \text{if } i \neq j \\ \dfrac{a^2}{2} J_{n+1}^2(\alpha_i), & \text{if } i = j \end{cases}$$

where α_i, α_j are the zeros at $J_n(x) = 0$, we have

$$100 \int_0^a rJ_0\left(\frac{\xi_n r}{a}\right) dr = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{\xi_n h}{a}\right) \frac{a^2}{2} J_1^2(\xi_n)$$

Therefore,

$$A_n = \frac{200}{a^2 \sinh\left(-\frac{\xi_n h}{a}\right) J_1^2(\xi_n)} \int_0^a rJ_0\left(\frac{\xi_n r}{a}\right) dr$$

Setting

$$\frac{\xi_n r}{a} = x, \qquad dr = \frac{a}{\xi_n} dx$$

the relation for A_n can also be written as

$$A_n = \frac{200}{\xi_n^2 \sinh\left(-\frac{\xi_n h}{a}\right) J_1^2(\xi_n)} \int_0^{\xi_n} xJ_0(x)\, dx$$

Using the recurrence relation

$$x^n J_{n-1}(x) = \frac{d}{dx}[x^n J_n(x)],$$

For $n = 1$, we get

$$\int xJ_0(x)\, dx = xJ_1(x)$$

Now, A_n can be written as

$$A_n = \left[\frac{200\, xJ_1(x)}{\xi_n^2 \sinh(-\xi_n h/a)\, J_1^2(\xi_n)}\right]_0^{\xi_n} = \frac{200}{\xi_n \sinh(-\xi_n h/a)\, J_1(\xi_n)}$$

Hence, the required temperature distribution inside the cylinder is

$$u(r, z) = 200 \sum_{n=1}^{\infty} \frac{J_0(\xi_n r/a) \sinh[(\xi_n/a)(z - h)]}{\xi_n \sinh(-\xi_n h/a)\, J_1(\xi_n)}$$

where ξ_n are the positive zeros of $J_0(\xi)$.

EXAMPLE 2.6 Find the potential u inside the cylinder $0 \le r \le a$, $0 \le \theta \le 2\pi$, $0 \le z \le h$, if the potential on the top $z = h$, and on the lateral surface $r = a$ is held at zero, while on the base $z = 0$, the potential is given by $u(r, \theta, 0) = V_0(1 - r^2/a^2)$, where V_0 is a constant; r, θ, z are cylindrical polar coordinates.

Solution The potential u must be a single-valued continuous function and satisfy the Laplace equation inside the cylinder. To compute the potential inside the cylinder, we have to solve the following BVP:

$$\text{PDE:} \quad \nabla^2 u = 0$$

$$\begin{aligned} \text{BCs:} \quad & u = 0 \quad \text{on } z = h, \\ & u = 0 \quad \text{on } r = a, \\ & u = V_0\left(1 - \frac{r^2}{a^2}\right) \quad \text{on } z = 0 \end{aligned}$$

In cylindrical coordinates, the general solution of the Laplace equation as given in Section 2.11 is

$$u(r, \theta, z) = J_n(\lambda r)(c_1 \cos n\theta + c_2 \sin n\theta)(c_3 e^{\lambda z} + c_4 e^{-\lambda z})$$

Since the face $z = 0$ has potential $V_0(1 - r^2/a^2)$, which is purely a function of r and is independent of θ and since the other faces of the cylinder are at zero potential, the potential at any point inside the cylinder will obviously be independent of θ. This is possible only when $n = 0$ in the general solution. Thus,

$$u(r, z) = J_0(\lambda r)(Ae^{\lambda z} + Be^{-\lambda z})$$

Using the BC: $u = 0$ on $z = h$, we obtain

$$0 = J_0(\lambda r)(Ae^{\lambda h} + Be^{-\lambda h})$$

implying $Ae^{\lambda h} + Be^{-\lambda h} = 0$, which yields

$$B = -\frac{Ae^{\lambda h}}{e^{-\lambda h}}$$

Hence, the solution is

$$u(r, z) = \frac{A}{e^{-\lambda h}} J_0(\lambda r)[e^{\lambda(z-h)} - e^{-\lambda(z-h)}]$$

or

$$u(r, z) = A_1 J_0(\lambda r) \sinh \lambda (z - h)$$

where $A_1 = A/e^{-\lambda h}$. Now, using the BC: $u = 0$ on the lateral surface, i.e., on $r = a$, we get

$$0 = A_1 J_0(\lambda a) \sinh \lambda (z - h)$$

implying $J_0(\lambda a) = 0$. This has infinitely many positive roots; denoting them by ξ_n we shall have

$$\xi_n = \lambda a \quad \text{or} \quad \lambda = \xi_n/a$$

The solution now takes the form

$$u(r, z) = A_1 J_0\left(\frac{\xi_n r}{a}\right) \sinh\left[\frac{\xi_n}{a}(z - h)\right], \qquad n = 1, 2, \ldots$$

The principle of superposition gives

$$u(r, z) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\xi_n r}{a}\right) \sinh\left[\frac{\xi_n}{a}(z - h)\right]$$

The last BC: $u = V_0\left(1 - \dfrac{r^2}{a^2}\right)$ on $z = 0$ yields

$$V_0\left(1 - \frac{r^2}{a^2}\right) = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{\xi_n h}{a}\right) J_0\left(\frac{\xi_n r}{a}\right)$$

This is a Fourier-Bessel's series. Multiplying both sides by $rJ_0(\xi_m r/a)$ and integrating, we get

$$V_0 \int_0^a \left(1 - \frac{r^2}{a^2}\right) r J_0\left(\frac{\xi_m r}{a}\right) dr = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{\xi_n h}{a}\right) \int_0^a r J_0\left(\frac{\xi_n r}{a}\right) J_0\left(\frac{\xi_m r}{a}\right) dr$$

Using the orthogonality property of the Bessel functions

$$\int_0^a x J_n(\alpha_i x)\, J_n(\alpha_j x)\, dx = \begin{cases} 0, & \text{if } i \neq j \\ \dfrac{a^2}{2} J_{n+1}^2(\alpha_i), & \text{if } i = j \end{cases}$$

where α_i, α_j are the zeros of $J_n(x) = 0$, we get

$$V_0 \int_0^a \left(1 - \frac{r^2}{a^2}\right) r J_0\left(\frac{\xi_n r}{a}\right) dr = \sum_{n=1}^{\infty} A_n \sinh\left(-\frac{\xi_n h}{a}\right) \frac{a^2}{2} J_1^2(\xi_n)$$

which gives

$$A_n = \frac{2V_0}{a^2 \sinh\left(-\dfrac{\xi_n h}{a}\right) J_1^2(\xi_n)} \int_0^a \left(1 - \frac{r^2}{a^2}\right) r J_0\left(\frac{\xi_n r}{a}\right) dr$$

By letting $\xi_n r/a = x$, this equation can be modified to

$$A_n = \frac{2V_0}{\xi_n^4 \sinh\left(-\dfrac{\xi_n h}{a}\right) J_1^2(\xi_n)} \int_0^{\xi_n} (\xi_n^2 - x^2)\, x J_0(x)\, dx$$

Using the well-known recurrence relation

$$x^\alpha J_{\alpha-1}(x) = \frac{d}{dx}[x^a J_\alpha(x)] \quad \text{for } \alpha = 1, 2, \ldots$$

we get

$$\int x J_0(x) = x J_1(x), \qquad \int x^2 J_1(x) = x^2 J_2(x)$$

From these relations, we obtain

$$A_n = \frac{2V_0}{\xi_n^4 \sinh\left(-\dfrac{\xi_n h}{a}\right) J_1^2(\xi_n)} \int_0^{\xi_n} (\xi_n^2 - x^2)\, d[x J_1(x)]$$

Integrating by parts, we get

$$A_n = \frac{4V_0}{\xi_n^4 \sinh(-\xi_n h/a)\, J_1^2(\xi_n)} \int_0^{\xi_n} x^2 J_1(x)\, dx$$

$$= \frac{4V_0}{\xi_n^4 \sinh(-\xi_n h/a)\, J_1^2(\xi_n)} \int_0^{\xi_n} d[x^2 J_2(x)]$$

$$= \frac{4V_0}{\xi_n^4 \sinh(-\xi_n h/a)\, J_1^2(\xi_n)} [x^2 J_2(x)]_0^{\xi_n}$$

Thus,

$$A_n = \frac{4V_0 J_2(\xi_n)}{\xi_n^2 \sinh(-\xi_n h/a)\, J_1^2(\xi_n)}$$

The recurrence relation

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

for $n = 1$ gives

$$J_0(\xi_n) + J_2(\xi_n) = \frac{2}{\xi_n} J_1(\xi_n)$$

Hence,

$$J_2(\xi_n) = \frac{2}{\xi_n} J_1(\xi_n)$$

since $J_0(\xi_n) = 0$. Therefore,

$$A_n = \frac{8V_0 J_1(\xi_n)}{\xi_n^3 \sinh(-\xi_n h/a)\, J_1^2(\xi_n)}$$

Thus, the required potential inside the cylinder is

$$u(r, z) = \sum_{n=1}^{\infty} \frac{8V_0 J_0\left(\frac{\xi_n r}{a}\right) \sinh\left[\frac{\xi_n}{a}(z-h)\right]}{\xi_n^3 J_1(\xi_n) \sinh\left(-\frac{\xi_n h}{a}\right)}$$

2.12 SOLUTION OF LAPLACE EQUATION IN SPHERICAL COORDINATES

In Example 2.3, the Laplace equation is expressed in spherical coordinates and has the following form:

$$\nabla^2 u = \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0 \tag{2.88}$$

Let us assume the separable solution in the form

$$u(r, \theta, \phi) = R(r)\, F(\theta, \phi) \tag{2.89}$$

Substituting Eq. (2.89) into Eq. (2.88), we get

$$F\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{R}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial F}{\partial\theta}\right) + \frac{R}{\sin^2\theta}\frac{\partial^2 F}{\partial\phi^2} = 0$$

Separation of variables gives

$$\frac{\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right)}{R} = \frac{-\frac{1}{\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial F}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 F}{\partial\phi^2}\right\}}{F} = -\mu$$

where μ is a separation constant. Therefore,

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\mu \tag{2.90}$$

$$\frac{1}{F\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 F}{\partial\phi^2}\right] = \mu \tag{2.91}$$

Equation (2.90) gives

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \mu R = 0$$

which is a Euler's equation. Hence, using the transformation $r = e^z$, the auxiliary equation can be written as

$$D(D-1) + 2D + \mu = D^2 + D + \mu = 0$$

where $D = d/dz$. Its roots are given by

$$D = \frac{-1 \pm \sqrt{1-4\mu}}{2}$$

Let $\mu = -\alpha(\alpha+1)$; then we get

$$D = \frac{\left\{-1 \pm 2\sqrt{\left(\alpha+\frac{1}{2}\right)^2}\right\}}{2} = -\frac{1}{2} \pm \left(\alpha + \frac{1}{2}\right)$$

Hence, $D = \alpha$ and $-(\alpha+1)$. Therefore, the solution of Euler's equation is

$$R = c_1 r^{\alpha} + c_2 r^{-(\alpha+1)} \tag{2.92}$$

Taking $-\mu = \alpha(\alpha+1)$, Eq. (2.91) becomes

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 F}{\partial\phi^2} + \alpha(\alpha+1)F\sin\theta = 0$$

Inserting

$$F = H(\theta)\Phi(\phi)$$

into the above equation and separating the variables, we obtain

$$\frac{\sin\theta}{H}\left[\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) + \alpha(\alpha+1)\sin\theta H\right] = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \nu^2$$

where ν^2 is another separation constant. Then

$$\frac{d^2\Phi}{d\phi^2} + \nu^2\Phi = 0 \tag{2.93}$$

$$\frac{\sin\theta}{H}\left[\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) + \alpha(\alpha+1)\sin(\theta)H\right] = \nu^2 \tag{2.94}$$

The general solution of Eq. (2.93) is

$$\Phi = c_3\cos\nu\phi + c_4\sin\nu\phi \tag{2.95}$$

provided $\nu \neq 0$. If $\nu = 0$, the solution is independent of ϕ which corresponds to the axisymmetric case. Equation (2.94) becomes, for the axisymmetric, case,

$$\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) + \alpha(\alpha+1)\sin(\theta)H = 0$$

Transforming the independent variable θ to x and by letting $x = \cos\theta$, the abvoe equation becomes

$$\frac{d}{dx}\left(\sin\theta\frac{dH}{dx}\frac{dx}{d\theta}\right)\frac{dx}{d\theta} + \alpha(\alpha+1)\sin(\theta)H = 0$$

i.e.,

$$\frac{d}{dx}\left[(1-\cos^2\theta)\frac{dH}{dx}\right] + \alpha(\alpha+1)H = 0$$

or

$$\frac{d}{dx}\left[(1-x^2)\frac{dH}{dx}\right] + \alpha(\alpha+1)H = 0 \tag{2.96}$$

This is the well-known Legendre equation. Its general solution is given by

$$H = c_5P_\alpha(x) + c_6Q_\alpha(x), \qquad -1 \le x \le 1 \tag{2.97}$$

where P_α, Q_α are Legendre functions of the first and second kind respectively. For convenience let α be a positive integer, say $\alpha = n$. Then

$$H = c_5P_n(\cos\theta) + c_6Q_n(\cos\theta) \tag{2.98}$$

Continuity of $H(\theta)$ at $\theta = 0, \pi$ implies the continuity of $H(x)$ at $x = \pm 1$. Since $Q_n(x)$ has a singularity at $x = 1$, we choose $c_6 = 0$. Therefore, in axisymmetric case the solution of Laplace equation in spherical coordinates is given by

$$u(r,\theta,\phi) = \{c_1r^\alpha + c_2r^{-(\alpha+1)}\}(c_3)[c_5P_n(\cos\theta)]$$

After renaming the constants and using the principle of superposition, we find the solution to be

$$u(r,\theta)=\sum_{n=0}^{\infty}[A_n r^n + B_n r^{-(n+1)}]\,P_n(\cos\theta) \tag{2.99}$$

EXAMPLE 2.7 In a solid sphere of radius '*a*', the surface is maintained at the temperature given by

$$f(\theta)=\begin{cases} k\cos\theta, & 0\le\theta<\pi/2 \\ 0, & \pi/2<\theta<\pi \end{cases}$$

Prove that the steady state temperature within the solid is

$$u(r,\theta)=k\left[\frac{1}{4}P_0(\cos\theta)+\frac{1}{2}\left(\frac{r}{a}\right)P_1(\cos\theta)+\frac{5}{16}\left(\frac{r}{a}\right)^2 P_2(\cos\theta)-\frac{3}{32}\left(\frac{r}{a}\right)^4 P_4(\cos\theta)+\cdots\right]$$

Solution It is known that the steady state temperature distribution is governed by the Laplace equation. In spherical polar coordinates, the axisymmetric solution of the Laplace equation in general with the assumption that the temperature should be finite at the origin is given by Eq. (2.99) in the form

$$u(r,\theta)=\sum_{n=0}^{\infty}A_n r^n P_n(\cos\theta) \tag{2.100}$$

Using the given BC: $u(a,\theta)=f(\theta)$, we have

$$u(a,\theta)=f(\theta)=\sum_{n=0}^{\infty}A_n a^n P_n(\cos\theta)=\sum_{n=0}^{\infty}b_n P_n(\cos\theta)$$

where $A_n a^n = b_n$. This is a Fourier-Legendre series, where

$$b_n=\frac{2n+1}{2}\int_{-1}^{1} f(\theta)\,P_n(\cos\theta)\,d\theta$$

In the present problem,

$$b_n=\frac{2n+1}{2}\int_{0}^{1} f(\theta)P_n(\cos\theta)\,d\theta$$

Let $\cos\theta = x$,

$$b_0=\frac{1}{2}\int_0^1 kx\cdot P_0(x)\,dx=\frac{1}{2}\int_0^1 kx\cdot 1\cdot dx=\frac{k}{4}$$

Hence, we get

$$A_0 = \frac{k}{4}$$

Also,

$$b_1 = \frac{3}{2}\int_0^1 kx \cdot x \cdot dx = \frac{k}{2} = A_1 a$$

Therefore,

$$A_1 = \frac{k}{2}\left(\frac{1}{a}\right)$$

$$b_2 = \frac{5}{2}\int_0^1 kxP_2(x)\,dx = \frac{5}{2}\int_0^1 kx\frac{3x^2-1}{2}dx = \frac{5}{16}k$$

Thus,

$$A_2 = \frac{5}{16}k \cdot \frac{1}{a^2}$$

Further,

$$b_3 = \frac{7}{2}\int_0^1 kxP_3(x)\,dx = \frac{7}{2}\int_0^1 kx\frac{5x^3-3x}{2}dx = 0$$

Similarly, noting that $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$, we get

$$b_4 = -\frac{3}{32}k = A_4 a^4$$

Hence,

$$A_4 = -\frac{3}{32}k \cdot \frac{1}{a^4}, \cdots$$

Substituting these values of A_0, A_1, A_2,… into Eq. (2.100), we obtain, finally, the required temperature as

$$u(r, \theta) = k\left[\frac{1}{4}P_0(\cos\theta) + \frac{1}{2}\left(\frac{r}{a}\right)P_1(\cos\theta)\right.$$
$$\left. + \frac{5}{16}\left(\frac{r}{a}\right)^2 P_2(\cos\theta) + \left(-\frac{3}{32}\right)\left(\frac{r}{a}\right)^4 P_4(\cos\theta) + \cdots\right].$$

EXAMPLE 2.8 Find the potential at all points of space inside and outside of a sphere of radius $R = 1$ which is maintained at a constant distribution of electric potential $u(R, \theta) = f(\theta) = \cos 2\theta$.

Solution It is known that the potential on the surface of a sphere is governed by the Laplace equation. The Laplace equation in spherical polar coordinates is

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0$$

The possible general solution by variables separable method, after using superposition principle, is given by Eq. (2.99). Thus we have two possible solutions:

$$u_1(r, \theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) \tag{2.101}$$

$$u_2(r, \theta) = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\theta) \tag{2.102}$$

For points inside the sphere, we take the series (2.101). Why is this so? Applying the BC: $u(R, \theta) = f(\theta) = \cos 2\theta$, we obtain

$$f(\theta) = \sum_{n=0}^{\infty} A_n R^n P_n(\cos\theta)$$

which is a generalized Fourier series of $f(\theta)$ in terms of the Legendre polynomials. Using the orthogonality property, we get

$$A_n R^n = \frac{2n+1}{2}\int_{-1}^{1} f(\theta)\, P_n(x)\, dx$$

Let $x = \cos\theta$. Then we have

$$A_n = \frac{2n+1}{2R^n}\int_0^{\pi} f(\theta) P_n(\cos\theta) \sin\theta\, d\theta$$

For points outside the sphere, we take the series (2.102). Why is this so? Using the BC: $u(R, \theta) = f(\theta)$, we get

$$f(\theta) = \sum_{n=0}^{\infty} \frac{B_n}{R^{n+1}} P_n(\cos\theta)$$

Again, using the orthogonality property of Legendre polynomials, we have

$$B_n = \frac{2n+1}{2} R^{n+1} \int_0^{\pi} f(\theta) P_n(\cos\theta) \sin\theta\, d\theta$$

In the present problem, it is assumed that at $R=1$, $f(\theta)=\cos 2\theta = 2\cos^2\theta - 1 = 2x^2 - 1$. Hence,

$$A_n = \frac{2n+1}{2}\int_{-1}^{1}(2x^2-1)\,P_n(x)\,dx$$

However,

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

Therefore,

$$2x^2 - 1 = \frac{4}{3}P_2(x) - \frac{1}{3}$$

Thus,

$$A_n = \frac{2n+1}{2}\left[\frac{4}{3}\int_{-1}^{1}P_2(x)\,P_n(x)\,dx - \frac{1}{3}\int_{-1}^{1}P_0(x)P_n(x)\,dx\right]$$

Using the orthogonality property of Legendre polynomials, all integrals vanish except those corresponding to $n=0$ and $n=2$. We obtain, therefore,

$$A_0 = -\frac{1}{2}\cdot\frac{1}{3}\int_{-1}^{1}P_0^2(x)\,dx = -\frac{1}{3}$$

$$A_2 = \frac{5}{2}\cdot\frac{4}{3}\int_{-1}^{1}P_2^2(x)\,dx = \frac{4}{3}$$

Also,

$$B_n = \frac{2n+1}{2}\int_{-1}^{1}(2x^2-1)\,P_n(x)\,dx$$

$$= \frac{2n+1}{2}\left[\frac{4}{3}\int_{-1}^{1}P_2(x)\,P_n(x)\,dx - \frac{1}{3}\int_{-1}^{1}P_0(x)\,P_n(x)\,dx\right]$$

which, on using the orthogonality property, gives the non-vanishing coefficients as

$$B_0 = -\frac{1}{3}, \qquad B_2 = \frac{4}{3}$$

Substituting these values of A_0 and A_2 into Eq. (2.101), we obtain

$$u_1(r,\theta) = -\frac{1}{3} + \frac{4}{3}r^2P_2(\cos\theta)$$

which gives the potential everywhere inside the sphere. Similarly, substituting the values of B_0 and B_2 into Eq. (2.102), we get

$$u_2(r,\theta) = -\frac{1}{3r} + \frac{4}{3r^3} P_2(\cos\theta)$$

which gives the potential outside the sphere.

EXAMPLE 2.9 Find a general spherically symmetric solution of the following Helmholtz equation:

$$(\nabla^2 - k^2)u = 0$$

Solution In spherical polar coordinates, the Helmholtz equation can be written as

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} - k^2 u = 0 \tag{2.103}$$

In view of spherical symmetry, we look for u to be a function of r alone. Hence, Eq. (2.103) becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} - k^2 u = 0$$

Therefore, we have to solve

$$r^2 \frac{\partial^2 u}{\partial r^2} + 2r\frac{\partial u}{\partial r} - k^2 r^2 u = 0 \tag{2.104}$$

Let

$$u = \frac{1}{\sqrt{r}} F(r)$$

Differentiating twice with respect to r and rearranging, we obtain

$$2r\frac{\partial u}{\partial r} = -\frac{F(r)}{\sqrt{r}} + 2\sqrt{r}F'(r)$$

$$r^2 \frac{\partial^2 u}{\partial r^2} = -\frac{3}{4} r^{-1/2} F(r) - r^{1/2} F'(r) + r^{3/2} F''(r)$$

Substituting the above relations, Eq. (2.104) becomes

$$r^2 F''(r) + rF'(r) - \left(k^2 r^2 + \frac{1}{4}\right) F(r) = 0$$

or

$$r^2 F''(r) + rF'(r) + \left[(ik)^2 r^2 - \left(\frac{1}{2}\right)^2\right] F(r) = 0$$

This is the Bessel equation whose solution is

$$F(r) = AJ_{1/2}(ikr) + BY_{1/2}(ikr)$$

where $J_{1/2}, Y_{1/2}$ are Bessel functions with imaginary arguments, and is rewritten as

$$F(r) = AI_{1/2}(kr) + BK_{1/2}(kr)$$

Therefore,

$$u(r) = r^{-1/2}[AI_{1/2}(kr) + BK_{1/2}(kr)]$$

But as $r \to \infty$, the solution should be finite, which is possible only if $A = 0$. It is also known that for large z,

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}$$

Thus the acceptable spherically symmetric solution of the Helmholtz equation is given by

$$u(r) = Br^{-1/2}\sqrt{\frac{\pi}{2kr}}e^{-kr} = \frac{c}{r}e^{-kr}$$

where

$$c = B\sqrt{\frac{\pi}{2k}}$$

2.13 MISCELLANEOUS EXAMPLES

EXAMPLE 2.10 Show that the velocity potential for an irrotational flow of an incompressible fluid satisfies the Laplace solution.

Solution Let us consider a closed surface S enclosing a fixed volume V in the region occupied by a moving fluid as shown in Fig. 2.5.

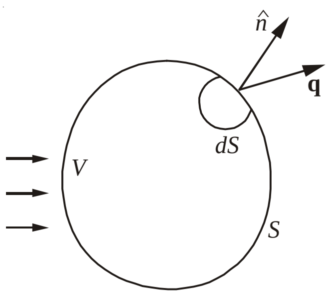

Fig. 2.5 Conservation of mass.

Let ρ be the density of the fluid. If $\hat{n}$ is a unit vector in the direction of the normal to the surface element dS and $\mathbf{q}$ the velocity of the fluid at that point, then the inward normal velocity is $(-\mathbf{q} \cdot \hat{n})$. Hence the mass of the fluid entering per unit time through the element dS is $(-\mathbf{q} \cdot \hat{n})\, dS$. It follows therefore that the mass of the fluid entering the surface S in unit time is

$$-\iint_S \rho (\mathbf{q} \cdot \hat{n})\, dS$$

Also, the mass of the fluid within S is

$$\iiint_V \rho \, dV$$

So the rate at which the mass goes on increasing is given by

$$\frac{\partial}{\partial t} \iiint_V \rho \, dV = \iiint_V \frac{\partial \rho}{\partial t} \, dV$$

By conservation of mass, the rate of generation of mass within a given volume under the assumption that no internal sources are present is equal to the net inflow of mass through the surface enclosing the given volume. Thus,

$$\iiint_V \frac{\partial \rho}{\partial t} \, dV = -\iint_S \rho (\mathbf{q} \cdot \hat{n})\, dS$$

$$= -\iiint_V \operatorname{div} (\rho \mathbf{q})\, dV \ \text{[using the divergence theorem]}$$

Therefore,

$$\iiint_V \left[\frac{\partial \rho}{\partial t} + \operatorname{div} (\rho \mathbf{q}) \right] dV = 0$$

Since the integrand is a continuous function and since this result is true for any arbitrary volume element dV, it follows that the integrand is zero. Therefore,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{q} = 0$$

which is called the equation of continuity. For an incompressible fluid, $\rho = \text{constant}$ and, therefore,

$$\nabla \cdot \mathbf{q} = 0$$

Further, if the flow is irrotational, i.e., there exists a velocity potential ϕ such that

$$\mathbf{q} = -\nabla \phi$$

Hence,

$$\nabla \cdot \mathbf{q} = \nabla \cdot \nabla \phi = \nabla^2 \phi = 0$$

Thus, an incompressible irrotational fluid satisfies the Laplace equation.

EXAMPLE 2.11 A thin rectangular homogeneous thermally conducting plate lies in the xy-plane defined by $0 \le x \le a,\ 0 \le y \le b$. The edge $y = 0$ is held at the temperature $Tx(x-a)$, where T is a constant, while the remaining edges are held at 0°. The other faces are insulated and no internal sources and sinks are present. Find the steady state temperature inside the plate.

Solution Since no heat sources and sinks are present in the plate, the steady state temperature u must satisfy $\nabla^2 u = 0$. Hence the problem is to solve

$$\text{PDE: } \nabla^2 u = 0$$

$$\text{BCs: } u(0, y) = 0, \qquad u(a, y) = 0, \qquad u(x, b) = 0, \qquad u(x, 0) = Tx(x-a)$$

This is a typical Dirichlet's problem. The general solution satisfying the first three BCs is given by Eq. (2.47). Therefore,

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left[\frac{n\pi}{a}(y-b)\right]$$

where

$$A_n \sinh\frac{-n\pi b}{a} = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi}{a}x\right) dx$$

Using the last BC: $u(x, 0) = Tx(x-a) = f(x)$, we get

$$\begin{aligned}
A_n \sinh\frac{-n\pi b}{a} &= \frac{2}{a}\int_0^a Tx(x-a) \sin\left(\frac{n\pi}{a}x\right) dx \\
&= \frac{2T}{a}\int_0^a x(x-a) \sin\left(\frac{n\pi}{a}x\right) dx \\
&= -\frac{a}{n\pi}\cdot\frac{2T}{a}\left[\int_0^a x(x-a)\, d\left\{\cos\left(\frac{n\pi}{a}x\right)\right\}\right] \\
&= -\frac{2T}{n\pi}\left[(x-a)\cos\left(\frac{n\pi}{a}x\right)\right]_0^a - \frac{a}{n\pi}\int_0^a (2x-a)\, d\left[\sin\left(\frac{n\pi}{a}x\right)\right] \\
&= \frac{2aT}{n^2\pi^2}\left[(2x-a)\sin\left(\frac{n\pi}{a}x\right)\right]_0^a - \int_0^a 2\sin\left(\frac{n\pi}{a}x\right) dx \\
&= \frac{2aT}{n^2\pi^2}\left\{a \sin n\pi + \frac{2a}{n\pi}\left[\cos\left(\frac{n\pi}{a}x\right)\right]_0^a\right\} \\
&= \frac{2aT}{n^2\pi^2}\,\frac{2a}{n\pi}(\cos n\pi - 1) = \frac{4a^2T}{n^3\pi^3}[(-1)^n - 1]
\end{aligned}$$

Thus the required temperature distribution is given by

$$u(x, y) = \sum_{n=1}^{\infty} \operatorname{cosech}\left(-\frac{n\pi}{a}b\right) \frac{4Ta^2}{n^3\pi^3}[(-1)^n - 1] \sin\left(\frac{n\pi}{a}x\right) \sinh\left[\frac{n\pi}{a}(y-b)\right]$$

EXAMPLE 2.12 Solve

$$\nabla^2 u = 0, \qquad 0 \le x \le a, \qquad 0 \le y \le b$$

satisfying the BCs:

$$u(0, y) = 0, \qquad u(x, 0) = 0, \qquad u(x, b) = 0$$

$$\frac{\partial u}{\partial x}(a, y) = T \sin^3 \frac{\pi y}{a}$$

Solution Using the variables separable method, one of the acceptable general solutions is given by Eq. (2.38). Hence

$$u(x, y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$$

Using the BC: $u(x, 0) = 0$, we get

$$0 = c_3(c_1 e^{px} + c_2 e^{-px})$$

implying $c_3 = 0$. Therefore,

$$u(x, y) = c_4 \sin py (c_1 e^{px} + c_2 e^{-px})$$

Now, using the BC: $u(x, b) = 0$, we obtain

$$0 = c_4 \sin pb (c_1 e^{px} + c_2 e^{-px})$$

$c_4 \ne 0$ (why?) implying $\sin pb = 0$ which gives

$$pb = n\pi \quad \text{or} \quad p = \frac{n\pi}{b}, \qquad n = 1, 2, 3, \ldots$$

Thus,

$$u(x, y) = c_4 \sin\left(\frac{n\pi}{b}y\right)(c_1 e^{px} + c_2 e^{-px})$$

Renaming the constants, we have

$$u(x, y) = \sin\left(\frac{n\pi}{b}y\right)\left[A \exp\left(\frac{n\pi}{b}x\right) + B \exp\left(-\frac{n\pi}{b}x\right)\right], \qquad n = 1, 2, \ldots$$

If we use the BC: $u(0, y) = 0$, we get

$$0 = \sin\left(\frac{n\pi}{b}y\right)(A + B)$$

giving $A + B = 0$; therefore, $A = -B$. Thus,

$$u(x, y) = A \sin\left(\frac{n\pi}{b} y\right)\left[\exp\left(\frac{n\pi}{b} x\right) - \exp\left(-\frac{n\pi}{b} x\right)\right]$$

$$= 2A \sin\left(\frac{n\pi}{b} y\right) \sinh\left(\frac{n\pi}{b} x\right), \qquad n = 1, 2, \ldots$$

Differentiating with respect to x, we obtain

$$\frac{\partial u}{\partial x} = 2A \frac{n\pi}{b} \sin\left(\frac{n\pi}{b} y\right) \cosh\left(\frac{n\pi}{b} x\right)$$

The last BC yields

$$T \sin^3 \frac{\pi y}{a} = 2A \frac{n\pi}{b} \sin\left(\frac{n\pi}{b} y\right) \cosh\left(\frac{n\pi}{b} a\right)$$

from which we can determine $2A$. Hence, the required solution is

$$u(x, y) = \frac{bT}{n\pi} \sin^3 \frac{\pi y}{a} \operatorname{sech} \frac{n\pi}{b} a \sinh\left(\frac{n\pi}{b} x\right)$$

The principle of superposition gives the required solution as

$$u(x, y) = \sum_{n=1}^{\infty} \frac{bT}{n\pi} \sin^3 \frac{\pi y}{a} \operatorname{sech}\left(\frac{n\pi}{b} a\right) \sinh\left(\frac{n\pi}{b} x\right)$$

EXAMPLE 2.13 Find the potential function $u(x, y, z)$ in a rectangular box defined by $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$ (see Fig. 2.6), if the potential is zero on all sides and the bottom, while $u = f(x, y)$ on the top of the box.

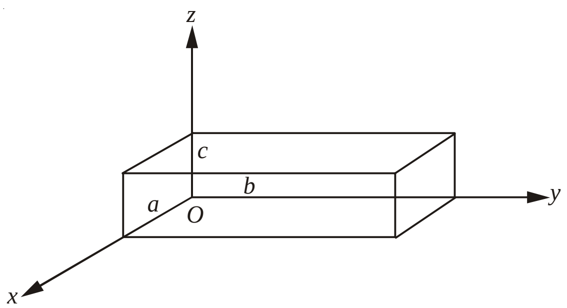

Fig. 2.6 Rectangular box.

Solution The potential distribution in the rectangular box satisfies the Laplace equation. Thus the problem is to solve

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$$

subject to the BCs:

$$u(0, y, z) = u(a, y, z) = 0$$

$$u(x, 0, z) = u(x, b, z) = 0$$

$$u(x, y, 0) = 0$$

$$u(x, y, c) = f(x, y)$$

Following the variables separable method, let us assume the solution in the form

$$u(x, y, z) = X(x)Y(y)Z(z)$$

Substituting into the Laplace equation, we get

$$X''(x)Y(y)Z(z) + X(x)Y''(y)Z(z) + X(x)Y(y)Z''(z) = 0$$

which can also be written as

$$\frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = -\frac{X''(x)}{X(x)} = \lambda_1^2$$

where λ_1 is a separation constant. Thus we have

$$X''(x) + \lambda_1^2 X(x) = 0 \tag{2.105}$$

After the second separation, we also have

$$\frac{Z''(z)}{Z(z)} - \lambda_1^2 = -\frac{Y''(y)}{Y(y)} = \lambda_2^2$$

$$Y''(y) + \lambda_2^2 Y(y) = 0 \tag{2.106}$$

$$Z''(z) - \lambda_3^2 Z(z) = 0 \tag{2.107}$$

where $\lambda_3^2 = \lambda_1^2 + \lambda_2^2$. The general solutions of Eqs. (2.105)–(2.107) are

$$X(x) = c_1 \cos \lambda_1 x + c_2 \sin \lambda_1 x$$

$$Y(y) = c_3 \cos \lambda_2 y + c_4 \sin \lambda_2 y$$

$$Z(z) = c_5 \cosh \lambda_3 z + c_6 \sinh \lambda_3 z$$

From the BCs,

$$X(0) = X(a) = 0$$

$$Y(0) = Y(b) = 0$$

$$Z(0) = 0$$

$$X(0) = 0 \text{ gives } c_1 = 0$$

$$X(a) = 0 \text{ gives } \lambda_1 a = m\pi.$$

Therefore,

$$\lambda_1 = \frac{m\pi}{a}, \qquad m = 1, 2, \ldots$$

Similarly,

$$Y(0) = 0 \text{ gives } c_3 = 0$$

$$Y(b) = 0 \text{ gives } \lambda_2 b = n\pi$$

Therefore,

$$\lambda_2 = \frac{n\pi}{b}, \qquad n = 1, 2, \ldots$$

Also, $Z(0) = 0$ gives $c_5 = 0$. Further, we note that

$$\lambda_3^2 = \lambda_1^2 + \lambda_2^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) = \lambda_{mn}^2 \quad \text{(say)}$$

Then

$$\lambda_3 = \pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} = \lambda_{mn}$$

The solutions now take the form

$$X(x) = c_{2m} \sin\frac{m\pi x}{a}, \qquad m = 1, 2, \ldots$$

$$Y(y) = c_{4n} \sin\frac{n\pi y}{b}, \qquad n = 1, 2, \ldots$$

$$Z(z) = c_{6mn} \sinh \lambda_{mn} z$$

Let $c_{mn} = c_{2m}c_{4n}c_{6mn}$; then, after using the principle of superposition, the required solution is

$$u(x, y, z) = X(x)Y(y)Z(z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sinh \lambda_{mn} z \qquad (2.108)$$

Using the final BC: $f(x, y) = u(x, y, c)$, we get

$$f(x, y) = \sum \sum c_{mn} \sinh \lambda_{mn} c \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

which is a double Fourier sine series. Thus, we have

$$c_{mn} \sinh \lambda_{mn} c = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx\, dy \tag{2.109}$$

Therefore, Eqs. (2.108) and (2.109) constitute the required potential.

EXAMPLE 2.14 Find the electrostatic potential u in the annular region bounded by the concentric spheres $r = a$, $r = b$, $0 < a < b$ (see Fig. 2.7), if the inner and outer surfaces are kept at constant potentials u_1 and u_2, $u_1 \neq u_2$.

Solution The electrostatic potential satisfies the Laplace equation

$$\nabla^2 u = 0$$

It is natural that we choose spherical polar coordinates. From the problem, it is evident that we are looking for a solution with spherical symmetry which is independent of θ and ϕ. Hence, $u = u(r)$.

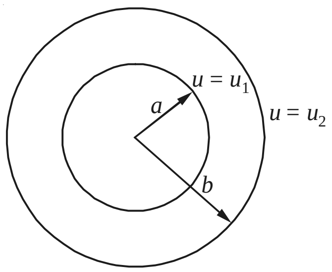

Fig. 2.7 Annular region.

Thus, we have to solve

$$\text{PDE: } \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = 0 \tag{2.110}$$

subject to

$$\text{BCs: } u = u_1 \quad \text{at } r = a$$

$$u = u_2 \quad \text{at } r = b$$

Integrating Eq. (2.110) with respect to r, we obtain

$$r^2 \frac{\partial u}{\partial r} = A \quad \text{(a constant of integration)}$$

Again, integrating, we get

$$u = -\frac{A}{r} + B$$

Now, using the BCs, we have

$$u_1 = -\frac{A}{a} + B, \qquad u_2 = -\frac{A}{b} + B$$

Solving these equations, we get

$$A = \frac{u_2 - u_1}{(1/a) - (1/b)}, \qquad B = \frac{(u_1/b) - (u_2/a)}{(1/b) - (1/a)}$$

Hence, using these values, the required potential is

$$u = \frac{1}{(1/a) - (1/b)}\left[u_1\left(\frac{1}{r} - \frac{1}{b}\right) + u_2\left(\frac{1}{a} - \frac{1}{r}\right)\right]$$

EXAMPLE 2.15 A thermally conducting solid bounded by two concentric spheres of radii a and b as shown in Fig. 2.8, $a < b$, is such that the internal boundary is kept at $f_1(\theta)$ and the outer boundary at $f_2(\theta)$. Find the steady state temperature in the solid.

Solution It is known that the steady temperature T satisfies the Laplace equation. In the present problem,

$$T = T(r, \theta)$$

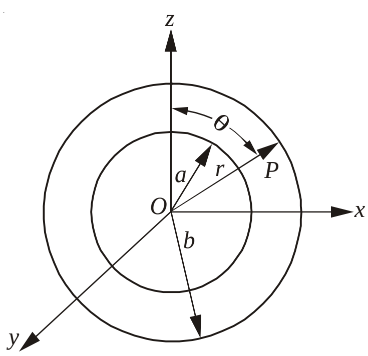

Fig. 2.8 Region bounded by two concentric spheres.

Thus, we have to solve

$$\text{PDE: } \nabla^2 T = 0$$

subject to the boundary conditions

$$T = f_1(\theta) \quad \text{at} \quad r = a$$

$$T = f_2(\theta) \quad \text{at} \quad r = b$$

In spherical polar coordinates, for axially symmetric case, the solution of the Laplace equation is given by Eq. (2.99) as follows:

$$T(r, \theta) = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos\theta)$$

Using the BCs:

$$f_1(\theta) = \sum_{n=0}^{\infty} \left(A_n a^n + \frac{B_n}{a^{n+1}} \right) P_n(\cos\theta) \tag{2.111}$$

$$f_2(\theta) = \sum_{n=0}^{\infty} \left(A_n b^n + \frac{B_n}{b^{n+1}} \right) P_n(\cos\theta) \tag{2.112}$$

In order to find the coefficients A_n and B_n, we have to express $f_1(\theta)$ and $f_2(\theta)$ in terms of Legendre polynomials and compare the coefficients. In this process, the following orthogonality relation is useful:

$$\int_0^{\pi} P_m(\cos\theta) P_n(\cos\theta) \sin\theta \, d\theta = \begin{cases} 0, & \text{if } m \neq n \\ \dfrac{2}{2n+1}, & \text{if } m = n \end{cases}$$

Thus, multiplying both sides of Eq. (2.111) by $P_m(\cos\theta)\sin\theta$ and integrating, we obtain

$$\begin{aligned} \int_0^{\pi} f_1(\theta) P_m(\cos\theta) \sin\theta \, d\theta &= \sum_{n=0}^{\infty} \left(A_n a^n + \frac{B_n}{a^{n+1}} \right) \int_0^{\pi} P_n(\cos\theta) P_m(\cos\theta) \sin\theta \, d\theta \\ &= \left(A_m a^m + \frac{B_m}{a^{m+1}} \right) \frac{2}{2m+1} \end{aligned} \tag{2.113}$$

Similarly, Eq. (2.112) gives

$$\begin{aligned} \int_0^{\pi} f_2(\theta) P_m(\cos\theta) \sin\theta \, d\theta &= \sum_{n=0}^{\infty} \left(A_n b^n + \frac{B_n}{b^{n+1}} \right) \int_0^{\pi} P_n(\cos\theta) P_m(\cos\theta) \sin\theta \, d\theta \\ &= \left(A_m b^m + \frac{B_m}{b^{m+1}} \right) \frac{2}{2m+1} \end{aligned} \tag{2.114}$$

Let

$$\frac{2m+1}{2}\int_0^{\pi} f_1(\theta)P_m(\cos\theta)\sin\theta\, d\theta = C_m$$

$$\frac{2m+1}{2}\int_0^{\pi} f_2(\theta)P_m(\cos\theta)\sin\theta\, d\theta = D_m$$

Then Eqs. (2.113) and (2.114) reduce to

$$A_m a^m + \frac{B_m}{a^{m+1}} = C_m$$

$$A_m b^n + \frac{B_m}{b^{m+1}} = D_m$$

Solving this pair of equations, we obtain

$$A_m = \frac{C_m a^{m+1} - D_m b^{m+1}}{a^{2m+1} - b^{2m+1}} \tag{2.115}$$

$$B_m = \frac{a^{m+1}b^{m+1}(C_m b^m - D_m a^m)}{b^{2m+1} - a^{2m+1}} \tag{2.116}$$

Hence, the required steady temperature is

$$T(r,\theta) = \sum_{m=0}^{\infty}\left(A_m r^m + \frac{B_m}{r^{m+1}}\right)P_m(\cos\theta)$$

where A_m and B_m are given by Eqs. (2.115) and (2.116).

EXAMPLE 2.16 A thin annulus occupies the region $0 < a \le r \le b,\ 0 \le \theta \le 2\pi$. The faces are insulated. Along the inner edge the temperature is maintained at 0°, while along the outer edge the temperature is held at $T = K\cos(\theta/2)$, where K is a constant. Determine the temperature distribution in the annulus.

Solution Mathematically, the problem is to solve

$$\text{PDE:}\quad \nabla^2 T = 0, \qquad a \le r \le b,\ 0 \le \theta \le 2\pi$$

$$\text{BCs:}\ T(a,\theta) = 0$$

$$T(b,\theta) = k\cos\theta/2$$

The required general solution is given by Eq. (2.57) in the form

$$T(r,\theta) = (c_1 r^n + c_2 r^{-n})(c_3\cos n\theta + c_4\sin n\theta)$$

Using the first BC, we get

$$0 = (c_1 a^n + c_2 a^{-n})\,(c_3 \cos n\theta + c_4 \sin n\theta)$$

implying thereby $c_1 a^n + c_2 a^{-n} = 0$, or $c_2 = -c_1 a^{2n}$. After adjusting the constants suitably, we have

$$T(r,\theta) = \left(r^n - \frac{a^{2n}}{r^n}\right)(A \cos n\theta + B \sin n\theta)$$

The principle of superposition gives

$$T(r,\theta) = \sum_{n=1}^{\infty} \left(r^n - \frac{a^{2n}}{r^n}\right)(A_n \cos n\theta + B_n \sin n\theta)$$

Now, using the second boundary condition, we obtain

$$T(b,\theta) = K \cos\frac{\theta}{2} = \sum_{n=1}^{\infty} (b^n - b^{-n} a^{2n})\,(A_n \cos n\theta + B_n \sin n\theta)$$

which is a full-range Fourier series. Hence,

$$A_n (b^n - b^{-n} a^{2n}) = \frac{1}{\pi}\int_0^{2\pi} K \cos\frac{\theta}{2} \cos n\theta\, d\theta$$

$$= \frac{k}{2\pi}\int_0^{2\pi} \left[\cos\left(n+\frac{1}{2}\right)\theta + \cos\left(n-\frac{1}{2}\right)\theta\right] d\theta$$

$$= \frac{k}{2\pi}\left[\frac{\sin\left(n+\frac{1}{2}\right)\theta}{n+\frac{1}{2}} + \frac{\sin\left(n-\frac{1}{2}\right)\theta}{n-\frac{1}{2}}\right]_0^{2\pi}$$

$$= 0$$

implying $A_n = 0$. Also,

$$B_n (b^n - b^{-n} a^{2n}) = \frac{k}{\pi}\int_0^{2\pi} \cos\frac{\theta}{2} \sin n\theta\, d\theta$$

$$= \frac{k}{2\pi}\int_0^{2\pi} \left[\sin\left(n+\frac{1}{2}\right)\theta + \sin\left(n-\frac{1}{2}\right)\theta\right] d\theta$$

$$= -\frac{k}{2\pi}\left[\frac{\cos\left(n+\frac{1}{2}\right)\theta}{n+\frac{1}{2}} + \frac{\cos\left(n-\frac{1}{2}\right)\theta}{n-\frac{1}{2}}\right]_0^{2\pi}$$

$$= -\frac{k}{2\pi}\left(-\frac{1}{n+\frac{1}{2}} - \frac{1}{n+\frac{1}{2}} - \frac{1}{n-\frac{1}{2}} - \frac{1}{n-\frac{1}{2}}\right)$$

$$= \frac{k}{\pi}\left(\frac{1}{n+\frac{1}{2}} + \frac{1}{n-\frac{1}{2}}\right) = \frac{k}{\pi}\,\frac{2n}{n^2-\frac{1}{4}}$$

or

$$B_n(b^n - b^{-n}a^{2n}) = \frac{8kn}{\pi(4n^2-1)}$$

Thus the temperature distribution in the annulus is given by

$$T(r,\theta) = \frac{8k}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\left[\frac{(r/a)^n-(a/r)^n}{(b/a)^n-(a/b)^{-n}}\right]\sin n\theta$$

EXAMPLE 2.17 V is a function of r and θ satisfying the equation

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

within the region of the plane bounded by $r = a$, $r = b$, $\theta = 0$, $\theta = \pi/2$. Its value along the boundary $r = a$ is $\theta(\pi/2 - \theta)$, along the other boundaries is zero. Prove that

$$V = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(r/b)^{4n-2}-(b/r)^{4n-2}}{(a/b)^{4n-2}-(b/a)^{4n-2}}\left[\frac{\sin(4n-2)\theta}{(2n-1)^3}\right]$$

Solution The task is to solve the PDE

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

subject to the following boundary conditions:

(i) $V(b,\theta) = 0,\qquad 0 < \theta < \pi/2$

(ii) $V(r,\pi/2) = 0,\qquad a < r \le b$

(iii) $V(r,0) = 0,\qquad a < r \le b$

(iv) $V(a,\theta) = \theta(\pi/2-\theta),\qquad 0 < \theta < \pi/2.$

The three possible solutions (see Section 2.8) are given as follows:

$$V = (c_1 r^p + c_2 r^{-p})\,(c_3 \cos p\theta + c_4 \sin p\theta)$$

$$V = [c_1 \cos(p \ln r) + c_2 \sin(p \ln r)]\,(c_3 e^{p\theta} + c_4 e^{-p\theta})$$

$$V = (c_1 \ln r + c_2)\,(c_3\theta + c_4)$$

Since the problem is not defined for $r = 0,\ \infty$, the second and third solutions are not acceptable. Hence, the generally acceptable solution is the first one. The boundary condition (iii) gives

$$0 = c_3 (c_1 r^p + c_2 r^{-p})$$

implying $c_3 = 0$. The boundary condition (ii) implies

$$0 = c_4 \sin p\frac{\pi}{2}(c_1 r^p + c_2 r^{-p})$$

Therefore,

$$\sin p\frac{\pi}{2} = 0 \quad \text{or} \quad p = 2n, \qquad n = 1, 2, \ldots$$

Thus, the possible solution of the given equation has the form

$$V(r, \theta) = c_4 \sin(2n\theta)(c_1 r^{2n} + c_2 r^{-2n})$$

Now, applying the boundary condition (i), we get

$$0 = c_4 \sin(2n\theta)\,(c_1 b^{2n} + c_2 b^{-2n})$$

which gives $c_2 = -c_1 b^{4n}$. Therefore,

$$V(r, \theta) = c_1 c_4 \sin(2n\theta)[r^{2n} - r^{-2n} b^{4n}]$$

Superposing all the solutions, we obtain

$$V(r, \theta) = \sum_{n=1}^{\infty} c_n \sin(2n\theta)(r^{2n} - r^{-2n} b^{4n})$$

Satisfying boundary conditions (iv), we get

$$\theta\left(\frac{\pi}{2} - \theta\right) = \sum c_n \sin(2n\theta)\left(\frac{a^{4n} - b^{4n}}{a^{2n}}\right)$$

which is a Fourier sine series. Thus, we have

$$\frac{2}{\pi/2}\int_0^{\pi/2} \theta\left(\frac{\pi}{2} - \theta\right)\sin(2n\theta) = c_n\left(\frac{a^{4n} - b^{4n}}{a^{2n}}\right)$$

Integrating by parts, we obtain

$$\left[\left(\frac{\pi}{2}\theta-\theta^2\right)\left\{-\frac{\cos(2n\theta)}{2n}\right\}-\left(\frac{\pi}{2}-2\theta\right)\left\{-\frac{\sin(2n\theta)}{4n^2}\right\}+(-2)\left\{\frac{\cos(2n\theta)}{8n^3}\right\}\right]_0^{\pi/2}=c_n\frac{\pi}{4}\left(\frac{a^{4n}-b^{4n}}{a^{2n}}\right)$$

On simplification, we get

$$-\frac{1}{4n^3}\{(-1)^n-1\}=\frac{\pi}{4}c_n\left(\frac{a^{4n}-b^{4n}}{a^{2n}}\right)$$

Thus,

$$\frac{\pi}{4}c_n\left(\frac{a^{4n}-b^{4n}}{a^{2n}}\right)=\begin{cases}\dfrac{1}{2n^3}, & \text{for } n \text{ odd}\\ 0, & \text{for } n \text{ even}\end{cases}$$

Hence, the required solution is

$$V(r,\theta)=\sum_1^{\infty}\frac{2}{\pi}\frac{1}{(2n-1)^3}\left(\frac{a}{r}\right)^{4n-2}\sin(4n-2)\theta\left(\frac{r^{8n-4}-b^{8n-4}}{a^{8n-4}-b^{8n-4}}\right)$$

which can be recast in the form given in the problem.

EXAMPLE 2.18 Determine the potential of a grounded conducting sphere in a uniform field defined by

$$\text{PDE: } \nabla^2 u=0, \qquad 0\le r<a,\ 0<\theta<\pi,\ 0\le\phi<2\pi$$

$$\text{BCs: (i) } u(a,\theta)=0.$$

$$\text{(ii) } u\to -E_0 r\cos\theta \text{ as } r\to\infty.$$

Solution In spherical polar coordinates, with axial symmetry, the solution of the Laplace equation is given by Eq. (2.99) in the form

$$u(r,\theta)=\sum_{n=0}^{\infty}\left(A_n r^n+\frac{B_n}{r^{n+1}}\right)P_n(\cos\theta)$$

Using the boundary condition (ii), we have

$$u(r,\theta)=\sum_{n=0}^{\infty}A_n r^n P_n(\cos\theta)=-E_0 r\cos\theta$$

which is true only for $n=1$, when $P_1(\cos\theta)=\cos\theta$. Also, $A_n=0$ for $n\ge 2$. Therefore,

$$u(r,\theta)=A_1 r\cos\theta=-E_0 r\cos\theta$$

implying $A_1 = -E_0$. Hence,

$$u(r,\theta) = -E_0 r\cos\theta + \sum_{n=1}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\theta)$$

Now, applying the boundary condition (i), we get

$$0 = -E_0 a\cos\theta + \sum_{n=1}^{\infty} \frac{B_n}{a^{n+1}} P_n(\cos\theta)$$

Multiplying both sides by $P_m(\cos\theta)\sin\theta$ and integrating between the limits 0 to π, we have

$$E_0 a\int_0^{\pi} \cos(\theta)\, P_m(\cos\theta)\sin\theta\, d\theta = \sum_{n=1}^{\infty} \frac{B_n}{a^{n+1}} \int_0^{\pi} P_n(\cos\theta)\, P_m(\cos\theta)\sin\theta\, d\theta \tag{2.117}$$

Using the orthogonality property

$$\int_0^{\pi} P_n(\cos\theta) P_m(\cos\theta)\sin\theta\, d\theta = \begin{cases} 0, & \text{for } m \neq n \\ \dfrac{2}{2m+1}, & \text{for } m = n \end{cases}$$

we obtain

$$\frac{B_m}{a^{m+1}}\frac{2}{2m+1} = E_0 a\int_0^{\pi} \cos(\theta)\, P_m(\cos\theta)\sin\theta\, d\theta$$

or

$$B_m = \frac{2m+1}{2} E_0 a^{m+2} \int_0^{\pi} \cos(\theta)\, P_m(\cos\theta)\sin\theta\, d\theta$$

It can be verified that the integral on the right-hand side of the Eq. (2.117) vanishes for all m except when $m = 1$, in which case

$$B_1 = E_0 a^3$$

Therefore, the required potential is given by

$$u(r,\theta) = -E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta$$

EXAMPLE 2.19 The steady, two-dimensional, incompressible viscous fluid flow past a circular cylinder, when the inertial terms are neglected (Stokes flow), is governed by the biharmonic PDE: $\nabla^4\psi = 0$, where ψ is the stream function. Find its solution subject to the BCs:

(i) $\psi(r,\theta) = \partial\psi/\partial r = 0$ on $r = 1$

(ii) $\psi(r,\theta) \to r\sin\theta$ as $r \to \infty$.

Solution In view of the cylindrical geometry, we can write

$$\nabla^4\psi = \nabla^2(\nabla^2\psi) = 0$$

where

$$\nabla^2\psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\psi$$

Using the variables separable method, let us look for a solution of the form

$$\psi(r, \theta) = f(r)\sin\theta$$

Therefore,

$$\frac{\partial\psi}{\partial r} = f'(r)\sin\theta, \qquad \frac{\partial\psi}{\partial\theta} = f(r)\cos\theta$$

$$\frac{\partial^2\psi}{\partial r^2} = f''(r)\sin\theta, \qquad \frac{\partial^2\psi}{\partial\theta^2} = -f(r)\sin\theta$$

Hence,

$$\nabla^2\psi = \left[f''(r) + \frac{1}{r}f'(r) - \frac{1}{r^2}f(r)\right]\sin\theta$$

which can also be written in the form

$$\nabla^2\psi = F(r)\sin\theta$$

where

$$F(r) = f''(r) + \frac{1}{r}f'(r) - \frac{1}{r^2}f(r)$$

Therefore,

$$\nabla^4\psi = \nabla^2(\nabla^2\psi) = \nabla^2[F(r)\sin\theta] = 0$$

i.e.,

$$\left[F''(r) + \frac{1}{r}F'(r) - \frac{1}{r^2}F(r)\right]\sin\theta = 0$$

implying

$$F''(r) + \frac{1}{r}F'(r) - \frac{1}{r^2}F(r) = 0$$

Introducing the transformation $r = e^z, D = d/dz,$ the above equation becomes

$$[D(D-1) + D - 1]F(r) = 0$$

or

$$(D^2 - 1)\, F(r) = 0$$

Its complementary function is

$$F(r) = Ae^z + Be^{-z} = Ar + \frac{B}{r}$$

or

$$f''(r) + \frac{1}{r} f'(r) - \frac{1}{r^2} f(r) = Ar + \frac{B}{r}$$

i.e.

$$r^2 f''(r) + rf'(r) - f(r) = Ar^3 + Br$$

which is a homogeneous ordinary differential equation. Again using the transformation $r = e^z$, $D = d/dz$, we get

$$[D(D-1) + D - 1]\, f = Ae^{3z} + Be^z$$

or

$$(D^2 - 1)\, f = Ae^{3z} + Be^z$$

Its complementary function is

$$f(r) = Ce^z + De^{-z}$$

while its particular integral is

$$\frac{1}{D^2 - 1}(Ae^{3z} + Be^z) = \frac{Ae^{3z}}{8} + \frac{Bze^z}{2}$$

Therefore,

$$f(r) = Cr + \frac{D}{r} + \frac{A}{8} r^3 + \frac{B}{2} r \ln r$$

Thus, we have

$$\psi = \left(\frac{A}{8} r^3 + \frac{B}{2} r \ln r + Cr + \frac{D}{r} \right) \sin \theta$$

Now to satisfy the BC: $\psi \to r \sin \theta$, as $r \to \infty$ and from physical considerations, we choose $A = 0$, Therefore,

$$\psi = \left(\frac{B}{2} r \ln r + Cr + \frac{D}{r} \right) \sin \theta$$

The boundary condition $\psi = 0$ on $r = 1$ gives $(C + D)\sin\theta = 0$, implying $C = -D$. Also, the boundary condition $\partial\psi/\partial r = 0$ on $r = 1$ gives

$$\frac{B}{2} + C - D = 0 = \frac{B}{2} - 2D$$

implying $B = 4D$. Hence, the general solution is

$$\psi = 2D\left(r\ln r - \frac{r}{2} + \frac{1}{2r}\right)\sin\theta$$

EXAMPLE 2.20 The problem of axisymmetric fluid flow in a semi-infinite or in a finite circular pipe of radius a is described as follows in cylindrical coordinates:

$$\text{PDE:}\quad \nabla^2 u = 0, \qquad 0 < r < a$$

$$\text{BCs:}\quad \text{(i)}\ \frac{\partial u}{\partial r} = 0 \quad \text{at } r = 0$$

$$\text{(ii)}\ \frac{\partial u}{\partial z} = 0, \qquad \frac{\partial u}{\partial r} = V(z) \quad \text{at } r = a$$

Show that the speed of suction is given by

$$V(z) = -\sum_{n=1}^{\infty} \alpha_n (A_n \cosh\alpha_n z + B_n \sinh\alpha_n z)\, J_1(\alpha_n a)$$

Solution In cylindrical coordinates (r, θ, z),

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

In axisymmetric case, the above equation becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{2.118}$$

Let $u(r, z) = f(r)\,\phi(z)$ which, when substituted into Eq. (2.118), gives

$$\frac{f'' + (1/r) f'}{f} = -\frac{\phi''}{\phi} = -\alpha^2 \quad \text{(say)}$$

Then

$$\phi'' - \alpha^2\phi = 0 \tag{2.119}$$

$$f'' + \frac{1}{r}f' + \alpha^2 f = 0 \tag{2.120}$$

The solution of Eq. (2.119) is

$$\phi = A\cosh \alpha z + B\sinh \alpha z$$

Equation (2.120) can be rewritten as

$$r^2 f'' + rf' + \alpha^2 r^2 f = 0$$

which is a Bessel's equation of zeroth order whose general solution is

$$f = J_0(\alpha r) + DY_0(\alpha r)$$

Here, $J_0(\alpha r)$ and $Y_0(\alpha r)$ are zeroth order Bessel functions of first and second kind respectively. Therefore, the typical solution is

$$u = (A\cosh \alpha z + B\sinh \alpha z)\,[J_0(\alpha r) + DY_0(\alpha r)]$$

Now, $Y_0(\alpha r)$ is infinite at $r = 0$, and hence $D = 0$. Therefore, the possible solution is

$$u = (A\cosh \alpha z + B\sinh \alpha z)\,J_0(\alpha r)$$

The condition $\partial u/\partial r = 0$ at $r = 0$ is automatically satisfied, since $J_0'(\alpha r) = 0$ at $r = 0$. Now the boundary condition $\partial u/\partial z = 0$ at $r = a$ gives $J_0(\alpha a) = 0$, implying that αa are the zeros of the Bessel function J_0. Let these zeros be $\alpha_n a\,(n = 0, 1, 2, \ldots)$. Thus the appropriate solution is

$$u(r, z) = \sum_{n=1}^{\infty} \alpha_n (A_n \cosh \alpha_n z + B_n \sinh \alpha_n z)\,J_0(\alpha_n r)$$

Using the fact that, $J_0' = -J_1$, the speed of suction is given by

$$V(z) = \left(\frac{\partial u}{\partial r}\right)_{r=a} = -\sum_{n=1}^{\infty} \alpha_n (A_n \cosh \alpha_n z + B_n \sinh \alpha_n z)\,J_1(\alpha_n a)$$

EXAMPLE 2.21 Solve the following Poisson equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2$$

subject to the boundary conditions

$$u(0, y) = u(5, y) = u(x, 0) = u(x, 4) = 0$$

Solution We assume the solution of the form

$$u = v + \omega \tag{2.121}$$

where v is a particular solution of the Poisson equation and ω is the solution of the corresponding homogeneous Laplace equation. That is,

$$\nabla^2 v = 2 \tag{2.122}$$

$$\nabla^2 \omega = 0 \tag{2.123}$$

It is customary to assume that v has the form

$$v(x, y) = a + bx + cy + dx^2 + exy + fy^2$$

Substituting this into Eq. (2.122), we get

$$2d + 2f = 2$$

Let $f = 0$. Then $d = 1$. The remaining coefficients can be chosen arbitrarily. Thus we take

$$v(x, y) = -5x + x^2 \tag{2.124}$$

so that v reduces to zero (satisfies the boundary conditions) on the sides $x = 0$ and $x = 5$.

Now, we shall find ω from

$$\nabla^2 \omega = 0, \qquad 0 < x < 5, \qquad 0 < y < 4 \tag{2.125}$$

satisfying

$$\omega(0, y) = -v(0, y) = 0$$

$$\omega(5, y) = -v(5, y) = 0$$

$$\omega(x, 0) = -v(x, 0) = -(-5x + x^2)$$

$$\omega(x, 4) = -v(x, 4) = -(-5x + x^2)$$

The above conditions are obtained by using Eqs. (2.121), (2.124) and the given boundary conditions. By using the superposition principle (see Section 2.5), the general solution of Eq. (2.125) is found to be

$$\omega(x, y) = \sum_{n=1}^{\infty} \sin(n\pi x/5)\,[a_n \exp(n\pi y/5) + b_n \exp(-n\pi y/5)] \tag{2.126}$$

Now, applying the non-homogeneous BC: $\omega(x, 0) = -(-5x + x^2)$, we get, after renaming the constants, the equation

$$\omega(x, 0) = -(-5x + x^2) = \sum A_n \sin(n\pi x/5)$$

Also, applying the BC: $\omega(x, 4) = -(-5x + x^2)$, Eq. (2.126) can be rewritten in the form

$$-(-5x + x^2) = \sum_{n=1}^{\infty} \left(a_n \cosh \frac{4n\pi}{5} + b_n \sinh \frac{4n\pi}{5} \right) \sin \frac{n\pi x}{5} \tag{2.127}$$

which gives

$$a_n = \frac{2}{5}\int_0^5 (5x - x^2)\sin\frac{n\pi x}{5}\,dx$$

Now, integrating by parts, the right-hand side yields

$$a_n = \frac{2}{5}\left[(5x - x^2)\left(-\frac{5}{n\pi}\cos\frac{n\pi}{5}x\right) - (5 - 2x)\left(-\frac{5^2}{n^2\pi^2}\sin\frac{n\pi x}{5}\right) + (-2)\left(\frac{5^3}{n^3\pi^3}\cos\frac{n\pi}{5}x\right)\right]_0^5$$

$$= \frac{2}{5}\left[-\frac{2(5^3)}{n^3\pi^3}\cos n\pi + \frac{2}{n^3}\left(\frac{5^3}{\pi^3}\right)\right]$$

$$= \frac{4(5^2)}{\pi^3}\left[\frac{1}{n^3} - \frac{\cos n\pi}{n^3}\right] = \frac{4(5)^2}{\pi^3}\left[\frac{1}{n^3} - \frac{(-1)^n}{n^3}\right]$$

Hence,

$$a_n = \begin{cases} \dfrac{8(5^2)}{\pi^3 n^3}, & \text{when } n \text{ is odd} \\[2ex] 0, & \text{when } n \text{ is even} \end{cases} \tag{2.128}$$

Also, from Eq. (2.127), we have

$$a_n \cosh\frac{4n\pi}{5} + b_n \sinh\frac{4n\pi}{5} = \frac{2}{5}\int_0^5 (5x - x^2)\sin\left(\frac{n\pi}{5}x\right)dx = a_n$$

Therefore,

$$b_n = \frac{a_n\left[1 - \cosh\left(\frac{4}{5}n\pi\right)\right]}{\sinh\left(\frac{4}{5}n\pi\right)} \tag{2.129}$$

Substituting a_n, b_n from Eqs. (2.128) and (2.129) into Eq. (2.126), we get

$$\omega(x, y) = \sum_{n=1}^{\infty} a_n\left[\cosh\left(\frac{n\pi}{5}y\right)\sinh\left(\frac{4}{5}n\pi\right) - \cosh\left(\frac{4}{5}n\pi\right)\sinh\left(\frac{n\pi}{5}y\right)\right.$$

$$\left. + \sinh\left(\frac{n\pi}{5}y\right)\sin\left(\frac{n\pi}{5}x\right)\Big/\sinh\left(\frac{4}{5}n\pi\right)\right]$$

or

$$\omega(x, y) = \sum_{n=1}^{\infty} \left[\sinh\left\{\frac{n\pi}{5}(4-y)\right\} + \sinh\left(\frac{n\pi}{5}y\right)\right] \sin\left(\frac{n\pi}{5}x\right) \Big/ \sinh\left(\frac{4}{5}n\pi\right) \quad (2.130)$$

Combining Eqs. (2.124), (2.128) and (2.130), the solution of the given Poisson equation is $u(x, y) =$

$$x(x-5) + \frac{8\times 5^2}{\pi^3} \sum_{n=1}^{\infty} \left[\frac{\sinh (2n-1)\,\pi(4-y)/5 + \sinh[(2n-1)\pi y/5]}{\sinh\left[(2n-1)\frac{4}{5}\pi\right]} \right] \times \left\{ \frac{\sin[(2n-1)\pi x/5]}{(2n-1)^3} \right\}$$

EXAMPLE 2.22 Let $\mathbb{R}$ be a region bounded by $\partial\mathbb{R}$. Let $P(x, y, z)$ be any point in the interior of $\mathbb{R}$, as shown in Fig. 2.9. Let ϕ be a harmonic function in $\overline{\mathbb{R}}$; also, let $\psi = 1/r$, where r is the distance from P. Applying Green's second identity, show that

$$\phi(P) = \frac{1}{4\pi} \iint_{\partial\mathbb{R}} \left[\frac{1}{r}\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] ds$$

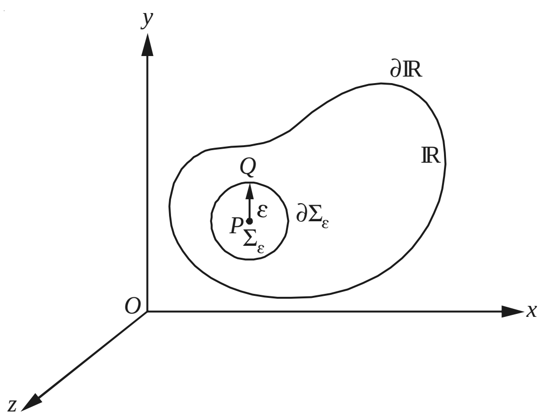

Fig. 2.9 An illustration of Example 2.22.

Solution Since ψ possesses a point of discontinuity in $\mathbb{R}$ at $P(x, y, z)$, Green's second identity cannot be directly applied to ϕ and ψ. However, $\psi = 1/r$ is bounded in $\mathbb{R} - \Sigma_\varepsilon$ with the boundary $\partial\mathbb{R} \cup \partial\Sigma_\varepsilon$, where Σ_ε is a sphere of radius ε with centre at P. Now applying Green's second identity (2.19) to functions ϕ and ψ in $\mathbb{R} - \Sigma_\varepsilon$, we get

$$\iiint_{\mathbb{R}-\Sigma_\varepsilon} \left[\phi\nabla^2(1/r) - \frac{1}{r}\nabla^2\phi\right] dV = \iint_{\partial\mathbb{R}} \left[\phi\frac{\partial}{\partial n}(1/r) - (1/r)\frac{\partial\phi}{\partial n}\right] dS$$

$$+ \iint_{\partial\Sigma_\varepsilon} \phi\frac{\partial}{\partial n}(1/r)\,dS - \iint_{\partial\Sigma_\varepsilon} (1/r)\left(\frac{\partial\phi}{\partial n}\right) dS \quad (2.131)$$

From the right-hand side of Eq. (2.131), we observe that the last two integrals depend only on ε. But in the direction of the exterior normal to $\partial\Sigma_\varepsilon$, we find that

$$\frac{\partial}{\partial n}(1/r)\bigg|_{\partial\Sigma_\varepsilon} = -\frac{\partial}{\partial r}(1/r)\bigg|_{r=\varepsilon} = \frac{1}{\varepsilon^2}$$

Therefore,

$$\iint_{\partial\Sigma_\varepsilon} \phi\frac{\partial}{\partial n}(1/r)\,dS = \frac{1}{\varepsilon^2}\iint_{\partial\Sigma_\varepsilon} \phi\,dS = \frac{4\pi\varepsilon^2}{\varepsilon^2}\phi(Q) = 4\pi\phi^*(Q)$$

where $\phi^*(Q)$ is the average value of $\phi(Q)$ on $\partial\Sigma_\varepsilon$. Further, the third integral

$$-\iint_{\partial\Sigma_\varepsilon} (1/r)\frac{\partial\phi}{\partial n}\,dS = -\frac{1}{\varepsilon}\iint_{\partial\Sigma_\varepsilon}\left(\frac{\partial\phi}{\partial n}\right)dS = -4\pi\varepsilon\left(\frac{\partial\phi^*}{\partial n}\right)$$

where $(\partial\phi/\partial n)^*$ is an average value of the normal derivative on $\partial\Sigma_\varepsilon$. Substituting these results and using the fact that $\nabla^2(1/r) = 0$ in $\mathbb{R} - \Sigma_\varepsilon$, we obtain

$$\iiint_{\mathbb{R}-\Sigma_\varepsilon} (-1/r)\nabla^2\phi\,dV = \iint_{\partial\mathbb{R}}\left[\phi\frac{\partial}{\partial n}(1/r) - (1/r)\frac{\partial\phi}{\partial n}\right]dS + 4\pi\phi^*(P) + 4\pi\varepsilon\left(\frac{\partial\phi}{\partial n}\right)^* \tag{2.132}$$

Now, taking the limit as $\varepsilon \to 0$, and using the fact that ϕ is harmonic in $\mathbb{R} - \Sigma_\varepsilon$, we arrive at the fundamental result

$$\phi(P) = \frac{1}{4\pi}\iint_{\partial\mathbb{R}}\left[(1/r)\frac{\partial\phi}{\partial n} - \phi\frac{\partial}{\partial n}(1/r)\right]dS \tag{2.133}$$

Thus, the value of a harmonic function at any point of $\mathbb{R}$ can be obtained in terms of the values of ϕ and $\partial\phi/\partial n$ on the boundary $\partial\mathbb{R}$ of the region $\mathbb{R}$.

EXAMPLE 2.23 Find the solution of the following Helmholtz equation, using separation of variables method:

$$\nabla^2 u + K^2 u = u_{xx} + u_{yy} + u_{zz} + K^2 u = 0 \tag{2.134}$$

Solution It may be noted that the Laplacian in cartesian coordinates is a PDE with constant coefficients, while in cylindrical or spherical coordinates, it is a PDE with variable coefficients. Thus, let us assume the solution of the given Helmholtz equation in the form

$$u(x, y, z) = X(x)\,Y(y)\,Z(z)$$

where $X(x)$ is a function of x alone, $Y(y)$ is a function of y alone, $Z(z)$ is a function of z only. Substituting into the given Helmholtz equation, we get

$$X''(x)\,Y(y)\,Z(z) + X(x)\,Y''(y)\,Z(z) + X(x)\,Y(y)\,Z''(z) + K^2X(x)\,Y(y)\,Z(z) = 0,$$

which can be rewritten as

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} + K^2 = 0 .$$

This equation is satisfied iff

$$\frac{X''(x)}{X(x)} = -K_1^2, \frac{Y''(y)}{Y(y)} = -K_2^2, \frac{Z''(z)}{Z(z)} = -K_3^2 \tag{2.135}$$

and
$$K^2 = K_1^2 + K_2^2 + K_3^2$$

The sign of these separation constants K_1, K_2 and K_3 need not be same, of course depends on the physical considerations. The solution of the three ODEs in Eq. (2.135) can be written in the form

$$\left.\begin{aligned} X(x) &= C_1 e^{iK_1 x} + C_2 e^{-iK_1 x} \\ Y(y) &= C_3 e^{iK_2 y} + C_4 e^{-iK_2 y} \\ Z(z) &= C_5 e^{iK_3 z} + C_6 e^{-iK_3 z} \end{aligned}\right\} \tag{2.136}$$

Hence, in general, the solution of Eq. (2.134) can be written as

$$u(x, y, z) = Ae^{iK\mathbf{r}} + Be^{-iK\mathbf{r}}.$$

However, if K^2 is positive, the solution is of the form

$$u(x, y, z) = A \cos(K_1x + K_2y + K_3z) + B \sin(K_1x + K_2y + K_3z),$$

while, if K^2 is negative, the solution is found to be

$$u(x, y, z) = A \cosh(K_1x + K_2y + K_3z) + B \sinh(K_1x + K_2y + K_3z).$$

EXERCISES

1. Solve the following boundary value problem:

$$\text{PDE: } \nabla^2 u = 0, \qquad 0 \le r \le 10, \ 0 \le \theta \le \pi$$

$$\text{BCs: } u(10, \theta) = \frac{400}{\pi}(\pi\theta - \theta^2)$$

$$u(r, 0) = 0 = u(r, \pi)$$

$$u(0, \theta) \text{ is finite}$$

2. A homogeneous thermally conducting solid is bounded by the concentric spheres $r = a$, $r = b$, $0 < a < b$. There are no heat sources within the solid. The inner surface $r = a$ is held at constant temperature T_1, and at the outer surface there is radiation into the medium $r > b$ which is at a constant temperature T_2. Find the steady temperature T in the solid.

3. A thermally conducting solid bounded by two concentric spheres of radii a and b, $a < b$, is such that the internal boundary is kept at T_1 and the outer boundary at $T_2(1-\cos\theta)$. Find the steady state temperature in the solid.

4. A thin annulus occupies the region $0 < a \le r \le b,\ 0 \le \theta \le 2\pi$, where $b > a$. The faces are insulated, and along the inner edge, the temperature is maintained at 0°, while along the outer edge, the temperature is held at 100°. Find the temperature distribution in the annulus.

5. A thermally conducting homogeneous disc with insulated faces occupies the region $0 \le r \le a$ in the xy-plane. The temperature u on the rim $r = a$, is

$$u = \begin{cases} C, & 0 < \theta < \alpha \\ 0, & \alpha < \theta < 2\pi \end{cases}$$

where α is a given angle $0 < \alpha < 2\pi$. Find the series expression for temperature at interior points of the disc. In particular, consider the case when $C = 100,\ \alpha = \pi/2$.

6. If ψ is a harmonic function which is zero on the cone $\theta = \alpha$ and takes the value $\Sigma\alpha_n r^n$ on the cone $\theta = \beta$, show that, when $\alpha < \theta < \beta$,

$$\psi = \sum_{n=0}^{\infty} \alpha_n \left\{ \frac{Q_n(\cos\alpha)\, P_n(\cos\theta) - P_n(\cos\alpha)\, Q_n(\cos\theta)}{Q_n(\cos\alpha) P_n(\cos\beta) - P_n(\cos\alpha)\, Q_n(\cos\beta)} \right\} r^n$$

7. Show that

$$\psi = \frac{q}{|\mathbf{r} - \mathbf{r}'|}, \qquad (q \text{ is constant})$$

is a solution of the Laplace equation.

8. Solve the following

$$\text{PDE: } \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

$$\text{BCs: } v_r = \frac{\partial\phi}{\partial r} = 0 \qquad \text{at } r = a$$

$$v_r = U_\infty \cos\theta \qquad \text{at } r = \infty$$

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U_\infty \sin\theta$$

9. In the theory of elasticity, the stress function ψ, in the problem of torsion of a beam satisfies the Poisson equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -2, \qquad 0 \le x \le 1,\ 0 \le y \le 1$$

with the boundary conditions $\psi = 0$ on sides $x = 0$, $x = 1$, $y = 0$ and on $y = 1$. Find the stress function ψ.

10. For an infinitely long conducting cylinder of radius a, with its axis coincident with its z-axis, the voltage $u(r, \theta)$ obeys the Laplace equation

$$\nabla^2 u = 0, \qquad 0 \le r \le \infty,\ 0 \le \theta \le 2\pi$$

Find the voltage $u(r, \theta)$ for $r \ge a$ if $\operatorname*{Lt}_{r \to \infty} u(r, \theta) = 0$, subject to the condition

$$\left.\frac{\partial u}{\partial r}\right|_{r=a} = \frac{u_0}{a} \sin 3\theta$$

11. Hadamard's example:

(a) Consider the Cauchy problem for the Laplace equation

$$u_{xx} + u_{yy} = 0 \tag{E11.1}$$

subject to $u(x, 0) = 0, u_y(x, 0) = \dfrac{1}{n} \sin nx$, where n is a positive integer. Show that its solution is

$$u_n(x, y) \ \frac{1}{n^2} \sinh ny \sin nx \tag{E11.2}$$

(b) Show that for large n, the absolute value of the initial data in (a) can be made arbitrarily small, while the solution (E11.2) takes arbitrarily large values even at the points (x, y) with $|y|$ as small as we want.

(c) Let f and g be analytic, and let u_1 be the solution to the Cauchy problem described by

$$u_{xx} + u_{yy} = 0$$

subject to

$$u(x, 0) = f(x), u_y(x, 0) = g(x) \tag{E11.3}$$

and let u_2 be the solution of the Laplace equation (E11.1) subject to $u(x, 0) = f(x), u_y(x, 0) = g(x) + (1/n) \sin nx$. Show that

$$u_2(x, y) - u_1(x, y) = \frac{1}{n^2} \sinh ny \sin nx \tag{E11.4}$$

(d) Conclude that the solution to the Cauchy problem for Laplace equation does not depend continuously on the initial data. In other words, the initial value problem (Cauchy problem) for the Laplace equation is not well-posed. It may be noted that a problem involving a PDE is well-posed if the following three properties are satisfied:

(i) The solution to the problem exists.

(ii) The solution is unique.

(iii) The solution depends continuously on the data of the problem.

Fortunately, many a physical phenomena give rise to initial or boundary or IBVPs which are well-posed.

12. Find the solution of the following PDE using separation of variables method

$$u_{xx} - u_y + u = 0.$$

CHAPTER 3

Parabolic Differential Equations

3.1 OCCURRENCE OF THE DIFFUSION EQUATION

The diffusion phenomena such as conduction of heat in solids and diffusion of vorticity in the case of viscous fluid flow past a body are governed by a partial differential equation of parabolic type. For example, the flow of heat in a conducting medium is governed by the parabolic equation

$$\rho C \frac{\partial T}{\partial t} = \operatorname{div}(K \nabla T) + H(\mathbf{r}, T, t) \tag{3.1}$$

where ρ is the density, C is the specific heat of the solid, T is the temperature at a point with position vector $\mathbf{r}$, K is the thermal conductivity, t is the time, and $H(\mathbf{r}, T, t)$ is the amount of heat generated per unit time in the element dV situated at a point (x, y, z) whose position vector is $\mathbf{r}$. This equation is known as diffusion equation or heat equation. We shall now derive the heat equation from the basic concepts.

Let V be an arbitrary domain bounded by a closed surface S and let $\bar{V} = V \cup S$. Let $T(x, y, z, t)$ be the temperature at a point (x, y, z) at time t. If the temperature is not constant, heat flows from a region of high temperature to a region of low temperature and follows the Fourier law which states that heat flux $\mathbf{q}(\mathbf{r}, t)$ across the surface element dS with normal $\hat{n}$ is proportional to the gradient of the temperature. Therefore,

$$\mathbf{q}(\mathbf{r}, t) = -K \nabla T(\mathbf{r}, t) \tag{3.2}$$

where K is the thermal conductivity of the body. The negative sign indicates that the heat flux vector points in the direction of decreasing temperature. Let $\hat{n}$ be the outward unit normal vector and $\mathbf{q}$ be the heat flux at the surface element dS. Then the rate of heat flowing out through the elemental surface dS in unit time as shown in Fig. 3.1 is

$$dQ = (\mathbf{q} \cdot \hat{n})\, dS \tag{3.3}$$

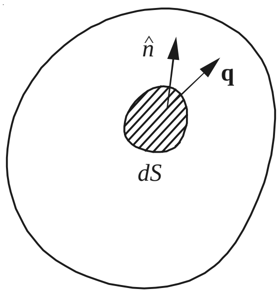

Fig. 3.1 The heat flow across a surface.

Heat can be generated due to nuclear reactions or movement of mechanical parts as in inertial measurement unit (IMU), or due to chemical sources which may be a function of position, temperature and time and may be denoted by $H(\mathbf{r}, T, t)$. We also define the specific heat of a substance as the amount of heat needed to raise the temperature of a unit mass by a unit temperature. Then the amount of heat dQ needed to raise the temperature of the elemental mass $dm = \rho\, dV$ to the value T is given by $dQ = C\rho T\, dV$. Therefore,

$$Q = \iiint_V C\rho T\, dV$$

$$\frac{dQ}{dt} = \iiint_V C\rho \frac{\partial T}{\partial t} dV$$

The energy balance equation for a small control volume V is: The rate of energy storage in V is equal to the sum of rate of heat entering V through its bounding surfaces and the rate of heat generation in V. Thus,

$$\iiint_V C\rho \frac{\partial T(\mathbf{r}, t)}{\partial t} dV = -\iint_S \mathbf{q} \cdot \hat{n}\, dS + \iiint_V H(\mathbf{r}, T, t)\, dV \tag{3.4}$$

Using the divergence theorem, we get

$$\iiint_V \left[C\rho \frac{\partial T}{\partial t}(\mathbf{r}, t) + \operatorname{div} \mathbf{q}(\mathbf{r}, t) - H(\mathbf{r}, T, t) \right] dV = 0 \tag{3.5}$$

Since the volume is arbitrary, we have

$$\rho C \frac{\partial T(\mathbf{r}, t)}{\partial t} = -\operatorname{div} \mathbf{q}(\mathbf{r}, t) + H(\mathbf{r}, T, t) \tag{3.6}$$

Substituting Eq. (3.2) into Eq. (3.6), we obtain

$$\rho C \frac{\partial T(\mathbf{r}, t)}{\partial t} = \nabla \cdot [K \nabla T(\mathbf{r}, t)] + H(\mathbf{r}, T, t) \tag{3.7}$$

If we define thermal diffusivity of the medium as

$$\alpha = \frac{K}{\rho C}$$

then the differential equation of heat conduction with heat source is

$$\frac{1}{\alpha}\frac{\partial T(\mathbf{r}, t)}{\partial t} = \nabla^2 T(\mathbf{r}, t) + \frac{H(\mathbf{r}, T, t)}{K} \tag{3.8}$$

In the absence of heat sources, Eq. (3.8) reduces to

$$\frac{\partial T(\mathbf{r}, t)}{\partial t} = \alpha \nabla^2 T(\mathbf{r}, t) \tag{3.9}$$

This is called Fourier heat conduction equation or diffusion equation. The fundamental problem of heat conduction is to obtain the solution of Eq. (3.8) subject to the initial and boundary conditions which are called initial boundary value problems, hereafter referred to as IBVPs.

3.2 BOUNDARY CONDITIONS

The heat conduction equation may have numerous solutions unless a set of initial and boundary conditions are specified. The boundary conditions are mainly of three types, which we now briefly explain.

Boundary Condition I: *The temperature is prescribed all over the boundary surface*. That is, the temperature $T(\mathbf{r}, t)$ is a function of both position and time. In other words, $T = G(\mathbf{r}, t)$ which is some prescribed function on the boundary. This type of boundary condition is called the *Dirichlet condition*. Specification of boundary conditions depends on the problem under investigation. Sometimes the temperature on the boundary surface is a function of position only or is a function of time only or a constant. A special case includes $T(\mathbf{r}, t) = 0$ on the surface of the boundary, which is called a *homogeneous boundary condition*.

Boundary Condition II: *The flux of heat, i.e., the normal derivative of the temperature $\partial T/\partial n$, is prescribed on the surface of the boundary*. It may be a function of both position and time, i.e.,

$$\frac{\partial T}{\partial n} = f(\mathbf{r}, t)$$

This is called the Neumann condition. Sometimes, the normal derivatives of temperature may be a function of position only or a function of time only. A special case includes

$$\frac{\partial T}{\partial n} = 0 \text{ on the boundary}$$

This homogeneous boundary condition is also called insulated boundary condition which states that the heat flow is zero.

Boundary Condition III: *A linear combination of the temperature and its normal derivative is prescribed on the boundary*, i.e.,

$$K\frac{\partial T}{\partial n} + hT = G(\mathbf{r}, t)$$

where K and h are constants. This type of boundary condition is called *Robin's condition*. It means that the boundary surface dissipates heat by convection. Following Newton's law of cooling, which states that the rate at which heat is transferred from the body to the surroundings is proportional to the difference in temperature between the body and the surroundings, we have

$$-K\frac{\partial T}{\partial n} = h(T - T_a)$$

As a special case, we may also have

$$K\frac{\partial T}{\partial n} + hT = 0$$

which is a homogeneous boundary condition. This means that heat is convected by dissipation from the boundary surface into a surrounding maintained at zero temperature.

The other boundary conditions such as the heat transfer due to radiation obeying the fourth power temperature law and those associated with change of phase, like melting, ablation, etc. give rise to non-linear boundary conditions.

3.3 ELEMENTARY SOLUTIONS OF THE DIFFUSION EQUATION

Consider the one-dimensional diffusion equation

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \qquad -\infty < x < \infty,\ t > 0 \tag{3.10}$$

The function

$$T(x, t) = \frac{1}{\sqrt{4\pi\alpha t}}\exp[-(x-\xi)^2/(4\alpha t)] \tag{3.11}$$

where ξ is an arbitrary real constant, is a solution of Eq. (3.10). It can be verified easily as follows:

$$\frac{\partial T}{\partial t} = \frac{1}{\sqrt{4\pi\alpha t}}\left[\frac{(x-\xi)^2}{4\alpha t^2} - \frac{1}{2t}\right]\exp[-(x-\xi)^2/(4\alpha t)]$$

$$\frac{\partial T}{\partial x} = \frac{1}{\sqrt{4\pi\alpha t}}\,\frac{-2(x-\xi)}{4\alpha t}\,\exp[-(x-\xi)^2/(4\alpha t)]$$

Therefore,

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\sqrt{4\pi\alpha t}}\left[-\frac{1}{2\alpha t} + \frac{(x-\xi)^2}{4\alpha^2 t^2}\right]\exp\left[-(x-\xi)^2/(4\alpha t)\right] = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

which shows that the function (3.11) is a solution of Eq. (3.10). The function (3.11), known as Kernel, is the elementary solution or the fundamental solution of the heat equation for the infinite interval. For $t > 0$, the Kernel $T(x, t)$ is an analytic function of x and t and it can also be noted that $T(x, t)$ is positive for every x. Therefore, the region of influence for the diffusion equation includes the entire x-axis. It can be observed that as $|x| \to \infty$, the amount of heat transported decreases exponentially.

In order to have an idea about the nature of the solution to the heat equation, consider a one-dimensional infinite region which is initially at temperature $f(x)$. Thus the problem is described by

$$\text{PDE: } \frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}, \qquad -\infty < x < \infty,\ t > 0 \tag{3.12}$$

$$\text{IC: } T(x, 0) = f(x), \qquad -\infty < x < \infty,\ t = 0 \tag{3.13}$$

Following the method of variables separable, we write

$$T(x, t) = X(x)\,\beta(t) \tag{3.14}$$

Substituting into Eq. (3.12), we arrive at

$$\frac{X''}{X} = \frac{1}{\alpha}\frac{\beta'}{\beta} = \lambda \tag{3.15}$$

where λ is a separation constant. The separated solution for β gives

$$\beta = Ce^{\alpha\lambda t} \tag{3.16}$$

If $\lambda > 0$, we have β and, therefore, T growing exponentially with time. From realistic physical considerations, it is reasonable to assume that $f(x) \to 0$ as $|x| \to \infty$, while $|T(x, t)| < M$ as $|x| \to \infty$. But, for $T(x, t)$ to remain bounded, λ should be negative and thus we take $\lambda = -\mu^2$. Now from Eq. (3.15) we have

$$X'' + \mu^2 X = 0$$

Its solution is found to be

$$X = c_1 \cos \mu x + c_2 \sin \mu x$$

Hence

$$T(x, t, \mu) = (A\cos \mu x + B\sin \mu x)\,e^{-\alpha\mu^2 t} \tag{3.17}$$

is a solution of Eq. (3.12), where A and B are arbitrary constants. Since $f(x)$ is in general not periodic, it is natural to use Fourier integral instead of Fourier series in the present case. Also, since A and B are arbitrary, we may consider them as functions of μ and take $A = A(\mu), B = B(\mu)$. In this particular problem, since we do not have any boundary conditions which limit our choice of μ, we should consider all possible values. From the principles of superposition, this summation of all the product solutions will give us the relation

$$T(x,t) = \int_0^\infty T(x,t,\mu)\,d\mu = \int_0^\infty [A(\mu)\cos\mu x + B(\mu)\sin\mu x]e^{-\alpha\mu^2 t}\,d\mu \tag{3.18}$$

which is the solution of Eq. (3.12). From the initial condition (3.13), we have

$$T(x,0) = f(x) = \int_0^\infty [A(\mu)\cos\mu x + B(\mu)\sin\mu x]\,d\mu \tag{3.19}$$

In addition, if we recall the Fourier integral theorem, we have

$$f(t) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(x)\cos\omega(t-x)\,dx\right] d\omega \tag{3.20}$$

Thus, we may write

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(y)\cos\mu(x-y)\,dy\right] d\mu$$

$$= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(y)\,(\cos\mu x\cos\mu y + \sin\mu x\sin\mu y)\,dy\right] d\mu$$

$$= \frac{1}{\pi}\int_0^\infty \left[\cos\mu x\int_{-\infty}^\infty f(y)\cos\mu y\,dy + \sin\mu x\int_{-\infty}^\infty f(y)\sin\mu y\,dy\right] d\mu \tag{3.21}$$

Let

$$A(\mu) = \frac{1}{\pi}\int_{-\infty}^\infty f(y)\cos\mu y\,dy$$

$$B(\mu) = \frac{1}{\pi}\int_{-\infty}^\infty f(y)\sin\mu y\,dy$$

Then Eq. (3.21) can be written in the form

$$f(x) = \int_0^\infty [A(\mu)\cos\mu x + B(\mu)\sin\mu x]\,d\mu \tag{3.22}$$

Comparing Eqs. (3.19) and (3.22), we shall write relation (3.19) as

$$T(x,0) = f(x) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^\infty f(y)\cos\mu(x-y)\,dy\right] d\mu \tag{3.23}$$

Thus, from Eq. (3.18), we obtain

$$T(x,t) = \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(y)\cos\mu(x-y)\exp(-\alpha\mu^2 t)\,dy\right]d\mu \tag{3.24}$$

Assuming that the conditions for the formal interchange of orders of integration are satisfied, we get

$$T(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(y)\left[\int_0^{\infty}\exp(-\alpha\mu^2 t)\cos\mu(x-y)\,d\mu\right]dy \tag{3.25}$$

Using the standard known integral

$$\int_0^{\infty}\exp(-s^2)\cos(2bs)\,ds = \frac{\sqrt{\pi}}{2}\exp(-b^2) \tag{3.26}$$

Setting $s = \mu\sqrt{\alpha t}$, and choosing

$$b = \frac{x-y}{2\sqrt{\alpha t}}$$

Equation (3.26) becomes

$$\int_0^{\infty} e^{-\alpha\mu^2 t}\cos\mu(x-y)\,d\mu = \frac{\sqrt{\pi}}{\sqrt{4\alpha t}}\exp[-(x-y)^2/(4\alpha t)] \tag{3.27}$$

Substituting Eq. (3.27) into Eq. (3.25), we obtain

$$T(x,t) = \frac{1}{\sqrt{4\alpha\pi t}}\int_{-\infty}^{\infty} f(y)\exp[-(x-y)^2/(4\alpha t)]\,dy \tag{3.28}$$

Hence, if $f(y)$ is bounded for all real values of y, Eq. (3.28) is the solution of the problem described by Eqs. (3.12) and (3.13).

EXAMPLE 3.1 In a one-dimensional infinite solid, $-\infty < x < \infty$, the surface $a < x < b$ is initially maintained at temperature T_0 and at zero temperature everywhere outside the surface. Show that

$$T(x,t) = \frac{T_0}{2}\left[\operatorname{erf}\left(\frac{b-x}{\sqrt{4\alpha t}}\right) - \operatorname{erf}\left(\frac{a-x}{\sqrt{4\alpha t}}\right)\right]$$

where erf is an error function.

Solution The problem is described as follows:

PDE: $T_t = \alpha T_{xx}, \quad -\infty < x < \infty$

IC: $T = T_0, \quad a < x < b$

$= 0$ outside the above region

The general solution of PDE is found to be

$$T(x,t) = \frac{1}{\sqrt{4\pi\alpha t}} \int_{-\infty}^{\infty} f(\xi) \exp[-(x-\xi)^2/(4\alpha t)]\, d\xi$$

Substituting the IC, we obtain

$$T(x,t) = \frac{T_0}{\sqrt{4\pi\alpha t}} \int_{a}^{b} \exp[-(x-\xi)^2/(4\alpha t)]\, d\xi$$

Introducing the new independent variable η defined by

$$\eta = -\frac{x-\xi}{\sqrt{4\alpha t}}$$

and hence

$$d\xi = \sqrt{4\alpha t}\, d\eta$$

the above equation becomes

$$T(x,t) = \frac{T_0}{\sqrt{\pi}} \int_{(a-x)/\sqrt{(4\alpha t)}}^{(b-x)/\sqrt{(4\alpha t)}} e^{-\eta^2}\, d\eta = \frac{T_0}{2}\left[\frac{2}{\sqrt{\pi}} \int_0^{(b-x)/\sqrt{(4\alpha t)}} e^{-\eta^2}\, d\eta - \frac{2}{\sqrt{\pi}} \int_0^{(a-x)/\sqrt{(4\alpha t)}} e^{-\eta^2}\, d\eta\right]$$

Now we introduce the error function defined by

$$\text{erf}\,(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-\eta^2)\, d\eta$$

Therefore, the required solution is

$$T(x,t) = \frac{T_0}{2}\left[\text{erf}\left(\frac{b-x}{\sqrt{4\alpha t}}\right) - \text{erf}\left(\frac{a-x}{\sqrt{4\alpha t}}\right)\right]$$

3.4 DIRAC DELTA FUNCTION

According to the notion in mechanics, we come across a very large force (ideally infinite) acting for a short duration (ideally zero time) known as impulsive force. Thus we have a function which is non-zero in a very short interval. The Dirac delta function may be thought of as a generalization of this concept. This Dirac delta function and its derivative play a useful role in the solution of initial boundary value problem (IBVP).

Consider the function having the following property:

$$\delta_\varepsilon(t) = \begin{cases} 1/2\varepsilon, & |t| < \varepsilon \\ 0, & |t| > \varepsilon \end{cases} \tag{3.29}$$

Thus,

$$\int_{-\infty}^{\infty} \delta_\varepsilon(t)\,dt = \int_{-\varepsilon}^{\varepsilon} \frac{1}{2\varepsilon}\,dt = 1 \tag{3.30}$$

Let $f(t)$ be any function which is integrable in the interval $(-\varepsilon, \varepsilon)$. Then using the Mean-value theorem of integral calculus, we have

$$\int_{-\infty}^{\infty} f(t)\,\delta_\varepsilon(t)\,dt = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(t)\,dt = f(\xi), \qquad -\varepsilon < \xi < \varepsilon \tag{3.31}$$

Thus, we may regard $\delta(t)$ as a limiting function approached by $\delta_\varepsilon(t)$ as $\varepsilon \to 0$, i.e.

$$\delta(t) = \underset{\varepsilon\to 0}{\text{Lt}}\ \delta_\varepsilon(t) \tag{3.32}$$

As $\varepsilon \to 0$, we have, from Eqs. (3.29) and (3.30), the relations

$$\delta(t) = \underset{\varepsilon\to 0}{\text{Lt}}\ \delta_\varepsilon(t) = \begin{cases} & \text{(in the sense of being very large)} \\ \infty, & \text{if } t = 0 \\ 0, & \text{if } t \neq 0 \end{cases} \tag{3.33}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \tag{3.34}$$

This limiting function $\delta(t)$ defined by Eqs. (3.33) and (3.34) is known as Dirac delta function or the unit impulse function. Its profile is depicted in Fig. 3.2. Dirac originally called it an improper function as there is no proper function with these properties. In fact, we can observe that

$$1 = \int_{-\infty}^{\infty} \delta(t)\,dt = \underset{\varepsilon\to 0}{\text{Lt}} \int_{|t|>\varepsilon} \delta_\varepsilon(t)\,dt = \underset{\varepsilon\to 0}{\text{Lt}}\ 0 = 0$$

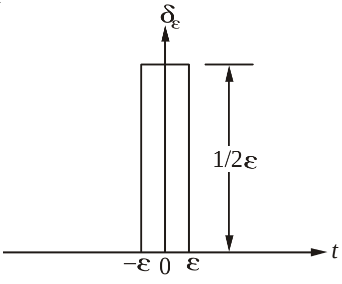

Fig. 3.2 Profile of Dirac delta function.

Obviously, this contradiction implies that $\delta(t)$ cannot be a function in the ordinary sense. Some important properties of Dirac delta function are presented now:

PROPERTY I: $$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

PROPERTY II: For any continuous function $f(t)$,

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\, dt = f(0)$$

Proof Consider the equation

$$\operatorname*{Lt}_{\varepsilon\to 0} \int_{-\infty}^{\infty} f(t)\,\delta_\varepsilon(t)\, dt = \operatorname*{Lt}_{\xi\to 0} f(\xi), \qquad -\varepsilon < \xi < \varepsilon$$

As $\varepsilon \to 0$, we have $\xi \to 0$. Therefore,

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\, dt = f(0)$$

PROPERTY III: Let $f(t)$ be any continuous function. Then

$$\int_{-\infty}^{\infty} \delta(t-a)\, f(t)\, dt = f(a)$$

Proof Consider the function

$$\delta_e(t-a) = \begin{cases} 1/\varepsilon, & a < t < a+\varepsilon \\ 0, & \text{elsewhere} \end{cases}$$

Using the mean-value theorem of integral calculus, we have

$$\int_{-\infty}^{\infty} \delta_\varepsilon(t-a)\, f(t)\, dt = \frac{1}{\varepsilon}\int_{a}^{a+\varepsilon} f(t)\, dt = f(a+\theta\varepsilon), \qquad 0<\theta<1$$

Now, taking the limit as $\varepsilon \to 0$, we obtain

$$\int_{-\infty}^{\infty} \delta(t-a)\, f(t)\, dt = f(a)$$

Thus, the operation of multiplying $f(t)$ by $\delta(t-a)$ and integrating over all t is equivalent to substituting a for t in the original function.

PROPERTY IV: $\delta(-t) = \delta(t)$

PROPERTY V: $\delta(at) = \dfrac{1}{a}\delta(t), \qquad a > 0$

PROPERTY VI: If $\delta(t)$ is a continuously differentiable. Dirac delta function vanishing for large t, then

$$\int_{-\infty}^{\infty} f(t)\,\delta'(t)\,dt = -f'(0)$$

Proof Using the rule of integration by parts, we get

$$\int_{-\infty}^{\infty} f(t)\,\delta'(t)\,dt = [f(t)\,\delta(t)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(t)\,\delta(t)\,dt$$

Using Eq. (3.33) and property (III), the above equations becomes

$$\int_{-\infty}^{\infty} f(t)\,\delta'(t)\,dt = -f'(0)$$

PROPERTY VII:
$$\int_{-\infty}^{\infty} \delta'(t-a)\,f(t)\,dt = -f'(a)$$

Having discussed the one-dimensional Dirac delta function, we can extend the definition to two dimensions. Thus, for every f which is continuous over the region S containing the point (ξ, η), we define $\delta(x-\xi, y-\eta)$ in such a way that

$$\iint_S \delta(x-\xi, y-\eta)\, f(x, y)\,d\sigma = f(\xi, \eta) \tag{3.35}$$

Note that $\delta(x-\xi, y-\eta)$ is a formal limit of a sequence of ordinary functions, i.e.,

$$\delta(x-\xi, y-\eta) = \operatorname*{Lt}_{\varepsilon\to 0} \delta_\varepsilon(r) \tag{3.36}$$

where $r^2 = (x-\xi)^2 + (y-\eta)^2$. Also observe that

$$\iint \delta(x-\xi)\,\delta(y-\eta)\, f(x, y)\,dx\,dy = f(\xi, \eta) \tag{3.37}$$

Now, comparing Eqs. (3.35) and (3.37), we see that

$$\delta(x-\xi, y-\eta) = \delta(x-\xi)\,\delta(y-\eta) \tag{3.38}$$

Thus, a two-dimensional Dirac delta function can be expressed as the product of two one-dimensional delta functions. Similarly, the definition can be extended to higher dimensions.

EXAMPLE 3.2 A one-dimensional infinite region $-\infty < x < \infty$ is initially kept at zero temperature. A heat source of strength g_s units, situated at $x = \xi$ releases its heat instantaneously at time $t = \tau$. Determine the temperature in the region for $t > \tau$.

Solution Initially, the region $-\infty < x < \infty$ is at zero temperature. Since the heat source is situated at $x = \xi$ and releases heat instantaneously at $t = \tau$, the released temperature at $x = \xi$ and $t = \tau$ is a δ- function type. Thus, the given problem is a boundary value problem described by

$$\text{PDE: } \frac{\partial^2 T}{\partial x^2} + \frac{g(x,t)}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \qquad -\infty < x < \infty,\ t > 0$$

$$\text{IC: } \quad T(x,t) = F(x) = 0, \qquad -\infty < x < \infty,\ t = 0$$

$$g(x,t) = g_s \delta(x-\xi)\,\delta(t-\tau)$$

The general solution to this problem as given in Example 7.25, after using the initial condition $F(x) = 0$, is

$$T(x,t) = \frac{\alpha}{k}\int_{t'=0}^{t} \frac{dt'}{\sqrt{4\pi\alpha(t-t')}} \int_{x'=-\infty}^{\infty} g(x',t') \exp[-(x-x')^2/\{4\alpha(t-t')\}]\,dx' \tag{3.39}$$

Since the heat source term is of the Dirac delta function type, substituting

$$g(x,t) = g_s \delta(x-\xi)\,\delta(t-\tau)$$

into Eq. (3.39), and integrating we get, with the help of properties of delta function, the relation

$$T(x,t) = \frac{\alpha}{k}\frac{g_s}{\sqrt{4\pi\alpha}}\int_0^t \frac{\exp[-(x-\xi)^2/\{4\alpha(t-t')\}]}{\sqrt{t-t'}}\,\delta(t-\tau)\,dt'$$

Therefore, the required temperature is

$$T(x,t) = \frac{\alpha g_s}{k}\,\frac{\exp[-(x-\xi)^2/\{4\alpha(t-\tau)\}]}{\sqrt{4\pi\alpha(t-\tau)}} \qquad \text{for } t > \tau$$

EXAMPLE 3.3 An infinite one-dimensional solid defined by $-\infty < x < \infty$ is maintained at zero temperature initially. There is a heat source of strength $g_s(t)$ units, situated at $x = \xi$, which releases constant heat continuously for $t > 0$. Find an expression for the temperature distribution in the solid for $t > 0$.

Solution This problem is similar to Example 3.2, except that $g(x,t) = g_s(t)\,\delta(x-\xi)$ is a Dirac delta function type. The solution to this IBVP is

$$T(x,t) = \frac{\alpha}{K}\int_{t'=0}^{t} \frac{g_s(t')}{\sqrt{4\pi\alpha(t-t')}} \exp[-(x-\xi)^2/\{4\alpha(t-t')\}]\,dt' \tag{3.40}$$

It is given as $g_s(t) = \text{constant} = g_s$ (say). Let us introduce a new variable η defined by

$$\eta = \frac{x-\xi}{\sqrt{4\alpha(t-t')}} \quad \text{or} \quad t - t' = \frac{1(x-\xi)^2}{\eta^2 4\alpha}$$

Therefore,

$$dt' = \frac{1}{\eta^3}\frac{(x-\xi)^2}{2\alpha}d\eta$$

Thus, Eq. (3.40) becomes

$$T(x,t) = g_s \frac{x-\xi}{2K\sqrt{\pi}}\int_{(x-\xi)/\sqrt{4\alpha t}}^{\infty} \frac{\exp(-\eta^2)}{\eta^2}d\eta$$

However,

$$\frac{d}{d\eta}\left(-\frac{e^{-\eta^2}}{\eta}\right) = \frac{e^{-\eta^2}}{\eta^2} + 2e^{-\eta^2}$$

Hence,

$$T(x,t) = g_s \frac{x-\xi}{2K\sqrt{\pi}}\left[\left(-\frac{e^{-\eta^2}}{\eta}\right)_{(x-\xi)/\sqrt{4\alpha t}}^{\infty} - 2\int_{(x-\xi)/\sqrt{(4\alpha t)}}^{\infty} e^{-\eta^2}d\eta\right]$$

Recalling the definitions of error function and its complement

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\eta^2}d\eta, \qquad \text{erf}(\infty) = 1$$

$$\text{erfc}(x) = 1 - \text{erf}(x) = \frac{2}{\sqrt{\pi}}\left(\int_0^{\infty}\exp(-\eta^2)d\eta - \int_0^x \exp(-\eta^2)d\eta\right)$$

$$= \frac{2}{\sqrt{\pi}}\int_x^{\infty}\exp(-\eta^2)d\eta$$

the temperature distribution can be expressed as

$$T(x,t) = \frac{\alpha g_s}{K}\left[\sqrt{\frac{t}{2\pi}}\exp[-(x-\xi)^2/(4\alpha t)] - \frac{|x-\xi|}{2\alpha}\left(1 - \text{erf}\frac{x-\xi}{\sqrt{4\alpha t}}\right)\right]$$

Alternatively, the required temperature is

$$T(x,t) = \frac{\alpha g_s}{K}\left[\sqrt{\frac{t}{2\pi}}\exp[-(x-\xi)^2/(4\alpha t)] - \frac{|x-\xi|}{2\alpha}\text{erfc}\frac{x-\xi}{\sqrt{4\alpha t}}\right]$$

3.5 SEPARATION OF VARIABLES METHOD

Consider the equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{3.41}$$

Among the many methods that are available for the solution of the above parabolic partial differential equation, the method of separation of variables is very effective and straightforward. We separate the space and time variables of $T(x, t)$ as follows: Let

$$T(x, t) = X(x)\beta(t) \tag{3.42}$$

be a solution of the differential Eq. (3.41). Substituting Eq. (3.42) into (3.41), we obtain

$$\frac{X''}{X} = \frac{1}{\alpha}\frac{\beta'}{\beta} = K, \text{ a separation constant}$$

Then we have

$$\frac{d^2 X}{dx^2} - KX = 0 \tag{3.43}$$

$$\frac{d\beta}{dt} - \alpha K \beta = 0 \tag{3.44}$$

In solving Eqs. (3.43) and (3.44), three distinct cases arise:

Case I When K is positive, say λ^2, the solution of Eqs. (3.43) and (3.44) will have the form

$$X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}, \qquad \beta = c_3 e^{\alpha\lambda^2 t} \tag{3.45}$$

Case II When K is negative, say $-\lambda^2$, then the solution of Eqs. (3.43) and (3.44) will have the form

$$X = c_1 \cos \lambda x + c_2 \sin \lambda x, \qquad \beta = c_3 e^{-\alpha\lambda^2 t} \tag{3.46}$$

Case III When K is zero, the solution of Eqs. (3.43) and (3.44) can have the form

$$X = c_1 x + c_2, \qquad \beta = c_3 \tag{3.47}$$

Thus, various possible solutions of the heat conduction equation (3.41) could be the following:

$$\begin{aligned} T(x, t) &= (c_1' e^{\lambda x} + c_2' e^{-\lambda x}) e^{\alpha\lambda^2 t} \\ T(x, t) &= (c_1' \cos \lambda x + c_2' \sin \lambda x) e^{-\alpha\lambda^2 t} \\ T(x, t) &= c_1' x + c_2' \end{aligned} \tag{3.48}$$

where

$$c_1' = c_1 c_3, \qquad c_2' = c_2 c_3$$

EXAMPLE 3.4 Solve the one-dimensional diffusion equation in the region $0 \le x \le \pi, t \ge 0$, subject to the conditions

(i) T remains finite as $t \to \infty$

(ii) $T = 0$, if $x = 0$ and π for all t

(iii) At $t = 0$, $T = \begin{cases} x, & 0 \le x \le \pi/2 \\ \pi - x, & \dfrac{\pi}{2} \le x \le \pi. \end{cases}$

Solution Since T should satisfy the diffusion equation, the three possible solutions are:

$$T(x, t) = (c_1 e^{\lambda x} + c_2 e^{-\lambda x}) e^{\alpha \lambda^2 t}$$

$$T(x, t) = (c_1 \cos \lambda x + c_2 \sin \lambda x) e^{-\alpha \lambda^2 t}$$

$$T(x, t) = (c_1 x + c_2)$$

The first condition demands that T should remain finite as $t \to \infty$. We therefore reject the first solution. In view of BC (ii), the third solution gives

$$0 = c_1 \cdot 0 + c_2, \qquad 0 = c_1 \cdot \pi + c_2$$

implying thereby that both c_1 and c_2 are zero and hence $T = 0$ for all t. This is a trivial solution. Since we are looking for a non-trivial solution, we reject the third solution also. Thus, the only possible solution satisfying the first condition is

$$T(x, t) = (c_1 \cos \lambda x + c_2 \sin \lambda x) e^{-\alpha \lambda^2 t}$$

Using the BC (ii), we have

$$0 = (c_1 \cos \lambda x + c_2 \sin \lambda x)\big|_{x=0}$$

implying $c_1 = 0$. Therefore, the possible solution is

$$T(x, t) = c_2 e^{-\alpha \lambda^2 t} \sin \lambda x$$

Applying the BC: $T = 0$ when $x = \pi$, we get

$$\sin \lambda \pi = 0 \Rightarrow \lambda \pi = n\pi$$

where n is an integer. Therefore,

$$\lambda = n$$

Hence the solution is found to be of the form

$$T(x, t) = c e^{-\alpha n^2 t} \sin nx$$

Noting that the heat conduction equation is linear, its most general solution is obtained by applying the principle of superposition. Thus,

$$T(x,t)=\sum_{n=1}^{\infty} c_n e^{-\alpha n^2 t}\sin nx$$

Using the third condition, we get

$$T(x,0)=\sum_{n=1}^{\infty} c_n \sin nx$$

which is a half-range Fourier-sine series and, therefore,

$$c_n=\frac{2}{\pi}\int_0^{\pi} T(x,0)\sin nx\,dx=\frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx\,dx+\int_{\pi/2}^{\pi}(\pi-x)\sin nx\,dx\right]$$

Integrating by parts, we obtain

$$c_n=\frac{2}{\pi}\left[\left(-x\frac{\cos nx}{n}-\frac{\sin nx}{n^2}\right)_0^{\pi/2}+\left\{-(\pi-x)\frac{\cos nx}{n}+\frac{\sin nx}{n^2}\right\}_{\pi/2}^{\pi}\right]$$

or

$$c_n=\frac{4\sin(n\pi/2)}{n^2\pi}$$

Thus, the required solution is

$$T(x,t)=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{e^{-\alpha n^2 t}\sin(n\pi/2)}{n^2}\sin nx$$

EXAMPLE 3.5 A uniform rod of length L whose surface is thermally insulated is initially at temperature $\theta=\theta_0$. At time $t=0$, one end is suddenly cooled to $\theta=0$ and subsequently maintained at this temperature; the other end remains thermally insulated. Find the temperature distribution $\theta(x, t)$.

Solution The initial boundary value problem IBVP of heat conduction is given by

$$\text{PDE:}\quad \frac{\partial\theta}{\partial t}=\alpha\frac{\partial^2\theta}{\partial x^2},\qquad 0\le x\le L,\ t>0$$

$$\text{BCs:}\quad \theta(0,t)=0,\qquad t\ge 0$$

$$\frac{\partial\theta}{\partial x}(L,t)=0,\qquad t>0$$

$$\text{IC:}\quad \theta(x,0)=\theta_0,\qquad 0\le x\le L$$

From Section 3.5, it can be noted that the physically meaningful and non-trivial solution is

$$\theta(x, t) = e^{-\alpha\lambda^2 t}(A\cos\lambda x + B\sin\lambda x)$$

Using the first boundary condition, we obtain $A = 0$. Thus the acceptable solution is

$$\theta = Be^{-\alpha\lambda^2 t}\sin\lambda x$$

$$\frac{\partial\theta}{\partial x} = \lambda Be^{-\alpha\lambda^2 t}\cos\lambda x$$

Using the second boundary condition, we have

$$0 = \lambda Be^{-\alpha\lambda^2 t}\cos\lambda L$$

implying $\cos\lambda L = 0$. Therefore,

The eigenvalues and the corresponding eigenfunctions are

$$\lambda_n = \frac{(2n+1)\pi}{2L}, \qquad n = 0, 1, 2, \ldots$$

Thus, the acceptable solution is of the form

$$\theta = B\exp[-\alpha\{(2n+1)/2L\}^2\pi^2 t]\sin\left(\frac{2n+1}{2L}\right)\pi x$$

Using the principle of superposition, we obtain

$$\theta(x, t) = \sum_{n=0}^{\infty} B_n \exp[-\alpha\{(2n+1)/2L\}^2\pi^2 t]\sin\left(\frac{2n+1}{2L}\pi x\right)$$

Finally, using the initial condition, we have

$$\theta_0 = \sum_{n=0}^{\infty} B_n \sin\left(\frac{2n+1}{2L}\pi x\right)$$

which is a half-range Fourier-sine series and, thus,

$$B_n = \frac{2}{L}\int_0^L \theta_0 \sin\left(\frac{2n+1}{2L}\pi x\right)dx$$

$$= \frac{2}{L}\left[-\theta_0\frac{2L}{(2n+1)\pi}\left\{\cos\left(\frac{2n+1}{2L}\pi x\right)\right\}_0^L\right]$$

$$= -\frac{4\theta_0}{(2n+1)\pi}[\cos\{(2n+1)\pi/2\} - \cos 0] = \frac{4\theta_0}{(2n+1)\pi}$$

Thus, the required temperature distribution is

$$\theta(x,t) = \sum_{n=0}^{\infty} \frac{4\theta_0}{(2n+1)\pi} \exp[-\alpha\{(2n+1)/2L\}^2 \pi^2 t] \sin\left(\frac{2n+1}{2L}\pi x\right)$$

EXAMPLE 3.6 A conducting bar of uniform cross-section lies along the x-axis with ends at $x = 0$ and $x = L$. It is kept initially at temperature 0° and its lateral surface is insulated. There are no heat sources in the bar. The end $x = 0$ is kept at 0°, and heat is suddenly applied at the end $x = L$, so that there is a constant flux q_0 at $x = L$. Find the temperature distribution in the bar for $t > 0$.

Solution The given initial boundary value problem can be described as follows:

$$\text{PDE:} \quad \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$\text{BCs:} \quad T(0,t) = 0, \qquad t > 0$$

$$\frac{\partial T}{\partial x}(L,t) = q_0, \qquad t > 0$$

$$\text{IC:} \quad T(x,0) = 0, \qquad 0 \le x \le L$$

Prior to applying heat suddenly to the end $x = L$, when $t = 0$, the heat flow in the bar is independent of time (steady state condition). Let

$$T(x,t) = T_{(s)}(x) + T_1(x,t)$$

where $T_{(s)}$ is a steady part and T_1 is the transient part of the solution. Therefore,

$$\frac{\partial^2 T_{(s)}}{\partial x^2} = 0$$

whose general solution is

$$T_{(s)} = Ax + B$$

when $x = 0, T_{(s)} = 0,$ implying $B = 0.$ Therefore,

$$T_{(s)} = Ax$$

Using the other BC: $\dfrac{\partial T_{(s)}}{\partial x} = q_0$, we get $A = q_0$. Hence, the steady state solution is

$$T_{(s)} = q_0 x$$

For the transient part, the BCs and IC are redefined as

(i) $T_1(0, t) = T(0, t) - T_{(s)}(0) = 0 - 0 = 0$

(ii) $\partial T_1(L, t)/\partial x = \partial T(L, t)/\partial x - \partial T_s(L, t)/\partial x = q_0 - q_0 = 0$

(iii) $T_1(x, 0) = T(x, 0) - T_{(s)}(x) = -q_0 x, 0 < x < L.$

Thus, for the transient part, we have to solve the given PDE subject to these conditions. The acceptable solution is given by Eq. (3.48), i.e.

$$T_1(x, t) = e^{-\alpha\lambda^2 t}(A\cos\lambda x + B\sin\lambda x)$$

Applying the BC (i), we get $A = 0$. Therefore,

$$T_1(x, t) = Be^{-\alpha\lambda^2 t}\sin\lambda x$$

and using the BC (ii), we obtain

$$\left.\frac{\partial T_1}{\partial x}\right|_{x=L} = B\lambda e^{-\alpha\lambda^2 t}\cos\lambda L = 0$$

implying $\lambda L = (2n-1)\dfrac{\pi}{2}, n = 1, 2, \ldots$ Using the superposition principle, we have

$$T_1(x, t) = \sum_{n=1}^{\infty} B_n \exp[-\alpha\{(2n-1)/2L\}^2\pi^2 t]\sin\left(\frac{2n-1}{2L}\pi x\right)$$

Now, applying the IC (iii), we obtain

$$T_1(x, 0) = -q_0 x = \sum_{n=1}^{\infty} B_n \sin\left(\frac{2n-1}{2L}\pi x\right)$$

Multiplying both sides by $\sin\left(\dfrac{2m-1}{2L}\pi x\right)$ and integrating between 0 to L and noting that

$$\int_0^L B_n \sin\left(\frac{2n-1}{2L}\right)\pi x \sin\left(\frac{2m-1}{2L}\pi x\right)dx = \begin{cases} 0, & n \neq m \\ \dfrac{B_m L}{2}, & n = m \end{cases}$$

we get at once, after integrating by parts, the equation

$$-q_0 \frac{4L^2}{(2m-1)^2\pi^2}\left[\sin\left(\frac{2m-1}{2}\pi\right)\right] = B_m\frac{L}{2}$$

or

$$-q_0 \frac{4L^2}{(2m-1)^2\pi^2}(-1)^{m-1} = B_m \frac{L}{2}$$

which gives

$$B_m = \frac{(-1)^m 8Lq_0}{(2m-1)^2\pi^2}$$

Hence, the required temperature distribution is

$$T(x,t) = q_0 x + \frac{8Lq_0}{\pi^2}\sum_{m=1}^{\infty}\left[\frac{(-1)^m}{(2m-1)^2}\exp\left[-\alpha\{(2m-1)/L\}^2\pi^2 t\right]\sin\left(\frac{2m-1}{2L}\pi x\right)\right]$$

EXAMPLE 3.7 The ends A and B of a rod, 10 cm in length, are kept at temperatures 0°C and 100°C until the steady state condition prevails. Suddenly the temperature at the end A is increased to 20°C, and the end B is decreased to 60°C. Find the temperature distribution in the rod at time t.

Solution The problem is described by

$$\text{PDE: } \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \qquad 0 < x < 10$$

$$\text{BCs: } T(0,t) = 0, \qquad T(10,t) = 100$$

Prior to change in temperature at the ends of the rod, the heat flow in the rod is independent of time as steady state condition prevails. For steady state,

$$\frac{d^2T}{dx^2} = 0$$

whose solution is

$$T_{(s)} = Ax + B$$

When $x = 0$, $T = 0$, implying $B = 0$. Therefore,

$$T_{(s)} = Ax$$

When $x = 10$, $T = 100$, implying $A = 10$. Thus, the initial steady temperature distribution in the rod is

$$T_{(s)}(x) = 10x$$

Similarly, when the temperature at the ends A and B are changed to 20 and 60, the final steady temperature in the rod is

$$T_{(s)}(x) = 4x + 20$$

which will be attained after a long time. To get the temperature distribution $T(x,t)$ in the intermediate period, counting time from the moment the end temperatures were changed, we assume that

$$T(x,t) = T_1(x,t) + T_{(s)}(x)$$

where $T_1(x,t)$ is the transient temperature distribution which tends to zero as $t \to \infty$. Now, $T_1(x,t)$ satisfies the given PDE. Hence, its general solution is of the form

$$T(x,t) = (4x+20) + e^{-\alpha\lambda^2 t}(B\cos\lambda x + c\sin\lambda x)$$

Using the BC: $T = 20$ when $x = 0$, we obtain

$$20 = 20 + Be^{-\alpha\lambda^2 t}$$

implying $B = 0$. Using the BC: $T = 60$ when $x = 10$, we get

$$\sin 10\lambda = 0, \text{ implying } \lambda = \frac{n\pi}{10}, \qquad n = 1, 2, \ldots$$

The principle of superposition yields

$$T(x,t) = (4x+20) + \sum_{n=1}^{\infty} c_n \exp[-\alpha(n\pi/10)^2 t]\sin\left(\frac{n\pi}{10}\right)x$$

Now using the IC: $T = 10x$, when $t = 0$, we obtain

$$10x = 4x + 20 + \sum c_n \sin\left(\frac{n\pi}{10}x\right)$$

or

$$6x - 20 = \sum c_n \sin\left(\frac{n\pi}{10}x\right)$$

where

$$c_n = \frac{2}{10}\int_0^{10}(6x-20)\sin\left(\frac{n\pi}{10}x\right)dx = -\frac{1}{5}\left[(-1)^n\frac{800}{n\pi} - \frac{200}{n\pi}\right]$$

Thus, the required solution is

$$T(x,t) = 4x + 20 - \frac{1}{5}\sum_{n=1}^{\infty}\left[(-1)^n\frac{800}{n\pi} - \frac{200}{n\pi}\right]\exp\left[-\alpha\left(\frac{n\pi}{10}\right)^2 t\right]\sin\left(\frac{n\pi}{10}x\right).$$

EXAMPLE 3.8 Assuming the surface of the earth to be flat, which is initially at zero temperature and for times $t > 0$, the boundary surface is being subjected to a periodic heat flux $g_0 \cos \omega t$. Investigate the penetration of these temperature variations into the earth's surface and show that at a depth x, the temperature fluctuates and the amplitude of the steady temperature is given by

$$\frac{g_0}{\sqrt{2}} \sqrt{\frac{2\alpha}{\omega}} \exp\left[-\sqrt{(\omega/2\alpha)}\, x\right]$$

Solution The given IBVP is described by

$$\text{PDE:} \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{3.49}$$

$$\text{BC:} -\frac{\partial T}{\partial x} = g_0 \cos \omega t \quad \text{at} \quad x = 0,\ t > 0 \tag{3.50}$$

$$\text{IC:} T(x, 0) = 0 \tag{3.51}$$

We shall introduce an auxiliary function $\tilde{T}$ satisfying Eqs. (3.49)–(3.51) and then define the complex function Z such that

$$Z = T + i\tilde{T}$$

We can easily verify that Z satisfies

$$\text{PDE:} \frac{\partial Z}{\partial t} = \alpha \frac{\partial^2 Z}{\partial x^2} \tag{3.52}$$

$$\text{BC:} -\frac{\partial Z}{\partial x} = g_0\, e^{i\omega t} \quad \text{at} \quad x = 0,\ t > 0$$

$$\text{IC:} Z = 0 \text{ in the region, } t = 0$$

Let us assume the solution of Eq. (3.52) in the form

$$Z = f(x) e^{i\omega t}$$

where $f(x)$ satisfies

$$\frac{d^2 f(x)}{dx^2} - i\frac{\omega}{\alpha} f(x) = 0 \tag{3.53}$$

$$-\frac{df(x)}{dx} = g_0 \qquad \text{at} \quad x = 0 \tag{3.54}$$

Also,

$$f(x) \text{ is finite for large } x. \tag{3.55}$$

The solution of Eq. (3.53), satisfying the BC (3.55), is

$$f(x) = A \exp[-\sqrt{(i\omega/\alpha)}\, x]$$

The constant A can be determined by using the BC (3.54). Therefore,

$$f(x) = \frac{1}{\sqrt{i}} g_0 \sqrt{\frac{\alpha}{\omega}} \exp[-\sqrt{(i\omega/\alpha)}\; x]$$

Thus,

$$Z = g_0 \sqrt{\frac{\alpha}{\omega}} \frac{1}{\sqrt{i}} \exp[i\omega t - \sqrt{(i\omega/\alpha)}\; x] \tag{3.56}$$

It can be shown for convenience that

$$\sqrt{i} = \frac{1+i}{\sqrt{2}}, \qquad \frac{1}{\sqrt{i}} = \frac{1-i}{\sqrt{2}}$$

Thus, Eq. (3.56) can be written in the form

$$Z = \frac{g_0}{2} \sqrt{\frac{2\alpha}{\omega}} \exp\left(-\sqrt{\frac{\omega}{2\alpha}} x\right)(1-i) \exp\left[i\left(\omega t - \sqrt{\frac{\omega}{2\alpha}} x\right)\right]$$

$$= \frac{g_0}{2} \sqrt{\frac{2\alpha}{\omega}} \exp\left(-\sqrt{\frac{\omega}{2\alpha}} x\right)(1-i)\left[\cos\left(\omega t - \sqrt{\frac{\omega}{2\alpha}} x\right) + i \sin\left(\omega t - \sqrt{\frac{\omega}{2\alpha}} x\right)\right]$$

Its real part gives the fluctuation in temperature is

$$T(x,t) = \frac{g_0}{2} \sqrt{\frac{2\alpha}{\omega}} \exp\left(-\sqrt{\frac{\omega}{2\alpha}} x\right)\left[\cos\left(\omega t - \sqrt{\frac{\omega}{2\alpha}} x\right) + \sin\left(\omega t - \sqrt{\frac{\omega}{2\alpha}} x\right)\right]$$

$$= \frac{g_0}{2} \sqrt{\frac{2\alpha}{\omega}} \exp\left(-\sqrt{\frac{\omega}{2\alpha}} x\right) \cos\left(\omega t - \sqrt{\frac{\omega}{2\alpha}} x - \frac{\pi}{4}\right)$$

Hence, the amplitude of the steady temperature is given by the factor

$$\frac{g_0}{\sqrt{2}} \sqrt{\frac{2\alpha}{\omega}} \exp\left(-\sqrt{\frac{\omega}{2\alpha}} x\right)$$

EXAMPLE 3.9 Find the solution of the one-dimensional diffusion equation satisfying the following BCs:

(i) T is bounded as $t \to \infty$

(ii) $\left.\dfrac{\partial T}{\partial x}\right|_{x=0} = 0, \quad \text{for all } t$

(iii) $\left.\dfrac{\partial T}{\partial x}\right|_{x=a} = 0, \quad \text{for all } t$

(iv) $T(x, 0) = x(a - x), \qquad 0 < x < a.$

Solution This is an example with insulated boundary conditions. From Section 3.5, it can be seen that a physically acceptable general solution of the diffusion equation is

$$T(x, t) = \exp(-\alpha\lambda^2 t)\,(A \cos \lambda x + B \sin \lambda x)$$

Thus,

$$\frac{\partial T}{\partial x} = \exp(-\alpha\lambda^2 t)\,(-A\lambda \sin \lambda x + B\lambda \cos \lambda x) \tag{3.57}$$

Using BC (ii), Eq. (3.57), gives $B = 0$. Since we are looking for a non-trivial solution, the use of BC (iii) into Eq. (3.57) at once gives

$$\sin \lambda a = 0 \text{ implying } \lambda a = n\pi, \qquad n = 0, 1, 2, \ldots$$

Using the principle of superposition, we get

$$T(x, t) = \sum A_n \exp(-\alpha\lambda^2 t) \cos \lambda x = \sum_{n=0}^{\infty} A_n \exp\left[-\alpha\left(\frac{n\pi}{a}\right)^2 t\right] \cos\left(\frac{n\pi}{a}\right) x.$$

The boundary condition (iv) gives

$$T(x, 0) = x(a - x) = A_0 + \sum_{n=1}^{\infty} A_n \exp\left[-\alpha\left(\frac{n\pi}{a}\right)^2 t\right] \cos\left(\frac{n\pi}{a}\right) x$$

where

$$A_0 = \frac{2}{a}\int_0^a (ax - x^2)\, dx = \frac{a^2}{6}$$

$$A_n = \frac{2}{a}\int_0^a (ax - x^2) \cos\left(\frac{n\pi}{a} x\right) dx$$

$$= \frac{2a^2}{n^2\pi^2}(1 + \cos n\pi) = \frac{2a^2}{n^2\pi^2}[1 + (-1)^n]$$

Therefore,

$$A_n = \begin{cases} -\dfrac{4a^2}{n^2\pi^2}, & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd} \end{cases}$$

Hence, the required solution is

$$T(x,t)=\frac{a^2}{6}-\frac{4a^2}{\pi^2}\sum_{n=2,4,\ldots\text{even}}^{\infty}\frac{1}{n^2}\cos\left(\frac{n\pi}{a}\right)x\exp\left[-\alpha\left(\frac{n\pi}{a}\right)^2 t\right]$$

EXAMPLE 3.10 The boundaries of the rectangle $0 \le x \le a,\ 0 \le y \le b$ are maintained at zero temperature. If at $t=0$ the temperature T has the prescribed value $f(x, y)$, show that for $t > 0$, the temperature at a point within the rectangle is given by

$$T(x,y,t)=\frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n)\exp(-\alpha\lambda_{mn}^2 t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

where

$$f(m,n)=\int_0^a\int_0^b f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

and

$$\lambda_{mn}^2=\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)$$

Solution The problem is to solve the diffusion equation described by

$$\text{PDE: } \frac{\partial T}{\partial t}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right), \qquad 0<x<a,\ 0<y<b,\ t>0$$

$$\text{BCs: } T(0,y,t)=T(a,y,t)=0, \qquad 0<y<b,\ t>0$$

$$T(x,0,t)=T(x,b,t)=0, \qquad 0<x<a,\ t>0$$

$$\text{IC: } T(x,y,0)=f(x,y), \qquad 0<x<a,\ 0<y<b$$

Let the separable solution be

$$T=X(x)Y(y)\beta(t)$$

Substituting into PDE, we get

$$\frac{X''}{X}+\frac{Y''}{Y}=\frac{1}{\alpha}\frac{\beta'}{\beta}=-\lambda^2$$

Then $\beta'+\alpha\lambda^2\beta=0$

$$\frac{X''}{X}=-\left(\lambda^2+\frac{Y''}{Y}\right)=-p^2 \text{ (say)}$$

Hence,

$$X'' + p^2 X = 0$$

$$\frac{Y''}{Y} = -\lambda^2 + p^2 = -q^2 \text{ (say)}$$

Therefore,

$$Y'' + q^2 Y = 0$$

Thus, the general solution of the given PDE is

$$T(x, y, t) = (A\cos px + B\sin px)(c\cos qy + D\sin qy)e^{-\alpha\lambda^2 t}$$

where

$$\lambda^2 = p^2 + q^2$$

Using the BC: $T(0, y, t) = 0$, we get $A = 0$. Then, the solution is of the form

$$T(x, y, t) = B\sin px(c\cos qy + D\sin qy)e^{-\alpha\lambda^2 t}$$

Applying the BC: $T(x, 0, t) = 0$, we get $c = 0$. Thus, the solution is given by

$$T(x, y, t) = BD\sin px\sin qye^{-\alpha\lambda^2 t}$$

Application of the BC: $T(a, y, t) = 0$ gives

$$\sin pa = 0, \text{ implying } pa = n\pi$$

or

$$p = \frac{n\pi}{a}, \qquad n = 1, 2, \ldots$$

Using the principle of superposition, the solution can be written in the form

$$T(x, y, t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right)\sin qye^{-\alpha\lambda^2 t}$$

Using the last BC: $T(x, b, t) = 0$, we obtain

$$q = \frac{m\pi}{b}, \qquad m = 1, 2, \ldots$$

Thus, the solution is found to be

$$T(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)e^{-\alpha\lambda^2 t}$$

where

$$\lambda^2 = p^2 + q^2 = \pi^2\left(\frac{m^2}{b^2} + \frac{n^2}{a^2}\right)$$

Finally, using the IC, we get

$$T(x, y, 0) = f(x, y) = \sum A_{mn} \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)$$

which is a double Fourier series, where

$$A_{mn} = \frac{2}{a}\cdot\frac{2}{b}\int_0^a \int_0^b f(x, y)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)dx\,dy$$

Hence, the required general solution is

$$T(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m, n)e^{-\alpha\lambda^2 t}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right)$$

where

$$f(m, n) = \frac{4}{ab}\int_0^a \int_0^b f(x, y)\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)dx\,dy$$

and

$$\lambda^2 = \pi^2\left(\frac{m^2}{b^2} + \frac{n^2}{a^2}\right)$$

3.6 SOLUTION OF DIFFUSION EQUATION IN CYLINDRICAL COORDINATES

Consider a three-dimensional diffusion equation

$$\frac{\partial T}{\partial t} = \alpha\nabla^2 T$$

In cylindrical coordinates (r, θ, z), it becomes

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \tag{3.58}$$

where $T = T(r, \theta, z, t)$.

Let us assume separation of variables in the form

$$T(r, \theta, z, t) = R(r)H(\theta)Z(z)\beta(t)$$

Substituting into Eq. (3.58), it becomes

$$R''HZ\beta+\frac{1}{r}R'HZ\beta+\frac{1}{r^2}H''RZ\beta+Z''RH\beta=\frac{\beta'}{\alpha}RHZ$$

or

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{H''}{H}+\frac{Z''}{Z}=\frac{1}{\alpha}\frac{\beta'}{\beta}=-\lambda^2$$

where $-\lambda^2$ is a separation constant. Then

$$\beta'+\alpha\lambda^2\beta=0 \tag{3.59}$$

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{H''}{H}+\lambda^2=-\frac{Z''}{Z}=-\mu^2 \text{ (say)}$$

Thus, the equations determining Z, R and H become

$$Z''-\mu^2Z=0 \tag{3.60}$$

$$\frac{R''}{R}+\frac{1}{r}\frac{R'}{R}+\frac{1}{r^2}\frac{H''}{H}+\lambda^2+\mu^2=0$$

or

$$r^2\frac{R''}{R}+r\frac{R'}{R}+(\lambda^2+\mu^2)r^2=-\frac{H''}{H}=v^2 \text{ (say)}$$

Therefore,

$$H''+v^2H=0 \tag{3.61}$$

$$R''+\frac{1}{r}R'+\left[(\lambda^2+\mu^2)-\frac{v^2}{r^2}\right]R=0 \tag{3.62}$$

Equations (3.59)–(3.61) have particular solutions of the form

$$\beta=e^{-\alpha\lambda^2 t}$$

$$H=c\cos v\theta+D\sin v\theta$$

$$Z=Ae^{\mu z}+Be^{-\mu z}$$

The differential equation (3.62) is called Bessel's equation of order v and its general solution is known as

$$R(r)=c_1J_v(\sqrt{\lambda^2+\mu^2 r})+c_2Y_v(\sqrt{\lambda^2+\mu^2 r})$$

where $J_\nu(r)$ and $Y_\nu(r)$ are Bessel functions of order ν of the first and second kind, respectively. Of course, Eq. (3.62) is singular when $r = 0$. The physically meaningful solutions must be twice continuously differentiable in $0 \le r \le a$. Hence, Eq. (3.62) has only one bounded solution, i.e.

$$R(r) = J_\nu(\sqrt{\lambda^2 + \mu^2} r)$$

Finally, the general solution of Eq. (3.58) is given by

$$T(r, \theta, z, t) = e^{-\alpha\lambda^2 t}[Ae^{\mu z} + Be^{-\mu z}][C \cos \nu\theta + D \sin \nu\theta] J_\nu(\sqrt{\lambda^2 + \mu^2} r)$$

EXAMPLE 3.11 Determine the temperature $T(r, t)$ in the infinite cylinder $0 \le r \le a$ when the initial temperature is $T(r, 0) = f(r)$, and the surface $r = a$ is maintained at 0° temperature.

Solution The governing PDE from the data of the problem is

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

where T is a function of r and t only. Therefore,

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{3.63}$$

The corresponding boundary and initial conditions are given by

$$\begin{aligned} \text{BC: } & T(a, t) = 0 \\ \text{IC: } & T(r, 0) = f(r) \end{aligned} \tag{3.64}$$

The general solution of Eq. (3.63) is

$$T(r, t) = A \exp(-\alpha\lambda^2 t) J_0(\lambda r)$$

Using the BC (3.64), we obtain

$$J_0(\lambda a) = 0$$

which has an infinite number of roots, $\xi_n a\,(n = 1, 2, \ldots, \infty)$. Thus, we get from the superposition principle the equation

$$T(r, t) = \sum_{n=1}^{\infty} A_n \exp(-\alpha\xi_n^2 t) J_0(\xi_n r)$$

Now using the IC: $T(r, 0) = f(r)$, we get

$$f(r) = \sum_{n=1}^{\infty} A_n J_0(\xi_n r)$$

To compute A_n, we multiply both sides of the above equation by $rJ_0(\xi_m r)$ and integrate with respect to r to get

$$\int_0^a rf(r)J_0(\xi_m r)\,dr = \sum_{n=1}^{\infty} A_n \int_0^a rJ_0(\xi_m r)\,J_0(\xi_n r)\,dr$$

$$= \begin{cases} 0 & \text{for } n \neq m \\ A_m\left(\dfrac{a^2}{2}\right)J_1^2(\xi_m a) & \text{for } n = m \end{cases}$$

which gives

$$A_m = \frac{2}{a^2 J_1^2(\xi_m a)} \int_0^a uf(u)J_0(\xi_m u)\,du$$

Hence, the final solution of the problem is given by

$$T(r,t) = \frac{2}{a^2}\sum_{m=1}^{\infty}\frac{J_0(\xi_m r)}{J_1^2(\xi_m a)}\exp(-\alpha\xi_m^2 t)\left[\int_0^a uf(u)J_0(\xi_m u)\,du\right]$$

3.7 SOLUTION OF DIFFUSION EQUATION IN SPHERICAL COORDINATES

In this section, we shall examine the solution of diffusion or heat conduction equation in the spherical coordinate system. Let us consider the three-dimensional diffusion Eq. (3.9), and let $T = T(r,\theta,\phi,t)$. In the spherical coordinate system, Eq. (3.9) can be written as

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{3.65}$$

This equation is separated by assuming the temperature function T in the form

$$T = R(r)\,H(\theta)\,\Phi(\phi)\,\beta(t) \tag{3.66}$$

Substituting Eq. (3.66) into Eq. (3.65), we get

$$\frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2\sin\theta}\frac{1}{H}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) + \frac{1}{\Phi r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = \frac{1}{\alpha}\frac{\beta'}{\beta} = -\lambda^2 \quad \text{(say)}$$

where λ^2 is a separation constant. Thus,

$$\frac{d\beta}{dt} + \lambda^2\alpha\beta = 0$$

whose solution is

$$\beta = c_1 e^{-\alpha\lambda^2 t} \tag{3.67}$$

Also,

$$r^2 \sin^2\theta \left[\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right) + \frac{1}{Hr^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) + \lambda^2\right] = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = m^2 \quad \text{(say)}$$

which gives

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$

whose solution is

$$\Phi(\phi) = c_1 e^{im\phi} + c_2 e^{-im\phi} \tag{3.68}$$

Now, the other separated equation is

$$\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right) + \frac{1}{Hr^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) + \lambda^2 = \frac{m^2}{r^2 \sin^2\theta}$$

or

$$\frac{r^2}{R}\left(R'' + \frac{2}{r}R'\right) + \lambda^2 r^2 = \frac{m^2}{\sin^2\theta} - \frac{1}{H \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right)$$

$$= n(n+1) \quad \text{(say)}$$

On re-arrangement, this equation can be written as

$$R'' + \frac{2}{r}R' + \left\{\lambda^2 - \frac{n(n+1)}{r^2}\right\}R = 0 \tag{3.69}$$

and

$$-\frac{1}{H \sin\theta}\left(\sin\theta\frac{d^2H}{d\theta^2} + \cos\theta\frac{dH}{d\theta}\right) + \frac{m^2}{\sin^2\theta} = n(n+1)$$

or

$$\frac{d^2H}{d\theta^2} + \cot\theta\frac{dH}{d\theta} + \left\{n(n+1) - \frac{m^2}{\sin^2\theta}\right\}H = 0 \tag{3.70}$$

Let $R = (\lambda r)^{-1/2}\psi(r)$; then Eq. (3.69) becomes

$$(\lambda r)^{-1/2}\left[\psi''(r) + \frac{1}{r}\psi'(r) + \left\{\lambda^2 - \frac{(n+1/2)^2}{r^2}\right\}\psi\right] = 0$$

Since $(\lambda r) \neq 0$, we have

$$\psi''(r) + \frac{1}{r}\psi'(r) + \left\{\lambda^2 - \frac{(n+1/2)^2}{r^2}\right\}\psi(r) = 0$$

which is Bessel's differential equation of order $(n+1/2)$, whose solution is

$$\psi(r) = AJ_{n+1/2}(\lambda r) + BY_{n+1/2}(\lambda r)$$

Therefore,

$$R(r) = (\lambda r)^{-1/2}[AJ_{n+1/2}(\lambda r) + BY_{n+1/2}(\lambda r)] \tag{3.71}$$

where J_n and Y_n are Bessel functions of first and second kind, respectively. Now, Eq. (3.70) can be put in a more convenient form by introducing a new independent variable

$$\mu = \cos\theta$$

so that

$$\cot\theta = \mu/\sqrt{1-\mu^2}$$

$$\frac{dH}{d\theta} = -\sqrt{1-\mu^2}\,\frac{dH}{d\mu}$$

$$\frac{d^2H}{d\theta^2} = (1-\mu^2)\frac{d^2H}{d\mu^2} - \mu\frac{dH}{d\mu}$$

Thus, Eq. (3.70) becomes

$$(1-\mu^2)\frac{d^2H}{d\mu^2} - 2\mu\frac{dH}{d\mu} + \left[n(n+1) - \frac{m^2}{1-\mu^2}\right]H = 0 \tag{3.72}$$

which is an associated Legendre differential equation whose solution is

$$H(\theta) = A'P_n^m(\mu) + B'Q_n^m(\mu) \tag{3.73}$$

where $P_n^m(\mu)$ and $Q_n^m(\mu)$ are associated Legendre functions of degree n and of order m, of first and second kind, respectively. Hence the physically meaningful general solution of the diffusion equation in spherical geometry is of the form

$$T(r,\theta,\phi,t) = \sum_{\lambda,m,n} A_{\lambda mn}(\lambda r)^{-1/2} J_{n+1/2}(\lambda r) P_n^m(\cos\theta) e^{\pm im\phi - \alpha\lambda^2 t} \tag{3.74}$$

In this general solution, the functions $Q_n^m(\mu)$ and $(\lambda r)^{-1/2}Y_{n+1/2}(\lambda r)$ are excluded because these functions have poles at $\mu = \pm 1$ and $r = 0$ respectively.

EXAMPLE 3.12 Find the temperature in a sphere of radius a, when its surface is kept at zero temperature and its initial temperature is $f(r, \theta)$.

Solution Here, the temperature is governed by the three-dimensional heat equation in spherical polar coordinates independent of ϕ. Therefore, the task is to find the solution of PDE

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) \tag{3.75}$$

subject to

$$\text{BC}: T(a, \theta, t) = 0 \tag{3.76}$$

$$\text{IC}: T(r, \theta, 0) = f(r, \theta) \tag{3.77}$$

The general solution of Eq. (3.75), with the help of Eq. (3.74), can be written as

$$T(r, \theta, t) = \sum_{\lambda, n} A_{\lambda n} (\lambda r)^{-1/2} J_{n+1/2}(\lambda r) P_n(\cos\theta) e^{-\alpha\lambda^2 t} \tag{3.78}$$

Applying the BC (3.76), we get

$$J_{n+1/2}(\lambda a) = 0$$

This equation has infinitely many positive roots. Denoting them by ξ_i, we have

$$T(r, \theta, t) = \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} A_{ni} (\xi_i r)^{-1/2} J_{n+1/2}(\xi_i r)\, P_n(\cos\theta) \exp(-\alpha \xi_i^2 t) \tag{3.79}$$

Now, applying the IC and denoting $\cos\theta$ by μ, we get

$$f(r, \cos^{-1}(\mu)) = \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} A_{ni} (\xi_i r)^{-1/2} J_{n+1/2}(\xi_i r)\, P_n(\mu).$$

Multiplying both sides by $P_m(\mu)\, d\mu$ and integrating between the limits, –1 to 1, we obtain

$$\begin{aligned}\int_{-1}^{1} f(r, \cos^{-1}(\mu))\, P_m(\mu)\, d\mu &= \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} A_{ni} (\xi_i r)^{-1/2} J_{n+1/2}(\xi_i r) \int_{-1}^{1} P_m(\mu) P_n(\mu)\, d\mu \\ &= \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} A_{ni} (\xi_i r)^{-1/2} J_{n+1/2}(\xi_i r) \left(\frac{2}{2n+1}\right)\end{aligned}$$

or

$$\left(\frac{2n+1}{2}\right)\int_{-1}^{1} P_n(\mu)\, f(r, \cos^{-1}(\mu))\, d\mu = \sum_{i=1}^{\infty} A_{ni}(\xi_i r)^{-1/2} J_{n+1/2}(\xi_i r) \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

Now, to evaluate the constants A_{ni}, we multiply both sides of the above equation by $r^{3/2} J_{n+1/2}(\xi_j r)$ and integrate with respect to r between the limits 0 to a and use the orthogonality property of Bessel functions to get

$$\xi_i^{1/2}\left(\frac{2n+1}{2}\right)\int_0^a r^{3/2} J_{n+1/2}(\xi_j r)\, dr \left[\int_{-1}^{1} P_n(\mu)\, f(r, \cos^{-1}(\mu))\, d\mu\right]$$

$$= \sum_{i=1}^{\infty} A_{ni} \int_0^a r J_{n+1/2}(\xi_i r)\, J_{n+1/2}(\xi_j r)\, dr$$

$$= \frac{1}{2}\sum_{i=1}^{\infty} A_{ni}\, [J'_{n+1/2}(\xi_i r)]^2, \quad n = 0, 1, 2, 3, \ldots \tag{3.80}$$

Thus, Eqs. (3.79) and (3.80) together constitutes the solution for the given problem.

3.8 MAXIMUM-MINIMUM PRINCIPLE AND CONSEQUENCES

Theorem 3.1 (Maximum-minimum principle).
Let $u(x, t)$ be a continuous function and a solution of

$$u_t = \alpha u_{xx} \tag{3.81}$$

for $0 \le x \le l,\ 0 \le t \le T$, where $T > 0$ is a fixed time. Then the maximum and minimum values of u are attained either at time $t = 0$ or at the end points $x = 0$ and $x = l$ at some time in the interval $0 \le t \le T$.

Proof To start with, let us assume that the assertion is false. Let the maximum value of $u(x, t)$ for $t = 0 (0 \le x \le l)$ or for $x = 0$ or $x = l\, (0 \le t \le T)$ be denoted by M. We shall assume that the function $u(x, t)$ attains its maximum at some interior point (x_0, t_0), in the rectangle defined by $0 \le x_0 \le l,\ 0 \le t_0 \le T$, and then arrive at a contradiction. This means that

$$u(x_0, t_0) = M + \varepsilon \tag{3.82}$$

Now, we shall compare the signs in Eq. (3.81) at the point (x_0, t_0). It is well known from calculus that the necessary condition for the function $u(x, t)$ to possess maximum at (x_0, t_0) is

$$\frac{\partial u}{\partial x}(x_0, t_0) = 0, \qquad \frac{\partial^2 u}{\partial x^2}(x_0, t_0) \le 0 \tag{3.83}$$

In addition, $u(x_0, t_0)$ attains maximum for $t = t_0$, implying

$$\frac{\partial u}{\partial t}(x_0, t_0) \geq 0 \tag{3.84}$$

Thus, with the help of Eqs. (3.83) and (3.84) we observe that the signs on the left- and right-hand sides of Eq. (3.81) are different. However, we cannot claim that we have reached a contradiction, since the left- and right-hand sides can simultaneously be zero.

To complete the proof, let us consider another point (x_1, t_1) at which $\partial^2 u/\partial x^2 \leq 0$ and $\partial u/\partial t > 0$.

Now, we construct an auxiliary function

$$v(x, t) = u(x, t) + \lambda(t_0 - t) \tag{3.85}$$

where λ is a constant. Obviously, $v(x_0, t_0) = u(x_0, t_0) = M + \varepsilon$ and $\lambda(t_0 - t) \leq \lambda T$. Suppose we choose $\lambda > 0$, such that $\lambda < \varepsilon/2T$; then the maximum of $v(x, t)$ for $t = 0$ or for $x = 0$, $x = l$ cannot exceed the value $M + \varepsilon/2$. But $v(x, t)$ is a continuous function and, therefore, a point (x_1, t_1) exists at which it assumes its maximum. It implies

$$M + \varepsilon/2 \leq v(x_1, t_1) \geq v(x_0, t_0) = M + \varepsilon$$

This pair of inequalities is inconsistent and therefore contradicts the assumption that v takes on its maximum at (x_0, t_0). Therefore, the assertion that u attains its maximum either at $t = 0$ or at the end points is true.

We can establish a similar result for minimum values of $u(x, t)$. If u satisfies Eq. (3.81), $-u$ also satisfies Eq. (3.81). Hence, both maximum and minimum values are attained either initially or at the end points. Thus the proof is complete. We shall give some of the consequences of the maximum-minimum principle in the following theorems.

Theorem 3.2 (Uniqueness theorem). Given a rectangular region defined by $0 \leq x \leq l$, $0 \leq t \leq T$, and a continuous function $u(x, t)$ defined on the boundary of the rectangle satisfying the heat equation

$$u_t = \alpha u_{xx}$$

This equation possesses one and only one solution satisfying the initial and boundary conditions

$$u(x, 0) = f(x)$$

$$u(0, t) = g_1(t),\ u(l, t) = g_2(t)$$

where $f(x)$, $g_1(t)$, $g_2(t)$ are continuous on their domains of definition.

Proof Suppose there are two solutions $u_1(x, t)$, $u_2(x, t)$ satisfying the heat equation as well as the same initial and boundary conditions. Now let us consider the difference

$$v(x, t) = u_2(x, t) - u_1(x, t)$$

It is also a solution of the heat conduction equation for $0 \le x \le l,\ 0 \le t \le T$ and is continuous in x and t. Also, $v(x, t) = 0,\ 0 \le x \le l$ and $v(0, t) = v(l, t) = 0,\ 0 \le t \le T$. Hence, $v(x, t)$ satisfies the conditions required for the application of maximum-minimum principle. Thus, $v(x, t) = 0$ in the rectangular region defined by $0 \le x \le l,\ 0 \le t \le T$. It follows therefore that $u_1(x, t) = u_2(x, t)$.

Another important consequence of the maximum-minimum principle is the stability property which is stated in the following theorem without proof.

Theorem 3.3 (Stability theorem). The solution $u(x, t)$ of the Dirichlet problem

$$\begin{aligned} u_t &= \alpha u_{xx}, && 0 \le x \le l,\ 0 \le t \le T \\ u(x, 0) &= f(x), && 0 \le x \le l \\ u(0, t) &= g(t), && u(l, t) = h(t),\ 0 \le t \le T \end{aligned}$$

depends continuously on the initial and boundary conditions.

3.9 NON-LINEAR EQUATIONS (MODELS)

Today various studies of fluid behaviour are available which encompass virtually any type of phenomena of practical importance. However, there are many unresolved important problems in fluid dynamics due to the non-linear nature of the governing PDEs and due to difficulties encountered in many of the conventional, analytical and numerical techniques in solving them. In the following, we shall present few non-linear model equations to have a feel for this vast field of study.

3.9.1 Semilinear Equations

Reaction–diffusion equations that appear in the literature are frequently semilinear and are of the form $u_t = \nabla^2 u + f(u, x, t)$.

Typically, they appear as models in population dynamics, with inhomogeneous term depending on the density of local population. In chemical engineering, f varies with temperature and/or chemical concentration in a reaction like $f(u) = \lambda u^N$.

3.9.2 Quasi-linear Equations

Many problems in fluid mechanics, when formulated mathematically, give rise to quasi-linear parabolic PDEs. A simple example concerns the flow of compressible fluid through a porous medium. Let ρ denotes fluid density. Following Darcy's law, which relates the velocity $\mathbf{V}$ to the pressure p as

$$\mathbf{V} = -\left(\frac{K}{\mu}\right)\nabla p.$$

Then, the equation of conservation of mass is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

and the equation of state $p = p(\rho)$ can be combined to get

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [K(\rho) \nabla \rho]$$

where $K(\rho)$ is proportional to $\rho \dfrac{dp}{d\rho}$. The model can then be written as the porous medium equation in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho^n \nabla \rho)$$

where n is a positive constant.

3.9.3 Burger's Equation

The well-known Burger's equation is non-linear and finding its solution has been the subject of active research for many years. For simplicity, let us consider one-dimensional Burger's equation in the form

$$u_t + uu_x = vu_{xx}$$

or

$$u_t - \left(vu_x - \frac{1}{2} u^2 \right)_x = 0 \tag{3.85a}$$

which is actually the non-linear momentum equation in *fluid mechanics* without the pressure term. v is the physical viscosity. Here vu_{xx} measures dissipative term and uu_x measures convective term, while u_t is the unsteady term. Hopf (1950) and Cole (1951) gave independently the analytical solution for a model problem using a two-step Hopf–Cole transformation described by

$$u(x, t) = \psi_x$$
$$\psi = -2v \log \phi(x, t)$$

That is,

$$u = -2v \frac{\phi_x}{\phi} \tag{3.85b}$$

Thus,

$$u_t = -2v \left(\frac{\phi_{xt}}{\phi} \right) + 2v \frac{\phi_x \phi_t}{\phi^2},$$

$$u_x = -2v\left(\frac{\phi_{xx}}{\phi}\right) + 2v\left(\frac{\phi_x^2}{\phi^2}\right)$$

$$u_{xx} = -2v\left(\frac{\phi_{xxx}}{\phi}\right) + 6v\left(\frac{\phi_x \phi_{xx}}{\phi^2}\right) - 4v\left(\frac{\phi_x^3}{\phi^3}\right).$$

Inserting, these derivative expressions into Eq. (3.85a) and on simplification, we arrive at

$$\frac{\phi_x}{\phi}(v\phi_{xx} - \phi_t) - (v\phi_{xx} - \phi_t)_x = 0. \tag{3.85c}$$

Therefore, we have to solve Eq. (3.85c) to find $\phi(x, t)$, and using this result in Eq. (3.85b), we obtain an expression for $u(x, t)$ which of course satisfies Eq. (3.85a). Thus, if $\phi(x, t)$ satisfies heat conduction equation

$$\phi_t = v\phi_{xx} \tag{3.85d}$$

which also means solving trivially Eq. (3.85c). This is also called *linearised Burger's equation*. Equivalently, we may introduce the transformation:

$$\left.\begin{aligned} \psi_x &= u, \\ \psi_t &= vu_x - \frac{u^2}{2} \end{aligned}\right\} \tag{3.85e}$$

in such a way, satisfying that $\psi_{xt} = \psi_{tx}$. Then, the above transformation can be rewritten as

$$\psi_t = v\psi_{xx} - \frac{\psi_x^2}{2} \tag{3.85f}$$

Also, Eq. (3.85b) can be recast in the form

$$\phi(x, t) = e^{[-\psi(x,\, t)/2v]} \tag{3.85g}$$

Thus, knowing $\phi(x, t)$, we can find $u(x, t)$ from Eq. (3.85b). It may also be observed that Eqs. (3.85d) and (3.85f) are equivalent.

Hence, the transformation of non-linear Burger's equation into heat conduction equation, made life easy to get analytical solution to the Burger's equation.

3.9.4 Initial Value Problem for Burger's Equation

The IVP for Burger's equation can be stated as follows. Solve

$$\text{PDE: } u_t + uu_x = vu_{xx}, \quad -\infty < x < \infty,\ t > 0$$

$$\text{IC: } u(x, 0) = f(x) \tag{3.85h}$$

Under the transformation defined by Eq. (3.85b) and using (3.85g), the given IVP can be restated as a Cauchy problem, described by

$$\text{PDE: } \phi_t = v\phi_{xx},$$

$$\text{IC: } \phi(x,\ 0) = \bar{\phi}(x) = e^{\left[-\frac{1}{2\nu}\int_0^x f(\eta)d\eta\right]} \tag{3.85i}$$

Using, the standard, separation of variables, method of solution, as given in Eq. (3.28), the solution to Eq. (3.85i) is found to be

$$\phi(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \bar{\phi}(x) \exp\left[-\frac{(x-\xi)^2}{4\nu t}\right] d\xi \tag{3.85j}$$

Substituting the expression for $\bar{\phi}(x)$, the above equation can be rewritten as

$$\phi(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \exp\left[-\frac{f(\xi, x, t)}{2\nu}\right] d\xi$$

where,

$$f(\xi, x, t) = \int_0^{\xi} f(\alpha)d\alpha + \frac{(x-\xi)^2}{2t}.$$

Finally, using Eq. (3.85b), the exact solution of the IVP for Burger's equation as stated in Eq. (3.85h) is found to be

$$u(x,t) = \frac{\left[\int_{-\infty}^{\infty} \frac{(x-\xi)}{t} \exp\left\{-\frac{f(\xi, x, t)}{2\nu}\right\} d\xi\right]}{\exp\left[-\frac{f(\xi, x, t)}{2\nu}\right] d\xi} \tag{3.85k}$$

Here, the function $f(\xi,\ x,\ t)$ is known as Hopf–Cole function.

3.10 MISCELLANEOUS EXAMPLES

EXAMPLE 3.13 A homogeneous solid sphere of radius R has the initial temperature distribution $f(r)$, $0 \le r \le R$, where r is the distance measured from the centre. The surface temperature is maintained at 0°. Show that the temperature $T(r,t)$ in the sphere is the solution of

$$T_t = c^2\left(T_{rr} + \frac{2}{r}T_r\right)$$

where c^2 is a constant. Show also that the temperature in the sphere for $t > 0$ is given by

$$T(r,t) = \frac{1}{r}\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{R}r\right) \exp(-\lambda_n^2 t), \qquad \lambda_n = \frac{cn\pi}{R}$$

Solution The temperature distribution in a solid sphere is governed by the parabolic heat equation

$$T_t = c^2 \nabla^2 T$$

From the data given, T is a function of r and t alone. In view of the symmetry of the sphere, the above equation with the help of Eq. (3.65) reduces to

$$T_t = c^2 \left(T_{rr} + \frac{2}{r} T_r \right) \tag{3.86}$$

Setting $v = rT$, the given BC gives

$$v(R, t) = rT(R, t) = 0$$

while the IC gives

$$v(r, 0) = rT(r, 0) = rf(r)$$

Since T must be bounded at $r = 0$, we require

$$v(0, t) = 0$$

Now,

$$v_t = rT_t, \qquad T_r = \left(\frac{v}{r} \right)_r = \frac{v_r r - v}{r^2}$$

Similarly, finding T_{rr} and substituting into Eq. (3.86), we obtain

$$v_t = c^2 v_{rr}$$

Using the variables separable method, we may write $v(r, t) = R(r)\tau(t)$ and get

$$R(r) = A \cos kr + B \sin kr$$

$$\tau(t) = \exp(-c^2 k^2 t)$$

Thus, using the principle of superposition, we get

$$v(r, t) = \sum_{n=1}^{\infty} (A_n \cos kr + B_n \sin kr) \exp(-c^2 k^2 t)$$

Also, using $v(0, t) = 0$, we have

$$(A_n \cos kr + B_n \sin kr)|_{r=0} = 0$$

implying $A_n = 0$. Also, $v(R, t) = 0$ gives $B_n \sin kR = 0$, implying $\sin kR = 0$, as $B_n \neq 0$. Therefore,

$$kR = n\pi, \qquad k = \frac{n\pi}{R}, \qquad n = 1, 2, \ldots$$

Thus, the possible solution is

$$v(r,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{R} r\right) \exp\left(-\frac{c^2 n^2 \pi^2 t}{R^2}\right)$$

Finally, applying the IC: $v(r, 0) = rf(r)$, we get

$$rf(r) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{R} r\right)$$

which is a half-range Fourier series. Therefore,

$$B_n = \frac{2}{R} \int_0^R rf(r) \sin\left(\frac{n\pi}{R} r\right) dr$$

But $v(r, t) = rT(r, t)$. Hence, the temperature in the sphere is given by

$$T(r,t) = \frac{1}{r} \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{R} r\right) \exp\left(-\frac{c^2 n^2 \pi^2 t}{R^2}\right)$$

EXAMPLE 3.14 A circular cylinder of radius a has its surface kept at a constant temperature T_0. If the initial temperature is zero throughout the cylinder, prove that for $t > 0$.

$$T(r,t) = T_0 \left\{ 1 - \frac{2}{a} \sum_{n=1}^{\infty} \frac{J_0(\xi_n a)}{\xi_n J_1(\xi_n a)} \exp(-\xi_n^2 kt) \right\}$$

where $\pm\xi_1, \pm\xi_2, \ldots, \pm\xi_n$ are the roots of $J_0(\xi a) = 0$, and k is the thermal conductivity which is a constant.

Solution It is evident that T is a function of r and t alone and, therefore, the PDE to be solved is

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{k}\frac{\partial T}{\partial t} \tag{3.87}$$

subject to

$$\text{IC}: T(r, 0) = 0, \qquad 0 \le r < a$$

$$\text{BC}: T(a, t) = T_0, \qquad t \ge 0$$

Let

$$T(r,t) = T_0 + T_1(r,t)$$

so that

$$T_1(r, 0) = -T_0 \tag{3.88}$$

$$T_1(a, t) = 0 \tag{3.89}$$

where T_1 is the solution of Eq. (3.87). By the variables separable method we have (see Example 3.11),

$$T_1(r, t) = AJ_0(\lambda r)\exp(-\lambda^2 kt)$$

Using the BC: $T_1(a, t) = 0$, we get

$$AJ_0(\lambda a)\exp(-\lambda^2 kt) = 0$$

which gives $J_0(\lambda a) = 0$ *as* $A \neq 0$. Let $\xi_1, \xi_2, \ldots, \xi_n$, be the roots of $J_0(\lambda a) = 0$. Then the possible solution using the superposition principle is

$$T_1(r, t) = \sum_{n=1}^{\infty} A_n J_0(\xi_n r)\exp(-\xi_n^2 kt) \tag{3.90}$$

Using the IC: $T_1(r, 0) = -T_0$ into Eq. (3.90), we obtain

$$\sum_{n=1}^{\infty} A_n J_0(\xi_n r) = -T_0$$

Multiplying both sides by $rJ_0(\xi_m r)$ and integrating, we get

$$-T_0 \int_0^a rJ_0(\xi_m r)\,dr = \sum_{n=1}^{\infty} A_n \int_0^a rJ_0(\xi_m r)\,J_0(\xi_n r)\,dr$$

$$= A_m \int_0^a rJ_0^2(\xi_m r) \quad \text{if } m = n; \text{ otherwise } 0$$

$$= A_m \frac{a^2}{2} J_1^2(\xi_m a)$$

But,

$$-T_0 \int_0^a rJ_0(\xi_m r)\,dr = -T_0 \int_0^{\xi_m a} \frac{x}{\xi_m} J_0(x)\frac{dx}{\xi_m} \qquad (x = \xi_m r)$$

$$= -\frac{T_0}{\xi_m^2}\int_0^{\xi_m a} \frac{d}{dx}[xJ_1(x)]\,dx$$

$$= -\frac{T_0}{\xi_m^2}[xJ_1(x)]_0^{\xi_m a} = -\frac{aT_0}{\xi_m} J_1(\xi_m a)$$

Therefore,

$$A_m \frac{a^2}{2} J_1^2(\xi_m a) = -\frac{aT_0}{\xi_m} J_1(\xi_m a)$$

or

$$A_n = -\frac{2T_0}{a\xi_n} \frac{1}{J_1(\xi_n a)}$$

Hence, Eq. (3.90) becomes

$$T_1(r, t) = -\frac{2}{a} T_0 \sum_{n=1}^{\infty} \frac{J_0(\xi_n r)}{J_1(\xi_n a)} \frac{\exp(-\xi_n^2 kt)}{\xi_n}$$

Finally, the complete solution is found to be

$$T(r, t) = T_0 \left[1 - \frac{2}{a} \sum_{n=1}^{\infty} \frac{J_0(\xi_n r)}{J_1(\xi_n a)} \frac{\exp(-\xi_n^2 kt)}{\xi_n} \right]$$

EXAMPLE 3.15 Determine the temperature in a sphere of radius a, when its surface is maintained at zero temperature while its initial temperature is $f(r, \theta)$.

Solution Here the temperature is governed by the three-dimensional heat equation in polar coordinates independent of ϕ, which is given by

$$\frac{1}{k}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) \tag{3.91}$$

Let

$$u(r, \theta, t) = R(r)H(\theta)T(t)$$

By the variables separable method (see Section 3.7), the general solution of Eq. (3.91) is found to be

$$u(r, \theta, t) = \sum_{\lambda}\sum_{n} A_{\lambda n}(\lambda r)^{-1/2} J_{n+1/2}(\lambda r)\, P_n(\cos\theta) \exp(-k\lambda^2 t) \tag{3.92}$$

In the present problem, the boundary and initial conditions are

$$\text{BC: } u(a, \theta, t) = 0 \tag{3.93}$$

$$\text{IC: } u(r, \theta, 0) = f(r, \theta) \tag{3.94}$$

Substituting the BC (3.93) into Eq. (3.92), we get

$$J_{n+1/2}(\lambda a) = 0 \tag{3.95}$$

Let $\xi_1 a, \xi_2 a, \ldots, \xi_i a, \ldots$ be the roots of Eq. (3.95). Then the general solution can be put in the form

$$u(r,\theta,t) = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} A_{ni}(\xi_i r)^{-1/2} J_{n+1/2}(\xi_i r) P_n(\cos\theta)\exp(-k\xi_i^2 t) \tag{3.96}$$

Now using the IC, we obtain

$$f(r,\theta) = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} A_{ni}(\xi_i r)^{-1/2} J_{n+1/2}(\xi_i r) P_n(\cos\theta)$$

Multiplying both sides by $P_n(\cos\theta)\, d(\cos\theta)$ and integrating, we have

$$\int_{-1}^{1} P_n(\cos\theta) f(r,\theta)\, d(\cos\theta) = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} A_{ni}(\xi_i r)^{-1/2} J_{n+1/2}(\xi_i r)\int_{-1}^{1} P_n^2(\cos\theta)\, d(\cos\theta)$$

Using the orthogonality property of Legendre polynomials, we get

$$\int_{-1}^{1} P_n(\cos\theta) f(r,\theta)\, d(\cos\theta) = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} A_{ni}(\xi_i r)^{-1/2} J_{n+1/2}(\xi_i r)\left(\frac{2}{2n+1}\right)$$

Rearranging and multiplying both sides of the above equation by $r^{3/2} J_{n+1/2}(\xi_i r)$ and integrating between the limits 0 to a with respect to r, we get

$$\frac{2n+1}{2}\int_0^a r^{3/2} J_{n+1/2}(\xi_i r)\, dr \int_{-1}^{1} P_n(\cos\theta)\; f(r,\theta)\, d(\cos\theta) = A_{ni}\int_0^a rJ_{n+1/2}^2(\xi_i r)\xi_i^{-1/2} dr$$

$$= A_{ni}\frac{a^2}{2}\{J'_{n+1/2}(\xi_i a)\}^2$$

Therefore,

$$A_{ni} = \frac{(2n+1)\xi_i^{1/2}}{a^2\{J'_{n+1/2}(\xi_i a)\}^2}\int_0^a r^{3/2} J_{n+1/2}(\xi_i r)\, dr \int_{-1}^{1} P_n(\cos\theta)\; f(r,\theta)\, d(\cos\theta) \tag{3.97}$$

Hence, we obtain the solution to the given problem from Eq. (3.96), where A_{ni} is given by Eq. (3.97).

EXAMPLE 3.16 The heat conduction in a thin round insulated rod with heat sources present is described by the PDE

$$u_t - \alpha u_{xx} = F(x,t)/\rho c, \qquad 0 < x < l,\ t > 0 \tag{3.98}$$

subject to

$$\begin{aligned} &\text{BCs: } u(0,t) = u(l,t) = 0 \\ &\text{IC: } \ u(x,0) = f(x), \qquad 0 \le x \le l \end{aligned} \tag{3.99}$$

where ρ and c are constants and F is a continuous function of x and t. Find $u(x,t)$.

Solution It can be noted that the boundary conditions are of homogeneous type. Let us consider the homogeneous equation

$$u_t - \alpha u_{xx} = 0 \tag{3.100}$$

Setting $u(x, t) = X(x)T(t)$, we get

$$\frac{T'}{\alpha T} = \frac{X''}{X} = -\lambda^2 \quad \text{(say)} \tag{3.101}$$

which gives $X'' + \lambda^2 X = 0$. The corresponding BCs are

$$X(0) = X(l) = 0$$

The solution of Eq. (3.101) gives the desired eigenfunctions and eigenvalues, which are

$$X_n(x) = \sin \lambda_n x, \qquad \lambda_n^2 = \left(\frac{n\pi}{l}\right)^2, \qquad n \ge 1 \tag{3.102}$$

For the non-homogeneous problem (3.98), let us propose a solution of the form

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t)\, X_n(x) \tag{3.103}$$

It is clear that Eq. (3.103) satisfies the BCs (3.99). From the orthogonality of eigenfunctions, it follows that

$$T_m(t) = \frac{2}{l}\int_0^l u(x, t)\, X_m(x)\, dx$$

However,

$$T_m(0) = \frac{2}{l}\int_0^1 f(x) \sin\left(\frac{m\pi}{l} x\right) dx \tag{3.104}$$

which is an IC for $T_m(t)$. Introducing Eq. (3.103) into the governing equation (3.98), we get

$$\sum_{n=1}^{\infty} T_n' X_n - \alpha \sum_{n=1}^{\infty} T_n X_n'' = \frac{F(x, t)}{\rho c} \tag{3.105}$$

Now, we shall expand $F(x, t)/\rho c$, so that it is represented by a convergent series on $0 < x < l$, $t > 0$ in the form

$$\frac{F}{\rho c} = \sum_{n=1}^{\infty} q_n(t)\, X_n(x) \tag{3.106}$$

where

$$q_n(t) = \frac{2}{l}\int_0^l \frac{F(x,t)}{\rho c}\sin\left(\frac{n\pi}{l}x\right)dx \tag{3.107}$$

Thus, $q_n(t)$ is known. Now, Eq. (3.105), with the help of Eq. (3.101), becomes

$$\sum_{n=1}^{\infty} X_n(T_n' + \lambda_n^2\alpha T_n - q_n) = 0$$

Therefore, it follows that

$$T_n'(t) + \lambda_n^2\alpha T_n(t) = q_n(t) \tag{3.108}$$

Its solution with the help of IC (3.104) is

$$T_n(t) = T_n(0)\exp(-\lambda_n^2\alpha t) + \int_0^t \exp[\lambda_n^2\alpha(\tau - t)]q_n(\tau)\,d\tau \tag{3.109}$$

From Eqs. (3.103) and (3.109), the complete solution is found to be

$$u(x,t) = \sum_{n=1}^{\infty}\left[T_n(0)\exp(-\lambda_n^2\alpha t) + \int_0^1 \exp[\lambda_n^2\alpha(\tau - t)\,q_n(\tau)\,d\tau\right]X_n(x)$$

In the expanded form, it becomes

$$u(x,t) = \sum_{n=1}^{\infty}\left[\left\{\frac{2}{l}\int_0^l f(\xi)X_n(\xi)\,d\xi\right\}\exp(-\lambda_n^2\alpha t)\right.$$

$$\left. + \frac{2}{l}\int_0^l \exp\left\{-\lambda_n^2\alpha(\tau - t)\right\}\int_0^l \frac{F(\xi,\tau)}{\rho c}X_n(\xi)\,d\xi\,d\tau\right]X_n(x) \tag{3.110}$$

It can be verified that the series in Eq. (3.110) converges uniformly for $t > 0$. By changing the order of integration and summation in Eq. (3.110), we get

$$u(x,t) = \int_0^l\left[\sum_{n=1}^{\infty}\frac{\exp(-\lambda_n^2\alpha t)\,X_n(x)X_n(\xi)}{1/2}\right]f(\xi)\,d\xi$$

$$+ \int_0^l\int_0^l\left[\sum_{n=1}^{\infty}\frac{\exp\{-\lambda_n^2\alpha(t-\tau)\}\,X_n(x)\,X_n(\xi)}{l/2}\right]\frac{F(\xi,t)}{\rho c}\,d\xi\,d\tau$$

which can also be written in the form

$$u(x,t) = \int_0^l G(x,\xi;t)\,f(\xi)\,d\xi + \int_0^l\int_0^l G(x,\xi;t-\tau)\frac{F(\xi,\tau)}{\rho c}\,d\xi\,d\tau \tag{3.111}$$

where

$$G(x,\xi;t)=\sum_{n=1}^{\infty}\frac{\exp(-\lambda_n^2\alpha t)\,X_n(x)\,X_n(\xi)}{l/2}$$

is called Green's function. More details on Green's function are given in Chapter 5.

EXAMPLE 3.17 The temperature distribution of a homogeneous thin rod, whose surface is insulated is described by the following IBVP:

$$\text{PDE: } v_t - v_{xx} = 0,\ 0 < x < L,\ 0 < t < \infty \tag{3.112}$$

$$\text{BCS: } v(0, t) = v(L, t) = 0 \tag{3.113}$$

$$\text{IC: } v(x, 0) = f(x),\ 0 \le x \le L \tag{3.114}$$

Find its formal solution.

Solution Let us assume the solution in the form

$$v(x, t) = X(x)T(t)$$

Eq. (3.112) gives

$$XT' = X''T$$

or

$$\frac{X''}{X}=\frac{T'}{T}=-\alpha^2 \text{ (say)}$$

where α is a positive constant. Then, we have

$$X'' + \alpha^2 X = 0$$

and

$$T' + \alpha^2 T = 0$$

From the BCS

$$v(0, t) = X(0)T(t) = 0,$$

and

$$v(L, t) = X(L)T(t) = 0,$$

we obtain, $X(0) = X(L) = 0$ for arbitrary t. Thus, we have to solve the eigenvalue problem

$$X'' + \alpha^2 X = 0$$

subject to $X(0) = X(L) = 0$.

The solution of the differential equation is

$$X(x) = A \cos \alpha x + B \sin \alpha x.$$

Since $X(0) = 0$, $A = 0$. The second condition yields

$$X(L) = B \sin \alpha L = 0$$

For non-trivial solution, $B \ne 0$ and therefore we have

$$\sin \alpha L = 0, \text{ implying } \alpha = n\pi/L, \text{ for } n = 1, 2, 3, \ldots$$

Thus, the solution is obtained as

$$X_n(x) = B_n \sin \frac{n\pi x}{L}.$$

Next, we consider the equation

$$T' + \alpha^2 T = 0$$

whose solution can be written as

$$T(t) = Ce^{-\alpha^2 t}$$

or
$$T_n(t) = C_n e^{-(n\pi/L)^2 t}.$$

Hence, the non-trivial solution of the given heat equation satisfying both the boundary conditions is found to be

$$v_n(x, t) = a_n e^{-(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right) \tag{3.115}$$

where $a_n = b_n c_n$ (arbitrary constant).

To satisfy the IC, we should have

$$v(x, 0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

which holds good, if $f(x)$ is representable as Fourier Sine series with Fourier coefficients

$$a_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Hence, the required formal solution is

$$v(x, t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(\tau)\sin\frac{n\pi\tau}{L}\,d\tau\right] e^{-(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right).$$

EXERCISES

1. A conducting bar of uniform cross-section lies along the x-axis, with its ends at $x = 0$ and $x = l$. The lateral surface is insulated. There are no heat sources within the body. The ends are also insulated. The initial temperature is $lx - x^2, 0 \le x \le l$. Find the temperature distribution in the bar for $t > 0$.
2. The faces $x = 0,\ x = a$ of a finite slab are maintained at zero temperature. The initial distribution of temperature in the slab is given by $T(x, 0) = f(x), 0 \le x \le a$. Determine the temperature at subsequent times.

3. Show that the solution of the equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

satisfying the conditions:

(i) $T \to 0$ as $t \to \infty$

(ii) $T = 0$ for $x = 0$ and $x = a$ for all $t > 0$

(iii) $T = x$ when $t = 0$ and $0 < x < a$

is

$$T(x, t) = \frac{2a}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin\left(\frac{n\pi}{a}\right) x \exp[-(n\pi/a)^2 t]$$

4. Solve the equation $\dfrac{\partial T}{\partial t} = \dfrac{\partial^2 T}{\partial x^2}$ satisfying the conditions:

(i) $T = 0$ when $x = 0$ and 1 for all t

(ii) $T = \begin{cases} 2x, & 0 \le x \le 1/2 \\ 2(1-x), & \dfrac{1}{2} \le x \le 1 \text{ when } t = 0. \end{cases}$

5. Solve the diffusion equation

$$\frac{\partial \theta}{\partial t} = v\left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r}\right)$$

subject to

$$r = 0, \quad \theta \text{ is finite}, \quad t > 0$$

$$r = a,\ \theta = 0, \quad t > 0$$

$$\theta = \frac{P}{4\mu}(a^2 - r^2), \quad t = 0$$

Here, P, μ and v are constants.

6. A homogeneous solid sphere of radius R has the initial temperature distribution $f(r), 0 \le r \le R,$ where r is the distance measured from the centre. The surface temperature is maintained at 0°. Show that the temperature $T(r, t)$ in the sphere is the solution of $T_t = c^2\left(T_{rr} + \dfrac{2}{r}T_r\right)$. Show that the temperature in the sphere for $t > 0$ is given by

$$T(r, t) = \frac{1}{r} \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{R} r\right) \exp(-\lambda_n^2 t)$$

where $\lambda_n = cn\pi/R$ and c^2 is a constant.

7. If $\phi(x)$ is bounded for all real values of x, show that

$$T(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \phi(\xi) \exp[-(x-\xi)^2/(4kt)]\, d\xi$$

is a solution of $T_t = kT_{xx}$ such that $T(x, 0) = \phi(x)$.

8. An infinite homogeneous solid circular cylinder of radius a is thermally insulated to prevent heat escape. At any time t, the temperature $T(r, t)$ at a distance r from the axis of symmetry is given by the heat conduction equation with axial symmetry. At time $t = 0$, the initial temperature distribution at a distance r from the axis is known to be a function of r. Find the temperature distribution at any subsequent time.

9. Let $\bar{r} = (x, y, z)$ represent a point in three-dimensional Euclidean space R_3. Find a formal solution $u(\bar{r}, t)$ which satisfies the diffusion equation

$$u_t = \alpha \nabla^2 u, \qquad t > 0$$

and the BC: $u(\bar{r}, 0) = f(\bar{r})$, where $\bar{r} \in R_3$.

10. Solve $\dfrac{\partial \theta}{\partial t} = \dfrac{\partial^2 \theta}{\partial x^2}$, $0 \le x \le a$, $t > 0$ subject to the conditions

$$\theta(0, t) = \theta(a, t) = 0 \text{ and } \theta(x, 0) = \theta_0 \text{ (constant).}$$

(GATE-Maths, 1996)

CHAPTER 4

Hyperbolic Differential Equations

4.1 OCCURRENCE OF THE WAVE EQUATION

One of the most important and typical homogeneous hyperbolic differential equations is the wave equation. It is of the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \tag{4.1}$$

where c is the wave speed. This differential equation is used in many branches of Physics and Engineering and is seen in many situations such as transverse vibrations of a string or membrane, longitudinal vibrations in a bar, propagation of sound waves, electromagnetic waves, sea waves, elastic waves in solids, and surface waves as in earthquakes. The solution of a wave equation is called a *wave function*.

An example for inhomogeneous wave equation is

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = F \tag{4.2}$$

where F is a given function of spatial variables and time. In physical problems F represents an external driving force such as gravity force. Another related equation is

$$\frac{\partial^2 u}{\partial t^2} + 2\gamma \frac{\partial u}{\partial t} - c^2 \nabla^2 u = F \tag{4.3}$$

where γ is a real positive constant. This equation is called a wave equation with damping term, the amplitude of which decreases exponentially as t increases. In Section 4.2, we shall derive the partial differential equation describing the transverse vibration of a string.

4.2 DERIVATION OF ONE-DIMENSIONAL WAVE EQUATION

Suppose a flexible string is stretched under tension τ between two points at a distance L apart as shown in Fig. 4.1. We assume the following:

1. The motion takes place in one plane only and in this plane each particle moves in a direction perpendicular to the equilibrium position of the string.
2. The tension τ in the string is constant.

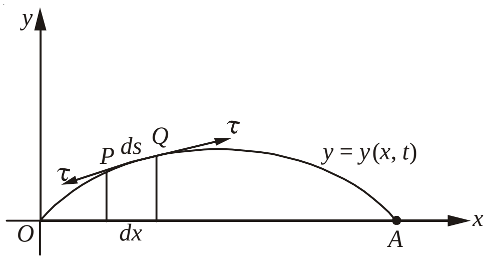

Fig. 4.1 Flexible string.

3. The gravitational force is neglected as compared with tension τ of the string.
4. The slope of the deflection curve is small.

Let the two fixed ends of the string be at the origin O and $A(L, 0)$ which lies along the x-axis in its equilibrium position. Consider an infinitesimal segment PQ of the string. Let ρ be the mass per unit length of the string. If the string is set vibrating in the xy-plane, the subsequent displacement, y from the equilibrium position of a point P of the string will be a function of x and time t, while an element of length dx is stretched into an element of length ds given by

$$ds = \sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\,dx \approx \left[1+\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right]dx$$

The elementary elongation is given by

$$dL = ds - dx = \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

while the work done by this element against the tension τ is

$$\frac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2 dx$$

Therefore, the total work done, W, for the whole string is

$$W = \frac{1}{2}\int_0^L \tau\left(\frac{\partial y}{\partial x}\right)^2 dx$$

If U is the potential energy of the string, then

$$U = W = \frac{1}{2}\int_0^L \tau\left(\frac{\partial y}{\partial x}\right)^2 dx$$

Also, the total kinetic energy T of the string is given by

$$T = \frac{1}{2}\int_0^L \rho\left(\frac{\partial y}{\partial t}\right)^2 dx$$

Using Hamilton's principle (See—Sankara Rao, 2005), we have

$$\delta\int_{t_0}^{t_1} (T - U)\, dt = 0$$

i.e.

$$\int_{t_0}^{t_1} (T - U)\, dt$$

is stationary. In other words,

$$\frac{1}{2}\int_{t_0}^{t_1}\int_0^L \left[\rho\left(\frac{\partial y}{\partial t}\right)^2 - \tau\left(\frac{\partial y}{\partial x}\right)^2\right] dx\, dt$$

is stationary, and is of the form

$$\iint F(x, t, y, y_x, y_t)\, dx\, dt$$

Noting that x and t are independent variables, from the Euler-Ostrogradsky equation, we have

$$\frac{\partial F}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial y_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y_x}\right) = 0$$

which gives

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial y}{\partial t}\right) - \frac{\partial}{\partial x}\left(\tau\frac{\partial y}{\partial x}\right) = 0$$

If the string is homogeneous, then ρ and τ are constants, in which case the governing equation representing the transverse vibration of a string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \tag{4.4}$$

where

$$c^2 = \tau/\rho \tag{4.5}$$

EXAMPLE 4.1 Consider Maxwell's equations of electromagnetic theory given by

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

$$\nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{4\pi i}{c} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$$

where **E** is an electric field, ρ is electric charge density, **H** is the magnetic field, i is the current density, and c is the velocity of light. Show that in the absence of charges, i.e., when $\rho = i = 0$, **E** and **H** satisfy the wave equations.

Solution Given

$$\text{curl}\,\mathbf{E} = \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}$$

Taking its curl again, we get

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left(-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\frac{1}{c}\nabla \times \mathbf{H}\right)$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Moreover, using the identity

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -\nabla^2\mathbf{E}$$

it follows directly that

$$\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Similarly, we can observe that the magnetic field **H** also satisfies

$$\nabla^2\mathbf{H} = \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}$$

which is also a wave equation.

4.3 SOLUTION OF ONE-DIMENSIONAL WAVE EQUATION BY CANONICAL REDUCTION

The one-dimensional wave equation is

$$u_{tt} - c^2 u_{xx} = 0 \tag{4.6}$$

Choosing the characteristic lines

$$\xi = x - ct, \quad \eta = x + ct \tag{4.7}$$

the chain rule of partial differentiation gives

$$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$$

$$u_t = u_\xi \xi_t + u_\eta \eta_t = c(u_\eta - u_\xi)$$

In the operator notation we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial t} = c\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \xi}\right)$$

Thus, we get

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)^2 u = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \tag{4.8}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) \tag{4.9}$$

Substituting Eqs. (4.8) and (4.9) into Eq. (4.6), we obtain

$$4u_{\xi\eta} = 0 \tag{4.10}$$

Integrating, we get

$$u(\xi, \eta) = \phi(\xi) + \psi(\eta),$$

where ϕ and ψ are arbitrary functions. Replacing ξ and η as defined in Eq. (4.7), we have the general solution of the wave equation (4.6) in the form

$$u(x, t) = \phi(x - ct) + \psi(x + ct) \tag{4.11}$$

The two terms in Eq. (4.11) can be interpreted as waves travelling to the right and left, respectively. Consider

$$u_1(x, t) = \phi(x - ct)$$

This represents a wave travelling to the right with speed c whose shape does not change as it travels, the initial shape being given by a known function $\phi(x)$. In fact, by setting $t = 0$ in the argument of ϕ, it can be observed that the initial wave profile is given by

$$u_1(x, 0) = \phi(x)$$

At time $t = 1/c$,

$$u_1(x, 1/c) = \phi(x - 1)$$

Let $x' = x - 1$. Then $\phi(x - 1) = \phi(x')$. That is, the same shape is retained even if the origin is shifted by one unit along the x-axis. In other words, the graph of $u_1(x, 1/c)$ is the same as the graph of the original wave profile translated one unit to the right. At $t = 2/c$, the graph of $u_1(x, 2/c)$ is the graph of the wave profile translated two units to the right. Thus, in particular, at $t = 1$, we have $u_1(x, 1) = \phi(x - c)$. Hence in one unit of time, the profile has moved c units to the right. Therefore, c is the wave speed or speed of propagation. Using similar argument, we can conclude that the equation $u_2(x, t) = \psi(x + ct)$ is also a wave profile travelling to the left with speed c along the x-axis. Hence the general solution (4.11) of one-dimensional wave equation represents the superposition of two arbitrary wave profiles, both of which are travelling with a common speed but in the opposite directions along the x-axis, while their forms remain unaltered as they travel. This situation is described in Fig. 4.2.

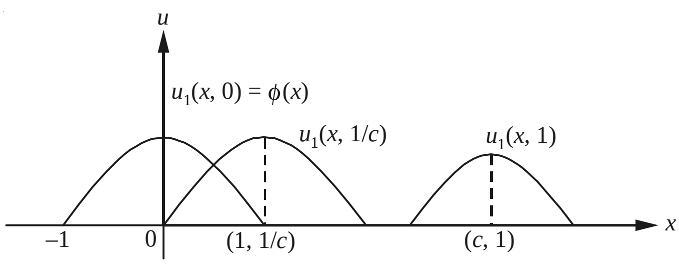

Fig. 4.2 Travelling wave profile.

Let k be an arbitrary real parameter. Consider then

$$u(x, t) = \phi[k(x - ct)] + \psi[k(x + ct)] \tag{4.12}$$

This is also a solution of the one-dimensional wave equation. Further, let $\omega = kc$. Then

$$u(x, t) = \phi[kx - \omega t] + \psi[kx + \omega t] \tag{4.13}$$

A function of the type given in Eq. (4.13) is a solution of one-dimensional wave equation iff $\omega = kc$. Therefore, waves travelling with speeds which are not the same as c cannot be described by the solution of the wave equation (4.6). Here, $(kx + \omega t)$ is called the phase for the left travelling wave. We have already noted that $x \pm ct$ are the characteristics of the one-dimensional wave equation.

EXAMPLE 4.2 Obtain the periodic solution of the wave equation in the form

$$u(x, t) = Ae^{i(kx \pm \omega t)}$$

where $i = \sqrt{-1}, k = \pm \omega/c$, A is constant; and hence define various terms involved in wave propagation.

Solution Let $u(x, t) = f(x)e^{\pm i\omega t}$ be a solution of the wave equation

$$u_{tt} = c^2 u_{xx}$$

Then

$$u_{xx} = f''(x)e^{\pm i\omega t}, \quad u_{tt} = -f(x)\omega^2 e^{\pm i\omega t}$$

Substituting into the wave equation, we get

$$f''(x) + \frac{\omega^2}{c^2} f(x) = 0$$

Its general solution is found to be

$$f(x) = c_1 \exp[i(\omega/c)x] + c_2 \exp[-i(\omega/c)x]$$

Therefore, the required solution of the wave equation is

$$u(x, t) = [c_1 \exp\{i(\omega/c)x\} + c_2 \exp\{-i(\omega/c)x\}]e^{\pm i\omega t}$$

Since $k = \pm \omega/c$, the time-dependent wave functions are of the form

$$u(x, t) = Ae^{i(kx \pm \omega t)}$$

Hence, $u(x, t) = Ae^{i(kx \pm \omega t)}$ is a solution of the wave equation, and is called a wave function. It is also called a plane harmonic wave or monochromatic wave. Here, A is called the amplitude, ω the angular or circular frequency, and k is the wave number, defined as the number of waves per unit distance. By taking the real and imaginary parts of the solution, we find the linear combination of terms of the form

$$A\cos(kx \pm \omega t), \quad A\sin(kx \pm \omega t)$$

representing periodic plane waves. For instance, consider the function $u(x, t) = A\sin(kx - \omega t)$. This is a sinusoidal wave profile moving towards the right along the x-axis with speed c. Defining the wave length λ as the length over which one full cycle is completed, we have $\lambda = 2\pi/k$, thereby implying that $k = 2\pi/\lambda$.

Suppose an observer is stationed at a fixed point x_0; then,

$$u\left(x_0, t + \frac{\lambda}{c}\right) = A\sin\left(kx_0 - \omega t - \omega\frac{\lambda}{c}\right)$$
$$= A\sin(kx_0 - \omega t - 2\pi) = A\sin(kx_0 - \omega t)$$

Thus, we have

$$u(x_0, t + \lambda/c) = u(x_0, t)$$

Hence, exactly one complete wave passes the observer in time $T = \lambda/c$, which is called the period of the wave. The reciprocal of the period is called frequency and is denoted by

$$f = 1/T$$

The function, $u = A\cos(kx - \omega t) = A\sin(\pi/2 + kx - \omega t)$, also represents a wave train except that it differs in phase by $\pi/2$ from the sinusoidal wave. Now consider the superposition of the sinusoidal waves having the same amplitude, speed, frequency, but moving in opposite directions. Thus, we have

$$u(x, t) = A\sin[k(x - ct)] + A\sin[k(x + ct)]$$
$$= 2A\sin kx\cos(kct) = 2A\cos(kct)\sin kx$$

Its amplitude factor $[2A\cos(kct)]$ varies sinusoidally with frequency ω. This situation is described as a standing wave. The points $x_n = n\pi/k$, $n = 0, \pm 1, \pm 2, \ldots$ are called nodes. No displacement takes place at a node. Therefore,

$$u(x_n, t) = 0 \quad \text{for all } t$$

The nth standing wave profile will have $(n-1)$ equally spaced nodes in a given interval as shown in Fig. 4.3.

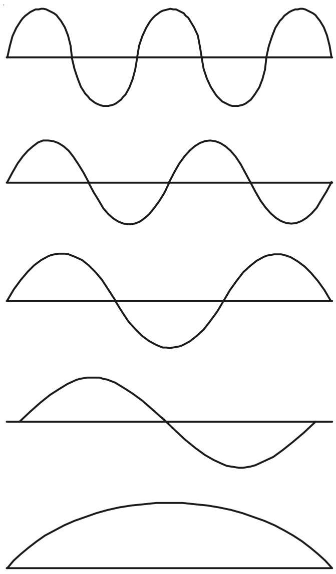

Fig. 4.3 Standing wave profiles.

4.4 THE INITIAL VALUE PROBLEM; D'ALEMBERT'S SOLUTION

Consider the initial value problem of Cauchy type described as

$$\text{PDE: } u_{tt} - c^2 u_{xx} = 0, \qquad -\infty < x < \infty, t \geq 0 \tag{4.14}$$

$$\text{ICs: } u(x, 0) = \eta(x), \qquad u_t(x, 0) = v(x) \tag{4.15}$$

where the curve on which the initial data $\eta(x)$ and $v(x)$ are prescribed is the x-axis. $\eta(x)$ and $v(x)$ are assumed to be twice continuously differentiable. Here, the string considered is of an infinite extent. Let $u(x, t)$ denote the displacement for any x and t. At $t = 0$, let the displacement and velocity of the string be prescribed. We have already noted in Section 4.3 that the general solution of the wave equation is given by

$$u(x, t) = \phi(x + ct) + \psi(x - ct) \tag{4.16}$$

where ϕ and ψ are arbitrary functions. Substituting the ICs (4.15) into Eq. (4.16), we obtain

$$\begin{aligned} \phi(x) + \psi(x) &= \eta(x) \\ c[\phi'(x) - \psi'(x)] &= v(x) \end{aligned} \tag{4.17}$$

Integrating the second equation of (4.17), we have

$$\phi(x) - \psi(x) = \frac{1}{c}\int_0^x v(\xi)\, d\xi$$

Addition and substraction of this equation with the first relation of Eqs. (4.17) yield

$$\phi(x) = \frac{\eta(x)}{2} + \frac{1}{2c}\int_0^x v(\xi)\, d\xi$$

$$\psi(x) = \frac{\eta(x)}{2} - \frac{1}{2c}\int_0^x v(\xi)\, d\xi$$

respectively. Substituting these relations for $\phi(x)$ and $\psi(x)$ into Eq. (4.16), we at once have

$$u(x, t) = \frac{1}{2}[\eta(x+ct) + \eta(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)\, d\xi \tag{4.18}$$

This is known as the D'Alembert's solution of the one-dimensional wave equation. If $v = 0$, i.e., if the string is released from rest, the required solution is

$$u(x, t) = \frac{1}{2}[\eta(x+ct) + \eta(x-ct)] \tag{4.19}$$

The D'Alembert's solution has an interesting interpretation as given in Fig. 4.4.

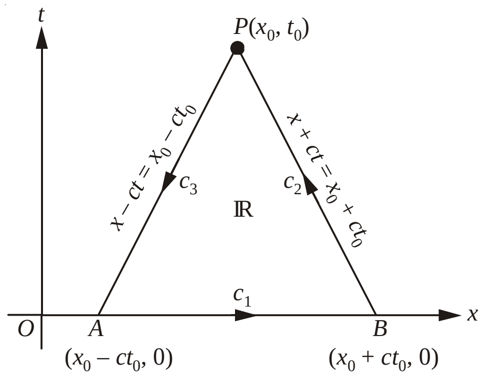

Fig. 4.4 Characteristic triangle.

Consider the xt-plane and a point $P(x_0, t_0)$. Draw two characteristics through P backwards, until they intersect the initial line, i.e., the x-axis at A and B. The equation of these two characteristics are

$$x \pm ct = x_0 \pm ct_0$$

Equation (4.18) reveals that the solution $u(x, t)$ at $P(x_0, t_0)$ can be obtained by averaging the value of η at A and B and integrating v along the x-axis between A and B. Thus, to find the solution of the wave equation at a given point P in the xt-plane, we should know the initial data on the segment AB of the initial line which is obtained by drawing the characteristics backward from P to the initial line. Here the segment AB of the initial line, on which the value of $u(x, t)$ at P depends, is called the domain of dependence of P, and the triangle PAB is called the characteristic triangle (see Fig. 4.4), which is also called the domain of determinancy of the interval.

EXAMPLE 4.3 A stretched string of finite length L is held fixed at its ends and is subjected to an initial displacement $u(x, 0) = u_0 \sin(\pi x/L)$. The string is released from this position with zero initial velocity. Find the resultant time dependent motion of the string.

Solution One of the practical applications of the theory of wave motion is the vibration of a stretched string, say, that of a musical instrument. In the present problem, let us consider a stretched string of finite length L, which is subjected to an initial disturbance. The governing equation of motion is

$$\text{PDE: } u_{tt} - c^2 u_{xx} = 0,\ 0 \le x \le L,\ t > 0 \tag{1}$$

$$\text{BCs: } u(0, t) = u(L, t) = 0 \tag{2}$$

$$\text{ICs: } u(x, 0) = u_0 \sin(\pi x/L), \tag{3}$$

$$\frac{\partial u}{\partial t}(x, 0) = 0 \tag{4}$$

In Section 4.3, we have shown the solution of the one-dimensional wave equation by canonical reduction as

$$u(x, t) = \phi(x - ct) + \psi(x + ct) \tag{5}$$

One of the known methods for solving this problem is based on trial function approach. Let us choose a trial function of the form

$$u(x, t) = A\left[\sin\frac{\pi}{L}(x + ct) + \sin\frac{\pi}{L}(x - ct)\right] \tag{6}$$

where A is an arbitrary constant. Now, we rewrite Eq. (6) as

$$u(x, t) = 2A \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{c\pi t}{L}\right) \tag{7}$$

obviously, Eq. (7) satisfies the initial condition (3) with $A = u_0/2$, while the second initial condition (4) is satisfied identically. In fact Eq. (7) also satisfies the boundary condition (2). Therefore, the final solution is found to be

$$u(x, t) = u_0 \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{c\pi t}{L}\right) \tag{8}$$

It may be noted that the trial function approach is easily adoptable if the initial condition is specified as a sin function. However, it is difficult if the initial conditions are specified as a general function such as $f(x)$. In such case, it is better to follow variables separable method as explained in Section 4.5.

EXAMPLE 4.4 Solve the following initial value problem of the wave equation (Cauchy problem), described by the inhomogeneous wave equation

$$\text{PDE: } u_{tt} - c^2 u_{xx} = f(x, t)$$

subject to the initial conditions

$$u(x, 0) = \eta(x), \qquad u_t(x, 0) = v(x)$$

Solution To make the task easy, we shall set $u = u_1 + u_2$, so that u_1 is a solution of the homogeneous wave equation subject to the general initial conditions given above. Then u_2 will be a solution of

$$\frac{\partial^2 u_2}{\partial t^2} - c^2 \frac{\partial^2 u_2}{\partial x^2} = f(x, t) \tag{4.20}$$

subject to the homogeneous ICs

$$u_2(x, 0) = 0, \quad \frac{\partial u_2}{\partial t}(x, 0) = 0 \tag{4.21}$$

To obtain the value of u at $P(x_0, t_0)$, we integrate the partial differential equation (4.20) over the region $\mathbb{R}$ as shown in Fig. 4.4, to obtain

$$\iint_{\mathbb{R}} \left(\frac{\partial^2 u_2}{\partial t^2} - c^2 \frac{\partial^2 u_2}{\partial x^2} \right) dx\, dt = \iint_{\mathbb{R}} f(x, t)\, dx\, dt$$

Using Green's theorem in a plane to the left-hand side of the above equation to replace the surface integral over $\mathbb{R}$ by a line integral around the boundary $\partial\mathbb{R}$ of $\mathbb{R}$, the above equation reduces to

$$-\iint_{\mathbb{R}} \left[\frac{\partial}{\partial x}\left(c^2 \frac{\partial u_2}{\partial x} \right) - \frac{\partial}{\partial t}\left(\frac{\partial u_2}{\partial t} \right) \right] dx\, dt = \iint_{\mathbb{R}} f(x, t)\, dx\, dt$$

and finally to

$$\int_{\partial\mathbb{R}} \left(\frac{\partial u_2}{\partial t} dx + c^2 \frac{\partial u_2}{\partial x} dt \right) = \iint_{\mathbb{R}} f(x, t)\, dx\, dt \tag{4.22}$$

Now, the boundary $\partial \mathbb{R}$ comprises three segments BP, PA and AB. Along BP, $dx/dt = -c$; along PB, $dx/dt = c$. Using these results, Eq. (4.22) becomes

$$\int_{BP} c\left(\frac{\partial u_2}{\partial t}dt + \frac{\partial u_2}{\partial x}dx\right) - \int_{PA} c\left(\frac{\partial u_2}{\partial t}dt + \frac{\partial u_2}{\partial x}dx\right)$$

$$-\int_{AB}\left(\frac{\partial u_2}{\partial t}dx + c^2\frac{\partial u_2}{\partial x}dt\right) = \iint_{\mathbb{R}} f(x,t)\,dx\,dt$$

The integrands of the first two integrals are simply the total differentials, while in the third integral, the first term vanishes on AB in view of the second IC in Eq. (4.21), and the second term also vanishes because AB is directed along the x-axis on which $dt/dx = 0$. Then we arrive at the result

$$\int_{BP} c\,du_2 - \int_{PA} c\,du_2 = \iint_{\mathbb{R}} f(x,t)\,dx\,dt$$

which can be rewritten as

$$cu_2(P) - cu_2(B) + cu_2(P) - cu_2(A) = \iint_{\mathbb{R}} f(x,t)dx\,dt \tag{4.23}$$

Using the first IC of Eq. (4.21), we get $u_2(A) = u_2(B) = 0$, and hence Eq. (4.23) becomes

$$u_2(P) = \frac{1}{2c}\iint_{\mathbb{R}} f(x,t)\,dx\,dt$$

with the help of Fig. 4.4, we deduce

$$u_2(P) = \frac{1}{2c}\int_0^{t_0}\int_{x_0-ct_0+ct}^{x_0+ct_0-ct} f(x,t)\,dx\,dt \tag{4.24}$$

Now, using the fact that $u = u_1 + u_2$, as also using Eq. (4.24) and D'Alembert's solution (4.18), the required solution of the inhomogeneous wave equation subject to the given ICs is given by

$$u(x,t) = \frac{1}{2}\{\eta(x+ct) + \eta(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} v(\xi)\,d\xi$$

$$+\frac{1}{2}\int_0^{t_0}\int_{x_0-ct_0+ct}^{x_0+ct_0-ct} f(x,t)\,dx\,dt \tag{4.25}$$

This solution is known as the Riemann-Volterra solution.

4.5 VIBRATING STRING—VARIABLES SEPARABLE SOLUTION

Following Tychonov and Samarski, it is known that transverse vibration of a string is normally generated in musical instruments. We distinguish the string instruments depending on whether the string is plucked as in the case of guitar or struck as in the case of harmonium or piano. In the case of strings which are struck we give a fixed initial velocity but does not undergo any initial displacement. In the case of plucked instruments, the strings vibrate from a fixed initial displacement without any initial velocity. The vibrations of stretched strings of musical instruments, vocal cards, power transmission cables, guy wires for antennae structures, etc. can be examined by considering the basic form of wave equations as discussed in Sections 4.3 and 4.4.

Let a thin homogeneous string which is perfectly flexible under uniform tension lie in its equilibrium position along the x-axis. The ends of the string are fixed at $x=0$ and $x=L$. The string is pulled aside a short distance and released. If no external forces are present which correspond to the case of free vibrations, the subsequent motion of the string is described by the solution $u(x,t)$ of the following problem:

$$\text{PDE: } u_{tt} - c^2 u_{xx} = 0, \qquad 0 \le x \le L,\ t > 0 \tag{4.26}$$

$$\text{BCs: } u(0,t) = 0, \qquad t > 0$$

$$u(L,t) = 0, \qquad t > 0 \tag{4.27}$$

$$\text{ICs: } u(x,0) = f(x), \qquad u_t(x,0) = g(x) \tag{4.28}$$

To obtain the variables separable solution, we assume

$$u(x,t) = X(x)T(t) \tag{4.29}$$

and substituting into Eq. (4.26), we obtain

$$X\frac{d^2T}{dt^2} = c^2 T\frac{d^2X}{dx^2}$$

i.e.,

$$\frac{d^2X/dx^2}{X} = \frac{d^2T/dt^2}{c^2T} = k \text{ (a separation constant)}$$

Case I When $k > 0$, we have $k = \lambda^2$. Then

$$\frac{d^2X}{dx^2} - \lambda^2 X = 0$$

$$\frac{d^2T}{dt^2} - c^2\lambda^2 T = 0$$

Their solution can be put in the form

$$X = c_1 e^{\lambda x} + c_2 e^{-\lambda x} \tag{4.30}$$

$$T = c_3 e^{c\lambda t} + c_4 e^{-c\lambda t} \tag{4.31}$$

Therefore,

$$u(x,\ t) = (c_1 e^{\lambda x} + c_2 e^{-\lambda x})(c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}) \tag{4.32}$$

Now, use the BCs:

$$u(0,t) = 0 = (c_1 + c_2)(c_3 e^{c\lambda t} + c_4 e^{-c\lambda t}) \tag{4.33}$$

which imply that $c_1 + c_2 = 0$. Also, $u(L,t) = 0$ gives

$$c_1 e^{-\lambda L} + c_2 e^{-\lambda L} = 0 \tag{4.34}$$

Equations (4.33) and (4.34) possess a non-trivial solution iff

$$\begin{vmatrix} 1 & 1 \\ e^{\lambda L} & e^{-\lambda L} \end{vmatrix} = e^{-\lambda L} - e^{\lambda L} = 0$$

or

$$1 - e^{2\lambda L} = 0 \text{ implying } e^{2\lambda L} = 1 \quad \text{or} \quad \lambda L = 0$$

This implies that $\lambda = 0$, since L cannot be zero, which is against the assumption as in Case I. Hence, this solution is not acceptable.

Case II Let $k = 0$. Then we have

$$\frac{d^2 X}{dx^2} = 0, \qquad \frac{d^2 T}{dt^2} = 0$$

Their solutions are found to be

$$X = Ax + B, \qquad T = ct + D$$

Therefore, the rquired solution of the PDE (4.26) is

$$u(x,t) = (Ax + B)(ct + D)$$

Using the BCs, we have

$$u(0,\ t) = 0 = B(ct + D), \quad \text{implying } B = 0$$

$$u(L,\ t) = 0 = AL(ct + D), \text{ implying } A = 0$$

Hence, only a trivial solution is possible. Since we are looking for a non-trivial solution, consider the following case.

Case III When $k < 0$, say $k = -\lambda^2$, the differential equations are

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0, \quad \frac{d^2T}{dt^2} + c^2\lambda^2 T = 0$$

Their general solutions give

$$u(x, t) = (c_1 \cos \lambda x + c_2 \sin \lambda x)(c_3 \cos c\lambda t + c_4 \sin c\lambda t) \tag{4.35}$$

Using the BC: $u(0, t) = 0$ we obtain $c_1 = 0$. Also, using the BC: $u(L, t) = 0$, we get $\sin \lambda L = 0$, implying that $\lambda_n = n\pi/L$, $n = 1, 2, \ldots$, which are the eigenvalues. Hence the possible solution is

$$u_n(x, t) = \sin \frac{n\pi x}{L}\left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L}\right), \quad n = 1, 2, \ldots \tag{4.36}$$

Using the superposition principle, we have

$$u(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L}\left(A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L}\right) \tag{4.37}$$

The initial conditions give

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$$

which is a half-range Fourier sine series, where

$$A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\,dx \tag{4.38}$$

Also,

$$u_t(x, 0) = g(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}\left(\frac{n\pi}{L}c\right)$$

which is also a half-range sine series, where

$$B_n = \frac{2}{n\pi c}\int_0^L g(x) \sin \frac{n\pi x}{L}\,dx \tag{4.39}$$

Hence the required physically meaningful solution is obtained from Eq. (4.37), where A_n and B_n are given by Eqs. (4.38) and (4.39). $u_n(x, t)$ given by Eq. (4.36) are called normal modes of vibration and $n\pi c/L = \omega_n, n = 1, 2, \ldots$ are called normal frequencies.

The following comments may be noted:

(i) The displaced form of the stretched string defined by Eq. (4.36) is referred to as the nth eigenfunction or the nth normal mode of vibration.

(ii) The period of the nth normal mode is $2L/nC$, which means $nC/2L$ cycles per second, called its frequency.

(iii) The frequency can be expressed as

$$f = \frac{n}{2L}\left(\frac{\tau}{\rho}\right)^{1/2}$$

Thus, the frequency can be increased either by reducing L or by increasing the tension τ.

(iv) For a given L, τ and ρ, the first normal mode $n = 1$, vibrates with the lowest frequency

$$f = \sqrt{\frac{\tau}{4L^2\rho}} = \frac{C}{2L},$$

is called the *fundamental frequency*.

(v) If the stretched string can be made to vibrate in a higher normal mode, the frequency is increased by an integer multiple. The deflected configuration of the stretched string corresponding to the given normal mode at a specified time $t = t^*$, can be obtained from Eq. (4.36). The deflected shapes corresponding to the first three normal modes and the associated frequencies are depicted in Fig. 4.4.

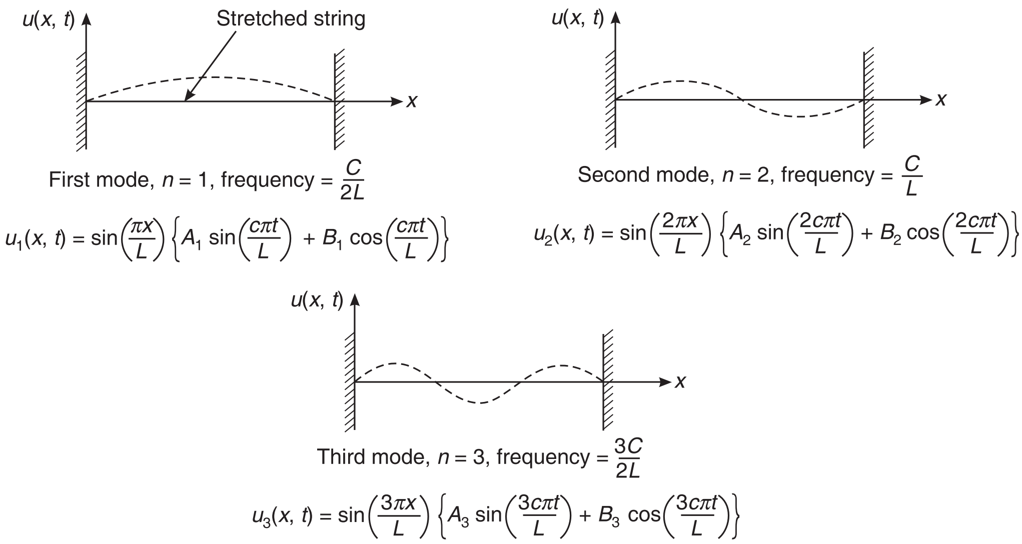

Fig. 4.4(a) Normal modes of a vibrating stretched string.

EXAMPLE 4.5 Obtain the solution of the wave equation

$$u_{tt} = c^2 u_{xx}$$

under the following conditions:

(i) $u(0, t) = u(2, t) = 0$

(ii) $u(x, 0) = \sin^3 \pi x/2$

(iii) $u_t(x, 0) = 0.$

Solution We have noted in Example 4.4 that the physically acceptable solution of the wave equation is given by Eq. (4.35), and is of the form

$$u(x, t) = (c_1 \cos \lambda x + c_2 \sin \lambda x)\,[c_3 \cos (c\lambda t) + c_4 \sin (c\lambda t)]$$

Using the condition $u(0, t) = 0$, we obtain $c_1 = 0$. Also, condition (iii) implies $c_4 = 0$. The condition $u(2, t) = 0$ gives

$$\sin 2\lambda = 0,$$

implying that

$$\lambda = n\pi/2, \qquad n = 1, 2, \ldots$$

Thus, the possible solution is

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{2} \cos \frac{n\pi ct}{2} \tag{4.40}$$

Finally, using condition (ii), we obtain

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{2} = \sin^3 \frac{\pi x}{2} = \frac{3}{4} \sin \frac{\pi x}{2} - \frac{1}{4} \sin \frac{3\pi x}{2}$$

which gives $A_1 = 3/4$, $A_3 = -1/4$, while all other A_n's are zero. Hence, the required solution is

$$u(x, t) = \frac{3}{4} \sin \frac{\pi x}{2} \cos \frac{\pi ct}{2} - \frac{1}{4} \sin \frac{3\pi x}{2} \cos \frac{3\pi ct}{2}$$

EXAMPLE 4.6 Prove that the total energy of a string, which is fixed at the points $x = 0$, $x = L$ and executing small transverse vibrations, is given by

$$\frac{1}{2} T \int_0^L \left[\left(\frac{\partial y}{\partial x} \right)^2 + \frac{1}{c^2} \left(\frac{\partial y}{\partial t} \right)^2 \right] dx$$

where $c^2 = T/\rho$, ρ is the uniform linear density and T is the tension. Show also that if $y = f(x - ct), 0 \le x \le L,$ then the energy of the wave is equally divided between potential energy and kinetic energy.

Solution The kinetic energy (KE) of an element dx of the string executing small transverse vibrations is given by (see Fig. 4.5)

$$\frac{1}{2}(\rho\, dx)\left(\frac{\partial y}{\partial t}\right)^2$$

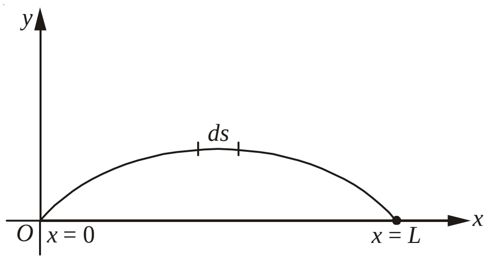

Fig. 4.5 Vibrating string.

Therefore,

$$\text{Total KE} = \frac{T}{2}\int_0^L \frac{1}{c^2}\left(\frac{\partial y}{\partial t}\right)^2 dx \tag{4.41}$$

However, $ds^2 = dx^2 + dy^2$, which gives

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \approx \left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right] dx$$

Hence, the stretch in the string is given by

$$ds - dx = \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx$$

Now, the potential energy (PE) of this element is given by

$$\text{PE} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 dx$$

Therefore,

$$\text{Total PE} = \frac{T}{2}\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx \tag{4.42}$$

When added, Eqs. (4.41) and (4.42) will yield the required total energy of the string. If $y = f(x - ct)$, then

$$\frac{\partial y}{\partial t} = -cf'(x - ct), \quad \frac{\partial y}{\partial x} = f'(x - ct)$$

$$\left(\frac{\partial y}{\partial t}\right)^2 = c^2 (f')^2$$

From Eq. (4.41),

$$\text{Total KE} = \frac{1}{2} T \int_0^L (f')^2 \, dx$$

From Eq. (4.42),

$$\text{Total PE} = \frac{1}{2} T \int_0^L (f')^2 \, dx$$

which clearly demonstrates that the total KE = total PE.

EXAMPLE 4.7 A string of length L is released from rest in the position $y = f(x)$. Show that the total energy of the string is

$$\frac{\pi^2 T}{4L} \sum_{n=1}^{\infty} s^2 k_s^2$$

where

$$k_s = \frac{2}{L} \int_0^L f(x) \sin(s\pi x/L) \, dx$$

T-tension in the string

If the mid-point of a string is pulled aside through a small distance and then released, show that in the subsequent motion the fundamental mode contributes $8/\pi^2$ of the total energy.

Solution If $f(x)$ can be expressed in Fourier series, then

$$f(x) = (a_0/2) + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Here, $(a_1 \cos x + b_1 \sin x)$ is called the fundamental mode. Following the variables separable method and using the superposition principle, the general solution of the wave equation is

$$y(x, t) = \sum_{n=1}^{\infty} k_n \sin \frac{n\pi x}{L} \cos \frac{cn\pi t}{L} \tag{4.43}$$

where

$$k_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

and the total energy is obtained from

$$E = \frac{T}{2}\int_0^L \left[\left(\frac{\partial y}{\partial x}\right)^2 + \frac{1}{c^2}\left(\frac{\partial y}{\partial t}\right)^2\right]dx \tag{4.44}$$

From Eq. (4.43),

$$\frac{\partial y}{\partial x} = \sum_{n=1}^{\infty} k_n \frac{n\pi}{L}\cos\frac{n\pi x}{L}\cos\frac{cn\pi t}{L}$$

$$\frac{\partial y}{\partial t} = -\sum_{n=1}^{\infty} k_n \frac{cn\pi}{L}\sin\frac{n\pi x}{L}\sin\frac{cn\pi t}{L}$$

Using the standard integrals

$$\int_0^{2\pi} \sin mx \sin nx\,dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

$$\int_0^{2\pi} \cos mx \cos nx\,dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

We have

$$\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 dx = \frac{\pi^2}{L^2}\int_0^L \sum k_n^2 n^2 \cos^2\frac{n\pi x}{L}\cos^2\frac{cn\pi t}{L}\,dx$$

$$= \frac{\pi^2}{2L}\sum k_n^2 n^2 \cos^2\frac{cn\pi t}{L} \tag{4.45}$$

Also,

$$\int_0^L \frac{1}{c^2}\left(\frac{\partial y}{\partial t}\right)^2 dx = \frac{\pi^2}{L^2}\sum k_n^2 n^2 \sin^2\frac{cn\pi t}{L}\left(\frac{L}{2}\right)$$

$$= \frac{\pi^2}{2L}\sum k_n^2 n^2 \sin^2\frac{cn\pi t}{L} \tag{4.46}$$

Substituting Eqs. (4.45) and (4.46) into Eq. (4.44), we obtain

$$E = \frac{T}{2}\frac{\pi^2}{2L}\sum_{n=1}^{\infty} n^2 k_n^2 \tag{4.47}$$

Now, the transverse motion of the string is described by the equation

$$y(x, t) = \sum_{n=1}^{\infty} k_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$$

where

$$k_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

But the equation of the line OP is (see Fig. 4.6)

$$y = \frac{2\varepsilon}{L}x, \quad 0 \le x \le L/2$$

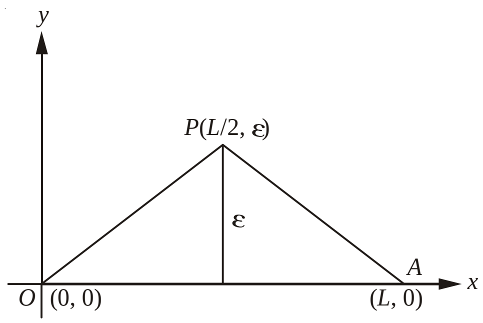

Fig. 4.6 An illustration of Example 4.7.

while the equation for the line of PA is

$$y = -\frac{2\varepsilon}{L}(x - L), \quad L/2 \le x \le L$$

Therefore,

$$k_n = \frac{2}{L}\left[\int_0^{L/2}\frac{2\varepsilon}{L}x\sin\frac{n\pi x}{L}dx + \int_{L/2}^{L} -\frac{2\varepsilon}{L}(x-L)\sin\frac{n\pi x}{L}dx\right]$$

Integration by parts yields

$$k_n = \frac{2}{L}\left[\frac{2\varepsilon}{L}\frac{L^2}{n^2\pi^2} + \frac{2\varepsilon}{L}\frac{L^2}{n^2\pi^2}\right]\sin\frac{n\pi}{2}$$

$$= \frac{8\varepsilon}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad \text{for } n \text{ odd} \tag{4.48}$$

Substituting this value of k_n into Eq. (4.47), we obtain

$$E = \frac{\pi^2 T}{4L} \sum_{n \text{ odd}} n^2 \frac{64\varepsilon^2}{\pi^4} \frac{1}{n^4} = \frac{16T\varepsilon^2}{L\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2}$$

but we know that

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$$

Hence,

$$\text{Total energy, } E = \frac{16T\varepsilon^2}{L\pi^2}\left(\frac{\pi^2}{8}\right)$$

while the total energy due to the fundamental mode is $16T\varepsilon^2/L\pi^2$. Hence the result.

4.6 FORCED VIBRATIONS—SOLUTION OF NON-HOMOGENEOUS EQUATION

Consider the problems of forced vibrations of a finite string due to an external driving force. If we assume that the string is released from rest, from its equilibrium position, the resulting motion of the string is governed by

$$\text{PDE: } u_{tt} - c^2 u_{xx} = F(x, t) \qquad 0 \le x \le L, \ t \ge 0 \tag{4.49}$$

$$\text{BCs: } u(0, t) = u(L, t) = 0, \qquad t \ge 0 \tag{4.50}$$

$$\text{ICs: } u(x, 0) = u_t(x, 0) = 0, \qquad 0 \le x \le L \tag{4.51}$$

Here, $F(x, t)$ is the external driving force. To obtain the solution of the above problem, we proceed as follows: Taking the solution of vibrating string in the absence of applied external forces as a guideline, we assume the solution to this case to be

$$u(x, t) = \sum_{n=1}^{\infty} \phi_n(t) \sin\frac{n\pi x}{L} \tag{4.52}$$

It can be seen easily that the BCs are satisfied. The function $u(x, t)$ defined by Eq. (4.52) also satisfies the ICs (4.51), provided.

$$\phi_n(0) = \phi_n'(0) = 0, \qquad n = 1, 2, \ldots \tag{4.53}$$

Substituting the assumed solution (4.52) into the governing PDE (4.49), we obtain

$$\sum_{n=1}^{\infty} \left[\ddot{\phi}_n(t) + \frac{n^2\pi^2}{L^2} c^2 \phi_n(t)\right] \sin\frac{n\pi x}{L} = F(x, t)$$

or

$$\sum_{n=1}^{\infty}\left[\ddot{\phi}_n(t)+\omega_n^2\,\phi_n(t)\right]\sin\frac{n\pi x}{L}=F(x,\,t) \tag{4.54}$$

where

$$\omega_n=\frac{n\pi c}{L} \tag{4.55}$$

and the dots over ϕ denote differentiation with respect to t. Multiplying Eq. (4.54) by $\sin k\pi x/L$ and integrating with respect to x from $x=0$ to $x=L$ and interchanging the order of summation and integration, we get

$$\sum_{n=1}^{\infty}[\ddot{\phi}_n(t)+\omega_n^2\phi_n(t)]\int_0^L\sin\frac{n\pi x}{L}\sin\frac{k\pi x}{L}\,dx=\int_0^L F(x,\,t)\sin\frac{k\pi x}{L}\,dx=\bar{F}_k(t)$$

From the orthogonality property of the function $\sin(n\pi x/L)$, we have

$$[\ddot{\phi}_k(t)+\omega_k^2\,\phi_k(t)]\int_0^L\sin^2\frac{k\pi x}{L}\,dx=\bar{F}_k(t)$$

or

$$[\ddot{\phi}_k(t)+\omega_k^2\phi_k(t)]=\frac{2}{L}\bar{F}_k(t),\quad k=1,2,\ldots \tag{4.56}$$

This is a linear second order ODE which, for instance, can be solved by using the method of variation of parameters. Thus, we solve

$$\ddot{\phi}_k(t)+\omega_k^2\,\phi_k(t)=\bar{F}_k(t) \tag{4.57}$$

subject to

$$\phi_k(0)=\dot{\phi}_k(0)=0$$

where

$$\bar{F}_k(t)=\frac{2}{L}\int_0^L F(x,\,t)\sin\frac{k\pi x}{L}\,dx$$

The complementary function for the homogeneous part is $A\cos\omega_k t+B\sin\omega_k t$. Taking A and B as functinos of t, let

$$\phi_k(t)=A(t)\cos\omega_k t+B(t)\sin\omega_k t$$

$$\dot{\phi}_k(t)=\dot{A}\cos\,\omega_k t+\dot{B}\sin\,\omega_k t-A\omega_k\sin\omega_k t+B\omega_k\cos\omega_k t$$

We choose A and B such that

$$\dot{A}\cos\omega_k t + \dot{B}\sin\omega_k t = 0 \tag{4.58}$$

Therefore,

$$\ddot{\phi}_k(t) = A\omega_k^2 \cos\omega_k t - B\omega_k^2 \sin\omega_k t - \dot{A}\omega_k \sin\omega_k t + \dot{B}\omega_k \cos\omega_k t$$

Substituting these expressions into Eq. (4.57), we get

$$\omega_k(\dot{B}\cos\omega_k t - \dot{A}\sin\omega_k t) = \bar{F}_k(t) \tag{4.59}$$

Solving Eqs. (4.58) and (4.59) for $\dot{A}$ and $\dot{B}$, we obtain

$$\dot{A}(t) = -\frac{\bar{F}_k(t)\sin\omega_k t}{\omega_k}$$

$$\dot{B}(t) = \frac{\bar{F}_k(t)\cos\omega_k t}{\omega_k}$$

Integrating, we get

$$A = -\frac{1}{\omega_k}\int_0^t \bar{F}_k(\xi)\sin\omega_k\xi\, d\xi$$

$$B = \frac{1}{\omega_k}\int_0^t \bar{F}_k(\xi)\cos\omega_k\xi\, d\xi$$

Thus,

$$\phi = \frac{1}{\omega_k}\int_0^t \bar{F}_k(\xi)\sin[\omega_k(t-\xi)]\, d\xi \tag{4.60}$$

It can be verified easily that zero ICs are also satisfied. Hence the formal solution to the problem described by Eqs. (4.49) to (4.51), using the superposition principle, is

$$u(x,\ t) = \sum_{n=1}^{\infty}\left\{\frac{1}{\omega_n}\int_0^t \bar{F}_n(\xi)\sin[\omega_n(t-\xi)]\,d\xi\right\}\sin\frac{n\pi x}{L} \tag{4.61}$$

Thus, if u_1 is a solution of the problem defined by Eqs. (4.26) to (4.28) and if u_2 is a solution of the problem described by Eqs. (4.49)–(4.51), then $(u_1 + u_2)$ is a solution of the IBVP described by

$$\text{PDE: } u_{tt} - c^2 u_{xx} = F(x,t), \qquad 0 \le x \le L, t \ge 0 \tag{4.62}$$

$$\text{BCs: } u(0,t) = u(L,t) = 0, \qquad t \ge 0 \tag{4.63}$$

$$\text{ICs: } u(x,0) = f(x), \qquad u_t(x,0) = g(x) \tag{4.64}$$

Hence, the solution of this nonhomogeneous problem is found to be

$$u(x, t) = \sum_{n=1}^{\infty} (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin \frac{n\pi x}{L}$$

$$+ \sum_{n=1}^{\infty} \left[\frac{1}{\omega_n} \int_0^t \bar{F}_n(\xi) \sin [\omega_n (t - \xi)]\, d\xi \right] \sin \frac{n\pi x}{L} \tag{4.65}$$

This solution may be termed as formal solution, because it has not been proved that the series actually converages and represents a function which satisfies all the conditions of the given physical problem.

4.7 BOUNDARY AND INITIAL VALUE PROBLEMS FOR TWO-DIMENSIONAL WAVE EQUATIONS—METHOD OF EIGENFUNCTION

Let $\mathbb{R}$ be a region in the *xy*-plane bounded by a simple closed curve $\partial \mathbb{R}$. Let $\bar{\mathbb{R}} = \mathbb{R}\, U \partial \mathbb{R}$. Consider the problem described by

$$\text{PDE:} \quad u_{tt} - c^2 \nabla^2 u = F(x, y, t), \qquad x, y \in \mathbb{R},\ t \geq 0 \tag{4.66}$$

$$\text{BCs:} \quad B(u) = 0 \text{ on } \partial \mathbb{R}, \qquad t \geq 0 \tag{4.67}$$

$$\text{ICs:} \quad u(x, y, 0) = f(x, y) \quad \text{in } \bar{\mathbb{R}}$$

$$u_t(x, y, 0) = g(x, y) \quad \text{in } \bar{\mathbb{R}} \tag{4.68}$$

where $B(u) = 0$ stands for any one of the following boundary conditions:

(i) $u = 0$ on $\partial \mathbb{R}$ (Dirichlet condition)

(ii) $\dfrac{\partial u}{\partial n} = 0$ on $\partial \mathbb{R}$ (Neumann condition)

(iii) $u = \dfrac{\partial u}{\partial n} = 0$ on $\partial \mathbb{R}$ (Robin/Mixed condition)

Before we discuss the method of eigenfunctions, it is appropriate to introduce Helmholtz equation or the space form of the wave equation. The wave equation in three dimensions may be written in vectorial form as

$$u_{tt} = c^2 \nabla^2 u$$

By the variables separable method, we assume the solution in the form

$$u(x, y, z\, t) = \phi(x, y, z)\, T(t)$$

Substituting into the above wave equation, we obtain

$$\phi T'' = c^2 T \nabla^2 \phi$$

which gives

$$\frac{T''}{c^2 T} = \frac{\nabla^2 \phi}{\phi} = -\lambda \text{ (a separation constant)}$$

thereby implying

$$T'' + \lambda c^2 T = 0 \tag{4.69}$$

$$\nabla^2 \phi + \lambda \phi = 0 \tag{4.70}$$

Equation (4.70) is the space form of the wave equation or Helmholtz equation. Of course, $\phi = 0$ is the trivial solution Eq. (4.70). But a nontrivial solution ϕ exists only for certain values of $\{\lambda_n\}$, called eigenvalues and the corresponding solution $\{\phi_n\}$, are the eigenfunctions. Corresponding to each eigenvalue λ_n, there exists at least one real-valued twice continuously differentiable function ϕ_n such that

$$\nabla^2 \phi_n + \lambda_n \phi_n = 0 \text{ in } \mathbb{R}$$
$$\phi_n = 0 \text{ on } \partial\mathbb{R}$$

It may be noted that the sequence of eigenfunctions (ϕ_n) satisfies the orthogonality property

$$\iint_{\mathbb{R}} \phi_n \phi_m \, dA = 0 \text{ for all } n \neq m$$

As in the case of one-dimensional wave equation, each continuously differentiable function in $\mathbb{R}$, which vanishes on $\partial\mathbb{R}$, can have a Fourier series expansion relative to the orthogonal set $\{\phi_n\}$. Thus the solution to the proposed problem can be written as

$$u(x, y, t) = \sum_{n=1}^{\infty} C_n(t) \phi_n(x, y) \tag{4.71}$$

where $C_n(t)$ has to be found out. Substitution of the Fourier series into the PDE (4.66) yields

$$\sum_{n=1}^{\infty} [\ddot{C}_n(t)\phi_n(x, y) - C^2 C_n(t)\nabla^2\phi_n(x, y)] = F(x, y, t)$$

But

$$\nabla^2 \phi_n = -\lambda_n \phi_n$$

Therefore,

$$\sum_{n=1}^{\infty} [\ddot{C}_n(t) + \omega_n^2 C_n(t)]\,\phi_n = F(x, y, t) \tag{4.72}$$

where

$$\omega_n^2 = C^2 \lambda_n, \quad n = 1, 2, \ldots \tag{4.73}$$

Multiplying both sides of Eq. (4.72) by ϕ_m and integrating over the ragion $\mathbb{R}$ and interchaning the order of integration and summation, we obtain

$$\sum_{n=1}^{\infty} [\ddot{C}_n(t) + \omega_n^2 C_n(t)] \iint_{\mathbb{R}} \phi_n(x, y)\phi_m(x, y)\,dA = \iint_{\mathbb{R}} F\phi_m\, dA$$

Using the orthogonality property, this equation can be reduced to

$$\ddot{C}_m(t) + \omega^2 C_m(t) = F_m(t) \tag{4.74}$$

where

$$F_m(t) = \frac{1}{\|\phi_m\|^2} \iint_{\mathbb{R}} F(x, y, t)\phi_m(x, y)\,dA \tag{4.75}$$

and

$$\|\phi_m\|^2 = \iint_{\mathbb{R}} |\phi_m|^2 dA \tag{4.76}$$

The series (4.71) satisfies the ICs (4.68) is

$$\sum C_n(0)\phi_n(x, y) = f(x, y)$$

$$\sum \dot{C}_n(0)\phi_n(x, y) = g(x, y)$$

In order to determine $C_n(0)$ and $\dot{C}_n(0)$, we multiply both sides of the above two equations by ϕ_m and integrate over $\mathbb{R}$ and use the orthogonality property to get

$$C_m(0) = \frac{1}{\|\phi_m\|^2} \iint_{\mathbb{R}} f(x, y)\phi_m(x, y)\,dA \tag{4.77}$$

$$\dot{C}_m(0) = \frac{1}{\|\phi_m\|^2} \iint_{\mathbb{R}} g(x, y)\phi_m(x, y)\,dA$$

Using the method of variation of parameters, the general solution of Eq. (4.74) is given by

$$C_m(t) = A_m \cos \omega_m t + B_m \sin \omega_m t + \frac{1}{\omega_n} \int_0^t F_m(\xi) \sin \omega_m (t - \xi)\, d\xi$$

Using Eq. (4.77), we obtain

$$A_m = \frac{1}{\|\phi_m\|^2} \iint_{\mathbb{R}} f(x, y) \phi_m(x, y)\, dA \tag{4.78}$$

$$B_m = \frac{1}{\omega_m \|\phi_m\|^2} \iint_{\mathbb{R}} g(x, y) \phi_m(x, y)\, dA, \quad m = 1, 2, \ldots$$

Hence, the formal series solution of the general problem represented by Eqs. (4.66)–(4.68) is

$$u(x, y, t) = \sum_{n=1}^{\infty} (A_n \cos \omega_n t + B_n \sin \omega_n t) \phi_n(x, y) + \sum_{n=1}^{\infty} \frac{1}{\omega_n} \left[\int_0^t F_n(\xi) \sin \{\omega_n (t - \xi)\}\, d\xi \right] \phi_n(x, y) \tag{4.79}$$

4.8 PERIODIC SOLUTION OF ONE-DIMENSIONAL WAVE EQUATION IN CYLINDRICAL COORDINATES

In cylindrical coordinates with u depending only on r, the one-dimensional wave equation assumes the following form:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \tag{4.80}$$

If we are looking for a periodic solution in time, we set

$$u = F(r)\, e^{i\omega t} \tag{4.81}$$

Then

$$\frac{\partial u}{\partial r} = F'(r) e^{i\omega t}, \quad \frac{\partial^2 u}{\partial t^2} = -\omega^2 F(r)\, e^{i\omega t}$$

Inserting these expressions, Eq. (4.80) reduces to

$$\frac{1}{r} \frac{\partial}{\partial r} [r F'(r) e^{i\omega t}] = -\frac{\omega^2}{c^2} F(r)\, e^{i\omega t}$$

or

$$F''(r) + \frac{F'(r)}{r} + \frac{\omega^2}{c^2} F(r) = 0 \tag{4.82}$$

which has the form of Bessel's equation and hence its solution can at once be written as

$$F = AJ_0\left(\frac{\omega r}{c}\right) + BY_0\left(\frac{\omega r}{c}\right) \tag{4.83}$$

In complex form, we can write this equation as

$$F = C_1\left[J_0\left(\frac{\omega r}{c}\right) + iY_0\left(\frac{\omega r}{c}\right)\right] + C_2\left[J_0\left(\frac{\omega r}{c}\right) - iY_0\left(\frac{\omega r}{c}\right)\right] \tag{4.84}$$

It can be rewritten as

$$F = C_1 H_0^{(1)}\left(\frac{\omega r}{c}\right) + C_2 H_0^{(2)}\left(\frac{\omega r}{c}\right) \tag{4.85}$$

where $H_0^{(1)}, H_0^{(2)}$ are Hankel functions defined by

$$H_0^{(1)} = J_0\left(\frac{\omega r}{c}\right) + iY_0\left(\frac{\omega r}{c}\right) \tag{4.86}$$

$$H_0^{(2)} = J_0\left(\frac{\omega r}{c}\right) - iY_0\left(\frac{\omega r}{c}\right) \tag{4.87}$$

which behave like damped trigonometric functions for large r. Thus the solution of one-dimensinoal wave equation becomes

$$u = C_1 e^{i\omega t} H_0^{(1)}\left(\frac{\omega r}{c}\right) + C_2 e^{i\omega t} H_0^{(2)}\left(\frac{\omega r}{c}\right) \tag{4.88}$$

Using asymptotic expressions, for $H_0^{(1)}$ and $H_0^{(2)}$ defined by

$$\left.\begin{aligned} H_0^{(1)}(x) &= \sqrt{\frac{2}{\pi x}}\, e^{i(x-\pi/4)} \\ H_0^{(2)}(x) &= \sqrt{\frac{2}{\pi x}}\, e^{-i(x-\pi/4)} \quad \text{for large } x \end{aligned}\right\} \tag{4.89}$$

the general periodic solution to the given wave equation in cylindrical coordinates is

$$u(r,t)=\sqrt{\frac{2c}{\pi\omega}}\left[C_1e^{-i\pi/4}\frac{\exp[i(\omega/c)(r+ct)]}{\sqrt{r}}+C_2e^{i\pi/4}\frac{\exp[i(\omega/c)(r-ct)]}{\sqrt{r}}\right] \tag{4.90}$$

4.9 PERIODIC SOLUTION OF ONE-DIMENSIONAL WAVE EQUATION IN SPHERICAL POLAR COORDINATES

In spherical polar coordinates, with u depending only on r, the source distance, the wave equation assumes the following form:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2},\qquad r>0 \tag{4.91}$$

We look for a periodic solution in time in the form

$$u=F(r)e^{i\omega t} \tag{4.92}$$

Then

$$\frac{\partial u}{\partial r}=F'(r)e^{i\omega t},\qquad \frac{\partial^2 u}{\partial t^2}=-\omega^2F(r)e^{i\omega t}$$

Substituting these derivatives into Eq. (4.91), we obtain

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2F'e^{i\omega t})=-\frac{\omega^2}{c^2}F(r)e^{i\omega t}$$

i.e.

$$\frac{1}{r^2}e^{i\omega t}[r^2F''+2rF']=-\frac{\omega^2}{c^2}e^{i\omega t}F$$

Therefore,

$$F''+\frac{2}{r}F'+\frac{\omega^2}{c^2}F=0 \tag{4.93}$$

Let

$$F=\left(\frac{\omega}{c}r\right)^{-1/2}\psi(r)$$

Then

$$F'=-\frac{\omega}{2c}\left(\frac{\omega}{c}r\right)^{-3/2}\psi(r)+\left(\frac{\omega}{c}r\right)^{-1/2}\psi'(r)$$

$$F''=\frac{3}{4}\left(\frac{\omega}{c}\right)^2\left(\frac{\omega}{c}r\right)^{-5/2}\psi(r)-\frac{\omega}{c}\left(\frac{\omega}{c}r\right)^{-3/2}\psi'(r)+\left(\frac{\omega}{c}r\right)^{-1/2}\psi''(r)$$

Substituting into Eq. (4.93), we obtain

$$\left(\frac{\omega}{c}r\right)^{-1/2}\left[\psi''(r)+\frac{1}{r}\psi'(r)+\left\{\left(\frac{\omega}{c}\right)^2-\left(\frac{1}{2r}\right)^2\right\}\psi(r)\right]=0$$

Since $\left(\frac{\omega}{c}r\right)\neq 0$, we have

$$\left[\psi''(r)+\frac{1}{r}\psi'(r)+\left\{\left(\frac{\omega}{c}\right)^2-\left(\frac{1}{2r}\right)^2\right\}\psi(r)\right]=0$$

which is a form of Bessel's equation, whose solution is given by

$$\psi(r)=A'J_{1/2}\left(\frac{\omega}{c}r\right)+B'J_{-1/2}\left(\frac{\omega}{c}r\right)$$

where A' and B' are constants. Therefore,

$$F(r)=\left(\frac{\omega}{c}r\right)^{-1/2}\left[A'J_{1/2}\left(\frac{\omega}{c}r\right)+B'J_{-1/2}\left(\frac{\omega}{c}r\right)\right] \tag{4.94}$$

or

$$F(r)=\frac{A}{\sqrt{r}}J_{1/2}\left(\frac{\omega}{c}r\right)+\frac{B}{\sqrt{r}}J_{-1/2}\left(\frac{\omega}{c}r\right) \tag{4.95}$$

But, we know that

$$J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$$

$$J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$$

Therefore,

$$F(r)=\sqrt{\frac{2c}{\pi\omega}}\left[A\frac{\sin(\omega r/c)}{r}+B\frac{\cos(\omega r/c)}{r}\right] \tag{4.96}$$

In complex form,

$$F(r)=C_1\frac{\exp(i\omega r/c)}{r}+C_2\frac{\exp(-i\omega r/c)}{r} \tag{4.97}$$

Thus, the required solution of the wave equation is

$$u(r, t) = C_1 \frac{\exp(i\omega/c)(r+ct)}{r} + C_2 \frac{\exp(-i\omega/c)(r-ct)}{r} \tag{4.98}$$

4.10 VIBRATION OF A CIRCULAR MEMBRANE

To find the solution of the wave equation representing the vibration of a circular membrane, it is natural that we introduce polar coordinates $(r, \theta), 0 \le r \le a, 0 \le \theta \le 2\pi$. Thus, the governing two-dimensional wave equation is given by

$$\text{PDE: } \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{4.99}$$

and the boundary and initial conditions are given by

$$\text{BCs: } u(a, \theta, t) = 0, \qquad t \ge 0 \tag{4.100}$$

i.e. the boundary is held fixed, and

$$\text{ICs: } u(r, \theta, 0) = f_1(r, \theta), \qquad \frac{\partial u}{\partial t}(r, \theta, 0) = f_2(r, \theta) \tag{4.101}$$

Let us look for a solution of Eq. (4.99) in the following variables separable form:

$$u = R(r)\ H(\theta)\ T(t) \tag{4.102}$$

Substituting into Eq. (4.99), we obtain

$$\frac{RHT''}{c^2} = R''HT + \frac{1}{r}R'HT + \frac{1}{r^2}RH''T$$

Dividing throughout by RHT/c^2, we get

$$\frac{T''}{T} = c^2\left[\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{H''}{H}\right] = -\mu^2 \text{(say)}$$

Then

$$T'' + \mu^2 T = 0 \tag{4.103}$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\mu^2}{c^2}r^2 = -\frac{H''}{H} = k^2 \text{(say)}$$

i.e.

$$r^2 R'' + rR' + \left(\frac{u^2}{c^2}r^2 - k^2\right)R = 0 \tag{4.104}$$

$$H'' + k^2 H = 0 \tag{4.105}$$

Here, μ^2 and k^2 are arbitrary separation constants. The general solutions of Eqs. (4.103)–(4.105) respectively are

$$T = A\cos\mu t + B\sin\mu t$$

$$R = PJ_k\left(\frac{\mu r}{c}\right) + QY_k\left(\frac{\mu r}{c}\right) \tag{4.106}$$

$$H = E\cos k\theta + F\sin k\theta$$

where J_k, Y_k are Bessel functions of first and second kind respectively of order k. Thus, the general solution of the wave equation (4.99) is

$$u(r,\theta,t) = (A\cos\mu t + B\sin\mu t)\left\{PJ_k\left(\frac{\mu r}{c}\right) + QY_k\left(\frac{\mu r}{c}\right)\right\}(E\cos k\theta + F\sin k\theta) \tag{4.107}$$

Since the deflection is a single-valued periodic function in θ of period 2π, k must be integral, say $k = n$. Also, since $Y_k(\mu r/c) \to -\infty$ as $r \to 0$, we can avoid infinite deflections at the centre $(r = 0)$ by taking $Q = 0$. Again noting that the BC: (4.100) implies that the deflection u is zero on the boundary of the circular membrane, we obtain

$$J_n\left(\frac{\mu a}{c}\right) = 0 \tag{4.108}$$

which has an inifinte number of positive zeros. These zeros (roots) are tabulated for several values of n in many handbooks. Their representation requires two indices. The first one indicates the order of the Bessel function, and the second, the solution. Thus denoting the roots by $\mu_{nm}(n = 0, 1, 2, \ldots; m = 1, 2, 3, \ldots)$, we have, after using the principle of superposition, the solution of the circular membrane in the form

$$u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} PJ_n\left(\frac{\mu_{nm} r}{c}\right)(A\cos\mu t + B\sin\mu t)\,(E\cos n\theta + F\sin n\theta)$$

Alternatively,

$$u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_n\left(\frac{\mu_{nm} r}{c}\right)\{[a_{nm}\cos n\theta + b_{nm}\sin n\theta]\cos\mu t + [c_{nm}\cos n\theta + d_{nm}\sin n\theta]\sin\mu t\} \tag{4.109}$$

Now, to determine the constants, we shall use the prescribed ICs which yield

$$f_1(r,\theta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(a_{nm}\cos n\theta + b_{nm}\sin n\theta)J_n\left(\frac{\mu_{nm}r}{c}\right) \tag{4.110}$$

$$f_2(r,\theta) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\mu_{nm}(c_{nm}\cos n\theta + d_{nm}\sin n\theta)J_n\left(\frac{\mu_{nm}r}{c}\right)$$

Hence, the solution of the circular membrance is given by Eq. (4.109), where

$$a_{nm} = \frac{2}{\pi a^2[J_n'(\mu_{nm})]^2}\int_0^{2\pi}\int_0^a f_1(r,\theta)\, J_n\left(\mu_{nm}\frac{r}{c}\right)\cos n\theta r\, dr\, d\theta$$

$$b_{nm} = \frac{2}{\pi a^2[J_n'(\mu_{nm})]^2}\int_0^{2\pi}\int_0^a f_1(r,\theta)\, J_n\left(\mu_{nm}\frac{r}{c}\right)\sin n\theta r\, dr\, d\theta$$

$$c_{nm} = \frac{2}{\pi a^2[J_n'(\mu_{nm})]^2}\int_0^{2\pi}\int_0^a f_2(r,\theta)\, J_n\left(\mu_{nm}\frac{r}{c}\right)\cos n\theta r\, dr\, d\theta,$$

$$d_{nm} = \frac{2}{\pi a^2[J_n'(\mu_{nm})]^2}\int_0^{2\pi}\int_0^a f_2(r,\theta)\, J_n\left(\mu_{nm}\frac{r}{c}\right)\sin n\theta r\, dr\, d\theta$$

4.11 UNIQUENESS OF THE SOLUTION FOR THE WAVE EQUATION

In Section 4.5, we have developed the variables separable method to find solutions to the wave equation with certain initial and boundary conditions. A formal solution for the non-homogeneous equation is also given in Section 4.6. In this section, we shall show that the solution to the wave equation is unique.

Uniqueness Theorem The solution to the wave equation

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0 \tag{4.111}$$

satisfying the ICs:

$$u(x, 0) = f(x), \quad 0 \le x \le L$$

$$u_t(x, 0) = g(x), \quad 0 \le x \le L$$

and the BCs:

$$u(0, t) = u(L, t) = 0$$

where $u(x, t)$ is twice continuously differentiable function with respect to x and t, is unique.

Proof Suppose u_1 and u_2 are two solutions of the given wave equation (4.111) and let $v = u_1 - u_2$. Obviously $v(x, t)$ is the solution of the following problem:

$$v_{tt} = c^2 v_{xx}, \quad 0 < x < L, \quad t > 0 \tag{4.112}$$

$$v(x, 0) = 0, \quad v_t(x, 0) = 0, \quad 0 \le x \le L$$

and

$$v(0, t) = v(L, t) = 0$$

It is required to prove that $v(x, t)$ is identically zero, implying $u_1 = u_2$. For, let us consider the function

$$E(t) = \frac{1}{2}\int_0^L (c^2 v_x{}^2 + v_t{}^2)\, dx \tag{4.113}$$

which, in fact, represents the total energy of the vibrating string at time t. It may be noted that $E(t)$ is differentiable with respect to t, as $v(x, t)$ is twice continuously differentiable. Thus,

$$\frac{dE}{dt} = \int_0^L \left[c^2 v_x v_{xt} + v_t\, v_{tt}\right] dx \tag{4.114}$$

Integrating by parts, the right-hand side of the above equation gives us

$$\int_0^L c^2 v_x v_{xt}\, dx = \left[c^2 v_x v_t\right]_0^L - \int_0^L c^2 v_t v_{xx}\, dx$$

But, $v(0, t) = 0$ implies $v_t(0, t) = 0$ for $t \ge 0$ and $v(L, t) = 0$ implying $v_t(L, t) = 0$ for $t \ge 0$. Hence, Eq. (4.114) reduces to

$$\frac{dE}{dt} = \int_0^L v_t (v_{tt} - c^2 v_{xx})\, dx = 0$$

In other words, $E(t) = \text{constant} = c$ (say). Since $v(x, 0) = 0$ implies $v_x(x, 0) = 0$, and $v_t(x, 0) = 0$, we can evaluate c and find that

$$E(0) = c = \int_0^L \left[c^2 v_x^2 + v_t^2\right]\Big|_{t=0} dx = 0$$

which gives $E(t) = 0$, which is possible if and only if $v_x \equiv 0$ and $v_t \equiv 0$ for all $t > 0, 0 \le x \le L$ which is possible only if $v(x, t) = \text{constant}$. However, since $v(x, 0) = 0$, we find $v(x, t) \equiv 0$. Hence, $u_1(x, t) = u_2(x, t)$. This means that the solution $u(x, t)$ of the given wave equation is unique.

4.12 DUHAMEL'S PRINCIPLE

With the help of Duhamel's principle, one can find the solution of an inhomogeneous equation, in terms of the general solution of the homogeneous equation. We shall illustrate this principle for wave equation. Let the Euclidean three-dimensional space be denoted by R_3, and a point in R_3 be represented by $X = (x_1, x_2, x_3)$. If $v(X, t, \tau)$ satisfies for each fixed τ the PDE

$$v_{tt}(X, t) - c^2\nabla^2 v(X, t) = 0, X \text{ in } R_3,$$

with the conditions

$$v(X, 0, \tau) = 0, \qquad v_t(X, 0, \tau) = F(X, \tau)$$

where $F(X, \tau)$ denotes a continuous function defined for X in R_3, and if u satisfies

$$u(X, t) = \int_0^t v(X, t - \tau, \tau)\, d\tau$$

then $u(X, t)$ satisfies

$$u_{tt} - c^2 \nabla^2 u = F(x, t), \quad X \text{ in } R_3, \ t > 0$$

$$u(X, 0) = u_t(X, 0) = 0$$

Proof Consider the equation

$$u_{tt} - c^2\nabla^2 u = F(X, t) \tag{4.115}$$

with

$$u(X, 0) = u_t(X, 0) = 0$$

Let us assume the solution of the problem (4.115) in the form

$$u(x, t) = \int_0^t v(X, t - \tau, \tau)\, d\tau \tag{4.116}$$

where $v(X, t - \tau, \tau)$ is a one-parameter family solution of

$$v_{tt} - c^2\nabla^2 v = 0 \quad \text{for all } \tau \tag{4.117}$$

Further, we assume that at $t = \tau$,

$$v(X, 0, \tau) = 0 \quad \text{for all values of } \tau \tag{4.118}$$

Now, differentiating with respect to t under integral sign and using the Liebnitz rule, from Eq. (4.116), we have

$$u_t = v(X, 0, t) + \int_0^t v_t(X, t - \tau, \tau)\, d\tau$$

Using Eq. (4.118), we get

$$u_t = \int_0^t v_t(X,\ t-\tau,\ \tau)\, d\tau$$

Differentiating this result once again with respect to t, we obtain

$$u_{tt} = v_t(X,\ 0,\ t) + \int_0^t v_{tt}(X,\ t-\tau,\ \tau)\, d\tau \tag{4.119}$$

Noting that u satisfies Eq. (4.115), v satisfies Eq. (4.117), and after using Eq. (4.117), the above equation reduces to

$$u_{tt} = v_t\,(X,\ 0,\ t) + \int_0^t c^2\nabla^2 v\, d\tau$$

Finally, using Eq. (4.116), the above equation reduces to

$$u_{tt} - c^2\nabla^2 u = v_t(X,\ 0,\ t) \tag{4.120}$$

Comparing Eqs. (4.115) and (4.120), we obtain

$$v_t(X,\ 0,\ t) = F(X,\ t) \tag{4.121}$$

Therefore, if v satisfies the equation

$$v_{tt} - c^2\nabla^2 v = 0$$

with the conditions

$$v(X, 0, \tau) = 0,\ v_t(X, 0, \tau) = F(X, \tau) \quad \text{at } t = \tau$$

then, u defined by Eq. (4.116) satisfies the given inhomogeneous equation (4.115) and the specified conditions. Here, the function $v(X,\ t)$ is called the pulse function or the force function.

EXAMPLE 4.8 Use Duhamel's principle to solve the heat equation problem described by

$$u_t(x,\ t) = ku_{xx}(x,\ t) + f(x,\ t), \qquad -\infty < x < \infty,\ t > 0 \tag{4.122}$$

$$u(x,\ 0) = 0, \qquad -\infty < x < \infty$$

Solution We have obtained, in Section 3.3, the unique solution of the problem

$$v_t(x,\ t) = kv_{xx}(x,\ t), \qquad -\infty < x < \infty,\ t > 0$$

$$v(x,\ 0) = f(x,\ \tau)$$

in the form

$$v(x,\ t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp[-(x-\xi)^2/(4kt)]\, f(\xi)\, d\xi$$

Hence, using Duhamel's principle, the solution of the corresponding inhomogeneous problem described by Eq. (4.122) is given by

$$u(x, t, \tau) = \int_0^t v(x, t-\tau, \tau)\, d\tau$$

or

$$u(x, t, \tau) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-\tau)}} \exp\left[\frac{-(x-\xi)^2}{4k(t-\tau)}\right] f(\xi)\, d\xi\, d\tau \tag{4.123}$$

4.13 MISCELLANEOUS EXAMPLES

EXAMPLE 4.9 A uniform string of line density ρ is stretched to tension ρc^2 and executes a small transverse vibration in a plane through the undisturbed line of string. The ends $x = 0, L$ of the string are fixed. The string is at rest, with the point $x = b$ drawn aside through a small distance ε and released at time $t = 0.$ Find an expression for the displacement $y(x, t)$.

Solution The transverse vibration of the string is described by

$$\text{PDE: } y_{xx} = \frac{1}{c^2} y_{tt} \tag{4.124}$$

The boundary and initial conditions are

$$\text{BCs: } y(0, t) = y(L, t) = 0$$

$$\text{IC: } y_t(x, 0) = 0$$

Using the variables separable method, let

$$y(x, t) = X(x)\, T(t)$$

then, we have from Eq. (4.124),

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \pm\lambda^2$$

The equation of the string is given by (see Fig. 4.7)

$$y(x, 0) = \begin{cases} \dfrac{\varepsilon x}{b}, & 0 \le x \le b \\[2ex] \dfrac{\varepsilon(x-L)}{(b-L)}, & b \le x \le L \end{cases}$$

The solution to the given problem is discussed now for various values of λ.

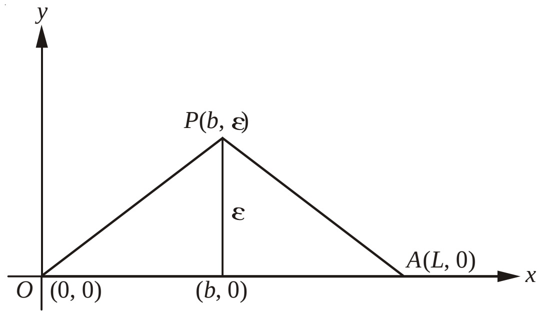

Fig. 4.7 Illustration of Example 4.9.

Case I Taking the constant $\lambda = 0,$ we have

$$X'' = T'' = 0$$

whose general solution is

$$X = Ax + B, \qquad T = Ct + D$$

Therefore,

$$y(x,t) = (Ax + B)\,(Ct + D)$$

Using the BCs at $x = 0, L,$ we can observe that $A = B = 0,$ implies a trivial solution.

Case II Taking the constant as $+\lambda^2$, we have

$$X'' - \lambda^2 X = 0 = T'' - c^2\lambda^2 T$$

Thus, the general solution is

$$y(x,\ t) = (A\cosh\lambda x + B\sinh\lambda x)\,(C\cosh c\lambda t + D\sinh c\lambda t)$$

Now the BCs:

$$y(0,\ t) = 0 \text{ gives } A = 0$$

and

$$y(L,\ t) = 0 \text{ gives } B\sinh\lambda L = 0$$

which is possible only if $B = 0$. Thus we are again getting only a trivial solution.

Case III If the constant is $-\lambda^2$, then we have

$$X'' + \lambda^2 X = 0 = T'' + c^2\lambda^2 T = 0$$

In this case, the general solution is

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)\,(P\cos c\lambda t + Q\sin c\lambda t)$$

using the BCs:

$$y(0, t) = 0 \text{ gives } A = 0$$

$$y(L, t) = 0 \text{ gives } B \sin \lambda L = 0$$

For a non-trivial solution, $B \neq 0 \Rightarrow \lambda L = n\pi$. Therefore, $\lambda = n\pi/L, n = 1, 2, \ldots$ Also, using the IC: $y_t(x, 0) = 0$, we can notice that $Q = 0$. Hence, the acceptable non-trivial solution is

$$y(x,\ t) = BP \sin \frac{n\pi x}{L} \cos \frac{cn\pi t}{L}, \qquad n = 1, 2, \ldots$$

Using the principle of superposition, we have

$$y(x,\ t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \cos \frac{cn\pi t}{L}$$

which gives

$$y(x,\ 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

This is half-range sine series, where

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L y(x,\ 0) \sin \frac{n\pi x}{L} dx \\ &= \frac{2}{L}\int_0^b \frac{\varepsilon}{b} x \sin \frac{n\pi x}{L} dx + \frac{2}{L}\int_0^L \frac{\varepsilon}{b-L}(x-L) \sin \frac{n\pi x}{L} dx \\ &= \frac{2\varepsilon}{Lb}\left[-\frac{\cos(n\pi x/L)}{n\pi/L} x\right]_0^b - \frac{2\varepsilon}{Lb}\left[-\frac{\sin(n\pi/L)\, x}{n^2\pi^2/L^2}\right]_0^b \\ &\quad + \frac{2\varepsilon}{L(b-L)}\left[-\frac{\cos(n\pi x/L)}{n\pi/L}(x-L)\right]_b^L - \frac{2\varepsilon}{L(b-L)}\left[-\frac{\sin(n\pi x/L)}{n^2\pi^2/L^2}\right]_b^L \end{aligned}$$

or

$$b_n = \frac{2\varepsilon L^2}{n^2\pi^2 b(L-b)} \sin \frac{n\pi b}{L}$$

Hence the subsequent motion of the string is given by

$$y(x,\ t) = \sum_{n=1}^{\infty} \frac{2\varepsilon L^2}{n^2\pi^2 b(L-b)} \sin \frac{n\pi b}{L} \sin \frac{n\pi x}{L} \cos \frac{cn\pi t}{L}.$$

EXAMPLE 4.10 Find a particular solution of the problem described by

$$\text{PDE: } y_{tt} - c^2 y_{xx} = g(x)\cos\omega t, \qquad 0 < x < L, \quad t > 0$$

$$\text{BCs: } y(0,\, t) = y(L,\, t) = 0, \qquad t > 0$$

where $g(x)$ is a piecewise smooth function and ω is a positive constant.

Solution Taking the clue from Example 4.9, we assume the solution in the form

$$y(x,\, t) = \sum_{n=1}^{\infty} A_n(t) \sin\frac{n\pi x}{L}$$

To determine $A_n(t)$, we consider the Fourier sine expansion of $g(x)$ in the form

$$g(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

and substitute into the given PDE which yields

$$\sum_{n=1}^{\infty} \left[A_n''(t) + \left(\frac{n\pi c}{L}\right)^2 A_n(t) \right] \sin\frac{n\pi x}{L} = \cos\omega t \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

Choosing $A_n(t)$ as the solution of the ODE

$$A_n''(t) + \left(\frac{n\pi c}{L}\right)^2 A_n(t) = B_n \cos\omega t$$

we have for any n, the particular solution

$$A_n(t) = A_n \cos\omega t \quad \text{if} \quad \omega \neq \frac{n\pi c}{L}$$

Therefore,

$$A_n \left[-\omega^2 + \left(\frac{n\pi c}{L}\right)^2 \right] = B_n$$

Hence, the required particular solution is given by

$$y(x,\, t) = \cos\omega t \sum_{n=1}^{\infty} \frac{B_n \sin(n\pi x/L)}{(n\pi c/L)^2 - \omega^2}$$

EXAMPLE 4.11 A rectangular membrane with fastened edges makes free transverse vibrations. Explain how a formal series solution can be found.

Solution Mathematically, the problem can be posed as follows:
Solve

$$\text{PDE: } u_{tt} - c^2(u_{xx} + u_{yy}) = 0, \quad 0 \le x \le a, 0 \le y \le b$$

subject to the BCs:

(i) $u(0, y, t) = 0$

(ii) $u(a, y, t) = 0$

(iii) $u(x, 0, t) = 0$

(iv) $u(x, b, t) = 0$

and ICs:

$$u(x, y, 0) = f(x, y), \quad u_t(x, y, 0) = g(x, y)$$

We look for a separable solution of the form

$$u(x, y, t) = X(x)\,Y(y)\,T(t)$$

Substituting into the given PDE, we obtain

$$\frac{1}{c^2}\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} = -\lambda^2 \quad \text{(say)}$$

Then

$$T'' + c^2\lambda^2 T = 0$$

$$\frac{X''}{X} + \lambda^2 = -\frac{Y''}{Y} = \mu^2 \quad \text{(say)}$$

thus yielding

$$Y'' + \mu^2 Y = 0, \qquad X'' + (\lambda^2 - \mu^2)\,X = 0$$

Let $\lambda^2 - \mu^2 = p^2$, $\mu^2 = q^2$. Then $\lambda^2 = p^2 + q^2 = r^2$. Therefore, we have

$$X'' + p^2 X = 0, \qquad Y'' + q^2 Y = 0, \qquad T'' + r^2 c^2 T = 0$$

The possible separable solution is

$$u(x, y, t) = (A\cos px + B\sin px)\,(C\cos qy + D\sin qy)\,(E\cos(rct) + F\sin(rct))$$

Using the BCs: $u(0, y, t) = 0$ gives $A = 0$

$u(x, 0, t) = 0$ gives $C = 0$

$u(a, y, t) = 0$ gives $p = m\pi/a, \quad m = 1, 2, \ldots$

$u(x, b, t) = 0$ gives $q = n\pi/b, \quad n = 1, 2, \ldots$

Using the principle of superposition, we get

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [A_{mn} \cos(rct) + B_{mn} \sin(rct)] \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{4.125}$$

where

$$r^2 = p^2 + q^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$$

Applying the initial condition: $u(x, y, 0) = f(x, y)$, Eq. (4.125) gives

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx\, dy \tag{4.126}$$

Finally, applying the initial condition: $u_t(x, y, 0) = g(x, y)$, Eq. (4.125) gives

$$g(x, y) = cr \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where

$$B_{mn} = \frac{4}{abcr} \int_0^a \int_0^b g(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx\, dy \tag{4.127}$$

Hence, the required series solution is given by Eq. (4.125), where A_{mn} and B_{mn} are given by Eqs. (4.126) and (4.127).

EXAMPLE 4.12 Solve the IVP described by

$$\text{PDE: } u_{tt} - c^2 u_{xx} = F(x, t), \qquad -\infty < x < \infty, t \ge 0$$

with the data

(i) $F(x, t) = 4x + t$, (ii) $u(x, 0) = 0$, (iii) $u_t(x, 0) = \cosh bx$.

Solution In Example 4.4, we have obtained the Riemann-Volterra solution for the inhomogeneous wave equation in the following form:

$$u(x, t) = \frac{1}{2}[\eta(x - ct) + \eta(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} v(\xi)\, d\xi + \frac{1}{2c} \iint_{\mathbb{R}} F(x, t)\, dx\, dt \tag{4.128}$$

Since $u(x, 0) = \eta(x) = 0$, the first term on the right-hand side of Eq. (4.128) vanishes. Also,

$$\frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)\, d\xi = \frac{1}{2c}\int_{x-ct}^{x+ct} \cosh b\xi\, d\xi$$

$$= \frac{1}{2c}\left(\frac{\sin b\xi}{b}\right)_{x-ct}^{x+ct} = \frac{1}{2bc}[\sinh b(x+ct) - \sinh b(x-ct)]$$

$$= (\cosh bx \sinh (bct))/bc$$

and

$$\frac{1}{2c}\iint_{\mathbb{R}} F(x,t)\, dx\, dt = \frac{1}{2c}\iint_{\mathbb{R}} (4x+t)\, dx\, dt$$

From Fig. 4.4, we can write the equation of the line PA in the form

$$t = \frac{x - x_0 + ct_0}{c}$$

or

$$x = x_0 + ct - ct_0$$

Similarly, the equation of the line PB is

$$x = x_0 + ct_0 - ct$$

Thus

$$\frac{1}{2c}\iint_{\mathbb{R}} F(x,t)\, dt = \int_0^{t_0}\int_{x_0+ct-ct_0}^{x_0+ct_0-ct} (4x+t)\, dx\, dt$$

$$= \int_0^{t_0} (4x_0t_0 - 4tx_0 + tt_0 - t^2)\, dt = 2x_0t_0^2 + t_0^3/6$$

The required solution at any point (x, t) is, therefore, given by

$$u(x, t) = \frac{\cosh bx \sinh (bct)}{bc} + 2xt^2 + \frac{t^3}{6}$$

EXAMPLE 4.13 Derive the wave equation representing the transverse vibration of a string in the form

$$\frac{\partial^2 u}{\partial t^2} = c^2\left\{1 + \left(\frac{\partial u}{\partial x}\right)^2\right\}^{-2}\frac{\partial^2 u}{\partial x^2}$$

Solution Consider the motion of an element $PQ = \delta s$ of the string as shown in Fig. 4.8. In equilibrium position, let the string lie along the x-axis, such that PQ is originally at $P'Q'$. Let the displacement of PQ from the x-axis, be denoted by u. Let T be the tension in the string and ρ be the density of the string. Writing down the equation of motion of the element PQ of the string in the u-direction, we have

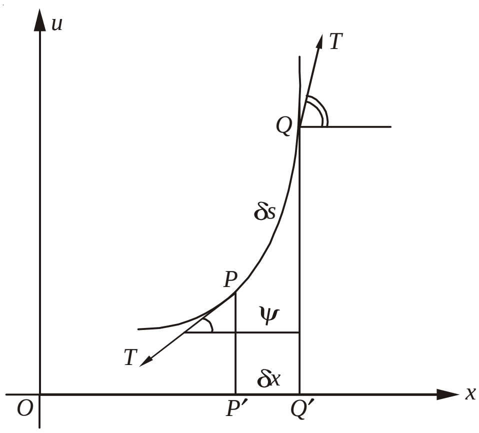

Fig. 4.8 An Illustration of Example 4.13.

$$T \sin(\psi + \delta\psi) - T \sin\psi = \rho\delta_s \frac{\partial^2 u}{\partial t^2}$$

Neglecting squares of small quantities, we get

$$T \cos\psi\delta\psi = \rho\delta_s \frac{\partial^2 u}{\partial t^2} \tag{4.129}$$

by noting that

$$\tan\psi = \frac{\partial u}{\partial x}, \quad \sec^2\psi\delta\psi = \frac{\partial^2 u}{\partial x^2}\delta x$$

Equation (4.129) becomes

$$\rho \frac{\partial^2 u}{\partial t^2} = T\cos^3\psi \frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial s} = T\cos^4\psi \frac{\partial^2 u}{\partial x^2} \tag{4.130}$$

but

$$\cos^2\psi = \frac{1}{1+\tan^2\psi} = \left\{1 + \left(\frac{\partial u}{\partial x}\right)^2\right\}^{-1} \tag{4.131}$$

Using Eq. (4.131) into Eq. (4.130), we get

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left[1+\left(\frac{\partial u}{\partial x}\right)^2\right]^{-2}\frac{\partial^2 u}{\partial x^2}$$

If we define $c^2 = T/\rho$, the required wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left[1+\left(\frac{\partial u}{\partial x}\right)^2\right]^{-2}\frac{\partial^2 u}{\partial x^2} \qquad (4.132)$$

This is a non-linear second order partial differential equation.

EXAMPLE 4.14 Using Duhamel's principle solve the following IBVP:

PDE: $u_t - u_{xx} = f(x, t),\ 0 < x < L,\ 0 < t < \infty$

BCS: $u(0, t) = u(L, t) = 0$

IC: $u(x, 0) = 0,\ 0 \le x \le L.$

Solution Using Duhamel's principle, the required solution is given by

$$u(x,t) = \int_0^t v(x, t-\tau, \tau)\, d\tau$$

where $v(x, t, \tau)$ is the solution of the homogeneous problem described as

$$v_t - v_{xx} = 0,\ 0 < x < L,\ 0 < t < \infty$$

$$v(0, t, \tau) = v(L, t, \tau) = 0$$

and $$u(x, 0, \tau) = f(x, \tau).$$

Now, recalling Example 3.17, the solution to this homogeneous problem is obtained as

$$v(x, t, \tau) = \sum_{n=1}^{\infty} a_n e^{-(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right).$$

Observe that, the Fourier coefficients a_n depends on the parameter τ, so that

$$a_n = a_n(\tau) = \frac{2}{L}\int_0^L f(x, \tau)\sin\left(\frac{n\pi x}{L}\right)dx$$

Hence, the solution to the gives IBVP is found to be

$$u(x,t) = \int_0^t \sum_{n=1}^{\infty} a_n(\tau) e^{-(n\pi/L)^2 (t-\tau)} \sin\left(\frac{n\pi x}{L}\right) d\tau.$$

EXERCISES

1. A homogeneous string is stretched and its ends are at $x = 0$ and $x = l$. Motion is started by displacing the string into the form $f(x) = u_0 \sin(\pi x/l)$, from which it is released at time $t = 0$. Find the displacement at any point x and time t.

2. Solve the boundary value problem described by

$$\begin{aligned}
&\text{PDE: } u_{tt} - c^2 u_{xx} = 0, && 0 \le x \le l, t \ge 0\\
&\text{BCs: } u(0, t) = u(l, t) = 0, && t \ge 0\\
&\text{ICs: } u(x, 0) = 10 \sin(\pi x/l), && 0 \le x \le l\\
&\qquad u_t(x, 0) = 0 &&
\end{aligned}$$

3. Solve the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}, \quad 0 \le x \le \pi, t \ge 0$$

subject to

$$\begin{aligned}
u &= 0 \quad \text{when } x = 0 \text{ and } x = \pi\\
u_t &= 0 \quad \text{when } t = 0 \text{ and } u(x, 0) = x, \quad 0 < x < \pi
\end{aligned}$$

4. Solve

$$u_{tt} = c^2 u_{xx}, \quad 0 \le x \le l, \ t \ge 0$$

subject to

$$u(0, t) = 0, \quad u(l, t) = 0 \ \text{ for all } t$$

$$u(x, 0) = 0, \quad u_t(x, 0) = b \sin^3(\pi x/l)$$

5. Solve the vibrating string problem described by

$$\begin{aligned}
&\text{PDE: } u_{tt} - c^2 u_{xx} = 0, && 0 < x < l, \ t > 0\\
&\text{BCs: } u(0, t) = u(l, t) = 0, && t > 0\\
&\text{ICs: } u(x, 0) = f(x), && 0 \le x \le l\\
&\qquad u_t(x, 0) = 0, && 0 \le x \le l
\end{aligned}$$

6. In spherical coordinates, if u is a spherical wave, i.e. $u = u(r, t)$, then the wave equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

which is called the Euler-Poisson-Darboux equation. Find its general solution.

7. Solve the initial value problem described by

$$\text{PDE}: u_{tt} - c^2 u_{xx} = e^x$$

with the given data

$$u(x,0) = 5, \qquad u_t(x,0) = x^2$$

8. Sovle the initial value problem described by

$$\text{PDE}: u_{tt} - c^2 u_{xx} = xe^t$$

with the data

$$u(x, 0) = \sin x, \quad u_t(x, 0) = 0$$

9. Solve the initial boundary value problem described by

$$\text{PDE}: u_{tt} = c^2 u_{xx}, \quad x > 0, \ t > 0$$

with the data:

$$u(x, 0) = 0, \qquad u_t(x, 0) = 0, \ x > 0$$

$$u(0, t) = \sin t, \quad t > 0$$

10. Determine the solution of the one-dimensional wave equation

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0, \quad 0 < x < a, t > 0$$

with c as a constant, under the following initial and boundary conditions:

(i) $\phi(x, 0) = f(x) = \begin{cases} x/b, & 0 \le x \le b \\ (a-x)/(a-b), & b < x \le a \end{cases}$

(ii) $\dfrac{\partial \phi}{\partial t}(x, 0) = 0, \ 0 < x < a$

(iii) $\phi(0, t) = \phi(a, t) = 0, \quad t \ge 0.$

11. A piano string of length L is fixed at both ends. The string has a linear density ρ and is under tension τ. At time $t = 0$, the string is pulled a distance s from equilibrium position at its mid-point so that it forms an isosceles triangle and is then released $(s \le L)$. Find the subsequent motion of the string.

12. Obtain the normal frequencies and normal modes for the vibrating string of Problem 11.

13. A flexible stretched string is constrained to move with zero slope at one end $x = 0$, while the other end $x = L$ is held fixed against any movement. Find an expression for the time-dependent motion of the string if it is subjected to the initial displacement given by

$$y(x, 0) = y_0 \cos\left(\frac{\pi x}{2L}\right)$$

and is released from this position with zero velocity.

14. Show that if f and g are arbitrary functions, then

$$u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y)$$

is a solution of the equation

$$u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt}$$

provided $\alpha^2 = 1 - v^2/c^2$.

Choose the correct answer in the following questions (15 and 16):

15. The solution of the initial value problem

$$u_{tt} = 4u_{xx}, \qquad t > 0, \qquad -\infty < x < \infty$$

satisfying the conditions $u(x, 0) = x, \quad u_t(x, 0) = 0$ is

(A) x (B) $x^2/2$ (C) $2x$ (D) $2t$ (GATE-Maths, 2001)

16. Let $u = \psi(x, t)$ be the solution to the initial value problem

$$u_{tt} = u_{xx} \qquad \text{for } -\infty < x < \infty, t > 0$$

with $u(x, 0) = \sin x, u_t(x, 0) = \cos x$, then the value of $\psi(\pi/2, \pi/6)$ is

(A) $\sqrt{3}/2$ (B) $1/2$ (C) $1/\sqrt{2}$ (D) 1 (GATE-Maths, 2003)

17. Solve the following IBVP

$$\text{PDE: } u_t = u_{xx} + f(x, t),\ 0 < x < \pi,\ 0 < t < \infty$$

$$\text{BCS: } u(0, t) = u(\pi, t) = 0$$

$$\text{IC: } u(x, 0) = 0,\ 0 \le x \le \pi$$

Using Duhamel's principle.

Chapter 5

Green's Function

5.1 INTRODUCTION

Consider the differential equation

$$Lu\,(x) = f\,(x) \tag{5.1}$$

where L is an ordinary linear differential operator, $f\,(x)$ is a known function, while $u\,(x)$ is an unknown function. To solve the above differential equation, one method is to find the operator L^{-1} in the form of an integral operator with a kernel $G\,(x,\ \xi)$ such that

$$u\,(x) = L^{-1} f\,(x) = \int G\,(x, \xi)\ f\,(\xi)\,d\xi \tag{5.2}$$

The kernel of this integral operator is called Green's function for the differential operator. Thus the solution to the non-homogeneous differential equation (5.1) can be written down, once the Green's function for the problem is known. Applying the differential operator L to both sides of Eq. (5.2), we get

$$f\,(x) = LL^{-1} f\,(x) = \int LG\,(x,\ \xi)\ f\,(\xi)\,d\xi \tag{5.3}$$

This equation is satisfied if we choose $G\,(x, \xi)$ such that (see propety III of Section 3.4)

$$LG\,(x, \xi) = \delta\,(x - \xi) \tag{5.4}$$

where $\delta\,(x - \xi)$ is a Dirac δ-function. The solution of Eq. (5.4) is called a singularity solution of Eq. (5.1). In Section 3.4, we have already introduced the concept of Dirac δ-function and studied its various properties. Now, they become handy to understand more about Green's function.

Definition 5.1 Let us consider an auxiliary function $\phi(x)$ of a real variable x which possesses derivatives of all orders and vanishes outside a finite interval. Such functions are called test functions.

Now we shall introduce the concept of the derivative of a δ-function in terms of the derivative of a test function. We say that $\delta'(x)$ is the derivative of $\delta(x)$ if

$$\int_{-\infty}^{\infty} \delta'(x)\,\phi(x)\,dx = -\phi'(0) \tag{5.5}$$

for every test function $\phi(x)$. Similarly, we define $\phi''(x)$ by

$$\int_{-\infty}^{\infty} \delta''(x)\,\phi(x)\,dx = \phi''(0) \tag{5.6}$$

With this definition of a derivative, we can show that the δ-function is the derivative of a Heaviside unit step function $H(x)$ defined by

$$H(x) = \begin{cases} 1 & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{5.7}$$

To see this result, we consider

$$\int_{-\infty}^{\infty} H'(x)\,\phi(x)\,dx = -\int_{-\infty}^{\infty} H(x)\,\phi'(x)\,dx = -\int_{0}^{\infty} \phi'(x)\,dx = \phi(0)$$

By comparing the above result with property III of Section 3.4, i.e., with

$$\int_{-\infty}^{\infty} \delta(x)\,\phi(x)\,dx = \phi(0)$$

we obtain

$$H'(x) = \delta(x) \tag{5.8}$$

Similarly, the notion of δ-function and its derivative enables us to give a meaning to the derivative of a function that has a Jump discontinuity at $x = \xi$ of magnitude unity. Let

$$H(x-\xi) = \begin{cases} 1, & x \ge \xi \\ 0, & x < \xi \end{cases}$$

Then, for any test function $\phi(x)$, we have

$$\int_{-\infty}^{\infty} H'(x-\xi)\,\phi(x)\,dx = \int_{-\infty}^{\infty} \delta(x-\xi)\,\phi(x)\,dx = \phi(\xi) \tag{5.9}$$

It can also be noted that

$$\frac{d}{dx}[(x-\xi)\,H\,(x-\xi)] = (x-\xi)\,H'(x-\xi) + H\,(x-\xi)$$

Integrating both sides with respect to x between the limits $-\infty$ to ∞, we get

$$(x-\xi)\,H(x-\xi) = \int_{-\infty}^{\infty} (x-\xi)\,H'(x-\xi)\,dx + \int_{-\infty}^{\infty} H\,(x-\xi)\,dx$$

$$= \int_{-\infty}^{\infty} H\,(x-\xi)\,dx \tag{5.10}$$

We now consider an example to illustrate the inversion of a differential operator by considering the BVP:

$$\frac{d^2u}{dx^2} = f\,(x), \qquad u\,(0) = u\,(1) = 0$$

In this case, Eq. (5.4) becomes

$$LG = \frac{d^2G}{dx^2} = \delta(x-\xi) \tag{5.11}$$

Noting that the δ-function is the derivative of the Heaviside unit step function and integrating Eq. (5.11), we get

$$\frac{d}{dx}[G\,(x,\xi)] = H\,(x-\xi) + C_1\,(\xi)$$

where $C_1\,(\xi)$ is an arbitrary function. Integrating the above result once again with respect to x, we get

$$G\,(x,\xi) = \int H\,(x-\xi)\,dx + C_1\,(\xi)\,x + C_2\,(\xi)$$

$$= (x-\xi)H\,(x-\xi) + C_1\,(\xi)\,x + C_2\,(\xi)$$

where $C_1\,(\xi)$ and $C_2\,(\xi)$ can be determined from the boundary conditions. Thus, from Eq. (5.2) we have

$$u\,(x) = \int_0^x (x-\xi)H(x-\xi)\,\,f\,(\xi)\,d\xi + x\int_{-\infty}^{\infty} C_1(\xi)\,\,f\,(\xi)\,d\xi + \int_{-\infty}^{\infty} C_2(\xi)\,\,f\,(\xi)\,d\xi$$

Now, using the boundary condition: $u(0)=0$, we get

$$0 = 0 + 0 + \int_{-\infty}^{\infty} C_2(\xi)\, f(\xi)\, d\xi$$

implying that $C_2(\xi)=0$. Using the second boundary condition: $u(1)=0$, we have

$$0 = \int_0^1 (1-\xi)\, f(\xi)\, d\xi + \int_{-\infty}^{\infty} C_1(\xi)\, f(\xi)\, d\xi$$

implying that $C_1(\xi) = -(1-\xi), 0 \le \xi \le 1,$ and zero for all other values of ξ. Hence,

$$u(x) = \int_0^x (x-\xi)\, H(x-\xi)\, f(\xi)\, d\xi - x\int_0^1 (1-\xi)\, f(\xi)\, d\xi \tag{5.12}$$

Comparing this result with Eq. (5.2), we have the kernel of the integral operator, which is known as Green's function or source function given by

$$G(x,\xi) = (x-\xi)\, H(x-\xi) - x(1-\xi), \qquad 0 \le \xi \le 1 \tag{5.13}$$

satisfying the boundary conditions: $G(0,\xi) = G(1,\xi) = 0$.

This concept can be extended to partial differential equations also. To make the ideas clearer, let us consider

$$L[u(\mathbf{X})] = f(\mathbf{X}) \tag{5.14}$$

where L is some linear partial differential operator in three independent variables x, y, z and $\mathbf{X}$ is a vector in three-dimensional space. Then the Green's function may be denoted by $G(\mathbf{X};\mathbf{X}')$ which satisfies the equation

$$L[G(\mathbf{X};\mathbf{X}')] = \delta(\mathbf{X}-\mathbf{X}') \tag{5.15}$$

On expansion, this equation becomes

$$L[G(x,y,z;x',y',z')] = \delta(x-x')\,\delta(y-y')\,\delta(z-z') \tag{5.16}$$

Here the expression $\delta(\mathbf{X}-\mathbf{X}')$ is the generalization of the concept of delta function in three-dimensional space $\mathbb{R}$ and $G(\mathbf{X};\mathbf{X}')$ represents the effect at the point $\mathbf{X}$ due to a source function or delta function input applied at $\mathbf{X}'$. Equation (5.15) has the following interpretation in heat conduction or electrostatics: $G(\mathbf{X};\mathbf{X}')$ can be viewed as the temperature (the electrostatc potential) at any point $\mathbf{X}$ in $\mathbb{R}$ due to a unit source (due to a unit charge) located at $\mathbf{X}'$. Multiplying Eq. (5.15) on both sides by $f(\mathbf{X}')$ and integrating over the volume V with respect to $\mathbf{X}'$, we get

$$L\left[\int_{V_{X'}} G(\mathbf{X};\mathbf{X}')\, f(\mathbf{X}')\, dV_{X'}\right] = \int_{V_{X'}} f(\mathbf{X}')\,\delta(\mathbf{X}-\mathbf{X}')\, dV_{X'} = f(\mathbf{X})$$

Comparing with Eq. (5.14), we arrive at

$$u(\mathbf{X}) = \int_{V_{X'}} G(\mathbf{X}; \mathbf{X}')\, f(\mathbf{X}')\, dV_{X'} \tag{5.17}$$

which is the solution of Eq. (5.14). This leads to the simple definition that a function $u(x, y, z; x', y', z')$ is a fundamental solution of the equation, for example, $\nabla^2 u = 0$, if u is a solution of the non-homogeneous equation

$$\nabla^2 u = \delta(x, y, z; x', y', z')$$

This idea can be easily extended to higher dimensions. Thus the Green's function technique can be applied, in principle, to find the solution of any linear non-homogeneous partial differential equation. Although a neat formula (5.17) has been given for solution of non-homogeneous PDE, in practice, it is not an easy task to construct the Green's function. We shall now present a few singularity solutions, also called fundamental solutions to the well-known operators. These will guide us to construct Green's function for the solutions of partial differential equations which occur most frequently in mathematical physics.

To start with, the fundamental solution for a three-dimensional potential problem satisfies

$$\nabla^2 u = \delta(\mathbf{X}) \tag{5.18}$$

or

$$\text{div grad } u = \delta(\mathbf{X})$$

where u can be interpreted, for example, as the electrostatic potential. We seek a solution which depends only on the source distance $r = |\mathbf{X}|$; thus, for $r > 0$, $u(r)$ satisfies

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = 0$$

On integration, we get

$$u = \frac{A}{r} + B$$

Using the fact that the potential vanishes at infinity, we have

$$u = A/r$$

Integrating Eq. (5.18) over a small sphere R_ε of radius ε, we have

$$\int_{R_\varepsilon} [\text{div grad } u]\, dV = 1$$

Using the divergence theorem, we obtain

$$\int_{\sigma_\varepsilon} \left.\frac{\partial u}{\partial r}\right|_{r=\varepsilon} dS = 1$$

where σ_ε is the surface of R_ε. Hence,

$$-A\int \frac{1}{r^2}\bigg|_\varepsilon dS = -\frac{A}{\varepsilon^2}\times 4\pi\varepsilon^2 = -4\pi A = 1$$

Therefore,

$$A = -1/4\pi$$

Thus, the singularity solution or the fundamental solution of $\nabla^2 u = 0$ is

$$u = -\frac{1}{4\pi r} \tag{5.19}$$

The two-dimensional case of Eq. (5.18) is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0, \qquad r > 0$$

On integration, we get

$$u = A\ln r + B$$

Integrating Eq. (5.18) over a disc R_ε of small radius ε, whose surface is σ_ε, we get

$$\int_{R_\varepsilon} \text{div grad } u = 1$$

Hence

$$\int_{\sigma_\varepsilon} \frac{\partial u}{\partial r}\bigg|_{r=\varepsilon} dS = \int \frac{A}{r}\bigg|_\varepsilon dS = \frac{A}{\varepsilon}\times 2\pi\varepsilon = 1$$

Therefore, $A = 1/2\pi$. The constant B remains arbitrary and can be set equal to zero for convenience. Thus, the fundamental solution is

$$u\,(r) = \frac{1}{2\pi}\ln r \tag{5.20}$$

If $r\,(x, y, z)$ and $r'\,(x', y', z')$ are two distinct points in three-dimensional space $\mathbb{R}$, then the singularity solution of Laplace equation is

$$u = \frac{1}{4\pi\,|\mathbf{r}-\mathbf{r}'|} \tag{5.21}$$

Similarly, the singularity solution for the diffusion equation

$$u_t - k\nabla^2 u = 0$$

in three-space variables assumes the form

$$\frac{1}{8[\pi k(t-\tau)]^{3/2}}\exp\left[\frac{-|\mathbf{r}-\mathbf{r}'|^2}{4k(t-\tau)}\right] \tag{5.22}$$

For Helmholtz equation in three space variables, viz.

$$\nabla^2 u + k^2 u = 0$$

the singularity solution is

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \tag{5.23}$$

Before we attempt to solve Eq. (5.17), we shall examine the form of three-dimensional δ-function in general curvilinear coordinates as a preparation to study the solution of partial differential equations in polar coordinates, spherical polar coordinates etc. Suppose we are looking for a transformation from Cartesian coordinates, x, y to curvilinear coordinates ξ, η through the relations

$$x = f(\xi, \eta), \qquad y = g(\xi, \eta) \tag{5.24}$$

where f and g are single-valued, continuously differentiable functions of their arguments. Suppose that under this transformation, $\xi = \beta_1$ and $\eta = \beta_2$ correspond to $x = \alpha_1$ and $y = \alpha_2$ respectively. Also,

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} f_\xi & f_\eta \\ g_\xi & g_\eta \end{pmatrix} \begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = \frac{\partial(f, g)}{\partial(\xi, \eta)} \begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = J \begin{pmatrix} d\xi \\ d\eta \end{pmatrix}$$

If we transform the coordinates following Eq. (5.24), the relation

$$\iint \phi(x, y)\, \delta(x-\alpha_1)\, \delta(y-\alpha_2)\, dx\, dy = \phi(\alpha_1, \alpha_2)$$

becomes

$$\iint \phi(f, g)\, \delta[f(\xi,\eta)-\alpha_1]\, \delta[g(\xi,\eta)-\alpha_2] |J|\, d\xi\, d\eta = \phi(\alpha_1, \alpha_2) \tag{5.25}$$

where J is the Jacobian of the transformation. Equation (5.25) states that the Dirac δ-function $\delta[f(\xi,\eta)-\alpha_1]\,\delta[g(\xi,\eta)-\alpha_2]|J|$ assigns to any test function $\phi(f, g)$ the value of that test function at the points where $f = \alpha_1, g = \alpha_2$, i.e. at the poins $\xi = \beta_1, \eta = \beta_2$. Thus we may write

$$\delta[f(\xi,\eta)-\alpha_1]\,\delta[g(\xi,\eta)-\alpha_2]|J| = \delta(x-\alpha_1)\,\delta(y-\alpha_2)|J| = \delta(\xi-\beta_1)\,\delta(\eta-\beta_2)$$

Hence,

$$\delta(x-\alpha_1)\,\delta(y-\alpha_2) = \frac{\delta(\xi-\beta_1)\,\delta(\eta-\beta_2)}{|J|} \tag{5.26}$$

In the next few sections, we shall duscuss the Green's function method for solving partial differential equations with particular emphasis on elliptic equations. The discussion on wave equation and heat equation is also included in Sections 5.5 and 5.6 respectively.

5.2 GREEN'S FUNCTION FOR LAPLACE EQUATION

To find analytical solution of the boundary value problems, the Green's function method is one of the convenient techniques. In this section, we shall give the definition of Green's function for the Laplace equation and study its basic properties. To begin with, we shall define Green's function for the Dirichlet problem, i.e., we shall find u such that $\nabla^2 u = 0$ is valid inside some finitely bounded region $\mathbb{R}$, enclosed by $\partial\mathbb{R}$, a sufficiently smooth surface, when $u = f$ is prescribed on the boundary $\partial\mathbb{R}$ (see Fig. 5.1).

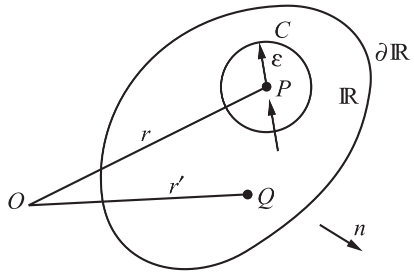

Fig. 5.1 Region and boundary surface.

Suppose that u is known at every point of the boundary $\partial\mathbb{R}$ and that it satisfies the relation

$$\nabla^2 u = 0 \quad \text{in } \mathbb{R}$$

The task is to find $u(P)$ when $P \in \mathbb{R}$. Let **OP** = **r**, and let C be a sphere with centre at P and radius ε. Also, let Σ be the new region exterior to C and interior to $\mathbb{R}$. Further, let the boundary of Σ be denoted by $\partial\Sigma$, and

$$u' = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \tag{5.27}$$

where $\mathbf{r}'$ is another point Q either in Σ or on the boundary $\partial\Sigma$. If u and u' are twice continuously differentiable functions in $\mathbb{R}$ and have first order derivatives on $\partial\mathbb{R}$, then by Green's theorem in the region $\mathbb{R}$ we have, from Eq. (2.19), the relation

$$\iiint_{\mathbb{R}} (u\,\nabla^2 u' - u'\,\nabla^2 u)\,dV = \iint_{\partial\mathbb{R}} \left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right) dS$$

Here, n is the unit vector normal to dS drawn outwards from $\mathbb{R}$ and $\partial/\partial n$ denotes differentiation in that direction. Since $\nabla^2 u = \nabla^2 u' = 0$ within $\partial\Sigma$, we have, in the region Σ, the relation

$$\int_C \left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right) dS' + \iint_{\partial\mathbb{R}} \left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right) dS = 0 \tag{5.28}$$

Inserting u' from relation (5.27), the above equation reduces to

$$\iint_C \left[u(\mathbf{r}')\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)-\frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial u}{\partial n}(\mathbf{r}')\right]dS'$$
$$+\iint_{\partial\mathbb{R}}\left[u(\mathbf{r}')\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)-\frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial u}{\partial n}(\mathbf{r}')\right]dS=0 \tag{5.29}$$

When Q is on C, we have

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{\varepsilon},\qquad \frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)=\frac{1}{\varepsilon^2}$$

Also, dS', the surface element on C, is $\varepsilon^2\sin\theta\, d\theta\, d\phi$. And on C,

$$u(\mathbf{r}')=u(\mathbf{r})+du$$

or

$$u(\mathbf{r}')=u(\mathbf{r})+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right)=u(\mathbf{r})+\varepsilon\left(\sin\theta\cos\phi\frac{\partial u}{\partial x}+\sin\theta\sin\phi\frac{\partial u}{\partial y}+\cos\theta\frac{\partial u}{\partial z}\right)$$

Therefore,

$$u(\mathbf{r}')=u(\mathbf{r})+O(\varepsilon)\qquad \text{on } C$$
$$\frac{\partial u}{\partial n}(\mathbf{r}')=\frac{\partial u}{\partial n}(\mathbf{r})+O(\varepsilon)$$

Now,

$$\iint_C u(\mathbf{r}')\frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dS'=\iint_C [u(\mathbf{r})+O(\varepsilon)]\frac{1}{\varepsilon^2}\times\varepsilon^2\sin\theta\, d\theta\, d\phi$$
$$=u(P)\iint_C \sin\theta\, d\theta\, d\phi+O(\varepsilon)$$
$$=u(P)\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\sin\theta\, d\theta\, d\phi+O(\varepsilon)$$
$$=4\pi u(\mathbf{r})+O(\varepsilon)$$

$$\iint_C \frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial}{\partial n}(\mathbf{r}')\, dS'=\frac{1}{\varepsilon}\iint\left[\frac{\partial u}{\partial n}(\mathbf{r})+0(\varepsilon)\right]\varepsilon^2\sin\theta\, d\theta\, d\phi+0(\varepsilon)$$

Employing these results and taking the limit as $\varepsilon \to 0$, Eq. (5.29) becomes

$$4\pi u(\mathbf{r}) + \iint_{\partial \mathbb{R}} \left[u(\mathbf{r}') \frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial}{\partial n} u(\mathbf{r}') \right] dS = 0$$

implying thereby

$$u(\mathbf{r}) = \frac{1}{4\pi} \iint_{\partial \mathbb{R}} \left[\frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial u}{\partial n}(\mathbf{r}') - u(\mathbf{r}') \frac{\partial}{\partial n}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\right] dS \tag{5.30}$$

Therefore, the value of u at an interior point of the region $\mathbb{R}$ is determined, if u and $\partial u/\partial n$ are known on the boundary, $\partial \mathbb{R}$. This leads to the conclusion that both the values of u and $\partial u/\partial n$ are required to obtain the solution of Dirichlet's problem. But this is not so, as can be seen from the concept of the Green's function defined as follows: Let $H(\mathbf{r}, \mathbf{r}')$ be a function harmonic in $\mathbb{R}$. Then the Green's function for the Dirichlet problem involving the Laplace operator is defined by G, the two point function of position, as

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} + H(\mathbf{r}, \mathbf{r}') \tag{5.31}$$

where $H(\mathbf{r}, \mathbf{r}')$ satifies the following

(i)

$$\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} \right) H(\mathbf{r}, \mathbf{r}') = 0 \tag{5.32}$$

(ii)

$$G = \frac{1}{|\mathbf{r}-\mathbf{r}'|} + H(\mathbf{r}, \mathbf{r}') = 0 \quad \text{on } \partial \mathbb{R} \tag{5.33}$$

Thus the Green's function for the Dirichlet problem involving the Laplace operator is a function $G(\mathbf{r}, \mathbf{r}')$ which satisfies the following properties:

(i) $$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') \quad \text{in } \mathbb{R} \tag{5.34}$$

(ii) $$G(\mathbf{r}, \mathbf{r}') = 0 \quad \text{on } \partial \mathbb{R} \tag{5.35}$$

(iii) G is symmetric, i.e.,

$$G(\mathbf{r}, \mathbf{r}') = G(\mathbf{r}', \mathbf{r}) \tag{5.36}$$

(iv) G is continuous, but $\partial G/\partial n$ has a discontinuity at the point $\mathbf{r}$, which is given by the equation

$$\operatorname*{Lt}_{\varepsilon \to 0} \iint_C \frac{\partial G}{\partial n}\, dS = 1 \tag{5.37}$$

Following the procedure adopted in the derivation of Eq. (5.30), replacing u' by $G(\mathbf{r}, \mathbf{r}')$, we can show that

$$u(\mathbf{r}) = -\frac{1}{4\pi} \iint_{\partial \mathbb{R}} \left[G(\mathbf{r}, \mathbf{r}') \frac{\partial u}{\partial n}(\mathbf{r}') - u(\mathbf{r}') \frac{\partial G}{\partial n}(\mathbf{r}, \mathbf{r}') \right] dS \tag{5.38}$$

From Eqs. (5.31) and (5.33) we can see that $G = 0$ on $\partial \mathbb{R}$. Thus the solution u at an interior point is given by the relation

$$u(\mathbf{r}) = -\frac{1}{4\pi} \iint_{\partial \mathbb{R}} u(\mathbf{r}') \frac{\partial G}{\partial n}(\mathbf{r}, \mathbf{r}')\, dS \tag{5.39}$$

and, therefore, the solution of the interior Dirichlet's problem is reduced to the determination of Green's function $G(\mathbf{r}, \mathbf{r}')$.

Green's function can be interpreted physically as follows: Let $\partial \mathbb{R}$ be a grounded electrical conductor (boundary potential zero) and if a unit charge is located at the source point $\mathbf{r}$, then G is the sum of the potential at the point $\mathbf{r}'$ due to the charge at the source point $\mathbf{r}$ in free space and the potential due to the charges induced on $\partial \mathbb{R}$. Thus,

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} + H(\mathbf{r}, \mathbf{r}') \tag{5.40}$$

Hence, property (i), viz, Eq. (5.34), essentially means that $\nabla^2 G = 0$ everywhere except at the source point $(\mathbf{r})$.

EXAMPLE 5.1 Consider a sphere with centre at the origin and radius 'a'. Apply the divergence theorem to the sphere and show that

$$\nabla^2 \left(\frac{1}{r} \right) = -4\pi \delta(r)$$

where $\delta(r)$ is a Dirac delta function.

Solution Applying the divergence theorem to

$$\text{grad} \left(\frac{1}{r} \right) = \nabla \left(\frac{1}{r} \right)$$

we get

$$\iiint_V \nabla \cdot \nabla \left(\frac{1}{r} \right) dV = \iint_S \nabla \left(\frac{1}{r} \right) \cdot \hat{n}\, dS$$

where $\hat{n}$ is an outward drawn normal. If $u = u(r, \theta, \phi)$, then

$$\text{grad } u = \hat{e}_r \frac{\partial u}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial u}{\partial \theta} + \hat{e}_\phi \frac{1}{r}\sin\theta \frac{\partial u}{\partial \phi}$$

Hence,

$$\iint_S \text{grad}\left(\frac{1}{r}\right)\cdot \hat{e}_r \, dS = \iint_S \frac{\partial}{\partial r}\left(\frac{1}{r}\right) dS = \iint_S \frac{1}{r^2}\, dS = -\frac{1}{a^2}\times 4\pi a^2 = -4\pi$$

Thus, we observe that $\nabla^2(1/r)$ has the following properties:

(i) It is undefined at the origin.

(ii) It vanishes if $r \neq 0$.

(iii) Its integral over any sphere with centre at the origin is -4π. Hence, we conclude that

$$\iiint_V \nabla^2\left(\frac{1}{r}\right) dV = -4\pi\delta(r)$$

Now we shall prove the symmetric property through the following theorem.

Theorem 5.1 Show that the Green's function $G(\mathbf{r}, \mathbf{r}')$ has the symmetric property. In other words, if P_1 and P_2 are two points within a fininte region $\mathbb{R}$ bounded by the surface $\partial\mathbb{R}$, then the value at P_2 of the Green's function for the point P_1 and the surface $\partial\mathbb{R}$ is equal to the value at P_1 of the Green's function for the point P_2 and the surface $\partial\mathbb{R}$.

Proof We have seen in Example 5.1 that $\nabla^2(1/r) = -4\pi\delta(r)$. If we define

$$G = \frac{1}{|\mathbf{r} - \mathbf{r}'|} + H$$

where H is harmonic, then

$$\nabla^2 G = \nabla^2\left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right) + \nabla^2 H = -4\pi\delta(\mathbf{r} - \mathbf{r}') + 0$$

Recalling Green's theorem

$$\iiint_{\mathbb{R}} (u\nabla^2 u' - u'\nabla^2 u)\, dV = \iint_{\partial\mathbb{R}} \left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right) dS \tag{5.41}$$

and taking $u = G(\mathbf{r}_1, \mathbf{r}') = (1/|\mathbf{r} - \mathbf{r}'|) + H$ (where $\mathbf{r}'$ is a position vector of Q, a variable point, and G is a Green's function for the point P_1 (see Fig. 5.2), we have

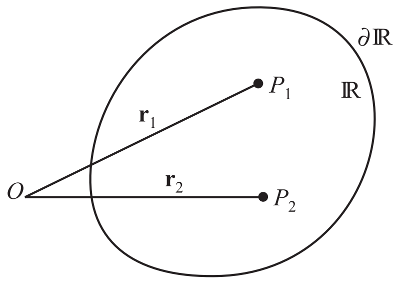

Fig. 5.2 An illustration of Theorem 5.1.

$$G(\mathbf{r}_1, \mathbf{r}') = 0 \text{ on } \partial \mathbb{R} \text{ when } \mathbf{r}' \in \mathbb{R}$$

Also,

$$\nabla^2 G(\mathbf{r}_1, \mathbf{r}') = \nabla^2 \left(\frac{1}{|\mathbf{r}_1 - \mathbf{r}'|} \right) = -4\pi\delta(\mathbf{r}_1 - \mathbf{r}')$$

Similarly, taking $u' = G(\mathbf{r}_2, \mathbf{r}')$, such that $\mathbf{r}' \in \mathbb{R}$, we get

$$G(\mathbf{r}_2, \mathbf{r}') = 0 \text{ on } \partial \mathbb{R}$$

$$\nabla^2 G(\mathbf{r}_2, \mathbf{r}') = -4\pi\delta(\mathbf{r}_2, \mathbf{r}')$$

Substituting these results into Eq. (5.41), we obtain

$$\iiint_{\mathbb{R}} [G(\mathbf{r}_1, \mathbf{r}')\nabla^2 G(\mathbf{r}_2, \mathbf{r}') - G(\mathbf{r}_2, \mathbf{r}')\nabla^2 G(\mathbf{r}_1, \mathbf{r}')]\, dV = \iint_{\partial \mathbb{R}} \left[G(\mathbf{r}_1, \mathbf{r}') \frac{\partial G}{\partial n}(\mathbf{r}_2, \mathbf{r}') - G(\mathbf{r}_2, \mathbf{r}') \frac{\partial G}{\partial n}(\mathbf{r}_1, \mathbf{r}') \right] dS$$

implying thereby

$$-4\pi \iiint_{\mathbb{R}} [G(\mathbf{r}_1, \mathbf{r}')\,\delta(\mathbf{r}_2 - \mathbf{r}') - G(\mathbf{r}_2, \mathbf{r}')\,\delta(\mathbf{r}_1 - \mathbf{r}')]\, dV = 0$$

Using the property of Dirac δ-function, we get immediately the relation

$$G(\mathbf{r}_1, \mathbf{r}_2) = G(\mathbf{r}_2, \mathbf{r}_1) \tag{5.42}$$

Property (iv) of the Green's function is proved in the following theorem.

Theorem 5.2 If G is continuous and $\partial G/\partial n$ has discontinuity at $\mathbf{r}$, in particular we can show that

$$\operatorname*{Lt}_{\varepsilon \to 0} \iint_{\partial C} \frac{\partial G}{\partial n}\, dS = 1$$

Proof Let C be a sphere with radius ε bounded by ∂C (see Fig. 5.1). We have already noted that G satisfies

$$\nabla^2 G = \delta(\mathbf{r} - \mathbf{r}')$$

Integrating both sides over the sphere C, we get

$$\iiint_C \nabla^2 G\, dV = 1$$

which can also be written as

$$\operatorname*{Lt}_{\varepsilon \to 0} \iiint_C \nabla^2 G\, dV = 1$$

Applying the divergence theorem, we get at once the result

$$\operatorname*{Lt}_{\varepsilon \to 0} \iint_{\partial C} \frac{\partial G}{\partial n}\, dS = 1 \tag{5.43}$$

Now, we shall present two well-known methods:

1. The method of images in Section 5.3.
2. The eigenfunction method in Section 5.4 for constructing Green's function for boundary value problems.

5.3 THE METHODS OF IMAGES

The method of images has been extensively used in the development of Electrostatics. It requires that we examine the effect of a certain source type of singularity at some point P of a given region $\mathbb{R}$, together with the influence of another source type of singularity located at a point P' outside the region of interest. Here, P' is the optical image of P in the boundary of the given region. This approach will work only for very simple geometries. We shall illustrate its application through the following examples for the construction of Green's function.

EXAMPLE 5.2 Use Green's function technique to solve the Dirichlet's problem for a semi-infinite space.

Solution Let the semi-infinite space be defined by $x \geq 0$, i.e., $0 \leq x \leq \infty, -\infty \leq y \leq \infty$, $-\infty \leq z \leq \infty$. We have to find a function u such that $\nabla^2 u = 0$ on $x \geq 0$, and $u = u\,(y, z)$ on $x = 0$; also $u \to 0$ as $r \to \infty$.

The Green's function $G(\mathbf{r}, \mathbf{r}')$ for the present problem must satisfy the following relations:

(i) $G(\mathbf{r}, \mathbf{r}') = \dfrac{1}{|\mathbf{r} - \mathbf{r}'|} + H$

(ii) $\left(\dfrac{\partial^2}{\partial x'^2} + \dfrac{\partial^2}{\partial y'^2} + \dfrac{\partial^2}{\partial z'^2}\right) H(\mathbf{r}, \mathbf{r}') = 0$

(iii) $G(\mathbf{r}, \mathbf{r}') = 0$ on the plane $x = 0$.

Let $P'(\boldsymbol{\rho})$ be the image of the point $P(\mathbf{r})$ in $x = 0$. If condition (ii) is satisfied

$$H(\mathbf{r}, \mathbf{r}') = -\frac{1}{|\boldsymbol{\rho} - \mathbf{r}'|}$$

Since $PQ = P'Q$ whenever Q lies on $x = 0$, we get (see Fig. 5.3)

$$|\boldsymbol{\rho} - \mathbf{r}'| = |\mathbf{r} - \mathbf{r}'|$$

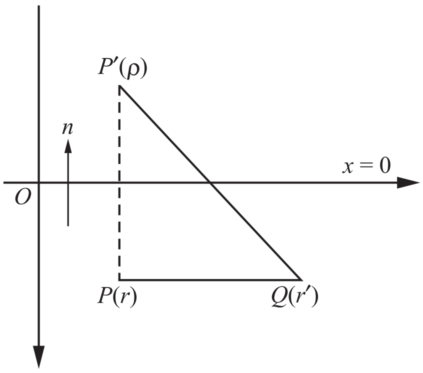

Fig. 5.3 An illustration of Example 5.2.

Then the required Green's function by the method of images is given by the equation

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{1}{|\boldsymbol{\rho} - \mathbf{r}'|} \tag{5.44}$$

which satisfies condition (iii). But from Eq. (5.39), we have

$$u(\mathbf{r}) = -\frac{1}{4\pi} \iint_{\partial \mathbb{R}} u(\mathbf{r}') \frac{\partial G}{\partial n}(\mathbf{r}, \mathbf{r}')\, dS \tag{5.45}$$

where dS is the surface element of $\partial\mathbb{R}$. Also,

$$\frac{\partial G}{\partial n} = -\frac{\partial}{\partial x'}\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x+x')^2+(y-y')^2+(z-z')^2}}\right]$$

$$\left(\frac{\partial G}{\partial x'}\right)_{x'=0} = \frac{2x}{[x^2+(y-y')^2+(z-z')^2]^{3/2}}$$

Substituting this result and noting that $u(\mathbf{r}') = f(y', z')$, Eq. (5.45) reduces to

$$u(x, y, z) = \frac{x}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(y', z')\,dy'\,dz'}{[x^2+(y-y')^2+(z-z')^2]^{3/2}} \tag{5.46}$$

which can be integrated when the nature of the function $f(y', z')$ is explicity given.

EXAMPLE 5.3 Obtain the solution of the interior Dirichlet problem for a sphere using the Green's function method and hence derive the Poisson integral formula.

Solution The task is to determine the function $u(r, \theta, \phi)$ satisfying

$$\nabla^2 u = 0, \quad 0 \le r \le a, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi \tag{5.47}$$

subject to

$$u(a, \theta, \phi) = f(\theta, \phi) \tag{5.48}$$

Green's function for a sphere can be expressed as

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|} + H(\mathbf{r}, \mathbf{r}') \tag{5.49}$$

where H is so chosen that the conditions

$$\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right)H(\mathbf{r}, \mathbf{r}') = 0 \tag{5.50}$$

$$G(\mathbf{r}, \mathbf{r}') = 0 \tag{5.51}$$

are satisfied on the surface of the sphere.

Let $P(r, \theta, \phi)$ be a point inside a sphere as shown in Fig. 5.4, where we place a unit charge with position vector $\mathbf{r}$ and let its inverse point with respect to the sphere be P' (see Fig. 5.4) such that

$$\mathbf{OP} \cdot \mathbf{OP'} = a^2$$

Let $\mathbf{OP} = \mathbf{r}, \mathbf{OP}' = \rho, \mathbf{OQ}' = \mathbf{r}'$, then $OP' = a^2/r$. If Q' be a variable point on the surface of the sphere, then from the similarity of triangles $OQ'P$ and $OQ'P'$, we have

$$\frac{PQ'}{P'Q'} = \frac{r}{a} = \frac{a}{\rho}$$

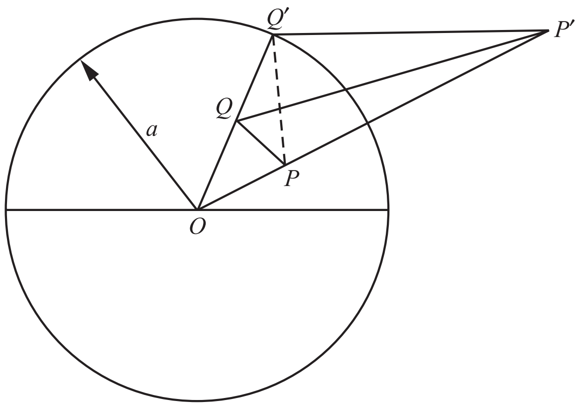

Fig. 5.4 Inverse image point in a sphere.

from which we obtain

$$PQ' = \frac{r}{a} P'Q'$$

This relation is valid for all points on the spherical surface. Therefore, the harmonic function is

$$H(r, r') = -\frac{a}{r} \frac{1}{|\mathbf{P'Q'}|} = -\frac{a}{r\,|\mathbf{OP'} - \mathbf{OQ'}|} = \frac{a}{r\,|\rho - \mathbf{r}'|}$$

which can also be written as

$$H(\mathbf{r}, \mathbf{r}') = \frac{-a}{r\left|\frac{a^2}{r} \cdot \frac{\mathbf{r}}{r} - \mathbf{r}'\right|} = \frac{-a}{r\left|\frac{a^2}{r^2}\mathbf{r} - \mathbf{r}'\right|} \tag{5.52}$$

This form of H satisfies the Laplace equation (5.50). Let $Q\,(r', \theta', \phi')$ be a variable point inside the sphere. When Q lies on the surface of the sphere, say Q', we can verify that

$$\frac{PQ}{P'Q} = \frac{r}{a} \tag{5.53}$$

Thus the Green's function to the present problem is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} - \frac{a/r}{\left|\frac{a^2}{r^2}\mathbf{r} - \mathbf{r}'\right|} \tag{5.54}$$

or

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{PQ}|} - \frac{a/r}{|\mathbf{P'Q}|} = \frac{1}{R} - \frac{a/r}{R'} \tag{5.55}$$

where $PQ = R, P'Q = R'$. On the surface of the sphere, using Eq. (5.53) we can verify that G vanishes, and hence G defined by Eq. (5.54) is the appropriate Green's function.

Using cosine law in solid geometry, we get

$$(PQ)^2 = r^2 + (r'^2) - 2rr' \cos\theta = R^2$$

$$(P'Q)^2 = (OP')^2 + (r')^2 - 2\,(OP')\,r' \cos\theta = (R')^2 \tag{5.56}$$

or

$$(P'Q)^2 = \frac{a^4}{r^2} + (r')^2 - \frac{2a^2}{r} r' \cos\theta = (R')^2$$

From Eq. (5.55), on the sphere,

$$\frac{\partial G}{\partial n} = \frac{\partial G}{\partial r'} = -\frac{1}{R^2}\frac{\partial R}{\partial r'} + \frac{a/r}{(R')^2}\frac{\partial R'}{\partial r'} = -\frac{1}{R^3}\left[R\frac{\partial R}{\partial r'} - \left(\frac{a}{r}\right)\frac{R^3}{(R')^3} R'\frac{\partial R'}{\partial r'}\right]$$

But, Eq. (5.53) gives

$$\frac{R}{R'} = \frac{r}{a}$$

Therefore,

$$\frac{\partial G}{\partial n} = -\frac{1}{R^3}\left[R\frac{\partial R}{\partial r'} - \left(\frac{a}{r}\right)\frac{r^3}{a^3} R'\frac{\partial R'}{\partial r'}\right]$$

Also, Eq. (5.56) yields

$$2R\frac{\partial R}{\partial r'} = 2r' - 2r\cos\theta$$

$$2R'\frac{\partial R'}{\partial r'} = 2r' - 2\frac{a^2}{r}\cos\theta$$

Hence,

$$\left.\frac{\partial G}{\partial n}\right|_{r'=a} = -\frac{1}{R^3}\left[(a - r\cos\theta) - \frac{r^2}{a^2}\left(a - \frac{a^2}{r}\cos\theta\right)\right]$$

$$= -\frac{1}{R^3}\left(a - \frac{r^2}{a}\right) = \frac{r^2 - a^2}{aR^3}$$

or

$$\left.\frac{\partial G}{\partial n}\right|_{r'=a} = \frac{r^2 - a^2}{a\,(r^2 + a^2 - 2ar\cos\theta)^{3/2}}$$

If $u = f(\theta, \phi)$ on the surface of the sphere, then the solution to the interior Dirichlet's problem for a sphere is

$$u(r, \theta, \phi) = \frac{-1}{4\pi}\iint_S \frac{f(\theta', \phi')(-1)(a^2 - r^2)}{a\,(r^2 + a^2 - 2ar\cos\theta)^{3/2}}\,dS'$$

But, $dS' = a^2 \sin\theta'\, d\theta'\, d\phi'$. Therefore,

$$u(r, \theta, \phi) = \frac{a\,(a^2 - r^2)}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{f(\theta', \phi')\sin\theta'\, d\theta'\, d\phi'}{(r^2 + a^2 - 2ar\cos\theta)^{3/2}} \tag{5.57}$$

which is called the Poisson integral formula.

EXAMPLE 5.4 Consider the case when $\mathbb{R}$ consists of the half-plane defined by $x \geq 0$, $-\infty < y < \infty$, and hence solve $\nabla^2 u = 0$ in the above region subject to the condition $u = f(y)$ on $x = 0$, using the Green's function technique.

Solution In this problem $x = 0$ is the boundary. Let $P'(-x, y)$ be the image point of $P(x, y)$. If $Q(x', y')$ is a point on the boundary $x = 0$ (see Fig. 5.5), then $PQ = P'Q$, and we construct the Green's function G such that

$$G = \ln\frac{1}{|\mathbf{r} - \mathbf{r}'|} + H = \ln\frac{1}{PQ} + H$$

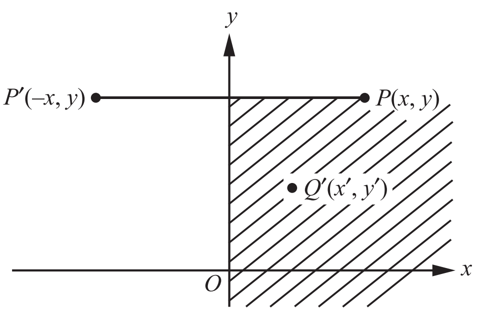

Fig. 5.5 An illustration of Example 5.4.

Let

$$H = -\ln \frac{1}{|\boldsymbol{\rho} - \mathbf{r}'|} = -\ln \frac{1}{P'Q}$$

Then, $G = \ln (P'Q/PQ)$.

Obviously, $\nabla^2 H = 0$ in the region $x \geq 0$ and on the boundary $x = 0$, $G = \ln 1 = 0$ is satisfied. Hence the required Green's function is given by

$$G(x, y; x', y') = \frac{1}{2} \ln \left[\frac{(x + x')^2 + (y - y')^2}{(x - x')^2 + (y - y')^2} \right]$$

Here, the outward drawn normal to the boundary is in the direction of the x-axis. Therefore,

$$\frac{\partial G}{\partial n} = -\left(\frac{\partial G}{\partial x'} \right)_{x'=0} = -\frac{1}{2} \left[\frac{2x}{(x + x')^2 + (y - y')^2} + \frac{2x}{(x - x')^2 + (y - y')^2} \right]_{x'=0} = -\frac{2x}{x^2 + (y - y')^2}$$

From Eq. (5.39),

$$u(x, y) = -\frac{1}{2\pi} \int f(y') \left[\frac{-2x}{x^2 + (y - y')^2} \right] dy'$$

or

$$u(x, y) = \frac{x}{\pi} \int_{-\infty}^{\infty} \left[\frac{f(y')}{x^2 + (y - y')^2} \right] dy' \qquad (5.58)$$

EXAMPLE 5.5 Determine the Green's function for the Dirichlet problem for a circle given by

$$\nabla^2 u = 0, \qquad r < a$$
$$u = f(\theta) \quad \text{on } r = a$$

Solution Let $P(r, \theta)$ and $Q(r', \theta')$ have position vectors $\mathbf{r}$ and $\mathbf{r}'$, and let P' be the inverse of P with respect to the circle so that $OP \cdot OP' = a^2$ and P' has coordinates $(a^2/r, \theta)$, as shown in Fig. 5.6.

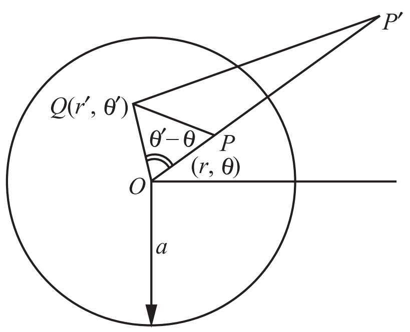

Fig. 5.6 An illustration of Example 5.5

Now, we construct the Green's function G such that

$$G = \ln \frac{1}{|\mathbf{r} - \mathbf{r}'|} + H$$

Let $H = \ln (r \cdot P'Q/a)$ (as for the sphere) so that it can be verified that

$$\nabla^2 H = 0$$

$$G = \ln \frac{r \cdot P'Q}{a \cdot PQ} \tag{5.59}$$

On the circle $r = a$,

$$G = \ln \frac{P'Q}{PQ} = \ln \frac{P'Q}{(r/a)P'Q} = \ln 1 = 0$$

However,

$$PQ^2 = r^2 + r'^2 - 2rr' \cos (\theta' - \theta) \tag{5.60}$$

$$P'Q^2 = r'^2 + \frac{a^4}{r^2} - 2r' \frac{a^2}{r} \cos (\theta' - \theta) \tag{5.61}$$

Substituting Eqs. (5.60) and (5.61) into Eq. (5.59), G can be written as

$$\begin{aligned} G &= \frac{1}{2} \ln \left[\frac{r^2}{a^2} \frac{\{r'^2 + a^4/r^2 - 2r'a^2 \cos (\theta' - \theta)/r\}}{\{r^2 + r'^2 - 2rr' \cos (\theta' - \theta)\}} \right] \\ &= \frac{1}{2} \ln \frac{a^2 + r^2 r'^2/a^2 - 2rr' \cos (\theta' - \theta)}{r^2 + r'^2 - 2rr' \cos (\theta' - \theta)} \end{aligned} \tag{5.62}$$

But, on the circle $r = a$,

$$\left(\frac{\partial G}{\partial n} \right)_{r'=a} = \left(\frac{\partial G}{\partial r'} \right)_{r'=a} = \frac{-(a^2 - r^2)}{a\,[a^2 - 2ar \cos (\theta' - \theta) + r^2]}$$

Therefore,

$$u(r, \theta) = \frac{a^2 - r^2}{2\pi a} \int_0^{2\pi} \frac{f(\theta')\, d\theta'}{\{a^2 - 2ar \cos (\theta' - \theta) + r^2\}}$$

5.4 THE EIGENFUNCTION METHOD

Let us consider the Dirichlet boundary value problem described by

$$\nabla^2 u = f \tag{5.63}$$

valid in certain region $\mathbb{R}$, subject to the boundary condition

$$u = g \tag{5.64}$$

on $\partial\mathbb{R}$, the boundary of $\mathbb{R}$.

From the definition of Green's function which has been introduced in Section 5.1. The Green's function must satisfy the relations

$$\nabla^2 G = \delta(x-\xi,\ y-\eta) \quad \text{in } \mathbb{R} \tag{5.65}$$

$$G = 0, \quad \text{on } \partial\mathbb{R} \tag{5.66}$$

Now, consider the eigenvalue problem associated with the operator ∇^2 in the domain $\mathbb{R}$, i.e.

$$\nabla^2\phi + \lambda\phi = 0 \quad \text{in } \mathbb{R} \tag{5.67}$$

$$\phi = 0 \quad \text{on } \partial\,\mathbb{R} \tag{5.68}$$

Let λ_{mn} be the eigenvalues and ϕ_{mn} be the corresponding eigenfunctions. Suppose we give Fourier series expansion to G and δ in terms of the eigenfunctions ϕ_{mn} in the following form:

$$G\,(x, y; \xi, \eta) = \sum_m \sum_n a_{mn}\,(\xi, \eta)\,\phi_{mn}(x, y) \tag{5.69}$$

$$\delta\,(x-\xi, y-\eta) = \sum_m \sum_n b_{mn}\,(\xi, \eta)\,\phi_{mn}(x, y) \tag{5.70}$$

where

$$b_{mn} = \frac{1}{\|\phi_{mn}\|^2} \iint_{\mathbb{R}} \delta\,(x-\xi, y-\eta)\,\phi_{mn}(x, y)\,dx\,dy = \frac{\phi_{mn}\,(\xi, \eta)}{\|\phi_{mn}\|^2} \tag{5.71}$$

$$\|\phi_{mn}\|^2 = \iint_{\mathbb{R}} \phi_{mn}^2\,dx\,dy$$

Now, substituting Eqs. (5.69) and (5.70) into Eqs. (5.65) and (5.66) and noting that Eq. (5.67) has the form

$$\nabla^2\phi_{mn} + \lambda_{mn}\,\phi_{mn} = 0 \tag{5.72}$$

we obtain

$$\nabla^2 \sum_m \sum_n a_{mn}(\xi, \eta)\,\phi_{mn}(x, y) = \sum_m \sum_n b_{mn}(\xi, \eta)\,\phi_{mn}(x, y)$$

Using Eqs. (5.71) and (5.72), the above equation reduces to

$$-\sum_m \sum_n \lambda_{mn} a_{mn}(\xi,\eta)\,\phi_{mn}(x,y) = \sum_m \sum_n \frac{\phi_{mn}(\xi,\eta)\,\phi_{mn}(x,y)}{\|\phi_{mn}\|^2}$$

from which we get

$$a_{mn}(\xi,\eta) = -\frac{\phi_{mn}(\xi,\eta)}{\lambda_{mn}\|\phi_{mn}\|^2} \tag{5.73}$$

Hence, Eqs. (5.69) and (5.73) give the required Green's function for the Dirichlet problem, in the form

$$G(x, y;\xi,\eta) = -\frac{\sum_m \sum_n \phi_{mn}(\xi,\eta)\,\phi_{mn}(x,y)}{\lambda_{mn}\|\phi_{mn}\|^2} \tag{5.74}$$

EXAMPLE 5.6 Find the Green's function for the Dirichlet problem on the rectangle $\mathbb{R}: 0 \le x \le a,\ \ 0 \le y \le b$, described by the PDE

$$(\nabla^2 + \lambda)\,u = 0 \quad \text{in } \mathbb{R} \tag{5.75}$$

and the BC $u = 0$ on $\partial \mathbb{R}$.

Solution The eigenfunctions of the given PDE can be obtained easily by using the variables separable method. Let us assume the solution of the given PDE in the form $u(x, y) = X(x)\,Y(y)$. Substituting into the given PDE, we obtain

$$\frac{X''}{X} = -\left(\frac{Y''}{Y} + \lambda\right) = -\nu \quad \text{(a separation constant)} \tag{5.76}$$

Since u is zero on the boundary $\partial\mathbb{R}$, X satisfies

$$X'' + \nu X = 0, \qquad X(0) = X(a) = 0 \tag{5.77}$$

Its solution, in general, is

$$X(x) = A\cos\sqrt{\nu}\,x + B\sin\sqrt{\nu}\,x$$

$X(0) = 0$ implies $A = 0$. $X(a) = 0$ gives $\sin\sqrt{\nu}\,a = 0$, implying $\sqrt{\nu} = n\pi/a$. The corresponding real valued eigenfunctions are $X_n = \sin(n\pi x/a), n = 1, 2, \ldots,$ while the eigenvalues are $\nu_n = n^2\pi^2/a^2, n = 1, 2, \ldots$ Now, the factor Y satisfies

$$Y'' + (\lambda - \nu_n)\,Y = 0, \qquad Y(0) = Y(b) = 0$$

Following the above procedure, we can show at once that the eigenfunctions are given by

$$Y_m = \sin \frac{m\pi y}{b}, \quad m = 1, 2, \ldots$$

and the corresponding eigenvalues are

$$\lambda - \nu_n = \frac{m^2 \pi^2}{b^2}, \quad m = 1, 2, \ldots$$

Thus, we obtain the eigenfunctions to the given problem in the form

$$\phi_{mn}(x, y) = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad m = 1, 2, \ldots; n = 1, 2, \ldots \tag{5.78}$$

while the eigenvalues are given by

$$\lambda_{mn} = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \tag{5.79}$$

Computation of $\|\phi_{mn}\|$ gives

$$\|\phi_{mn}\|^2 = \int_0^a \int_0^b \sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} \, dx \, dy = \frac{ab}{4} \tag{5.80}$$

Hence, the Green's function for the given Dirichlet problem can be obtained from Eq. (5.74) with the help of Eqs. (5.79) and (5.80) as

$$G(x, y; \xi, \eta) = -\frac{4ab}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin(m\pi x/a) \sin(n\pi y/b) \sin(m\pi\xi/a) \sin(n\pi\eta/b)}{m^2b^2 + n^2a^2} \tag{5.81}$$

5.5 GREEN'S FUNCTION FOR THE WAVE EQUATION—HELMHOLTZ THEOREM

In finding the solution of the wave equation by the variables separable method, we have observed that the function depending on spatial coordinates satisfies the Helmholtz equation which is also called the spatial form of the wave equation. The solution of the Helmholtz equation under certain boundary conditions can be made to depend on how the appropriate Green's function is determined, in terms of which the solution to the wave equation can be obtained.

Let u be a solution of the Helmholtz equation

$$\nabla^2 u + k^2 u = 0 \tag{5.82}$$

in the region $\mathbb{R}$. Also, let all the singularities of u lie outside the closed region $\mathbb{R}$ (see Fig. 5.7), the boundary of which is denoted by $\partial \mathbb{R}$. Consider the singularity solution of the Helmholtz equation given by

$$u' = \exp\{ik|\mathbf{r} - \mathbf{r}'|\}/|\mathbf{r} - \mathbf{r}'| \tag{5.83}$$

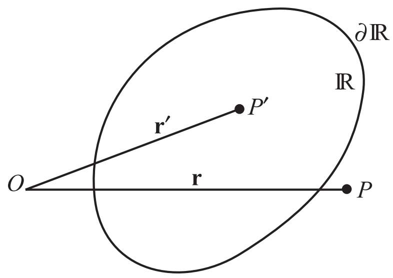

Fig. 5.7 A closed region.

Recalling the Green's theorem (2.19), we have

$$\iiint_{\mathbb{R}} (u\,\nabla^2 u' - u'\nabla^2 u)\,dV = \iint_{\partial\mathbb{R}} \left(u\frac{\partial u'}{\partial n} - u'\frac{\partial u}{\partial n}\right) dS \tag{5.84}$$

where n is the outward normal to $\partial\mathbb{R}$. Since $P(\mathbf{r})$ lines outside $\mathbb{R}$, $|\mathbf{r}-\mathbf{r}'| \neq 0$, and it is possible to calculate $\nabla^2 u'$ for all $\mathbf{r}' \in R$. Thus, setting $u = \psi$ and $u' = \psi'$ into the left-hand side of Eq. (5.84), we obtain, after using Eqs. (5.82) and (5.83), relation

$$\iiint_{\mathbb{R}} (\psi\,\nabla^2\psi' - \psi'\,\nabla^2\psi)\,dV = \iiint_{\mathbb{R}} \left[\psi\nabla^2\left\{\frac{\exp(ik\,|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\right\} - \frac{\exp(ik\,|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}(-k^2\psi)\right] dV$$

$$= \iiint_{\mathbb{R}} \left[\psi(ik)^2\frac{\exp(ik\,|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} + k^2\psi\frac{\exp(ik\,|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\right] dV = 0$$

Therefore,

$$\iiint_{\partial\mathbb{R}} \left[\psi(\mathbf{r}')\frac{\partial}{\partial n}\left\{\frac{\exp(ik\,|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\right\} - \frac{\exp(ik\,|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial}{\partial n}\psi(\mathbf{r}')\right] dS = 0 \tag{5.85}$$

If $P(\mathbf{r})$ lies inside $\mathbb{R}$, we surround P by a sphere of radius ε. Now applying Green's theorem to the region Σ bounded externally by $\partial\mathbb{R}$ and internally by C as in Fig. 5.1, and noting that $|\mathbf{r}-\mathbf{r}'| \neq 0$, we get

$$\left[\iint_C + \iint_{\partial\mathbb{R}}\right]\left\{\psi(\mathbf{r}')\frac{\partial}{\partial n}\left(\frac{\exp(ik\,|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\right) - \frac{\exp(ik\,|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial}{\partial n}\psi(\mathbf{r}')\right\} dS = 0$$

However,

$$\frac{\partial}{\partial n}\left\{\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\right\}=\frac{\partial}{\partial \varepsilon}\left\{\frac{\exp(ik\varepsilon)}{\varepsilon}\right\}=\frac{ik\exp(ik\varepsilon)}{\varepsilon}-\frac{\exp(ik\varepsilon)}{\varepsilon^2}=\frac{\exp(ik\varepsilon)}{\varepsilon}\left(ik-\frac{1}{\varepsilon}\right)$$

$$=\left(ik-\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}$$

Hence,

$$\iint_{\partial \mathbb{R}}\left[\psi(\mathbf{r}')\frac{\partial}{\partial n}\left\{\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\right\}-\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial}{\partial n}\psi(\mathbf{r}')\right]dS$$

$$=-\iint_{C}\left\{\left(ik-\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\psi(\mathbf{r}')-\frac{\partial}{\partial n}\psi(\mathbf{r}')\right\}\cdot\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\,dS \qquad (7.86)$$

Now using the relations

$$\psi(\mathbf{r}')=\psi(\mathbf{r})+0(\varepsilon)$$

$$dS=\varepsilon^2\sin\theta\,d\theta\,d\phi$$

$$\frac{\partial\psi}{\partial n}=\left(\frac{\partial\psi}{\partial n}\right)_P+0(\varepsilon)$$

Equation (5.86) can be rewritten as

$$\iint_{C}\left\{\left(ik-\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)[\psi(\mathbf{r})+0(\varepsilon)]-\left(\frac{\partial\psi}{\partial n}\right)_P+0(\varepsilon)\right\}\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\varepsilon^2\sin\theta\,d\theta\,d\phi$$

Taking the limit as $\varepsilon\to 0$, we get

$$-\iint_{C}\psi(\mathbf{r})\sin\theta\,d\theta\,d\phi=-4\pi\psi(\mathbf{r})$$

Hence,

$$\iint_{\partial \mathbb{R}}\left[\psi(\mathbf{r}')\frac{\partial}{\partial n}\left\{\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\right\}-\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial}{\partial n}\psi(\mathbf{r}')\right]dS=-4\pi\psi(\mathbf{r}) \qquad (5.87)$$

Thus, combining the results (5.85) and (5.87), we have the Helmholtz theorem which states that if $\psi(\mathbf{r})$ is a solution of the spatial form of the wave equation $\nabla^2\psi+k^2\psi=0$, possessing continuous first and second order partial derivatives in $\mathbb{R}$ bounded by $\partial\mathbb{R}$, then

$$\frac{1}{4\pi}\iint_{\partial \mathbb{R}} \left\{\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial\psi(\mathbf{r}')}{\partial n}-\psi(\mathbf{r}')\frac{\partial}{\partial n}\left(\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\right)\right\} dS = \begin{cases}\psi(\mathbf{r}') & \text{if } \mathbf{r}\in R\\ 0 & \text{if } \mathbf{r}\notin R\end{cases} \tag{5.88}$$

From this result, it appears as though the values of ψ and $\partial\psi/\partial n$ on the surface $\partial\mathbb{R}$ bounding the region $\mathbb{R}$ have to be prescribed; it also appears that these values can be considered arbitrarily and independently of each other. But, it can be seen that this is not true if we introduce the Green's function $G(\mathbf{r},\mathbf{r}')$. Let $G(\mathbf{r},\mathbf{r}')$ be a Green's function such that

(i) $\nabla^2 G(\mathbf{r},\mathbf{r}')+k^2 G(\mathbf{r},\mathbf{r}')=0$, and

(ii) G is finite and continuous with respect to both the variables $\mathbf{r}$ and $\mathbf{r}'$.

If we replace $\exp(ik|\mathbf{r}-\mathbf{r}'|)/|\mathbf{r}-\mathbf{r}'|$ by G, we get from Eq. (5.88) the relations

$$\psi(\mathbf{r}) = \frac{1}{4\pi}\iint_{\partial\mathbb{R}}\left[G(\mathbf{r},\mathbf{r}')\frac{\partial\psi(\mathbf{r}')}{\partial n}-\psi(\mathbf{r}')\frac{\partial G}{\partial n}(\mathbf{r},\mathbf{r}')\right] dS$$

where n is the outward drawn normal to $\partial\mathbb{R}$. If $G=G_1$ such that G_1 satisfies (i) and (ii) and $G_1(\mathbf{r},\mathbf{r}')=0$ on $\partial\mathbb{R}$, then

$$\psi(\mathbf{r}) = -\frac{1}{4\pi}\iint_{\partial\mathbb{R}}\psi(\mathbf{r}')\frac{\partial G}{\partial n}(\mathbf{r},\mathbf{r}')\, dS \tag{5.89}$$

EXAMPLE 5.7 Determine the Green's function for the Helmholtz equation for the half-space $z\ge 0$.

Solution Here the boundary is the xoy-plane. Let $P(\mathbf{r})$ be a point and let P' be its image in the plane $z=0$ (see Fig. 5.8). Also, let $\mathbf{r}=(x,y,z)$; then $\boldsymbol{\rho}=(x,y,-z)$; again, let $\mathbf{r}'=(x',y',z')$. When $\mathbf{r}'$ lies on the boundary, i.e., on the xy plane, $\mathbf{r}'=(x',y',0)$ and $|\mathbf{P}-\mathbf{r}'|=|\mathbf{r}-\mathbf{r}'|$.

Let

$$G(\mathbf{r},\mathbf{r}')=\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}-\frac{\exp(ik|\boldsymbol{\rho}-\mathbf{r}'|)}{|\boldsymbol{\rho}-\mathbf{r}'|} \tag{5.90}$$

If $\mathbf{r}'$ lies on the boundary, then

$$\frac{\partial G}{\partial n}=\left(-\frac{\partial G}{\partial z'}\right)_{\text{on the } xy\text{-plane}}$$

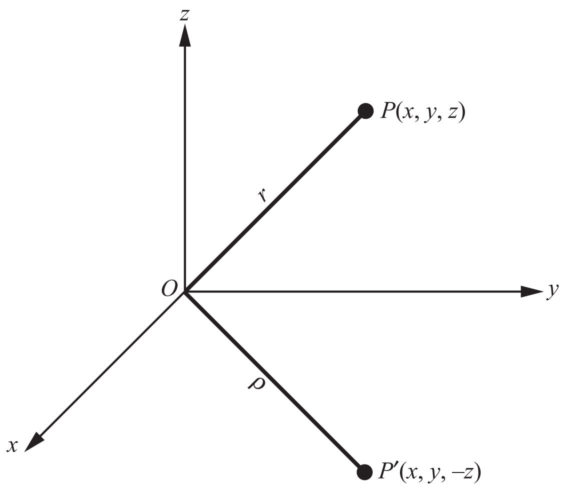

Fig. 5.8 Illustration of Example 5.7.

But

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$$

$$|\boldsymbol{\rho}-\mathbf{r}'| = \sqrt{(x-x')^2+(y-y')^2+(z+z')^2}$$

Therefore,

$$\left.\frac{\partial G}{\partial z'}\right|_{z'=0} = \left[-\frac{ze^{ikR}}{R^3}+\frac{ikze^{ikR}}{R^2}-\frac{ze^{ikR}}{R^3}+\frac{ikze^{ikR}}{R^2}\right]$$

$$= \frac{2ze^{ikR}}{R^2}\left(ik-\frac{1}{R}\right) = 2\frac{\partial}{\partial z}\left(\frac{e^{ikR}}{R}\right)$$

where $R^2 = (x-x')^2+(y-y')^2+z^2$

From the result (5.89), the required Green's function is

$$\psi(\mathbf{r}) = -\frac{1}{2\pi}\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y')\frac{e^{ikR}}{R}dx'\,dy' \tag{5.91}$$

where $f(x', y')$, is the value of ψ on the boundary $z = 0$.

EXAMPLE 5.8 Solve the following one-dimensional wave equation using the Green's function method:

$$\text{PDE}: u_{tt} - c^2u_{xx} = 0, \qquad 0 \le x \le L,\ t \ge 0$$

$$\text{BCs}: u\,(0, t) = u\,(L, t) = 0 \qquad \text{for } t \ge 0$$

$$\text{ICs}: u\,(x, 0) = f\,(x)$$

$$u_t\,(x, 0) = g\,(x), \qquad 0 \le x \le L$$

Solution In Section 4.5, the general solution of the one-dimensional wave equation was obtained as

$$u(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left(A_n\cos\frac{n\pi ct}{L}+B_n\sin\frac{n\pi ct}{L}\right) \tag{5.92}$$

where

$$A_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$B_n=\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

Now, let us define a function $G(x,t;\xi,\tau)$ as follows:

$$G(x,t;\xi,\tau)=\frac{2}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{L}\sin\frac{n\pi\xi}{L}\sin\left[\frac{n\pi c}{L}(t-\tau)\right] \tag{5.93}$$

It can be shown that these series converge for all values of x, t, ξ, τ. It can also be noted that G as a function of x satisfies the boundary conditions, i.e.,

$$G(0,t,\xi,\tau)=0, \quad G(L,t,\xi,\tau)=0, \quad t\ge 0$$

Also,

$$\begin{aligned}
&G(x,\tau;\xi,\tau)=0, && 0\le x\le L\\
&G(\xi,t;x,\tau)=G(x,t;\xi,\tau) && \text{for all } x \text{ and } \xi\\
&G(x,\tau;\xi,t)=-G(x,t;\xi,\tau) && \text{for all } t \text{ and } \tau
\end{aligned}$$

Thus, the function G defined by Eq. (5.93) is called the Green's function of the given IBVP.

Substituting the series expression for G and formally interchanging the operations of summation and integration, we can verify that the series solution (5.92) of the given problem can be rewritten in terms of Green's function of the form

$$u(x,t)=\int_0^L G_t(x,t;\xi,0)\,f(\xi)\,d\xi+\int_0^L G(x,t;\xi,0)\,g(\xi)\,d\xi$$

5.6 GREEN'S FUNCTION FOR THE DIFFUSION EQUATION

Consider the diffusion equation (also known as heat conduction equation) with no sources present:

$$\frac{\partial u}{\partial t}=k\nabla^2 u \tag{5.94}$$

subject to the boundary condition

$$u(\mathbf{r}, t) = \theta(\mathbf{r}, t), \quad \mathbf{r} \in S \tag{5.95}$$

and the initial condition

$$u(\mathbf{r}, 0) = f(\mathbf{r}), \quad \mathbf{r} \in V \tag{5.96}$$

It is necessary for us to find $u(\mathbf{r}, t)$ in the volume V bounded by the surface S by using the Green's function technique.

We shall define the Green's function $G(\mathbf{r}, \mathbf{r}', t - t')$, $t > t'$, where t' is a parameter, such that the following conditions are satisfied:

(i)
$$\frac{\partial G}{\partial t} = k\nabla^2 G \tag{5.97}$$

(ii)
$$\text{BC: } G(\mathbf{r}, \mathbf{r}', t - t') = 0, \ \mathbf{r}' \in S \tag{5.98}$$

(iii) The initial condition $\underset{t \to t'}{\text{Lt}}\, G = 0$ for all points in the volume V except at the point $\mathbf{r}$, where G assumes the singularity solution as given in the introduction in the form

$$\frac{1}{8[\pi k(t - t')]^{3/2}} \exp\left[-\frac{|\mathbf{r} - \mathbf{r}'|^2}{4k(t - t')}\right] \tag{5.99}$$

It is easy to note that G depends only on t through the term $(t - t')$. Hence, equivalently, Eq. (5.97) can also be written as

$$\frac{\partial G}{\partial t'} = -k\nabla^2 G \tag{5.100}$$

Here, G can be interpreted as the temperature at the point $\mathbf{r}'$ at time t, corresponding to a source of unit strength generated at $t = t'$. Initially, the solid with volume V and surface S is at zero temperature. Since t' must lie within the time interval for t for which Eqs. (5.94) and (5.95) are valid, these equations may be rewritten as

$$\frac{\partial u}{\partial t'} = -k\nabla^2 u, \quad t' < t \tag{5.101}$$

$$u(\mathbf{r}', t') = \theta(\mathbf{r}', t'), \ \mathbf{r}' \in S \tag{5.102}$$

Also,

$$\frac{\partial}{\partial t'}(uG) = G\frac{\partial u}{\partial t'} + u\frac{\partial G}{\partial t'}$$

Now, using Eqs. (5.100) and (5.101), we have

$$\frac{\partial}{\partial t'}(uG) = Gk\nabla^2 u - ku\nabla^2 G$$

If ε is an arbitrary positive constant, then

$$\int_0^{t-\varepsilon}\left\{\iiint_V \frac{\partial}{\partial t'}(uG)\,d\tau'\right\}dt' = k\int_0^{t-\varepsilon}\left\{\iiint_V (G\nabla^2 u - u\nabla^2 G)\,d\tau'\right\}dt' \qquad (5.103)$$

Interchanging the order of integration on the left-hand side of the above equation, we have

$$\iiint_V\left\{\int_0^{t-\varepsilon}\frac{\partial}{\partial t'}(uG)\,dt'\right\}d\tau' = \iiint_V (uG)_0^{t-\varepsilon}d\tau' = \iiint_V (uG)_{t'=t-\varepsilon}d\tau' - \iiint_V (uG)_{t'=0}d\tau'$$

But

$$[u\,(\mathbf{r}',\ t')]_{t'=t-\varepsilon} = u\,(\mathbf{r}',\ t-\varepsilon)$$

and we want values of u for times greater than some initial time. Hence, $u\,(\mathbf{r}', t-\varepsilon)$ can be taken as $u\,(\mathbf{r}, t')$. After using IC (5.96), the left-hand side of Eq. (5.103) becomes

$$u\,(\mathbf{r},\ t)\iiint_V G\,(\mathbf{r}, \mathbf{r}',\ t-t')|_{t'=t-\varepsilon}d\tau' - \iiint_V G\,(\mathbf{r}, \mathbf{r}',\ t)\ f\,(\mathbf{r}')d\tau'$$

From the expression (5.99) for $G(\mathbf{r}, \mathbf{r}', t-t')$, we can show that

$$\iiint_V G\,(\mathbf{r}, \mathbf{r}', t-t')|_{t'=t-\varepsilon}^{d\tau'} = \iiint_V \frac{1}{8(\pi k\varepsilon)^{3/2}}\exp\left[-\frac{|\mathbf{r}-\mathbf{r}'|^2}{4k\varepsilon}\right] = 1$$

as $\varepsilon \to 0$. After applying Green's theorem, the right-hand side of Eq. (5.103) beomes

$$k\int_0^{t-\varepsilon}\left\{\iiint_V (G\nabla^2 u - u\nabla^2 G)\,d\tau'\right\}dt' = k\int_0^{t-\varepsilon}\left\{\iint_S \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n}\right)dS'\right\}dt'$$

But $G = 0$ on S. Now taking the limit as $\varepsilon \to 0$, the above equation becomes

$$k\int_0^{t}\left\{-\iint_S \theta(\mathbf{r}', t)\frac{\partial G}{\partial n}dS'\right\}dt'$$

Finally, Eq. (5.103) reduces to the form

$$u\,(\mathbf{r}, t) = \iiint_V f\,(\mathbf{r}')\,G(\mathbf{r}, \mathbf{r}', t)\,d\tau' - k\int_0^t dt'\iint_S \theta(\mathbf{r}', t)\,\frac{\partial G}{\partial n}dS' \qquad (5.104)$$

which is the required solution to the boundary value problem described by Eqs. (5.94)–(5.96).

EXAMPLE 5.9 Determine Green's function for the problem of heat flow in an infinite rod described by

$$\text{PDE: } u_t = \alpha u_{xx}, \qquad -\infty < x < \infty, \quad t > 0$$

$$\text{IC: } \quad u(x, 0) = f(x), \qquad -\infty < x < +\infty$$

Solution The variables separable method of solution as discussed in Section 3.3 to the given problem is

$$u(x, t) = \frac{1}{\sqrt{4\pi\alpha t}} \int_{-\infty}^{\infty} f(y) \exp\{-(x-y)^2/(4\alpha t)\}\, dy = \int_{-\infty}^{\infty} G(x-y, \alpha t) f(y)\, dy \qquad (5.105)$$

where

$$G(x-y, \alpha t) = \frac{1}{\sqrt{4\pi t\alpha}} [\exp\{-(x-y)^2/(4t\alpha)\}] \qquad (5.106)$$

is called the Green's function for heat transfer in infinite rod.

EXAMPLE 5.10 Find a Green's function for the heat flow problem in a finite rod described by

$$\text{PDE: } u_t = \alpha u_{xx}, \qquad 0 \le x \le L, \ t > 0$$

$$\text{BCs: } u(0, t) = u(L, t) = 0, \qquad t > 0$$

$$\text{IC: } u(x, 0) = f(x), \qquad 0 \le x \le L$$

Solution In Example 3.5, we have obtained the variables separable solution to the given PDE; its general form is

$$u(x, t) = Ce^{-\alpha\lambda^2 t} \sin \lambda x$$

Applying the BCs: $u(0, t) = u(L, t) = 0$, we get

$$u(L, t) = Ce^{-\alpha\lambda^2 t} \sin \lambda L = 0$$

which means that $\sin \lambda L = 0$. Therefore, $\lambda = n\pi/L, n = 1, 2, \ldots$

Using the superposition principle, we obtain

$$u(x, t) = \sum_{n=1}^{\infty} C_n \exp(-\alpha\lambda_n^2 t) \sin \lambda_n x, \qquad \lambda_n = n\pi/L \qquad (5.107)$$

Now, using the IC: $u(x, 0) = f(x)$, we get

$$f(x) = \sum_{n=1}^{\infty} C_n \sin \lambda_n x$$

which is a half-range Fourier sine series. Therefore,

$$C_n = \frac{2}{L}\int_0^L f(y)\sin\lambda_n y\,dy \tag{5.108}$$

Inserting Eq. (5.108) into Eq. (5.107), we obtain the following integral representation of the solution:

$$u(x,t) = \int_0^L \left[\frac{2}{L}\sum_{n=1}^{\infty} \exp(-\alpha\lambda_n^2 t)\sin\lambda_n y\sin\lambda_n x\right] f(y)\,dy \tag{5.109}$$

If we define

$$G(x,y,t) = \frac{2}{L}\sum_{n=1}^{\infty} \exp(-\alpha\lambda_n^2 t)\sin\lambda_n y\sin\lambda_n x \tag{5.110}$$

for $0 \le x, y \le L$ and $t > 0$, then solution (5.109) can be expressed as

$$u(x,t) = \int_0^L G(x,y,t)\,f(y)\,dy \qquad \text{for } 0 \le x \le L,\ t > 0 \tag{5.111}$$

The function $G(x,y,t)$ defined in Eq. (5.110) is called the Green's function for the given heat equation.

It can be observed that this function has the following properties:

(i) $$G_t(x,y,t) = \alpha G_{xx}(x,y,t) = \alpha G_{yy}(x,y,t) \qquad \text{for } 0 \le x, y \le L,\ t > 0 \tag{5.112}$$

(ii) $$G(0,y,t) = G(L,y,t) = 0 \qquad \text{for } 0 \le y \le L,\ t > 0 \tag{5.113}$$

(iii) $$G(x,y,t) = G(y,x,t) \qquad \text{for } 0 \le x, y \le L,\ t > 0 \tag{5.114}$$

Equation (5.112) follows from Eq. (5.110) by term-by-term differentiation. Thus the Green's function satisfies the heat equation. In fact, it is symmetric and satisfies the boundary conditions.

EXERCISES

1. Show that the three-dimensional Dirac δ-function can be written as

$$\delta(x,y,z;\xi,\eta,\zeta) = \delta(x-\xi)\,\delta(y-\eta)\,\delta(z-\zeta)$$

2. Let $P(x_0, y_0)$ be a point in rectangular coordinates corresponding to the point $P(r_0, \theta_0)$ in polar coordinates. Then show that

$$\delta(x-x_0)\,\delta(y-y_0) = \delta(r-r_0)\,\delta(\theta-\theta_0)\,r$$

3. Let $P(x_1, y_1, z_1)$ be a point in rectangular coordinates corresponding to the point $OP\,(r_1, \theta_1, \phi_1)$ in spherical polar coordinates. Then, show that

$$\delta(x-x_1)\,\delta(y-y_1)\,\delta(z-z_1) = \frac{\delta(r-r_1)\,\delta(\theta-\theta_1)\,\delta(\phi-\phi_1)}{r^2 \sin\theta}$$

4. Use the method of images and show that the harmonic Green's function for the half-space $z \geq 0$ is

$$G\,(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi r} - \frac{1}{4\pi r'}$$

where

$$r^2 = (x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2$$

$$r'^2 = (x-x_1)^2 + (y-y_1)^2 + (z+z_1)^2$$

5. Solve

$$\nabla^2 u = 0$$

in the upper half-plane defined by $y \geq 0, -\infty < x < \infty,$ using Green's function method, subject to the condition

$$u = f(x) \qquad \text{on } y = 0$$

6. Determine the Green's function for the Robin's problem on the quarter infinite plane described by

$$\nabla^2 u = \phi\,(x, y), \qquad x > 0, \qquad y > 0$$

subject to the conditions

$$u = f(y) \qquad \text{on } x = 0$$

$$\frac{\partial u}{\partial n} = g(x) \qquad \text{on } y = 0$$

7. Show that the Green's function for the heat flow problem in semi-infinite rod described by

$$\text{PDE}: u_t = \alpha u_{xx}, \qquad x > 0,\ t > 0$$

$$\text{BC}: u\,(0, t) = 0, \qquad t > 0$$

$$\text{IC}: u\,(x, 0) = f(x), \qquad x > 0$$

is given in the form

$$G(x, y, t) = \frac{1}{\sqrt{4\pi t}}\,[\exp\,\{-(x-y)^2/4t\} - \exp\,\{-(x+y)^2/4t\}]$$

CHAPTER 6

Laplace Transform Methods

6.1 INTRODUCTION

Laplace transform is essentially a mathematical tool which can be used to solve several problems in science and engineering. This transform was first introduced by Laplace, a French mathematician, in the year 1790 in his work on probability theorem. This technique became popular when Heaviside applied to the solution of an ordinary differential equation referred hereafter as ODE, representing a problem in electrical engineering. To the basic question as to why one should learn Laplace transform technique when other techinques are available, the answer is very simple. Transforms are used to accomplish the solution of certain problems with less effort and in a simple routine way. To illustrate, consider the problem of finding the value of x from the equation

$$x^{1.85} = 3 \tag{6.1}$$

It is an extremely tedious task to solve this problem algebraically. However, taking logarithms on both sides, we have the transformed equation as

$$1.85 \ln x = \ln 3 \tag{6.2}$$

In this transformed equation, the algebraic operation and exponentiation have been changed to multiplication which immediately gives

$$\ln x = \frac{\ln 3}{1.85}$$

To get the required result, it is enough if we take the antilogarithm on both sides of the above equation, which yields

$$x = \ln^{-1}\left(\frac{\ln 3}{1.85}\right)$$

With the help of any ordinary calculator, we can now compute x. Following this simple example, the Laplace transform method reduces the solution of an ODE to the solution of an algebraic equation. In fact, this method has a particular advantage in finding the solution of an ODE with appropriate ICs, without first finding the general solution and then using ICs for evaluating the arbitrary constants. Also, when the Laplace transform technique is applied to a PDE, it reduces the number of independent variables by one.

Definition 6.1 Suppose $f(t)$ is a piecewise continuous function and if it has an additional property that there exists a real number γ_0 and a finite positive number M such that

$$\underset{t\to\infty}{\text{Lt}}\ |f(t)|e^{-\gamma_t} \le M \quad \text{for} \quad \gamma > \gamma_0$$

and the limit does not exist when $\gamma < \gamma_0$, then such a function is said to be of exponential order γ_0, also written as

$$|f(t)| = 0(e^{\gamma_0 t})$$

Variables such as velocity and current are always finite; which means that $f(t)$ is bounded. Thus for any bounded function $f(t), |f(t)|e^{-\gamma t} \to 0$ for all $\gamma > 0$. The order of such a function is zero. However, variables such as electrical charge and mechanical displacement may increase without limit but of course proportional to t. Such functions are also of exponential order.

For illustration, let us consider the following examples:

(i) $\underset{t\to\infty}{\text{Lt}}\ te^{-\gamma_t} = 0$

The fact that t^n is of exponential order zero can be seen as follows:

$$\underset{t\to\infty}{\text{Lt}}\ t^n e^{-\gamma t} = \underset{t\to\infty}{\text{Lt}}\left(\frac{t^n}{e^{\gamma t}}\right) = \underset{t\to\infty}{\text{Lt}}\left(\frac{nt^{n-1}}{\gamma e^{\gamma t}}\right) \text{ (using L'Hospital's rule)}$$

Applying the L'Hospital's rule repeatedly, we get

$$\underset{t\to\infty}{\text{Lt}}\ t^n e^{-\gamma t} = \underset{t\to\infty}{\text{Lt}}\left(\frac{n!}{\gamma^n e^{\gamma t}}\right) = 0$$

(ii) In an unstable system a function may increase as e^{at} and we can see that

$$\underset{t\to\infty}{\text{Lt}}\ e^{at}e^{-\gamma t} = 0 \quad \text{if} \quad \gamma > a$$

Thus the function e^{at} is of exponential order a.

(iii) exp $(t^n)(n > 1)$ is not of exponential order, since

$$\underset{t\to\infty}{\text{Lt}}\ \exp(t^n)e^{-\gamma t} = \underset{t\to\infty}{\text{Lt}}\ \exp[t(t^{n-1} - \gamma)] = \infty$$

for any finite value of γ.

Definition 6.2 Let $f(t)$ be a continuous and single-valued function of the real variable t defined for all $t, 0 < t < \infty$, and is of exponential order. Then the Laplace transform of $f(t)$ is defined as a function $F(s)$ denoted by the integral

$$L[f(t); s] = F(s) = \int_0^\infty e^{-st} f(t)\, dt \tag{6.3}$$

over that range of values of s for which the integral exists. Here, s is a parameter, real or complex. Obviously, $L[f(t); s]$ is a function of s. Thus,

$$L[f(t); s] = F(s)$$

$$f(t) = L^{-1}[F(s); t]$$

where L is the operator which transforms $f(t)$ into $F(s)$, called Laplace transform operator, and L^{-1} is the inverse Laplace transform operator.

The Laplace transform belongs to the family of "integral transforms''. An integral transform $F(s)$ of the function $f(t)$ is defined by an integral of the form

$$\int_a^b k(s, t)\, f(t)\, dt = F(s) \tag{6.4}$$

where $k(s, t)$, a function of two variables s and t, is called the kernel of the integral transform. The kernels and limits of integration for various integral transforms are given in Table 6.1 (which is not exhaustive).

Table 6.1 Kernels and Limits for Various Integral Transforms

Name of the transform	$k(s, t)$	a	b
Laplace transform	e^{-st}	0	∞
Fourier transform	$e^{ist}/\sqrt{2\pi}$	$-\infty$	∞
Fourier sine transform	$\sqrt{\frac{2}{\pi}}\sin st$	0	∞
Fourier cosine transform	$\sqrt{\frac{2}{\pi}}\cos st$	0	∞
Hankel transform	$tJ_n(st)$	0	∞
Mellin transform	t^{s-1}	0	∞

The integral transforms defined above are applicable, either for semi-infinite or infinite domains. Similarly, finite integral transforms can be defined on finite domains.

Now, we are in a position to verify the following important result.

Theorem 6.1 If $f(t)$ is piecewise continuous in the range $t \geq 0$ and is of exponential order γ, then the Laplace transform $F(s)$ of $f(t)$ exists for all $s > \gamma$.

Proof From the definition of Laplace transform,

$$L[f(t); s] = \int_0^\infty e^{-st} f(t)\, dt = \int_0^T e^{-st} f(t)\, dt + \int_T^\infty e^{-st} f(t)\, dt = I_1 + I_2$$

Since $f(t)$ is piecewise continuous on every finite interval $0 < t < T, I_1$ exists, whereas

$$|I_2| \leq \int_T^\infty |e^{-st} f(t)|\, dt$$

But $f(t)$ is a function of exponential order; therefore,

$$|f(t)| < Me^{\gamma t} \quad \text{for} \quad \gamma \text{ real}$$

Hence,

$$|e^{-st} f(t)| < Me^{-(s-\gamma)t}$$

Thus,

$$|I_2| = \int_T^\infty e^{-(s-\gamma)t} M\, dt = \frac{Me^{-(s-\gamma)T}}{s-\gamma}, \quad s > \gamma$$

In other words, I_2 can be made as small as we like provided T is large enough and, therefore, I_2 exists. Hence, $L[f(t); s]$ exists for $s > \gamma$.

6.2 TRANSFORM OF SOME ELEMENTARY FUNCTIONS

Following the definition of Laplace transform by the integral (6.3), we shall compute the Laplace transform of some elementary functions.

EXAMPLE 6.1 Find the Laplace transform of

(i) 1, (ii) 0, (iii) t, (iv) e^{at}, (v) e^{-at}.

Solution Using the definition of Laplace transform, we have

(i) $$L[1; s] = \int_0^\infty e^{-st}(1)\, dt = \left(\frac{e^{-st}}{-s}\right)_0^\infty = \frac{1}{s} \quad \text{if} \quad s > 0$$

(ii) $$L[0; s] = \int_0^\infty e^{-st}(0)\, dt = 0$$

(iii) $$L[t; s] = \int_0^\infty e^{-st} \cdot t\, dt = \int_0^\infty t d\left(\frac{e^{-st}}{-s}\right) = \left(t\frac{e^{-st}}{-s}\right)_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} dt = \frac{1}{s^2}$$

(iv) $L[e^{at}; s] = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t} dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a},\ s > a$

(v) $L[e^{-at}; s] = \int_0^\infty e^{-st} e^{-at}\, dt = \int_0^\infty e^{-(s+a)t} dt = \frac{1}{s+a}$

EXAMPLE 6.2 Find the Laplace transform of

(i) cos at, (ii) sin at.

Solution Following the definition of Laplace transform, we have

(i) $L[\cos at; s] = \int_0^\infty e^{-st} \cos at\, dt = \operatorname{Re} \int_0^\infty e^{-st} e^{iat} dt = \operatorname{Re} L[e^{iat}; s]$

$$\operatorname{Re} \frac{1}{s-ia} = \operatorname{Re} \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2}$$

(ii) $L[\sin at; s] = \operatorname{Im} L[e^{iat}; s] = \operatorname{Im} \frac{s+ia}{s^2+a^2} = \frac{a}{s^2+a^2}$

EXAMPLE 6.3 Find the Laplace transform of

(i) cosh at, (ii) sinh at.

Solution Using the results established in Example 6.1, we have

(i) $L[\cosh at; s] = L\left[\frac{e^{at}+e^{-at}}{2}; s\right] = \frac{1}{2}\{L[e^{at}; s] + L[e^{-at}; s]\}$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2-a^2}$$

(ii) $L[\sinh at; s] = L\left[\frac{e^{at}-e^{-at}}{2}; s\right] = \frac{1}{2}\{L[e^{at}; s] - L[e^{-at}; s]\}$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) = \frac{a}{s^2-a^2}$$

EXAMPLE 6.4 Find the Laplace transform of t^n, where n is a positive integer.

Solution Using the definition of Laplace transform, we have

$$L[t^n; s] = \int_0^\infty e^{-st} t^n dt = \int_0^\infty t^n d\left(\frac{e^{-st}}{-s}\right) = \left(t^n \frac{e^{-st}}{-s}\right)_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt = \frac{n}{s}\int_0^\infty t^{n-1} e^{-st} dt$$

Hence,

$$L[t^n; s] = \frac{n}{s} L[t^{n-1}; s]$$

Similarly, we can prove the following:

$$L[t^{n-1}; s] = \frac{n-1}{s} L[t^{n-2}; s]$$

$$L[t^{n-2}; s] = \frac{n-2}{s} L[t^{n-3}; s]$$

$$\vdots$$

$$L[t^2; s] = \frac{2}{s} L[t; s]$$

$$L[t; s] = \frac{1}{s^2}$$

Therefore,

$$L[t^n; s] = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdots \frac{2}{s} \cdot \frac{1}{s^2} = \frac{n!}{s^{n+1}}$$

which can be expressed in Gamma function as

$$L[t^n; s] = \frac{\lfloor n+1}{s^{n+1}}$$

6.3 PROPERTIES OF LAPLACE TRANSFORM

We present a few important properties of the Laplace transform in the following theorems which will enable us to find the Laplace transform of a combination of functions whose transforms are known.

Theorem 6.2 (Linearity property). If c_1 and c_2 are any two constants and if $F_1(s)$ and $F_2(s)$ are the Laplace transforms, respectively of $f_1(t)$ and $f_2(t)$, then

$$L[\{c_1 f_1(t) + c_2 f_2(t)\}; s] = c_1 L[f_1(t); s] + c_2 L[f_2(t); s] = c_1 F_1(s) + c_2 F_2(s)$$

Proof Following the definition of Laplace transform, we have

$$L[\{c_1 f_1(t) + c_2 f_2(t)\}; s] = \int_0^\infty e^{-st}\{c_1 f_1(t) + c_2 f_2(t)\}\, dt$$

$$= c_1 \int_0^\infty e^{-st} f_1(t)\, dt + c_2 \int_0^\infty e^{-st} f_2(t)\, dt$$

$$= c_1 L[f_1(t); s] + c_2 L[f_2(t); s]$$

$$= c_1 F_1(s) + c_2 F_2(s)$$

Theorem 6.3 (Shifting property). If a function is multiplied by e^{at}, the transform of the resultant is obtained by replacing s by $(s-a)$ in the transform of the original function. That is, if

$$L[f(t); s] = F(s)$$

then

$$L[e^{at} f(t); s] = F(s-a)$$

Proof From the definition of Laplace transform,

$$L[e^{at} f(t); s] = \int_0^\infty e^{-st} e^{at} f(t)\, dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt = F(s-a)$$

Similarly,

$$L[e^{-at} f(t); s] = F(s+a) \tag{6.5}$$

Theorem 6.4 (Multiplication by power of t). If

$$L[f(t); s] = F(s)$$

then

$$L[t^n f(t); s] = (-1)^n \frac{d^n}{ds^n} F(s) = (-1)^n F^{(n)}(s)$$

where $n = 1, 2, 3, \ldots$

Proof From the definition of Laplace transform

$$F(s) = L[f(t); s] = \int_0^\infty e^{-st} f(t)\, dt$$

Hence,

$$\frac{d}{ds}[F(s)] = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)\, dt\right]$$

Interchanging the operations of differentiation and integration for which we assume that the necessary conditions are satisfied, and since there are two variables s and t, we use the notation of partial differentiation and obtain

$$\frac{d}{ds}[F(s)] = \int_0^\infty \frac{\partial}{\partial s}\{e^{-st} f(t)\}\, dt = -\int_0^\infty e^{-st} t f(t)\, dt = -L[tf(t); s]$$

Therefore,

$$L[tf(t); s] = -\frac{d}{ds} F(s)$$

By repeated application of the above result, it can be shown that

$$L[t^n f(t); s] = (-1)^n \frac{d^n}{ds^n} F(s) = (-1)^n F^{(n)}(s) \tag{6.6}$$

Theorem 6.5 (Differentiation property). If

$$L[f(t); s] = F(s)$$

then

$$L[f^{(n)}(t); s] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - sf^{n-2}(0) - f^{n-1}(0)$$

Proof From the definition of Laplace transform, we have

$$L[f'(t); s] = \int_0^\infty e^{-st} f'(t)\, dt = [e^{-st} f(t)]_0^\infty + s\int_0^\infty e^{-st} f(t)\, dt$$

$$= -f(0) + sL[f(t); s] = sF(s) - f(0)$$

Similarly, it can be shown that

$$L[f''(t); s] = sL[f'(t); s] - f'(0) = s\{sF(s) - f(0)\} - f'(0) = s^2 F(s) - sf(0) - f'(0)$$

$$L[f'''(t); s] = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$$

Thus, in general,

$$L[f^{(n)}(t); s] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{(n-1)}(0) \tag{6.7}$$

This property is very useful for solving differential equations.

EXAMPLE 6.5 Find the Laplace transform of

(i) $e^{at} \cos bt$, (ii) $e^{at} \sin bt$, (iii) $e^{at} \cosh bt$, (iv) $e^{at} t^n$, and (v) $\cos at \cosh bt$.

Solution Using the shifting property

(i) $$L[e^{at} \cos bt; s] = \left.\frac{s}{s^2 + b^2}\right|_{s \to (s-a)} = \frac{s-a}{(s-a)^2 + b^2}$$

(ii) $$L[e^{at} \sin bt; s] = \left.\frac{b}{s^2 + b^2}\right|_{s \to (s-a)} = \frac{b}{(s-a)^2 + b^2}$$

(iii) $$L[e^{at} \cosh bt; s] = \left.\frac{s}{s^2 - b^2}\right|_{s \to (s-a)} = \frac{s-a}{(s-a)^2 - b^2}$$

(iv) $$L[e^{at}t^n; s] = \left.\frac{n!}{s^{n+1}}\right|_{s\to(s-a)} = \frac{n!}{(s-a)^{n+1}}$$

(v) $$L[\cos at \cosh bt; s] = L\left[\cos at\left(\frac{e^{bt}-e^{-bt}}{2}\right); s\right]$$

$$= \frac{1}{2}\{L[e^{bt}\cos at; s] - L[e^{-bt}\cos at; s]\}$$

$$= \frac{1}{2}\left\{\frac{s-b}{(s-b)^2+a^2} - \frac{s+b}{(s+b)^2+a^2}\right\}$$

EXAMPLE 6.6 Fing the Laplace transform of the following:

(i) t^2e^{at}, (ii) $t\sin at$, (iii) $t^2\cos at$, (iv) t^ne^{-at}.

Solution Using the result established in Theorem 6.4, we have

(i) $$L[t^2e^{at}; s] = (-1)^2\frac{d^2}{ds^2}L[e^{at}; s] = \frac{d^2}{ds^2}\left(\frac{1}{s-a}\right) = \frac{d}{ds}\left(\frac{-1}{(s-a)^2}\right) = \frac{2}{(s-a)^3}$$

Alternatively,

$$L[e^{at}t^2; s] = \left.\frac{2!}{s^3}\right|_{s\to(s-a)} = \frac{2}{(s-a)^3} \quad \text{(using the shifting property)}$$

(ii) $$L[t\sin at; s] = (-1)^1\frac{d}{ds}L[\sin at; s] = -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \frac{2as}{(s^2+a^2)^2}$$

(iii) $$L[t^2\cos at; s] = (-1)^2\frac{d^2}{ds^2}L[\cos at; s] = \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{d}{ds}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right] = \frac{2s^3-6sa^2}{(s^2+a^2)^3}$$

(iv) $$L[e^{-at}t^n; s] = \left.\frac{n!}{s^{n+1}}\right|_{s\to(s+a)} = \frac{n!}{(s+a)^{n+1}} \quad \text{(using the shifting property)}$$

EXAMPLE 6.7 Find the Laplace transform of

(i) $te^{-4t}\sin 3t$, (ii) $\sin 2t\sin 3t$, (iii) $\sin^3 2t$.

Solution We may note that

(i) $L[te^{-4t}\sin 3t;\, s] = L[e^{-4t}(t\sin 3t);\, s]$

Using the result of Theorem 6.4, we get

$$L[t\sin 3t;\, s] = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = \frac{6s}{(s^2+9)^2}$$

Now, using the shifting property, we obtain

$$L[e^{-4t}(t\sin 3t);\, s] = \left.\frac{6s}{(s^2+9)^2}\right|_{s\to(s+4)} = \frac{6(s+4)}{\{(s+4)^2+9\}^2} = \frac{6(s+4)}{(s^2+8s+25)^2}$$

(ii) Since $\sin 2t \sin 3t = \frac{1}{2}(\cos t - \cos 5t)$,

$$L[\sin 2t\sin 3t;\, s] = \frac{1}{2}\{L[\cos t;\, s] - L[\cos 5t;\, s]\}$$

$$= \frac{1}{2}\left(\frac{s}{s^2+1} - \frac{s}{s^2+25}\right) = \frac{12s}{(s^2+1)(s^2+25)}$$

(iii) Since $\sin 6t = \sin 3(2t) = 3\sin 2t - 4\sin^3 2t$, we have $\sin^3 2t = \frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t$.

Thus,

$$L[\sin^3 2t;\, s] = \frac{3}{4}L[\sin 2t;\, s] - \frac{1}{4}L[\sin 6t;\, s] = \frac{3}{4}\left(\frac{2}{s^2+4}\right) - \frac{1}{4}\left(\frac{6}{s^2+36}\right)$$

$$= \frac{48}{(s^2+4)(s^2+36)}$$

EXAMPLE 6.8 Find the Laplace transform of $f(t)$ defined as

$$f(t) = \begin{cases}\sin t, & 0 < t < \pi\\ 0, & t > \pi\end{cases}$$

Solution Using the definition of Laplace transform, we have

$$L[f(t);\, s] = \int_0^{\pi} e^{-st}\sin t\, dt + \int_{\pi}^{\infty} e^{-st}(0)\, dt = \text{Im}\int_0^{\pi} e^{-st}e^{it}\, dt = \text{Im}\int_0^{\pi} e^{(i-s)t}\, dt$$

$$= \text{Im}\left[\frac{e^{(i-s)t}}{i-s}\right]_0^{\pi} = \text{Im}\left[\frac{e^{(i-s)\pi}-1}{i-s}\right] = \text{Im}\frac{i+s}{s^2+1}[1 - e^{-s\pi}(\cos\pi + i\sin\pi)]$$

$$= \text{Im}\frac{i+s}{s^2+1}[1+e^{-s\pi}] = \frac{1+e^{-s\pi}}{s^2+1}$$

Theorem 6.6 (Initial value theorem). If $f(t)$ and $f'(t)$ are Laplace transformable and $F(s)$ is the Laplace transform of $f(t)$, then the behaviour of $f(t)$ in the neighbourhood of $t = 0$ corresponds to the bebaviour of $sF(s)$ in the neighbourhood of $s = \infty$. Mathematically,

$$\operatorname*{Lt}_{t\to 0} f(t) = \operatorname*{Lt}_{s\to\infty} sF(s)$$

Proof From the property of derivative, we have

$$L[f'(t); s] = sF(s) - f(0) \tag{6.8}$$

Taking the limit as $s \to \infty$ on both sides, we get

$$\operatorname*{Lt}_{s\to\infty} \int_0^\infty e^{-st} f'(t)\, dt = \operatorname*{Lt}_{s\to\infty} sF(s) - \operatorname*{Lt}_{s\to\infty} f(0) \tag{6.9}$$

since s is independent of t, we can take the limit before integrating the left-hand side of Eq. (6.9), thus getting

$$\operatorname*{Lt}_{s\to\infty} \int_0^\infty e^{-st} f'(t)\, dt = \int_0^\infty \left[\operatorname*{Lt}_{s\to\infty} e^{-st} f'(t)\right] dt = 0$$

and Eq. (6.9) becomes

$$\operatorname*{Lt}_{s\to\infty} sF(s) = f(0) = \operatorname*{Lt}_{t\to 0} f(t)$$

Hence the result. For example, let $f(t)$ be a polynomial of degree n of the form

$$f(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

Its Laplace transform is

$$F(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2a_2}{s^3} + \cdots + \frac{n!a_k}{s^{n+1}}$$

Now, taking the limit on both sides as $s \to \infty$, we obtain

$$\operatorname*{Lt}_{s\to\infty} sF(s) = a_0 = f(0)$$

EXAMPLE 6.9 Verify the initial value theorem for the function

$$f(t) = 1 + e^{-t}(\sin t + \cos t)$$

Solution Given $f(t) = 1 + e^{-t}(\sin t + \cos t)$, we have

$$F(s) = L[f(t); s] = L[1; s] + L[(e^{-t}\sin t + e^{-t}\cos t); s]$$
$$= \frac{1}{s} + \left(\frac{1}{s^2+1} + \frac{s}{s^2+1}\right)_{s \to (s+1)}$$
$$= \frac{1}{s} + \left(\frac{2+s}{s^2+2s+2}\right)$$

Hence,

$$sF(s) = 1 + \frac{2s+s^2}{s^2+2s+2}$$

Therefore,

$$\operatorname*{Lt}_{s\to\infty} sF(s) = \operatorname*{Lt}_{s\to\infty} 1 + \frac{2/s+1}{1+2/s+2/s^2} = 1+1 = 2$$

But $f(0) = 1+1 = 2.$ Thus,

$$\operatorname*{Lt}_{s\to\infty} sF(s) = f(0)$$

Hence the result.

Theorem 6.7 (Final value theorem). If $f(t)$ and $f'(t)$ are Laplace transformable and $F(s)$ is the Laplace transform of $f(t)$, then the behaviour of $f(t)$ in the neighbourhood of $t = \infty$ corresponds to the behaviour of $sF(s)$ in the neighbourhood of $s = 0.$ Mathematically,

$$\operatorname*{Lt}_{t\to\infty} f(t) = \operatorname*{Lt}_{s\to 0} sF(s)$$

Proof From the property of derivative, we have

$$L[f'(t); s] = sL[f(t); s] - f(0)$$

i.e.

$$\int_0^\infty e^{-st} f'(t)\, dt = sF(s) - f(0)$$

Taking the limit as $s \to 0$ on both sides of the above equation, we have

$$\operatorname*{Lt}_{s\to 0} \int_0^\infty e^{-st} f'(t)\, dt = \operatorname*{Lt}_{s\to 0} sF(s) - \operatorname*{Lt}_{s\to 0} f(0)$$

But,

$$\int_0^\infty \operatorname*{Lt}_{s\to 0} e^{-st} f'(t)\, dt = \int_0^\infty f'(t)\, dt = [f(t)]_0^\infty = \operatorname*{Lt}_{t\to\infty} f(t) - f(0)$$

Using this result in the above equation, we get

$$\operatorname*{Lt}_{s\to 0} sF(s) = \operatorname*{Lt}_{t\to\infty} f(t)$$

Theorem 6.8 (Division by t). If

$$L[f(t); s] = F(s)$$

then

$$L\left[\frac{f(t)}{t}; s\right] = \int_s^\infty F(s)\, ds \tag{6.10}$$

Proof From the definition of Laplace transform

$$L[f(t); s] = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Integrating the above equation with respect to s between the limits s to ∞, we get

$$\int_s^\infty F(s)\, ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)\, dt\right] ds = \int_0^\infty f(t)\left(\int_s^\infty e^{-st} ds\right) dt \quad \text{(by changing the order of integration)}$$

$$= \int_0^\infty f(t)\left(\frac{e^{-st}}{-t}\right)_s^\infty dt = \int_0^\infty \frac{f(t)}{t} e^{-st} dt = L\left[\frac{f(t)}{t}; s\right]$$

Hence the result.

Note: In applying this rule, one should be careful. Since $f(t)/t$ may have an infinite discontinuity at $t = 0$, it may not be integrable. If $f(t)/t$ is not integrable, then its Laplace transform does not exist. For example, at $t = 0$, the function $\sin t/t$ does not have an infinite discontinuity, while the function $\cos t/t$ has an infinite discontinuity.

EXAMPLE 6.10 Find the Laplace transform of

(i) $\dfrac{1-\cos t}{t}$, (ii) $\dfrac{\cos 2t - \cos 3t}{t}$.

Solution Using the result of Theorem 6.8, we have

(i) $L\left[\dfrac{1-\cos t}{t}; s\right] = \displaystyle\int_s^\infty F(s)\, ds$

where,

$$F(s) = L[(1-\cos t); s] = L[1; s] - L[\cos t; s] = \frac{1}{s} - \frac{s}{s^2+1}$$

Therefore,

$$L\left[\frac{1-\cos t}{t}; s\right] = \int_s^{\infty}\left(\frac{1}{s}-\frac{s}{s^2+1}\right) ds$$

$$= \left[\ln s - \frac{1}{2}\ln (s^2+1)\right]_s^{\infty} = \left[\ln \frac{s}{(s^2+1)^{1/2}}\right]_s^{\infty}$$

$$= \left[\ln \frac{1}{(1+1/s^2)^{1/2}}\right]_s^{\infty} = 0 - \ln \frac{s}{(s^2+1)^{1/2}}$$

$$= \ln \frac{(s^2+1)^{1/2}}{s}$$

(ii) $L\left[\left(\dfrac{\cos 2t - \cos 3t}{t}\right); s\right] = \displaystyle\int_s^{\infty} F(s)\, ds$

where

$$F(s) = L[(\cos 2t - \cos 3t); s] = \frac{s}{s^2+4} - \frac{s}{s^2+9}$$

Therefore,

$$L\left[\frac{\cos 2t - \cos 3t}{t}; s\right] = \int_s^{\infty}\left(\frac{s}{s^2+4}-\frac{s}{s^2+9}\right) ds$$

$$= \frac{1}{2}\left[\ln\left(\frac{s^2+4}{s^2+9}\right)\right]_s^{\infty} = \frac{1}{2}\left(\ln \frac{1+4/s^2}{1+9/s^2}\right)_s^{\infty}$$

$$= \frac{1}{2}\ln\left(\frac{s^2+9}{s^2+4}\right)$$

6.4 TRANSFORM OF A PERIODIC FUNCTION

A function $f(t)$ is called periodic with period T, if $f(t+T) = f(t)$ for all values of t and $T > 0$. For example, the trigonometric functions $\sin t$ and $\cos t$ are periodic functions of period 2π. Periodic functions occur very often in a variety of engineering problems.

Theorem 6.9 If $f(t)$ is a periodic function with period T, then

$$L[f(t); s] = \int_0^T e^{-st} f(t)\, dt/(1-e^{-st}) \tag{6.11}$$

Proof From the definition of Laplace transform, we have

$$L[f(t); s] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$$

If we substitute $t = u + T$ in the second integral on the right-hand side and write $dt = du$, we obtain

$$L[f(t); s] = \int_0^T e^{-st} f(t)\,dt + \int_0^\infty e^{-s(u+T)} f(u+T)\,du$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT} \int_0^\infty e^{-su} f(u)\,du$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT} L[f(t); s]$$

Rearranging, we get

$$(1 - e^{-sT}) L[f(t); s] = \int_0^T e^{-st} f(t)\,dt$$

Thus,

$$L[f(t); s] = \int_0^T e^{-st} f(t)\,dt/(1 - e^{-sT})$$

Hence the result.

EXAMPLE 6.11 Obtain the Laplace transform of the periodic saw-tooth wave function given by

$$f(t) = \frac{t}{T} \text{ of period } T,\ 0 < t < T$$

Solution The graph of the periodic saw-tooth function is described in Fig. 6.1. Since $f(t)$ is periodic with period T, we have

$$L[f(t); s] = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} \frac{t}{T}\,dt = \frac{1}{T(1 - e^{-sT})} \int_0^T e^{-st} t\,dt = \frac{1}{T(1 - e^{-sT})} \int_0^T t\,d\left(\frac{e^{-st}}{-s}\right)$$

$$= \frac{1}{T(1 - e^{-sT})} \left[\left(\frac{te^{-st}}{-s} \right)_0^T + \frac{1}{s} \int_0^T e^{-st}\,dt \right]$$

$$= \frac{1}{T(1 - e^{-sT})} \left[\frac{Te^{-sT}}{-s} - \frac{1}{s^2}(e^{-sT} - 1) \right]$$

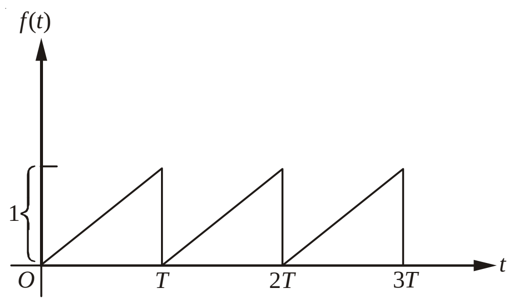

Fig. 6.1 Saw-tooth function with period T.

Therefore,

$$L[f(t); s] = \frac{1}{s^2T} - \frac{e^{-sT}}{s(1-e^{-sT})}$$

EXAMPLE 6.12 Find the Laplace transform of the following full wave rectifier function:

$$f(t) = \begin{cases} E \sin \omega t, & 0 < t < \lambda/\omega \\ 0, & \lambda/\omega < t < 2\lambda/\omega \end{cases}$$

Given that

$$f\left(t + \frac{2\lambda}{\omega}\right) = f(t)$$

Solution Since the given function $f(t)$ is periodic with period $2\lambda/\omega$, we have

$$L[f(t); s] = \frac{1}{1-e^{-2\lambda s/\omega}} \int_0^{2\lambda/\omega} e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2\lambda s/\omega}} \left[\int_0^{\lambda/\omega} e^{-st} E \sin \omega t\, dt + \int_{\lambda/\omega}^{2\lambda/\omega} e^{-st} (0)\, dt \right]$$

$$= \frac{E}{1-e^{-2\lambda s/\omega}} \left[\frac{e^{-st}}{s^2+\omega^2} (s \sin \omega t + \omega \cos \omega t) \right]_0^{\lambda/\omega}$$

Therefore,

$$L[f(t); s] = \frac{E}{1-e^{-2\lambda s/\omega}} \left[\frac{e^{-s\lambda/\omega}}{s^2+\omega^2} (s \sin \lambda + \omega \cos \lambda) - \frac{\omega}{s^2+\omega^2} \right]$$

EXAMPLE 6.13 Find the Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \le t < 2 \\ -1, & 2 \le t < 4 \end{cases}$$

$$f(t+4) = f(t)$$

Solution In this problem, $f(t)$ is a periodic function of period 4; we, therefore, have

$$L[f(t); s] = \frac{1}{1-e^{-4s}} \int_0^4 e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-4s}} \left(\int_0^2 e^{-st}(1)\, dt + \int_2^4 e^{-st}(-1)\, dt \right)$$

$$= \frac{1}{1-e^{-4s}} \left(\frac{-2e^{-2s}}{s} + \frac{e^{-4s}}{s} + \frac{1}{s} \right)$$

6.5 TRANSFORM OF ERROR FUNCTION

The error function denoted by erf (t) is defined as

$$\text{erf}\ (t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du \tag{6.12}$$

This function occurs in many branches of science and engineering; for example, in probability theory, the theory of heat conduction, and so on. In terms of the power series, we have

$$\text{erf}\ (t) = \frac{2}{\sqrt{\pi}} \int_0^t \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} u^{2n} du \tag{6.13}$$

Alternatively, it can be written as

$$\text{erf}\ (t) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{n!\,(2n+1)} \tag{6.14}$$

We can easily verify that these series converge everywhere and, therefore, erf (t) is an entire function. From the definition (6.12), it can be verified at once that

$$\text{erf}\ (0) = 0 \tag{6.15}$$

$$\text{erf}\ (\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} du = \frac{\boxed{1/2}}{\sqrt{\pi}} = 1 \tag{6.16}$$

The graph of error function is shown in Fig. 6.2.

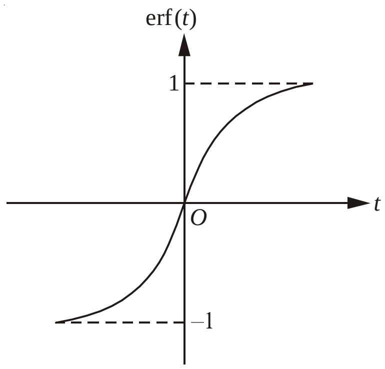

Fig. 6.2 Error function.

In solving heat conduction equation, it has been found useful to introduce the complementary error function defined as

$$\operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-u^2} du = \frac{2}{\sqrt{\pi}} \left(\int_0^\infty e^{-u^2} du - \int_0^t e^{-u^2} du \right) \tag{6.17}$$

Therefore,

$$\operatorname{erfc}(t) = 1 - \operatorname{erf}(t) \tag{6.18}$$

Now we shall find the Laplace transform of erf (t): From the definition of Laplace transform,

$$L[\operatorname{erf}(t); s] = \int_0^\infty e^{-st} \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du\, dt$$

Changing the order of integration, we obtain

$$\begin{aligned} L[\operatorname{erf}(t); s] &= \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} \int_0^\infty e^{-st} dt\, du \\ &= \frac{2}{s\sqrt{\pi}} \int_0^\infty e^{-(u^2+su)} du \\ &= \frac{2}{s\sqrt{\pi}} e^{s^2/4} \int_0^\infty e^{-(u+s/2)^2} du \end{aligned}$$

Setting $x = u + s/2$, we get

$$\begin{aligned} L[\operatorname{erf}(t); s] &= \frac{2}{s\sqrt{\pi}} e^{s^2/4} \int_{s/2}^\infty e^{-x^2} dx \\ &= \frac{1}{s} e^{s^2/4} \operatorname{erfc}(s/2) \end{aligned} \tag{6.19}$$

EXAMPLE 6.14 Find the Laplace transform of erf $(t^{1/2})$.

Solution From the definition of Laplace transform, we have

$$L[\text{erf }(t^{1/2}); s] = \int_0^\infty e^{-st} \frac{2}{\sqrt{\pi}} \int_0^{t^{1/2}} e^{-u^2} du\, dt$$

Changing the order of integration, we get

$$L[\text{erf }(t^{1/2}); s] = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} du \int_{u^2}^\infty e^{-st} dt$$

$$= \frac{2}{s\sqrt{\pi}} \int_0^\infty e^{-u^2 - su^2} du$$

$$= \frac{2}{s\sqrt{\pi}} \int_0^\infty e^{-(1+s)u^2} du$$

Setting $(1+s)u^2 = x^2$ or $\sqrt{1+s}u = x$, we have

$$du = dx/\sqrt{1+s}$$

Then

$$L[\text{erf }(t^{1/2}); s] = \frac{2}{s\sqrt{\pi}\sqrt{1+s}} \int_0^\infty e^{-x^2} dx = \frac{1}{s\sqrt{1+s}} \left(\frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx \right)$$

or

$$L[\text{erf }(t^{1/2}); s] = \frac{1}{s\sqrt{1+s}} \tag{6.20}$$

EXAMPLE 6.15 Find the Laplace transform of

$$\frac{\cos\sqrt{t}}{\sqrt{t}}$$

Solution Let $f(t) = \sin\sqrt{t}$; then

$$f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$$

Using the property of the Laplace transform of the derivative of a function, we have

$$L[f'(t); s] = sL[f(t)] - f(0)$$

Therefore,

$$L\left[\frac{\cos\sqrt{t}}{2\sqrt{t}};s\right] = sL[\sin\sqrt{t};s] \tag{6.21}$$

But,

$$L[\sin\sqrt{t};s] = L\left[\left(\sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \cdots\right);s\right]$$

$$= L\left[\left(t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \cdots\right);s\right]$$

$$= \left[\frac{\Gamma(3/2)}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma(5/2)}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma(7/2)}{s^{7/2}} - \cdots\right]$$

$$= \left[\frac{1/2\sqrt{\pi}}{s^{3/2}} - \frac{1}{6}\frac{3/2\cdot 1/2\sqrt{\pi}}{s^{5/2}} + \frac{1}{120}\frac{5/2\cdot 3/2\cdot 1/2\sqrt{\pi}}{s^{7/2}} - \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \left(\frac{1}{4s}\right) + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}e^{-1/4s} \tag{6.22}$$

Now substituting Eq. (6.22) into Eq. (6.21), we get

$$L\left[\frac{\cos\sqrt{t}}{2\sqrt{t}};s\right] = \frac{\sqrt{\pi}}{2s^{1/2}}e^{-1/4s}$$

Therefore,

$$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}};s\right] = \sqrt{\frac{\pi}{s}}\,e^{-1/4s}$$

Hence the result.

6.6 TRANSFORM OF BESSEL'S FUNCTION

Bessel functions arise in several problems involving circular or cylindrical geometry. It is therefore useful to find the Laplace transform of Bessel functions of the first kind.

EXAMPLE 6.16 Find the Laplace transform of

(i) $J_0(t)$, (ii) $tJ_0(t)$, (iii) $e^{-at}J_0(t)$.

Solution (i) From the definition of the Bessel function, we have

$$J_n(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{t}{2}\right)^{n+2r}$$

For $n = 0$, we have

$$J_0(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(r+1)}\left(\frac{t}{2}\right)^{2r} = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2}\left(\frac{t}{2}\right)^{2r}$$

$$= 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 \times 4^2} - \frac{t^6}{2^2 \times 4^2 \times 6^2} + \cdots$$

Thus,

$$L[J_0(t); s] = L[1; s] - \frac{1}{2^2}L[t^2; s] + \frac{1}{2^2 \times 4^2}L[t^4; s] - \cdots$$

$$= \frac{1}{s} - \frac{1}{2^2}\frac{2!}{s^3} + \frac{1}{2^2 \times 4^2}\frac{4!}{s^5} - \frac{1}{2^2 \times 4^2 \times 6^2}\frac{6!}{s^7} + \cdots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1 \times 3}{2 \times 4}\left(\frac{1}{s^2}\right)^2 - \frac{1 \times 3 \times 5}{2 \times 4 \times 6}\left(\frac{1}{s^2}\right)^3 + \cdots\right]$$

$$= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2} = \frac{1}{\sqrt{1+s^2}}$$

Hence,

$$L[J_0(t); s] = \frac{1}{\sqrt{1+s^2}}$$

(ii) From the properties of Laplace transform, we have

$$L[t^n f(t); s] = (-1)^n \frac{d^n}{ds^n} F(s)$$

Therefore,

$$L[tJ_0(t);s] = (-1)\frac{d}{ds}[L[J_0(t);s]] = -\frac{d}{ds}\left(\frac{1}{\sqrt{1+s^2}}\right)$$

Thus,

$$L[tJ_0(t);s] = \frac{s}{(1+s^2)^{3/2}}$$

(iii) From the shifting property of the Laplace transform, we have

$$L[e^{-at}f(t);s] = F(s+a)$$

Therefore,

$$L[e^{-at}J_0(t);s] = \frac{1}{\sqrt{1+s^2}}\bigg|_{s\to(s+a)} = \frac{1}{\sqrt{1+(s+a)^2}}$$

6.7 TRANSFORM OF DIRAC DELTA FUNCTION

The concept of *impulse function* or *Dirac delta function* has been introduced in Chapter 3 itself. In certain applications involving a sudden excitation of a system or a large voltage over a short interval of time, the Laplace transform of Dirac Delta function is useful. From the property of Dirac Delta function, we have

$$\int_0^\infty \delta(t-a)\, f(t)\, dt = f(a)$$

In particular, if $f(t) = e^{-st}$, then

$$L[\delta(t-a);s] = \int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-as}, \quad a > 0 \tag{6.23}$$

6.8 INVERSE TRANSFORM

So far we have discussed various properties of the Laplace transform and studied the Laplace transform of some simple functions. However, if the Laplace transform technique is to be useful in applications, we have to consider the reverse problem too, i.e., we have to find the original function $f(t)$ when we know its Laplace transform $F(s)$. Thus, if

$$L[f(t);s] = F(s)$$

then

$$f(t) = L^{-1}[F(s);t] \tag{6.24}$$

In other words, the inverse Laplace transform of a given function $F(s)$ is that function $f(t)$ whose Laplace transform is $F(s)$. It can be established that $f(t)$ is unique. Here, L^{-1} is known as inverse Laplace transform operator. From the elementary definition (6.24) and from the results obtained thus far in finding the Laplace transform of some elementary functions, we can immediately generate the following table of transforms:

Table 6.2 Table of Laplace Transform

$f(t)$	$L[f(t); s] = F(s)$	$F(s)$	$L^{-1}[F(s); t] = f(t)$
0	0	0	0
1	$1/s$	$1/s$	1
e^{at}	$1/(s-a)$	$1/(s-a)$	e^{at}
e^{-at}	$1/(s+a)$	$1/(s+a)$	e^{-at}
t	$1/s^2$	$1/s^2$	t
t^n	$n!/s^{n+1}$	$1/s^{n+1}$	$t^n/n!$
$\sin at$	$\dfrac{a}{s^2+a^2}$	$\dfrac{a}{s^2+a^2}$	$\sin at$
$\cos at$	$\dfrac{s}{s^2+a^2}$	$\dfrac{s}{s^2+a^2}$	$\cos at$
$\sinh at$	$\dfrac{a}{s^2-a^2}$	$\dfrac{a}{s^2-a^2}$	$\sinh at$
$\cosh at$	$\dfrac{s}{s^2-a^2}$	$\dfrac{s}{s^2-a^2}$	$\cosh at$
$t \sin at$	$\dfrac{2as}{(s^2+a^2)^2}$	$\dfrac{2as}{(s^2+a^2)^2}$	$t \sin at$
$t \cos at$	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$	$\dfrac{s^2-a^2}{(s^2+a^2)^2}$	$t \cos at$

In most of the problems we have considered earlier, $L[f(t); s]$ is a simple rational function. The linearity property holds true even in the case of inverse transform. That is, if $F_1(s)$ and $F_2(s)$ are the Laplace transform of $f_1(t)$ and $t_2(t)$, and if c_1 and c_2 are any two constants, then

$$L^{-1}[\{c_1F_1(s) \pm c_2F_2(s)\}; t] = c_1L^{-1}[F_1(s); t] \pm c_2L^{-1}[F_2(s); t]$$

By expressing $L[f(t); s]$ as partial fractions, we should be able to recognise them as the transform of some known functions, with the help of which we can write down the inverse transform. Similarly, shifting property is also useful in constructing the inverse transform of some functions, which is stated in the following theorem:

Theorem 6.10 If $L[f(t); s] = F(s)$, then

$$L^{-1}[F(s+a); t] = e^{-at} L^{-1}[F(s); t] \tag{6.25}$$

Proof Since $L[f(t); s] = F(s)$, we have

$$L^{-1}[F(s); t] = f(t)$$

Recalling the shifting property of Laplace transform, we find that

$$L[e^{-at} f(t); s] = F(s+a)$$

$$L^{-1}[F(s+a); t] = e^{-at} f(t)$$

Thus,

$$L^{-1}[F(s+a); t] = e^{-at} L^{-1}[F(s); t]$$

EXAMPLE 6.17 Obtain the inverse Laplace transform of

$$\frac{4s^2 - 3s + 5}{(s+1)(s^2 - 2s + 2)}$$

Solution Using partial fraction expansion, we can write

$$\frac{4s^2 - 3s + 5}{(s+1)(s^2 - 2s + 2)} = \frac{A}{s+1} + \frac{Bs + C}{s^2 - 2s + 2}$$

Therefore,

$$4s^2 - 3s + 5 = A(s^2 - 2s + 2) + (Bs + C)(s+1)$$

Let $s = -1$; then $A = 12/5$. Equating the coefficient of s on both sides, we have

$$B + C = 9/5$$

Equating the coefficient of constant on both sides, we get $2A + C = 5$ which gives $C = 1/5$, and hence $B = 8/5$. The given expression can now be written as

$$\frac{4s^2 - 3s + 5}{(s+1)(s^2 - 2s + 2)} = \frac{12}{5}\frac{1}{s+1} + \frac{8}{5}\frac{s}{(s-1)^2 + 1^2} + \frac{1}{5}\frac{1}{(s-1)^2 + 1^2}$$

Thus, we find that

$$L^{-1}\left[\frac{4s^2-3s+5}{(s+1)(s^2-2s+2)}; t\right] = \frac{12}{5}L^{-1}\left[\frac{1}{s+1}; t\right] + \frac{8}{5}L^{-1}\left[\frac{(s-1)+1}{(s-1)^2+1^2}; t\right] + \frac{1}{5}L^{-1}\left[\frac{1}{(s-1)^2+1^2}; t\right]$$

$$= \frac{12}{5}e^{-t}L^{-1}\left[\frac{1}{s}; t\right] + \frac{8}{5}e^{t}L^{-1}\left[\frac{s+1}{s^2+1}; t\right]$$

$$+ \frac{1}{5}e^{t}L^{-1}\left[\frac{1}{s^2+1}; t\right] \text{(by using the shifting property)}$$

$$= \frac{12}{5}e^{-t} + \frac{8}{5}e^{t}(\cos t + \sin t) + \frac{1}{5}e^{t}\sin t$$

$$= \frac{12}{5}e^{-t} + \frac{8}{5}e^{t}\cos t + \frac{9}{5}e^{t}\sin t$$

Theorem 6.11 If $f(t)$ is a piecewise continuous function and satisfies the condition of exponential order c_0 such that $\operatorname*{Lt}_{t\to 0} f(t)/t$ exists, then for $s > c_0$,

$$L^{-1}\left[\frac{F(s)}{s}; t\right] = \int_0^t f(x)\,dx$$

Proof Let $G(t) = \int_0^t f(x)\,dx$. Then $G(0) = 0$ and $G'(t) = f(t)$; Also,

$$L[G'(t); s] = sL[G(t); s] - G(0) = sL[G(t); s]$$

i.e.,

$$F(s) = sL[G(t); s]$$

Therefore,

$$L[G(t); s] = \frac{F(s)}{s}$$

Hence,

$$L^{-1}\left[\frac{F(s)}{s}; t\right] = G(t) = \int_0^t f(x)\,dx \tag{6.26}$$

This result can be generalized to show that

$$L^{-1}\left[\frac{F(s)}{s^n}; t\right] = \int_0^t \int_0^t \int_0^t \cdots \int_0^t f(t)\,dt^n \tag{6.27}$$

Theorem 6.12 (Change of scale property). If

$$L^{-1}[F(s), t] = f(t)$$

then

$$L^{-1}[F(\alpha s); t] = \frac{1}{\alpha} f\left(\frac{t}{\alpha}\right)$$

Proof From the definition of Laplace transform, we have

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Therefore,

$$F(\alpha s) = \int_0^\infty e^{-\alpha st} f(t)\, dt$$

Let $\alpha t = x$, so that $dt = dx/\alpha$. Then we get

$$F(\alpha s) = \frac{1}{\alpha}\int_0^\infty e^{-sx} f\left(\frac{x}{\alpha}\right) dx = \frac{1}{\alpha} L\left[f\left(\frac{x}{\alpha}\right); s\right] = \frac{1}{\alpha} L\left[f\left(\frac{t}{\alpha}\right); s\right]$$

Thus,

$$L^{-1}[F(\alpha s); t] = \frac{1}{\alpha} f\left(\frac{t}{\alpha}\right) \tag{6.28}$$

EXAMPLE 6.18 Find the inverse Laplace transform of

(i) $\dfrac{1}{(s+a)^n}$, (ii) $\dfrac{s+2}{s^2-4s+13}$, (iii) $\dfrac{2s-3}{s^2-s-3/4}$, (iv) $\dfrac{s^3}{(s^2+a^2)^2}$.

Solution (i) Using the shifting property, we at once have

$$L^{-1}\left[\frac{1}{(s+a)^n}; t\right] = e^{-at} L^{-1}\left[\frac{1}{s^n}; t\right] = \frac{e^{-at} t^{n-1}}{(n-1)!}$$

(ii) $$\frac{s+2}{s^2-4s+13} = \frac{(s-2)+4}{(s-2)^2+3^2}$$

Therefore,

$$L^{-1}\left[\frac{s+2}{s^2-4s+13}; t\right] = L^{-1}\left[\frac{(s-2)}{(s-2)^2+3^2}; t\right] + 4L^{-1}\left[\frac{1}{(s-2)^2+3^2}; t\right]$$

$$= e^{2t} L^{-1}\left[\frac{s}{s^2+3^2}; t\right] + 4e^{2t} L^{-1}\left[\frac{1}{s^2+3^2}; t\right] \quad \text{(by using the shifting property)}$$

Thus,

$$L^{-1}\left[\frac{s+2}{s^2-4s+13}; s\right] = e^{2t}\cos 3t + (4/3)e^{2t}\sin 3t$$

(iii) $$\frac{2s-3}{s^2-s-3/4} = \frac{2s-3}{(s-1/2)^2-1} = \frac{2(s-1/2)-2}{(s-1/2)^2-1}$$

Thus,

$$L^{-1}\left[\frac{2s-3}{s^2-s-3/4}; t\right] = 2L^{-1}\left[\frac{(s-1/2)}{(s-1/2)^2-1}; t\right] - 2L^{-1}\left[\frac{1}{(s-1/2)^2-1}; t\right]$$

$$= 2e^{t/2}L^{-1}\left[\frac{s}{s^2-1^2}; t\right] - 2e^{t/2}L^{-1}\left[\frac{1}{s^2-1^2}; t\right]$$ (by using the shifting property)

Therefore,

$$L^{-1}\left[\frac{2s-3}{s^2-s-3/4}; t\right] = 2e^{t/2}\cosh t - 2e^{t/2}\sinh t$$

(iv) $$\frac{s^3}{(s^2+a^2)^2} = \frac{s(s^2+a^2-a^2)}{(s^2+a^2)^2} = \frac{s}{s^2+a^2} - \frac{a^2 s}{(s^2+a^2)^2}$$

Hence,

$$L^{-1}\left[\frac{s^3}{(s^2+a^2)^2}; t\right] = L^{-1}\left[\frac{s}{s^2+a^2}; t\right] - a^2 L^{-1}\left[\frac{s}{(s^2+a^2)^2}; t\right]$$

Thereore,

$$L^{-1}\left[\frac{s^3}{(s^2+a^2)^2}; t\right] = \cos at - \frac{a}{2}t\sin at$$

EXAMPLE 6.19 Find the inverse Laplace transform of

(i) $\ln\dfrac{s^2+1}{s(s+1)}$, (ii) $\cot^{-1}\left(\dfrac{s}{k}\right)$.

Solution (i) From Theorem 6.4, we have

$$L[t^n f(t); s] = (-1)^n F^{(n)}(s)$$

In particular, $n=1$ gives

$$L[tf(t);s]=-F'(s)$$

i.e.

$$\frac{d}{ds}F(s)=-L[tf(t);s] \tag{6.29}$$

Let

$$L[f(t);s]=\ln\frac{s^2+1}{s(s+1)}=F(s)$$

Then,

$$\frac{d}{ds}F(s)=\frac{d}{ds}[\ln(s^2+1)-\ln s-\ln(s+1)]$$

$$=\frac{2s}{s^2+1}-\frac{1}{s}-\frac{1}{s+1}=-L[tf(t);s]$$

Now, using Eq. (6.29), we get

$$L[tf(t);s]=\frac{1}{s}+\frac{1}{s+1}-\frac{2s}{s^2+1}$$

Hence,

$$tf(t)=L^{-1}\left(\frac{1}{s};t\right)+L^{-1}\left(\frac{1}{s+1};t\right)-2L^{-1}\left(\frac{s}{s^2+1};t\right)$$

$$=1+e^{-t}-2\cos t$$

Therefore,

$$f(t)=L^{-1}\left[\ln\frac{s^2+1}{s(s+1)};t\right]=\frac{1+e^{-t}-2\cos t}{t}$$

(ii) Let $L[f(t);s]=\cot^{-1}\left(\frac{s}{k}\right)=F(s)$

Then,

$$\frac{d}{ds}F(s)=\frac{d}{ds}\left(\cot^{-1}\frac{s}{k}\right)=-\frac{k}{k^2+s^2}$$

Now using Eq. (6.29), we obtain

$$\frac{d}{ds}F(s) = \frac{k}{k^2+s^2} = L[tf(t); s]$$

Therefore,

$$tf(t) = L^{-1}\left[\frac{k}{k^2+s^2}; t\right] = \sin kt$$

Hence,

$$f(t) = L^{-1}\left[\cot^{-1}\left(\frac{s}{k}\right); t\right] = \frac{\sin kt}{t}$$

6.9 CONVOLUTION THEOREM (FALTUNG THEOREM)

We often come across functions which are not the transform of some known function, but then, they can possibly be expressed as a product of two functions, each of which is the transform of a known function. Thus we may be able to write the given function as $F(s)\,G(s)$, where $F(s)$ and $G(s)$ are known to be transforms of the functions $f(t)$ and $g(t)$, respectively.

Theorem 6.13 If $F(s)$ and $G(s)$ are the Laplace transforms of $f(t)$ and $g(t)$ respectively, then $F(s)\,G(s)$ is the Laplace transform of

$$\int_0^t f(t-u)\,g(u)\,du$$

i.e.

$$L^{-1}[F(s)\,G(s)] = \int_0^t f(t-u)\,g(u)\,du \tag{6.30}$$

This integral is called the convolution of f and g and is denoted by the symbol $f * g$.

Proof From the definition of Laplace transform, we have

$$F(s)G(s) = \left[\int_0^\infty e^{-sv} f(v)\,dv\right]\left[\int_0^\infty e^{-su} g(u)\,du\right]$$

$$= \int_0^\infty \int_0^\infty e^{-s(v+u)} f(v)\,g(u)\,dv\,du$$

$$= \int_0^\infty g(u)\left\{\int_0^\infty e^{-(v+u)} f(v)\,dv\right\} du$$

Let $u+v=t$ in the inner integral. Then,

$$F(s)G(s)=\int_0^{\infty} g(u)\left\{\int_u^{\infty} e^{-st} f(t-u)\, dt\right\} du$$

Change the order of integration as shown in Fig. 6.3.

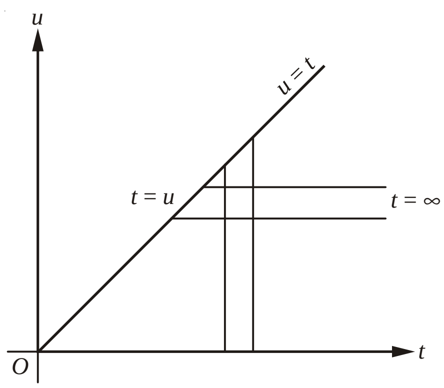

Fig. 6.3 Convolution integral.

Then, we get

$$F(s)G(s)=\int_0^{\infty} \left\{\int_0^{t} e^{-st} f(t-u)\, g(u)\, du\right\} dt$$

$$=\int_0^{\infty} e^{-st}\left\{\int_0^{t} f(t-u)\, g(u)\, du\right\} dt$$

$$=L\left[\int_0^{t} f(t-u)\, g(u)\, du; t\right]$$

Hence the result.

Definition 6.3 We define the Laplace convolution of $f(t)$ and $g(t)$ by the integral

$$f*g=\int_0^{t} f(t-u)\, g(u)\, du \tag{6.31}$$

It can be verified that f and g can be interchanged in the convolution, i.e., f and g are commutative. Let $t-u=v$ in Eq. (6.31) so that $-du=dv$. Then,

$$f*g=-\int_t^{0} f(v)g(t-v)\, dv=\int_0^{t} g(t-v)\, f(v)\, dv$$

Therefore,

$$f*g=g*f \tag{6.32}$$

EXAMPLE 6.20 Apply the convolution theorem to evaluate

(i) $L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}; t\right]$ (ii) $L^{-1}\left[\dfrac{s}{(s+a)(s^2+1)}; t\right]$ (iii) $L^{-1}\left[\dfrac{1}{s^2(s+1)}; t\right]$.

Solution Consider

(i) $$\frac{s}{(s^2+a^2)^2} = \frac{1}{a}\left[\frac{s}{s^2+a^2}\,\frac{a}{s^2+a^2}\right]$$

Now choosing

$$F(s) = \frac{s}{s^2+a^2}, \qquad G(s) = \frac{a}{s^2+a^2},$$

we can write the inverse transform for $F(s)$ and $G(s)$ as

$$L^{-1}\left(\frac{s}{s^2+a^2}; t\right) = \cos at = f(t)$$

$$L^{-1}\left(\frac{a}{s^2+a^2}; t\right) = \sin at = g(t)$$

Hence, using the convolution theorem, we obtain

$$\begin{aligned}
L^{-1}\left[\frac{s}{(s^2+a^2)^2}; t\right] &= \frac{1}{a}\int_0^t \cos a(t-u)\sin au\, du \\
&= \frac{1}{a}\int_0^t (\cos at\cos au + \sin at\sin au)\sin au\, du \\
&= \frac{1}{2a}\cos at\int_0^t \sin 2au\, du + \frac{\sin at}{2a}\int_0^t (1-\cos 2au)\, du \\
&= \frac{\cos at}{2a}\left(-\frac{\cos 2au}{2a}\right)_0^t + \frac{\sin at}{2a}\left(u - \frac{\sin 2au}{2a}\right)_0^t \\
&= -\frac{1}{4a^2}(\cos at\cos 2at + \sin at\sin 2at) + \frac{\cos at}{4a^2} + \frac{t\sin at}{2a} \\
&= \frac{t\sin at}{2a}
\end{aligned}$$

Therefore,

$$L^{-1}\left[\frac{s}{(s^2-a^2)^2}; t\right] = \frac{t\sin at}{2a}$$

(ii) Consider

$$\frac{s}{(s+a)(s^2+1)} = \frac{1}{s+a}\frac{s}{s^2+1}$$

Now choosing

$$F(s) = \frac{1}{s+a}, \qquad G(s) = \frac{s}{s^2+1}$$

the inverse transforms for this pair are obtained as

$$L^{-1}\left[\frac{1}{s+a}; t\right] = e^{-at} = f(t)$$

$$L^{-1}\left[\frac{s}{s^2+1}; t\right] = \cos at = g(t)$$

Hence, using the convolution theorem, we obtain

$$L^{-1}\left[\frac{1}{(s+a)(s^2+1)}; t\right] = \int_0^t e^{-a(t-u)} \cos u \, du$$

$$= e^{-at}\int_0^t e^{au} \cos u \, du \qquad \text{(a standard integral)}$$

$$= e^{-at}\left[\frac{e^{au}}{a^2+1}(\sin u + a\cos u)\right]_0^t$$

Therefore,

$$L^{-1}\left[\frac{s}{(s+a)(s^2+1)}; t\right] = \frac{1}{a^2+1}(a\cos t + \sin t - ae^{-at})$$

(iii) We shall write

$$\frac{1}{s^2(s+1)^2} = \frac{1}{s^2}\frac{1}{(s+1)^2}$$

For this pair, we can write inverse transforms as

$$L^{-1}\left[\frac{1}{s^2}; t\right] = t = f(t)$$

$$L^{-1}\left[\frac{1}{(s+1)^2}; t\right] = e^{-t}L^{-1}\left[\frac{1}{s^2}; t\right] = e^{-t}t = g(t)$$

Hence, using the convolution theorem, we obtain

$$L^{-1}\left[\frac{1}{s^2(s+1)^2};t\right]=\int_0^t (t-u)ue^{-u}du$$

$$=t\int_0^t e^{-u}u\,du-\int_0^t u^2e^{-u}\,du$$

$$=-t\left[(ue^{-u})_0^t-\int_0^t e^{-u}du\right]+[u^2e^{-u}]_0^t-2\int_0^t ue^{-u}du$$

$$=(t+2)e^{-t}+t-2$$

Therefore

$$L^{-1}\left[\frac{1}{s^2(s+1)^2};t\right]=(t+2)e^{-t}+t-2$$

EXAMPLE 6.21 Prove that

$$\int_0^t J_0(t)J_0(t-u)\,du=\sin t$$

Solution From Table 6.2, we note that

$$L^{-1}\left[\frac{1}{s^2+1};t\right]=\sin t$$

We shall also write

$$\frac{1}{s^2+1}=\frac{1}{\sqrt{s^2+1}}\frac{1}{\sqrt{s^2+1}}$$

Now taking

$$F(s)=\frac{1}{\sqrt{s^2+1}},\quad G(s)=\frac{1}{\sqrt{s^2+1}}$$

their inverse transforms give

$$L^{-1}\left[\frac{1}{\sqrt{s^2+1}};t\right]=J_0(t)=f(t)=g(t)$$

Hence, using the convolution theorem, we have

$$L^{-1}\left[\frac{1}{s^2+1};t\right]=\int_0^t J_0(t)\,J_0(t-u)\,du=\sin t$$

6.10 TRANSFORM OF UNIT STEP FUNCTION

Definition 6.4 The unit step function or Heaviside unit function is defined as

$$H(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t \geq a,\ a \geq 0 \end{cases}$$

Graphically, it can be depicted as in Fig. 6.4.

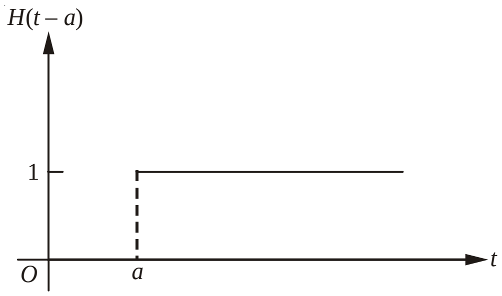

Fig. 6.4 Illustration of Heaviside unit function.

EXAMPLE 6.22 Find the Laplace transform of unit step function.

Solution From the definition of Laplace transform, we have

$$L[H(t-a); s] = \int_0^\infty e^{-st} H(t-a)\, dt$$

$$= \int_0^a e^{-st} \cdot 0 \cdot dt + \int_a^\infty e^{-st} \quad 1\, dt$$

$$= \frac{e^{-as}}{s}, \qquad s > 0$$

Theorem 6.14 (Second shifting property). If $L[f(t); s] = F(s)$, then

$$L[f(t-a)\, H(t-a); s] = e^{-as} F(s)$$

$$L^{-1}[e^{-as} F(s); t] = f(t-a)\, H(t-a)$$

Proof From the definition of Laplace transform, we have

$$L[f(t-a)H(t-a); s] = \int_0^\infty e^{-st} f(t-a)\, H(t-a)\, dt$$

$$= \int_0^a 0 \cdot dt + \int_a^\infty e^{-st} f(t-a)\, dt$$

Let $t - a = v$, and hence $dt = dv$, the right-hand side of the above equation becomes

$$\int_0^\infty e^{-s(a+v)} f(v)\, dv = e^{-as} \int_0^\infty e^{-sv} f(v)\, dv$$

$$= e^{-as} F(s)$$

Therefore,

$$L[f(t-a)\, H(t-a); s] = e^{-as} F(s) \tag{6.33}$$

$$L^{-1}[e^{-as} F(s); t] = f(t-a)\, H(t-a) \tag{6.34}$$

EXAMPLE 6.23 Find the inverse Laplace transform of

$$\frac{e^{-as}}{s^2+1}, \quad a > 0$$

Solution From the second shifting property, we have

$$L^{-1}[e^{-as} F(s); t] = f(t-a)\, H(t-a)$$

In the given problem,

$$F(s) = \frac{1}{s^2+1}$$

Hence,

$$f(t) = L^{-1}[F(s); t] = L^{-1}\left[\frac{1}{s^2+1}; t\right] = \sin t$$

Therefore,

$$L^{-1}\left[e^{-as} \frac{1}{s^2+1}\right] = \sin(t-a)\, H(t-a)$$

$$= \begin{cases} 0, & t < a \\ \sin(t-a), & t \ge a \end{cases}$$

Theorem 6.15 (Heaviside expansion theorem). Let $F(s)$ and $G(s)$ be two polynomials in s where the degree of $F(s)$ is lower than that of $G(s)$ and if $G(s)$ has n distinct roots $\alpha_i\, (i = 1, \ldots, n)$, then

$$L^{-1}\left[\frac{F(s)}{G(s)};t\right]=\sum_{i=1}^{n}\frac{F(\alpha_i)}{G'(\alpha_i)}e^{\alpha_i t} \tag{6.35}$$

Proof Since $G(s)$ is a polynomial in s having n distinct roots and $F(s)$ is a polynomial whose degree is less than that of $G(s)$, we have

$$\frac{F(s)}{G(s)}=\frac{F(s)}{\prod_{i=1}^{n}(s-\alpha_i)}=\frac{A_1}{s-\alpha_1}+\frac{A_2}{s-\alpha_2}+\cdots+\frac{A_n}{s-\alpha_n} \quad \text{(using partial fractions)}$$

Multiplying both sides by $(s-\alpha_i)$ and taking the limit as $s\to\alpha_i$, we obtain the coefficients

$$A_i=\underset{s\to\alpha_i}{\text{Lt}}\frac{F(s)\,(s-\alpha_i)}{G(s)}=F(\alpha_i)\underset{s\to\alpha_i}{\text{Lt}}\frac{s-\alpha_i}{G(s)}$$

which take indeterminate form and, therefore, using L'Hospital's rule, we get

$$A_i=F(\alpha_i)\underset{s\to\alpha_i}{\text{Lt}}\frac{1}{G'(s)}=\frac{F(\alpha_i)}{G'(\alpha_i)}$$

Hence,

$$\frac{F(s)}{G(s)}=\frac{F(\alpha_1)}{G'(\alpha_1)}\frac{1}{(s-\alpha_1)}+\frac{F(\alpha_2)}{G'(\alpha_2)}\frac{1}{(s-\alpha_2)}+\cdots+\frac{F(\alpha_n)}{G'(\alpha_n)}\frac{1}{(s-\alpha_n)}$$

Thus,

$$L^{-1}\left[\frac{F(s)}{G(s)};t\right]=\frac{F(\alpha_1)}{G'(\alpha_1)}L^{-1}\left[\frac{1}{s-\alpha_1};t\right]+\frac{F(\alpha_2)}{G'(\alpha_2)}L^{-1}\left[\frac{1}{s-\alpha_2};t\right]+\cdots$$

$$+\frac{F(\alpha_n)}{G'(\alpha_n)}L^{-1}\left[\frac{1}{s-\alpha_n};t\right]=\sum_{i=1}^{n}\frac{F(\alpha_i)}{G'(\alpha_i)}e^{\alpha_i t} \tag{6.36}$$

Hence the result.

EXAMPLE 6.24 Using the method of Heaviside expansion theorem, find the Laplace inverse of

$$\frac{s^2+1}{s^3+3s^2+2s}$$

Solution We shall take $F(s)=s^2+1$, and

$$G(s)=s^3+3s^2+2s=s(s+1)\,(s+2)$$

Here, $G(s)$ has three distinct roots $0, -1, -2$ and the degree of $F(s)$ is lower than that of $G(s)$. Hence, using the Heaviside expansion theorem, we have

$$L^{-1}\left[\frac{s^2+1}{s^3+3s^2+2s}; t\right] = \frac{F(0)}{G'(0)}e^{0r} + \frac{F(-1)}{G'(-1)}e^{-t} + \frac{F(-2)}{G'(-2)}e^{-2t}$$

But, $G'(s) = 3s^2 + 6s + 2$. Therefore,

$$L^{-1}\left[\frac{s^2+1}{s^3+3s^2+2s}; t\right] = \frac{1}{2} - 2e^{-t} + \frac{5}{2}e^{-2t}$$

6.11 COMPLEX INVERSION FORMULA (MELLIN-FOURIER INTEGRAL)

In solving some complicated problems using the Laplace transform method, the direct approach so far followed may not be helpful in finding the inverse Laplace transform. Methods based on complex variable theory may come in handy for finding the inverse transform. Also, it can be noted that the Laplace transform of $f(t)$ is expressed as an integral. Similarly, the inverse Laplace transform of $F(s)$ can be expressed as in integral which is known as inverse integral. This integral can be evaluated by using contour integration methods. The complex inversion formula is stated in the following theorem.

Theorem 6.16 Let $f(t)$ and $f'(t)$ be continuous functions on $t \geq 0$ and $f(t) = 0$ for $t < 0$. In addition, if $f(t)$ is $0(e^{\gamma_0 t})$ and

$$F(s) = L[f(t); s]$$

Then

$$L^{-1}[F(s); t] = f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\, ds$$

$t > 0$ and γ is a positive constant.

Proof Let $g(t)$ and $g'(t)$ be continuous functions and if $\int_{-\infty}^{\infty} g(t)\, dt$ converges absolutely and uniformly then $g(t)$ may be represented by the Fourier integral formula

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(v)\left[\int_{-\infty}^{\infty} \cos\omega(t-v)\, d\omega\right] dv$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g(v)\cos\omega(t-v)\, dv\right] d\omega \tag{6.37}$$

Since $\sin \omega(t-v)$ is an odd function of ω, we have

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g(v) \sin \omega (t-v)\, dv \right] d\omega = 0$$

Combining this expression with Eq. (6.37), we can write

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g(v)\, e^{i\omega(t-v)} dv \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \left[\int_{-\infty}^{\infty} g(v)\, e^{-i\omega v}\, dv \right] d\omega \qquad (6.38)$$

In addition, we assume that $g(t)$ is of exponential order $= 0(e^{\gamma_0 t})$.

Now we consider the function

$$g(t) = \begin{cases} e^{-\gamma t} f(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

where γ is a real number greater than γ_0. Thus, $g(t)$ satisfies all the conditions required by the Fourier integral theorem and, therefore, we have from Eq. (6.38), for $t \geq 0$ the relation

$$e^{-\gamma t} f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \left[\int_{-\infty}^{\infty} e^{\gamma v} f(v)\, e^{-i\omega v}\, dv \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \left[\int_{-\infty}^{\infty} e^{-(\gamma + i\omega)v} f(v)\, dv \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} F(\gamma + i\omega)\, d\omega \quad \text{[From the definition of Laplace transform]}$$

Let $\gamma + i\omega = s$, so that $d\omega = ds/i$. It follows that

$$e^{-\gamma t} f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{t(s-\gamma)} F(s)\, ds$$

Therefore,

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\, ds, \qquad t \geq 0 \qquad (6.39)$$

Hence the proof.

EXAMPLE 6.25 Find the inverse Laplace transform of

$$\frac{1}{(s+1)\,(s-2)^2}$$

Solution From complex inversion formula, we get

$$L^{-1}\left[\frac{1}{(s+1)\,(s-2)^2};t\right] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st}\,ds}{(s+1)\,(s-2)^2}$$

$$= \text{sum of the residues of } \frac{e^{st}}{(s+1)\,(s-2)^2}$$

at the simple pole $s=-1$ and double pole $s=2$. Therefore,

$$L^{-1}\left[\frac{1}{(s+1)\,(s-2)^2};t\right] = \underset{s\to-1}{\text{Lt}}\,\frac{(s+1)\,e^{st}}{(s+1)\,(s-2)^2} + \underset{s\to2}{\text{Lt}}\,\frac{d}{ds}\left(\frac{e^{st}}{s+1}\right)$$

$$= \frac{e^{-t}}{9} + \underset{s\to2}{\text{Lt}}\left[\frac{e^{st}\{(s+1)\,t-1\}}{(s+1)^2}\right]$$

$$= \frac{e^{-t}}{9} + \frac{t}{3}e^{2t} - \frac{1}{9}e^{2t}$$

EXAMPLE 6.26 Find the inverse Laplace transform of

$$\frac{\sinh\,(x\sqrt{s})}{\sinh\,(l\sqrt{s})}, \qquad 0<x<l$$

Solution Let

$$F(s) = \frac{\sinh\,(x\sqrt{s})}{\sinh\,(l\sqrt{s})}$$

Then,

$$L^{-1}[F(s);t] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\,\frac{\sinh\,(x\sqrt{s})}{\sinh\,(l\sqrt{s})}\,ds = \frac{1}{2\pi i}\int_C e^{+st}\,\frac{\sinh\,(x\sqrt{s})}{\sinh\,(l\sqrt{s})}\,ds$$

= sum of the residues of the integrand at infinitely many simple poles given by the roots of transcendental equation $\sinh\,(l\sqrt{s})=0$

i.e.,

$$e^{l\sqrt{s}} - e^{-l\sqrt{s}} = 0, \text{ implying } 2l\sqrt{s} = 1 = e^{2n\pi i}.$$

Hence, $sl^2 = -n^2\pi^2$, i.e.,

$$s = -\frac{n^2\pi^2}{l^2}, \qquad n = 0, 1, 2, \ldots$$

Thus, we have to compute the residues at the poles

$$s = 0, \qquad s = s_n = \frac{-n^2\pi^2}{l^2}, \qquad n = 1, 2, \ldots$$

Now, the residue at $s = 0$ is

$$\underset{s\to 0}{\text{Lt}}\; se^{st}\frac{\sinh(x\sqrt{s})}{\sinh(l\sqrt{s})} = 0$$

The residues at $s = s_n$ are obtained as

$$\underset{s\to s_n}{\text{Lt}}\;(s - s_n)\,e^{st}\frac{\sinh(x\sqrt{s})}{\sinh(l\sqrt{s})} = \frac{2\sqrt{s}}{l\cosh(l\sqrt{s})}e^{st}\sinh(x\sqrt{s})\Big|_{s=s_n}$$

$$= \frac{2\sqrt{\dfrac{-n^2\pi^2}{l^2}}}{l\cosh\left(l\sqrt{\dfrac{-n^2\pi^2}{l^2}}\right)}\exp\left(\frac{-n^2\pi^2 t}{l^2}\right)\sinh\left(x\sqrt{\frac{-n^2\pi^2}{l^2}}\right)$$

$$= \frac{2in\pi}{l^2}\exp\left(\frac{-n^2\pi^2 t}{l^2}\right)\frac{\sinh[i(n\pi/l)x]}{\cosh(in\pi)}$$

$$= \frac{2n\pi(-1)}{l^2}\exp\left(\frac{-n^2\pi^2 t}{l^2}\right)\frac{\sin(n\pi x/l)}{\cos n\pi}$$

$$= \frac{(-1)^{n+1}\cdot 2n\pi}{l^2}\sin\left(\frac{n\pi}{l}x\right)\exp\left(\frac{-n^2\pi^2 t}{l^2}\right); \quad n = 1, 2, \ldots$$

Therefore,

$$L^{-1}\left[\frac{\sinh(x\sqrt{s})}{\sinh(l\sqrt{s})}; t\right] = \frac{2\pi}{l^2}\sum_{n=1}^{\infty}(-1)^{n+1}n\sin\left(\frac{n\pi}{l}x\right)\exp\left(\frac{-n^2\pi^2 t}{l^2}\right)$$

6.12 SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

The Laplace transform technique is one of the powerful tools for solving physical problems involving ordinary differential equations (ODE), particularly initial value problems. It reduces the solution of ODE to the solution of an algebraic equation. This method has a particular advantage in finding the solution of an initial value problem, without first finding the general solution and then using the given ICs for evaluating the arbitrary constants. We shall first apply the Laplace transform technique to find the solution of a typical initial value problem and demonstrate the steps involved. Consider a second order linear differential equation

$$\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + \beta y = f(t) \tag{6.40}$$

subject to the ICs

$$y(0) = A, \quad y'(0) = B \tag{6.41}$$

where α and β are constants. Taking the Laplace transform of (6.40) on both sides, we obtain

$$L[\{y'' + \alpha y' + \beta y\}; s] = L[f(t); s]$$

Using the property of the Laplace transform of the derivatives, we get

$$\{s^2L[y(t); s] - sy(0) - y'(0)\} + \alpha\{sL[y(t); s] - y(0)\} + \beta L[y(t); s] = L[f(t); s]$$

Using the ICs, we have, after regrouping the terms,

$$(s^2 + \alpha s + \beta)\, L[y(t); s] = L[f(t); s] + (s + \alpha)A + B$$

Alternatively,

$$(s^2 + \alpha s + \beta)\, Y(s) = F(s) + (s + \alpha)A + B$$

i.e.

$$Y(s) = \frac{(s+\alpha)\, A + B}{s^2 + \alpha s + \beta} + \frac{F(s)}{s^2 + \alpha s + \beta} \tag{6.42}$$

Taking the inverse Laplace transform on both sides, we get the solution in the form

$$y(t) = L^{-1}\left[\frac{(s+\alpha)\, A + B}{s^2 + \alpha s + \beta}; t\right] + L^{-1}\left[\frac{F(s)}{s^2 + \alpha s + \beta}; t\right] \tag{6.43}$$

EXAMPLE 6.27 Solve the equation using the Laplace transform method

$$y'' + 4y' + 8y = \cos 2t$$

Given that $y = 2$ and $y' = 1$ when $t = 0$.

Solution Taking the Laplace transform of the given ODE, we obtain

$$\{s^2Y(s) - sy(0) - y'(0)\} + 4\{sY(s) - y(0)\} + 8Y(s) = \frac{s}{s^2+4}$$

Using the initial conditions, we get

$$s^2Y(s) - 2s - 1 + 4sY(s) - 8 + 8Y(s) = \frac{s}{s^2+4}$$

$$(s^2 + 4s + 8)\,Y(s) = \frac{s}{s^2+4} + 2s + 9$$

Therefore,

$$Y(s) = \frac{(s/20 + 1/5)}{s^2+4} - \frac{(s/20 + 2/5)}{s^2+4s+8} + \frac{2s}{s^2+4s+8} + \frac{9}{s^2+4s+8}$$

$$= \frac{1}{20} \times \frac{s}{s^2+4} + \frac{1}{5}\frac{1}{s^2+4} - \frac{1}{20} \times \frac{(s+2)-2}{(s+2)^2+2^2} - \frac{2}{5}$$

$$\times \frac{1}{(s+2)^2+2^2} + \frac{2(s+2)-4}{(s+2)^2+2^2} + \frac{9}{(s+2)^2+2^2}$$

Taking the inverse Laplace transform, we obtain

$$y(t) = \frac{1}{20}\cos 2t + \frac{1}{10}\sin 2t - \frac{1}{20}e^{-2t}\cos 2t + \frac{1}{20}e^{-2t}\sin 2t$$

$$-\frac{2}{10}e^{-2t}\sin 2t + 2e^{-2t}\cos 2t - 2e^{-2t}\sin 2t + \frac{9}{2}e^{-2t}\sin 2t$$

On simplification, we get

$$y(t) = \frac{e^{-2t}}{20}(39\cos 2t + 47\sin 2t) + \frac{1}{20}(\cos 2t + 2\sin 2t)$$

EXAMPLE 6.28 Using the Laplace transform technique, solve

$$\ddot{x} + 3\dot{x} + 2x = te^{-t}$$

Given

$$x(0) = 1 \text{ and } \dot{x}(0) = 0$$

Solution Taking the Laplace transform of the given ODE, we get

$$\{s^2X(s) - sx(0) - x'(0)\} + 3\{sX(s) - x(0)\} + 2X(s) = \frac{1}{(s+1)^2}$$

Applying the ICs, we have

$$(s^2+3s+2)\,X(s)=\frac{1}{(s+1)^2}+s+3$$

Therefore,

$$X(s)=\frac{1}{(s+1)^3(s+2)}+\frac{s+3}{(s+1)\,(s+2)}$$

$$=\frac{1}{2}\times\frac{1}{s+1}-\frac{1}{2}\times\frac{1}{(s+1)^2}+\frac{1}{2}\times\frac{1}{(s+1)^3}$$

$$-\frac{1}{2}\times\frac{1}{s+2}+\frac{2}{s+1}-\frac{1}{s+2}\qquad\text{(using partial fractions)}$$

The inverse Laplace transform yields

$$x(t)=L^{-1}\left[\left\{\frac{5}{2}\times\frac{1}{s+1}-\frac{1}{2}\times\frac{1}{(s+1)^2}+\frac{1}{2}\times\frac{1}{(s+1)^3}-\frac{3}{2}\times\frac{1}{s+2}\right\};t\right]$$

$$=\frac{5}{2}e^{-t}-\frac{1}{2}e^{-t}t+\frac{1}{2}e^{-t}\frac{t^2}{2}-\frac{3}{2}e^{-2t}=\frac{e^{-t}}{2}\left(5-t+\frac{t^2}{2}\right)-\frac{3}{2}e^{-2t}$$

EXAMPLE 6.29 Using the Laplace transform technique, solve the following initial value problem:

$$ty''+y'+ty=0,\qquad y(0)=1,\qquad y'(0)=0$$

Solution This is an example of ODE with variable coefficients. Taking the Laplace transform on both sides of the given ODE and using Eq. (6.7), we get

$$-\frac{d}{ds}L[y'';s]+L[y';s]-\frac{d}{ds}L[y;s]=0$$

$$-\frac{d}{ds}\{s^2Y(s)-sy(0)-y'(0)\}+\{sY(s)-y(0)\}-\frac{d}{ds}\{Y(s)\}=0$$

Applying the ICs, we obtain

$$\frac{d}{ds}\{s^2Y(s)-s\}+\{sY(s)-1\}-\frac{d}{ds}Y(s)=0$$

or

$$(s^2+1)\frac{dY}{ds}+sY=0$$

which is a first order ODE. On rewriting, we get

$$\frac{dY}{Y} + \frac{s\,ds}{s^2+1} = 0$$

Integrating, we see that

$$\ln Y + \frac{1}{2}\ln(s^2+1) = \ln c$$

$$Y = \frac{c}{\sqrt{s^2+1}}$$

Now, taking the inverse Laplace transform, we find $y(t) = cJ_0(t)$, where $J_0(t)$ is a Bessel function of order zero. Since $y(0) = 1 = cJ_0(0) = c$, the required solution is

$$y(t) = J_0(t)$$

EXAMPLE 6.30 Solve the simultaneous equations

$$\frac{dx}{dt} - y = e^t, \qquad \frac{dy}{dt} + x = \sin t$$

using the Laplace transform method which satisfies the conditions $x(0) = 1,\ y(0) = 0.$

Solution Taking the Laplace transform of both the equations and using the notation $Y = L[y; s]$, we have

$$sX - x(0) - Y = L[e^t; s] = \frac{1}{s-1}$$

$$sY - y(0) + X = L[\sin t; s] = \frac{1}{s^2+1}$$

Using the given ICs, the above equations reduce to

$$sX - Y = \frac{s}{s-1} \tag{6.44}$$

$$X + sY = \frac{1}{s^2+1} \tag{6.45}$$

Solving Eqs. (6.44) and (6.45), we get

$$X = \frac{s^2}{(s-1)(s^2+1)} + \frac{1}{(s^2+1)^2} \tag{6.46}$$

$$Y = \frac{-s^3 + s^2 - 2s}{(s-1)(s^2+1)^2} \tag{6.47}$$

$$= \frac{s}{(s^2+1)^2} - \frac{s}{(s-1)(s^2+1)} \tag{6.48}$$

Using partial fractions, we get

$$\frac{s^2}{(s-1)(s^2+1)} = \frac{A}{(s-1)} + \frac{Bs+C}{s^2+1} = \frac{1}{2}\left(\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right)$$

Hence Eq. (6.46) gives

$$X = \frac{1}{2}\left(\frac{1}{s-1} + \frac{s}{s^2+1} + \frac{1}{s^2+1}\right) + \frac{1}{(s^2+1)^2}$$

Taking inverse Laplace transform, we obtain

$$x = \frac{1}{2}[e^t + \cos t + \sin t + (\sin t - t\cos t)] \tag{6.49}$$

Also, from Eq. (6.48), we have

$$Y = \frac{s}{(s^2+1)^2} - \frac{1}{2}\left(\frac{1}{s-1} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\right)$$

Again, taking inverse Laplace transform, we get

$$y = \frac{1}{2}(t\sin t - e^t + \cos t - \sin t) \tag{6.50}$$

Equations (6.49) and (6.50) constitute the solution of the given system.

6.13 SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

A large number of problems in science and engineering involve the solution of linear partial differential equations. A function of two or more variables may also have a Laplace transform. Suppose x and t are two independent variables; consider t as the principal variable and x as the secondary variable. When the Laplace transform is applied with t as a variable, the PDE is reduced to an ordinary differential equation of the t-transform $U(x, s)$, where x is the independent variable. The general solution $U(x, s)$ of the ODE is then fitted to the BCs of the original problem. Finally, the solution $u(x, t)$ is obtained by using the complex inversion formula. Thus, the Laplace transform is specially suited to solving initial boundary value problems (IBVP), when conditions are prescribed at $t = 0$. We have already noted in Chapters 3 and 4 that such situations naturally arise in the case of heat conduction equation and wave equation. Before we consider the Laplace transform method of solution of IBVP, we consider the following example to prove some of the useful elementary results.

EXAMPLE 6.31 If $u(x, t)$ is a function of two variables x and t, prove that

(i) $L\left[\frac{\partial u}{\partial t}; s\right] = sU(x, s) - u(x, 0)$

(ii) $L\left[\frac{\partial^2 u}{\partial t^2}; s\right] = s^2 U(x, s) - su(x, 0) - u_t(x, 0)$

(iii) $L\left[\frac{\partial u}{\partial x}; s\right] = \frac{dU(x, s)}{dx}$

(iv) $L\left[\frac{\partial^2 u}{\partial x^2}; s\right] = \frac{d^2}{dx^2} U(x, s)$

(v) $L\left[\frac{\partial^2 u}{\partial x \partial t}; s\right] = s\frac{d}{dx} U(x, s) - \frac{d}{dx} u(x, 0)$.

where $U(x, s) = L[u(x, t); s]$.

Proof Following the definition of the Laplace transform, we have

(i) $$L\left[\frac{\partial u}{\partial t}; s\right] = \int_0^\infty e^{-st} \frac{\partial u}{\partial t}\, dt = \underset{p\to\infty}{\text{Lt}} \int_0^p e^{-st} \frac{\partial u}{\partial t}\, dt$$

$$= \underset{p\to\infty}{\text{Lt}} \left[\{e^{-st} u(x, t)\}_0^p + s\int_0^p e^{-st} u(x, t)\, dt\right]$$

$$= -u(x, 0) + s\int_0^\infty e^{-st} u(x, t)\, dt$$

Therefore,

$$L\left[\frac{\partial u}{\partial t}; s\right] = sU(x, s) - u(x, 0)$$

(ii) $$L\left[\frac{\partial^2 u}{\partial t^2}; s\right] = L\left[\frac{\partial V}{\partial t}; s\right], \quad V = \frac{\partial u}{\partial t}$$

$$= sL[V; s] - V(x, 0)$$

$$= s\{sU(x, s) - u(x, 0)\} - u_t(x, 0)$$

Thus,

$$L\left[\frac{\partial^2 u}{\partial t^2}; s\right] = s^2 U(x, s) - su(x, 0) - u_t(x, 0)$$

(iii) $$L\left[\frac{\partial u}{\partial x}; s\right] = \int_0^\infty e^{-st}\frac{\partial u}{\partial x}dt = \frac{d}{dx}\int_0^\infty e^{-st}u(x, t)dt = \frac{dU}{dx}(x, s)$$

(iv) $$L\left[\frac{\partial^2 u}{\partial x^2}; s\right] = L\left[\frac{\partial \bar{u}}{\partial x}; s\right] = \frac{d}{dx}\left(\frac{dU}{dx}\right) = \frac{d^2U}{dx^2}(x, s), \quad \bar{u} = \frac{\partial u}{\partial x}$$

(v) $$L\left[\frac{\partial^2 u}{\partial x\, \partial t}; s\right] = \frac{d}{dx}[sU(x, s) - u(x, 0)] = s\frac{dU}{dx}(x, s) - \frac{du}{dx}(x, 0)$$

Thus, we notice from the above results that the partial derivatives are transformed into ordinary derivatives.

6.13.1 Solution of Diffusion Equation

EXAMPLE 6.32 Solve the following IBVP using the Laplace transform technique:

$$\text{PDE: } u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0$$

$$\text{BCs: } u(0, t) = 1, \quad u(1, t) = 1, \quad t > 0$$

$$\text{IC: } u(x, 0) = 1 + \sin \pi x, \quad 0 < x < 1$$

Solution Taking the Laplace transform of both sides of the given PDE, we have

$$sU(x, s) - u(x, 0) = \frac{d^2U}{dx^2}$$

Thus, the solution of the second order PDE reduces to the solution of second order ODE given by

$$\frac{d^2U}{dx^2} - sU(x, s) = -1(1 + \sin \pi x) \text{ (after using IC)} \tag{6.51}$$

The general solution of Eq. (6.51) is found to be

$$U(x, s) = Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x} + \frac{1}{s} + \frac{\sin \pi x}{\pi^2 + s} \tag{6.52}$$

But, $u(0, t) = 1, u(1, t) = 1,$ and their Laplace transforms are

$$U(0, s) = \frac{1}{s}, \quad U(1, s) = \frac{1}{s}$$

From Eq. (6.52), we have

$$A+B+\frac{1}{s}=\frac{1}{s}$$

Hence, $A+B=0,$ and

$$Ae^{\sqrt{s}}+Be^{-\sqrt{s}}+\frac{1}{s}=\frac{1}{s}$$

Therefore,

$$Ae^{\sqrt{s}}+Be^{-\sqrt{s}}=0$$

This is a homogeneous system; the determinant of the coefficient matrix is

$$\begin{vmatrix} 1 & 1 \\ e^{\sqrt{s}} & e^{-\sqrt{s}} \end{vmatrix} = e^{-\sqrt{s}}-e^{\sqrt{s}}\neq 0$$

Thus, the only possible solution is the trivial solution and, therefore,

$$A=B=0$$

From Eq. (6.52), we now have

$$U(x,s)=\frac{1}{s}+\frac{\sin \pi x}{\pi^2+s} \tag{6.53}$$

Taking the inverse Laplace transform of Eq. (6.53), we get

$$u(x,t)=L^{-1}\left(\frac{1}{s};t\right)+L^{-1}\left[\frac{\sin \pi x}{\pi^2+s};t\right]$$

Thus,

$$u(x,t)=1+\sin \pi x\, e^{-\pi^2 t} \tag{6.54}$$

is the required solution.

EXAMPLE 6.33 Using Laplace transform, solve the following initial boundary value problem:

PDE: $ku_t=u_{xx}, \quad 0<x<1, \quad 0<t<\infty$

BCs: $u(0,t)=0, \quad u(l,t)=g(t), \quad 0<t<\infty$

IC: $u(x,0)=0, \quad 0<x<l$

Solution Taking the Laplace transform of both sides of the PDE, we get

$$K[sU(x,s)-u(x,0)]=\frac{d^2U}{dx^2}$$

Thus the solution of PDE reduces to the solution of ODE which, after using the IC, can be rewritten as

$$\frac{d^2U}{dx^2} - KsU(x, s) = 0 \tag{6.55}$$

Its general solution is found to be

$$U = \cosh(\sqrt{ks}\ x) + B\sinh(\sqrt{ks}\ x) \tag{6.56}$$

Taking the Laplace transform of the BCs, we have

$$U(0, s) = 0, \qquad U(l, s) = G(s) \tag{6.57}$$

Applying Eq. (6.57) in Eq. (6.56), we get $0 = A$. Therefore,

$$G(s) = B\sinh(\sqrt{ks}\ l)$$

which implies

$$B = \frac{G(s)}{\sinh(\sqrt{ks}\ l)} \tag{6.58}$$

Taking the inverse Laplace transform, we obtain

$$u(x, t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} G(s)\, e^{st}\frac{\sinh(\sqrt{ks}\ x)}{\sinh(\sqrt{ks}\ l)}\, ds \tag{6.59}$$

To evaluate the integral on the right-hand side of this equation, we use the method of residues for which we note that the integrand has poles given by

$$\sinh(\sqrt{ks}\ l) = 0 = e^{\sqrt{ks}\ l} - e^{-\sqrt{ks}\ l}$$

$$e^{2\sqrt{ks}\ l} = 1 = e^{2n\pi i}$$

implying the simple poles at

$$s_n = -\frac{n^2\pi^2}{Kl^2}, \qquad n = 0, \pm 1, \pm 2, \ldots. \tag{6.60}$$

Now,

$$\frac{1}{2\pi i}\int_{\gamma-\infty}^{\gamma+\infty} G(s)e^{st}\frac{\sinh(\sqrt{ks}\ x)}{\sinh(\sqrt{ks}\ l)}\, ds = \text{ sum of the residues of the integrand at the poles.}$$

But the residue at the pole $s = 0$ is zero. The residues at $s = s_n$ are

$$\underset{s \to s_n}{\text{Lt}} \; G(s)\, e^{st} \frac{\sinh(\sqrt{ks}\; x)}{\dfrac{d}{ds}[\cosh(\sqrt{ks}\; l)]}$$

$$= \underset{s \to s_n}{\text{Lt}} \frac{2\sqrt{s}G(s)}{l\sqrt{k}\cosh(\sqrt{ks}\; l)} e^{st} \sinh(\sqrt{ks}\; x)$$

$$= \frac{2\sqrt{\dfrac{-n^2\pi^2}{kl^2}}\, G\left(\dfrac{-n^2\pi^2}{kl^2}\right)}{l\sqrt{k}\cosh\left(\sqrt{\dfrac{-n^2\pi^2}{l^2}}\, l\right)} \exp\left(\frac{-n^2\pi^2}{kl^2}\right) \sinh\left(\sqrt{\frac{-n^2\pi^2}{l^2}}\, x\right), \qquad n = 1, 2, 3, \ldots$$

$$= \frac{2in\pi}{l^2 k} \frac{G\left(\dfrac{-n^2\pi^2}{kl^2}\right)}{\cosh(in\pi)} \sinh\left(\frac{in\pi}{l} x\right) \exp\left(\frac{-n^2\pi^2 t}{kl^2}\right), \qquad n = 1, 2, \ldots$$

Using the fact that $\cosh(in\pi) = \cos n\pi$, and

$$\sinh\left(\frac{in\pi}{l} x\right) = i \sin\left(\frac{n\pi}{l} x\right)$$

the above expression becomes

$$u(x, t) = \frac{2n\pi(-1)^n}{l^2 k} \sin\left(\frac{n\pi}{l} x\right) G\left(\frac{-n^2\pi^2}{kl^2}\right) \exp\left(\frac{-n^2\pi^2}{kl^2} t\right), \qquad n = 1, 2, \ldots$$

Therefore, the required solution is

$$u(x, t) = \frac{2\pi}{l^2 k} \sum_{n=1}^{\infty} (-1)^n n G\left(\frac{n^2\pi^2}{kl^2}\right) \sin\left(\frac{n\pi}{l} x\right) \exp[(-n^2\pi^2/kl^2)t] \tag{6.61}$$

EXAMPLE 6.34 Find the solution of the BVP given by

$$\text{PDE: } \frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}, \qquad 0 \le x \le a, t > 0$$

$$\text{BCs: } u(0, t) = f(t), \qquad u(a, t) = 0$$

$$\text{IC: } u(x, 0) = 0$$

using the Laplace transform method.

Solution Taking the Laplace transform of the given PDE, we have

$$\frac{d^2U}{dx^2} = \frac{1}{k}[sU(x, s) - u(x, 0)] \tag{6.62}$$

Using the initial condition $u(x, 0) = 0$, we get

$$\frac{d^2U}{dx^2} - \frac{s}{k}U(x, s) = 0 \tag{6.63}$$

Its general solution is found to be

$$U(x, s) = Ae^{\sqrt{(s/k)}x} + Be^{-\sqrt{(s/k)}x} \tag{6.64}$$

Now taking the Laplace transform of the BCs, we have

$$U(0, s) = F(s) \tag{6.65}$$

$$U(a, s) = 0 \tag{6.66}$$

Using Eqs. (6.64) and (6.65), we obtain

$$F(s) = A + B \tag{6.67}$$

Substituting Eq. (6.66) into Eq. (6.64), we get

$$U(a, s) = A \exp[\sqrt{(s/k)}\, a] + B \exp[-\sqrt{(s/k)}\, a] \tag{6.68}$$

Combining Eqs. (6.67) and (6.68), we get

$$F(s) = A\,(1 - e^{2a\sqrt{s/k}})$$

Therefore,

$$A = F(s)/(1 - e^{2a\sqrt{s/k}})$$

and

$$B = F(s) - \frac{F(s)}{(1 - e^{2a\sqrt{s/k}})} - \frac{F(s)e^{2a\sqrt{s/k}}}{1 - e^{2a\sqrt{s/k}}}$$

or

$$A = F(s)\, e^{-a\sqrt{s/k}}/(e^{-a\sqrt{s/k}} - e^{a\sqrt{s/k}})$$

and

$$B = F(s)\, e^{a\sqrt{s/k}}/(e^{a\sqrt{s/k}} - e^{-a\sqrt{s/k}})$$

Therefore,

$$U(x, s) = \frac{F(s)}{(e^{a\sqrt{s/k}} - e^{-a\sqrt{s/k}})}\left\{e^{\sqrt{s/k}\,(a-x)} - e^{-\sqrt{s/k}\,(a-x)}\right\}$$

$$= F(s)\frac{\sinh\left[\sqrt{\frac{s}{k}}(a-x)\right]}{\sinh\left(\sqrt{\frac{s}{k}}a\right)} \tag{6.69}$$

Its inverse Laplace transform yields

$$u(x,t) = L^{-1}\left[\frac{F(s)\sinh\left[\sqrt{\frac{s}{k}}(a-x)\right]}{\sinh\left(\sqrt{\frac{s}{k}}a\right)}; t\right] \tag{6.70}$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}F(s)\sinh\left[\sqrt{\frac{s}{k}}(a-x)\right]}{\sinh\left(\sqrt{\frac{s}{k}}a\right)}ds \tag{6.71}$$

6.13.2 Solution of Wave Equation

EXAMPLE 6.35 Using the Laplace transform method, solve the IBVP described as

PDE: $u_{xx} = \frac{1}{c^2}u_{tt} - \cos \omega t, \qquad 0 \le x < \infty, \qquad 0 \le t < \infty$

BCs: $u(0,t) = 0, \qquad u$ is bounded as x tends to ∞

ICs: $u_t(x,0) = u(x,0) = 0$

Solution Taking the Laplace transform of PDE, we obtain

$$\frac{d^2U}{dx^2} = \frac{1}{c^2}[s^2U(x,s) - su\,(x,0) - u_t(x,0)] - \frac{s}{s^2+\omega^2}$$

Using the ICs, we get

$$\frac{d^2U}{dx^2} - \frac{s^2}{c^2}U(x,s) = -\frac{s}{s^2+\omega^2} \tag{6.72}$$

Its general solution is found to be

$$U(x,s) = Ae^{(s/c)x} + Be^{-(s/c)x} + \frac{c^2}{s(s^2+\omega^2)}$$

As $x \to \infty$, the transform should also be bounded which is possible if $A = 0$; thus,

$$U(x, s) = Be^{-(s/c)x} + \frac{c^2}{s(s^2 + \omega^2)} \tag{6.73}$$

Taking the Laplace transform of the BC, we get

$$U(0, s) = 0$$

Using this result in Eq. (6.73), we have

$$B = -\frac{c^2}{s(s^2 + \omega^2)}$$

Hence,

$$U(x, s) = \frac{c^2}{s(s^2 + \omega^2)}[1 - e^{-(s/c)x}] \tag{6.74}$$

Now, taking its inverse Laplace transform, we get

$$u(x, t) = c^2 L^{-1}\left[\frac{1}{s(s^2 + \omega^2)}; t\right] - c^2 L^{-1}\left[\frac{e^{-(s/c)x}}{s(s^2 + \omega^2)}; t\right] \tag{6.75}$$

But,

$$L^{-1}\left[\frac{1}{s(s^2 + \omega^2)}; t\right] = \frac{1}{\omega^2}\left\{L^{-1}\left[\frac{1}{s}; t\right] - L^{-1}\left[\frac{s}{s^2 + \omega^2}; t\right]\right\} = \frac{1}{\omega^2}(1 - \cos \omega t),$$

$$L^{-1}\left[\frac{e^{-(x/c)s}}{s(s^2 + \omega^2)}; t\right] = \frac{1}{\omega^2}\left\{1 - \cos \omega\left(t - \frac{x}{c}\right)\right\} H\left(t - \frac{x}{c}\right),$$

where H is the Heaviside unit function. Substituting these results in Eq. (6.75), we arrive at

$$u(x, t) = \frac{c^2}{\omega^2}(1 - \cos \omega t) - \frac{c^2}{\omega^2}\left[\left\{1 - \cos \omega\left(t - \frac{x}{c}\right)\right\} H\left(t - \frac{x}{c}\right)\right]$$

EXAMPLE 6.36 Solve the IBVP described by

PDE: $u_{tt} = u_{xx}$, $\quad 0 < x < 1, \quad t > 0$

BCs: $u(0, t) = u(1, t) = 0, \quad t > 0$

ICs: $u(x, 0) = \sin \pi x, \quad u_t(x, 0) = -\sin \pi x, \quad 0 < x < 1$

Solution Taking the Laplace transform of the given PDE, we get

$$\frac{d^2U}{dx^2} = s^2 U(x, s) - su(x, 0) - u_t(x, 0)$$

Using the initial conditions, this equation becomes

$$\frac{d^2U}{dx^2} - s^2U = (1-s)\sin \pi x \tag{6.76}$$

Its general solution is found to be

$$U(x, s) = Ae^{sx} + Be^{-sx} + \frac{(s-1)\sin \pi x}{\pi^2 + s^2} \tag{6.77}$$

The Laplace transform of the BCs gives

$$U(0, s) = 0, \quad U(1, s) = 0 \tag{6.78}$$

Using Eq. (6.78) into Eq. (6.77), we find $A = B = 0$. Hence, we obtain

$$U(x, s) = \frac{(s-1)\sin \pi x}{\pi^2 + s^2} \tag{6.79}$$

Taking the inverse Laplace transform, we get

$$u(x, t) = \sin \pi x \left\{ L^{-1}\left[\frac{s-1}{s^2+\pi^2}; t\right]\right\} = \sin \pi x \left\{ L^{-1}\left[\frac{s}{s^2+\pi^2}; t\right] - L^{-1}\left[\frac{1}{s^2+\pi^2}; t\right]\right\}$$

$$= \sin \pi x \left(\cos \pi t - \frac{\sin \pi t}{\pi}\right)$$

Hence the required solution of the given IBVP is

$$u(x, t) = \sin \pi x \left(\cos \pi t - \frac{\sin \pi t}{\pi}\right)$$

EXAMPLE 6.37 If the function $u(x, t)$ satisfies the following:

PDE: $x_{xx} = \frac{1}{c^2}u_{tt} + k, \qquad 0 \le x \le l, \qquad t > 0$

BCs: $u(0, t) = u_x(l, t) = 0, \qquad t > 0,$

ICs: $u(x, 0) = u_t(x, 0), \qquad 0 \le x \le l$

then, show that

$$u(x, t) = L^{-1}\left[\frac{kc^2}{s^3}\left(\frac{\cosh\{s(l-x)/c\}}{\cosh\{sl/c\}} - 1\right); t\right]$$

and find the solution.

Solution Taking the Laplace transform of the given PDE, we get

$$\frac{d^2U}{dx^2} = \frac{1}{c^2}[s^2U(x, s) - su\,(x, 0) - u_t\,(x, 0)] + \frac{k}{s}$$

Using the initial conditions, we obtain

$$\frac{d^2U}{dx^2} - \frac{s^2}{c^2}U(x, s) = \frac{k}{s} \tag{6.80}$$

Its general solution is found to be

$$U(x, s) = Ae^{(s/c)x} + Be^{-(s/c)x} - \frac{kc^2}{s^3} \tag{6.81}$$

Taking the Laplace transform of the BCs, we get

$$U(0, s) = \frac{dU}{dx}(l, s) = 0 \tag{6.82}$$

Using these boundary conditions, Eq. (6.81) gives

$$A + B = kc^2/s^3$$

$$\frac{As}{c}e^{(s/c)l} - B\frac{s}{c}e^{-(s/c)l} = 0$$

Eliminating B, we get

$$A\frac{s}{c}[e^{(s/c)l} + e^{-(s/c)l}] = \frac{kc}{s^2}\,e^{-(s/c)l}$$

which gives

$$A = \frac{kc^2}{2s^3}\,e^{-(s/c)l}/\cosh\left(\frac{s}{c}l\right)$$

$$B = \frac{kc^2}{2s^3}\,e^{(s/c)l}/\cosh\left(\frac{s}{c}l\right)$$

Hence, from Eq. (6.81), we have

$$U(x, s) = \frac{kc^2}{s^3}\frac{\cosh\left[\frac{s}{c}(l - x)\right]}{\cosh\left(\frac{s}{c}l\right)} - \frac{kc^2}{s^3} \tag{6.83}$$

Applying the complex inversion formula, we obtain

$$u(x,t)=kc^2L^{-1}\left[\frac{\cosh\left[\frac{s}{c}(l-x)\right]}{s^3\cosh\left(\frac{s}{c}l\right)};t\right]-kc^2L^{-1}\left[\frac{1}{s^3};t\right] \tag{6.84}$$

$$=\frac{kc^2}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st}\cosh\left[\frac{s}{c}(l-x)\right]}{s^3\cosh\left(\frac{s}{c}l\right)}ds-\frac{kc^2t^2}{2} \tag{6.85}$$

But

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st}\cosh\left[\frac{s}{c}(l-x)\right]}{s^3\cosh\left(\frac{s}{c}l\right)}ds=$$ sum of the residues at the poles $s=0$ of order three and at the poles of $\cosh\,(s/c)l$, i.e., at $e^{(2s/c)l}=-1=e^{i(2n\pi+\pi)}$

or at

$$s=\frac{(2n+1)c\pi i}{2l}, \qquad n=0,\pm1,\pm2,\ldots$$

Now the residue at the pole $s = 0$ of order 3 is

$$\underset{s\to0}{\mathrm{Lt}}\frac{1}{2}\frac{d^2}{ds^2}\left[\frac{e^{st}\cosh\left\{\frac{s}{c}(l-x)\right\}}{\cosh\left(\frac{s}{c}l\right)}\right]=\frac{1}{2}\underset{s\to0}{\mathrm{Lt}}\frac{d}{ds}\left[te^{st}\frac{\cosh\left\{\frac{s}{c}(l-x)\right\}}{\cosh\left(\frac{s}{c}l\right)}+e^{xt}\frac{l-x}{c}\right.$$

$$\left.\times\frac{\sinh\left\{\frac{s}{c}(l-x)\right\}}{\cosh\left(\frac{s}{c}l\right)}-e^{st}\left(\frac{l}{c}\right)\frac{\cosh\left\{\frac{s}{c}(l-x)\right\}\sinh\left(\frac{s}{c}l\right)}{\cosh^2\left(\frac{s}{c}l\right)}\right]$$

Again differentiating and taking the limit, we get

$$\frac{1}{2}\left[t^2+\frac{(l-x)^2}{c^2}-\frac{l^2}{c^2}\right]=\frac{t^2}{2}+\frac{x^2}{2c^2}-\frac{2lx}{2c^2}$$

Also, the residue at the poles

$$s_n = \frac{i(2n+1)\pi c}{2l}, \quad n = 0, 1, 2, \ldots$$

$$= \frac{1}{s_n^3} \frac{1}{\frac{l}{c}\sinh\left(\frac{l}{c}s_n\right)} \exp(s_n t)\cosh\left(\frac{l-x}{c}s_n\right), \quad n = 0, 1, 2, \ldots$$

$$= \frac{c\exp\left\{\left(\frac{2n+1}{2l}\right)\pi ict\right\}\cosh\left\{\left(\frac{2n+1}{2l}\right)\pi ic\left(\frac{l-x}{c}\right)\right\}}{\frac{(2n+1)^3\pi^3 i^3 c^3}{2^3 l^3}\, l\sinh\left\{\frac{(2n+1)}{2l}\pi ic\frac{l}{c}\right\}}$$

$$= \frac{8l^2 c\exp\left\{\left(\frac{2n+1}{2l}c\right)\pi ict\right\}\cosh\left\{\left(\frac{2n+1}{2l}\right)\pi i(l-x)\right\}}{(2n+1)^3\pi^3(-i)c^3\sinh\left\{\frac{(2n+1)}{2}\pi i\right\}}$$

$$= \frac{8l^2\exp\left\{i\left(\frac{2n+1}{2l}\right)\pi ct\right\}\cos\left\{\left(\frac{2n+1}{2l}\right)\pi(l-x)\right\}}{(2n+1)^3\pi^3c^2\sin(2n+1)\frac{\pi}{2}}$$

(since $\sinh i\theta = i\sin\theta$
$\cosh i\theta = \cos\theta$)

Its real part is

$$\frac{8l^2\cos\left(\frac{2n+1}{2l}\right)\pi ct\cos\left\{\left(\frac{2n+1}{2l}\right)\pi(l-x)\right\}}{(-1)^n(2n+1)^3\pi^3c^2}, \quad n = 0, 1, 2, \ldots$$

Therefore,

$$L^{-1}\left[\frac{\cosh\left\{\frac{s}{c}(l-x)\right\}}{s^3\cosh\left(\frac{s}{c}l\right)}\right] = \frac{t^2}{2} + \frac{x^2}{2c^2} - \frac{lx}{c^2}$$

$$+ \sum_{n=0}^{\infty} \frac{(-1)^n 8l^2\left[\cos\left(\frac{2n+1}{2l}\right)\pi ct\right]\cos\left[\left(\frac{2n+1}{2l}\right)\pi(l-x)\right]}{(2n+1)^3\pi^3c^2}$$

Thus, the required solution from Eq. (6.84) is

$$u(x,t)=\frac{kx}{2}(x-2l)+\frac{8kl^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\cos\left[\left(\frac{2n+1}{2l}\right)\pi ct\right]\cos\left[\left(\frac{2n+1}{2l}\right)\pi(l-x)\right]$$

EXAMPLE 6.38 A string is stretched and fixed between two points (0, 0) and (l, 0). Motion is initiated by displacing the string in the form $u=\lambda\sin(\pi x/l)$ and released from rest at time $t=0$. Find the displacement of any point on the string at any time t.

Solution The displacement $u(x,t)$ of the string is governed by

PDE: $u_{tt}=c^2u_{xx}$, $\quad 0<x<l, \quad t>0$

BCs: $u(0,t)=u(l,t)=0$

ICs: $u(x,0)=\lambda\sin(\pi x/l), \quad u_t(x,0)=0$

Taking the Laplace transform of the given PDE, we have

$$s^2U(x,s)-su(x,0)-u_t(x,0)=c^2\frac{d^2U}{dx^2}$$

Using the ICs, we get

$$\frac{d^2U}{dx^2}-\frac{s^2U}{c^2}=-\frac{\lambda s}{c^2}\sin\frac{\pi x}{l} \tag{6.86}$$

Its general solution is found to be

$$U(x,s)=Ae^{(s/c)x}+Be^{-(s/c)x}+\frac{\lambda s\sin(\pi/l)x}{s^2+\pi^2c^2/l^2} \tag{6.87}$$

The Laplace transform of the BCs is given by

$$U(0,s)=0, \qquad U(l,s)=0$$

Applying these conditions in Eq. (6.87), we obtain

$$A+B=0$$

$$Ae^{sl/c}+Be^{-sl/c}=0$$

On solving the above set of equations, we get only the trivial solution, viz.

$$A=B=0$$

Thus,

$$U(x,s)=\frac{\lambda s\sin(\pi/l)x}{s^2+\pi^2c^2/l^2}$$

Its inverse Laplace transform yields

$$u(x,t) = \lambda L^{-1}\left[\frac{s}{s^2+\pi^2c^2/l^2}; t\right]\sin\frac{\pi x}{l} = \lambda\cos\left(\frac{\pi c}{l}t\right)\sin\frac{\pi x}{l}$$

which gives the displacement of the string for a given time.

EXAMPLE 6.39 An infinitely long string having one end at $x = 0$ is initially at rest on the x-axis. The end $x = 0$ undergoes a periodic transverse displacement described by $A_0 \sin \omega t, t > 0$. Find the displacement of any point on the string at any time t.

Solution Let $u(x, t)$ be the transverse displacement of the string at any point x and at any time t. Then the transverse displacement of the string is described by the

$$\begin{aligned} &\text{PDE: } u_{tt} = c^2 u_{xx}, && x > 0, \quad t > 0 \\ &\text{BCs: } u(0,t) = A_0 \sin \omega t, && t > 0 \ (A_0 = \text{constant}) \\ &\text{ICs: } u(x,0) = 0, && u_t(x,0) = 0, \quad x \ge 0 \end{aligned}$$

The Laplace transform of the PDE gives

$$s^2U - su\,(x,0) - u_t(x,0) = c^2\frac{d^2U}{dx^2} \tag{6.88}$$

After using the ICs, Eq. (6.88) becomes

$$\frac{d^2U}{dx^2} - \frac{s^2}{c^2}U(x,s) = 0 \tag{6.89}$$

Its general solution is found to be

$$U(x,s) = Ae^{(s/c)x} + Be^{-(s/c)x}$$

Since the displacement $U(x, s)$ is bounded as $x \to \infty$, we get $A = 0$ and, therefore,

$$U(x,s) = Be^{-(s/c)x} \tag{6.90}$$

Now taking the Laplace transform of the BC, we obtain

$$U(0,s) = \frac{A_0\omega}{s^2+\omega^2}$$

Using this expression, from Eq. (6.90), we have

$$B = \frac{A_0\omega}{s^2+\omega^2}$$

Hence the solution of Eq. (6.89) is found to be

$$U(x, s) = \frac{A_0\omega}{s^2 + \omega^2} e^{-(s/c)x} \tag{6.91}$$

Finally, taking the inverse Laplace transform of Eq. (6.91), we get

$$u(x, t) = L^{-1}\left[\frac{A_0\omega}{s^2 + \omega^2} e^{-(s/c)x}; t\right]$$

$$= \begin{cases} A_0 \sin \omega\left(t - \dfrac{x}{c}\right) & \text{if } t > \dfrac{x}{c} \\ 0 & \text{if } t < \dfrac{x}{c} \end{cases}$$

6.14 MISCELLANEOUS EXAMPLES

EXAMPLE 6.40 Find the Laplace transform of erf $\left(\dfrac{1}{\sqrt{t}}\right)$.

Solution Following the definition of Laplace transform, we have

$$L[\text{erf }(1/\sqrt{t}); s] = \int_0^\infty e^{-st} \frac{2}{\sqrt{\pi}} \int_0^{1/\sqrt{t}} e^{-u^2} du\, dt$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty \int_0^{1/\sqrt{t}} e^{-st-u^2} du\, dt$$

Now, changing the order of integration (see Fig. 6.5), we get

$$L[\text{erf }(1/\sqrt{t}); s] = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2} du \int_0^{1/u^2} e^{-st} dt$$

$$= -\frac{2}{s\sqrt{\pi}} \int_0^\infty (e^{-u^2 - s/u^2} - e^{-u^2})\, du \tag{6.92}$$

To evaluate this integral, consider an integral of the form

$$I(a, b) = \int_0^\infty e^{-a^2u^2 - b/u^2} du \tag{6.93}$$

We can easily verify that

$$I(a, 0) = \int_0^\infty e^{-a^2u^2} du = \frac{\sqrt{\pi}}{2a} \tag{6.94}$$

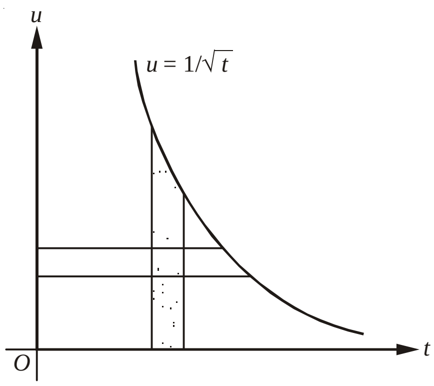

Fig. 6.5 Curve of $1/\sqrt{t}$.

since

$$\frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-u^2}\,du = 1$$

Differentiating Eq. (6.93) partially with respect to b, we get

$$\frac{\partial I}{\partial b} = -2b\int_0^{\infty} u^{-2} e^{-a^2u^2 - b^2/u^2}\,du \tag{6.95}$$

Let $au = x$; then $du = dx/a$, and we can easily verify that

$$I(a, b) = \frac{1}{a}\int_0^{\infty} \exp\,(-x^2 - a^2b^2/x^2)\,dx = \frac{1}{a} I(1, ab) \tag{6.96}$$

Also, let $b/u = x$; then we find from Eq. (6.95) that

$$\begin{aligned}
\frac{\partial I(a, b)}{\partial b} &= 2b\int_{\infty}^{0} \frac{x^2}{b^2} \exp\left(\frac{-a^2b^2}{x^2} - x^2\right)\frac{b}{x^2}\,dx \\
&= -2\int_0^{\infty} \exp\left(\frac{-x^2 - a^2b^2}{x^2}\right) dx \\
&= -2I(1, ab) \qquad (6.97)
\end{aligned}$$

From Eqs. (6.96) and (6.97), we can eliminate $I(1, ab)$ and arrive at

$$\frac{\partial I}{\partial b} = -2aI$$

On integration, we get

$$\ln I = -2ab + \ln c, \qquad I(a, b) = ce^{-2ab}$$

Thus,

$$I(a, b) = I(a, 0)\, e^{-2ab}$$

$$\int_0^\infty e^{-a^2u^2 - b^2/u^2}\, du - \frac{\sqrt{\pi}}{2a}\, e^{-2ab} \tag{6.98}$$

By making use of Eq. (6.98), the Laplace transform of the given function is obtained from Eq. (6.92) as

$$L[\text{erf }(1/\sqrt{t});\, s] = -\frac{2}{s\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2}e^{-2\sqrt{s}} - \frac{\sqrt{\pi}}{2}\right) = \frac{1}{s}(1 - e^{-2\sqrt{s}})$$

EXAMPLE 6.41 Find the inverse Laplace transform of $\dfrac{e^{-1/s}}{\sqrt{s}}$.

Solution The given function can be rewritten as

$$\frac{1}{\sqrt{s}}e^{-1/s} = \frac{1}{\sqrt{s}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\, s^n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\, s^{n+1/2}}$$

Therefore,

$$L^{-1}\left[\frac{e^{-1/s}}{\sqrt{s}};\, t\right] = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}L^{-1}\left[\frac{1}{s^{n+1/2}};\, t\right] = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{t^{n-1/2}}{\Gamma(n+1/2)}$$

But,

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{2^{2n}\, n!}\sqrt{\pi}$$

Hence,

$$L^{-1}\left[\frac{e^{-1/s}}{\sqrt{s}};\, t\right] = \sum_{n=0}^{\infty}\frac{(-1)^n\, 2^{2n}\, t^{n-1/2}}{(2n)!\sqrt{\pi}}$$

$$= \frac{t^{-1/2}}{\sqrt{\pi}} - \frac{4t^{1/2}}{2!\sqrt{\pi}} + \frac{2^4\, t^{3/2}}{4!\sqrt{\pi}}\cdots$$

$$= \frac{1}{\sqrt{\pi t}}\left[1 - \frac{(2\sqrt{t})^2}{2!} + \frac{(2\sqrt{t})^4}{4!} - \cdots\right]$$

$$= \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$$

EXAMPLE 6.42 Solve

$$\dot{x} + ky = a \sin kt, \qquad \dot{y} - kx = a \cos kt$$

using Laplace transform technique, given that

$$x(0) = 0, \qquad y(0) = b$$

Solution Taking the Laplace transform of the given equations, we get

$$sX - x(0) + kY = \frac{ak}{s^2 + k^2} \tag{6.99}$$

$$sY - y(0) - kX = \frac{as}{s^2 + k^2} \tag{6.100}$$

Utilizing the given ICs, we obtain

$$sX + kY = \frac{ak}{s^2 + k^2} \tag{6.101}$$

$$sY - kX = \frac{as}{s^2 + k^2} + b \tag{6.102}$$

Solving Eqs. (6.101) and (6.102) and using Cramer's rule, we obtain

$$X = \begin{vmatrix} \dfrac{ak}{s^2 + k^2} & k \\ \dfrac{as}{s^2 + k^2} + b & s \end{vmatrix} \Bigg/ \begin{vmatrix} s & k \\ -k & s \end{vmatrix} = -\frac{bk}{s^2 + k^2}$$

Similarly, we can show that

$$Y = \frac{a}{s^2 + k^2} + \frac{bs}{s^2 + k^2}$$

Taking the inverse Laplace transform of the above two equations, we get

$$x = -b \sin kt, \qquad y = \frac{a}{k} \sin kt + b \cos kt$$

which is the required solution.

EXAMPLE 6.43 A beam which is coincident with the x-axis is simply supported at its end $x = 0$ and is clamped at the other end $x = l$. Let a vertical load W act transversely on the beam at $x = l/4$. The differential equation for deflection at any point is given by

$$\frac{d^4 y}{dx^4} = \frac{W}{EI} \delta(x - l/4)$$

where EI is the flexural rigidity of the beam. Find the deflection at any point.

Solution Taking the Laplace transform on both sides of the governing differential equation and noting the transform of Dirac delta function from Eq. (6.23), we obtain

$$s^4Y - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = \frac{W}{EI}\exp[-(l/4)s] \tag{6.103}$$

Since the beam is simply supported at $x = 0$, we have

$$y(0) = y''(0) = 0 \tag{6.104}$$

Also, it is given that the beam is clamped at $x = l$, which means that

$$y(l) = y'(l) = 0 \tag{6.105}$$

Let $y'(0) = A$ and $y'''(0) = B$. Then using Eq. (6.104) into Eq. (6.103), we get

$$s^4Y - As^2 - B = \frac{W}{EI}e^{-(l/4)s}$$

$$Y = \frac{A}{s^2} + \frac{B}{s^4} + \frac{W}{EI}\frac{e^{-(l/4)s}}{s^4}$$

Taking the inverse Laplace transform, we obtain

$$y = Ax + \frac{Bx^3}{3!} + \frac{W}{EI}L^{-1}\left[\{e^{-(l/4)s}\}\cdot\frac{1}{s^4}; x\right] \tag{6.106}$$

But, from the second shifting property (Theorem 6.14), we have

$$L^{-1}\left[\{e^{-(l/4)s}\}\cdot\frac{1}{s^4}; x\right] = f(x - l/4)\,H(x - l/4)$$

where

$$f(x) = L^{-1}\left[\frac{1}{s^4}; x\right] = \frac{x^3}{3!}$$

Hence Eq. (6.106) becomes

$$y = Ax + \frac{B}{3!}x^3 + \frac{W}{EI}\frac{(x - l/4)^3}{3!}H(x - l/4)$$

Thus, the deflection is given by

$$\begin{aligned} y &= Ax + \frac{B}{6}x^3 && \text{for } 0 < x \le l/4 \\ &= Ax + \frac{B}{6}x^3 + \frac{W}{EI}\frac{(x - l/4)^3}{6} && \text{for } l/4 < x < l \end{aligned}$$

The unknowns A and B can now be determined by using the conditions (6.105). Thus, we have

$$0 = Al + \frac{B}{6}l^3 + \frac{9\,Wl^3}{128EI}$$

$$0 = A + \frac{B}{2}l^2 + \frac{9\,Wl^2}{32EI}$$

whose solution gives

$$A = \frac{9\,Wl^2}{256EI}, \quad B = -\frac{81\,W}{128EI}$$

Hence the resulting deflection y is given as

$$y = \begin{cases} \dfrac{9\,Wl^3}{256EI} - \dfrac{27\,Wl^3}{256EI}, & 0 < x \le l/4 \\[2ex] -\dfrac{18\,Wl^3}{256EI} + \dfrac{W}{6EI}(x - l/4)^3, & l/4 < x < l \end{cases}$$

EXAMPLE 6.44 Find the solution of the

$$\text{PDE: } u_t = ku_{xx}, \quad 0 < x < \infty, \quad t > 0$$

subject to

$$\text{BCs: } u(x, t) \to 0 \text{ as } x \to \infty$$

$$u(0, t) = g(t)$$

$$\text{IC: } u(x, 0) = 0$$

assuming

$$L^{-1}\left[\exp\left(-\sqrt{\frac{s}{k}}x\right); t\right] = \frac{x}{2\sqrt{k\pi t^3}}\exp\left(-\frac{x^2}{4kt}\right)$$

Solution Taking the Laplace transform of the PDE, we have

$$sU(x, s) - u(x, 0) = k\frac{d^2U}{dx^2} \tag{6.107}$$

Using IC: $u(x, 0) = 0$, we obtain

$$\frac{d^2U}{dx^2} - \frac{s}{k}U(x, s) = 0 \tag{6.108}$$

whose general solution is given by

$$U(x, s) = A\exp\left(\sqrt{\frac{s}{k}}x\right) + B\exp\left(-\sqrt{\frac{s}{k}}x\right)$$

The Laplace transform of the first BC gives:

$$U \to 0 \text{ as } x \to \infty$$

Using this in the solution, we get $A = 0$ and

$$U(x, s) = B\exp\left(-\sqrt{\frac{s}{k}}x\right) \tag{6.109}$$

The Laplace transform of the second BC gives

$$U(0, s) = G(s)$$

Using this condition, Eq. (6.109) becomes

$$U(x, s) = G(s)\exp\left(-\sqrt{\frac{s}{k}}x\right) \tag{6.110}$$

Taking inverse Laplace transform, we get

$$u(x, t) = L^{-1}\left[G(s)\exp\left(-\sqrt{\frac{s}{k}}x\right); s\right] = L^{-1}\left[L[g(t); s]\, L\left[\frac{x}{2\sqrt{k\pi t^3}}\exp(-x^2/4kt); s\right]\right]$$

Finally, through the use of the convolution theorem, we arrive at the result

$$u(x, t) = \int_0^t \frac{x\exp[-x^2/4k(t-u)]}{2\sqrt{\pi k}\,(t-u)^{3/2}}\, g(u)\, du$$

EXAMPLE 6.45 Find the solution of IBVP described by

$$\begin{aligned}
&\text{PDE: } u_t(x, t) = u_{xx}(x, t) - hu(x, t),\\
&\qquad h = \text{constant},\ 0 < x < \pi,\ t > 0\\
&\text{BCs: } u(0, t) = 0, \qquad t > 0\\
&\qquad u(\pi, t) = 1, \qquad t > 0\\
&\text{IC: } u(x, 0) = 0, \qquad t = 0
\end{aligned}$$

Solution Taking the Laplace transform of the PDE with respect to the variable t, we have

$$sU(x, s) - u(x, 0) = \frac{d^2U}{dx^2} - hU(x, s) \tag{6.111}$$

Using the IC: $u(x, 0) = 0$, we get

$$\frac{d^2U}{dx^2} - (h + s)U(x, s) = 0 \tag{6.112}$$

Its general solution is found to be

$$U(x, s) = A \cosh (\sqrt{h+s}\ x) + B \sinh (\sqrt{h+s}\ x) \tag{6.113}$$

Taking the Laplace transform of BCs, we obtain

$$U(0, s) = 0, \quad U(\pi, s) = 1/s \tag{6.114}$$

Using these BCs into Eq. (6.113), we have

$$U(x, s) = B \sinh (\sqrt{h+s}\ x)$$

$$\frac{1}{s} = B \sinh (\sqrt{h+s}\ \pi)$$

implying thereby

$$B = \frac{1}{s} \frac{1}{\sinh (\sqrt{h+s}\ \pi)}$$

Hence, the required solution of Eq. (6.112) is

$$U(x, s) = \frac{1}{s} \frac{\sinh (\sqrt{h+s}\ x)}{\sinh (\sqrt{h+s}\ \pi)}$$

By means of complex inversion formula, we get

$$u(x, t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st} \sinh (\sqrt{h+s}\ x)}{s \sinh (\sqrt{h+s}\ \pi)}$$

$$= \text{sum of the contributions from all the poles of the integrand}$$

The poles are given by $s = 0$, and $\sinh (\sqrt{h+s}\ \pi) = 0 = \sin (i\sqrt{h+s}\ \pi)$. Therefore,

$$i\sqrt{h+s}\ \pi = n\pi, \text{ implying } h + s = -n^2$$

Thus, the integrand has poles at $s = 0$, and

$$s = -n^2 - h, \qquad n = 1, 2, 3, \ldots$$

The residue of the expression

$$\frac{e^{st} \sinh (\sqrt{h+s}\, x)}{s \sinh (\sqrt{h+s}\, \pi)} \quad \text{at } s = 0$$

is

$$\frac{\sinh (\sqrt{h}\, x)}{\sinh (\sqrt{h}\, \pi)}$$

The residue of the integrand at $s = -n^2 - h$ is

$$\frac{e^{(-n^2-h)t} \sinh (\sqrt{-n^2}\, x)}{\sinh \sqrt{-n^2}\, \pi + \dfrac{(-n^2-h)\pi}{2\sqrt{-n^2}} \cosh (\sqrt{-n^2}\, \pi)}, \qquad n = 1, 2, 3, \ldots$$

Using the relations $\sinh x = i \sin ix$, and $\cosh x = \cos ix$, the above residues become

$$\frac{\exp [-(n^2+h)t] i \sin nx}{\dfrac{-(n^2+h)\pi}{2in} \cos n\pi} = \frac{2n \exp [-(n^2+h)t] \sin nx}{(n^2+h)\pi \cos n\pi}$$

$$= \frac{2n \exp [-(n^2+h)t] \sin nx}{(n^2+h)\pi \, (-1)^n}$$

Hence, the required solution is

$$u(x, t) = \frac{\sinh (\sqrt{h}\, x)}{\sinh (\sqrt{h}\, \pi)} + \frac{2e^{-ht}}{\pi} \sum_{n=1}^{\infty} \frac{n(-1)^n \exp (-n^2 t) \sin nx}{n^2 + h}$$

EXERCISES

1. Find the Laplace transform of the following:

(i) $t \cos at$, (ii) $te^{-t} \sin t$,

(iii) $te^{t} \sin 2t$, (iv) $(1+t)^2 e^{-at}$.

2. Find the Laplace transform of $f(t)$ defined as

(i) $f(t) = \begin{cases} t/\tau, & 0 < t < \tau \\ 1, & t > \tau \end{cases}$

(ii) $f(t) = \begin{cases} t, & 0 < t < 1 \\ (1-t), & 1 < t < 2 \\ 0, & t > 2 \end{cases}$

3. Find the Laplace transform of

(i) $\dfrac{1-e^t}{t}$, (ii) $\dfrac{\cos 2t - \sin 3t}{t}$.

4. Obtain the Laplace transform of

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}$$

given $f(t+2) = f(t)$ for $t > 0$.

5. Find the Laplace transform of

$$f(t) = \begin{cases} t, & 0 \le t < 6 \\ 12 - t, & 6 \le t < 12 \end{cases}$$

given $f(t+12) = f(t)$ for $t > 0$.

6. Find the Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \le t < b \\ -1, & b \le t < 2b \end{cases}$$

given $f(t+2b) = f(t)$.

7. Find the Laplace transform of erfc $(1/\sqrt{t})$.

8. Find the Laplace transform of

(i) $J_1(t)$, (ii) $tJ_1(t)$.

9. Find the inverse Laplace transform of

(i) $\dfrac{1}{s(s+1)(s+2)}$, (ii) $\dfrac{s}{(s-2)^3}$

(iii) $\dfrac{1}{s(s^2+1)}$, (iv) $\dfrac{4s+5}{(s-1)^2(s+2)}$,

(v) $\dfrac{s+3}{(s^2+6s+13)^2}$, (vi) $\dfrac{s^2+1}{s^3+3s^2+2s}$.

10. Find the inverse Laplace transform of

(i) $\ln \dfrac{s+a}{s+b}$, (ii) $\tan^{-1}\left(\dfrac{s}{k}\right)$,

(iii) $\ln \dfrac{s^2+1}{(s-1)^2}$, (iv) $\ln \dfrac{s-4}{4+s^2}$.

11. State and prove the convolution theorem for Laplace transforms.

12. Using the convolution theorem, find the inverse Laplace transform of

(i) $\dfrac{1}{s^2(s^2+a^2)}$, (ii) $\dfrac{1}{\sqrt{s}(s-a)}$, (iii) $\dfrac{s}{(s^2+4)^3}$.

13. Find the inverse Laplace transform of

$$e^{-\pi s}/(s^2+1)$$

14. Using the Heaviside expansion theorem, find the inverse Laplace transform of

(i) $\dfrac{1}{s^3+1}$, (ii) $\dfrac{3s+1}{(s-1)(s^2+1)}$.

15. Find the inverse Laplace transform of

$$F(s) = \frac{\cosh (x\sqrt{s})}{s \cosh \sqrt{s}}, \quad 0 < x < 1$$

using the complex inversion formula.

16. Solve the following ODE using the Laplace transform

$$y'' - 3y' + 2y = 4e^{2t}$$

given that $y(0) = -3$, $y'(0) = 5$.

17. Solve the ODE by the Laplace transform method

$$\frac{d^4y}{dt^4} - k^4 y = 0$$

with the initial conditions $y(0) = 1$, $y'(0) = y''(0) = 0$, $y'''(0) = 0$.

18. Solve the initial value problem using the method of Laplace transform

$$y'' + ty' + y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

19. Using the Laplace transform method, solve the system of equations

$$\frac{dx}{dt} + \frac{dy}{dt} + x + y = 1$$

$$\frac{dy}{dt} = 2x + y$$

given that $x = 0$ and $y = 1$ when $t = 0$.

20. Applying the Laplace transform technique, solve the system:

$$\frac{dx}{dt}+\frac{dy}{dt}+x+e^{-t}=0$$

$$\frac{dx}{dt}+2\frac{dy}{dt}+2x+2y=0$$

given that $x=-1$ and $y=1$ when $t=0$.

21. Using the Laplace transform, find the solution of the system of ODEs

$$\frac{d^2x}{dt^2}+y=1, \qquad \frac{d^2x}{dt^2}+x=0$$

satisfying the ICs $x(0)=y(0)=x'(0)=y'(0)=0$.

22. Show that, by means of the Laplace transform, the solution $\theta(x,t)$ of one-dimensional diffusion equation

$$\frac{\partial^2\theta}{\partial x^2}=\frac{1}{k}\frac{\partial\theta}{\partial t}, \qquad 0\le x\le\pi, \qquad t>0$$

satisfying the boundary conditions

$$\theta(0,t)=1-e^{-t}, \qquad t>0$$

$$\theta(\pi,t)=0, \qquad t>0$$

$$\theta(x,0)=0, \qquad t=0,\ 0\le x\le\pi$$

is given by the formula

$$\theta(x,t)=\frac{\pi-x}{\pi}-\frac{e^{-t}\sin\{(\pi-x)/\sqrt{k}\}}{\sin(\pi/\sqrt{k})}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx\exp(-n^2kt)}{n(n^2k-1)}$$

23. Using the Laplace transform method, solve

PDE: $u_t=3u_{xx}$

BCs: $u\left(\frac{\pi}{2},t\right)=0, \qquad u_x(0,t)=0$

IC: $u(x,0)=30\cos 5x$

24. Using the Laplace transform, solve the following problem in wave propagation:

PDE: $c^2u_{xx}=u_{tt}, \qquad 0<x<l, \qquad t>0$

BCs: $u=0$ at $x=0$

$Eu_t=P$ at $x=l, \qquad t>0$ (E, P are constants)

ICs: $u=u_t(x,0)=0, \qquad 0<x<l$

25. Solve the BVP using the Laplace transform method

$$\text{PDE: } u_{tt} = c^2 u_{xx}, \qquad 0 \le x < \infty, \quad 0 \le t < \infty$$

$$\text{BC: } u(0, t) = 0$$

$$\text{ICs: } u(x, 0) = 0,$$

$$u_t(x, 0) = 1$$

26. If ϕ is the potential, i the current, l the inductance per unit length of cable, c the capacitance to ground per unit length, then ϕ satisfies the wave equation

$$\phi_{xx} = lc\phi_{tt}$$

Initially, the line is considered to be dead, i.e.

$$\phi(x, 0) = \phi_t(x, 0) = 0$$

The other boundary conditions are

$$\phi(0, t) = f(t), \qquad t > 0$$

$$\phi_x(l, t) = \phi(l, t) = 0, \qquad t > 0$$

Find the potential at any point on the cable at any time t, assuming $l = \infty$.

CHAPTER 7

Fourier Transform Methods

7.1 INTRODUCTION

Joseph Fourier, a French mathematician, had invented a method called Fourier transform in 1801, to explain the flow of heat around an anchor ring. Since then, it has become a powerful tool in diverse fields of science and engineering. It can provide a means of solving unwieldy equations that describe dynamic responses to electricity, heat or light. In some cases, it can also identify the regular contributions to a fluctuating signal, thereby helping to make sense of observations in astronomy, medicine and chemistry. Fourier transform has become indispensable in the numerical calculations needed to design electrical circuits, to analyze mechanical vibrations, and to study wave propagation.

Fourier transform techniques have been widely used to solve problems involving semi-infinite or totally infinite range of the variables or unbounded regions. In order to deal with such problems, it is necessary to generalize Fourier series to include infinite intervals and to introduce the concept of Fourier integral. In this chapter, we deal with Fourier integral representations and Fourier transforms together with some applications to Diffusion, Wave and Laplace equations.

7.2 FOURIER INTEGRAL REPRESENTATIONS

Definition 7.1 (Dirichlet's conditions). A function $f(x)$ is said to have satisfied Dirichlet's conditions in the interval $(-L, L)$, provided $f(x)$ is periodic, piecewise continuous, and has a finite number of relative maxima and minima in $(-L, L)$.

Let a function $f(x)$ be periodic with period $2L$, i.e., $f(x+2L) = f(x)$, and satisfy Dirichlet's conditions in the interval $(-L, L)$. Then $f(x)$ has a Fourier series representation for every x in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \tag{7.1}$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{n\pi t}{L} dt, \qquad n = 0, 1, 2, \ldots \tag{7.2}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi t}{L} dt, \qquad n = 1, 2, \ldots \tag{7.3}$$

Here, a_n, b_n are called Fourier coefficients. Fourier series representation, however, can be extended to some non-periodic functions also, provided the integral of the modulus of such a function $f(t)$ satisfies the condition

$$\int_{-\infty}^{\infty} | f(t) | \, dt$$

is finite.

Substituting Eqs. (7.2) and (7.3) into Fourier series (7.1), we get

$$f(x) = \frac{1}{2L} \int_{-L}^{L} f(t)\, dt + \sum_{n=1}^{\infty} \left[\frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{n\pi t}{L} \cos \frac{n\pi x}{L} dt + \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi t}{L} \sin \frac{n\pi x}{L} dt \right]$$

Noting that $\cos (A - B) = \cos A \cos B + \sin A \sin B,$ and interchanging the order of summation and integration, we obtain

$$f(x) = \frac{1}{2L} \int_{-L}^{L} f(t)\, dt + \frac{1}{L} \int_{-L}^{L} f(t) \sum_{n=1}^{\infty} \cos \frac{n\pi(t - x)}{L} dt \tag{7.4}$$

Further, if we assume that the function $f(x)$ is absolutely integrable, and allowing L to tend to infinity, i.e.,

$$\int_{-\infty}^{\infty} | f(t) | \, dt < \infty \tag{7.5}$$

we get

$$\operatorname*{Lt}_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} f(t)\, dt = 0 \tag{7.6}$$

In the remaining part of the infinite sum of Eq. (7.4), if we set $\Delta s = \pi/L,$ the equation reduces to

$$f(x) = \operatorname*{Lt}_{\Delta s \to 0} \frac{1}{\pi} \int_{-\pi/\Delta s}^{\pi/\Delta s} f(t) \sum_{n=1}^{\infty} \cos \{n\Delta s\, (t - x)\} dt \tag{7.7}$$

As $L \to \infty, \Delta s \to 0$, implying that Δs is a small positive number and the points $n\Delta s$ are equally spaced along the s-axis. The series under the integral can be approximated by an integral of the form (as $\Delta s \to 0$)

$$\int_0^\infty \cos\{s(t-x)\}\, ds$$

Thus, Eq. (7.7) can be rewritten as

$$f(x) = \frac{1}{\pi}\int_{-\infty}^\infty f(t)\int_0^\infty \cos\{s(t-x)\}\, ds\, dt \tag{7.8}$$

$$= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\{s(t-x)\}\, dt\, ds \tag{7.9}$$

which is the Fourier integral representation of $f(x)$.

7.2.1 Fourier Integral Theorem

Theorem 7.1 If $f(x)$ satisfies Dirichlet's conditions for $-\infty < x < \infty$ and if the integral $\int_{-\infty}^\infty f(x)\, dx$ is absolutely convergent, then

$$\frac{1}{\pi}\int_0^\infty d\alpha \int_{-\infty}^\infty f(t)\cos\alpha(t-x)\, dt = \frac{1}{2}[f(x+0)+f(x-0)] \tag{7.10}$$

To establish this result, the central results required are the Riemann-Lebesgue lemma and the Riemann localization lemma; first we shall state and prove the former.

Riemann-Lebesgue lemma If $f(x)$ satisfies Dirichlet's conditions in the interval (a, b), then each of the integrals

$$\int_a^b f(x)\sin Nx\, dx, \quad \int_a^b f(x)\cos Nx\, dx$$

tends to zero as $N \to \infty$.

Proof Suppose $a_1, a_2, \ldots, a_p$ are the points in (a, b) taken in the order at which the function $f(x)$ has either a turning value or a finite discontinuity. Let $a = a_0$, and $b = a_{p+1}$. Then we may write

$$\int_a^b f(x)\sin Nx\, dx = \sum_{r=0}^{p}\int_{a_r}^{a_{r+1}} f(x)\sin Nx\, dx \tag{7.11}$$

Now in each of the sub-intervals $(a_r, a_{r+1}), r = 0, 1, 2, ..., p, f(x)$ is a continuous function and is either monotonically increasing or decreasing. Thus, aplying the second Mean Value Theorem of integral calculus to each of these intervals, we have

$$\int_{a_r}^{a_{r+1}} f(x) \sin Nx\, dx = f(a_r) \int_{a_r}^{\xi} \sin Nx\, dx + f(a_{r+1}) \int_{\xi}^{a_{r+1}} \sin Nx\, dx \tag{7.12}$$

where ξ is some value of x in the range (a_r, a_{r+1}). Carrying out the integrations on the right-hand side of Eq. (7.12), we get

$$f(a_r)\left(\frac{\cos Na_r - \cos N\xi}{N}\right) + f(a_{r+1})\left(\frac{\cos N\xi - \cos Na_{r+1}}{N}\right)$$

Now taking the limit as $N \to \infty$, we get

$$\underset{N\to\infty}{\mathrm{Lt}} \int_{a_r}^{a_{r+1}} f(x) \sin Nx\, dx = 0$$

Since the number of terms on the right-hand side of Eq. (7.11) is finite, interchanging of summation and limit process can be carried out and, therefore,

$$\underset{N\to\infty}{\mathrm{Lt}} \int_a^b f(x) \sin Nx\, dx = \underset{N\to\infty}{\mathrm{Lt}} \sum_{r=0}^{p} \int_{a_r}^{a_{r+1}} f(x) \sin Nx\, dx = \sum_{r=0}^{p} \underset{N\to\infty}{\mathrm{Lt}} \int_{a_r}^{a_{r+1}} f(x) \sin Nx\, dx = 0$$

i.e.,

$$\underset{N\to\infty}{\mathrm{Lt}} \int_a^b f(x) \sin Nx\, dx = 0 \tag{7.13}$$

which is the Riemann-Lebesgue lemma.

Riemann localization lemma If $f(t)$ satisfies Dirichlet's conditions in the interval $(0, a)$, where a is finite, then

$$\int_0^a f(t) \frac{\sin Nt}{t} dt \to \frac{\pi}{2} f(0+)$$

as $N \to \infty$.

Proof We may write

$$\int_0^a f(t) \frac{\sin Nt}{t} dt = f(0+) \int_0^a \frac{\sin Nt}{t} dt + \int_0^a \{f(t) - f(0+)\} \frac{\sin Nt}{t} dt = I_1 + I_2$$

Since the function $f'(t)$ is continuous in (0, a), from the definition of derivative

$$\frac{f(t)-f(0+)}{t}$$

is continuous in (0, a). By the Riemann-Lebesgue lemma (since the integrand of I_2 is bounded as $N \to \infty$) $I_2 \to 0$ as $N \to \infty$. Hence,

$$\underset{N\to\infty}{\text{Lt}} \int_0^a f(t)\frac{\sin Nt}{t}\,dt = f(0+)\int_0^a \frac{\sin Nt}{t}\,dt = f(0+)\int_0^{Na} \frac{\sin u}{u}\,du$$

$$= f(0+)\int_0^\infty \frac{\sin u}{u}\,du = \frac{\pi}{2} f(0+) \tag{7.14}$$

which is the Riemann localization lemma.

Proof (Fourier integral theorem). Since the integral $\int_{-\infty}^{\infty} f(x)\,dx$ is absolutely convergent, $\int_{-\infty}^{\infty} |f(x)|\,dx$ is finite and converges for all α in the range (0, N). Also, $|\cos\alpha\,(t-x)| \le 1$, implying that the integral

$$\int_{-\infty}^{\infty} f(t)\cos\alpha\,(t-x)\,dt$$

converges and is independent of α and x. Thus, after changing the order of integration, the double integral.

$$I = \int_0^N \left[\int_{-\infty}^{\infty} f(t)\cos\alpha\,(t-x)\,dt\right] d\alpha \tag{7.15}$$

can be expressed as

$$I = \int_{-\infty}^{\infty}\left[\int_0^N f(t)\cos\alpha\,(t-x)\,d\alpha\right]dt = \int_{-\infty}^{\infty} f(t)\left[\frac{\sin N(t-x)}{t-x}\right]dt$$

Let $v = t - x$. Then the above integral becomes

$$I = \int_{-\infty}^{\infty} f(v+x)\frac{\sin Nv}{v}\,dv = \left(\int_{-\infty}^{-\delta} + \int_{-\delta}^{0} + \int_0^{\delta} + \int_{\delta}^{\infty}\right) f(v+x)\frac{\sin Nv}{v}\,dv = I_1 + I_2 + I_3 + I_4$$

When $N \to \infty$, I_1 and I_4 both tend to zero in view of the Riemann-Lebesgue lemma. Thus the only contribution to the integral will be from the neighbourhood of $v = 0$. Using the Riemann localization lemma, we get

$$I_3 = \underset{N \to \infty}{\text{Lt}} \int_0^{\delta} f(v+x) \frac{\sin Nv}{v} dv = \frac{\pi}{2} f(x+) \tag{7.16}$$

and the second integral

$$I_2 = \int_{-\delta}^{0} f(v+x) \frac{\sin Nv}{v} dv = \int_0^{\delta} f(x-v) \frac{\sin Nv}{v} dv = \frac{\pi}{2} f(x-) \tag{7.17}$$

Incorporating these results into Eq. (7.15), we obtain

$$\int_0^{\infty} \left[\int_{-\infty}^{\infty} f(t) \cos \alpha (t-x)\, dt \right] d\alpha = \frac{\pi}{2} [f(x+) + f(x-)] \tag{7.18}$$

If f is a continuous function of x, then

$$f(x+) = f(x-) = f(x)$$

and Eq. (7.18) reduces to

$$f(x) = \frac{1}{\pi} \int_0^{\infty} d\alpha \left[\int_{-\infty}^{\infty} f(t) \cos \alpha (t-x)\, dt \right] \tag{7.19}$$

If x is a point of discontinuity, then

$$f(x) = \frac{1}{2} [f(x+) + f(x-)] \tag{7.20}$$

i.e., the intergral (7.19) converges to the average value of the right- and left-hand limits. Thus, the proof of the Fourier integral theorem is complete.

In order to bring out the analogy between Fourier series and Fourier integral theorem, we rewirte Eq. (7.19) as

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) (\cos \alpha t \cos \alpha x + \sin \alpha t \sin \alpha x)\, dt\, d\alpha$$

If we define

$$A(\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \alpha t\, dt \tag{7.21}$$

$$B(\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \alpha t\, dt \tag{7.22}$$

the above equation can also be written as

$$f(x) = \int_0^{\infty} [A(\alpha) \cos \alpha x + B(\alpha) \sin \alpha x]\, dt\, d\alpha \tag{7.23}$$

7.2.2 Sine and Cosine Integral Representations

If $f(-x) = -f(x)$, i.e., if $f(x)$ is an odd function, Eq. (7.21) gives

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \cos \alpha x \, dx$$

If $f(x)$ is an odd function, then

$$A(\alpha) = -\frac{1}{\pi}\int_{\infty}^{-\infty} f(-x) \cos \alpha x \, dx = +\frac{1}{\pi}\int_{\infty}^{-\infty} f(x) \cos \alpha x \, dx$$

$$= -\frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \cos \alpha x \, dx = -A(\alpha)$$

implying $2A(\alpha) = 0$ or $A(\alpha) = 0$, i.e.,

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \cos \alpha x \, dx = 0 \tag{7.24}$$

Also,

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \sin \alpha x \, dx$$

$$= -\frac{1}{\pi}\int_{\infty}^{-\infty} f(x) \sin \alpha x \, dx \quad \text{if } f(x) \text{ is an odd function}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x) \sin \alpha x \, dx = \frac{2}{\pi}\int_{0}^{\infty} f(x) \sin \alpha x \, dx \tag{7.25}$$

Thus, Eq. (7.23) reduces to

$$f(x) = \int_{0}^{\infty} B(\alpha) \sin \alpha x \, d\alpha \tag{7.26}$$

which is the Fourier sine integral representation, where $A(\alpha)$ and $B(\alpha)$ are defined by the relations (7.24) and (7.25). Similarly, if $f(x)$ is an even function, i.e., if $f(-x) = f(x)$, then we obtain the Fourier cosine integral representation

$$f(x) = \int_{0}^{\infty} A(\alpha) \cos \alpha x \, d\alpha \tag{7.27}$$

where

$$B(\alpha) = 0, \qquad A(\alpha) = \frac{2}{\pi}\int_{0}^{\infty} f(x) \cos \alpha x \, dx \tag{7.28}$$

7.3 FOURIER TRANSFORM PAIRS

From the Fourier Integral Theorem 7.1, we have

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\alpha(t-x)\,dt\,d\alpha \tag{7.29}$$

In terms of the complex exponential function, Eq. (7.29) takes the form

$$f(x)=\frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\,[e^{i\alpha(t-x)}+e^{-i\alpha(t-x)}]\,dt\,d\alpha$$

$$=\frac{1}{2\pi}\left[\int_0^{\infty}\int_{-\infty}^{\infty} f(t)e^{i\alpha(t-x)}dt\,d\alpha+\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\,e^{-i\alpha(t-x)}dt\,d\alpha\right]$$

Let $\alpha=-\alpha$ in the second integral; then it becomes

$$-\int_0^{-\infty}\int_{-\infty}^{\infty} f(t)e^{i\alpha(t-x)}dt\,d\alpha=\int_{-\infty}^{0}\int_{-\infty}^{\infty} f(t)\,e^{i\alpha(t-x)}dt\,d\alpha$$

Hence,

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\,e^{i\alpha(t-x)}dt\,d\alpha \tag{7.30}$$

This is the exponential form of the Fourier integral theorem. Equation (7.30) can be rewritten as

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{i\alpha t}dt\right]e^{-i\alpha x}d\alpha \tag{7.31}$$

Thus, we formally define the Fourier transform pair as follows:

Definition 7.2 Let $f(x)$ be a function defined on $(-\infty,\infty)$ and is piecewise continuous, differentiable in each finite interval and is absolutely integrable on $(-\infty,\infty)$. From Eq. (7.31), if

$$F(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\alpha t}dt \tag{7.32}$$

then we have, for all x,

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha)e^{-i\alpha x}d\alpha \tag{7.33}$$

Here, $F(\alpha)$ defined by Eq. (7.32) is the Fourier transform of $f(x)$, and $f(x)$ defined by Eq. (7.33) is called the Inverse Fourier transform of $F(\alpha)$ and is denoted by

$$F(\alpha) = \mathcal{F}[f(t); \alpha] \tag{7.34}$$

$$f(x) = \mathcal{F}^{-1}[F(\alpha); x] \tag{7.35}$$

which constitute the Fourier transform pair.

We have seen in Section 7.2.2 that if $f(x)$ is an odd function, the Fourier integral representation of $f(x)$ reduces to

$$f(x) = \int_0^\infty B(\alpha) \sin \alpha x \, d\alpha$$

or

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin \alpha x \int_0^\infty f(t) \sin \alpha t \, dt \, d\alpha \tag{7.36}$$

If

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \alpha t \, dt = \mathcal{F}_s[f(t); \alpha] \tag{7.37}$$

then

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\alpha) \sin \alpha x \, d\alpha = \mathcal{F}_s^{-1}[F_s(\alpha); x] \tag{7.38}$$

Here, Eq. (7.37) is the Fourier sine transform of $f(x)$ and its inverse sine transform is given by Eq. (7.38).

Similarly, when $f(x)$ is an even function, we can obtain the Fourier cosine transform and the corresponding inverse as

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \alpha t \, dt = \mathcal{F}_c[f(t); \alpha] \tag{7.39}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\alpha) \cos \alpha x \, d\alpha = \mathcal{F}_c^{-1}[F_c(\alpha); x] \tag{7.40}$$

7.4 TRANSFORM OF ELEMENTARY FUNCTIONS

EXAMPLE 7.1 Find the Fourier transform of

$$f(x) = e^{-x^2/2}$$

Solution Following the definition of Fourier transform, we have

$$\mathcal{F}[f(x);\alpha] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2} e^{i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-i\alpha)^2/2}\; e^{-\alpha^2/2} dx$$

Let

$$\frac{x - i\alpha}{\sqrt{2}} = t$$

Then

$$\frac{dx}{\sqrt{2}} = dt$$

Thus

$$\mathcal{F}[f(x);\alpha] = \frac{e^{-\alpha^2/2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2} dt$$

or

$$F(\alpha) = \frac{e^{-\alpha^2/2}}{\sqrt{\pi}}\sqrt{\pi} = e^{-\alpha^2/2}$$

EXAMPLE 7.2 Find the Fourier transform of

$$f(x) = e^{-a|x|}, \qquad -\infty < x < \infty$$

Solution We know that

$$|x| = \begin{cases} -x, & x < 0 \\ \ \ x, & x \ge 0 \end{cases}$$

Therefore,

$$F[f(x);\alpha] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{ax} e^{i\alpha x} dx + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-ax} e^{i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(a+i\alpha)x} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{(-a+i\alpha)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{a+i\alpha} - \frac{1}{-a+i\alpha} \right) = \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2+\alpha^2} \right)$$

EXAMPLE 7.3 Find the Fourier transform of $f(x)$ defined by

$$f(x) = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$$

and hence evaluate

$$\int_{-\infty}^{\infty} \frac{\sin \alpha a \cos \alpha x}{\alpha} d\alpha, \ \int_{0}^{\infty} \frac{\sin \alpha a}{\alpha} d\alpha$$

Solution From the definition of the Fourier transform,

$$F(\alpha) = \mathscr{F}[f(x); \alpha] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{i\alpha x} dx = \frac{1}{\sqrt{2\pi}} \left(\frac{e^{i\alpha x}}{i\alpha} \right)_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{i\alpha a}}{i\alpha} - \frac{e^{-i\alpha a}}{i\alpha} \right) = \frac{2}{\alpha\sqrt{2\pi}} \left(\frac{e^{i\alpha a} - e^{-i\alpha a}}{2i} \right)$$

Therefore,

$$F(\alpha) = \begin{cases} \dfrac{2 \sin \alpha a}{\alpha\sqrt{2\pi}}, & \alpha > 0 \\ \underset{\alpha \to 0}{\text{Lt}} \dfrac{2a}{\sqrt{2\pi}} \dfrac{\sin \alpha a}{\alpha a} = \dfrac{2a}{\sqrt{2\pi}}, & \alpha = 0 \end{cases}$$

Now,

$$f(x) = \mathscr{F}^{-1}[F(\alpha); x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha)\, e^{-i\alpha x} d\alpha$$

Thus,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2 \sin \alpha a}{\alpha\sqrt{2\pi}} e^{-i\alpha x} d\alpha = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$$

i.e.,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin \alpha a\,(\cos \alpha x - i \sin \alpha x)}{\alpha}\, dx = \begin{cases} 1, & |x| \le a \\ 0, & |x| > a \end{cases}$$

Hence,

$$\int_{-\infty}^{\infty} \frac{\sin \alpha a \cos \alpha x}{\alpha}\, d\alpha = \begin{cases} \pi, & |x| \le a \\ 0, & |x| > a \end{cases}$$

Also, by setting $x = 0$ in the above equations, we obtain

$$\int_{-\infty}^{\infty} \frac{\sin \alpha a}{\alpha}\, d\alpha = \pi$$

Since the integrand is even, we can have

$$\int_{0}^{\infty} \frac{\sin \alpha a}{\alpha}\, d\alpha = \frac{\pi}{2}$$

EXAMPLE 7.4 Find the Fourier cosine and sine transforms of e^{-bx} and evaluate the integrals

(i) $\displaystyle\int_{0}^{\infty} \frac{\cos \alpha x}{\alpha^2 + b^2}\, d\alpha$

(ii) $\displaystyle\int_{0}^{\infty} \frac{\alpha \sin \alpha x}{\alpha^2 + b^2}\, d\alpha$

Solution Given $f(x) = e^{-bx}$ and following the definitions of Fourier cosine and sine transforms, viz.

$$F_c(\alpha) = \mathscr{F}_c[f(x); \alpha] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos \alpha x\, dx$$

$$F_s(\alpha) = \mathscr{F}_s[f(x); \alpha] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin \alpha x\, dx$$

we obtain

$$\mathscr{F}_c[e^{-bx}; \alpha] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-bx} \cos \alpha x\, dx \tag{7.41}$$

$$\mathscr{F}_s[e^{-bx}; \alpha] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-bx} \sin \alpha x\, dx \tag{7.42}$$

Let

$$I_1 = \int_0^\infty e^{-bx} \cos \alpha x \, dx$$

$$I_2 = \int_0^\infty e^{-bx} \sin \alpha x \, dx$$

Integrating I_1 by parts, we have

$$I_1 = \left(-\frac{1}{b} e^{-bx} \cos \alpha x \right)_0^\infty - \frac{\alpha}{b} \int_0^\infty e^{-bx} \sin \alpha x \, dx = \frac{1}{b} - \frac{\alpha}{b} I_2 \tag{7.43}$$

Integrating I_2 by parts, we have

$$I_2 = \left(-\frac{1}{b} e^{-bx} \sin \alpha x \right)_0^\infty - \frac{\alpha}{b} \int_0^\infty e^{-bx} \cos \alpha x \, dx = \frac{\alpha}{b} I_1 \tag{7.44}$$

Solving Eqs. (7.43) and (7.44) for I_1 and I_2, we obtain

$$I_1 = \frac{b}{\alpha^2 + b^2}, \qquad I_2 = \frac{\alpha}{\alpha^2 + b^2}$$

Hence,

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \frac{b}{\alpha^2 + b^2}, \qquad F_s(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2 + b^2}$$

Then,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\alpha) \cos \alpha x \, d\alpha$$

i.e.

$$e^{-bx} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{b}{\alpha^2 + b^2} \cos \alpha x \, d\alpha$$

or

$$\int_0^\infty \frac{\cos \alpha x}{\alpha^2 + b^2} = \frac{\pi}{2b} e^{-bx}$$

Similarly, it can be shown that

$$\int_0^\infty \frac{\alpha \sin \alpha x}{\alpha^2 + b^2} = \frac{\pi}{2} e^{-bx}$$

EXAMPLE 7.5 Find the Fourier sine transform of $f(x)$, if

$$f(x) = \begin{cases} 0, & 0 < x < a \\ x, & a \le x \le b \\ 0, & x > b \end{cases}$$

Solution Following the definition of the Fourier sine transform, we have

$$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_a^b x \sin \alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(-\frac{x \cos \alpha x}{\alpha} \right)_a^b + \frac{1}{\alpha} \int_a^b \cos \alpha x \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{a \cos \alpha a - b \cos \alpha b}{\alpha} + \frac{\sin \alpha b - \sin \alpha a}{\alpha^2} \right)$$

7.5 PROPERTIES OF FOURIER TRANSFORM

In many practical situations, determination of Fourier transform of certain functions is very complex. Once we know the transform of some elementary functions, we can find the transform of many other functions with the help of the properties associated with the Fourier transform. We now discuss some of the important properties of the Fourier transform.

Theorem 7.2 (Linearity property). If $F(\alpha)$ and $G(\alpha)$ are the Fourier transforms of $f(x)$ and $g(x)$ respectively, then

$$\mathcal{F}[c_1 f(x) + c_2 g(x); \alpha] = c_1 F(\alpha) + c_2 G(\alpha)$$

$$\mathcal{F}^{-1}[c_1 F(\alpha) + c_2 G(\alpha); x] = c_1 f(x) + c_2 g(x)$$

where c_1 and c_2 are constants.

Proof

$$\mathcal{F}[c_1 f(x) + c_2 g(x); \alpha] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} [c_1 f(x) + c_2 g(x)] \, dx$$

$$= \frac{c_1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} f(x) \, dx + \frac{c_2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} g(x) \, dx$$

$$= c_1 F(\alpha) + c_2 G(\alpha)$$

Theorem 7.3 (Change of scale). If $F(\alpha)$ is the Fourier transform of $f(x)$, then the Fourier transform of $f(ax)$ is

$$\frac{1}{a}F\left(\frac{\alpha}{a}\right)$$

Proof From the definition of the Fourier transform, we have

$$F(\alpha) = \mathcal{F}[f(x);\alpha] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} f(x)\,dx$$

Hence,

$$\mathcal{F}[f(ax);\alpha] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} f(ax)\,dx$$

Setting $ax = t$, we have $dx = dt/a$. Therefore,

$$\mathcal{F}[f(ax);\alpha] = \frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left(i\frac{\alpha}{a}t\right) f(t)\,dt$$

Hence,

$$\mathcal{F}[f(ax);\alpha] = \frac{1}{a}F(\alpha/a), \qquad a > 0 \tag{7.45}$$

Similarly, it can be shown that

$$\mathcal{F}[f(ax);\alpha] = -\frac{1}{a}F(\alpha/a), \qquad a < 0 \tag{7.46}$$

Combining Eqs. (7.45) and (7.46), we have the property

$$\mathcal{F}[f(ax);\alpha] = \frac{1}{|a|}F(\alpha/a), \qquad a \neq 0 \tag{7.47}$$

It can also be established that

$$\mathcal{F}_s[f(ax);\alpha] = \frac{1}{a}F_s(\alpha/a), \qquad a > 0 \tag{7.48}$$

$$\mathcal{F}_c[f(ax);\alpha] = \frac{1}{a}F_c(\alpha/a), \qquad a > 0 \tag{7.49}$$

Theorem 7.4 (Shifting property). If $F(\alpha)$ is the Fourier transform of $f(x)$, then the Fourier transform of $f(x-a)$, i.e.,

$$\mathcal{F}[f(x-a);\alpha] = e^{i\alpha a}F(\alpha)$$

Proof From the definition of Fourier transform

$$F(\alpha) = \mathcal{F}[f(x); \alpha] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} f(x)\, dx$$

Therefore,

$$\mathcal{F}[f(x-a); \alpha] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} f(x-a)\, dx$$

Setting

$$x - a = t,$$

we have $dx = dt$. Then

$$\mathcal{F}[f(x-a); \alpha] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha(a+t)} f(t)\, dt$$

Hence,

$$\mathcal{F}[f(x-a); \alpha] = e^{i\alpha a} F(\alpha) \tag{7.50}$$

Similarly, it can be shown that

$$\mathcal{F}^{-1}[e^{i\alpha a} F(\alpha); x] = f(x-a) \tag{7.51}$$

Theorem 7.5 (Modulation property). If $F(\alpha)$ is the Fourier transform of $f(x)$, then the Fourier transform of $f(x) \cos ax$ is

$$\frac{1}{2}[F(\alpha - a) + F(\alpha + a)]$$

Proof From the definition of Fourier transform, we have

$$F(\alpha) = \mathcal{F}[f(x); \alpha] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} f(x)\, dx$$

Therefore,

$$\begin{aligned} \mathcal{F}[f(x) \cos ax; \alpha] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} f(x) \left(\frac{e^{iax} + e^{-iax}}{2} \right) dx \\ &= \frac{1}{2} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(\alpha + a)x} f(x)\, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(\alpha - a)x} f(x)\, dx \right] \\ &= \frac{1}{2}[F(\alpha + a) + F(\alpha - a)] \end{aligned} \tag{7.52}$$

In the same fashion, it can be established that

$$\mathcal{F}_s[f(x)\cos ax;\alpha] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos ax \sin \alpha x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{1}{2}\int_0^\infty f(x)[\sin(\alpha+a)x + \sin(\alpha-a)x]\,dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\left\{\int_0^\infty f(x)\sin(\alpha+a)x\,dx + \int_0^\infty f(x)\sin(\alpha-a)x\,dx\right\}\right]$$

$$= \frac{1}{2}[F_s(\alpha+a)+F_s(\alpha-a)] \tag{7.53}$$

$$\mathcal{F}_c[f(x)\cos ax;\alpha] = \frac{1}{2}[F_c(\alpha+a)+F_c(\alpha-a)] \tag{7.54}$$

$$\mathcal{F}_s[f(x)\sin ax;\alpha] = \frac{1}{2}[F_c(\alpha-a)-F_c(\alpha+a)] \tag{7.55}$$

$$\mathcal{F}_c[f(x)\sin ax;\alpha] = \frac{1}{2}[F_s(\alpha+a)-F_s(\alpha-a)] \tag{7.56}$$

Theorem 7.6 (Differentiation). If $f(x)$ and its first $(r-1)$ derivatives are continuous, and if its rth derivative is piecewise continuous, then

$$\mathcal{F}[f^{(r)}(x);\alpha] = (-i\alpha)^r\,\mathcal{F}[f(x);\alpha], \qquad r = 0, 1, 2, \ldots$$

provided f and its derivatives are absolutely integrable. In addition, we assume that $f(x)$ and its first $(r-1)$ derivatives vanish as $x \to \pm\infty$.

Proof From the definition, we have the Fourier transform of $d^r f/dx^r$ as

$$\mathcal{F}[f^{(r)}(x);\alpha] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \frac{d^r f}{dx^r}e^{i\alpha x}dx = F^{(r)}(\alpha) \qquad \text{(say)}$$

Integrating by parts, we get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \frac{d^r f}{dx^r}e^{i\alpha x}dx = \frac{1}{\sqrt{2\pi}}\left(\frac{d^{r-1}f}{dx^{r-1}}e^{i\alpha x}\right)_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \frac{d^{r-1}f}{dx^{r-1}}(i\alpha)e^{i\alpha x}\,dx$$

If we assume that d^{r-1}/dx^{r-1} tends to zero as $x \to \pm\infty$, we may write the above result in the form

$$F^{(r)}(\alpha) = -(i\alpha)F^{(r-1)}(\alpha) = (-i\alpha)^2 F^{(r-2)}(\alpha)\cdots = (-i\alpha)^r F(\alpha)$$

Hence,

$$F^{(r)}(\alpha) = (-i\alpha)^r F(\alpha)$$

and, therefore,

$$\mathcal{F}[f^{(r)}(x); \alpha] = (-i\alpha)^r F(\alpha) \tag{7.57}$$

EXAMPLE 7.6 Let $F_c^{(r)}(\alpha)$ be the Fourier cosine transform of $d^r f/dx^r$ and $F_s^{(r)}(\alpha)$ be the Fourier sine transform of $d^r f/dx^r$. Then prove that

$$F_c^{(2r)}(\alpha) = -\sum_{n=0}^{r-1} (-1)^n a_{2r-2n-1}\alpha^{2n} + (-1)^r \alpha^{2r} F_c(\alpha)$$

$$F_c^{(2r+1)}(\alpha) = -\sum_{n=0}^{r-1} (-1)^n a_{2r-2n}\alpha^{2n} + (-1)^r \alpha^{2r+1} F_s(\alpha)$$

assuming

$$\underset{x\to\infty}{\mathrm{Lt}} \left(\frac{d^{r-1} f}{dx^{r-1}}\right) = 0$$

$$\underset{x\to 0}{\mathrm{Lt}} \sqrt{\frac{2}{\pi}} \frac{d^{r-1} f}{dx^{r-1}} = a_{r-1} \qquad \text{(say)}$$

Solution From the definition, we have

$$F_c^{(r)}(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d^r f}{dx^r} \cos \alpha x \, dx \tag{7.58}$$

$$F_s^{(r)}(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d^r f}{dx^r} \sin \alpha x \, dx \tag{7.59}$$

Integrating Eq. (7.58) by parts, we get

$$F_c^{(r)}(\alpha) = \sqrt{\frac{2}{\pi}} \left\{ \left[\frac{d^{r-1} f}{dx^{r-1}} \cos (\alpha x) \right]_0^\infty + \int_0^\infty \frac{d^{r-1}}{dx^{r-1}} \sin (\alpha x) \alpha \, dx \right\}$$

Now, we assume that

$$\underset{x\to\infty}{\mathrm{Lt}} \left(\frac{d^{r-1} f}{dx^{r-1}}\right) = 0$$

$$\underset{x \to 0}{\text{Lt}} \sqrt{\frac{2}{\pi}} \frac{d^{r-1} f}{dx^{r-1}} = a_{r-1} \qquad \text{(say)}$$

Then

$$F_c^{(r)}(\alpha) = -a_{r-1} + \alpha F_s^{(r-1)}(\alpha) \tag{7.60}$$

Integrating Eq. (7.59) by parts, we have

$$F_s^{(r)}(\alpha) = \sqrt{\frac{2}{\pi}} \left(\frac{d^{r-1} f}{dx^{r-1}} \sin \alpha x \right)_0^\infty - \int_0^\infty \frac{d^{r-1}}{dx^{r-1}} \cos(\alpha x) \alpha \, dx = -\alpha F_c^{r-1}(\alpha) \tag{7.61}$$

Substituting Eq. (7.61) into Eq. (7.60), we get

$$F_c^{(r)}(\alpha) = -a_{r-1} - \alpha^2 F_c^{(r-2)}(\alpha)$$

By repeated application of these results, we can show $F_c^{(r)}(\alpha)$ to be a sum of a's and either $F_c^{(1)}(\alpha)$ or $F_c(\alpha)$, depending on whether r is odd or even. $F_c^{(1)}$ will be present if r is odd and may be replaced by $a_0 + \alpha F_s(\alpha)$. Similar arguments give us the formulae

$$F_c^{(2r)}(\alpha) = -\sum_{n=0}^{r-1} (-1)^n a_{2r-2n-1} \alpha^{2n} + (-1)^{r+1} \alpha^{2r} F_c(\alpha)$$

$$F_c^{(2r+1)}(\alpha) = -\sum_{n=0}^{r-1} (-1)^n a_{2r-2n} \alpha^{2n} + (-1)^{r+1} \alpha^{2r+1} F_s(\alpha)$$

Note: When $x = 0$ and $\dfrac{df}{dx} = \dfrac{d^3 f}{dx^3} = 0,$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d^2 f}{dx^2} \cos \alpha x \, dx = -\alpha^2 F_c(\alpha)$$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d^4 f}{dx^4} \cos \alpha x \, dx = \alpha^4 F_c(\alpha)$$

Similarly, when $x = 0$ and $f = \dfrac{d^2 f}{dx^2} = 0,$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d^2 f}{dx^2} \sin \alpha x \, dx = -\alpha^2 F_s(\alpha)$$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d^4 f}{dx^4} \sin \alpha x \, dx = \alpha^4 F_s(\alpha)$$

EXAMPLE 7.7 Find the Fourier cosine transform of $\exp(-at^2)$.

Solution We have from the definition of Fourier cosine transform,

$$\mathscr{F}_c[e^{-at^2}; \alpha] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at^2} \cos \alpha t \, dt = I \quad \text{(say)} \tag{7.62}$$

Differentiating with respect to α, we obtain

$$\frac{dI}{d\alpha} = -\sqrt{\frac{2}{\pi}} \int_0^\infty t e^{-at^2} \sin \alpha t \, dt = \frac{1}{2a}\sqrt{\frac{2}{\pi}} \int_0^\infty \sin \alpha t \, d(e^{-at^2})$$

$$= \frac{1}{2a}\sqrt{\frac{2}{\pi}} \left\{ [(e^{-at^2} \sin \alpha t)]_0^\infty - \alpha \int_0^\infty e^{-at^2} \cos \alpha t \, dt \right\}$$

Therefore,

$$\frac{dI}{d\alpha} = -\frac{\alpha}{2a} I$$

i.e.,

$$\frac{dI}{I} = -\frac{\alpha}{2a} d\alpha$$

On integration, we get

$$I = c e^{-\alpha^2/4a} \tag{7.63}$$

But when $\alpha = 0$, from Eq. (7.62) we have

$$I = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at^2} dt$$

From Example 7.1, using change of scale property, we obtain

$$I = \sqrt{\frac{2}{\pi}} \frac{1}{a} \frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2a}}$$

From Eq. (7.63) we get $c = \dfrac{1}{\sqrt{2a}}$. Hence,

$$\mathscr{F}_c[e^{-at^2}; \alpha] = \frac{1}{\sqrt{2a}} e^{-\alpha^2/4a}$$

EXAMPLE 7.8 If the Fourier sine transform of $f(x)$ is $\alpha/(1+\alpha^2)$, find $f(x)$.

Solution From the definition, we have

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\alpha}{1+\alpha^2}\sin \alpha x \, d\alpha$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{(\alpha^2+1)-1}{\alpha(1+\alpha^2)}\sin \alpha x \, d\alpha$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin \alpha x}{\alpha}\, d\alpha - \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin \alpha x}{\alpha(1+\alpha^2)}\, d\alpha$$

But

$$\int_0^\infty \frac{\sin \alpha x}{\alpha}\, d\alpha = \frac{\pi}{2}$$

Hence,

$$f(x) = \frac{\pi}{2} - \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin \alpha x}{\alpha(1+\alpha^2)}\, d\alpha \tag{7.64}$$

$$\frac{df}{dx} = -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos \alpha x}{1+\alpha^2}\, d\alpha \tag{7.65}$$

$$\frac{d^2 f}{dx^2} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\alpha \sin \alpha x}{1+\alpha^2}\, d\alpha \tag{7.66}$$

From Eqs. (7.64) and (7.66), it follows that

$$\frac{d^2 f}{dx^2} - f = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin \alpha x}{\alpha}\, d\alpha - \sqrt{\frac{\pi}{2}} = 0$$

whose solution is found to be

$$f = c_1 e^x + c_2 e^{-x} \tag{7.67}$$

Therefore,

$$\frac{df}{dx} = c_1 e^x - c_2 e^{-x}$$

When, $x = 0$, from Eq. (7.64) we have $f(0) = \sqrt{\pi/2}$, and from Eq. (7.65),

$$\frac{df(0)}{dx} = -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{d\alpha}{1+\alpha^2} = -\sqrt{\frac{\pi}{2}}$$

Also, from Eq. (7.67), using these results, we get

$$c_1 + c_2 = \sqrt{\frac{\pi}{2}}, \qquad c_1 - c_2 = -\sqrt{\frac{\pi}{2}}$$

Solving the above two equations, we get $c_1 = 0, c_2 = \sqrt{\pi/2}$. Thus, $f(x) = \sqrt{\pi/2}\, e^{-x}$.

EXAMPLE 7.9 If the Fourier cosine transform of $f(x)$ is $\alpha^n e^{-a\alpha}$, find $f(x)$.

Solution Using the definition, we have

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \alpha^n e^{-a\alpha} \cos \alpha x \, d\alpha \tag{7.68}$$

But we know from calculus that

$$\int_0^\infty e^{-a\alpha} \cos \alpha x \, d\alpha = \frac{a}{a^2 + x^2} \tag{7.69}$$

Differentiating this result n times with respect to a, we get

$$\begin{aligned}
(-1)^n \int_0^\infty \alpha^n e^{-a\alpha} \cos \alpha x \, d\alpha &= \frac{d^n}{da^n}\left(\frac{a}{a^2 + x^2}\right) \\
&= \frac{1}{2} \frac{d^n}{da^n}\left(\frac{1}{a - ix} + \frac{1}{a + ix}\right) \\
&= \frac{1}{2}[(-1)^n n!(a - ix)^{-n-1} + (-1)^n n!(a + ix)^{-n-1}] \\
&= \frac{(-1)^n n!}{2}[(a - ix)^{-n-1} + (a + ix)^{-n-1}]
\end{aligned}$$

Let

$$a + ix = r(\cos\theta + i \sin\theta)$$

Then

$$a = r\cos\theta, \qquad x = r\sin\theta$$

Therefore,

$$r^2 = a^2 + x^2, \qquad \tan\theta = \frac{x}{a}$$

Thus,

$$(a + ix)^{-n-1} = r^{-n-1}[\cos(-n-1)\theta + i \sin(-n-1)\theta]$$

$$(a - ix)^{-n-1} = r^{-n-1}[\cos(-n-1)\theta - i \sin(-n-1)\theta]$$

Then

$$\int_0^\infty \alpha^n e^{-a\alpha} \cos \alpha x \, d\alpha = \frac{n! \cos (n+1)\theta}{(a^2 + x^2)^{(n+1)/2}}$$

Hence from Eq. (7.68),

$$f(x) = \sqrt{\frac{2}{\pi}} \frac{n! \cos (n+1)\theta}{(a^2 + x^2)^{(n+1/2)}}$$

EXAMPLE 7.10 Find the Fourier transform of

(i) $\partial^n u/\partial x^n$ of the function $u(x, t)$ assuming that u and its first $(n - 1)$ derivatives with respect to x vanish as $x \to \pm\infty$.

(ii) $\partial u/\partial t$.

Also, find the sine and cosine Fourier transforms of $\partial^2 u/\partial x^2$ of the function $u(x, t)$.

Solution (a) We shall adopt the following notation: The Fourier transform of $u(x, t)$ with respect to the variable x is defined as

$$\mathcal{F}[u(x,t); x \to \alpha] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} u(x,t) \, dx = U(\alpha, t) \tag{7.70}$$

Then the Fourier transform of $\partial u/\partial x$ is

$$\mathcal{F}\left[\frac{\partial u}{\partial x}(x,t); x \to \alpha\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{i\alpha x} \, dx$$

Integration by parts yields

$$\frac{1}{\sqrt{2\pi}} \left\{ [u(x,t) \, e^{i\alpha x}]_{-\infty}^{\infty} - i\alpha \int_{-\infty}^{\infty} u(x,t) \, e^{i\alpha x} dx \right\}$$

If we assume

$$\underset{x \to \pm\infty}{\text{Lt}} \; u(x,t) = 0$$

then we find that

$$\mathcal{F}\left[\frac{\partial u}{\partial x}(x,t); x \to \alpha\right] = -i\alpha \mathcal{F}[u(x,t); x \to \alpha] = -i\alpha \, U(\alpha, t) \tag{7.71}$$

Similarly, the Fourier transform of $\partial^2 u/\partial x^2$ is

$$\mathcal{F}\left[\frac{\partial^2 u(x,t)}{\partial x^2}; x \to \alpha\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{i\alpha x} dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} d\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left(\frac{\partial u}{\partial x} e^{i\alpha x}\right)_{-\infty}^{\infty} - i\alpha(e^{i\alpha x} u)_{-\infty}^{\infty} + (i\alpha)^2 \int_{-\infty}^{\infty} e^{i\alpha x} u\, dx\right]$$

Assuming that both u and $\partial u/\partial t$ tend to zero as $x \to \pm\infty$, we have

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}(x,t); x \to \alpha\right] = (-1)^2 (i\alpha)^2 \mathcal{F}[u(x,t); x \to \alpha] = (-1)^2 (i\alpha)^2 U(\alpha, t) \quad (7.72)$$

Thus, in general, the Fourier transform of the nth derivative of $u(x,t)$ is given by

$$\mathcal{F}\left[\frac{\partial^n u(x,t)}{\partial x^n}; x \to \alpha\right] = (-1)^n (i\alpha)^n \mathcal{F}[u(x,t); x \to \alpha] = (-1)^n (i\alpha)^n U(\alpha, t) \quad (7.73)$$

(b) Now

$$\mathcal{F}\left[\frac{\partial u}{\partial t}(x,t); x \to \alpha\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} \frac{\partial u}{\partial t}(x,t)\, dx = \frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} e^{i\alpha x} u(x,t)\, dx = U_t(\alpha, t)$$

Therefore,

$$\mathcal{F}\left[\frac{\partial u}{\partial t}(x,t); x \to \alpha\right] = U_t(\alpha, t) \quad (7.74)$$

(c) In the case of Fourier sine and cosine transforms, we have

$$\mathcal{F}_s\left[\frac{\partial^2 u}{\partial x^2}(x,t); x \to \alpha\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \sin \alpha x \frac{\partial^2 u}{\partial x^2} dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\partial u}{\partial x} \sin \alpha x\right]_0^{\infty} - \sqrt{\frac{2}{\pi}}\alpha \int_0^{\infty} \cos \alpha x \frac{\partial u}{\partial x} dx$$

We assume that $\dfrac{\partial u}{\partial x} \to 0$ as $x \to \infty$. Then the RHS term of the above equation becomes

$$-\sqrt{\frac{2}{\pi}}\alpha \int_0^{\infty} \cos \alpha x \frac{\partial u}{\partial x} dx = -\sqrt{\frac{2}{\pi}}\alpha \left\{[u(x,t) \cos \alpha x]_0^{\infty} + \alpha \int_0^{\infty} u(x,t) \sin \alpha x\, dx\right\}$$

Also, assuming that $u(x, t) \to 0$ as $x \to \infty$, this equation becomes

$$\sqrt{\frac{2}{\pi}}\, \alpha u\,(x, t)\Big|_{x=0} - \alpha^2 \mathscr{F}_s[u\,(x, t);\, x \to \alpha]$$

Hence,

$$\mathscr{F}_s\left[\frac{\partial^2 u}{\partial x^2}(x, t);\, x \to \alpha\right] = \sqrt{\frac{2}{\pi}}\, \alpha u\,(x, t)\Big|_{x=0} - \alpha^2 \mathscr{F}_s[u\,(x, t);\, x \to \alpha] \tag{7.75}$$

Similarly, it can be shown that if

$$u\,(x, t) \to 0 \;\text{ and }\; \frac{\partial u}{\partial x} \to 0 \;\text{ as }\; x \to \infty$$

then

$$\mathscr{F}_c\left[\frac{\partial^2 u}{\partial x^2}(x, t);\, x \to \alpha\right] = -\sqrt{\frac{2}{\pi}}\, \frac{\partial u\,(x, t)}{\partial x}\Big|_{x=0} - \alpha^2 \mathscr{F}_c[u\,(x, t);\, x \to \alpha] \tag{7.76}$$

Obviously, the choice of the sine or cosine transform is dccided by the form of the boundary condition at the lower limit of the variable selected for exclusion. Thus, we observe that for the exclusion of $\partial^2 u/\partial x^2$ from a given PDE, we require

$u|_{x=0}$ in the case of sine transform

$\left.\dfrac{\partial u}{\partial x}\right|_{x=0}$ in the case of cosine transform

7.6 CONVOLUTION THEOREM (FALTUNG THEOREM)

If $F(\alpha)$ and $G(\alpha)$ are the Fourier transforms of the functions $f(x)$ and $g(x)$, then the product $F(\alpha)\, G(\alpha)$ is the Fourier transform of the convolution product $f * g$.

Proof The product

$$f * g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\,(u)\; g\,(x - u)\; du \tag{7.77}$$

is called the convolution or Faltung of the functions f and g over the interval $(-\infty, \infty)$. Then the Fourier transform of this convolution integral yields

$$\begin{aligned} \mathscr{F}[(f * g);\, \alpha] &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\alpha x} \int_{-\infty}^{\infty} f\,(u)\; g(x - u)\; du\; dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\alpha x} f\,(u)\; g(x - u)\; du\; dx \end{aligned} \tag{7.78}$$

Since f and g are absolutely integrable, the order of integration can be interchanged and, therefore,

$$\mathcal{F}[(f*g);\alpha]=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} g(x-u)\,e^{i\alpha(x-u)}e^{i\alpha u}\,dx\right]du$$

Let $x-u=y$. Then $dx=dy$. Therefore,

$$\begin{aligned}\mathcal{F}[(f*g);\alpha]&=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[e^{i\alpha u}\int_{-\infty}^{\infty} g(y)\,e^{i\alpha y}\,dy\right]du\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha u} f(u)\,du\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha y} g(y)\,dy\\&=F(\alpha)G(\alpha)\end{aligned}\tag{7.79}$$

Hence the theorem is proved.

We can verify that

$$f*g=g*f,\tag{7.80}$$

i.e.

$$f*g=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,g(x-u)\,du\tag{7.81}$$

Setting $x-u=\alpha$, we have $du=-d\alpha$. Then

$$f*g=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\alpha)\,g(\alpha)\,d\alpha=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(u)\,f(x-u)\,du=g*f\tag{7.82}$$

Hence, the convolution is commutative.

Special cases (sine and cosine convolution integrals)

(i)

$$\begin{aligned}\int_0^{\infty} F_c(\alpha)G_c(\alpha)\cos\alpha x\,d\alpha&=\sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(\alpha)\cos\alpha x\,d\alpha\int_0^{\infty} g(\eta)\cos\eta\alpha\,d\eta\\&=\sqrt{\frac{2}{\pi}}\int_0^{\infty} g(\eta)\,d\eta\int_0^{\infty} F_c(\alpha)\cos x\alpha\cos\eta\alpha\,d\alpha\\&=\frac{1}{\sqrt{2\pi}}\int_0^{\infty} g(\eta)\,d\eta\int_0^{\infty} F_c(\alpha)\,[\cos|x-\eta|\,\alpha+\cos(x+\eta)\,\alpha]\,d\alpha\end{aligned}$$

But,

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\alpha) \cos x\alpha \, d\alpha = \frac{1}{2} \int_0^\infty g(\eta) \, d\eta \, [f(|x-\eta|) + f(x+\eta)] \tag{7.83}$$

(ii) If $F_s(\alpha)$ and $G_s(\alpha)$ are the Fourier sine transforms of $f(x)$ and $g(x)$, then we can show that

$$\int_0^\infty F_s(\alpha) G_s(\alpha) \sin x\alpha \, d\alpha = \frac{1}{2} \int_0^\infty f(\eta) [g(|x-\eta|) - g(x+\eta)] d\eta \tag{7.84}$$

(iii) $\int_0^\infty F_c(\alpha) G_s(\alpha) \sin x\alpha \, d\alpha$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(\alpha) \sin x\alpha \, d\alpha \int_0^\infty g(\eta) \sin \eta\alpha \, d\eta$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty g(\eta) \, d\eta \int_0^\infty F_c(\alpha) \sin x\alpha \sin \eta\alpha \, d\alpha$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty g(\eta) \, d\eta \int_0^\infty F_c(\alpha) [\cos |x-\eta| \alpha - \cos (x+\eta) \alpha] \, d\alpha$$

$$= \frac{1}{2} \int_0^\infty g(\eta) [f(|x-\eta|) - f(x+\eta)] d\eta \tag{7.85}$$

(iv) $$\int_0^\infty F_s(\alpha) G_c(\alpha) \sin x\alpha \, d\alpha = \frac{1}{2} \int_0^\infty f(\eta) [g(|x-\eta|) - g(x+\eta)] d\eta \tag{7.86}$$

EXAMPLE 7.11 If $F_c(\alpha)$ and $G_c(\alpha)$ are the Fourier cosine transforms of $f(x)$ and $g(x)$ respectively, show that $F_c(\alpha) G_c(\alpha)$ is the transform of

$$\frac{1}{2} \int_0^\infty f(u) [g(|x-u|) + g(x+u)] du$$

Solution Refer Eq. (7.83).

7.7 PARSEVAL'S RELATION

From the Fourier convolution theorem, we have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-i\alpha x} F(\alpha) G(\alpha) \, d\alpha = f * g = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(u) \, g(x-u) \, du$$

If we set $x = 0$, the above equation reduces to

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha)\, G(\alpha)\, d\alpha = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, g(-u)\, du$$

For the case $g(-u) = f^*(u)$, where $f^*(u)$ is the complex conjugate of the function $f(u)$, we have

$$G(\alpha) = \mathcal{F}[g(u);\alpha] = \mathcal{F}[f^*(-u);\alpha] + F^*(\alpha)$$

Thus,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha) F^*(\alpha)\, d\alpha = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u) f^*(u)\, du \tag{7.87}$$

Therefore,

$$\int_{-\infty}^{\infty} |F(\alpha)|^2 d\alpha = \int_{-\infty}^{\infty} |f(u)|^2 du \tag{7.88}$$

Equation (7.88) is known as Parseval's relation.

EXAMPLE 7.12 Using the Fourier cosine transform of e^{-ax} and e^{-bx}, show that

$$\int_0^{\infty} \frac{d\alpha}{(a^2+\alpha^2)(b^2+\alpha^2)} = \frac{\pi}{2ab\,(a+b)}, \qquad a > 0,\ b > 0 \tag{7.89}$$

Solution Let $f(x) = e^{-bx}$, $g(x) = e^{-ax}$; then

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos\alpha x\, dx = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-bx}\cos\alpha x\, dx = \sqrt{\frac{2}{\pi}}\,\frac{b}{b^2+\alpha^2}$$

Similarly, it can be shown that

$$G_c(\alpha) = \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+\alpha^2}$$

However,

$$\begin{aligned}
\int_0^{\infty} F_c(\alpha)\, G_c(\alpha)\, d\alpha &= \int_0^{\infty} F_c(\alpha)\, d\alpha \sqrt{\frac{2}{\pi}}\int_0^{\infty} g(x)\cos\alpha x\, dx \\
&= \int_0^{\infty} g(x)\, dx \sqrt{\frac{2}{\pi}}\int_0^{\infty} F_c(\alpha)\cos\alpha x\, d\alpha \\
&= \int_0^{\infty} f(x) g(x)\, dx
\end{aligned}$$

i.e.

$$\int_0^{\infty} F_c(\alpha)\, G_c(\alpha)\, d\alpha = \int_0^{\infty} f(x) g(x)\, dx$$

Hence, it follows that

$$\int_0^\infty F_c(\alpha)\, G_c(\alpha)\, d\alpha = \int_0^\infty e^{-(a+b)x} dx = \frac{1}{a+b}$$

Therefore,

$$\int_0^\infty \frac{d\alpha}{(a^2+\alpha^2)(b^2+\alpha^2)} = \frac{\pi}{2ab\,(a+b)}, \qquad a > 0,\ b > 0$$

7.8 TRANSFORM OF DIRAC DELTA FUNCTION

The Dirac delta function has been defined in Chapter 3. The details of its properties can be found in Section 3.4. We may recall the shifting property of the delta function, i.e.

$$\int_{-\infty}^\infty \delta(t-a)\, f(t)\, dt = f(a)$$

and then obtain its Fourier transform as

$$\mathcal{F}[\delta(t-a); \alpha] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{i\alpha t} \delta(t-a)\, dt = e^{i\alpha a}/\sqrt{2\pi} \tag{7.90}$$

When $a = 0$, we obtain the formal result

$$\mathcal{F}[\delta(t); \alpha] = \frac{1}{\sqrt{2\pi}} \tag{7.91}$$

That is, the Fourier transform of the Dirac delta function $\delta(t)$ is constant and equal to $1/\sqrt{2\pi}$. It then follow that

$$\mathcal{F}^{-1}[1; t] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-i\alpha t} d\alpha = \sqrt{2\pi}\,\delta(t) \tag{7.92}$$

7.9 MULTIPLE FOURIER TRANSFORMS

As already discussed, the idea of a Fourier transform and its inverse can be extended to functions involving two or more variables.

Let $f(x, y)$ be a function of two variables x and y. Let it be piecewise continuous and satisfy the condition

$$\int_{-\infty}^\infty \int_{-\infty}^\infty |\, f(x, y)\,|\, dx\, dy < \infty \tag{7.93}$$

Then the Fourier transform pair can be written as

$$F(\alpha, \beta) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty e^{i(\alpha x + \beta y)}\, f(x, y)\, dx\, dy \tag{7.94}$$

$$f(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\alpha x + \beta y)} F(\alpha, \beta)\, d\alpha\, d\beta \tag{7.95}$$

We can split Eq. (7.94) into two steps: first by treating $f(x, y)$ as a function of x and then treating the result as a function of y sequentially, i.e.,

$$f^*(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} f(x, y)\, dx$$

$$F(\alpha, \beta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\beta y} f^*(\alpha, y)\, dy$$

Similarly, the inversion formula (7.95), can be written as

$$f(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha x} f^*(\alpha, y)\, d\alpha$$

$$f^*(\alpha, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\beta y} F(\alpha, \beta)\, d\beta$$

Assuming that the partial derivatives of f occurring in the equation are absolutely integrable and that $f, \partial f/\partial x, \partial f/\partial y$ tend to zero at infinity, the double Fourier transform of derivatives yield the following results:

$$\mathcal{F}\left[\frac{\partial f}{\partial x}(x, y);\, x \to \alpha, y \to \beta\right] = -i\alpha F(\alpha, \beta) \tag{7.96}$$

$$\mathcal{F}\left[\frac{\partial^2 f}{\partial x \partial y}(x, y);\, x \to \alpha, y \to \beta\right] = -\alpha\beta F(\alpha, \beta) \tag{7.97}$$

The convolution property leads to the following results:

$$\mathcal{F}^{-1}[F(\alpha, \beta)\, G(\alpha, \beta);\, \alpha \to x, \beta \to y] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x - u)\, g(x - u)\, g(u, v)\, du\, dv \tag{7.98}$$

The Fourier transform in the case of three variables is

$$F(\alpha, \beta, \gamma) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\alpha x + \beta y + \gamma z)} f(x, y, z)\, dx\, dy\, dz \tag{7.99}$$

7.10 FINITE FOURIER TRANSFORM

The Fourier transform technique outlined so far is applicable to problems involving infinite or semi-infinite domains. But, in many practical situations, we come across finite intervals in boundary value problems. Therefore, it is natural to extend Fourier transform method to problems in which the range of an independent variable is finite. It is then possible to find their inverses from the well-known theory of Fourier series. It may be recalled that if a function $f(x)$ satisfies Dirichlet conditions in the interval $0 \le x \le \pi$, then it has Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \tag{7.100}$$

in which

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx, \qquad n = 1, 2, 3, \ldots \tag{7.101}$$

The Fourier series in Eq. (7.100) converges pointwise to $f(x)$ at points where $f(x)$ is continuous and to the value $(1/2)[f(x+) + f(x-)]$ at other points. Similarly, $f(x)$ has Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{7.102}$$

in which

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx, \qquad n = 0, 1, 2, \ldots \tag{7.103}$$

7.10.1 Finite Sine Transform

If $f(x)$ satisfies Dirichlet conditions in the interval $0 \le x \le \pi$, then we define finite sine transform of $f(x)$ by

$$\mathcal{F}_s[f(x); n] = F_s(n) = \int_0^{\pi} f(x) \sin nx \, dx, \qquad n = 1, 2, 3, \ldots \tag{7.104}$$

which is a sequence of numbers. Comparing Eqs. (7.101) and (7.104), we notice that

$$b_n = \frac{2}{\pi} F_s(n), \qquad n = 1, 2, 3, \ldots \tag{7.105}$$

Now from Eq. (7.100) we can have the result

$$\mathcal{F}_s^{-1}[F_s(n); x] = f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} F_s(n) \sin nx \tag{7.106}$$

which is the inverse finite sine transform. Similarly, we can define finite sine transform when the independent variable x lies in the interval $(0, L)$,

$$\mathcal{F}_s[f(x); n] = F_s(n) = \int_0^{L} f(x) \sin \frac{n\pi x}{L} \, dx, \qquad n = 1, 2, 3, \ldots \tag{7.107}$$

with

$$\mathcal{F}_s^{-1}[F_s(n); x] = f(x) = \frac{2}{L}\sum_{n=1}^{\infty} F_s(n)\sin\frac{n\pi x}{L}, \qquad 0 < x < L \tag{7.108}$$

as the corresponding inversion formula.

7.10.2 Finite Cosine Transform

If $f(x)$ satisfies the Dirichlet conditions in the interval $0 \le x \le \pi$, we define the finite cosine transform of $f(x)$ as

$$\mathcal{F}_c[f(x); n] = F_c(n) = \int_0^{\pi} f(x)\cos nx\, dx, \qquad n = 0, 1, 2, \ldots \tag{7.109}$$

The corresponding inversion formula is

$$\mathcal{F}_c^{-1}[F_c(n); x] = f(x) = \frac{F_c(0)}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} F_c(n)\cos nx \tag{7.110}$$

which holds at each point in the interval $(0, \pi)$ at which $f(x)$ is continuous. Similarly, when the independent variable x lies in the interval $(0, L)$, the corresponding pair assumes the form

$$\mathcal{F}_c[f(x); n] = F_c(n) = \int_0^{L} f(x)\cos\frac{n\pi x}{L}\, dx, \qquad n = 0, 1, 2, \ldots \tag{7.111}$$

$$\mathcal{F}_c^{-1}[F_c(n); x] = f(x) = \frac{F_c(0)}{\pi} + \frac{2}{L}\sum_{n=1}^{\infty} F_c(n)\cos\frac{n\pi x}{L} \tag{7.112}$$

Having defined the finite cosine transform, we shall attempt to find some results involving derivatives up to Fourth order to facilitate the solution of a few boundary value problems, which are actually presented under miscellaneous examples. For instance, if f is a function of x and t, $0 < x < L, t > 0$, then we have

$$\mathcal{F}_s\left[\frac{\partial f}{\partial x}; n\right] = \int_0^L \frac{\partial f}{\partial x}\sin\frac{n\pi x}{L}\, dx = \left[f(x, t)\sin\frac{n\pi x}{L}\right]_0^L - \frac{n\pi}{L}\int_0^L f\cos\frac{n\pi x}{L}\, dx$$

Therefore,

$$\mathcal{F}_s\left[\frac{\partial f}{\partial x}; n\right] = -\frac{n\pi}{L}\mathcal{F}_c[f; n] = -\frac{n\pi}{L}F_c(n) \tag{7.113}$$

$$\mathcal{F}_c\left[\frac{\partial f}{\partial x}; n\right] = \int_0^L \frac{\partial f}{\partial x}\cos\frac{n\pi x}{L}\, dx = \left[f(x, t)\cos\frac{n\pi x}{L}\right]_0^L + \frac{n\pi}{L}\int_0^L f\sin\frac{n\pi x}{L}\, dx$$

$$= \frac{n\pi}{L}\mathcal{F}_s[f; n] - \{f(0, t) - f(L, t)\cos n\pi\} \tag{7.114}$$

Considering second order derivatives, we obtain

$$\mathcal{F}_s\left[\frac{\partial^2 f}{\partial x^2}; n\right] = \int_0^L \frac{\partial^2 f}{\partial x^2} \sin\frac{n\pi x}{L}\,dx = \left(\frac{\partial f}{\partial x}\sin\frac{n\pi x}{L}\right)_0^L - \frac{n\pi}{L}\int_0^L \frac{\partial f}{\partial x}\cos\frac{n\pi x}{L}\,dx$$

$$= -\frac{n\pi}{L}\left\{\left(f\cos\frac{n\pi x}{L}\right)_0^L + \frac{n\pi}{L}\int_0^L f\sin\frac{n\pi x}{L}\,dx\right\}$$

$$= -\frac{n\pi}{L}\left[\frac{n\pi}{L}\mathcal{F}_s[f; n] - [f(0,t) - f(L,t)\cos n\pi]\right]$$

Therefore,

$$\mathcal{F}_s\left[\frac{\partial^2 f}{\partial x^2}; n\right] = -\frac{n^2\pi^2}{L^2}\mathcal{F}_s[f; n] + \frac{n\pi}{L}\{f(0,t) - f(L,t)\cos n\pi\} \tag{7.115}$$

In particular, if $f(0,t) = f(L,t) = 0$, this result simplifies to

$$\mathcal{F}_s\left[\frac{\partial^2 f}{\partial x^2}; n\right] = -\frac{n^2\pi^2}{L^2}\mathcal{F}_s[f; n] \tag{7.116}$$

Similarly, it can be shown that

$$\mathcal{F}_c\left[\frac{\partial^2 f}{\partial x^2}; n\right] = -\frac{n^2\pi^2}{L^2}\mathcal{F}_c[f; n] - \{f_x(0,t) - f_x(L,t)\cos n\pi\} \tag{7.117}$$

In case $\partial f/\partial x$ vanishes at the ends $x = 0$ and $x = L$, it simplifies to

$$\mathcal{F}_c\left[\frac{\partial^2 f}{\partial x^2}; n\right] = -\frac{n^2\pi^2}{L^2}\mathcal{F}_c[f; n] \tag{7.118}$$

By repeatedly applying these results, we can deduce that

$$\mathcal{F}_s\left[\frac{\partial^4 f}{\partial x^4}; n\right] = +\frac{n^4\pi^4}{L^4}\mathcal{F}_s[f; n], \tag{7.119}$$

if f_x and f_{xx} vanish at both the ends $x = 0, x = L,$ and

$$\mathcal{F}_c\left[\frac{\partial^4 f}{\partial x^4}; n\right] = +\frac{n^4\pi^4}{L^4}\mathcal{F}_c[f; n] \tag{7.120}$$

provided

$$f_x = 0 = f_{xxx} \quad \text{when } x = 0;\ x = L$$

7.11 SOLUTION OF DIFFUSION EQUATION

Let us consider the problem of flow of heat in an infinite medium $-\infty < x < \infty$, when the initial temperature distribution $f(x)$ is known and no heat sources are present. Mathematically, we have to solve the problem described in the following example.

EXAMPLE 7.13 Solve the heat conduction equation given by

$$\text{PDE: } k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad -\infty < x < \infty, \qquad t > 0$$

subject to

$$\text{BCs: } u(x, t) \text{ and } u_x(x, t) \text{ both} \to 0 \text{ as } |x| \to \infty$$

$$\text{IC:} \quad u(x, 0) = f(x), \qquad -\infty < x < \infty$$

Solution Taking the Fourier transform* of PDE, we get

$$-k\alpha^2 \mathcal{F}[u(x, t); x \to \alpha] = \mathcal{F}[u_t(x, t); x \to \alpha]$$

or

$$U_t(\alpha, t) + k\alpha^2 U(\alpha, t) = 0 \tag{7.121}$$

In deriving this, the BCs are already utilized (as can be seen from Eqs. (7.72) and (7.74). The Fourier transform of the IC gives

$$U(\alpha, 0) = F(\alpha), \qquad -\infty < \alpha < \infty \tag{7.122}$$

The solution of Eq. (7.121) can be readily seen to be

$$U = Ae^{-k\alpha^2 t}$$

When $t = 0$, we have from Eq. (7.122) the relation $U = F(\alpha)$, implying $A = F(\alpha)$. Therefore,

$$U(\alpha, t) = F(\alpha)e^{-k\alpha^2 t} \tag{7.123}$$

Inverting this relation, we obtain

$$u(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha)\, e^{-k\alpha^2 t} e^{-i\alpha x}\, d\alpha = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\alpha) \exp(-k\alpha^2 t - i\alpha x)\, d\alpha \tag{7.124}$$

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*When the range of spatial variable is infinite, the Fourier exponential transform is used rather than the sine or cosine transform.

The product form of the integrand in Eq. (7.124) suggests the use of convolution. If the Fourier transform of $g(x)$ is $e^{-k\alpha^2 t}$, then $g(x)$ will be given by

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-k\alpha^2 t} e^{-i\alpha x}\, d\alpha$$

But, if $a > 0, b$ is real or complex, and we know that

$$\int_{-\infty}^{\infty} \exp\,(-ax^2 - 2bx)\, dx = \frac{\sqrt{\pi}}{\sqrt{a}} \exp\,(b^2/a) \tag{7.125}$$

Here, $a = kt, 2b = ix.$ Therefore,

$$g(x) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{\pi}}{\sqrt{kt}} \exp\left(-\frac{x^2}{4kt}\right) = \frac{1}{\sqrt{2kt}} \exp\left(-\frac{x^2}{4kt}\right)$$

Using the convolution theorem, we have

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\alpha)\, g(x - \alpha)\, d\alpha \tag{7.126}$$

Hence,

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\alpha) \frac{1}{\sqrt{2kt}} \exp\left[\frac{-(x-\alpha)^2}{4kt}\right] d\alpha \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(\alpha) \exp\left[-\frac{(x-\alpha)^2}{4kt}\right] d\alpha \end{aligned} \tag{7.127}$$

Introducing the change of variable

$$Z = \frac{\alpha - x}{\sqrt{4kt}}$$

we can rewrite solution (7.127) in the form

$$u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + \sqrt{4kt}\, z)\, e^{-z^2} dz \tag{7.128}$$

EXAMPLE 7.14 (Flow of heat in a semi-infinite medium). Solve the heat conduction problem described by

$$\text{PDE: } k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < \infty, \qquad t > 0$$

$$\text{BC: } u(0, t) = u_0, \qquad t \ge 0$$

$$\text{IC: } u(x, 0) = 0, \qquad 0 < x < \infty$$

u and $\partial u/\partial x$ both tend to zero as $x \to \infty$.

Solution Since u is specified at $x = 0$, the Fourier sine transform is applicable to this problem. Taking the Fourier sine transform of the given PDE and using the notation.

$$U_s(\alpha, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x, t) \sin \alpha x \, dx$$

we obtain from Eqs. (7.74) and (7.75) the relation

$$k\left[\sqrt{\frac{2}{\pi}} \alpha u(x, t)\Big|_{x=0} - \alpha^2 \mathcal{F}_s[u(x, t); x \to \alpha]\right] = \frac{\partial U_s}{\partial t}(\alpha, t)$$

or

$$\frac{dU_s}{dt} + k\alpha^2 U_s = \sqrt{\frac{2}{\pi}} k\alpha u_0 \qquad (7.129)$$

Its general solution is found to be

$$U_s(\alpha, t) = \sqrt{\frac{2}{\pi}} \frac{u_0}{\alpha}(1 - e^{-k\alpha^2 t}) \qquad (7.130)$$

Inverting by Fourier inverse sine transform, we obtain

$$u(\alpha, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty U_s(\alpha, t) \sin \alpha x \, d\alpha$$

Therefore,

$$u(x, t) = \frac{2}{\pi} u_0 \int_0^\infty \frac{\sin \alpha x}{\alpha}(1 - e^{-k\alpha^2 t}) \, d\alpha \qquad (7.131)$$

Noting that

$$\text{erf}\,(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-u^2} du$$

and using the standard integral

$$\int_0^\infty e^{-\alpha^2} \frac{\sin(2\alpha y)}{\alpha} d\alpha = \frac{\pi}{2} \text{erf}\,(y)$$

we have solution (7.131) in the form

$$u(x, t) = \frac{2u_0}{\pi}\left[\frac{\pi}{2} - \frac{\pi}{2} \text{erf}\left(\frac{x}{\sqrt{2kt}}\right)\right] \qquad (7.132)$$

Finally, the solution of the heat conduction problem is

$$u(x, t) = u_0\left(1 - \frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{2kt}} e^{-u^2} du\right) = u_0 \,\text{erfc}\left(\frac{x}{\sqrt{2kt}}\right) \qquad (7.133)$$

EXAMPLE 7.15 Determine the temperature distribution in the semi-infinite medium $x \geq 0$, when the end $x = 0$ is maintained at zero temperature and the initial temperature distribution is $f(x)$.

Solution The given problem is described by

$$\text{PDE:} \frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < \infty, \quad t > 0 \tag{7.134}$$

$$\text{BC:} u(0, t) = 0, \qquad t > 0 \tag{7.135}$$

$$\text{IC:} u(x, 0) = f(x), \qquad 0 < x < \infty \tag{7.136}$$

and, $u, \partial u/\partial x$, both tend to zero as $x \to \infty$. Taking the Fourier sine transform of Eq. (7.134) and denoting

$$\mathcal{F}_s[u(x, t); x \to \alpha] \text{ by } U_s$$

we have

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t} \sin \alpha x \, dx = \sqrt{\frac{2}{\pi}} K \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin \alpha x \, dx$$

which becomes

$$\frac{dU_s}{dt} = K[\alpha u\,(0, t) - \alpha^2 U_s]$$

Using the BC (7.135), we obtain

$$\frac{dU_s}{dt} + K\alpha^2 U_s = 0 \tag{7.137}$$

Also, taking the Fourier sine transform of IC (7.136), we get

$$U_s = F_s(\alpha) \quad \text{at } t = 0 \tag{7.138}$$

Now, Eq. (7.137) can be rewritten as

$$\frac{d}{dt}(U_s e^{K\alpha^2 t}) = 0 \tag{7.139}$$

Integrating, we get

$$U_s e^{K\alpha^2 t} = \text{const}.$$

Using Eq. (7.138), we note that $F_s(\alpha) = \text{constant}$. Therefore,

$$U_s e^{K\alpha^2 t} = F_s(\alpha)$$

$$U_s = F_s(\alpha) e^{-K\alpha^2 t} \tag{7.140}$$

Finally, taking the inverse Fourier sine transform of Eq. (7.140), we obtain

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\alpha) e^{-K\alpha^2 t} \sin \alpha x \, d\alpha$$

7.12 SOLUTION OF WAVE EQUATION

Wave motions that occur in nature, viz., sound waves, surface waves, transverse vibrations of an infinite string, and of mechanical systems are governed by the wave equation. As our first example, we shall consider the transverse displacements of an infinite string.

EXAMPLE 7.16 Compute the displacement $u(x,t)$ of an infinite string using the method of Fourier transform given that the string is initially at rest and that the initial displacement is $f(x), -\infty < x < \infty$.

Solution Displacement of an infinite string is governed by the PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty \tag{7.141}$$

and ICs

$$u(x,0) = f(x), \quad -\infty < x < \infty \tag{7.142}$$

$$u_t(x,0) = 0 \tag{7.143}$$

In view of two ICs, the given problem is a properly posed problem. Taking the Fourier transform of PDE, we have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2} e^{i\alpha x} dx = \frac{c^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{dx^2} e^{i\alpha x} dx$$

$$\frac{\partial^2}{\partial t^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{i\alpha x} \, dx = -c^2 \alpha^2 U(\alpha, t)$$

i.e.,

$$\frac{d^2 U}{dt^2} + c^2 \alpha^2 U = 0 \tag{7.144}$$

Its general solution is found to be

$$U(\alpha, t) = A \cos(c\alpha t) + B \sin(c\alpha t) \tag{7.145}$$

The Fourier transform of the ICs gives

$$\frac{dU}{dt}(\alpha, t) = 0, \quad U = F(\alpha) \tag{7.146}$$

i.e., $-Ac\alpha \sin(c\alpha t) + Bc\alpha \cos(c\alpha t)_{t=0} = 0$, implying $Bc\alpha = 0$ or $B = 0$. Also, Eqs. (7.145) and (7.146) yield

$$A = F(\alpha) \tag{7.147}$$

Thus,

$$U(\alpha, t) = F(\alpha)\cos(c\alpha t) \tag{7.148}$$

Taking its inverse Fourier transform, we obtain

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha)\cos(c\alpha t)\, e^{-i\alpha x}\, d\alpha \tag{7.149}$$

If we simplify Eq. (7.149) further, an interesting result emerges, i.e.,

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(u)\, e^{i\alpha u}\, du\right]\left(\frac{e^{ic\alpha t} + e^{-ic\alpha t}}{2}\right) e^{-i\alpha x}\, d\alpha \\ &= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(u)\, e^{isu}\, du\right\}\right](e^{-icst} + e^{icst})\, e^{-isx}\, ds \\ &= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{-isu}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{is(x-ct)}\, ds\right\} du \right. \\ &\quad \left. + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{-isu}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{is(x+ct)}\, ds\right) du\right] \end{aligned}$$

Using the Fourier integral formula, we arrive at the result

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$$

which is the well-known D'Alembert's solution of the wave equation.

EXAMPLE 7.17 Obtain the solution of free vibrations of a semi-infinite string governed by

$$\text{PDE:}\ u_{tt} = c^2 u_{xx}, \quad 0 < x < \infty, \quad t > 0 \tag{7.150}$$

$$\text{ICs:}\ u(x,0) = f(x) \tag{7.151}$$

$$u_t(x,0) = g(x) \tag{7.152}$$

Solution Taking the Fourier sine transform of PDE, we have

$$\int_0^{\infty} \frac{\partial^2 u}{\partial t^2} \sin \alpha x\, dx = c^2 \int_0^{\infty} \frac{\partial^2 u}{\partial x^2} \sin \alpha x\, dx \tag{7.153}$$

Now consider

$$\int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin \alpha x \, dx$$

Integrating by parts, we get

$$\left(\frac{\partial u}{\partial x} \sin \alpha x - \alpha u \cos \alpha x \right)_0^\infty - \alpha^2 \int_0^\infty u \sin \alpha x \, dx$$

Now, since the string is fixed at $x = 0$ for all t and we assume that u and $\partial u/\partial x$ both tend to zero as $x \to \infty$, we arrive at

$$\int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin \alpha x \, dx = -\alpha^2 \int_0^\infty u \sin \alpha x \, dx = -\alpha^2 U$$

Hence, Eq. (7.153) reduces to

$$\frac{d^2 U}{dt^2} + c^2 \alpha^2 U = 0 \tag{7.154}$$

Its general solution is known to be

$$U = A(\alpha) \cos (c\alpha t) + B(\alpha) \sin (c\alpha t) \tag{7.155}$$

where $A(\alpha)$ and $B(\alpha)$ have to be determined. Now, at $t = 0$, we have

$$U = \int_0^\infty f(x) \sin \alpha x \, dx = F(\alpha)$$

$$\frac{\partial U}{\partial t} = \int_0^\infty g(x) \sin \alpha x \, dx = G(\alpha)$$

In the solution (7.155), if we take $t = 0$, then we have

$$U = A(\alpha) = F(\alpha)$$

$$\frac{\partial U}{\partial t} = c\alpha B(\alpha) = G(\alpha)$$

Substituting $A(\alpha)$ and $B(\alpha)$ into Eq. (7.155), we get

$$U = F(\alpha) \cos (c\alpha t) + \frac{G(\alpha)}{c\alpha} \sin (c\alpha t)$$

Taking the inverse Fourier sine transform of this relation, we obtain

$$u(x,t)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}U(\alpha,t)\sin\alpha x\,d\alpha$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\left[F(\alpha)\cos(c\alpha t)\sin\alpha x+\frac{G(\alpha)}{c\alpha}\sin(c\alpha t)\sin\alpha x\right]d\alpha$$

$$=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}F(\alpha)[\sin\alpha(x+ct)+\sin\alpha(x-ct)]\,d\alpha$$

$$+\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\frac{G(\alpha)}{c\alpha}[\cos\alpha(x-ct)-\cos\alpha(x+ct)]\,d\alpha$$

Since

$$f(x+ct)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}F(\alpha)\sin\alpha(x+ct)\,d\alpha$$

$$g(u)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}G(\alpha)\sin u\alpha\,d\alpha$$

$$\int_{x-ct}^{x+ct}g(u)\,du=\sqrt{\frac{2}{\pi}}\int_0^{\infty}G(\alpha)\,d\alpha\int_{x-ct}^{x+ct}\sin\alpha u\,du$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty}G(\alpha)\,d\alpha\left(-\frac{\cos\alpha u}{\alpha}\right)_{x-ct}^{x+ct}$$

$$=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{G(\alpha)}{\alpha}\,d\alpha[\cos\alpha(x-ct)-\cos\alpha(x+ct)]$$

we arrive at the solution

$$u(x,t)=\frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(u)\,du \tag{7.156}$$

7.13 SOLUTION OF LAPLACE EQUATION

One of the most important PDEs that occurs in many applications is the Laplace equation. Steady-state heat conduction, the electric potential in the steady flow of currents in solid conductors, the velocity potential of inviscid, irrotational fluids, the gravitational potential at an exterior point due to ellipsoidal Earth and so on, are all governed by Laplace equation. We shall now consider a few related examples.

EXAMPLE 7.18 Solve the following boundary value problem in the half-plane $y > 0$, described by

$$\text{PDE}: u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0$$
$$\text{BCs}: u(x, 0) = f(x), \quad -\infty < x < \infty,$$

u is bounded as $y \to \infty$; u and $\partial u/\partial x$ both vanish as $|x| \to \infty$.

Solution Since x has an infinite range of values, we take the Fourier exponential transform of PDE in the variable x to get

$$\mathcal{F}[u_{xx}; x \to \alpha] + \mathcal{F}[u_{yy}; x \to \alpha] = 0$$

Since u and $\partial u/\partial x$ both vanish as $|x| \to \infty$, we have

$$-\alpha^2 U(\alpha, y) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_{yy} e^{i\alpha x}\, dx = 0$$

$$-\alpha^2 U(\alpha, y) + \frac{\partial^2}{\partial y^2}\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u\,(x, y)\, e^{i\alpha x}\, dx\right] = 0$$

i.e.,

$$\frac{d^2 U\,(\alpha, y)}{dy^2} - \alpha^2 U\,(\alpha, y) = 0 \tag{7.157}$$

Its general solution is known to be

$$U\,(\alpha, y) = A\,(\alpha)\, e^{\alpha y} + B\,(\alpha)\, e^{-\alpha y} \tag{7.158}$$

Since u must be bounded as $y \to \infty$, $U\,(\alpha, y)$ and its Fourier transform also should be bounded as $y \to \infty$, implying $A\,(\alpha) = 0$ for $\alpha > 0$; but if $\alpha < 0$, $B\,(\alpha) = 0$; thus for any α,

$$U\,(\alpha, y) = \text{constant}\,(e^{-|\alpha| y}) \tag{7.159}$$

Now the Fourier transform of the BC yields

$$U(\alpha, 0) = \mathcal{F}[f(x); x \to \alpha] = F(\alpha) \tag{7.160}$$

From Eqs. (7.159) and (7.160), we find that

$$F(\alpha) = \text{const}. \tag{7.161}$$

Hence,

$$U\,(\alpha, y) = F(\alpha)\, e^{-|\alpha| y} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\,(x)\, e^{-|\alpha| y} e^{i\alpha x}\, dx$$

Taking its Fourier inverse transform, we obtain, after replacing the dummy variable x by ξ, the equation

$$u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)\, e^{-|\alpha| y} e^{i\alpha\xi}\, d\xi \right] e^{-i\alpha x}\, d\alpha$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi)\, d\xi \int_{-\infty}^{\infty} \exp\,[\{\alpha[i(\xi - x)] - |\alpha|\, y\}]\, d\alpha \tag{7.162}$$

But

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\,\{\alpha[i\,(\xi - x)] - |\alpha|\, y\}\, d\alpha$$

$$= \frac{1}{2\pi} \int_{-\infty}^{0} \exp\,\{\alpha[y + i\,(\xi - x)]\}\, d\alpha + \frac{1}{2\pi} \int_{0}^{\infty} \exp\,\{-\alpha\,[y - i\,(\xi - x)]\}\, d\alpha$$

$$= \frac{1}{2\pi} \left[\frac{\exp\{\alpha[y + i\,(\xi - x)]\}}{y + i\,(\xi - x)} \right]_{-\infty}^{0} - \frac{1}{2\pi} \left[\frac{\exp\{-\alpha\,[y - i\,(\xi - x)]\}}{y - i\,(\xi - x)} \right]_{0}^{\infty}$$

$$= \frac{1}{2\pi} \left[\frac{1}{y + i\,(\xi - x)} + \frac{1}{y - i\,(\xi - x)} \right]$$

$$= \frac{1}{\pi} \frac{y}{(\xi - x)^2 + y^2} \tag{7.163}$$

Substituting Eq. (7.163) into Eq. (7.162), the required solution is found to be

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)\, d\xi}{(\xi - x)^2 + y^2} \tag{7.164}$$

This solution is a well-known Poisson integral formula and is valid for $y > 0$, when $f(x)$ is bounded and piecewise continuous for all real x.

EXAMPLE 7.19 Solve the following Neumann problem described by

$$\text{PDE:}\; u_{xx}(x, y) + u_{yy}(x, y) = 0, \qquad -\infty < x < \infty, \qquad y > 0$$

$$\text{BC:}\; u_y(x, 0) = f(x), \qquad -\infty < x < \infty$$

$$u \text{ is bounded as } y \to \infty$$

$$u \text{ and } \partial u/\partial x \text{ both vanish as } |x| \to \infty$$

Solution Let us define a function $\phi(x, y) = u_y(x, y)$. Then

$$\phi_{xx} + \phi_{yy} = \frac{\partial}{\partial y}(u_{xx} + u_{yy}) = 0 \tag{7.165}$$

$$\text{BC}: \phi(x, 0) = u_y(x, 0) = f(x) \tag{7.166}$$

Thus, the function $\phi(x, y)$ is a solution of the problem described in Example 7.18 and, therefore, the solution of Eq. (7.165) subject to Eq. (7.166) is of the form

$$\phi(x, y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)\, d\xi}{(\xi - x)^2 + y^2}, \qquad y > 0 \tag{7.167}$$

However,

$$u(x, y) = \int \phi(x, y)\, dy = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi) \int \frac{y\, dy}{(x - \xi)^2 + y^2} d\xi$$

Therefore,

$$u(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi) \log[(x - \xi)^2 + y^2]\, d\xi + \text{const}. \tag{7.168}$$

is the required solution.

7.14 MISCELLANEOUS EXAMPLES

EXAMPLE 7.20 Find the Fourier transform of the normal density function

$$f(x) = \exp[-(x - m)^2/2\sigma^2]/\sqrt{2\pi\sigma^2}$$

Solution From the definition of Fourier transform we have

$$\mathcal{F}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\exp[-(x - m)^2/2\sigma^2]}{\sqrt{2\pi\sigma^2}} e^{i\alpha x}\, dx$$

Let $\dfrac{x - m}{\sigma} = z$; then

$$\mathcal{F}(\alpha) = \int_{-\infty}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} e^{i\alpha(m + \sigma z)} dz$$

Expanding $e^{i\alpha\sigma z}$ in the form

$$e^{i\alpha m}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-z^2/2}}{\sqrt{2\pi}}\frac{(i\alpha\sigma z)^n}{n!}dz \tag{7.169}$$

and denoting

$$I_n=\int_{-\infty}^{\infty}\frac{z^n e^{-z^2/2}}{\sqrt{2\pi}}dz, \qquad n=0,1,2,\dots \tag{7.170}$$

we obtain

$$\sqrt{2\pi}F(\alpha)=e^{i\alpha m}\sum_{n=0}^{\infty}\frac{(i\alpha\sigma)^n}{n!}I_n \tag{7.171}$$

For $n=0$, we have

$$(I_0)^2=\left(\int_{-\infty}^{\infty}\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz\right)^2=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp(-z_1^2/2)\exp(-z_2^2/2)}{2\pi}dz_1dz_2$$

$$=\int_0^{2\pi}\int_0^{\infty}\frac{e^{-r^2/2}}{2\pi}r\,dr\,d\theta \quad \text{[in polar coordinates]}$$

$$=1$$

To compute $I_n, n>0,$ we can integrate by parts to get

$$\sqrt{2\pi}I_n=-\int_{-\infty}^{\infty}z^{n-1}d(e^{-z^2/2})=(n-1)\int_{-\infty}^{\infty}z^{n-2}e^{-z^2/2}dz=(n-1)\sqrt{2\pi}I_{n-2}$$

Thus, we get a recurrence formula

$$I_n=(n-1)I_{n-2}$$

Finally, we note that $I_n=0,$ when n is odd, since $z^n\cdot e^{-z^2/2}$ is an odd function. In that case we have

$$\begin{aligned}I_{2n}&=(2n-1)I_{2n-2}\\&=(2n-1)(2n-3)I_{2n-4}\\&\ \ \vdots\\&=(2n-1)(2n-3)\dots3\times1\end{aligned}$$

However,

$$(2n-1)(2n-3)\ldots 3\times 1=\frac{(2n)(2n-1)(2n-2)\ldots 3\times 2+1}{(2n)(2n-2)\ldots 4\times 2}$$

$$=\frac{(2n)!}{2^n n!}$$

Finally, Eq. (7.171) becomes

$$\sqrt{2\pi}F(\alpha)=e^{i\alpha m}\sum_{n=0}^{\infty}\frac{(i\alpha\sigma)^{2n}}{(2n)!}\frac{(2n)!}{2^n n!}=e^{i\alpha m}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-\alpha^2\sigma^2}{2}\right)^n=e^{i\alpha m}\exp(-\alpha^2\sigma^2/2)$$

Thus,

$$F(\alpha)=\mathscr{F}[f(x);\alpha]=\frac{\exp(i\alpha m-\alpha^2\sigma^2/2)}{\sqrt{2\pi}}$$

EXAMPLE 7.21 Using the Fourier cosine transform, find the temperature $u(x,t)$ in a semi-infinite rod $0\le x\le\infty$, determined by the PDE

$$u_t=ku_{xx},\quad 0<x<\infty,\quad t>0$$

subject to

$$\text{IC}: u(x,0)=0,\quad 0\le x\le\infty,$$

$$\text{BC}: u_x(0,t)=-u_0 \text{ (a constant) when } x=0 \text{ and } t>0$$

u, $\partial u/\partial x$ both tend to zero as $x\to\infty$.

Solution Since $\partial u/\partial x$ is given at the lower limit, we take the Fourier cosine transform of the given PDE, to obtain

$$\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{\partial u}{\partial t}\cos\alpha x\,dx=k\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{\partial^2 u}{\partial x^2}\cos\alpha x\,dx$$

$$=k\sqrt{\frac{2}{\pi}}\left[\frac{\partial u}{\partial x}\cos\alpha x\right]_0^\infty+k\alpha\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{\partial u}{\partial x}\sin\alpha x\,dx$$

From physical considerations, we expect $\partial u/\partial x\to 0$ as $x\to\infty$. Therefore,

$$\frac{d}{dt}(U_c)=-k\sqrt{\frac{2}{\pi}}\left(\frac{\partial u}{\partial x}\right)_{x=0}+k\alpha\sqrt{\frac{2}{\pi}}(u\sin\alpha x)_0^\infty-k\alpha^2\sqrt{\frac{2}{\pi}}\int_0^\infty u\cos\alpha x\,dx$$

If $u \to 0$ as $x \to \infty$, we have

$$\frac{d}{dt}(U_c) = \sqrt{\frac{2}{\pi}}\, ku_0 - k\alpha^2 U_c$$

$$\frac{d}{dt}(U_c) + k\alpha^2 U_c = \sqrt{\frac{2}{\pi}}\, ku_0$$

which is a linear ODE, whose general solution is found to be of the form

$$\frac{d}{dt}(e^{k\alpha^2 t} U_c) = \sqrt{\frac{2}{\pi}}\, ku_0 e^{k\alpha^2 t}$$

On integration, we obtain

$$e^{k\alpha^2 t} U_c = \sqrt{\frac{2}{\pi}}\, ku_0 \int e^{k\alpha^2 t} dt + \text{const.} \tag{7.172}$$

Taking the cosine transform of the IC, we get

$$U_c = 0 \quad \text{when } t = 0$$

Using this condition in Eq. (7.172), we have

$$0 = \sqrt{\frac{2}{\pi}} \frac{u_0}{\alpha^2} + \text{const.}$$

which yields

$$\text{const.} = -\frac{u_0}{\alpha^2} \sqrt{\frac{2}{\pi}}$$

Hence,

$$U_c = \sqrt{\frac{2}{\pi}} \frac{u_0}{\alpha^2} (1 - e^{-k\alpha^2 t}) \tag{7.173}$$

Finally, taking the Fourier inverse cosine transform of Eq. (7.173), we have

$$u(x, t) = \frac{2u_0}{\pi} \int_0^\infty \frac{\cos \alpha x}{\alpha^2} (1 - e^{-k\alpha^2 t})\, d\alpha$$

EXAMPLE 7.22 Using the method of integral transform, solve the following potential problem in the semi-infinite strip described by

$$\text{PDE:}\, u_{xx} + u_{yy} = 0, \quad 0 < x < \infty, \quad 0 < y < a$$

subject to

$$\text{BCs}: u(x, 0) = f(x)$$

$$u(x, a) = 0$$

$$u(x, y) = 0, \quad 0 < y < a, \quad 0 < x < \infty$$

and

$$\partial u/\partial x \text{ tends to zero as } x \to \infty.$$

Solution Taking the Fourier sine transform of the above PDE, we obtain

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin \alpha x \, dx + \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial y^2} \sin \alpha x \, dx = 0$$

Integrating by parts, we get

$$\sqrt{\frac{2}{\pi}} \left(\frac{\partial u}{\partial x} \sin \alpha x \right)_0^\infty - \alpha \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial x} \cos \alpha x \, dx + \frac{d^2 U_s}{dy^2} = 0$$

Using the fact that $\partial u/\partial x \to 0$ as $x \to \infty$ and integrating again by parts, we obtain

$$-\alpha \sqrt{\frac{2}{\pi}} (u \cos \alpha x)_0^\infty - \alpha^2 \sqrt{\frac{2}{\pi}} \int_0^\infty u \sin \alpha x \, dx + \frac{d^2 U_s}{dy^2} = 0$$

Using the BC

$$u(x, y) = 0, \quad 0 < y < a, \quad 0 < x < \infty$$

we have

$$\frac{d^2 U_s}{dy^2} - \alpha^2 U_s = 0$$

Its general solution is found to be

$$U_s = A(\alpha) \cosh \alpha y + B(\alpha) \sinh \alpha y \tag{7.174}$$

Now, the Fourier sine transform of the first two BCs gives

$$U_s(\alpha, 0) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(\xi) \sin \alpha \xi \, d\xi, \quad U_s(\alpha, a) = 0$$

Using these two relations in Eq. (7.174), we obtain

$$A(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(\xi) \sin \alpha \xi \, d\xi \tag{7.175}$$

$$0 = A(\alpha) \cosh \alpha a + B(\alpha) \sinh \alpha a$$

Therefore,

$$B(\alpha) = -\frac{\cosh \alpha a}{\sinh \alpha x}\sqrt{\frac{2}{\pi}}\int_0^\infty f(\xi)\sin \alpha\xi \, d\xi \tag{7.176}$$

Thus, from Eqs. (7.174) to (7.176), we obtain

$$U_s = \sqrt{\frac{2}{\pi}}\left[\cosh \alpha y \int_0^\infty f(\xi)\sin \alpha\xi \, d\xi - \frac{\cosh \alpha a \sinh \alpha y}{\sinh \alpha a}\int_0^\infty f(\xi)\sin \alpha\xi \, d\xi\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^\infty f(\xi)\sin \alpha\xi \frac{\sinh (a-y)\alpha}{\sin a\alpha} d\xi\right] \tag{7.177}$$

Finally, taking the inverse Fourier sine transform of Eq. (7.177), we have

$$u(x, y) = \frac{2}{\pi}\int_0^\infty f(\xi)\, d\xi \int_0^\infty \frac{\sinh (a-y)\alpha}{\sin a\alpha}\sin a\xi \sin \alpha x \, d\alpha$$

EXAMPLE 7.23 Using finite sine transform to solve the following BVP described by

$$\text{PDE:}\; u_{xx} = \frac{1}{k}u_t, \quad 0 < x < L, \quad t > 0$$

$$\text{BC:}\; u(0, t) = u(L, t) = 0 \quad \text{for all } t$$

$$\text{IC:}\; u(x, 0) = \begin{cases} 2u_0 \dfrac{x}{L}, & 0 \le x \le L/2 \\ 2u_0\left(1 - \dfrac{x}{L}\right), & \dfrac{L}{2} \le x \le L \end{cases}$$

Solution Taking the finite Fourier sine transform of the given PDE, we have

$$\int_0^L \frac{\partial^2 u}{\partial x^2}\sin \frac{n\pi x}{L} dx = \frac{1}{k}\int_0^L \frac{\partial u}{\partial t}\sin \frac{n\pi x}{L} dx \tag{7.178}$$

Integrating the left-hand side by parts, we get

$$\left(\frac{\partial u}{\partial x}\sin \frac{n\pi x}{L}\right)_0^L - \frac{n\pi}{L}\int_0^L \frac{\partial u}{\partial x}\cos \frac{n\pi x}{L} dx$$

Again integrating by parts, we obtain

$$-\frac{n\pi}{L}\left[\left(u \cos \frac{n\pi x}{L}\right)_0^L + \frac{n\pi}{L}\int_0^L u \sin \frac{n\pi x}{L} dx\right] = \frac{n^2\pi^2}{L^2}U_s(n) + \frac{n\pi}{L}\{u(0, t) - u(L, t)\cos n\pi\}$$

$$= -\frac{n^2\pi^2}{L^2}U_s(n) \quad \text{[after using the BCs]}$$

Dropping the subscript s, Eq. (7.178) becomes

$$\frac{dU}{dt}+\frac{kn^2\pi^2}{L^2}U=0 \tag{7.179}$$

Its general solution is known to be

$$U=U_s(n,t)=A\exp\left(-\frac{kn^2\pi^2}{L^2}t\right) \tag{7.180}$$

where A is a constant to be determined from the ICs. Taking the finite Fourier sine transform of the given IC, we have

$$U_s(n,0)=\int_0^{L/2}2u_0\frac{x}{L}\sin\frac{n\pi x}{L}dx+\int_{L/2}^{L}2u_0\left(1-\frac{x}{L}\right)\sin\frac{n\pi x}{L}dx$$

$$=\left[-2u_0\frac{x}{L}\cos\frac{n\pi x}{L}\Big/\frac{n\pi}{L}\right]_0^{L/2}+\frac{2u_0}{L}\frac{1}{n\pi/L}\int_0^{L/2}\cos\frac{n\pi x}{L}dx$$

$$+\left[-2u_0\left(1-\frac{x}{L}\right)\cos\frac{n\pi x}{L}\Big/\frac{n\pi}{L}\right]_{L/2}^{L}-\frac{2u_0}{L}\frac{1}{n\pi/L}\int_{L/2}^{L}\cos\frac{n\pi x}{L}dx$$

or

$$U_s(n,0)=\frac{4u_0L}{n^2\pi^2}\sin\frac{n\pi}{2}$$

From Eq. (7.180) when $t=0$, we get

$A=U_s(n,0)$ and, therefore,

$$A=\begin{cases}0 & \text{if } n \text{ is even}\\ \dfrac{4u_0L}{(2r+1)^2\pi^2}\sin\dfrac{(2r+1)\pi}{2} & \text{if } n \text{ is odd}\end{cases}$$

i.e.

$$A=\frac{(-1)^r\,4u_0L}{(2r+1)^2\pi^2}$$

Thus, from Eq. (7.180), we have

$$U_s(n,t)=\frac{(-1)^r\,4u_0L}{(2r+1)^2\pi^2}\exp\left[-\frac{k\,(2r+1)^2\pi^2t}{L^2}\right] \tag{7.181}$$

Applying the inversion theorem to Eq. (7.181), we obtain

$$U(x,t)=\frac{2}{L}\sum_{n=1}^{\infty}U_s(n,t)\sin\frac{n\pi x}{L}=\frac{8u_0}{\pi^2}\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r+1)^2}\exp\left[-\frac{k\,(2r+1)^2\pi^2}{L^2}t\right]\sin\frac{(2r+1)\pi x}{L}$$

EXAMPLE 7.24 Find the steady-state temperature distribution $u\,(x,y)$ in a long square bar of side π with one face maintained at constant temperature u_0 and the other faces at zero temperature.

Solution Mathematically, the above problem can be posed as the following BVP:

$$\text{PDE}: u_{xx}+u_{yy}=0,\qquad 0<x<\pi,\ 0<y<\pi$$
$$\text{BCs}: u\,(0,y)=u\,(\pi,y)=0$$
$$u\,(x,0),\ u(x,\pi)=u_0$$

Taking the finite Fourier sine transform with respect to the variable x, we have

$$\int_0^{\pi}\frac{\partial^2 u}{\partial x^2}\sin nx\,dx+\int_0^{\pi}\frac{\partial^2 u}{\partial y^2}\sin nx\,dx=0$$

$$\left(\frac{\partial u}{\partial x}\sin nx\right)_0^{\pi}-n\int_0^{\pi}\frac{\partial u}{\partial x}\cos nx\,dx+\frac{\partial^2}{\partial y^2}\int_0^{\pi}u\sin nx\,dx=0$$

$$-n\,(u\cos nx)_0^{\pi}-n^2\int_0^{\pi}u\sin nx\,dx+\frac{d^2U_s}{dy^2}=0$$

Therefore,

$$\frac{d^2U_s}{dy^2}-n^2U_s=0$$

Its general solution is known to be

$$U_s=A\cosh ny+B\sinh ny \tag{7.182}$$

Taking the finite Fourier sine transform of the second set of BCs, we have

$$U_s\,(n,0)=0$$

$$U_s\,(n,\pi)=\int_0^{\pi}u_0\sin nx\,dx=u_0\left(-\frac{\cos nx}{n}\right)_0^{\pi}=u_0\left(\frac{1-\cos n\pi}{n}\right)$$

Using these results in Eq. (7.182), we obtain $A = 0$ and

$$B \sinh n\pi = u_0 \left(\frac{1 - \cos n\pi}{n} \right)$$

Hence,

$$U_s = \frac{u_0}{\sinh n\pi} \left(\frac{1 - \cos n\pi}{n} \right) \sinh ny$$

Finally, taking the finite Fourier sine inverse transform, we have the result

$$u(x, y) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{u_0}{\sinh n\pi} \left(\frac{1 - \cos n\pi}{n} \right) \sinh ny \sin nx$$

Thus, the required temperature distribution is

$$u(x, y) = \frac{4u_0}{\pi} \sum_{r=0}^{\infty} \frac{\sinh (2r+1)\, y \sin (2r+1)\, x}{(2r+1) \sinh (2r+1)\, \pi} \qquad \text{when } n = \text{odd}$$

$$= 0 \qquad \text{when } n = \text{even}$$

EXAMPLE 7.25 A one-dimensional infinite solid, $-\infty < x < \infty$, is initially at temperature $F(x)$. For times $t > 0$, heat is generated within the solid at a rate of $g(x, t)$ units. Determine the temperature in the solid for $t > 0$.

Solution The problem can be described as follows:

$$\text{PDE: } \frac{\partial^2 T}{\partial x^2} + \frac{[g(x, t)]}{k} = \frac{1}{k} \frac{\partial T}{\partial t}, \qquad -\infty < x < \infty, \; t > 0$$

$$\text{IC: } T(x, 0) = F(x), \qquad -\infty < x < \infty$$

The exponential Fourier transform of PDE with respect to x is given by

$$\int_{-\infty}^{\infty} e^{i\alpha x} \frac{\partial^2 T}{\partial x^2} dx + \frac{1}{k} \int_{-\infty}^{\infty} e^{i\alpha x} g(x, t)\, dx = \frac{1}{k} \int_{-\infty}^{\infty} e^{i\alpha x} \frac{\partial T}{\partial t} dx$$

which on rewriting becomes

$$\int_{-\infty}^{\infty} e^{i\alpha x} \frac{\partial^2 T}{\partial x^2} dx + \frac{1}{k} \bar{g}(\alpha, t) = \frac{1}{k} \frac{d\bar{T}(\alpha, t)}{dt}$$

or

$$-\alpha^2 \bar{T}(\alpha, t) + \frac{1}{k}\bar{g}(\alpha, t) = \frac{1}{k}\frac{d\bar{T}(\alpha, t)}{dt} \tag{7.183}$$

The Fourier transform of the IC is given by

$$\bar{T}(\alpha, 0) = \int_{-\infty}^{\infty} e^{i\alpha x} F(x)\, dx = \bar{F}(\alpha, 0)$$

Therefore,

$$\bar{T}(\alpha, t)\big|_{t=0} = \bar{T}(\alpha, 0) = \int_{-\infty}^{\infty} e^{i\alpha x'} F(x')\, dx' = \bar{F}(\alpha, 0)$$

Using this result, Eq. (7.183) can be modified to

$$\frac{d\bar{T}(\alpha, t)}{dt} + k\alpha^2 \bar{T}(\alpha, t) = \bar{g}(\alpha, t)$$

The solution of this equation is found to be

$$\bar{T} \exp\left(\int k\alpha^2 dt\right) = \int \bar{g}(\alpha, t)\, e^{k\alpha^2 t} dt + c$$

Using the IC, we get

$$c = \bar{F}(\alpha, 0)$$

Hence, using the IC, the solution of Eq. (7.183) is obtained as

$$\bar{T}(\alpha, t) = e^{-k\alpha^2 t}\left[\bar{F}(\alpha, 0) + \int_{t'=0}^{t} \bar{g}(\alpha, t')\, e^{k\alpha^2 t'} dt'\right]$$

Now, taking its inverse exponential Fourier transform, we get

$$T(x, t) = \frac{1}{2\pi}\int_{\alpha=-\infty}^{\infty} \exp(-k\alpha^2 t - i\alpha x)\left[\bar{F}(\alpha) + \int_{t'=0}^{t} e^{k\alpha^2 t'} \bar{g}(\alpha, t')\, dt'\right] d\alpha \tag{7.184}$$

where

$$\bar{F}(\alpha) = \int_{x'=-\infty}^{\infty} e^{i\alpha x'} F(x')\, dx'$$

$$\bar{g}(\alpha, t') = \int_{x'=-\infty}^{\infty} e^{i\alpha x'} g(x', t')\, dx'$$

After changing the order of integration and rewriting, Eq. (7.184) becomes

$$T(x, t) = \frac{1}{2\pi}\int_{x'=-\infty}^{\infty} F(x')\, dx' \int_{\alpha=-\infty}^{\infty} \exp[-k\alpha^2 t - i\alpha(x - x')]\, d\alpha$$

$$+\frac{1}{2\pi}\int_{t'=0}^{t} dt' \int_{x'=-\infty}^{\infty} g(x', t')\, dx' \int_{\alpha=-\infty}^{\infty} \exp\,[-k\alpha^2 (t-t') - i\alpha\,(x-x')]\, d\alpha \qquad (7.185)$$

Now,

$$\int_{\alpha=-\infty}^{\infty} \exp\,[-k\alpha^2 (t) - i\alpha\,(x-x')]\, d\alpha = \int_{\alpha=-\infty}^{\infty} \exp\left[\left(-\frac{x-x_1}{2\sqrt{kt}} + i\alpha\sqrt{kt}\right)^2 - \left(\frac{x-x'}{2\sqrt{kt}}\right)^2\right] d\alpha$$

Let

$$-\frac{x-x'}{2\sqrt{kt}} + i\alpha\sqrt{kt} = i\eta$$

Then the above equation can be written as

$$\int_{\alpha=-\infty}^{\infty} \exp\,[-k\alpha^2 t - i\alpha\,(x-x')]\, d\alpha = \frac{1}{\sqrt{kt}} \exp\,[-(x-x')^2/(4kt)] \int_{-\infty}^{\infty} \exp\,(-\eta^2)\, d\eta \qquad (7.186)$$

But,

$$\int_{-\infty}^{\infty} e^{-\eta^2}\, dn = \sqrt{\pi} \qquad \text{(Standard integral)} \qquad (7.187)$$

Using Eqs. (7.186) and (7.187), the required temperature is obtained from Eq. (7.185) in the form

$$T(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} F\,(x') \exp\,[-(x-x')^2/(4kt)]\, dx'$$

$$+ \int_{t'=0}^{t} \frac{dt'}{\sqrt{4\pi k(t-t')}} \int_{-\infty}^{\infty} g\,(x',\, t') \exp\,[-(x-x')^2/4\alpha(t-t')]\, dx'$$

EXAMPLE 7.26 A uniform string of length L is stretched tightly between two fixed points at $x = 0$ and $x = L$. If it is displaced a small distance ε at a point $x = b, 0 < b < L$, and released from rest at time $t = 0$, find an expression for the displacement at subsequent times.

Solution Let $u\,(x, t)$ denote the displacement of the string. Then at $t = 0$, the equation for OA is: $y = (\varepsilon/b)\, x$, (see Fig. 7.1) while the equation for AB is given by

$$y = \frac{\varepsilon}{b-L}(x-L)$$

Now, the IBVP is described by the PDE:

$$u_{tt} = c^2 u_{xx}$$

$$\text{BCs}: u(0,t) = u\,(L, t) = 0, \qquad t \ge 0,$$

$$\text{ICs}:\ u\,(x, 0) = \begin{cases} \dfrac{\varepsilon x}{b}, & 0 < x < b \\ \dfrac{\varepsilon\,(x - L)}{b - L}, & b < x < L \end{cases}$$

$$u_t\,(x, 0) = 0$$

Taking the finite Fourier sine transform of the PDE, we have

$$\int_0^L \frac{\partial^2 u}{\partial t^2} \sin\frac{n\pi x}{L}\,dx = c^2 \int_0^L \frac{\partial^2 u}{\partial x^2} \sin\frac{n\pi x}{L}\,dx$$

or

$$\frac{d^2 U_s}{dt^2} = c^2 \left\{ \left(\frac{\partial u}{\partial x} \sin\frac{n\pi x}{L} \right)_0^L - \frac{n\pi}{L} \int_0^L \frac{\partial u}{\partial x} \cos\frac{n\pi x}{L}\,dx \right\}$$

$$= -\frac{n\pi}{L} c^2 \left[\left(u \cos\frac{n\pi x}{L} \right)_0^L + \frac{n\pi}{L} \int_0^L u \sin\frac{n\pi x}{L}\,dx \right]$$

$$= \frac{-n^2\pi^2 c^2}{L^2} U_s + \frac{n\pi c^2}{L} \{ u\,(0, t) - u\,(L, t) \cos n\pi \}$$

$$= \frac{-n^2\pi^2 c^2}{L^2} U_s \qquad \text{(after using BCs)}$$

Therefore,

$$\frac{d^2 U_s}{dt^2} + \frac{n^2\pi^2 c^2}{L^2} U_s = 0$$

Its general solution is found to be

$$U_s\,(n, t) = A \cos\frac{n\pi ct}{L} + B \sin\frac{n\pi ct}{L} \tag{7.188}$$

Now, taking finite Fourier sine transform of ICs, we obtain

$$U_s\ (n, 0) = \int_0^b \frac{\varepsilon x}{b} \sin\frac{n\pi x}{L}\,dx + \int_b^L \frac{\varepsilon\,(x - L)}{b - L} \sin\frac{n\pi x}{L}\,dx$$

$$= \frac{\varepsilon}{b} \frac{L}{n\pi} \left(-x \cos\frac{n\pi x}{L} \right)_0^b + \frac{\varepsilon}{b} \times \frac{L}{n\pi} \int_0^b \cos\frac{n\pi x}{L}\,dx$$

$$+ \frac{\varepsilon L}{n\pi\,(b - L)} \left[-(x - L) \cos\frac{n\pi x}{L} \right]_0^L + \frac{\varepsilon L}{n\pi\,(b - L)} \int_b^L \cos\frac{n\pi x}{L}\,dx$$

$$= \frac{\varepsilon L^2}{n^2\pi^2 b} \sin\frac{n\pi b}{L} - \frac{\varepsilon L^2}{n^2\pi^2\,(b - L)} \sin\frac{n\pi b}{L}$$

or

$$U_s(n,0)=\frac{\varepsilon L^3}{n^2\pi^2 b(L-b)}\sin\frac{n\pi b}{L}$$

From Eq. (7.188), when $t=0$, we have

$$U_s(n,0)=A=\frac{\varepsilon L^3}{n^2\pi^2 b(L-b)}\sin\frac{n\pi b}{L}$$

Further, taking finite Fourier sine transform of the second IC, we get

$$\frac{dU_s}{dt}=0$$

From Eq. (7.188) it can be easily seen that $B=0.$ Thus, we obtain from Eq. (7.188) the relation

$$U_s(n,t)=\frac{\varepsilon L^3}{n^2\pi^2 b(L-b)}\sin\frac{n\pi b}{L}\cos\frac{n\pi ct}{L}$$

Finally, inverting, we have the required, displacement as

$$u(x,t)=\frac{2\varepsilon L^2}{\pi^2 b(L-b)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi b}{L}\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$$

EXERCISES

1. Find the Fourier transform of

$$f(x)=\begin{cases}1-x^2, & |x|\le 1\\ 0, & |x|>1\end{cases}$$

and hence evaluate

$$\int_0^{\infty}\left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}\,dx$$

2. Find Fourier sine and cosine transforms of e^{-x}.

3. Find the Fourier cosine transform of

$$f(x)=\frac{1}{1+x^2}$$

4. Using Parseval's relation for the Fourier cosine transform of

$$g(x) = e^{-ax}$$

$$f(x) = \begin{cases} 1, & 0 < x < \lambda \\ 0, & x > \lambda \end{cases}$$

show that

$$\int_0^\infty \frac{\sin \lambda\alpha \, d\alpha}{\alpha(a^2 + \alpha^2)} = \frac{\pi}{2}\left(\frac{1 - e^{-\lambda a}}{a^2}\right)$$

5. Verify the following relations:

(i) $\int_0^\infty F_s(\alpha)\, G_s(\alpha)\, d\alpha = \int_0^\infty f(x)\, g(x)\, dx$

(ii) $\int_0^\infty [F_c(\alpha)]^2 d\alpha = \int_0^\infty [f(x)]^2\, dx$

(iii) $\int_0^\infty [F_s(\alpha)]^2 d\alpha = \int_0^\infty [f(x)]^2 dx.$

6. If $a > 0$, b is any real or complex, show that

$$\int_{-\infty}^{\infty} e^{-ax^2 - 2bx} dx = \frac{\sqrt{\pi}}{\sqrt{a}} e^{b^2/a}$$

7. Solve the problem described by

$$\text{PDE}: u_t = u_{xx} + \delta(x)\,\delta(t), \qquad -\infty < x < \infty$$

$$\text{BC}: \quad \underset{|x|\to\infty}{\text{Lt}} \; u(x, t) = 0$$

$$\text{IC}: \quad u(x, 0) = \delta(x)$$

8. If $v(x, t)$ denotes the solution of

$$v_t - v_{xx} = 0, \qquad -\infty < x < \infty, \; t > 0$$

$$v(x, 0) = f(x), \qquad -\infty < x < \infty$$

show that

$$u(x, t) = \int_0^t v(x, t - \tau)\, d\tau$$

is a solution of

$$u_t - u_{xx} = f(x), \quad -\infty < x < \infty, \; t > 0$$
$$u(x, 0) = 0, \quad -\infty < x < \infty$$

and hence, write down the Green's function for the above non-homogeneous PDE with the homogeneous initial condition.

[Duhamel's principle]

9. The temperature $\theta(x, t)$ in the semi-infinite rod $x \geq 0$ is determined from the differential equation

$$\frac{\partial \theta}{\partial t} = \alpha^2 \frac{\partial^2 \theta}{\partial x^2}$$

subject to

$$\text{IC}: \theta(x, 0) = 0$$
$$\text{BC}: \theta(0, t) = \phi(t) \text{ for } t > 0$$

Using the Fourier sine transform, derive the solution in the form

$$\theta(x, t) = \frac{2}{\sqrt{\pi}} \int_{x/(2a\sqrt{t})}^{\infty} \phi\left(t - \frac{x^2}{4\alpha^2 u^2}\right) e^{-u^2} du$$

10. If $\nabla^2 u = 0$, for $x \geq 0$ and if $u = f(y)$ on $x = 0$, show that by using Fourier transform technique,

$$u(x, y) = \frac{x}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)\, d\xi}{x^2 + (y - \xi)^2}$$

11. Using the Fourier transform method, show that the solution of the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is valid in the half-plane $y > 0$, subject to the condition

$$u(x, 0) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

and

$$\underset{x^2 + y^2 \to \infty}{\text{Lt}} u(x, y) = 0 \text{ in the above half-plane.}$$

12. Using the finite Fourier transform, solve the BVP described by

$$\text{PDE}: \frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2}, \qquad 0 < x < 6, t > 0$$

subject to

$$\text{BC}: V_x(0, t) = 0 = V_x\,(6, t)$$

$$\text{IC}: V(x, 0) = 2x$$

13. Using the finite Fourier transform, solve the two-dimensional Laplace equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad 0 < x < \pi, 0 < y < y_0$$

subject to

$$V(0, y) = 0, \qquad V(\pi, y) = 1$$

$$V_y(x, 0) = 0, \quad V(x, y_0) = 1.$$

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Answers and Keys to Exercises

Chapter 0

1. $px - qy = x - y$
2. $pq = 4xyz$
3. $F\left(\frac{xy}{z}, \frac{x-y}{z}\right) = 0$
4. (i) $F(x^3 - y^3, x^2 - z^2) = 0$

   (ii) $F(x^2 - y^2 + 2x - 2y, z(x - y)) = 0$
5. $x^2 + y^2 z^4 - z^2 = 0$
6. $x^2 + y^2 - 2x = z^2 - 4z$
7. $(x - y + z)^2 + z^4 (x + y + z)^2 - 2z^2 (x - y + z) - 2z^4 (x + y + z) = 0$
8. $x = s(2e^t - 1),\ y = s(e^t - 1)$

   $p = 2s(e^t - 1),\ q = s(e^t + 1)$

   $z = \frac{5}{2} s^2 (e^{2t} - 1) - 3s^2 (e^t - 1)$

   and the equation of the integral surface is

$$z = y(4x - 3y)/2.$$

9. $x = \frac{s}{2}(e^t + e^{-t}),\ y = (e^t - e^{-t})/2$

$p = (e^t + e^{-t})/2,\ \ q = s(e^t - e^{-t})/2$

$z = \frac{s}{4}e^{2t} + \frac{s}{4}e^{-2t} + \frac{s}{2}$

The equation of the integral surface is

$z^2 = x^2(1 + y^2)$

10. $x = 2s(2 - e^{-t}),\ y = 2\sqrt{2}\ s(e^{-t} - 1)$

$z = -s^2 e^{-2t}$

and the equation of the integral surface is

$$4z + (x + \sqrt{2}y)^2 = 0$$

11. $z^2 = 2xy + c_1$

12. $z^2 = x^2 + (y + c)^2$

13. $z^2 = a^2x^2 + (ay + b)^2$

14. (i) $z = \frac{2}{3}(y + a)^{3/2} + \frac{1}{9} + \frac{1}{3x^2} + be^{3/x^2}$

(ii) $z = \frac{ax}{y^2} + \frac{b}{y} - \frac{a^2}{4y^3}$

15. $z = ax + (a/(a-1))\,y + c$

16. (i) $z^2 = 2(a+1)\,(x + y/a) + b$

(ii) $z = \frac{1}{3}(x^2 + a^2)^{3/2} + (y^2 - a^2)^{1/2} + b$

17. $z = ax + by - \sin(ab)$

18. (i) The given PDE is of Clairaut's form

$$z = xp + yq + \left(\frac{p}{q^2} + \frac{q}{p^2}\right)$$

Its complete integral is

$$z = ax + by + \left(\frac{a}{b^2} + \frac{b}{a^2}\right)$$

(ii) $z = ax + by + \left(\dfrac{a^4 + b^4}{ab}\right)$

19. The required surface is

$$x^2 + y^2 - 2z^3 - z^2 = -2$$

20. Similar to Example 0.16.

21. The auxiliary equations for the given PDE are

$$\frac{dx}{x - y} = \frac{dy}{-(x - y + z)} = \frac{dz}{z} \tag{1}$$

from which we observe that $dx + dy + dz = 0$.

On integration, we obtain

$$x + y + z = c_1 \tag{2}$$

We also observe that

$\dfrac{dx - dy + dz}{x - y + x - y + z + z} = \dfrac{dz}{z}$ which on integration yields

$$\frac{1}{2} \ln (x - y + z) = \ln (z) + \ln c_2$$

or

$$(x - y + z)/z^2 = c_2 \tag{3}$$

Given $z = 1$, and $x^2 + y^2 = 1$. Therefore, Eqs. (2) and (3) become

$$x + y + 1 = c_1, \;\; x - y + 1 = c_2$$

whose solution is found to be

$$x = \frac{c_1 + c_2}{2} - 1 \quad \text{and} \quad y = \frac{c_1 - c_2}{2} \tag{4}$$

But, we are given $x^2 + y^2 = 1$

Therefore,

$\left[\dfrac{c_1 + c_2}{2} - 1\right]^2 + \left[\dfrac{c_1 - c_2}{2}\right]^2 = 1$ which simplifies to

$$c_1^2 + c_2^2 - c_1 - c_2 = 0$$

or

$$(x+y+z)^2+\left[\frac{x-y+z}{z^2}\right]^2-(x+y+z)-\left[\frac{x-y+z}{z^2}\right]=0$$

is the required integral surface.

22. Given $u = w/y$, which implies $u_x = w_x/y, u_y = -\frac{w}{y^2}+\frac{w_y}{y}$

Substituting in the given PDE, we get

$$\frac{x}{y}w_x = u + y\left(\frac{w}{y^2}+\frac{w_y}{y}\right)$$

That is,

$$\frac{x}{y}w_x = \frac{w}{y}+w_y-\frac{w}{y} \text{ or } xw_x - yw_y = 0$$

This is a Lagrange's equation, whose auxiliary equations are

$$\frac{dx}{x}=\frac{dy}{-y}=\frac{dw}{0}$$

The first two members give us $x\,dy + y\,dx = 0$, which on integration yields $xy = c_1$. The last equation is $dw = 0$, which on integration gives $w = c_2$. Hence, the solution is $w = f(xy)$. Therefore, the correct choice is (D).

23. The given differential equation can be recast as

$$z = xp + yq + \left(\frac{p}{q^2}+\frac{q}{p^2}\right)$$

which is of Clairauts form. Hence, its complete integral is

$$z = ax + by + (ab^{-2}+ba^{-2})$$

Therefore, the correct choice is (A)

24. $\frac{\partial z}{\partial x} = p = 1 + yA'(xy), \frac{\partial z}{\partial y} = q = 1 + xA'(xy)$

Eliminating A' in the above pair of equations, we get

$$px - qy = x - y$$

Hence, the correct choice is (B).

25. The given PDE is in Clairaut's form. Hence, the correct choice is (B).

Chapter 1

1. The given PDE is hyperbolic if the determinant

$$B^2 - 4AC = 4(x-y)^2[2-(x-y)^2] > 0$$

That is, when $2-(x-y)^2 > 0$ as $x-y \neq 0$. This means that $(x-y)^2 < 2$, implying $|y-x| < \sqrt{2}$; in other words, $-\sqrt{2} < y-x < \sqrt{2}$. Hence, the given PDE is hyperbolic between the straight lines $y = x+\sqrt{2}$ and $y = x-\sqrt{2}$, but not on the line $y = x$.

2. The discriminant $B^2 - 4AC$ is $4(1-x^2)$. Thus the given PDE is

Hyperbolic	Elliptic
If $1-x^2 > 0$ or $\|x\| < 1$	If $1-x^2 < 0$ or $\|x\| > 1$
The characteristic equations are	The characteristic equations are
$\dfrac{dy}{dx} = \dfrac{B \pm \sqrt{B^2-4AC}}{2A} = \pm\dfrac{1}{\sqrt{1-x^2}}$	$\dfrac{dy}{dx} = \pm i \times \dfrac{1}{\sqrt{x^2-1}}$
On integration, we obtain	On integration, we get
$y = \pm\sin^{-1}x + c$	$y = \pm i\cosh^{-1}x + c$
Hence, the characteristic equations are	Hence, the characteristic equations are
$\xi = y - \sin^{-1}x$	$\xi = y + i\cosh^{-1}x$
$\eta = y + \sin^{-1}x$	$\eta = y - i\cosh^{-1}x$

3. The discriminant $B^2 - 4AC = -4xy < 0$ is true when $x > 0$, $y > 0$ and when $x < 0$, $y < 0$. In either of these cases, the given PDE is of elliptic type and

$$\frac{dy}{dx} = \pm i\sqrt{xy}$$

On integration, we get $2\sqrt{y} - (2/3)\,ix^{3/2} = c_1$ and $2\sqrt{y} + (2/3)\,ix^{3/2} = c_2$. If we put $\eta = (2/3)x^{3/2}$, $\xi = 2y^{1/2}$, we get the required canonical form

$$u_{\xi\xi} + u_{\eta\eta} = \frac{1}{\xi}u_{\xi} - \frac{1}{3\eta}u_{\eta}$$

4. (a) Hyperbolic in the first quadrant. The characteristic equations are:

$$\xi = x^2 + y^2, \qquad \eta = x^2 - y^2$$

The canonical form is

$$2(\xi^2 - \eta^2)u_{\xi\eta} - \eta u_\xi + \xi u_\eta = 0$$

(b) Parabolic type.

Characteristic equation: $\xi = y - x, \quad \eta = y$

Canonical form: $u_{\eta\eta} = 0.$

(c) Elliptic for finite values of x and y. The canonical form is

$$u_{\alpha\alpha} + u_{\beta\beta} = u - \frac{u_\alpha}{\alpha} - \frac{u_\beta}{\beta}$$

(d) Parabolic everywhere. The characteristic equations are:

$$\xi = y/x, \quad \eta = y$$

Canonical form: $u_{\eta\eta} = 0$

(e) The equation is hyperbolic. The characteristic equations are:

$$\xi = y - x, \qquad \eta = y - (x/4)$$

Canonical form: $u_{\xi\eta} = \frac{1}{3}u_\eta - \frac{8}{9}$

5. The equation is hyperbolic. The characteristic equations are:

$$\xi = y - 3x, \quad \eta = y - \frac{1}{3}x$$

Canonical equation: $u_{\xi\eta} = 0.$

The general solution is

$$u(x, y) = f(y - 3x) + g\left(y - \frac{x}{3}\right)$$

7. $L^*(v) = \dfrac{\partial^2}{\partial x^2}(Av) + \dfrac{\partial^2}{\partial x \partial y}(Bv) + \dfrac{\partial^2}{\partial y^2}(Cv) - \dfrac{\partial}{\partial x}(Dv) - \dfrac{\partial}{\partial y}(Ev) + (Fv)$

9. Comparing with the general second order PDE, that is,

$$AZ_{xx} + BZ_{xy} + CZ_{yy} + DZ_x + EZ_y + FZ = G$$

From the given PDE, we find that $A=1, B=2, C=\cos^2 x.$ Then, the discriminant is given by $B^2-4AC=4-4\cos^2 x=4\sin^2 x.$ Therefore, the given PDE is of hyperbolic type if $\sin x>0.$ On integration, the required characteristics are found to be

$$y-x+\cos x=c_1 \text{ and } y-x-\cos x=c_2$$

10. Let us introduce the transformation $\eta=x+y$ and $\xi=x-y.$ Then, $u_x=u_\eta\eta_x+u_\xi\xi_x,$ $u_y=u_\eta\,\eta_y+u_\xi\xi_y$ and $u_{xx}=u_{\eta\eta}+2u_{\eta\xi}+u_{\xi\xi}, u_{yy}=u_{\eta\eta}-2u_{\eta\xi}+u_{\xi\xi}, u_{xy}=u_{\eta\eta}-u_{\xi\xi}.$ Substituting these results in the given PDE, we get $u_{\xi\xi}=0,$ which is the required canonical form.

11. Comparing the given PDE with the standard PDE, obtain $A=y^3, B=0, C=-(x^2-1).$ Therefore, the discriminant of the given PDE is given as $B^2-4AC=4y^3(x^2-1).$ Evidently, the given PDE is hyperbolic in $\{(x,y), y>0\}$. Thus, the correct choice is (B).

12. Comparing the given PDE with the standard PDE, we find the discriminant as

$$\begin{aligned} B^2-4AC &= x^2(y^2-1)^2-4x^2(y-1)\,y(y^2-1) \\ &= x^2(y^2-1)\,[y^2-1-4y\,(y-1)] \\ &= x^2(y^2-1)[-3y^2+4y-1]>0. \end{aligned}$$

Hence, the given PDE is hyperbolic everywhere except along $x=0,$ that is, except along y-axis. Therefore, the correct choice is (B).

13. Comparing the given PDE with the standard PDE of second order, we find that the characteristic equations are given by

$$\frac{dy}{dx}=\frac{-B\pm\sqrt{B^2-4AC}}{2A},$$

where $A=x^2, B=0, C=-y^2.$

Thus,

$$\frac{dy}{dx}=\pm\frac{\sqrt{4x^2y^2}}{2x^2}=\pm\frac{y}{x}.$$

Taking –ve sign we have,

$$\frac{dy}{y}=-\frac{dx}{x}.$$

On integration, we get

$$\ln y=-\ln x+\ln C.$$

That is,

$$xy = c,$$

which is a rectangular hyperbola. Hence, the correct choice is (A). If we take +ve sign, the correct choice is (D).

14. Comparing the given PDE with the standard PDE of second order, we observe that $A = y, B = 2xy, C = x$. Hence its discriminant $B^2 - 4AC = 4x^2y^2 - 4xy$ should be positive. For hyperbolic case, we should have $xy > 1$. Hence, the correct choice is (C).

15. (i) $u_{CF} = e^x\phi_1(y + x) + e^{2x}\phi_2(y + x)$

(ii) $u_{CF} = e^x\phi_1(x - y) + e^{3x}\phi_2(2x - y)$.

16. $u_{CF} = \sum_{i=1}^{\infty} c_i e^{a_i x + b_i y}$

where $F(a_i, b_i) = a_i^2 + a_ib_i + a_i + b_i + 1 = 0$.

17. (i) $u = e^x\phi_1(y) + e^{-x}\phi_2(y + x) + \frac{1}{2}\sin(x + 2y)$

(ii) $u = \phi_1(x) + e^{3x}\phi_2(y + 2x) - \frac{1}{6}e^{x+2y}$

(iii) $u = e^y[y\phi_2(x - 2y) + \phi_1(x - 2y) + e^y[y^2\psi_3(2x + y) + y\psi_2(2x + y) + \psi_1(2x + y)]$.

18. (i) $u = \psi_1(xy) + x^2\psi_2(xy) + \frac{y}{x}$

(ii) $u = \phi_1(y) + e^{3x}\phi_2(y + 2x) - \frac{1}{6}e^{x+2y}$.

19. (i) $u = \phi_1(y - x) + \phi_2(y - 2x) + \frac{x^2y}{2} - \frac{x^3}{3}$

(ii) $u = \phi_1(y + ix) + \phi_2(y - ix) - \frac{1}{(p^2 + q^2)}\cos px \cos qy$

(iii) $u = \phi_1(y - x) + \phi_2(y + 2x) + ye^x$

(iv) $u = \phi_1(2y + x) + x\phi_2(2y + x) + 2x^2\log(x + 2y)$

(v) $u = \phi_1(y + x) + \phi_2(y + 2x) + \frac{1}{4}e^{2x+3y} - \frac{1}{15}\sin(x - 2y)$.

Chapter 2

1. It has been shown in Example 2.4 that the possible solution of the given BVP satisfying all the BCs except the first one is

$$u(r,\theta) = \sum_{n=1}^{\infty} B_n r^n \sin n\theta$$

Using the BC:

$$u(10,\theta)=\frac{400}{\pi}(\pi\theta-\theta^2)$$

we have

$$\sum 10^n B_n \sin n\theta=\sum b_n \sin n\theta$$

where

$$b_n=\frac{2}{\pi}\int_0^{\pi}\frac{400}{\pi}(\pi\theta-\theta^2)\sin n\theta\, d\theta$$

$$=\frac{800}{\pi^2}\int_0^{\pi}(\pi\theta-\theta^2)\sin n\theta\, d\theta \qquad \text{for } n \text{ odd}$$

Integrating by parts, we obtain

$$10^n B_n=\frac{800}{\pi^2}\left[\frac{2}{n^3}-\frac{2}{n^3}(-1)^n\right]=\frac{1600}{\pi^2 n^3}[1-(-1)^n] \text{ for } n \text{ odd}$$

Therefore,

$$B_n=(1600\times 2)/\pi^2(2n-1)^3 10^{2n-1}$$

Hence the required solution is

$$u(r,\theta)=\frac{3200}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left(\frac{r}{10}\right)^{2n-1}\sin(2n-1)\theta$$

2. The problem reduces to the solution of

$$\text{PDE}: \frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)=0$$

subject to

$$\text{BCs}: T(a)=T_1,\quad \frac{\partial T}{\partial r}+h(T-T_2)=0 \text{ at } r=b$$

Integration of PDE with respect to r gives

$$\frac{\partial T}{\partial r}=\frac{c_1}{r^2} \tag{i}$$

Using the second BC, we obtain

$$h(T_2-T_b)=\frac{c_1}{b^2} \tag{ii}$$

Integrating again, we get from equation (i) the relation

$$T = -\frac{c_1}{r} + c_2 \tag{iii}$$

Using the first BC, we have

$$T_1 = -\frac{c_1}{a} + c_2 \tag{iv}$$

From equations (ii)–(iv),

$$\frac{c_1}{b^2} = hT_2 - hT(b) = hT_2 - h\left(-\frac{c_1}{b} + T_1 + \frac{c_1}{a}\right)$$

which gives

$$c_1 = \frac{h[T_2 - T_1]\,ab^2}{a + hb(b-a)}$$

Finally, the required steady temperature T can be obtained from equations (iii) and (iv) as

$$T = \frac{h(T_1 - T_2)\,ab^2}{r[a + hb(b-a)]} + T_1 + \frac{h(T_2 - T_1)\,ab^2}{a[a + hb(b-a)]}$$

$$= T_1 + \frac{h(T_2 - T_1)\,ab^2}{[a + hb(b-a)]}\left(\frac{1}{a} - \frac{1}{r}\right)$$

3. It has been shown in Example 2.15 that

$$T(r, \theta) = \sum_{n=0}^{\infty} (A_n r^n + B_n / r^{n+1})\, P_n(\cos\theta)$$

using the BCs:

$$T_1 = \sum_{n=0}^{\infty} (A_n a^n + B_n / a^{n+1})\, P_n(\cos\theta)$$

For $n = 0$,

$$T_1 = A_0 + B_0/a \tag{i}$$

For $n = 1$,

$$0 = A_1 a + B_1/a^2 \tag{ii}$$

Also using the second BC, we have

$$T_2(1-\cos\theta) = \sum (A_n b^n + B_n/b^{n+1})\, P_n(\cos\theta)$$

For $n = 0$,

$$T_2 = A_0 + B_0/b$$

For $n = 1$,

$$-T_2 = A_1 b + B_1/b^2$$

Solving (i) and (iii), we get

$$A_0 = T_2 - \frac{T_1 - T_2}{b^2 - ab}, \quad B_0 = \frac{T_1 - T_2}{1/a - 1/b}$$

Similarly, solving (ii) and (iv), we obtain B_1 and A_1. For $n = 2, 3, \ldots$, we get the following homogeneous system

$$A_n a^n + B_n/a^{n+1} = 0,$$

$$A_n b^n + B_n/b^{n+1} = 0$$

which gives $A_n = B_n = 0$ for $n = 2, 3, \ldots$

4. Solve the PDE:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}$$

subject to the BCs:

$$T(a,\theta) = 0°, \quad T(b,\theta) = 100°, \quad 0 \le \theta \le 2\pi$$

The possible solution is

$$T(r) = c_1 \ln r + c_2 \tag{i}$$

Using the BCs, we obtain $0 = c_1 \ln a + c_2$. Therefore, $c_2 = -c_1 \ln a$. Also,

$$100 = c_1 \ln b - c_1 \ln a = c_1 \ln (b/a)$$

which gives

$$c_1 = \frac{100}{\ln(b/a)}, \; c_2 = \frac{-100 \ln a}{\ln (b/a)}$$

Hence the temperature distribution in the annulus is obtained from equation (i) as

$$T = 100 \frac{\ln (r/a)}{\ln (b/a)}$$

5. Following the interior Dirichlet's problem, the solution of Laplace equation in polar coordinates is

$$u(r,\theta) = \frac{A_0}{2} + \sum r^n (A_n \cos n\theta + B_n \sin n\theta) \qquad \text{(i)}$$

Using the prescribed BC, we have

$$u(a,\theta) = \frac{A_0}{2} + \sum a^n (A_n \cos n\theta + B_n \sin n\theta) \qquad \text{(ii)}$$

which is a full-range Fourier series, where

$$A_0 = \frac{1}{\pi}\int_0^{\alpha} c\, d\theta = \frac{c\alpha}{\pi}$$

$$a^n A_n = \frac{1}{\pi}\int_0^{\alpha} c \cos n\theta \, d\theta = \frac{c \sin n\alpha}{n\pi}$$

Similarly,

$$a^n B_n = \frac{1}{\pi}\int_0^{\alpha} c \sin n\theta \, d\theta = -\frac{c}{\pi}\left(\frac{\cos n\alpha}{n} - \frac{1}{n}\right)$$

Thus the temperature distribution at the interior points of the disc can be obtained from equation (i) as

$$u(r,\theta) = \frac{c\alpha}{2\pi} + \sum \frac{r^n}{a^n}\left[\frac{c}{n\pi}\sin n\alpha \cos n\theta + \frac{c \sin n\theta}{n\pi} - \frac{c}{n\pi}\frac{\cos n\alpha \sin n\theta}{1}\right]$$

$$= \frac{c\alpha}{2\pi} + \sum \left(\frac{r}{a}\right)^n \frac{c}{n\pi}[\sin n(\alpha - \theta) + \sin n\theta)$$

or

$$u(r,\theta) = \frac{c}{\pi}\left[\frac{\alpha}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \frac{\sin n(\alpha-\theta) + \sin n\theta}{n}\right]$$

6. Solve the PDE:

$$\nabla^2 \psi = 0 \qquad \text{(i)}$$

subject to the BCs:

$$\theta = \alpha, \quad \psi = 0 \qquad \text{(ii)}$$

$$\theta = \beta, \quad \psi = \sum \alpha_n r^n \qquad \text{(iii)}$$

Using spherical polar coordinates and in view of symmetry of the cone, the solution of equation (i) is

$$\psi = [c_1 r^n + c_2 r^{-(n+1)}][c_3 P_n(\cos\theta) + c_4 Q_n(\cos\theta)] \tag{iv}$$

When $r = 0, \psi$ must be finite, implying $c_2 = 0$. Hence the solution may be of the form

$$\psi = r^n[AP_n(\cos\theta) + BQ_n(\cos\theta)]$$

Using the BC (ii), we get

$$B = -\frac{AP_n(\cos\alpha)}{Q_n(\cos\alpha)}$$

Therefore,

$$\psi(r,\theta) = \sum_{n=0}^{\infty} \frac{r^n A_n}{Q_n(\cos\alpha)}[P_n(\cos\theta)Q_n(\cos\alpha) - P_n(\cos\alpha)Q_n(\cos\theta)]$$

Now, applying the BC (iii), we obtain

$$A_n = \frac{\alpha_n}{[P_n(\cos\beta)Q_n(\cos\alpha) - P_n(\cos\alpha)Q_n(\cos\beta)]}$$

Hence, the required solution is

$$\psi = \sum_{n=0}^{\infty} \alpha_n \left[\frac{P_n(\cos\theta)Q_n(\cos\alpha) - P_n(\cos\alpha)Q_n(\cos\theta)}{P_n(\cos\beta)Q_n(\cos\alpha) - P_n(\cos\alpha)Q_n(\cos\beta)}\right] r^n$$

7. Let

$$\psi = \frac{q}{|\mathbf{r} - \mathbf{r}'|} = \frac{q}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

then

$$\frac{\partial\psi}{\partial x} = -\frac{q(x-x')}{|\mathbf{r} - \mathbf{r}'|^3}$$

$$\frac{\partial^2\psi}{\partial x^2} = -\frac{q}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{3q(x-x')^2}{|\mathbf{r} - \mathbf{r}'|^5}$$

Similarly,

$$\frac{\partial^2\psi}{\partial y^2} = -\frac{q}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{3q(y-y')^2}{|\mathbf{r} - \mathbf{r}'|^5}$$

$$\frac{\partial^2 \psi}{\partial z^2} = -\frac{q}{|\mathbf{r} - \mathbf{r}'|^3} + \frac{3q(z - z')^2}{|\mathbf{r} - \mathbf{r}'|^5}$$

Therefore,

$$\nabla^2 \psi = 0$$

Hence,

$$\psi = \frac{q}{|\mathbf{r} - \mathbf{r}'|}$$

is one of the elementary solutions of the Laplace equation.

8. Assume $\phi = R(r) F(\theta)$, by variables separable method, we have

$$\frac{r^2 R'' + rR'}{R} = -\frac{F''}{F} = \lambda^2$$

Since the velocity components must be periodic with respect to θ, we have

$$F = c \cos \lambda\theta + D \sin \lambda\theta \tag{i}$$

The ODE for R becomes

$$r^2 R'' + rR' - n^2 R = 0, \qquad \lambda = n \tag{ii}$$

This is Euler's homogeneous equation whose solution is

$$R = Ar^n + Br^{-n}$$

For special case when $\lambda = n = 0$, the solution of (i) and (ii) are

$$R = k_1 \ln r + k_2, \qquad F = M\theta + N$$

Therefore, the general solution of the given PDE is

$$\phi = (k_1 \ln r + k_2)(M\theta + N) + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) + \sum_{n=1}^{\infty} r^{-n} (c_n \cos n\theta + D_n \sin n\theta) \tag{iii}$$

Now the BCs

$$v_r = \frac{\partial \phi}{\partial r} = \frac{k_1}{r}(M\theta + N) + \sum_{n=1}^{\infty} nr^{n-1} (A_n \cos n\theta + B_n \sin n\theta) - \sum nr^{-n-1} (c_n \cos n\theta + D_n \sin n\theta)$$

To satisfy the BC at infinity, i.e.

$$\left. \frac{\partial \phi}{\partial r} \right|_{r=\infty} = U_\infty \cos \theta$$

we must choose

$$A_1 = U_\infty, \quad A_n = 0\ (n \ge 2), \quad B_n = 0$$

Similarly, the BC: $\left.\dfrac{\partial \phi}{\partial r}\right|_{r=a} = 0$ gives, for all values of θ, the relations

$$0 = \frac{k_1}{a}(M\theta + N) + U_\infty \cos\theta - \sum_{n=1}^{\infty} n a^{-n-1}(c_n \cos n\theta + D_n \sin n\theta)$$

implying thereby

$$k_1 = 0, \quad c_1 = a^2 U_\infty, \quad c_n = 0\ (n \ge 2), \quad D_n = 0$$

Hence, the required solution is

$$\phi = U_\infty r\left(1 + \frac{a^2}{r^2}\right)\cos\theta + k\theta$$

9. $v(x, y) = x - x^2$ satisfies $\nabla^2 v = -2$, and the boundary condition $v = 0$ along $x = 0$ and $x = 1$. Let $\psi = v + \omega$. Now construct a function ω such that $\nabla^2 \omega = 0$, satisfying

$$\omega(0, y) = \omega(1, y) = 0, \qquad 0 \le y \le 1$$

$$\omega(x, 0) = \omega(x, 1) = x^2 - x, \qquad 0 \le x \le 1$$

Following Example 2.21, it can be shown that

$$\psi(x, y) = x - x^2 - \frac{8}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin[2n-1)\pi x]}{(2n-1)^3}\left\{\frac{\sinh[(2n-1)\pi y] + \sinh[(2n-1)\pi(1-y)]}{\sinh(2n-1)\pi}\right\}$$

10. Since u is a function of r and θ alone, in cylindrical polar coordinates, the solution of Laplace equation satisfying $\operatorname*{Lt}_{r\to\infty} u(r,\theta) = 0$ is given by

$$u(r,\theta) = \sum_n r^{-n}(A_n \cos n\theta + B_n \sin n\theta)$$

The given boundary condition

$$\left.\frac{\partial u}{\partial r}\right|_{r=a} = -\sum n a^{-n-1}(A_n \cos n\theta + B_n \sin n\theta) = \frac{u_0}{a}\sin 3\theta$$

will be satisfied only if $A_n = 0$ for all n. Thus, we have

$$\frac{u_0}{a}\sin 3\theta = -\sum n a^{-n-1} B_n \sin n\theta$$

This implies that all the constants B_n will be zero, except, B_3 which is given by

$$-3a^{-4}B_3 \sin 3\theta = \frac{u_0}{a} \sin 3\theta$$

i.e.,

$$B_3 = -\frac{u_0}{3} a^3$$

Therefore, the required solution is

$$u(r,\theta) = -\frac{u_0}{3}\left(\frac{a}{r}\right)^3 \sin 3\theta \;(a \le r \le \infty)$$

12. $u = e^{(1+\lambda)y}(Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x})$

Chapter 3

1. Mathematically, the problem is described as

$$\text{PDE}: T_t = \alpha^2 T_{xx}$$

$$\text{BCs}: T_x = 0 \text{ at } x = 0, \qquad x = l \text{ for } t \ge 0$$

$$\text{IC}: T(x,0) = lx - x^2, \qquad 0 \le x \le l$$

Using the variables separable method, the physically meaningful non-trivial general solution is

$$T(x,t) = (A \cos \lambda x + B \sin \lambda x)\, e^{-\alpha^2\lambda^2 t} \tag{i}$$

$$\frac{\partial T(x,t)}{\partial x} = (-A\lambda \sin \lambda x + B\lambda \cos \lambda x)\, e^{-\alpha^2\lambda^2 t} \tag{ii}$$

The first BC gives from (ii), $B = 0$. The second BC gives from (ii) $(A\lambda \sin \lambda l)\, e^{-\alpha^2\lambda^2 t} = 0$, implying $\sin \lambda l = 0$ or $\lambda = n\pi/l,\ n = 1, 2, \ldots$

Hence the required solution is

$$T(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{l} x\right) \exp(-\alpha^2 n^2 \pi^2 t/l^2) \tag{iii}$$

Finally, use the IC:

$$lx - x^2 = \sum A_n \cos\left(\frac{n\pi}{l} x\right)$$

where

$$A_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\left(\frac{n\pi}{l}x\right)dx$$

2. Solve

$$\text{PDE}: T_t = \alpha^2 T_{xx}, \quad 0 \le x \le a$$

$$\text{BCs}: T(0, t) = 0, \quad T(a, t) = 0$$

$$\text{IC}: T(x, 0) = f(x)$$

The general solution is

$$T(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\alpha^2\lambda^2 t} \tag{i}$$

The first BC gives $A = 0$; the second BC gives

$$\sin\lambda a = 0, \text{ implying } \lambda = \frac{n\pi}{a}, \quad n = 1, 2, \ldots$$

Therefore,

$$T(x,t) = \sum B_n \sin\left(\frac{n\pi}{a}x\right)e^{-\alpha^2\lambda^2 t} \tag{ii}$$

Now the IC:

$$f(x) = \sum B_n \sin\left(\frac{n\pi}{a}x\right)$$

Hence the required solution is

$$T(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{a}x\right)\exp\left[-\alpha^2\left(\frac{n\pi}{a}\right)^2 t\right]$$

where

$$B_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi}{a}x\right)dx$$

4. The general solution is

$$T(x,t) = (A\cos\lambda x + B\sin\lambda x)\, e^{-\lambda^2 t}$$

Using the BCs and the superposition principle, the solution is

$$T(x,t) = \sum B_n\, e^{-n^2\pi^2 t}\sin(n\pi x)$$

Now use the IC:

$$B_n = 2\int_0^1 T(x, 0) \sin(n\pi x)\, dx$$

$$= 2\left[\int_0^{1/2} 2x \sin(n\pi x)\, dx + \int_{1/2}^1 2(1-x)\sin(n\pi x)\, dx\right]$$

$$= \frac{4}{n\pi}\int_0^{1/2} x d\,[-\cos(n\pi x)] + \frac{4}{n\pi}\int_{1/2}^1 (1-x)\, d\,[-\cos(n\pi x)]$$

Integration by parts gives

$$B_n = -\frac{4}{n\pi}\left[\left\{-\frac{\sin(n\pi x)}{n\pi}\right\}_0^{1/2} + \left\{\frac{\sin(n\pi x)}{n\pi}\right\}_{1/2}^1\right] = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Hence the required solution is

$$T(x,t) = \frac{8}{\pi^2}\sum \frac{1}{n^2}\sin\frac{n\pi}{2}\sin(n\pi x)\, e^{-n^2\pi^2 t}$$

5. Let $\theta(r,t) = R(r)\,T(t)$. Substituting into the given PDE, we get

$$\frac{T'}{T} = -\nu p^2 \tag{i}$$

$$R'' + \frac{1}{r}R' + p^2 R = 0, \quad -p^2 = \text{const}. \tag{ii}$$

Integration of (i) gives

$$T = c_1 \exp(-\nu p^2 t) \tag{iii}$$

Equation (ii) is a Bessel's equation of order zero whose solution is

$$R = c_2 J_0(pr) + c_3 Y_0(pr)$$

Thus the general solution is

$$\theta(r,t) = [AJ_0(pr) + BY_0(pr)]\exp(-\nu p^2 t) \tag{iv}$$

In view of the first BC, the solution is

$$\theta(r,t) = AJ_0(pr)\exp(-\nu p^2 t)$$

The second BC gives $J_0(pa) = 0$ which has an infinite number of zeros, say, $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ Therefore,

$$p = \frac{\lambda_n}{a}, \quad n = 1, 2, \ldots$$

Using the principle of superposition, we get the solution

$$\theta(r,t) = \sum A_n J_0\left(\frac{\lambda_n}{a} r\right) \exp\left(\frac{-\nu \lambda_n^2}{a^2} t\right)$$

Now using the IC, we get

$$\frac{P}{4\mu}(a^2 - r^2) = \sum A_n J_0\left(\frac{\lambda_n}{a} r\right) \tag{v}$$

which is a Fourier-Bessel series. Multiplying both sides by $rJ_0\,(\lambda_m r/a)$ and integration from 0 to a and using the orthogonality property.

$$\int_0^a rJ_0(\lambda_m r)\, J_0(\lambda_n r)\, dr = \begin{cases} 0 & \text{if } n \neq m \\ \dfrac{a^2}{2} J_1^2(\lambda_m a) & \text{if } n = m \end{cases}$$

we obtain

$$A_m = \frac{P}{2\mu a^2 J_1^2(\lambda_m)} \int_0^a r(a^2 - r^2)\, J_0\left(\frac{\lambda_m r}{a}\right) dr$$

which on evalution reduces to

$$A_m = \frac{2Pa^2}{\lambda_m^3\, \mu\, J_1(\lambda_m)}$$

Hence the required solution is

$$\theta(r,t) = \sum_{m=1}^{\infty} \frac{2Pa^2}{\mu\, \lambda_m^3\, J_1(\lambda_m)} J_0\left(\frac{\lambda_m}{a} r\right) \exp\,(-\nu \lambda_m^2\, t/a^2)$$

6. In view of spherical symmetry, the governing PDE:

$$T_t = c^2\left(T_{rr} + \frac{2}{r} T_r\right) \tag{i}$$

Setting $v = rT$, we have

$$v_t = rT_t, \quad T_r = \left(\frac{v}{r}\right)_r = \frac{v_r r - v}{r^2}$$

$$T_{rr} = \frac{\{(v_r\, r)_r - v_r)\} r^2 - 2r\,(v_r r - v)}{r^4}$$

Equation (i) reduces to

$$v_t = c^2 v_{rr} \tag{ii}$$

The corresponding boundary and initial conditions are

$$v(0,t) = v(R,t) = 0 \tag{iii}$$

$$v(r,0) = rf(r) \tag{iv}$$

respectively. The possible solution of (ii) using BCs is

$$v(r,t) = \sum B_n \sin\left(\frac{n\pi}{R}r\right) \exp(-c^2 n^2 \pi^2 t/R^2) \tag{v}$$

Now using the IC, we get

$$rf(r) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{R}r\right)$$

which is a half-range Fourier series of $rf(r)$. Hence

$$B_n = \frac{2}{R}\int_0^R rf(r) \sin\frac{n\pi r}{R}\, dr \tag{vi}$$

The required solution is

$$T(r,t) = \frac{v(r,t)}{r} = \frac{1}{r}\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{R}r\right) \exp\left(-\frac{c^2 n^2 \pi^2 t}{R^2}\right) \tag{vii}$$

8. The governing differential equation is

$$T_t = k\left(T_{rr} + \frac{1}{r}T_r\right), \quad 0 \le r \le a, t \ge 0$$

$$\text{BC:} \left.\frac{\partial T}{\partial r}\right|_{r=a} = 0 \quad \text{(thermally insulated)}$$

$$\text{IC:} T(r,0) = f(r)$$

The required solution is

$$T(r,t) = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r) \exp(-k\lambda_n^2 t)$$

where

$$A_n = \frac{\int_0^a rf(r) J_0(\lambda_n r)\, dr}{\int_0^a rJ_0^2(\lambda_n r)\, dr}$$

9. The formal solution is found to be

$$u(\mathbf{r},t) = \iiint\limits_{R_3} k(\mathbf{r}-\mathbf{r}',\alpha t)\, f(\mathbf{r}')\, d\mathbf{r}'$$

where

$$k(\mathbf{r},t) = (4\pi\alpha t)^{-3/2} \exp\left(-\frac{|\mathbf{r}|^2}{4\alpha t}\right)$$

Here, the function $k(\mathbf{r},t)$ is called a fundamental solution of the diffusion equation.

10. Using variables separable method, the only solution for the given PDE, which is physically acceptable is of the form

$$\theta(x,t) = (c_1 \cos\lambda x + c_2 \sin\lambda x)\, e^{-\lambda^2 t} \tag{1}$$

Applying the BC: $x = 0, \theta = 0$, Eq. (1) becomes

$$\theta(x,t) = c_1 \cos\lambda x\, e^{-\lambda^2 t} \tag{2}$$

Applying the second BC: $x = a, \theta = 0$, we find $c_1 = 0$.
Therefore, the possible solution is

$$\theta(a,t) = c_2 \sin\lambda \alpha e^{-\lambda^2 t} = 0 \text{ for all } t$$

Since $c_2 \neq 0, \sin\lambda a = 0$, implies $\lambda a = n\pi$, therefore, $\lambda = n\pi/a$, where n is any integer. Hence, the possible solution is

$$\theta(x,t) = b_n \sin\left(\frac{n\pi x}{a}\right) e^{\frac{-n^2\pi^2}{a^2}t} \tag{3}$$

where $b_n = c_2$. Adding all possible solutions the most general solution satisfying the given BCs: is

$$\theta(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right) \exp\left(\frac{-n^2\pi^2}{a^2}t\right) \tag{4}$$

Finally, we have to satisfy the IC: $\theta(x,0) = \theta_0$ (constant). Therefore,

$$\theta_0 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{a}\right), \quad \text{which is a half range}$$

Fourier sine series and hence, the required solution is given by Eq. (4) where b_n is given by

$$b_n = \frac{2}{a}\theta_0 \int_0^a \sin\left(\frac{n\pi x}{a}\right)dx \tag{v}$$

Chapter 4

1. From D'Alembert's method, the required solution is of the form

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$$

Given

$$u = f(x) = u_0 \sin\frac{\pi x}{l}$$

it follows that

$$u(x,t) = \frac{u_0}{2}\left[\sin\frac{\pi(x+ct)}{l} + \sin\frac{\pi(x-ct)}{l}\right]$$

Using

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

we have

$$u(x,t) = u_0 \sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}$$

2. The general solution is given by Eq. (4.37). Using the IC: $u_t(x,0) = 0$, we have $B_n = 0$. The use of another IC: $u(x,0) = 10\sin(\pi x/l)$ yields

$$10\sin\frac{\pi x}{l} = \sum A_n \sin\frac{n\pi x}{l}$$

implying thereby $A_1 = 10$, $A_2 = A_3 = \cdots = 0$. Hence the required solution is

$$u(x,t) = 10\cos\frac{\pi ct}{l}\sin\frac{\pi x}{l}$$

3. The general solution is given by Eq. (4.35):

$$u = 0,\ \ x = 0 \Rightarrow c_1 = 0,\ \ u_t = 0 \text{ gives } c_4 = 0$$

The solution reduces to: $u(x,t) = c_3 \sin \lambda x \cos c\lambda t$. $u = 0$ at $x = \pi \Rightarrow \lambda = n$. Thus the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \cos (nct)$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx = \frac{2}{n}(-1)^{n+1}$$

4. $u(x,t) = \dfrac{bl}{12c\pi}\left[9 \sin \dfrac{\pi x}{l} \sin \dfrac{c\pi t}{l} - \sin \dfrac{3\pi x}{l} \sin \dfrac{3c\pi t}{l}\right]$

5. From Eq. (4.37), the possible solution is

$$u(x,t) = \sum \sin \frac{n\pi x}{l}\left[A_n \cos\left(\frac{n\pi}{l}ct\right) + B_n \sin\left(\frac{n\pi}{l}ct\right)\right]$$

Applying the IC: $u_t(x,0) = 0$, the possible solution is of the form

$$u(x,t) = \sum B_n \left(\frac{n\pi c}{l}\right) \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} = \sum_{n=1}^{\infty} B_n' \cos \frac{n\pi ct}{l} \sin \frac{n\pi x}{l} \tag{i}$$

To find the coefficients B_n' we use the IC: $u(x,0) = f(x)$ to get

$$f(x) = \sum_{n=1}^{\infty} B_n' \sin \frac{n\pi x}{l}$$

where

$$B_n' = \frac{2}{l}\int_0^{l} f(x) \sin \frac{n\pi x}{l} dx$$

This form of the solution does not convey the fact that the initial disturbance will be propagated as some sort of wave motion. But if we rewrite equation (i) in the alternative form

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n' \left[\sin\left\{\frac{n\pi}{l}(x+ct)\right\} + \sin\left\{\frac{n\pi}{l}(x-ct)\right\}\right],$$

it gives a feeling that $u(x,t)$ is a sum of two travelling waves, one moving to the right and the other moving to the left with speed c.

6. Since u is a function of r and t only, the given equation reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Let $u = \phi/r$, the governing equation further simplifies to the form

$$\frac{\partial^2 \phi}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}$$

the solution of which is

$$\phi = f(r - ct) + g(r + ct)$$

Hence the general solution of the given equation is

$$u = [f(r - ct) + g(r + ct)]/r$$

7. $u(x,t) = 5 + x^2 t + \frac{1}{3}c^2 t^3 + \frac{1}{2t^2}(e^{x+ct} + e^{x-ct} - 2e^x).$

8. $u(x,t) = \sin x \cos ct + (e^t - 1)(xt + x) - xte^t.$

9. We notice that one boundary condition is a function of time. Using D'Alemberts method, the solution of the wave equation is $u(x,t) = \psi(x + ct) + \phi(x - ct)$. The boundary conditions give $\psi(x) + \phi(x) = 0$ and $\psi(x) - \phi(x) = A$ (the constant of integration). Their solution gives $\psi(x) = A/2$, $\phi(x) = -A/2$ when their arguments are positive. Now applying the IC, we get $\phi(-ct) = \sin t - \psi(ct)$. But for $t > 0, \psi(ct) = A/2$. Therefore, $\phi(-ct) = \sin t - A/2$ for $t > 0$. Now, when $x - ct < 0$ or $t > x/c$, we have

$$\phi(x - ct) = \phi\left[-c\left(t - \frac{x}{c}\right)\right] = \sin\left(t - \frac{x}{c}\right) - A/2$$

Combining these results, we get

$$u(x,t) = \begin{cases} 0 & \text{when } x < ct \\ \sin\left(t - \frac{x}{c}\right) & \text{when } x > ct \end{cases}$$

10. Similar to Example 4.9 with $\varepsilon = 1, \phi = y$ etc.

11. The vibration of the string is described by

$$\text{PDE}: y_{tt} = c^2 y_{xx},\ 0 \le x \le L,\ t > 0$$

where $c^2 = \tau/\rho$, subject to

$$\text{BCs}: y(0,t) = y(L,t) = 0 \text{ for all } t \text{ and}$$

$$\text{ICs}: y_t(x,0) = 0\ .$$

The initial displacement of the string is given by

$$y(x,0) = \begin{cases} 2sx/L, & 0 < x \le L/2 \\ 2s(L - x)/L, & L/2 \le x \le L \end{cases}$$

Using variables separable method, the displacement of the string at any time is given by

$$y(x, t) = \sum_{n=1}^{\infty} \frac{8s}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi ct}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

where $c^2 = \tau/\rho$.

or

$$y(x, t) = \frac{8s}{\pi^2}\left[\frac{1}{l^2}\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi ct}{L}\right) - \frac{1}{3^2}\sin\left(\frac{3\pi x}{L}\right) \times \cos\left(\frac{3\pi ct}{L}\right) + \frac{1}{5^2}\sin\left(\frac{5\pi x}{L}\right)\cos\left(\frac{5\pi ct}{L}\right) - \cdots\right]$$

12. The first term in the solution of problem 11 is the normal mode corresponding to the lowest normal frequency often called Fundamental frequency. That is,

$$\frac{8s}{\pi^2}\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi ct}{L}\right).$$

Let f_1 be the corresponding frequency, then

$$2\pi f_1 = \pi c/L \quad \text{or} \quad f_1 = c/2L = \frac{1}{2L}\sqrt{\tau/\rho}$$

Since $-1 \le \cos \le 1$, the mode is such that, the string oscillates as shown in Fig. 4.9(a) from the heavy to the dotted curve and back to the heavy curve and so on.

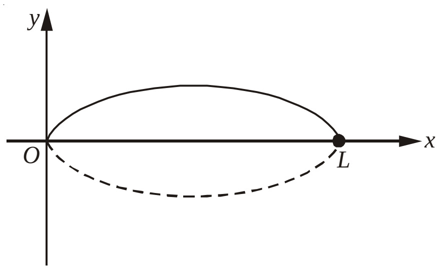

Fig. 4.9(a) First normal mode.

The next higher frequency is given by the mode which corresponds to the term

$$\frac{8s}{9\pi^2}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3\pi ct}{L}\right)$$

Let f_3 be the corresponding frequency, then

$$2\pi f_3 = 3\pi c/L \quad \text{or} \quad f_3 = \frac{3c}{2L} = \frac{3}{2L}\sqrt{\tau/\rho}$$

The mode in this case is depicted in Fig. 4.9(b).

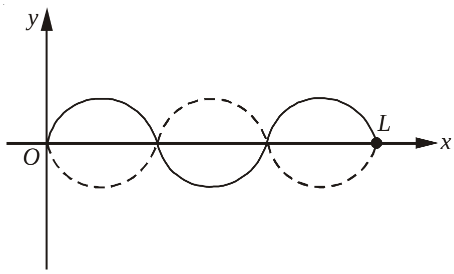

Fig. 4.9(b) The third normal mode.

Similarly, we find the higher normal frequencies

$$f_5 = \frac{5}{2L}\sqrt{\tau/\rho},\ f_7 = \frac{7}{2L}\sqrt{\tau/\rho} \quad \text{and so on.}$$

However, we observe that the even frequencies f_2, f_4 etc. are absent. But, it should be noted that both even and odd frequencies will be present in a general displacement.

13. The governing PDE: $y_{tt} = c^2 y_{xx},\ 0 \le x \le L$ subject to the BCs:

$$\left.\frac{\partial y}{\partial x}\right|_{x=0} = 0, \quad y(L, t) = 0$$

and the ICs:

$$y(x, 0) = y_0 \cos(\pi x/2L), \left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$$

The time-dependent displacement of the string is found to be

$$y(x, t) = y_0 \cos\left(\frac{\pi x}{2L}\right)\sin\left(\frac{c\pi t}{2L}\right).$$

14. Let $u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y)$.
Then,

$$u_x = f' + g', \qquad u_{xx} = f'' + g'' \tag{1}$$

$$u_y = i\alpha f' - i\alpha g', \qquad u_{yy} = -\alpha^2 f'' - \alpha^2 g'' \tag{2}$$

$$f_t = -v f' - v g', \qquad f_{tt} = v^2 f'' + v^2 g'' \tag{3}$$

Now, substituting expressions (1), (2) and (3) in the given PDE, we find that

$$f'' + g'' - \alpha^2 f'' - \alpha^2 g'' = \frac{v^2}{c^2}(f'' + g'')$$

or

$$(1-\alpha^2)\, f'' + (1-\alpha^2)\, g'' = \frac{v^2}{c^2}(f'' + g'')$$

is satisfied, provided

$$1-\alpha^2 = v^2/c^2 \quad \text{or} \quad \alpha^2 = 1 - v^2/c^2.$$

15. The D'Alembert's solution for the IVP defined as

$$\text{PDE}: u_{tt} = c^2 u_{xx}, \qquad t > 0, \quad -\infty < x < \infty$$

$$\text{ICs}: u(x,0) = \eta(x), \qquad u_t\,(x,0) = v(x) \text{ is}$$

$$u(x,t) = \frac{1}{2}[\eta(x+ct) + \eta(x-ct)] = \frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)\, d\xi$$

In the given problem, $v(x) = 0$, $c = 2$ and $\eta\,(x) = x.$ Therefore, the solution to the given problem is

$$u(x,t) = \frac{1}{2}[\eta\,(x+2t) + \eta\,(x-2t)] = \frac{1}{2}[x + ct + x - ct] = x$$

Hence, the correct choice is (A).

16. D'Alembert's solution is given as

$$u\,(x,t) = \frac{1}{2}[\eta\,(x+t) + \eta\,(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \cos\xi\, d\xi$$

$$= \frac{1}{2}[\sin\,(x+t) + \sin\,(x-t)] + \frac{1}{2}[\sin\,(x+t) - \sin\,(x-t)] = \sin\,(x+t)$$

Therefore,

$$u\left(\frac{\pi}{2}, \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{2} + \frac{\pi}{6}\right) = \sin\left(\frac{2}{3}\pi\right) = \sin 120 = \sqrt{3}/2$$

Hence, the correct choice is (A).

17. $u(x,t) = \int_0^t \left[\sum a_n(\tau)\, e^{-n^2(t-\tau)} \sin nx\right] d\tau$

Chapter 5

1. Let $\delta(x,y,z;\xi,\eta,\zeta)$ be a Dirac δ-function defined by the equation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta,\zeta)\,\delta\,(x,y,z;\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta = f(x,y,z)$$

which is valid for all continuous functions $f(x, y, z)$ having compact support in the (x, y, z)-space. If $\delta(x-\xi), \delta(y-\eta), \delta(z-\zeta)$ are three one-dimensional δ-functions, then we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta, \zeta)(x-\xi)(y-\eta)(z-\zeta)\, d\xi\, d\eta\, d\zeta = f(x, y, z)$$

Hence comparing the above two equations, we get

$$\delta(x, y, z; \xi, \eta, \zeta) = \delta(x-\xi)\,\delta(y-\eta)\,\delta(z-\zeta)$$

2. Suppose we transform from cartesian coordinates x, y to curvilinear coordinates ξ, η satisfying the realtions $x = f(\xi, \eta)$; $y = g(\xi, \eta)$, where f and g are single-valued continuously differentiable functions. Also suppose that $\xi = \beta_1, \eta = \beta_2$, corresponding to $x = \alpha_1, y = \alpha_2$. Then the equation

$$\iint \phi(x, y)\,\delta(x-\alpha_1)\,\delta(y-\alpha_2)\,dx\,dy = \phi(\alpha_1, \alpha_2)$$

becomes

$$\iint \phi(f, g)\,\delta[f(\xi, \eta)-\alpha_1]\,\delta[g(\xi, \eta)-\alpha_2]\,|J|\,d\xi\,d\eta = \phi(\alpha_1, \alpha_2)$$

where

$$J = \begin{vmatrix} f_\xi & f_\eta \\ g_\xi & g_\eta \end{vmatrix} = \frac{\partial(f, g)}{\partial(\xi, \eta)}$$

The above equation indicates

$$\delta[f(\xi, \eta)-\alpha_1]\,\delta[g(\xi, \eta)-\alpha_2]\,|J| = \delta(x-\alpha_1)\,\delta(y-\alpha_2)\,|J| = \delta(\xi-\beta_1)\,\delta(\eta-\beta_2)$$

In polar coordinates, let $x = r\cos\theta, y = r\sin\theta$. Then $x = f(r, \theta), y = g(r, \theta)$. Thus,

$$|J| = \begin{vmatrix} f_r & f_\theta \\ g_r & g_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Hence

$$\delta(x-x_0)\,\delta(y-y_0) = \frac{\delta(r-r_0)\,\delta(\theta-\theta_0)}{r}$$

3. Let $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$. That is, $x = f(r, \theta, \phi)$, $y = g(r, \theta, \phi)$, $z = h(r, \theta)$. Then

$$|J| = \begin{vmatrix} f_r & f_\theta & f_\phi \\ g_r & g_\theta & g_\phi \\ h_r & h_\theta & h_\phi \end{vmatrix} = r^2\sin\theta$$

It therefore, follows that

$$\delta(x-x_1)\,\delta(y-y_1)\,\delta(z-z_1)=\frac{\delta(r-r_1)\,\delta(\theta-\theta_1)\,\delta(\phi-\phi_1)}{r^2\sin\theta}$$

5. $u(x,y)=\dfrac{y}{\pi}\displaystyle\int_{-\infty}^{\infty}\frac{f(x')\,dx'}{(x-x')^2+y^2}.$

6. In the two-dimensional case, we can place singularity at (x, y) and consider its image over one side, again reflect the image over the image of that side, and so on until we return, after an even number of reflections to the original domain. Let $(-x', y')$ be the image point of (x', y') in the y-axis and let $(-x', -y'), (x', -y')$ be the image of the above two points in the x-axis. Then G may be constructed by means of three suitably placed unit charges, a positive charge at $(-x', y')$ and negative charges at $(-x', -y')$ and $(x', -y')$. Thus

$$G(x,y)=\frac{1}{4\pi}\ln\left[\frac{\{(x-x')^2+(y-y')^2\}\{(x-x')^2+(y+y')^2\}}{\{(x+x')^2+(y-y')^2\}\{(x+x')+(y+y')^2\}}\right]$$

This function satisfies $\nabla^2 G=0$ except at the source point (x', y') and $G = 0$ on $x = 0$; also, $\partial G/\partial n=0$ on $y = 0$.

Chapter 6

1. (i) $(s^2-a^2)/(s^2+a^2)^2$; (ii) $2(s+1)/(s^2+2s+1)^2$; (iii) $4(s-1)/(s^2-2s+5)^2$;
   (iv) $\dfrac{1}{(s+a)}+\dfrac{2}{(s+a)^2}+\dfrac{3}{(s+a)^3}.$

2. (i) $(1-e^{-s\tau})/s^2\tau.$

3. (i) $\ln\left(\dfrac{s-1}{s}\right)$; (ii) $\dfrac{1}{2}[\ln(s^2+4)]_s^\infty-[\tan^{-1}(s/3)]_s^\infty.$

4. $\dfrac{1}{1-e^{-2s}}\left[-\dfrac{e^{-s}}{s}-\dfrac{1}{s^2}(e^{-s}-1)\right]$

5. $(1-e^{-6s})/s^2(1+e^{-6s})$

6. $(1-e^{-bs})/s(1+e^{-bs})$

7. $e^{-2\sqrt{s}}/s$

8. (i) $(1-s/\sqrt{s^2+1})$; (ii) $\dfrac{1}{(s^2+1)^{3/2}}$

9. (i) $\dfrac{1}{2}-e^{-t}+\dfrac{1}{2}e^{-2t}$; (ii) $e^{2t}(t+t^2)$; (iii) $1-\cos t$;

(iv) $\dfrac{1}{3}e^t+3te^t-\dfrac{1}{3}e^{-2t}$; (v) $\dfrac{e^{-3t}}{4}[t\sin 2t]$; (vi) $\dfrac{1}{2}-2e^{-t}+\dfrac{5}{2}e^{-2t}$

10. (i) $(e^{-bt}-e^{-at})/t$; (ii) $-\dfrac{\sin kt}{t}$;

(iii) $\dfrac{2e^t-2\cos t}{t}$; (iv) $\dfrac{2\cos 2t-e^{4t}}{t}$.

12. (i) $\dfrac{at-\sin at}{a^3}$; (ii) $\dfrac{e^{at}}{\sqrt{a}}\,erf\,(\sqrt{at})$;

(iii) $\dfrac{t}{64}(\sin 2t-2t\cos 2t)$.

13. $L^{-1}[e^{-\pi s}/(s^2+1)]=\begin{cases}0, & t<\pi\\ -\sin t, & t\ge\pi\end{cases}$

14. (i) $\dfrac{1}{3}\left[e^{-t}-e^{-t/2}\left(-\sqrt{3}\sin\dfrac{\sqrt{3}t}{2}+\cos\dfrac{\sqrt{3}t}{2}\right)\right]$;

(ii) $2e^t+\sin t-2\cos t$

15. $1+\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{(2n-1)}\cos[(n-1/2)\pi x]\,e^{-(n-1/2)^2\pi^2 t}$

16. $y=-7e^t+4e^{2t}+4te^{2t}$

17. $y=\dfrac{1}{2}(\cos kt+\cosh kt)$

18. $y=e^{-t^2/2}$

19. $x=e^{-t}-1,\ y=2-e^{-t}$

20. $x=3e^{-t}\sin t-e^{-t}\cos t$;

$y=e^{-t}[2\cos t-1-\sin t]$

21. $x = \frac{e^t}{4} + \frac{e^{-t}}{4} - \frac{\cos t}{2}, \quad y = 1 - \frac{e^t}{4} - \frac{e^{-t}}{4} - \frac{\cos t}{2}$

22. The Laplace transform of the given PDE yields

$$\frac{d^2H}{dx^2} = \frac{s}{k}H$$

The Laplace transform of the BCs gives

$$H = \frac{1}{s(s+1)} \quad \text{when} \quad x = 0$$

$$H = 0 \quad \text{when} \quad x = \pi$$

Hence the solution of the above ODE is

$$H = \frac{1}{s(s+1)} \frac{\sinh\left[\sqrt{\frac{s}{k}}(\pi - x)\right]}{\sinh\left(\sqrt{\frac{s}{k}}x\right)}$$

Taking the inverse Laplace transform using inversion formula, we get

$$\theta(x,t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}\sinh\left[\sqrt{\frac{s}{k}}(\pi - x)\right]}{s(s+1)\sinh\left(\sqrt{\frac{s}{k}}\pi\right)} ds$$

The integrand has simple poles at $s = 0, -1$ and $-n^2k, n = 1, 2, \ldots$

23. After taking the Laplace transform of the given PDE and using the IC, we obtain

$$\frac{d^2U}{dx^2} - \frac{s}{3}U(x,s) = -10\cos 5x$$

Its general solution is

$$U(x,s) = Ae^{\sqrt{s/3}x} + B\exp\left(-\sqrt{\frac{s}{3}}x\right) + \frac{30\cos 5x}{75 + s}$$

Taking the Laplace transform of the BCs and utilizing them into the above solution, we have only the trivial solution $A = B = 0.$ Thus we get

$$U(x,s) = \frac{30\cos 5x}{75 + s}$$

its inverse Laplace transform yields

$$u(x,t) = L^{-1}\left[\frac{30}{75+s}\cos 5x; t\right] = 30e^{-75t}\cos 5x$$

24. Taking the Laplace transform of the given PDE and using the ICs, we obtain

$$\frac{d^2U}{dx^2} - \frac{s^2}{c^2}U(x,s) = 0$$

Its general solution is

$$U(x,s) = A\cosh\left(\frac{s}{c}x\right) + B\sinh\left(\frac{s}{c}x\right)$$

Taking the Laplace transform of the BCs, we get

$$U(0,s) = 0, \quad U_x(l,s) = \frac{P}{Es}$$

Using these expressions, the general solution reduces to

$$U(x,s) = \frac{PC}{Es^2\cosh\left(\frac{s}{c}l\right)}\sinh\left(\frac{s}{c}x\right)$$

Taking inverse Laplace transform and using complex inversion formula, we get

$$u(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{PCe^{st}}{Es^2\cosh\left(\frac{s}{c}l\right)}\sinh\left(\frac{s}{c}x\right)ds$$

$$= \frac{Pl}{E}\left[\frac{x}{l} - \frac{8}{\pi^2}\sum_{n=0}^{\infty}(-1)^n(2n+1)^{-2}\sin(s_n x)\cos(s_n ct)\right]$$

where

$$s_n = \frac{2n+1}{2l}\pi$$

25. Taking the Laplace transform of the given PDE, we obtain

$$s^2\,U(x,s) - su(x,0) - u_t(x,0) = c^2\frac{d^2U}{dx^2}$$

Using the initial conditions, we get

$$\frac{d^2U}{dx^2} - \frac{s^2}{c^2}U = -\frac{1}{c^2}$$

whose general solution is

$$U(x, s) = A \exp\left(-\frac{s}{c}x\right) + B \exp\left(\frac{s}{c}x\right) + \frac{1}{s^2}$$

As $x \to \infty$, the transform should be bounded, which is possible only if $B = 0$. Also, using the Laplace transform of the BC, i.e., $U(0, s) = 0$, we get $A = -1/s^2$. Hence,

$$U(x, s) = \frac{1}{s^2} - \frac{1}{s^2}\exp\left(-\frac{s}{c}x\right)$$

Taking the inverse Laplace transform of the above equation, we obtain

$$u(x, t) = t - \left[\left(t - \frac{x}{c}\right) H\left(t - \frac{x}{c}\right)\right]$$

where H is the Heaviside unit function.

26. $$\phi(x, t) = \begin{cases} f\left(t - \dfrac{x}{a}\right), & \text{if } t > \dfrac{x}{a} \\ 0, & \text{if } t < \dfrac{x}{a} \end{cases}$$

where $a = 1/\sqrt{LC}$.

Chapter 7

1. $$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2)\, e^{i\alpha x} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \frac{1 - x^2}{i\alpha}\, d(e^{i\alpha x})$$

$$= -\frac{2}{\sqrt{2\pi}\alpha^2}\left[2\left(\frac{e^{i\alpha} + e^{-i\alpha}}{2}\right) - \frac{2}{\alpha}\left(\frac{e^{i\alpha} - e^{-i\alpha}}{2i}\right)\right]$$

$$= -2\sqrt{\frac{2}{\pi}}\left(\frac{\alpha\cos\alpha - \sin\alpha}{\alpha^3}\right)$$

Then,

$$f(x) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}}\left(\frac{\alpha\cos\alpha - \sin\alpha}{\alpha^3}\right) e^{-i\alpha x} dx$$

$$= \begin{cases} 1 - x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

Therefore,

$$-\int_{-\infty}^{\infty} \frac{\alpha \cos \alpha - \sin \alpha}{\alpha^3} \cos \alpha x \, d\alpha = \begin{cases} \frac{\pi}{2}(1-x^2), & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

Set $x = \frac{1}{2}$; we then have

$$-\int_{-\infty}^{\infty} \frac{\alpha \cos \alpha - \sin \alpha}{\alpha^3} \cos \frac{\alpha}{2} \, d\alpha = \frac{\pi}{2}\left(1 - \frac{1}{4}\right) = \frac{3}{8}\pi$$

Therefore,

$$\int_0^{\infty} \frac{x \cos x - \sin x}{x^3} \cos \frac{x}{2} \, dx = -\frac{3\pi}{6}$$

2. $$F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \sin \alpha x \, dx = \frac{\alpha}{1+\alpha^2} \sqrt{\frac{2}{\pi}}$$

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \cos \alpha x \, dx = \frac{1}{1+\alpha^2} \sqrt{\frac{2}{\pi}}$$

3. $$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \, e^{-\alpha}$$

4. From Examples 7.3 and 7.4, we have

$$G_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos \alpha x \, dx = \sqrt{\frac{2}{\pi}} \frac{\alpha}{a^2 + \alpha^2}$$

Also,

$$F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \alpha x \, dx = \sqrt{\frac{2}{\pi}} \int_0^{\lambda} \cos \alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin \alpha x}{\alpha}$$

using Parseval's relation

$$\int_0^{\infty} F_c(\alpha) \, G_c(\alpha) \, d\alpha = \int_0^{\lambda} e^{-ax} dx$$

or

$$\frac{2}{\pi}\int_0^\infty \frac{a \sin \alpha\lambda}{\alpha\,(a^2+\alpha^2)}\,d\alpha = \int_0^\lambda e^{-ax}dx$$

Therefore,

$$\int_0^\infty \frac{\sin \alpha\lambda}{\alpha\,(a^2+\alpha^2)}\,d\alpha = \frac{\pi}{2}\left(\frac{1-e^{-\lambda a}}{a^2}\right)$$

7. Taking the Fourier transform of PDE, we get

$$\frac{dU}{dt}+\alpha^2 U = \frac{1}{\sqrt{2\pi}}\,\delta(t)$$

After applying the transform of IC

$$U\,(\alpha, 0) = \frac{1}{\sqrt{2\pi}}$$

its solution

$$U\,(\alpha, t) = \frac{1}{\sqrt{2\pi}}e^{-\alpha^2 t}$$

Taking the inverse transform, of this result, we obtain

$$u\,(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha^2 t}e^{-i\alpha x}d\alpha = \frac{1}{2}\frac{1}{\pi t}e^{-x^2/4t}$$

8. Duhamel's principle is: The solution of the homogeneous PDE with a non-homogeneous initial condition is equivalent to solving non-homogeneous PDE with a homogeneous initial condition.

In fact, from Example 7.13, the solution of the initial value problem

$$v_t - v_{xx} = 0, \qquad -\infty < x < \infty,\; t > 0$$
$$v\,(x, 0) = f\,(x), \qquad -\infty < x < \infty$$

is

$$v\,(x, t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} f\,(\alpha)\exp\left[-\frac{(x-\alpha)^2}{4t}\right]d\alpha$$

Now, applying Duhamel's principle, the solution of the non-homogeneous problem

$$v_t - v_{xx} = f\,(x), \qquad -\infty < x < \infty,\; t > 0$$
$$v\,(x, 0) = 0, \qquad -\infty < x < \infty$$

is

$$v(x,t)=\int_0^t v(x,t-\tau)\,d\tau=\int_0^t\int_{-\infty}^{\infty} f(\alpha)\frac{1}{\sqrt{4\pi(t-\tau)}}\exp\left[-\frac{(x-\alpha)^2}{4(t-\tau)}\right]d\alpha\,d\tau$$

Therefore, the Green's function for the non-homogeneous problem is

$$G(x,t;\alpha,\tau)=\frac{1}{\sqrt{4\pi(t-\tau)}}\exp\left[-\frac{(x-\alpha)^2}{4(t-\tau)}\right],\qquad t>\tau>0$$

11. From Example 7.18, the solution of Laplace equation when $u(x,0)=f(x)$ is given in Eq. (7.164). Therefore, in the present problem

$$f(x)=\begin{cases}0, & x<0\\ 1, & x>0\end{cases}$$

the required solution is given by

$$u(x,y)=\frac{y}{\pi}\left[\int_{-\infty}^{0} 0\cdot d\xi+\int_0^{\infty}\frac{1d\xi}{(\xi-x)^2+y^2}\right]$$

or

$$u(x,y)=\frac{y}{\pi}\left[\frac{1}{y}\tan^{-1}\left(\frac{\xi-x}{y}\right)\right]_0^{\infty}=\frac{1}{\pi}\left[\tan^{-1}\infty-\tan^{-1}\left(-\frac{x}{y}\right)\right]=\frac{1}{\pi}\left[\frac{\pi}{2}+\tan^{-1}\left(\frac{x}{y}\right)\right]$$

12. Finally, taking finite cosine Fourier transform, we have, after inversion, the relation

$$V(x,t)=6+\frac{24}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{\cos n\pi-1}{n^2}\right)\exp\left(-\frac{n^2\pi^2 t}{36}\right)\cos\frac{n\pi x}{6}$$

13. $V(x,y)=\dfrac{2}{\pi}+\dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\cosh ny}{\cosh ny_0}\sin nx.$

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