Krishna's

TEXT BOOK on

GEOMETRY AND VECTOR ANALYSIS

(For B.A. and B.Sc. Ist Semester students of Kumaun University)

Kumaun University Semester Syllabus *w.e.f.* 2016-17

By

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Dedicated

to

Lord

Krishna

Authors & Publishers

Preface

This book on **Geometry and Vector Analysis** has been specially written according to the latest **Syllabus** to meet the requirements of **B.A. and B.Sc. Semester-I Students** of all colleges affiliated to Kumaun University.

The subject matter has been discussed in such a simple way that the students will find no difficulty to understand it. The proofs of various theorems and examples have been given with minute details. Each chapter of this book contains complete theory and a fairly large number of solved examples. Sufficient problems have also been selected from various university examination papers. At the end of each chapter an exercise containing objective questions has been given.

We have tried our best to keep the book free from misprints. The authors shall be grateful to the readers who point out errors and omissions which, inspite of all care, might have been there.

The authors, in general, hope that the present book will be warmly received by the students and teachers. We shall **indeed** be very thankful to our colleagues for their recommending this book to their students.

The authors wish to express their thanks to **Mr. S.K. Rastogi**, *M.D.,* **Mr. Sugam Rastogi**, *Executive Director,* **Mrs. Kanupriya Rastogi**, *Director* and **entire team** of **KRISHNA Prakashan Media (P) Ltd.**, **Meerut** for bringing out this book in the present nice form.

The authors will feel amply rewarded if the book serves the purpose for which it is meant. Suggestions for the improvement of the book are always welcome.

—Authors

Syllabus

GEOMETRY AND VECTOR ANALYSIS

B.A./B.Sc. I Semester *w.e.f.* 2016-17

Kumaun University, Nainital

First Semester – Third Paper

B.A./B.Sc. Paper-I

PAPER III: GEOMETRY AND VECTOR ANALYSIS

Polar Equation of conics: Polar coordinate system, Distance between two points, Polar equation of a Straight line, Polar equation of a circle, Polar equation of a conic, Chords, Tangent and Normal to a conic, Chord of contact, Polar of a point.

Vector Algebra and its Applications to geometry (Plane and Straight Line):

Triple product, Reciprocal vectors, Product of four vectors. General equation of a Plane, Normal and Intercept forms, Two sides of a plane, Length of perpendicular from a point to a plane, Angle between two planes, System of planes.

Direction Cosines and Direction ratios of a line, Projection on a straight line, Equation of a line, Symmetrical and unsymmetrical forms, Angle between a line and a plane, Coplanar lines, Lines of shortest distance, Length of perpendicular from a point to a line, Intersection of three planes, Transformation of coordinates.

Vector Differentiation: Ordinary differentiation of vectors, Applications to mechanics, Velocity and Acceleration, Differential operator-Del, Gradient, Divergence and Curl, **Vector Integration:** Line, Surface and volume integrals, Simple applications of Gauss divergence theorem,Green 's theorem and Stokes theorem (without proof).

Brief Contents

Krishna's

GEOMETRY

Chapters

Chapter

1

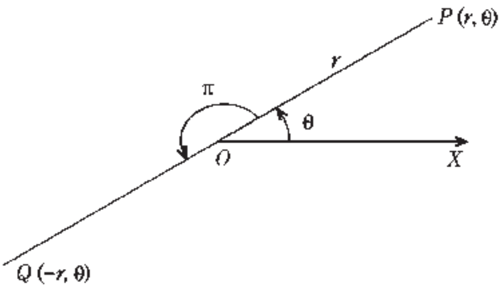
Polar Equation of a Conic

1 Conic Section

Definition: *A* **conic section**, *or* **conic** *is the locus of a point which moves so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.* The fixed point is called the *focus*, the fixed straight line is called the *directrix* and the constant ratio is called the *eccentricity* of the conic.

2 Polar Coordinates

(Agra 2006)



In this chapter we shall discuss another system of coordinates, known as **polar system.** In polar system, the position of a point in a plane is determined by its distance '*r*' from a fixed point *O*, called the **pole** or **origin,** and the angle 'θ' that the line joining the pole to the point makes with a fixed line OX through the pole, called the **initial line.** The angle θ is called the **vectorial angle** and is taken to be positive if measured in anti-clockwise direction, otherwise negative. The distance r is called the **radius vector** and is taken to be positive if measured along the line bounding the vectorial angle, and negative if measured in the opposite direction.

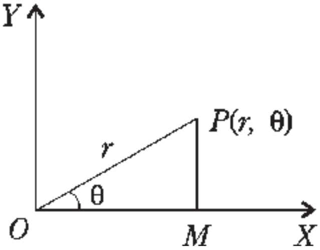
In the figure, O is the pole and OX the initial line.

Let $OP = r$ and $\angle XOP = \theta$. Then the polar coordinates of P are (r, θ). If $OQ = OP$, then Q is the point $(-r, \theta)$.

Also the points (r, θ), $(r, \theta \pm 2\pi)$, $(-r, \theta + \pi)$ etc. are all coincident. Thus in the polar system the coordinates of a point are not unique but can be expressed in an infinite number of ways.

3 Relation between Cartesian and Polar Coordinates

Take the pole O as origin, the initial line as the positive direction of x-axis, and the line through O making angle $\frac{\pi}{2}$ with OX in the anti-clockwise direction as the positive direction of y-axis. Suppose (r, θ) are the polar and (x, y) are the cartesian coordinates of any point P. Draw PM perpendicular to OX. Then $OM = x$ and $MP = y$.



From Δ OPM, we have

$$\mathbf{x = r\cos\theta}, \qquad \ldots(1)$$

$$\mathbf{y = r\sin\theta}. \ldots(2)$$

Squaring and adding (1) and (2), we get

$$x^2 + y^2 = r^2$$

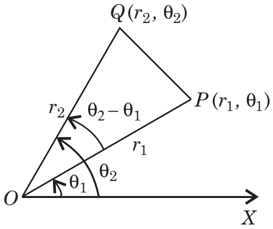
and dividing (2) by (1), we get

$$\tan\theta = \frac{y}{x} \quad \textbf{or} \quad \theta = \tan^{-1}\frac{y}{x}.$$

4 Distance between Two Points

Referred to some origin O. Let $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$ be two given points. Here

$$OP = r_1, OQ = r_2 \text{ and } \angle POQ = \theta_2 - \theta_1.$$



From $\Delta\,OPQ$, we get

$$\cos(\theta_2 - \theta_1) = \cos POQ$$

$$= \frac{OP^2 + OQ^2 - PQ^2}{2OP.OQ}$$

$$= \frac{r_1^2 + r_2^2 - PQ^2}{2r_1 r_2}$$

or $$2r_1r_2\cos(\theta_2-\theta_1)=r_1^2+r_2^2-PQ^2$$

or $$PQ^2=r_1^2+r_2^2-2r_1r_2\cos(\theta_2-\theta_1)$$

or $$PQ=\sqrt{\{r_1^2+r_2^2-2r_1r_2\cos(\theta_2-\theta_1\}}.$$

5 Polar Equation of a Straight Line

Let PQ be a straight line whose equation is required, intersect the initial line OX at A. Draw OM perpendicular upon the given line PQ. Let $OM = p$ and $\angle XOM = \alpha$. If r, θ be the co-ordinates of the point P, then

$$OP = r,\ \angle XOP = \theta \text{ and } \angle POM = \alpha - \theta.$$

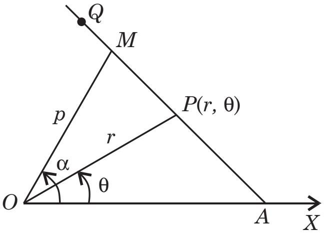
From the right angled triangle OMP, we have

$$OM = OP\cos POM$$

or $$p = r\cos(\alpha-\theta)$$

or $$r\cos(\theta-\alpha) = p,$$

which is the required equation of the straight line.



Remark 1: The polar equation of a straight line passing through the pole and inclined at an angle α to the initial line is

$$\theta = \alpha.$$

The equation of straight line passing through origin and making an angle α with the x-axis is

$$y = x\tan\alpha.$$

Changing to polar co-ordinates, we have

$$r\sin\theta = (r\cos\theta)\tan\alpha$$

or $$\tan\theta = \tan\alpha$$

or $$\theta = \alpha.$$

Remark 2: In cartesian form the general equation is

$$ax + by + c = 0.$$

Changing to polar coordinates, we have

$$ar\cos\theta + br\sin\theta + c = 0$$

or $$\frac{-c}{r} = a\cos\theta + b\sin\theta$$

or $$\frac{1}{r} = A\cos\theta + B\sin\theta,$$

which is equation of straight line in polar co-ordinates.

6 Polar Equation of a Straight Line Perpendicular to the given Line

The polar equation of the straight line

$$ax + by + c = 0 \text{ is}$$

$$\frac{1}{r} = A\cos\theta + B\sin\theta.$$

The cartesian equation of a line perpendicular to this line is

$$bx - ay + \lambda = 0.$$

Changing to polar co-ordinates, we have

$$r\,(b\cos\theta - a\sin\theta) = -\lambda$$

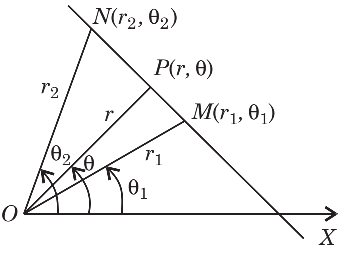
or $$b\cos\theta - a\sin\theta = \frac{-\lambda}{r}$$

or $$\frac{l}{r} = a\cos\left(\frac{\pi}{2}+\theta\right) + b\sin\left(\frac{\pi}{2}+\theta\right),$$

which is the required equation of the perpendicular straight line.

Thus in polar co-ordinates, the equation of a line perpendicular to the given line is obtain by replacing θ by $\frac{\pi}{2}+\theta$ in the given equation and changing the coefficient of $\frac{1}{r}$ to a new constant.

7 Polar Equation of a Straight Line through Two Points



Let $M\,(r_1, \theta_1)$ and $N\,(r_2, \theta_2)$ be two given points and $P\,(r, \theta)$ be any point on the line joining M and N. We have

$$\text{area of } \Delta\, MON = \text{area of } \Delta\, MOP + \text{area of } \Delta\, PON.$$

$\therefore$ $$\frac{1}{2} r_1 r_2 \sin\angle MON = \frac{1}{2} r r_1 \sin\angle MOP + \frac{1}{2} r r_2 \sin\angle PON$$

or $$r_1 r_2 \sin(\theta_2 - \theta_1) = r r_1 \sin(\theta - \theta_1) + r r_2 \sin(\theta_2 - \theta)$$

or $$\frac{\sin(\theta_2 - \theta_1)}{r} = \frac{\sin(\theta - \theta_1)}{r_2} + \frac{\sin(\theta_2 - \theta)}{r_1},$$

which is the required equation.

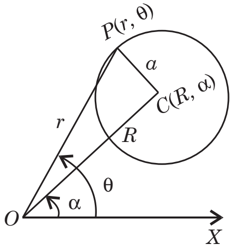
8 Polar Equation of a Circle

Let $P(r, \theta)$ be any point on the circle with centre $C(R, \alpha)$ and radius a.

We have

$$OC = R, \angle XOC = \alpha$$

$$OP = r, \angle COP = \theta - \alpha.$$



From ΔCOP, we have

$$\cos(\theta - \alpha) = \frac{r^2 + R^2 - a^2}{2rR}$$

or $$r^2 - 2rR\cos(\theta - \alpha) + R^2 = a^2,$$

which is the required general equation of the circle.

9 Particular Cases of the General Equation of the Circle

1. Centre *C* of the circle lies on the initial line:

In this case we have $\alpha = 0$ and so the equation of the circle is

$$r^2 - 2Rr\cos\theta + R^2 = a^2.$$

2. Pole *O* lies on the circle:

In this case $OC = R = a$ and so the equation of the circle is

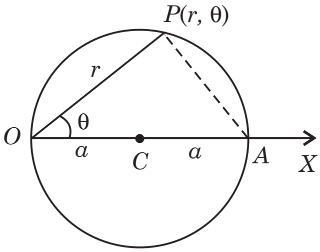
$$r^2 - 2ar\cos(\theta - \alpha) + a^2 = a^2$$

or $$r = 2a\cos(\theta - \alpha).$$

3. Pole lies on the circumference and centre on the initial line:

Let $P(r, \theta)$ be any point on the circle and OCA be a diameter.

We have



$$\angle AOP = \theta,$$

$$\angle APO = 90°$$

and $$OA = 2a.$$

From ΔAOP, we have

$$r = 2a\cos\theta,$$

which is the required equation of the circle.

4. Pole is the centre of the circle:

In this case for any point $P(r, \theta)$ on the circumference, we have

$$r = a,$$

which is the required equation of the circle of radius a.

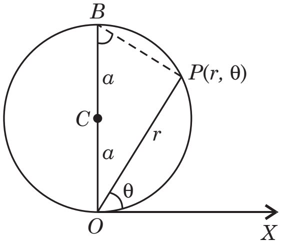
5. Initial line is the tangent at the pole:

Let $P(r, \theta)$ and B be any points on the circumference such that OCB is the diameter, where C is the centre of the circle with radius a. We have

$$OB = 2a,$$

$$\angle OPB = 90°$$

and $$\angle XOP = \theta = \angle OBP.$$



From $\Delta\, OPB$, we have

$$r = 2a \sin\theta,$$

which is the required equation of the circle.

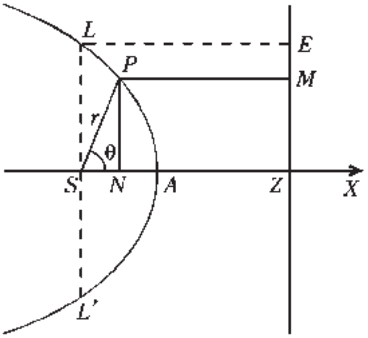
10 The Polar Equation of a Conic

To find the polar equation of a conic with its latus rectum of length 2l, eccentricity e and the focus being the pole. **(Kumaun 2001, 07; Gorakhpur 05; Rohilkhand 06; Kanpur 11)**

Let S be the focus which is taken as the pole. Let ZM be the directrix of the conic. Draw SZ perpendicular to the directrix, and take SZ as the initial line SX.

Consider a point P on the conic and let its polar coordinates be (r, θ) so that

$$SP = r, \angle XSP = \theta.$$



Let LSL' be the latus rectum of length $2l$ so that the semi-latus rectum $SL = l$.

Since the point P is on the conic, therefore by the definition of a conic, we have

$$SP = e\,.\,PM = e\,.\,NZ = e\,(SZ - SN).$$

$$\therefore \quad r = e\,.\,SZ - e\,.\,SN$$

$$= e\,.\,SZ - e\,.\,SP\cos\theta \qquad [\because \; SN = SP\cos\theta]$$

$$= e\,.\,SZ - er\cos\theta. \qquad \ldots(1)$$

But the point L is also on the conic. Therefore

$$l = SL = e\,.\,LE = e\,.\,SZ.$$

Putting $e\,.\,SZ = l$ in (1), we have

$$r = l - er\cos\theta$$

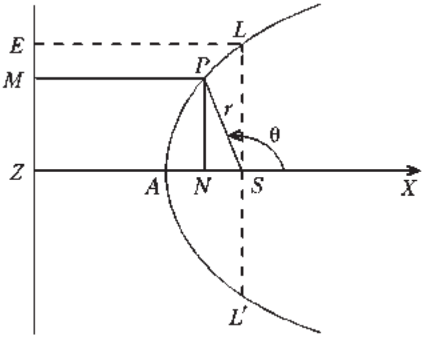
or $$r\,(1 + e\cos\theta) = l$$

or $$\frac{l}{r} = 1 + e\cos\theta. \qquad \ldots(2)$$

The equation (2) is the required polar equation of a conic referred to the focus as the pole and the axis of the conic as the initial line.

Remark: While deriving the equation (2) the positive direction of the initial line has been taken as the direction directed from the focus towards the directrix. However, as shown in the adjoining figure, if we take the positive direction of the initial line opposite to the direction directed from the focus towards the directrix, the equation of the conic will come out to be

$$\frac{l}{r} = 1 - e \cos \theta. \qquad \ldots(3)$$



We clearly see that if we rotate the initial line through an angle π in the equation (2) *i.e.*, if we replace θ by $\pi + \theta$ in the equation (2), we get the equation (3). Hence any result for the conic (3) can be obtained from the corresponding result for the conic (2) by increasing each vectorial angle by π, *i.e.*, by writing $\theta + \pi$ for θ, $\alpha + \pi$ for α, $\beta + \pi$ for β etc., where α, β etc. are vectorial angles.

Corollary 1: If the conic is a parabola, then $e = 1$.

The equation (2) becomes

$$l/r = 1 + \cos \theta = 2 \cos^2 \frac{1}{2} \theta$$

or $$r = (l/2) \sec^2 \frac{1}{2} \theta \qquad \ldots(4)$$

and the equation (3) becomes

$$r = (l/2) \operatorname{cosec}^2 \frac{1}{2} \theta. \qquad \ldots(5)$$

Corollary 2: From the equation (2), we have

$$r = l/(1 + e \cos \theta).$$

Therefore the coordinates of a point $P(r, \theta)$ on the conic (2) may be written as $\left(\dfrac{l}{1 + e \cos \theta}, \theta \right)$. This is called the point 'θ'.

11 The Equation to the Directrix of the Conic

(Avadh 2009; Meerut 12)

To find the equation to the directrix of the conic $l/r = 1 + e \cos \theta$.

Let S be the focus. Let P be any point on the directrix ZM. Let the polar coordinates of P be (r, θ), so that

$$\angle ZSP = \theta, SP = r.$$

We have $$\frac{SZ}{SP} = \cos \theta$$

or $SZ = r\cos\theta$. ...(1)

Also, $SL = e\,.\,LE$

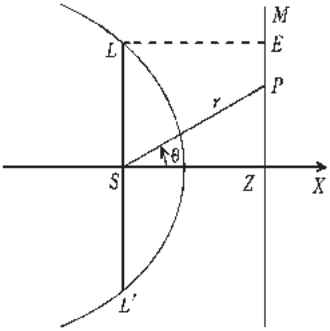
$= e\,.\,SZ$

$= er\cos\theta$. [Using (1)]

$\therefore$ $l = er\cos\theta$

or $\dfrac{l}{r} = e\cos\theta$,

which is the required equation of the directrix.

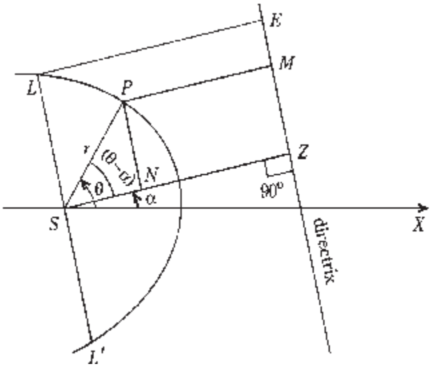


12 The Polar Equation of a Conic with its Focus as the Pole and its Axis Inclined at an Angle α to the Initial Line

(Rohilkhand 2005; Kumaun 08, 10, 15)

To find the polar equation of a conic with its focus as the pole and its axis inclined at an angle α to the initial line.

Let SZ, the axis of the conic, be inclined to the initial line at an angle α. Consider a point P with coordinates (r, θ) on the conic. We have



$r = SP = e\,.\,PM$

[By the definition of a conic]

$= e\,.\,NZ = e\,(SZ - SN)$

$= e\,(LE - SN)$

$= e\left\{\dfrac{LS}{e} - SP\cos(\theta-\alpha)\right\}$

$= e\left\{\dfrac{l}{e} - r\cos(\theta-\alpha)\right\} = l - e\,r\cos(\theta-\alpha)$

or $l = r + e\,r\cos(\theta-\alpha)$

or $l/r = 1 + e\cos(\theta-\alpha)$, is the required equation.

Corollary: The equation of the directrix in this case is given by $l/r = e\cos(\theta-\alpha)$, as in article 11.

Illustrative Examples

Example 1: *Show that the equations* $l/r = 1 + e\cos\theta$ *and* $l/r = -1 + e\cos\theta$ *represent the same conic.* **(Agra 2006, 10; Bundelkhand 04, 13; Kanpur 09, 11; Meerut 07B, 10, 12B; Kashi 13, 14; Purvanchal 11; Rohilkhand 05, 06, 09, 10; Kumaun 10, 14)**

Solution: The given equations are

$$l/r = 1 + e\cos\theta, \qquad \ldots(1)$$

and $$l/r = -1 + e\cos\theta. \qquad \ldots(2)$$

First we shall show that every point on the curve (1) also lies on the curve (2). Let $P(r_1, \theta_1)$ be any point on the curve (1) so that

$$l/r_1 = 1 + e\cos\theta_1. \qquad \ldots(3)$$

Now the coordinates of the point P can also be expressed as $(-r_1, \theta_1 + \pi)$ instead of (r_1, θ_1). These coordinates will satisfy the equation (2) if

$$l/(-r_1) = -1 + e\cos(\theta_1 + \pi)$$

i.e., if $-l/r_1 = -1 - e\cos\theta_1$ *i.e.*, if $l/r_1 = 1 + e\cos\theta_1$ which is true by virtue of (3). Thus every point P on the curve (1) also lies on the curve (2). Similarly we can show that every point on the curve (2) is also a point on the curve (1). Hence the equations (1) and (2) represent the same conic.

Example 2: *In a conic prove the following :—*

(i) The semi-latus rectum is the harmonic mean between the segments of a focal chord.

(Kashi 2013; Rohilkhand 11, 12; Purvanchal 13; Bundelkhand 13)

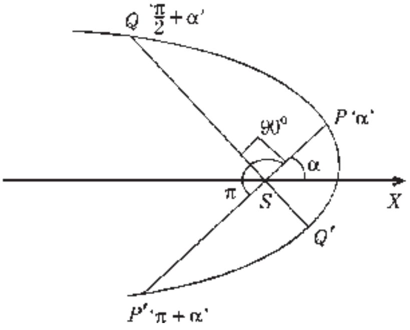
(ii) The sum of the reciprocals of the segments of any focal chord is constant

i.e., $\dfrac{1}{SP} + \dfrac{1}{SP'} = \dfrac{2}{l}$ *(constant).*

(iii) The sum of the reciprocals of two perpendicular focal chords is constant

i.e., $\dfrac{1}{PP'} + \dfrac{1}{QQ'} =$ *constant.*

(Meerut 2004B; Bundelkhand 06; Kanpur 06, 08, 10; Purvanchal 13)



Solution: (i) Let the equation of the conic be

$$l/r = 1 + e\cos\theta. \qquad \ldots(1)$$

Let PSP' be a focal chord. If the vectorial angle of P is α, the vectorial angle of P' is $\pi + \alpha$. Thus the coordinates of P and P' are respectively (SP, α) and $(SP', \alpha + \pi)$.

$$\therefore \qquad \frac{l}{SP} = 1 + e\cos\alpha, \qquad \ldots(2)$$

and $$\frac{l}{SP'} = 1 + e\cos(\alpha + \pi)$$

$$= 1 - e\cos\alpha. \qquad \ldots(3)$$

Adding (2) and (3), we have

$$\frac{l}{SP} + \frac{l}{SP'} = 2$$

or $$\frac{1}{2}\left[\frac{1}{SP}+\frac{1}{SP'}\right]=\frac{1}{l}. \qquad \dots(4)$$

$\therefore$ $1/l$ is the arithmetic mean of $1/SP$ and $1/SP'$ and so l is the harmonic mean of SP and SP'.

(ii) The equation (4) implies that the sum of the reciprocals of the segments of any focal chord is $2/l$ which is a constant.

(iii) Now suppose QQ' is a focal chord at right angles to PP' *i.e.*, PP' and QQ' are two perpendicular focal chords. Hence if the vectorial angle of P is α, then the vectorial angle of Q is $\frac{1}{2}\pi+\alpha$. Also the vectorial angle of P' is $\pi+\alpha$.

From (2) and (3), we have

$$PP'=SP+SP'=\frac{l}{1+e\cos\alpha}+\frac{l}{1-e\cos\alpha}=\frac{2l}{1-e^2\cos^2\alpha}$$

or $$\frac{1}{PP'}=\frac{1-e^2\cos^2\alpha}{2l}. \qquad \dots(5)$$

Now if the vectorial angle of the extremity P of the focal chord PSP' is α, then the vectorial angle of the extremity Q of the focal chord QSQ' is $\frac{1}{2}\pi+\alpha$. So replacing PP' by QQ' and α by $\frac{1}{2}\pi+\alpha$ in the relation (5), we have

$$\frac{1}{QQ'}=\frac{1-e^2\cos^2\left(\frac{1}{2}\pi+\alpha\right)}{2l}=\frac{1-e^2\sin^2\alpha}{2l} \qquad \dots(6)$$

Adding (5) and (6), we have

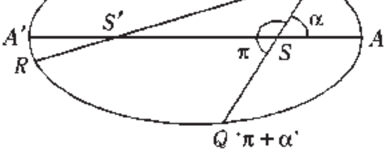
$$\frac{1}{PP'}+\frac{1}{QQ'}=\frac{2-e^2}{2l}$$ which is a constant.

Example 3: *If PSQ and $PS'R$ be two chords of an ellipse through the foci S and S', show that $\frac{PS}{SQ}+\frac{PS'}{S'R}$ is independent of the position of P.*

(Bundelkhand 2005, 12; Purvanchal 08)

Solution: Let the polar equation of the ellipse be

$$l/r=1+e\cos\theta. \qquad \dots(1)$$



Since PSQ is a focal chord, therefore if the vectorial angle of P is α, then that of Q is $\pi+\alpha$.

$\therefore$ $$l/SP=1+e\cos\alpha, \qquad \dots(2)$$

and $$l/SQ=1+e\cos(\pi+\alpha)=1-e\cos\alpha. \qquad \dots(3)$$

Adding (2) and (3), we get

$$\frac{l}{SP} + \frac{l}{SQ} = 2$$

or $$\frac{1}{SP} + \frac{1}{SQ} = \frac{2}{l}. \qquad \ldots(4)$$

Multiplying both sides of (4) by SP, we have

$$\frac{SP}{SQ} = \frac{2}{l}\, SP - 1. \qquad \ldots(5)$$

Similarly for the focal chord $PS'R$, we have

$$\frac{S'P}{S'R} = \frac{2}{l}\, S'P - 1. \qquad \ldots(6)$$

Adding (5) and (6), we get

$$\frac{SP}{SQ} + \frac{S'P}{S'R} = \frac{2}{l}\,(SP + S'P) - 2.$$

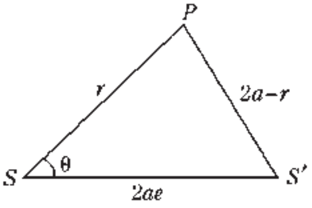
But in the ellipse,

$$\begin{aligned} SP + S'P &= \text{the sum of the focal distances of the point } P \\ &= \text{the length of the major axis} = 2\,a, \text{ say.} \end{aligned}$$

$\therefore$ $$\frac{SP}{SQ} + \frac{S'P}{S'R} = \frac{2\,.\,2a}{l} - 2\,,$$

which is a constant and so is independent of the position of P.

Example 4: *A point P moves, so that the sum of its distances from two fixed points S, S′ is constant and equal to 2a. Show that P lies on the conic* $\frac{a\,(1 - e^2)}{r} = 1 - e\cos\theta$ *referred to S as pole and SS′ as initial line, SS′ being equal to 2ae.*



Solution: Taking S as the pole and SS' as the initial line, let the polar coordinates of P be (r, θ).

$\therefore$ $PS = r$, and $PS' = 2a - r$

because $PS + PS' = 2a$ (given).

Also $SS' = 2ae$ (given).

Now from $\Delta\, PSS'$, we have

$$\cos\theta = \frac{r^2 + (2ae)^2 - (2a - r)^2}{2\,.\,r\,.\,(2ae)} \qquad \text{[By cosine formula]}$$

$$= \frac{r + a\,(e^2 - 1)}{er}.$$

$\therefore$ $$e\cos\theta = 1 + \frac{a\,(e^2 - 1)}{r}$$

or $$\frac{a\,(e^2 - 1)}{r} = 1 - e\cos\theta,$$

which is the required locus of P and is a conic.

Comprehensive Exercise 1

1. If PSP' and QSQ' be two perpendicular focal chords of a conic, prove that $\dfrac{1}{SP\,.\,SP'}+\dfrac{1}{SQ\,.\,SQ'}$ is constant. **(Meerut 2004, 06B; Rohilkhand 06; Kanpur 07; Bundelkhand 14; Kashi 14; Kumaun 09, 13, 15)**
2. Prove that the perpendicular focal chords of a rectangular hyperbola are equal.
3. A chord PQ of a conic whose eccentricity is e and semi-latus rectum l subtends a right angle at the focus S, show that $\left(\dfrac{1}{SP}-\dfrac{1}{l}\right)^2+\left(\dfrac{1}{SQ}-\dfrac{1}{l}\right)^2=\dfrac{e^2}{l^2}\cdot$ **(Kumaun 2012)**
4. PSP' is a focal chord of a conic. Prove that the locus of its middle point is a conic of the same kind as the original conic.
5. A circle passing through the focus of a conic whose latus rectum is $2l$ meets the conic in four points whose distances from the focus are r_1, r_2, r_3 and r_4, prove that $\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}+\dfrac{1}{r_4}=\dfrac{2}{l}\cdot$
6. If the circle $r+2a\cos\theta=0$ cuts the conic $l/r=1+e\cos(\theta-\alpha)$ in four points, find the equation in r which determines the distances of these four points from the pole. Show that if the algebraic sum of these four distances is equal to $2a$, the eccentricity is equal to $2\cos\alpha$.
7. Show that the equation of the directrix of the conic $l/r=1+e\cos\theta$ corresponding to the focus other than the pole is $\dfrac{l}{r}=-\dfrac{1-e^2}{1+e^2}e\cos\theta$. **(Kumaun 2013)**
8. A circle of given radius passing through the focus S of a given conic intersects it in A, B, C, D; show that $SA\,.\,SB\,.\,SC\,.\,SD$ is constant. **(Kumaun 2012)**

13 Chord Joining any Two Points on the Conic

To find the equation of the chord of the conic $l/r=1+e\cos\theta$, *whose extremities are* (r_1,θ_1) *and* (r_2,θ_2). **(Meerut 2005B, 06, 07)**

The equation of the conic is

$$l/r=1+e\cos\theta. \qquad \ldots(1)$$

Suppose the points $P(r_1,\theta_1)$ and $Q(r_2,\theta_2)$ lie on (1). Then

$$l/r_1=1+e\cos\theta_1 \quad\text{and}\quad l/r_2=1+e\cos\theta_2.$$

Let the polar equation of any line be

$$A\cos\theta + B\sin\theta = l/r. \qquad \ldots(2)$$

[The equation (2) represents a straight line because if we change it to cartesian coordinates we get $Ax + By = l$ which is linear in x and y].

If the straight line (2) passes through the point $P\,(r_1, \theta_1)$, we have

$$A\cos\theta_1 + B\sin\theta_1 = l/r_1 = 1 + e\cos\theta_1$$

or $$(A - e)\cos\theta_1 + B\sin\theta_1 - 1 = 0. \qquad \ldots(3)$$

Similarly if the straight line (2) passes through the point $Q\,(r_2, \theta_2)$, we have

$$(A - e)\cos\theta_2 + B\sin\theta_2 - 1 = 0. \qquad \ldots(4)$$

Solving (3) and (4) for $A - e$ and B, we get

$$\frac{A-e}{\sin\theta_2 - \sin\theta_1} = \frac{B}{\cos\theta_1 - \cos\theta_2}$$

$$= \frac{1}{\sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1} = \frac{1}{\sin(\theta_2 - \theta_1)}$$

or $$\frac{A-e}{2\cos\frac{1}{2}(\theta_1+\theta_2)\sin\frac{1}{2}(\theta_2-\theta_1)} = \frac{B}{2\sin\frac{1}{2}(\theta_1+\theta_2)\sin\frac{1}{2}(\theta_2-\theta_1)}$$

$$= \frac{1}{2\sin\frac{1}{2}(\theta_2-\theta_1)\cos\frac{1}{2}(\theta_2-\theta_1)}$$

or $$\frac{A-e}{\cos\frac{1}{2}(\theta_1+\theta_2)} = \frac{B}{\sin\frac{1}{2}(\theta_1+\theta_2)} = \frac{1}{\cos\frac{1}{2}(\theta_2-\theta_1)}.$$

$\therefore$ $$A = \cos\frac{1}{2}(\theta_1+\theta_2)\sec\frac{1}{2}(\theta_2-\theta_1) + e,$$

$$B = \sin\frac{1}{2}(\theta_1+\theta_2)\sec\frac{1}{2}(\theta_2-\theta_1).$$

Putting these values of A and B in (2), the equation of the chord PQ is given by

$$l/r = \{\cos\frac{1}{2}(\theta_1+\theta_2)\sec\frac{1}{2}(\theta_2-\theta_1) + e\}\cos\theta$$

$$+ \{\sin\frac{1}{2}(\theta_1+\theta_2)\sec\frac{1}{2}(\theta_2-\theta_1)\}\sin\theta$$

or $$\mathbf{l/r = e\cos\theta + \sec\frac{1}{2}(\theta-\theta_1)\cos\left\{\theta - \frac{1}{2}(\theta_1+\theta_2)\right\}} \qquad \ldots(5)$$

Thus remember that (5) is the equation of the chord joining the points 'θ_1' and 'θ_2' on the conic $l/r = 1 + e\cos\theta$.

Corollary: If the points P and Q are such that their vectorial angles are $\alpha - \beta$ and $\alpha + \beta$, so that the sum of the angles is 2α and their difference is 2β, then the equation (5) of the chord PQ becomes

$$\mathbf{l/r = e\cos\theta + \sec\beta\cos(\theta - \alpha).} \qquad \ldots(6)$$

14 Tangent to the Conic at a Given Point on it

To find the equation of the tangent at the point (r_1, θ_1) *of the conic*

$$l/r = 1 + e\cos\theta.$$

(Rohilkhand 2005, 07; Meerut 07B; Bundelkhand 10; Kumaun 07, 09, 14)

Let P be a given point (r_1, θ_1) on the conic $l/r = 1 + e\cos\theta$. Take another point $Q\,(r_2, \theta_2)$ on the conic. Proceeding as in article 13, we get the equation of the chord joining the points P and Q as

$$l/r = e\cos\theta + \sec\frac{1}{2}(\theta_2 - \theta_1)\cos\{\theta - \frac{1}{2}(\theta_1 + \theta_2)\}.$$

[Derive the equation here].

Now the tangent at P to the conic $l/r = 1 + e\cos\theta$ is the limiting position of the chord PQ as $Q \to P$ *i.e.*, as $\theta_2 \to \theta_1$. So taking the limit of the equation of the chord PQ as $\theta_2 \to \theta_1$, we get the equation of the tangent to the conic $l/r = 1 + e\cos\theta$ at the point whose vectorial angle is θ_1 as

$$l/r = e\cos\theta + \cos(\theta - \theta_1). \qquad \ldots(1)$$

Corollary 1: If the conic is $l/r = 1 + e\cos(\theta - \alpha)$, the tangent at the point 'θ_1' is given by

$$l/r = e\cos(\theta - \alpha) + \cos(\theta - \theta_1). \qquad \ldots(2)$$

Corollary 2: If the conic is $l/r = 1 - e\cos\theta$, the tangent at the point θ_1 is

$$l/r = e\cos(\pi + \theta) + \cos\{(\pi + \theta) - (\pi + \theta_1)\}$$

i.e.,

$$l/r = -e\cos\theta + \cos(\theta - \theta_1). \qquad \ldots(3)$$

(Bundelkhand 2008)

Corollary 3: To find the slope of the tangent (1). The equation (1) may be written as

$$l = r\cos\theta\cos\theta_1 + r\sin\theta\sin\theta_1 + e\,.\,r\cos\theta$$

or $$l = x\cos\theta_1 + y\sin\theta_1 + ex$$

or $$y\sin\theta_1 = -x(e + \cos\theta_1) + l.$$

$\therefore$ the slope of the tangent (1) $= -\dfrac{e + \cos\theta_1}{\sin\theta_1}$. ...(4)

15 Asymptotes

To find the equation of the asymptotes of the conic $l/r = 1 + e\cos\theta$.

(kumaun 2013, 15)

The equation of the conic is

$$l/r = 1 + e\cos\theta. \qquad \ldots(1)$$

Suppose (r', α) is a point on (1), so that

$$l/r' = 1 + e\cos\alpha. \qquad \ldots(2)$$

The equation of the tangent to the conic (1) at the point (r', α) on it is

$$l/r = e\cos\theta + \cos(\theta - \alpha). \qquad \ldots(3)$$

We know that an asymptote is the limiting position of the tangent as the point of contact tends to infinity. Hence (3) will tend to an asymptote if r' tends to infinity. Now if r' tends to infinity, we have from (2),

$$0 = 1 + e\cos\alpha.$$

$$\therefore \qquad \cos\alpha = -1/e \text{ and } \sin\alpha = \pm\sqrt{\{1 - (1/e^2)\}} \qquad \ldots(4)$$

From (3), we have

$$l/r = e\cos\theta + \cos\theta\cos\alpha + \sin\theta\sin\alpha.$$

Substituting for $\cos\alpha$ and $\sin\alpha$ from (4) in it, we have

$$l/r = e\cos\theta + \cos\theta\,.\,(-1/e) \pm \sin\theta\sqrt{\{1 - (1/e^2)\}}$$

or

$$le/r = (e^2 - 1)\cos\theta \pm \sqrt{(e^2 - 1)}\sin\theta. \qquad \ldots(5)$$

The equation (5) represents the equations of the two asymptotes of the conic (1). Clearly the asymptotes are real only when $e > 1$ *i.e.*, the conic (1) is a hyperbola.

Illustrative Examples

Example 5: *A chord of a conic subtends a constant angle at a focus of the conic. Show that the chord touches another conic.*

Solution: Referred to the focus S as the pole, let the equation of the conic be

$$l/r = 1 + e\cos\theta. \qquad \ldots(1)$$

Suppose a chord PQ of the conic (1) subtends a constant angle 2β at the focus S. Let $\alpha - \beta$ and $\alpha + \beta$ be the vectorial angles of the extremities of the chord PQ. Then the equation of the chord PQ is

$$l/r = e\cos\theta + \sec\beta\cos(\theta - \alpha)$$

or

$$(l\cos\beta)/r = e\cos\beta\cos\theta + \cos(\theta - \alpha). \qquad \ldots(2)$$

Obviously the straight line (2) is the tangent to the conic $(l\cos\beta)/r = 1 + (e\cos\beta)\cos\theta$ at the point whose vectorial angle is α.

Hence the proposition.

Example 6: *Find the condition that the line $l/r = A\cos\theta + B\sin\theta$ may be a tangent to the conic $l/r = 1 + e\cos\theta$.* **(Meerut 2004B, 06B, 07, 11, 12B; Kanpur 07, 09; Kumaun 08, 13, 15; Bundelkhand 04; Avadh 07, 09, 13; Purvanchal 08, 13; Rohilkhand 13)**

Solution: Suppose the line

$$l/r = A\cos\theta + B\sin\theta \qquad \ldots(1)$$

is a tangent to the conic

$$l/r = 1 + e\cos\theta \qquad \ldots(2)$$

at the point whose vectorial angle is α. The equation of the tangent to (2) at the point 'α' is

$$l / r = \cos(\theta - \alpha) + e \cos\theta$$

or $$l / r = (e + \cos\alpha)\cos\theta + \sin\theta\sin\alpha. \qquad \ldots(3)$$

The equations (1) and (3) should represent the same line. So comparing the coefficients of $1/r$, $\cos\theta$ and $\sin\theta$, we have

$$1 = \frac{e + \cos\alpha}{A} = \frac{\sin\alpha}{B}$$

or $$\cos\alpha = A - e \text{ and } \sin\alpha = B.$$

Squaring and adding, we have

$$(A - e)^2 + B^2 = 1.$$

This is the required condition.

Example 7: *A conic is described having the same focus and eccentricity as the conic* $l / r = 1 + e\cos\theta$, *and the two conics touch at the point* $\theta = \alpha$; *prove that the length of its latus rectum is* $2l(1 - e^2)/(e^2 + 2e\cos\alpha + 1)$. **(Avadh 2007)**

Solution: The equation of the given conic is

$$l / r = 1 + e\cos\theta \qquad \ldots(1)$$

the focus being at the pole.

Let the equation of the conic having the same focus and eccentricity as the given conic be

$$l_1 / r = 1 + e\cos(\theta - \gamma), \qquad \ldots(2)$$

where γ is the angle of inclination of its axis to the initial line and l_1 is its semi-latus rectum. Since the conics (1) and (2) touch at the point $\theta = \alpha$, the tangents to them at the point α are the same lines.

The equation of the tangent to (1) at the point α is

$$l / r = \cos(\theta - \alpha) + e\cos\theta$$

or $$l / r = (\cos\alpha + e)\cos\theta + \sin\alpha\sin\theta. \qquad \ldots(3)$$

The equation of the tangent to (2) at the point α is

$$l_1 / r = \cos(\theta - \alpha) + e\cos(\theta - \gamma)$$

or $$l_1 / r = (\cos\alpha + e\cos\gamma)\cos\theta + (\sin\alpha + e\sin\gamma)\sin\theta. \qquad \ldots(4)$$

Now the equations (3) and (4) are identical because they represent the same line. So comparing (3) and (4), we have

$$\frac{l}{l_1} = \frac{\cos\alpha + e}{\cos\alpha + e\cos\gamma} = \frac{\sin\alpha}{\sin\alpha + e\sin\gamma}.$$

$\therefore$ $$el\cos\gamma = (l_1 - l)\cos\alpha + el_1, \qquad \ldots(5)$$

and $$el\sin\gamma = (l_1 - l)\sin\alpha. \qquad \ldots(6)$$

To eliminate γ, squaring (5) and (6), and adding, we get

$$e^2 l^2 = (l_1 - l)^2 + e^2 l_1^2 + 2el_1 (l_1 - l) \cos \alpha$$

or $$-e^2 (l_1^2 - l^2) = (l_1 - l)^2 + 2el_1 (l_1 - l) \cos \alpha.$$

Since $l_1 \neq l$, therefore, dividing both sides by $l_1 - l$, we have

$$-e^2 (l_1 + l) = (l_1 - l) + 2el_1 \cos \alpha$$

or $$l_1 (1 + 2e \cos \alpha + e^2) = l (1 - e^2).$$

$\therefore$ the length of the latus rectum of the conic (2)

$$= 2l_1 = \frac{2\, l\, (1 - e^2)}{(1 + 2\, e \cos \alpha + e^2)}.$$

Example 8: *Two equal ellipses of eccentricity e, are placed with their axes at right angles and they have one focus S in common. If PQ be a common tangent, show that the angle PSQ is equal to 2 $\sin^{-1} (e/\sqrt{2})$.*

Solution: Take the common focus S as the pole and the axis of one ellipse as the initial line so that the axis of the other ellipse makes an angle $\frac{1}{2}\pi$ with the initial line. Let the equations to the two equal ellipses be

$$l/r = 1 + e \cos \theta \qquad \ldots(1)$$

and $$l/r = 1 + e \cos (\theta - \pi/2) \quad \text{or} \quad l/r = 1 + e \sin \theta. \qquad \ldots(2)$$

It is given that PQ is a common tangent to the two ellipses. Let the vectorial angles of P, a point on (1), and Q, a point on (2), be α and β respectively. Therefore the tangent to (1) at the point α, *i.e.*, $l/r = \cos (\theta - \alpha) + e \cos \theta$

or $$l/r = (\cos \alpha + e) \cos \theta + \sin \alpha \sin \theta \qquad \ldots(3)$$

and the tangent to (2) at the point β *i.e.*, $l/r = \cos (\theta - \beta) + e \sin \theta$

or $$l/r = \cos \beta \cos \theta + (\sin \beta + e) \sin \theta \qquad \ldots(4)$$

should be identical. Hence comparing (3) and (4), we have

$$1 = \frac{\cos \alpha + e}{\cos \beta} = \frac{\sin \alpha}{\sin \beta + e}.$$

$\therefore$ $\cos \alpha + e = \cos \beta$ *i.e.*, $\cos \beta - \cos \alpha = e$

and $\sin \alpha = \sin \beta + e$ *i.e.*, $\sin \alpha - \sin \beta = e.$

Squaring and adding, we get $2 - 2 (\cos \alpha \cos \beta + \sin \alpha \sin \beta) = 2e^2$

or $$\cos (\alpha - \beta) = 1 - e^2$$

or $$1 - 2 \sin^2 \frac{1}{2} (\alpha - \beta) = 1 - e^2$$

or $$\sin^2 \frac{1}{2} (\alpha - \beta) = \frac{1}{2} e^2 \quad \text{or} \quad \sin \frac{1}{2} (\alpha - \beta) = \frac{e}{\sqrt{2}}.$$

$\therefore$ $$\frac{1}{2} (\alpha - \beta) = \sin^{-1} (e/\sqrt{2}).$$

$\therefore$ $$\angle PSQ = \alpha - \beta = 2 \sin^{-1} (e/\sqrt{2}).$$

Example 9: *A focal chord PSP′ of an ellipse is inclined at an angle α to the major axis. Show that the perpendicular from the focus on the tangent at P makes an angle*

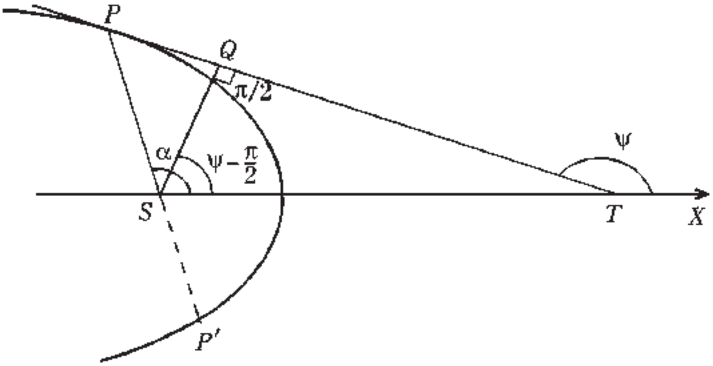
$$\tan^{-1} \{\sin \alpha / (e + \cos \alpha)\} \text{ with the axis.}$$

Solution: Let the conic (given to be an ellipse) be

$$l / r = 1 + e \cos \theta,$$

the focus S being at the pole.

Since the focal chord PSP' makes an angle α with the major axis *i.e.*, the initial line, therefore the vectorial angle of P is α.



The tangent at P is

$$l / r = \cos (\theta - \alpha) + e \cos \theta$$

or $$l = (\cos \alpha + e) (r \cos \theta) + \sin \alpha \,.\, (r \sin \theta)$$

or $$l = (\cos \alpha + e)\, x + \sin \alpha \,.\, y. \qquad \ldots(1)$$

If the tangent (1) makes an angle ψ with the major axis *i.e.*, with the initial line, then

$$\tan \psi = \text{the slope of the line (1)}$$

$$= -\frac{\cos \alpha + e}{\sin \alpha} \qquad \ldots(2)$$

Let SQ be the perpendicular from S to the tangent (1).

Then the angle which SQ makes with the major axis is $\psi - \frac{1}{2}\pi$. (See the figure.)

Now $$\tan (\psi - \frac{1}{2}\pi) = -\cot \psi$$

$$= \frac{\sin \alpha}{\cos \alpha + e}, \text{ using (2).}$$

∴ $$\text{the required angle} = \psi - \frac{1}{2}\pi$$

$$= \tan^{-1} \{\sin \alpha / (e + \cos \alpha)\}.$$

Comprehensive Exercise 2

1. Prove that the condition that the line $l / r = A \cos \theta + B \sin \theta$ may touch the conic $l / r = 1 + e \cos (\theta - \alpha)$ is

$$A^2 + B^2 - 2e (A \cos \alpha + B \sin \alpha) + e^2 - 1 = 0.$$

(Meerut 2009, 10B; Avadh 13)

2. Prove that the line $\dfrac{l}{r} = \cos (\theta - \alpha) + e \cos (\theta - \gamma)$ is the tangent to the conic $\dfrac{l}{r} = 1 + e \cos (\theta - \gamma)$ at the point for which $\theta = \alpha$.

3. Show that the two conics $l_1 / r = 1 + e_1 \cos \theta$ and $l_2 / r = 1 + e_2 \cos (\theta - \alpha)$ will touch one another if $l_1^2 (1 - e_2^2) + l_2^2 (1 - e_1^2) = 2 l_1 l_2 (1 - e_1 e_2 \cos \alpha)$.

(Bundelkhand 2006, 07, 08; Purvanchal 07, 10)

4. PSP' is a focal chord of a conic; prove that the angle between the tangents at P and P' is $\tan^{-1} \left(\dfrac{2 e \sin \alpha}{1 - e^2} \right)$ where α is the angle between the chord and the major axis.

(Meerut 2005, 06; Kanpur 06, 08, 14; Purvanchal 07, 13)

5. Show that the locus of the point of intersection of two tangents to the parabola $l / r = 1 + \cos \theta$, which cut one another at a constant angle α is the hyperbola, $l / r = \cos \alpha + \cos \theta$.

(Rohilkhand 2007)

6. Prove that the portion of the tangent intercepted between the conic and the directrix subtends a right angle at the corresponding focus.

or

Let the tangent at any point P on a conic whose focus is S meet the directrix in K, show that the angle PSK is a right angle.

7. PSP' is a focal chord of the conic. Prove that the tangents at P and P' intersect on the directrix.

8. Two conics have a common focus; prove that two of their common chords pass through the intersection of their directrices.

9. If the tangent at any point of an ellipse makes an angle α with its major axis and an angle β with the focal radius to the point of contact, show that $e \cos \alpha = \cos \beta$.

10. QR, a chord of the conic $l / r = 1 - e \cos \theta$, subtends a constant angle 2α at its focus S, and SP, the bisector of the angle QSR, meets QR in P. Show that the locus of P is the conic $(l \cos \alpha) / r = 1 - e \cos \alpha \cos \theta$.

11. Prove that two points on the conic $l/r = 1 + e\cos\theta$ whose vectorial angles are α and β respectively will be the extremities of a diameter if

$$\frac{e+1}{e-1} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\cdot$$

12. If POP' be a chord of a conic through a fixed point O, prove that $\tan\frac{1}{2}P'SO\tan\frac{1}{2}PSO$ is constant, S being a focus of the conic.

13. If the tangent from a point P to the conic $l/r = 1 + e\cos\theta$ subtends the fixed angle β at the focus, prove that the locus of the middle point of SP is a conic of eccentricity $e\sec\beta$.

16 Auxiliary Circle

Definition: *The locus of the foot of the perpendicular from the focus on any tangent to a conic (ellipse or hyperbola) is a circle called the auxiliary circle of the conic.*

The equation of the auxiliary circle: *To find the locus of the foot of the perpendicular from the focus of the conic $l/r = 1 + e\cos\theta$ on a tangent to it.*

Or

To find the polar equation of the auxiliary circle of the conic $l/r = 1 + e\cos\theta$.

(Gorakhpur 2006)

The equation of the conic is

$$l/r = 1 + e\cos\theta. \qquad \ldots(1)$$

Consider a point 'α' on (1). The equation of the tangent at the point 'α' is

$$l/r = \cos(\theta-\alpha) + e\cos\theta. \qquad \ldots(2)$$

Changing (2) to cartesian coordinates, we have

$$l = (\cos\alpha + e)\,x + \sin\alpha\;y. \qquad \ldots(2')$$

The equation of the line perpendicular to (2′) and passing through the focus (*i.e.*, the pole or origin) is

$$0 = \sin\alpha\,.\,x - (\cos\alpha + e)\,y.$$

Changing it to polars, we have

$$0 = \sin\alpha\,.\,r\cos\theta - (\cos\alpha + e)\,r\sin\theta$$

or $$\sin(\theta-\alpha) = -\,e\sin\theta. \qquad \ldots(3)$$

Now the foot of the perpendicular from the focus S to the tangent (2) is given by the intersection of (2) and (3) , and hence its locus is obtained by eliminating the variable 'α' between (2) and (3). The equations (2) and (3) may be rewritten as

$$\frac{l}{r} - e\cos\theta = \cos(\theta-\alpha) \quad \text{and} \quad -\,e\sin\theta = \sin(\theta-\alpha).$$

Squaring and adding these equations, we have

$$\left(\frac{l}{r} - e\cos\theta\right)^2 + e^2 \sin^2 \theta = 1$$

or $$\frac{l^2}{r^2} - 2\,\frac{le}{r}\cos\theta + e^2 - 1 = 0$$

or $$(e^2 - 1)\,r^2 - 2\,ler\cos\theta + l^2 = 0. \qquad \ldots(4)$$

This is the required equation of the auxiliary circle.

Particular case: If the conic be a parabola *i.e.*, $e = 1$, the equation (4) becomes

$$-2\,ler\cos\theta + l^2 = 0$$

or $$l/r = 2\cos\theta$$

or $$l/r = \cos(\theta - 0) + 1\,.\cos\theta$$

which is the equation of the tangent to the parabola $l/r = 1 + \cos\theta$ at the vertex (*i.e.*, at the point $\theta = 0$). **(Gorakhpur 2006)**

17 The Point of Intersection of the Two Tangents

To find the point of intersection of the two tangents at the points α and β on the conic

$$l/r = 1 + e\cos\theta.$$

The equation of the conic is

$$l/r = 1 + e\cos\theta. \qquad \ldots(1)$$

The equations of the tangents to (1) at the points α and β are respectively

$$l/r = \cos(\theta - \alpha) + e\cos\theta, \qquad \ldots(2)$$

and $$l/r = \cos(\theta - \beta) + e\cos\theta. \qquad \ldots(3)$$

To find the points of intersection of (2) and (3), subtracting (3) from (2), we have

$$0 = \cos(\theta - \alpha) - \cos(\theta - \beta),$$

or $$\cos(\theta - \alpha) = \cos(\theta - \beta).$$

$\therefore$ $$\theta - \alpha = \pm(\theta - \beta).$$

If we take the +ive sign, we get $\alpha = \beta$ which is inadmissible.

So taking the – ive sign, we get

$$\theta - \alpha = -\theta + \beta \quad \text{or} \quad \theta = \frac{1}{2}(\alpha + \beta). \qquad \ldots(4)$$

Putting the value of θ from (4) in (2) or (3), we have

$$l/r = \cos\left\{\frac{1}{2}(\alpha + \beta) - \alpha\right\} + e\cos\frac{1}{2}(\alpha + \beta)$$

$$= \cos\frac{1}{2}(\alpha - \beta) + e\cos\frac{1}{2}(\alpha + \beta).$$

If the point of intersection is (r', θ'), then we have

$$\theta' = \frac{1}{2}(\alpha + \beta)$$

and $$l/r' = \cos\frac{1}{2}(\alpha-\beta) + e\cos\frac{1}{2}(\alpha+\beta). \qquad \ldots(5)$$

Particular case: If the conic is a parabola *i.e.*, $e = 1$, then from (5),

$$l/r' = \cos\frac{1}{2}(\alpha-\beta) + \cos\frac{1}{2}(\alpha+\beta) = 2\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta$$

or $$r' = (l/2)\sec\frac{1}{2}\alpha\sec\frac{1}{2}\beta, \text{ and } \theta' = \frac{1}{2}(\alpha+\beta). \qquad \ldots(6)$$

Note: Students are advised not to use the results of equations (5) or (6) directly in solving the problems.

18 Director Circle

Definition: *The locus of the point of intersection of two perpendicular tangents to a conic, is called the director circle of the conic.*

The equation of the director circle: *To find the equation of the director circle of the conic* $l/r = 1 + e\cos\theta$. **(Rohilkhand 2005; Kumaun 08; Avadh 09; Purvanchal 12, 13)**

The equation of the conic is

$$l/r = 1 + e\cos\theta. \qquad \ldots(1)$$

The director circle of the conic (1) is the locus of the point of intersection of perpendicular tangents to the conic (1).

The equations of the tangents to (1) at the points α and β are

$$l/r = \cos(\theta-\alpha) + e\cos\theta, \qquad \ldots(2)$$

and $$l/r = \cos(\theta-\beta) + e\cos\theta \qquad \ldots(3)$$

respectively.

To find the point of intersection of (2) and (3), subtracting (3) from (2), we have

$$0 = \cos(\theta-\alpha) - \cos(\theta-\beta) \quad\text{or}\quad \cos(\theta-\alpha) = \cos(\theta-\beta).$$

$$\therefore \qquad \theta - \alpha = -(\theta-\beta) \quad\text{or}\quad \theta = \frac{1}{2}(\alpha+\beta).$$

Putting $\theta = \frac{1}{2}(\alpha+\beta)$ in (2), we have

$$l/r = \cos\left\{\frac{1}{2}(\alpha+\beta) - \alpha\right\} + e\cos\frac{1}{2}(\alpha+\beta)$$

$$= \cos\frac{1}{2}(\alpha-\beta) + e\cos\frac{1}{2}(\alpha+\beta).$$

Therefore if (r', θ') be the point of intersection of the tangents (2) and (3), we have

$$\theta' = \frac{1}{2}(\alpha+\beta) \text{ and } l/r' = \cos\frac{1}{2}(\alpha-\beta) + e\cos\frac{1}{2}(\alpha+\beta). \qquad \ldots(4)$$

Changing the equation (2) of the tangent at the point α to cartesian form, we have

$$l = (\cos\alpha + e)\,x + (\sin\alpha)\,y.$$

$\therefore$ m_1 = slope of the tangent (2) $= -(\cos\alpha + e)/(\sin\alpha)$.

Similarly m_2 = slope of the tangent (3) $= -(\cos\beta + e)/(\sin\beta)$.

The tangents (2) and (3) are perpendicular, if $m_1 m_2 = -1$

or $$\left[-\frac{(\cos\alpha + e)}{\sin\alpha}\right]\left[-\frac{(\cos\beta + e)}{\sin\beta}\right] = -1$$

or $$(\cos\alpha\cos\beta + \sin\alpha\sin\beta) + e(\cos\alpha + \cos\beta) + e^2 = 0$$

or $$\cos(\alpha - \beta) + 2e\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) + e^2 = 0$$

or $$2\cos^2\left(\frac{\alpha - \beta}{2}\right) - 1 + 2e\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) + e^2 = 0 \qquad \ldots(5)$$

Now from (4), we have

$$\frac{1}{2}(\alpha + \beta) = \theta' \text{ and } \cos\frac{1}{2}(\alpha - \beta) = l/r' - e\cos\theta'. \qquad \ldots(6)$$

Eliminating α and β with the help of (5) and (6), we have

$$2\left(\frac{l}{r'} - e\cos\theta'\right)^2 - 1 + 2e\cos\theta' \cdot \left(\frac{l}{r'} - e\cos\theta'\right) + e^2 = 0$$

or $$(1 - e^2)\, r'^2 + 2\, ler'\cos\theta' - 2l^2 = 0.$$

$\therefore$ the locus of (r', θ') is

$$(1 - e^2)\, r^2 + 2\, ler\cos\theta - 2l^2 = 0, \qquad \ldots(7)$$

which is the required equation of the director circle.

Particular case: If the conic is a parabola *i.e.*, if $e = 1$, the equation (7) becomes

$$2\, lr\cos\theta - l^2 = 0$$

or $$l/r = \cos\theta,$$

which is the equation of the directrix of the parabola $l/r = 1 + \cos\theta$.

Hence in the case of a parabola the locus of the point of intersection of perpendicular tangents is the directrix of the parabola.

Illustrative Examples

Example 10: *Show that the locus of the feet of perpendiculars from the focus S of a conic on chords subtending a constant angle 2γ at S is the circle whose polar equation referred to S as pole is* $r^2(e^2 - \sec^2\gamma) - 2ler\cos\theta + l^2 = 0$

where 2l is the latus rectum and e the eccentricity of the conic. **(Kumaun 2015)**

Solution: Referred to the focus S as the pole let the equation of the conic be

$$l/r = 1 + e\cos\theta. \qquad \ldots(1)$$

Let PQ be a chord of (1) subtending an angle 2γ at the focus S. Let $\alpha - \gamma$ and $\alpha + \gamma$ be the vectorial angles of the extremities of the chord PQ so that $\angle QSP = 2\gamma$; here α is a parameter. Then the equation of the chord PQ is

$$l/r = e \cos\theta + \sec\gamma \cos(\theta - \alpha) \qquad \ldots(2)$$

or $$l/r = e\cos\theta + \sec\gamma\cos\theta\cos\alpha + \sec\gamma\sin\theta\sin\alpha$$

or $$l = (e + \sec\gamma\cos\alpha)\, r\cos\theta + (\sec\gamma\sin\alpha)\, r\sin\theta. \qquad \ldots(2')$$

Mentally transforming (2) to cartesians, we see that the equation of the perpendicular drawn from the focus S (*i.e.*, the pole or origin) to the line (2′) is

$$0 = (e + \sec\gamma\cos\alpha)\, r\sin\theta - (\sec\gamma\sin\alpha)\, r\cos\theta$$

or $$-\,e\sin\theta = \sec\gamma\sin(\theta - \alpha). \qquad \ldots(3)$$

The foot of the perpendicular drawn from the focus S to the chord (2) is the point of intersection of the lines (2) and (3). To find its locus we have to eliminate the variable α between (2) and (3).

The equation (2) can be written as

$$(l/r - e\cos\theta) = \sec\gamma\cos(\theta - \alpha). \qquad \ldots(4)$$

Squaring and adding (3) and (4), we get

$$e^2\sin^2\theta + (l/r - e\cos\theta)^2 = \sec^2\gamma$$

or $$e^2 + \frac{l^2}{r^2} - \frac{2\,le}{r}\cos\theta = \sec^2\gamma$$

or $$r^2(e^2 - \sec^2\gamma) - 2\,ler\cos\theta + l^2 = 0,$$

which is the required locus.

Example 11: *If A, B, C be any three points on a parabola, and the tangents at these points form a triangle A′B′C′, show that SA . SB . SC = SA′ . SB′ . SC′, S being the focus of the parabola.*

Solution: Let the equation of the parabola be $l/r = 1 + \cos\theta$, referred to the focus S as the pole.

Let the vectorial angles of A, B, C be α, β, γ respectively. The equations of the tangents at these points are

$$l/r = \cos(\theta - \alpha) + \cos\theta, \qquad \ldots(1)$$

$$l/r = \cos(\theta - \beta) + \cos\theta, \qquad \ldots(2)$$

and $$l/r = \cos(\theta - \gamma) + \cos\theta. \qquad \ldots(3)$$

If C' is the point of intersection of the tangents (1) and (2), then the vectorial angle of $C' = \frac{1}{2}(\alpha + \beta)$ and the radius vector of C'

i.e., $$SC' = (l/2)\sec\tfrac{1}{2}\alpha\sec\tfrac{1}{2}\beta. \qquad \text{[See equation (6) of article 17]}$$

Similarly $\quad SA' = (l/2) \sec \frac{1}{2}\beta \sec \frac{1}{2}\gamma$ and $SB' = (l/2) \sec \frac{1}{2}\gamma \sec \frac{1}{2}\alpha.$

$$\therefore \quad SA' \,.\, SB' \,.\, SC' = (l^3/8) \sec^2 \frac{1}{2}\alpha \sec^2 \frac{1}{2}\beta \sec^2 \frac{1}{2}\gamma. \qquad \ldots(4)$$

Again since the point A whose vectorial angle is α and radius vector is SA lies on (1), therefore

$$l/SA = 1 + \cos\alpha = 2\cos^2 \frac{1}{2}\alpha \quad \text{or} \quad SA = (l/2)\sec^2 \frac{1}{2}\alpha.$$

Similarly $\quad SB = (l/2) \sec^2 \frac{1}{2}\beta$ and $SC = (l/2)\sec^2 \frac{1}{2}\gamma.$

$$\therefore \quad SA \,.\, SB \,.\, SC = (l^3/8) \sec^2 \frac{1}{2}\alpha \sec^2 \frac{1}{2}\beta \sec^2 \frac{1}{2}\gamma. \qquad \ldots(5)$$

From (4) and (5) the required result follows.

Example 12: *Find the equation of the circle circumscribing the triangle formed by tangents at three given points of a parabola.*

Solution: Let the three points on the parabola

$$l/r = 1 + \cos\theta$$

be A, B, C and let α, β, γ be their vectorial angles. Also let A', B', C' be the points of intersection of these tangents. Then as in Example 11,

the point A' is $\left(\frac{l}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}, \frac{\beta+\gamma}{2}\right)$,

the point B' is $\left(\frac{l}{2}\sec\frac{\alpha}{2}\sec\frac{\gamma}{2}, \frac{\alpha+\gamma}{2}\right)$,

and the point C' is $\left(\frac{l}{2}\sec\frac{\alpha}{2}\sec\frac{\beta}{2}, \frac{\alpha+\beta}{2}\right)$.

By actual substitution we see that the three points A', B', C' lie on the curve

$$r = \frac{l}{2}\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}\cos\left(\theta - \frac{\alpha+\beta+\gamma}{2}\right). \qquad \ldots(1)$$

The equation (1) is of the type

$$r = 2a\cos(\theta - \lambda)$$

which is the equation of a circle passing through the pole, the diameter through the pole making an angle λ with the initial line and the length of the diameter equal to $2a$.

Hence the equation (1) is the equation of the circumcircle of the triangle $A'B'C'$, the length of the diameter of the circumcircle being equal to

$$(l/2)\sec\frac{1}{2}\alpha\sec\frac{1}{2}\beta\sec\frac{1}{2}\gamma$$

and the diameter through the pole making an angle $1/2\,(\alpha+\beta+\gamma)$ with the initial line.

Remark: The centre of the circle (1) is the middle point of its diameter passing through the pole. So the vectorial angle of the centre of the circle (1) is $1/2\ (\alpha+\beta+\gamma)$ and the radius vector is $(l/4)\sec\frac{1}{2}\alpha\sec\frac{1}{2}\beta\sec\frac{1}{2}\gamma$.

Comprehensive Exercise 3

1. Prove that the centres of the four circles circumscribing the four triangles formed by the four tangents drawn to a parabola at points whose vectorial angles are $\alpha, \beta, \gamma, \delta$ lie on another circle which passes through the focus of the parabola.
2. If the tangents at any two points P and Q of a conic meet in a point T, and if the chord PQ meets the directrix corresponding to S in a point K, prove that $\angle KST$ is a right angle.
3. PQ is a variable chord of a conic having S for focus and angle PSQ is constant. Prove that the locus of the point of intersection of tangents at P and Q is a conic having S for a focus and the corresponding directrix is common with the given conic.
4. Prove that, if chords of a conic subtend a constant angle at a focus, the tangents at the ends of the chord will meet on a fixed conic and the chord will touch (or envelope) another fixed conic.
5. Prove that the locus of the point of intersection of tangents at the extremities of perpendicular focal radii of a conic is another conic having the same focus.
6. Prove that the radius vector TS of the point of intersection of tangents at P and Q bisects the angle between the radii vectors of P and Q (i.e., between PS and QS).
7. A chord PQ of a conic subtends a constant angle 2γ at the focus S and tangents at P and Q meet in T; prove that
$$\frac{1}{SP}+\frac{1}{SQ}-\frac{2\cos\gamma}{ST}=\frac{2\sin^2\gamma}{l}.$$
8. Show that the locus of the intersection of two perpendicular tangents one drawn to each of the two parabolas with a common focus whose axes are neither coincident nor perpendicular is a conic.
9. P, Q, R are three points on the conic $l/r=1+e\cos\theta$ the focus S being the pole. The tangent at Q meets SP and SR in M and N so that $SM=SN=l$. Prove that the chord PR touches the conic $l/r=1+2e\cos\theta$.

10. If PQ is the chord of contact of tangents drawn from a point T to a conic whose focus is S, prove that

(i) $ST^2 = SP \,.\, SQ$, if the conic is a parabola;

(ii) $\dfrac{1}{SP \,.\, SQ} - \dfrac{1}{ST^2} = \dfrac{1}{b^2} \sin^2 \dfrac{PSQ}{2}$ if the conic is a central conic and b is its semi-minor axis.

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19 Pair of Tangents

To prove that the equation of the pair of tangents drawn to the conic $l / r = 1 + e \cos \theta$ *from the point* (r', θ') *is* $(S^2 - 1)(S'^2 - 1) = P^2$,

where $S \equiv l / r - e \cos \theta,\ S' \equiv l / r' - e \cos \theta'$

and $P \equiv (l / r - e \cos \theta)(l / r' - e \cos \theta') - \cos (\theta - \theta')$.

Hence to prove that the equations of the asymptotes of the conic are

$$le / r = (e^2 - 1) \cos \theta \pm \sqrt{(e^2 - 1)} \sin \theta.$$

Consider a point P with vectorial angle α on the conic

$$l / r = 1 + e \cos \theta \qquad \ldots(1)$$

The tangent to (1) at the point α is

$$l / r = \cos (\theta - \alpha) + e \cos \theta. \qquad \ldots(2)$$

If the tangent (2) passes through the point (r', θ'), we have

$$l / r' = \cos (\theta' - \alpha) + e \cos \theta'. \qquad \ldots(3)$$

The required equation of the pair of tangents is obtained by eliminating α between (2) and (3).

We have

$$\begin{aligned}(S^2 - 1)(S'^2 - 1) &= \{(l / r - e \cos \theta)^2 - 1\}\{(l / r' - e \cos \theta')^2 - 1\} \\ &= \{\cos^2 (\theta - \alpha) - 1\}\{\cos^2 (\theta' - \alpha) - 1\}, \qquad \text{[Using (2) and (3)]} \\ &= \{-\sin^2 (\theta - \alpha)\}\{-\sin^2 (\theta' - \alpha)\} \\ &= \sin^2 (\theta - \alpha) \sin^2 (\theta' - \alpha). \qquad \ldots(4)\end{aligned}$$

Also

$$\begin{aligned}P &= (l / r - e \cos \theta)(l / r' - e \cos \theta') - \cos (\theta - \theta') \\ &= \cos (\theta - \alpha) \cos (\theta' - \alpha) - \cos (\theta - \theta') \qquad \text{[Using (2) and (3)]} \\ &= \frac{1}{2} \{2 \cos (\theta - \alpha) \cos (\theta' - \alpha)\} - \cos (\theta - \theta') \\ &= \frac{1}{2} \{\cos (\theta + \theta' - 2\alpha) + \cos (\theta - \theta')\} - \cos (\theta - \theta') \\ &= \frac{1}{2} \{\cos (\theta + \theta' - 2\alpha) - \cos (\theta - \theta')\}\end{aligned}$$

$$= \frac{1}{2} \{2 \sin (\theta - \alpha) \sin (\alpha - \theta')\}$$

$$= - \sin (\theta - \alpha) \sin (\theta' - \alpha).$$

$$\therefore \qquad P^2 = \sin^2 (\theta - \alpha) \sin^2 (\theta' - \alpha). \qquad ...(5)$$

From (4) and (5), we have

$$(S^2 - 1)(S'^2 - 1) = P^2. \qquad ...(6)$$

Since the equation (6) does not contain α, therefore it is the required equation of the pair of tangents drawn from the point (r', θ') to the conic $l/r = 1 + e \cos \theta$.

To find the asymptotes: The asymptotes can be regarded as the pair of tangents drawn from the centre of the conic. The coordinates of the centre referred to the focus S as pole are (ae, π) *i.e.*, $\left(\frac{el}{1-e^2}, \pi\right)$.

Therefore to find the asymptotes the point (r', θ') is to be taken as the point $\left(\frac{el}{1-e^2}, \pi\right)$.

$$\therefore \qquad r' = \frac{el}{1-e^2} \quad \text{and} \quad \theta' = \pi. \qquad ...(7)$$

Taking these values of r' and θ', we have

$$\cos (\theta - \theta') = \cos (\theta - \pi) = - \cos \theta,$$

$$\left.\begin{aligned} S' &= \frac{1}{r'} - e \cos \theta' = \frac{1-e^2}{e} - e \cos \pi = \frac{1-e^2}{e} + e = \frac{1}{e}, \\ \text{and} \quad P &= \left(\frac{l}{r} - e \cos \theta\right)\frac{1}{e} + \cos \theta = \frac{l}{er}. \end{aligned}\right\} \qquad ...(8)$$

Putting the values from (8) in (6), the equation of the asymptotes is

$$\left\{\left(\frac{l}{r} - e \cos \theta\right)^2 - 1\right\}\left\{\frac{1}{e^2} - 1\right\} = \left(\frac{l}{er}\right)^2$$

or $$\left\{\frac{l^2}{r^2} - \frac{2\,le}{r} \cos \theta + e^2 \cos^2 \theta - 1\right\}(1 - e^2) = \frac{l^2}{r^2}$$

or $$\frac{l^2}{r^2} - \frac{e^2 l^2}{r^2} - \frac{2\,le}{r}(1 - e^2) \cos \theta + e^2 (1 - e^2) \cos \theta - (1 - e^2) = \frac{l^2}{r^2}$$

or $$\frac{e^2 l^2}{r^2} + \frac{2\,le}{r}(1 - e^2) \cos \theta = e^2 (1 - e^2) \cos^2 \theta - (1 - e^2),$$

by transposition of terms.

Adding the term $(1 - e^2)^2 \cos^2 \theta$ to both sides, we have

$$\frac{e^2 l^2}{r^2} + \frac{2\,le}{r}(1 - e^2) \cos \theta + (1 - e^2)^2 \cos^2 \theta$$
$$= (1 - e^2)^2 \cos^2 \theta + e^2 (1 - e^2) \cos^2 \theta - (1 - e^2)$$

or $$\left\{\frac{el}{r} + (1-e^2)\cos\theta\right\}^2 = (1-e^2)\cos^2\theta\,\{1-e^2+e^2\} - (1-e^2)$$

$$= (1-e^2)(\cos^2\theta - 1)$$

$$= -(1-e^2)\sin^2\theta$$

$$= (e^2-1)\sin^2\theta.$$

Taking square root of both sides, we get

$$(le/r) + (1-e^2)\cos\theta = \pm\sqrt{(e^2-1)}\sin\theta$$

or $$le/r = (e^2-1)\cos\theta \pm \sqrt{(e^2-1)}\sin\theta.$$

These are the equations of the asymptotes of the conic

$$l/r = 1 + e\cos\theta.$$

20 Chord of Contact

To find the polar equation of the chord of contact of the point $T(r', \theta')$ with respect to the conic $l/r = 1 + e\cos\theta$.

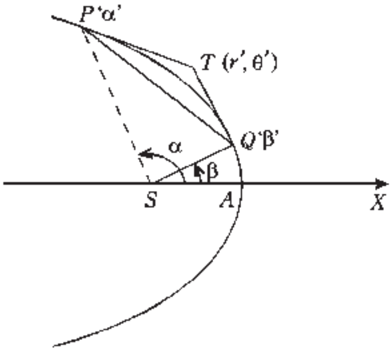
The given conic is

$$l/r = 1 + e\cos\theta. \qquad \ldots(1)$$

Let P and Q be the points of contact of the tangents drawn from the point $T(r', \theta')$ to the conic (1).

Then the chord PQ is the chord of contact of the point T with respect to the conic (1).

Let the vectorial angles of P and Q be α and β respectively.



The point of intersection $T(r', \theta')$ of the tangents at P and Q is given by

$$\theta' = \frac{1}{2}(\alpha+\beta), \qquad \ldots(2)$$

and $$l/r' = \cos\frac{1}{2}(\alpha-\beta) + e\cos\frac{1}{2}(\alpha+\beta). \qquad \ldots(3)$$

[See article 17]

The equation of the chord PQ joining the points α and β is

$$l/r = \sec\frac{1}{2}(\alpha-\beta)\cos\left\{\theta - \frac{1}{2}(\alpha+\beta)\right\} + e\cos\theta. \qquad \ldots(4)$$

The equation (4) will become the equation of the chord of contact if α, β are eliminated with the help of (2) and (3).

The equation (4) may be written as

$$\frac{l}{r} - e\cos\theta = \frac{\cos\left\{\theta - \frac{1}{2}(\alpha+\beta)\right\}}{\cos\frac{1}{2}(\alpha-\beta)}$$

or $$\frac{l}{r} - e\cos\theta = \frac{\cos(\theta-\theta')}{(l/r' - e\cos\theta')}, \text{using (2) and (3)}$$

or $$\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{r'} - e\cos\theta'\right) = \cos(\theta-\theta').$$

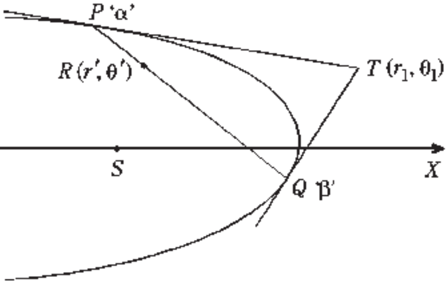
This is the required equation of the chord of contact of the point (r', θ') with respect to the conic $l/r = 1 + e\cos\theta$.

21 Polar of a Point

To find the equation of the polar of a point (r', θ') w.r.t. the conic $l/r = 1 + e\cos\theta$.

(Avadh 2006; Purvanchal 09; Kumaun 08)

Suppose we want to find the equation of the polar of a given point $R(r', \theta')$ with respect to the conic $l/r = 1 + e\cos\theta$.



If we draw chords of the given conic passing through the point R, then the locus of the point of intersection of the tangents at the extremities of these chords is said to be the polar of R with respect to the given conic.

Let PQ be any such chord meeting the conic in points P and Q whose vectorial angles are α and β respectively.

If $P(r_1, \theta_1)$ is the point of intersection of the tangents at the points P 'α' and Q 'β', then proceeding as in article 17, we have

$$\theta_1 = \frac{1}{2}(\alpha+\beta), \qquad \text{...(1)}$$

and $$l/r_1 = \cos\frac{1}{2}(\alpha-\beta) + e\cos\frac{1}{2}(\alpha+\beta)$$

i.e., $$l/r_1 - e\cos\theta_1 = \cos\frac{1}{2}(\alpha-\beta). \qquad \text{...(2)}$$

The equation of the chord PQ joining the points α and β is

$$l/r = \sec\frac{1}{2}(\alpha-\beta)\cos\left\{\theta - \frac{1}{2}(\alpha+\beta)\right\} + e\cos\theta.$$

Since it passes through the point $R(r', \theta')$, therefore

$$l/r' = \sec\frac{1}{2}(\alpha-\beta)\cos\left\{\theta' - \frac{1}{2}(\alpha+\beta)\right\} + e\cos\theta'$$

or $$l/r' - e\cos\theta' = \frac{\cos\left\{\theta' - \frac{1}{2}(\alpha+\beta)\right\}}{\cos\frac{1}{2}(\alpha-\beta)}$$

or $$(l/r' - e\cos\theta') = \frac{\cos(\theta' - \theta_1)}{(l/r_1 - e\cos\theta_1)} \qquad \text{[Using (1) and (2)]}$$

or $$(l/r_1 - e\cos\theta_1)(l/r' - e\cos\theta') = \cos(\theta' - \theta_1). \qquad \ldots(3)$$

∴ the polar of the point (r', θ') *i.e.*, the locus of the point (r_1, θ_1) is

$$(l/r - e\cos\theta)(l/r' - e\cos\theta') = \cos(\theta' - \theta)$$

[Replacing r_1 by r and θ_1 by θ in (3)]

or $$(l/r - e\cos\theta)(l/r' - e\cos\theta') = \cos(\theta - \theta'). \qquad \ldots(4)$$

Remark 1: The pole of a line is the point of intersection of the tangents at its extremities.

Remark 2: Articles 20 and 21 show that the polar of a point with respect to a given conic is the same as the chord of contact of the tangents drawn from that point to the conic. But here the point must lie outside the conic.

22 Perpendicular Lines

Let the equation of a straight line be

$$l/r = A\cos\theta + B\sin\theta. \qquad \ldots(1)$$

Multiplying both sides by r the equation (1) may be written as

$$l = Ar\cos\theta + Br\sin\theta.$$

Changing to cartesians this equation becomes

$$l = Ax + By. \qquad \ldots(2)$$

The equation of any line perpendicular to the line (2) is

$$Bx - Ay = L, \quad \text{where } L \text{ is any real number}$$

or $$Br\cos\theta - Ar\sin\theta = L, \text{ changing to polars}$$

or $$L/r = A\cos\left(\frac{1}{2}\pi + \theta\right) + B\sin\left(\frac{1}{2}\pi + \theta\right). \qquad \ldots(3)$$

Thus (3) is the equation of any line which is perpendicular to the line (1). In the equation (3) L is any real number.

Rule: *The equation of any line perpendicular to* $l/r = A\cos\theta + B\sin\theta$ *is obtained by writing* $\theta + \frac{1}{2}\pi$ *for* θ *and changing* l *to a new constant, say,* L.

23 Normal to the Conic

To find the equation of the normal at a point 'α' on the conic $l/r = 1 + e\cos\theta$.

(Rohilkhand 2005; Bundelkhand 07, 09; Kumaun 11, 14; Kashi 13; Avadh 14)

Let $P(r_1, \alpha)$ be a point on the conic

$$l/r = 1 + e\cos\theta. \qquad \ldots(1)$$

The tangent to (1) at the point $P(r_1, \alpha)$ is

$$l/r = \cos(\theta - \alpha) + e\cos\theta. \qquad \ldots(2)$$

The normal to (1) at the point P is the line perpendicular to the line (2) and passing through the point P.

The equation of any line perpendicular to the line (2) is

$$L/r = \cos\left(\theta + \frac{1}{2}\pi - \alpha\right) + e\cos\left(\theta + \frac{1}{2}\pi\right)$$

or $$L/r = -\sin(\theta - \alpha) - e\sin\theta. \qquad \ldots(3)$$

The line (3) will be the normal at the point P if it passes through the point $P(r_1, \alpha)$. So putting $r = r_1$ and $\theta = \alpha$ in (3), we have

$$L/r_1 = -\sin(\alpha - \alpha) - e\sin\alpha = -e\sin\alpha$$

or $$L = -r_1 e\sin\alpha.$$

Now putting $L = -r_1 e\sin\alpha$ in (3), the equation of the normal to (1) at the point $P(r_1, \alpha)$ is

$$-(r_1 e\sin\alpha)/r = -\sin(\theta - \alpha) - e\sin\theta$$

or $$\frac{er_1\sin\alpha}{r} = \sin(\theta - \alpha) + e\sin\theta. \qquad \ldots(4)$$

Again the point $P(r_1, \alpha)$ lies on (1).

$\therefore$ $$l/r_1 = 1 + e\cos\alpha \quad i.e., \quad r_1 = l/(1 + e\cos\alpha).$$

Putting the value of r_1 in (4), the equation of the normal at the point P in terms of α alone, is given by

$$\frac{le\sin\alpha}{(1 + e\cos\alpha)}\cdot\frac{1}{r} = \sin(\theta - \alpha) + e\sin\theta. \qquad \ldots(5)$$

Corollary : In the case of the **parabola** $l/r = 1 + \cos\theta$, we have $e = 1$ and so the equation (5) of the normal at the point 'α' becomes

$$\frac{l\sin\alpha}{1 + \cos\alpha}\cdot\frac{1}{r} = \sin(\theta - \alpha) + \sin\theta. \qquad \ldots(6)$$

(Kashi 2013)

Remark: The equation of the normal at the point $P(r_1, \alpha)$ to the conic $l/r = 1 - e\cos\theta$ is

$$\frac{-le\sin\alpha}{1 - e\cos\alpha}\cdot\frac{1}{r} = \sin(\theta - \alpha) - e\sin\theta.$$

Illustrative Examples

Example 13: *If the normals at* α, β, γ *on the parabola* $l/r = 1 + \cos\theta$ *meet at a point* (ρ, ϕ), *prove that* $2\phi = \alpha + \beta + \gamma$. **(Purvanchal 2009)**

Solution: The equation of the normal to the parabola $l/r = 1 + \cos\theta$ at the point λ on it is
$$\frac{l \sin\lambda}{1 + \cos\lambda} \cdot \frac{1}{r} = \sin(\theta - \lambda) + \sin\theta.$$

If it passes through the point (ρ, ϕ), we have
$$\frac{l \sin\lambda}{1 + \cos\lambda} \cdot \frac{1}{\rho} = \sin(\phi - \lambda) + \sin\phi.$$

Changing $\cos\lambda, \sin\lambda$ to half angles this equation takes the form
$$l \tan^3 \frac{1}{2}\lambda + 0 \,.\, \tan^2 \frac{1}{2}\lambda + (l + 2\rho\cos\phi)\tan\frac{1}{2}\lambda - 2\rho\sin\phi = 0. \qquad \ldots(1)$$

This equation being a cubic in $\tan\frac{1}{2}\lambda$ gives three values of $\tan\frac{1}{2}\lambda$ and hence three values of λ. Thus there are three points on the parabola the normals at which pass through the point (ρ, ϕ). If these points are α, β, γ, then $\tan\frac{1}{2}\alpha, \tan\frac{1}{2}\beta, \tan\frac{1}{2}\gamma$ are the roots of the cubic (1) in $\tan\frac{1}{2}\lambda$.

By the theory of equations, we have for the cubic (1),
$$\tan\frac{1}{2}\alpha + \tan\frac{1}{2}\beta + \tan\frac{1}{2}\gamma = \Sigma \tan\frac{1}{2}\alpha = 0,$$
$$\Sigma \tan\frac{1}{2}\alpha \tan\frac{1}{2}\beta = (l + 2\rho\cos\phi)/l,$$
$$\tan\frac{1}{2}\alpha \tan\frac{1}{2}\beta \tan\frac{1}{2}\gamma = 2\rho\sin\phi/l.$$

Now from trigonometry, we have
$$\tan\left(\frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{2}\gamma\right) = \frac{\Sigma \tan\frac{1}{2}\alpha - \tan\frac{1}{2}\alpha \tan\frac{1}{2}\beta \tan\frac{1}{2}\gamma}{1 - \Sigma \tan\frac{1}{2}\alpha \tan\frac{1}{2}\beta}$$
$$= \frac{0 - (2\rho\sin\phi/l)}{1 - \{(l + 2\rho\cos\phi)/l\}}$$
$$= \frac{-2\rho\sin\phi/l}{1 - \{1 + (2\rho\cos\phi)/l\}} = \tan\phi.$$

$$\therefore \qquad \frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{2}\gamma = \phi \quad \text{or} \quad \alpha + \beta + \gamma = 2\phi.$$

Example 14: *Find the locus of the pole of a chord of the conic* $l/r = 1 + e \cos\theta$ *which subtends a constant angle* 2γ *at the focus.* **(Meerut 2006, 09B; Kumaun 10, 11)**

Solution: The pole of a chord is the point of intersection of the tangents at its extremities.

Let PQ be a variable chord of the conic $l/r = 1 + e\cos\theta$ which subtends a constant angle 2γ at the focus S of the conic. Let α, β be the vectorial angles of the points P and Q respectively. Then $\angle PSQ = \alpha - \beta$. But according to the question $\angle PSQ = 2\gamma$. Therefore

$$\alpha - \beta = 2\gamma \qquad \ldots(1)$$

If $T\ (r', \theta')$ be the point of intersection of the tangents at P and Q, then proceeding as in article 17, we have

$$\theta' = \frac{\alpha + \beta}{2}, \qquad \ldots(2)$$

and
$$\frac{l}{r'} = \cos\left(\frac{\alpha - \beta}{2}\right) + e\cos\left(\frac{\alpha + \beta}{2}\right). \qquad \ldots(3)$$

The pole of the chord PQ is the point T and we have to find the locus of the point T. The locus of T will be obtained by eliminating α and β between (1), (2) and (3). Substituting the values of $\alpha - \beta$ and $\alpha + \beta$ from (1) and (2) in (3), we have

$$l/r' = \cos\gamma + e\cos\theta'$$

or
$$(l\sec\gamma)/r' = 1 + (e\sec\gamma)\cos\theta'.$$

$\therefore$ the locus of $T\ (r', \theta')$ is

$$(l\sec\gamma)/r = 1 + (e\sec\gamma)\cos\theta,$$

which is a conic whose focus is the pole S *i.e.*, the focus of the given conic.

Comprehensive Exercise 4

1. Show that three normals can be drawn from a point (ρ, ϕ) to a parabola.
2. If the normals at three points of the parabola $r = a \operatorname{cosec}^2 1/2\,\theta$ whose vectorial angles are α, β *and* γ meet in a point whose vectorial angle is ϕ, prove that $2\phi = \alpha + \beta + \gamma - \pi$.
3. If the tangent and normal at any point P of a conic meet the transverse axis in T and G respectively and if S be the focus, then show that $\dfrac{1}{SG} - \dfrac{1}{ST}$ is constant.

4. If the normals at $\alpha, \beta, \gamma, \delta$ on the conic $l/r = 1 + e\cos\theta$ meet at a point (ρ, ϕ) (*i.e.*, these normals are concurrent), prove that

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2} + \left(\frac{1+e}{1-e}\right)^2 = 0,$$

and $\alpha + \beta + \gamma + \delta - 2\phi = (2n+1)\pi$ *i.e.*, an odd multiple of π radians.

(Rohilkhand 2010)

5. If the normal at L, one of the extremities of the latus rectum of the conic $l/r = 1 + e\cos\theta$, meets the curve again at Q, show that

$$SQ = l(1 + 3e^2 + e^4)/(1 + e^2 - e^4).$$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The polar equation of the directrix of the conic $\frac{l}{r} = 1 + e\cos\theta$ corresponding to the focus taken as pole is

(a) $\frac{l}{r} = -\frac{1-e^2}{1+e^2}e\cos\theta$ (b) $\frac{l}{r} = e\cos\theta$

(c) $\frac{l}{r} = -e\cos\theta$ (d) $\frac{l}{r} = e\sec\theta$.

(Bundelkhand 2005, 07)

2. The equation of the tangent to the conic $\frac{l}{r} = 1 - e\cos\theta$ at the point 'α' on it is

(a) $\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha)$ (b) $\frac{l}{r} = -e\cos\theta + \cos(\theta - \alpha)$

(c) $\frac{l}{r} = -e\cos\theta - \cos(\theta - \alpha)$ (d) $\frac{l}{r} = e\cos\theta - \cos(\theta - \alpha)$

3. The conic $l/r = 1 + e\cos\theta$ represents a hyperbola if

(a) $e = 0$ (b) $e = 1$

(c) $e < 1$ (d) $e > 1$

(Rohilkhand 2005; Agra 14)

4. For a second degree curve if $e = 1$, then the curve shall be

(a) a hyperbola (b) an ellipse

(c) a parabola (d) none of these

(Kumaun 2015)

5. For a second degree curve if $e < 1$, then the curve shall be

(a) Hyperbola (b) Parabola
(c) Ellipse (d) Circle **(Kumaun 2007, 11)**

6. The polar equation of the circle, when its radius is a and its circumference lies on the pole and the centre on initial line is

(a) $r = 2a$ (b) $r = 2a\sin\theta$
(c) $r = 2a\cos\theta$ (d) $2a = r\cos\theta$ **(Kumaun 2007)**

7. The polar equation of a circle of radius a whose centre is at the pole is

(a) $r = 2a\cos\theta$ (b) $r = 2a\sin\theta$
(c) $r = a$ (d) $r^2 = a^2\cos\theta$ **(Kumaun 2009)**

8. Conic $\frac{8}{r} = 4 - 5\cos\theta$ is

(a) a parabola (b) a straight line
(c) an ellipse (d) a hyperbola **(Kumaun 2010)**

9. $r = 2a\cos\theta$ represents

(a) an ellipse (b) a parabola
(c) a circle (d) none of these
(Kumaun 2013)

10. For a second degree curve if $e > 1$, then the curve shall be

(a) a hyperbola (b) an ellipse
(c) a parabola (d) none of these
(Kumaun 2014)

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. A conic is the locus of a point which moves so that its distance from a fixed point is in a …… to its perpendicular distance from a fixed straight line.

2. The length of the latus rectum of the conic $l / r = 1 + e\cos\theta$ is …… .

3. If $e < 1$, the conic $l / r = 1 + e\cos\theta$ is an …… .

4. The polar equation of a conic with its focus as the pole and its axis inclined at an angle α to the initial line is $\frac{l}{r} = 1 + e$ ……

where l is the length of the semi-latus rectum of the conic and e is the eccentricity of the conic.

5. The equation of the tangent at the point (r_1, θ_1) of the conic $\frac{l}{r} = 1 + e\cos\theta$ is …… .

6. The slope of the tangent to the conic $\frac{l}{r} = 1 + e\cos\theta$ at the point 'α' on it is …… .

7. The locus of the foot of the perpendicular from the focus on any tangent to a conic (ellipse or hyperbola) is a circle called the ...... of the conic.
8. The locus of the point of intersection of two perpendicular tangents to a conic, is called the ...... of the conic.
9. The polar equation of the chord of contact of the point (r', θ') with respect to the conic $\frac{l}{r} = 1 + e\cos\theta$ is $\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{r'} - e\cos\theta'\right) = \ldots\ldots.$
10. The equation of the normal at a point 'α' on the conic $\frac{l}{r} = 1 + e\cos\theta$ is $\frac{le\sin\alpha}{1 + e\cos\alpha} \cdot \frac{1}{r} = \ldots\ldots.$

True or False

Write 'T' for true and 'F' for false statement.

1. The conic $\frac{l}{r} = 1 + 3\cos\theta$ is an ellipse.
2. The conic $\frac{l}{r} = 1 + \cos\theta$ is a parabola.
3. The equation of the tangent to the conic $\frac{l}{r} = 1 + e\cos(\theta - \alpha)$ at the point 'β' on it is $\frac{l}{r} = e\cos(\theta - \beta) + \cos(\theta - \alpha)$.
4. The equation of the director circle of the conic $\frac{l}{r} = 1 + e\cos\theta$ is $(1 - e^2)r^2 + 2ler\cos\theta - 2l^2 = 0$.
5. The equation of the polar of a point (r', θ') with respect to the conic $\frac{l}{r} = 1 + e\cos\theta$ is $\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{r'} - e\cos\theta'\right) = \sin(\theta - \theta')$.
6. The equation of the chord of the conic $\frac{l}{r} = 1 + e\cos\theta$ whose extremities are the points (r_1, θ_1) and (r_2, θ_2) is $\frac{l}{r} = e\cos\theta + \sec\left(\frac{\theta_2 - \theta_1}{2}\right)\cos\left(\theta - \frac{\theta_1 + \theta_2}{2}\right)$.

Answers

Multiple Choice Questions

1.	(b)	2.	(b)	3.	(d)	4.	(c)	5.	(c)
6.	(c)	7.	(c)	8.	(c)	9.	(c)	10.	(a)

Fill in the Blank(s)

1. constant ratio
2. $2\,l$
3. ellipse
4. $\cos(\theta - \alpha)$
5. $\frac{l}{r} = e\cos\theta + \cos(\theta - \theta_1)$
6. $-\frac{e + \cos\alpha}{\sin\alpha}$
7. auxiliary circle
8. director circle
9. $\cos(\theta - \theta')$
10. $\sin(\theta - \alpha) + e\sin\theta$

True or False

1. *F*
2. *T*
3. *F*
4. *T*
5. *F*
6. *T*

☐

Chapter 2

Systems of Co-ordinates

1 Introduction

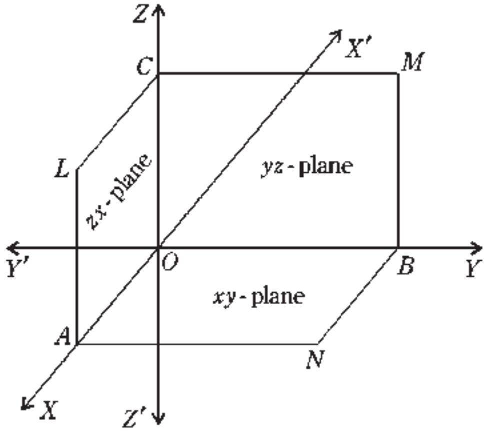
Students know well that in *co-ordinate geometry of two dimensions* (*i.e.*, *plane analytical geometry*) the position of a point in a plane is referred to two intersecting lines (in the plane of the point) called **the axes of reference** and their point of intersection called **the origin of co-ordinates.** The axes are called **rectangular axes** if they are at right angles, otherwise they are called **oblique axes**. Whatever the axes may be, they divide the plane into four quadrants called the first, second, third and fourth quadrants respectively.

But it is not always possible to determine the positions of all the points we can imagine with reference to above co-ordinate axes. For example, consider the five corners of a rectangular parallelopiped, they do not lie in one plane. Such points are called *points in space*. A point in space can be **demonstrated** as follows :

Consider your study room and let dimensions of the room be that of a rectangular parallelopiped. Now consider any particle in air, then this particle in air is a point in space.

The geometry of such points in space is discussed in "**Analytical geometry of three dimensions**" also called "**Solid geometry.**"

2 Definitions



Analytical geometry is that branch of mathematics which treats geometry algebraically *i.e.*, we have equations of geometric curves which reveal to us their nature and properties.

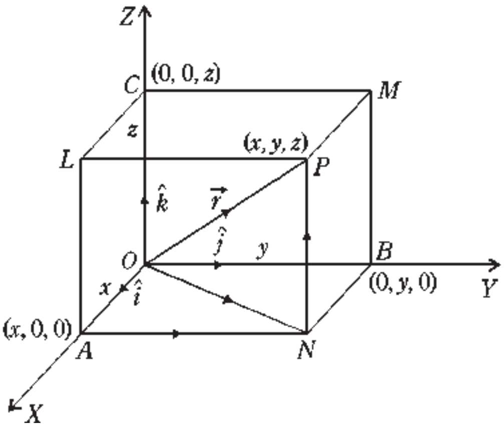
Origin, Coordinate Axes and Coordinate Planes. Let O be the point of intersection of two mutually perpendicular straight lines $X'OX$ and $Y'OY$ drawn in the plane of paper. Imagine a third straight line $Z'OZ$ passing through O and perpendicular to both $X'OX$ and $Y'OY$. Thus OZ is perpendicular to both OX and OY *i.e.*, OZ is perpendicular to the plane of the paper. Also suppose that OZ points in that direction in which a right handed screw will translate if rotated from OX to OY. Such a system of three mutually perpendicular lines namely $X'OX, Y'OY$ and $Z'OZ$ is called a *right handed system of three dimensional rectangular coordinate axes.* These lines are called x-axis, y-axis and z-axis respectively. The point O is called the *origin.* OX, OY and OZ are taken to be *positive directions* whereas OX', OY' and OZ' as *negative directions* of x-axis, y-axis and z-axis respectively.

We get three planes XOY (xy-plane), YOZ (yz-plane) and ZOX (zx-plane) if the above three axes are taken in pairs. They are called the *coordinate planes*. The three coordinate planes divide the space into eight parts, called *octants*. The octant $OXYZ$ is called the *positive octant.*

3 Coordinates of a Point in Space

Let P be a point in space. Through P draw three planes $PLAN$, $PNBM$ and $PLCM$ parallel to yz-plane, zx-plane and xy-plane respectively and meeting the x-axis, y-axis and z-axis in the points A, B and C respectively. Complete the parallelopiped whose coterminous edges are OA, OB and OC respectively. Let $OA = x, OB = y$ and $OC = z$, where x, y, z are taken with proper signs by the rule explained in article 2. Then x is called the x-coordinate of P, y is called the y-coordinate of P and z is called the z-coordinate of P. These coordinates are written in the form of the ordered triad (x, y, z) and we say that the *coordinates of P are* (x, y, z).

Obviously the x-coordinate of P is the algebraic distance of P from the yz-plane, the y-coordinate of P is the algebraic distance of P from the zx-plane and the



z-coordinate of P is the algebraic distance of P from the xy-plane. These coordinates are taken positive or negative in the sense explained in article 2. ***Thus the perpendicular distances of a point P with proper signs from the three coordinate planes respectively are the coordinates of the point P.***

Obviously the coordinates of A are $(x, 0, 0)$, those of B are $(0, y, 0)$ and those of C are $(0, 0, z)$.

The line PA is in the plane $LANP$ which is parallel to the yz-plane *i.e.*, perpendicular to OX and so PA is perpendicular to OX. Similarly PB is perpendicular to OY and PC is perpendicular to OZ. Thus if the coordinates of P are (x, y, z) and if the perpendiculars from P on x-axis, y-axis and z-axis meet them at A, B and C respectively, then

$$OA = x,\ OB = y,\ OC = z.$$

The z-coordinate of P is zero if and only if P lies in the xy-plane, the y-coordinate of P is zero if and only if P lies in the zx-plane and the x-coordinate of P is zero if and only if P lies in the yz-plane. The coordinates of the origin O are $(0, 0, 0)$. The y and z coordinates of each point on x-axis are both zero, the z and x coordinates of each point on y-axis are both zero and the x and y coordinates of each point on z-axis are both zero.

Relation between the position vector of a point in space and the coordinates of that point.

Let $\vec{r}$ be the position vector of the point $P\ (x, y, z)$ *i.e.*, $\overrightarrow{OP} = \vec{r}$. Let $\hat{i}$, $\hat{j}$ and $\hat{k}$ denote the unit vectors in the directions OX, OY and OZ respectively. Draw PN perpendicular from P to the xy-plane so that $NP = z$ with proper sign and $\overrightarrow{NP} = z\ \hat{k}$. Through N draw a straight line NA parallel to y-axis and meeting the x-axis at A and a straight line NB parallel to x-axis and meeting the y-axis at B. Then $OA = x$ with proper sign and $OB = y$ with proper sign.

We have $\overrightarrow{OA} = x\hat{i}$ and $\overrightarrow{OB} = y\hat{j} = \overrightarrow{AN}$.

Now $\vec{r} = \overrightarrow{OP} = \overrightarrow{ON} + \overrightarrow{NP} = \overrightarrow{OA} + \overrightarrow{AN} + \overrightarrow{NP}$ $[\because \overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}]$

$= x\hat{i} + y\hat{j} + z\hat{k}.$

Instead of writing $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, we often find it convenient to write it as $\vec{r} = (x, y, z)$.

Thus the position vector of a point P is the vector $x\hat{i} + y\hat{j} + z\hat{k}$ *if and only if the coordinates of the point P are* (x, y, z).

4 Octants

The three co-ordinate planes namely yz-plane, zx-plane and xy-plane divide the space into eight parts called the octants, and to which octant the point P belongs is determined by the signs of the co-ordinates of the point P. [See figure of article 2.] The octant $OXYZ$ in which the three co-ordinates are all positive is called the **first octant.** The following table determines the signs in eight octants :

Octant	$OXYZ$	$OXY'Z$	$OXY'Z'$	$OXYZ'$	$OX'YZ$	$OX'Y'Z$	$OX'YZ'$	$OX'Y'Z'$
x	+	+	+	+	–	–	–	–
y	+	–	–	+	+	–	+	–
z	+	+	–	–	+	+	–	–

Illustrative Examples

Example 1: *What are the positions of the following points ?*

(i) $(1, 2, 3)$, *(ii)* $(1, -2, 3)$, *(iii)* $(0, 0, -3)$,
(iv) $(-1, -2, 0)$, *(v)* $(2, 0, 0)$, *(vi)* $(-1, -2, -3)$.

Solution: (i) $(1, 2, 3)$ is a point in the octant $OXYZ$ and its distances from the co-ordinate planes yz, zx and xy are 1, 2 and 3 respectively.

(ii) $(1, -2, 3)$ is a point in the octant $OXY'Z$ and its distances from the co-ordinate planes yz, zx and xy are 1, 2 and 3 respectively.

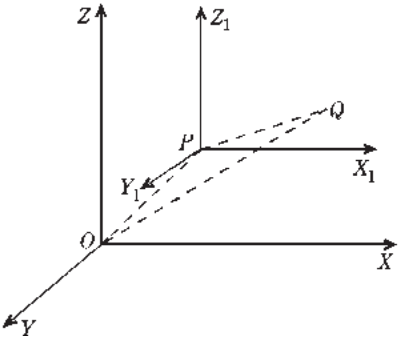
(iii) $(0, 0, -3)$ is a point on OZ' *i.e.*, on the –ve side of the z-axis situated at a distance 3 from the origin O.

(iv) $(-1, -2, 0)$ is a point in the co-ordinate plane xy since its z-coordinate is zero. It lies in the octant $OX'Y'Z$ and its distances from the co-ordinate planes yz and zx are 1 and 2 respectively.

(v) $(2, 0, 0)$ is a point on the positive side of the x-axis situated at a distance 2 from the origin O.

(vi) $(-1, -2, -3)$ is a point in the octant $OX'Y'Z'$ and its distances from the co-ordinate planes yz, zx and xy are 1, 2 and 3 respectively.

5 Change of Origin



Let OX, OY, OZ be a rectangular set of axes. Referred to these axes let the co-ordinates of two points P and Q be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Suppose we want to shift the origin from O to the point P *i.e.*, we want to find the co-ordinates of Q referred to P as origin.

Draw the new axes PX_1, PY_1 and PZ_1 parallel to the original axes OX, OY and OZ respectively.

The position vectors of the points P and Q with respect to O as origin are given by

$$\overrightarrow{OP} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k},$$

$$\overrightarrow{OQ} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}.$$

Also the position vector of the point Q with respect to P as origin is $\overrightarrow{PQ}$. Now we have

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) - (x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k})$$

$$= (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k}$$

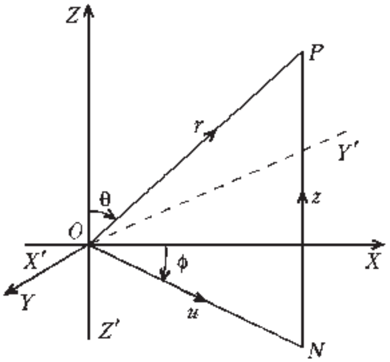
$$= (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

Therefore, the co-ordinates of the point Q with respect to the new origin P are

$$(x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

6 Spherical Polar Co-ordinates

Let $X'OX, Y'OY$ and $Z'OZ$ be the set of rectangular axes. Let P be a point in space. Draw PN perpendicular from P to the xy-plane. The position of P is determined if the length OP, angles ZOP and XON are known. Suppose $OP = r$, $\angle ZOP = \theta$ and $\angle XON = \phi$, measured positively in the directions shown by arrows in the figure. The quantities r, θ, ϕ defined as above, are called the **spherical polar co-ordinates** of P and are written as (r, θ, ϕ).



Now we shall find relations between these co-ordinates and cartesian co-ordinates. Let (x, y, z) be the cartesian co-ordinates of P. Hence we have

$$z = PN = OP \cos(\angle OPN) = r\cos(\angle ZOP) = r\cos\theta. \qquad \ldots(1)$$

Also $\quad ON = OP \sin \angle OPN = r \sin \theta \qquad [\because \angle ONP = 90°]$

$\therefore \quad x = ON \cos \phi = r \cos \phi \sin \theta, \qquad \ldots(2)$

and $\quad y = ON \sin \phi = r \sin \phi \sin \theta. \qquad \ldots(3)$

Thus relations (2), (3) and (1) give the relations between x, y, z and r, θ, ϕ.

Now squaring the relations (2) and (3) and adding, we get

$$x^2 + y^2 = ON^2$$

or $\quad u^2 = x^2 + y^2, \quad$ where $u = ON$

or $\quad \sqrt{(x^2 + y^2)} = u = r \sin \theta. \qquad \ldots(4)$

Dividing (4) by (1), we get

$$\tan \theta = \sqrt{(x^2 + y^2)} / z.$$

Dividing (3) by (2), we get

$$\tan \phi = y / x.$$

Squaring (1) and (4) and adding, we get

$$x^2 + y^2 + z^2 = r^2.$$

Thus the relations between spherical polar co-ordinates and cartesian co-ordinates are

$$x = r \cos \phi \sin \theta, \ y = r \sin \phi \sin \theta, \ z = r \cos \theta$$

$$x^2 + y^2 + z^2 = r^2, \ \tan \theta = \sqrt{(x^2 + y^2)} / z, \ \tan \phi = y / x.$$

7 Cylindrical Co-ordinates

See figure of article 6. Let P be a point in space. The position of P can also be determined if the measures of $ON, \angle XON$ and NP are known. Suppose $ON = u, \angle XON = \phi, NP = z$. The quantities u, ϕ, z are called the cylindrical co-ordinates of P and are written as (u, ϕ, z).

Let (x, y, z) be the cartesian co-ordinates of P, then N has the co-ordinates $(x, y, 0)$. Hence, we have

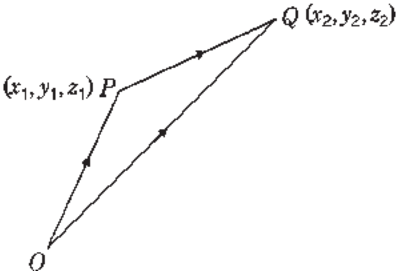
$$x = ON \cos \phi = u \cos \phi, \ y = u \sin \phi, \ z = z.$$

Also $\quad u^2 = x^2 + y^2, \ \tan \phi = y / x.$

We observe that the z-coordinate is the same in the two systems *i.e.*, cartesian and cylindrical.

8 Formula for Distance between Two given Points

Theorem: *To show that the distance between the points* $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$ *is given by* $\quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.



Proof: Referred to some origin O let the coordinates of two given points P and Q be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

Then $\overrightarrow{OP}$ = position vector of the point P

$$= x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$

and $\overrightarrow{OQ}$ = position vector of the point Q

$$= x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\ .$$

We have $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$

$$= (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}\ .$$

$\therefore$ distance $PQ = |\overrightarrow{OQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\ .$

Hence distance between the points $P\ (x_1, y_1, z_1)$ and $Q\ (x_2, y_2, z_2)$

$$= PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}\ .$$

Corollary: *The distance of the point $P\ (x, y, z)$ from the origin $O\ (0, 0, 0)$ is*

$$OP = \sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2} = \sqrt{x^2 + y^2 + z^2}\ .$$

Illustrative Examples

Example 2: *Show that the points $(0, 7, 10)$, $(-1, 6, 6)$, $(-4, 9, 6)$ form an isosceles right angled triangle.*

Solution: Let ABC be a given triangle and let the coordinates of the vertices A, B and C be $(0, 7, 10)$, $(-1, 6, 6)$ and $(-4, 9, 6)$ respectively. We have

$$AB = \sqrt{(-1 - 0)^2 + (6 - 7)^2 + (6 - 10)^2} = \sqrt{1 + 1 + 16} = \sqrt{18}$$

$$BC = \sqrt{(-4 + 1)^2 + (9 - 6)^2 + (6 - 6)^2} = \sqrt{9 + 9 + 0} = \sqrt{18}$$

and $CA = \sqrt{(0 + 4)^2 + (7 - 9)^2 + (10 - 6)^2} = \sqrt{16 + 4 + 16}$

$$= \sqrt{36} = 6.$$

Since $AB = BC$, therefore $\Delta\ ABC$ is an isosceles triangle.

Again $AB^2 + BC^2 = 18 + 18 = 36 = CA^2$. $\quad\therefore\ \angle ABC = 90^\circ$.

Hence $\Delta\ ABC$ is also a right angled triangle. Therefore the given triangle is an isosceles right angled triangle.

Example 3: *P is a variable point and the coordinates of two given points A and B are $(-2, 2, 3)$ and $(13, -3, 13)$ respectively. Find the locus of P if $3PA = 2PB$.*

Solution: Let the coordinates of P be (x, y, z). Then

$$PA = \sqrt{(x+2)^2 + (y-2)^2 + (z-3)^2}$$

and $$PB = \sqrt{(x-13)^2 + (y+3)^2 + (z-13)^2}\ .$$

Now it is given that $3PA = 2PB$.

$\therefore$ $$9PA^2 = 4PB^2$$

$\Rightarrow$ $$9\{(x+2)^2 + (y-2)^2 + (z-3)^2\} = 4\{(x-13)^2 + (y+3)^2 + (z-13)^2\}$$

$\Rightarrow$ $$9(x^2 + y^2 + z^2 + 4x - 4y - 6z + 17) = 4(x^2 + y^2 + z^2 - 26x + 6y - 26z + 347)$$

$\Rightarrow$ $$5x^2 + 5y^2 + 5z^2 + 140x - 60y + 50z - 1235 = 0$$

$\Rightarrow$ $$x^2 + y^2 + z^2 + 28x - 12y + 10z - 247 = 0.$$

Hence the required locus of the point P is the surface

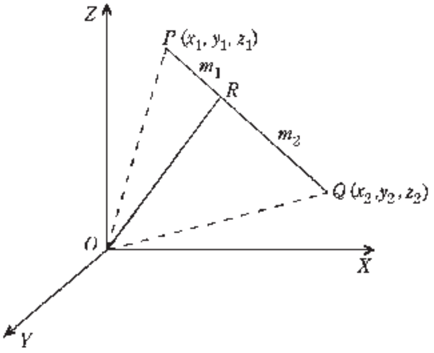
$$x^2 + y^2 + z^2 + 28x - 12y + 10z - 247 = 0.$$

9 Section Formulae

To find the co-ordinates of the point which divides the straight line joining two given points.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be the given points.

Let (x, y, z) be the required co-ordinates of R, the point which divides the join of the line joining the two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally in the ratio $m_1 : m_2$. The position vectors of the points $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ and $R(x, y, z)$ are given by



$$\vec{OP} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}, \qquad \ldots(1)$$

$$\vec{OQ} = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}, \qquad \ldots(2)$$

$$\vec{OR} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \qquad \ldots(3)$$

Now the point R divides the join of P and Q in the ratio $m_1 : m_2$, so that

$$\frac{m_1}{m_2} = \frac{PR}{RQ} \quad \text{or} \quad m_1\,(RQ) = m_2\,(PR).$$

Hence $$m_2\,\vec{PR} = m_1\,\vec{RQ}$$

$$\Rightarrow \quad m_2\,(\overrightarrow{OR} - \overrightarrow{OP}) = m_1\,(\overrightarrow{OQ} - \overrightarrow{OR})$$

$$\Rightarrow \quad (m_1 + m_2)\,\overrightarrow{OR} = m_1\,\overrightarrow{OQ} + m_2\,\overrightarrow{OP}$$

$$\Rightarrow \quad \overrightarrow{OR} = \frac{m_1\,\overrightarrow{OQ} + m_2\,\overrightarrow{OP}}{m_1 + m_2}$$

$$\Rightarrow \quad x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \frac{m_1\,(x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}) + m_2\,(x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k})}{m_1 + m_2}$$

[Using (1), (2) and (3)]

$$= \frac{(m_1x_2 + m_2x_1)\,\mathbf{i} + (m_1y_2 + m_2y_1)\,\mathbf{j} + (m_1z_2 + m_2z_1)\,\mathbf{k}}{m_1 + m_2}.$$

Equating the coefficients of **i, j, k ,** we get

$$x = \frac{m_1x_2 + m_2x_1}{m_1 + m_2},\quad y = \frac{m_1y_2 + m_2y_1}{m_1 + m_2},\quad z = \frac{m_1z_2 + m_2z_1}{m_1 + m_2}.$$

Corollary 1: Mid-point formula. *The co-ordinates of the mid-point of the join of* (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *are*

$$x = \frac{x_1 + x_2}{2},\quad y = \frac{y_1 + y_2}{2},\quad z = \frac{z_1 + z_2}{2}. \qquad [\text{Putting } m_1 = m_2]$$

Corollary 2: *If* $m_1 : m_2 = \lambda : 1$, *then the co-ordinates of the point R are*

$$\left(\frac{x_1 + \lambda\,x_2}{\lambda + 1}, \frac{y_1 + \lambda\,y_2}{\lambda + 1}, \frac{z_1 + \lambda\,z_2}{\lambda + 1}\right).$$

These are called **general coordinates of a point on the line *PQ*.**

Corollary 3: If the ratio (m_1 / m_2) is **positive**, then the point *R* divides *PQ* internally and if it is **negative** then externally.

The co-ordinates of the point $R\,(x, y, z)$ which divides the join of the line joining the two points $P\,(x_1, y_1, z_1)$ and $Q\,(x_2, y_2, z_2)$ **externally** in the ratio $m_1 : m_2$ are

$$\left(\frac{m_1\,x_2 - m_2\,x_1}{m_1 - m_2}, \frac{m_1\,y_2 - m_2\,y_1}{m_1 - m_2}, \frac{m_1\,z_2 - m_2\,z_1}{m_1 - m_2}\right).$$

General coordinates of a point on a line:

Let $P\,(x_1, y_1, z_1)$ and $Q\,(x_2, y_2, z_2)$ be any two given points. Let *R* be any point on the line *PQ* . Suppose *R* divides *PQ* in the ratio $\lambda : 1$. Then the coordinates of *R* are

$$\left(\frac{\lambda\,x_2 + x_1}{\lambda + 1}, \frac{\lambda\,y_2 + y_1}{\lambda + 1}, \frac{\lambda\,z_2 + z_1}{\lambda + 1}\right), \lambda \neq -1.$$

These are called the general coordinates of a point on the line *PQ* . If λ is positive, then *R* divides *PQ* internally and if λ is negative, then *R* divides *PQ* externally.

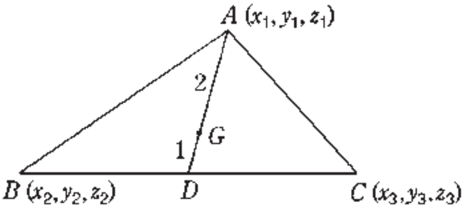
10 Centroid of a Triangle and Centroid of a Tetrahedron

Theorem 1: *To show that the centroid of the triangle with vertices* $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ *and* $C(x_3, y_3, z_3)$ *is the point*

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right).$$

Proof: Let D be the middle point of BC. Then the coordinates of D are

$$\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}, \frac{z_2 + z_3}{2}\right).$$



The centroid of the $\Delta\ ABC$ is the point G on the median AD dividing AD internally in the ratio $2:1$. So the x-coordinate of G

$$= \frac{2 \cdot \left(\frac{x_2 + x_3}{2}\right) + 1 \cdot x_1}{2 + 1}$$

$$= \frac{x_1 + x_2 + x_3}{3}.$$

Similarly the y-coordinate of G

$$= \frac{y_1 + y_2 + y_3}{3}$$

and the z-coordinate of G

$$= \frac{z_1 + z_2 + z_3}{3}.$$

Hence the coordinates of the centroid of the triangle ABC are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right).$$

Theorem 2: *To show that the centroid of the tetrahedron with vertices* $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ *and* $D(x_4, y_4, z_4)$ *is the point*

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right).$$

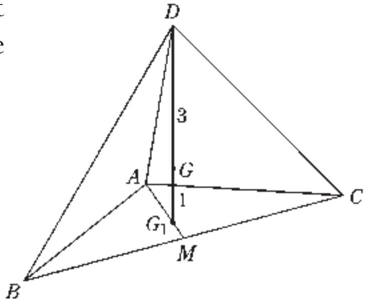
Proof: Let G_1 be the centroid of the face ABC of the tetrahedron $DABC$. Then the coordinates of G_1 are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right).$$

The centroid of the tetrahedron *DABC* is the point G on the line DG_1 dividing DG_1 internally in the ratio 3 : 1. So the x-coordinate of G

$$= \frac{3\,.\left(\frac{x_1 + x_2 + x_3}{3}\right) + 1\,.\,x_4}{3 + 1}$$

$$= \frac{x_1 + x_2 + x_3 + x_4}{4}\,.$$



Similarly we can find the y and z coordinates of G .

Hence the centroid of the tetrahedron *DABC* is the point

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right).$$

Remark: Remember the following facts about different types of quadrilaterals. A quadrilateral *ABCD* is

(i) a parallelogram if $AB = CD$ and $BC = AD$

(ii) a rhombus if $AB = BC = CD = DA$

(iii) a rectangle if $AB = CD$, $BC = AD$ and $AC = BD$

(iv) a square if $AB = BC = CD = DA$ and $AC = BD$.

Illustrative Examples

Example 4: *Find the coordinates of the point which divides the join of the points* $A\ (3, 1, -2)$ *and* $B\ (1, -3, -1)$.

(i) *internally in the ratio* 2 : 3 (*ii*) *externally in the ratio* 3 : 1.

Solution: A is the point (3, 1, – 2) and B is the point (1, – 3, – 1).

(i) Coordinates of the point dividing *AB* internally in the ratio 2 : 3 are

$$\left(\frac{3\,(3) + 2\,(1)}{2+3}, \frac{3\,(1) + 2\,(-3)}{2+3}, \frac{3\,(-2) + 2\,(-1)}{2+3}\right) \quad i.e., \quad \left(\frac{11}{5}, -\frac{3}{5}, -\frac{8}{5}\right).$$

(ii) Coordinates of the point dividing *AB* externally in the ratio 3 : 1 are

$$\left(\frac{1\,(3) - 3\,(1)}{1-3}, \frac{1\,(1) - 3\,(-3)}{1-3}, \frac{1\,(-2) - 3\,(-1)}{1-3}\right) \quad i.e., \quad \left(0, -5, -\frac{1}{2}\right).$$

Example 5: *A point P lies on the line whose end points are* $A\ (1, 2, 3)$ *and* $B\ (2, 10, 1)$. *If the z-coordinate of P is* 7, *find its other coordinates.*

Solution: Let the coordinates of the point P be (x, y, z) and let it divide the join of $A\ (1, 2, 3)$ and $B\ (2, 10, 1)$ in the ratio $\lambda : 1$. Then

$$z = \frac{\lambda\,.\,1 + 1\,.\,3}{\lambda + 1} = \frac{\lambda + 3}{\lambda + 1}\,.$$

But it is given that the z-coordinate of P is 7.

$$\therefore \qquad 7 = \frac{\lambda + 3}{\lambda + 1} \quad \text{or} \quad 7\lambda + 7 = \lambda + 3 \quad \text{or} \quad 6\lambda = -4 \quad \text{or} \quad \lambda = -2/3$$

$$\therefore \qquad x = \frac{\lambda \cdot 2 + 1 \cdot 1}{\lambda + 1} = \frac{2(-2/3) + 1}{(-2/3) + 1} = \frac{(-4/3) + 1}{1/3} = -1$$

and
$$y = \frac{\lambda \cdot 10 + 1 \cdot 2}{\lambda + 1} = \frac{10(-2/3) + 2}{(-2/3) + 1} = \frac{(-20/3) + 2}{1/3} = -14.$$

Example 6: *Find the ratio in which the join of* $A(2, 1, 5)$ *and* $B(3, 4, 3)$ *is divided by the plane* $2x + 2y - 2z = 1$. *Also find the coordinates of the point of division.*

Solution: Suppose the plane $2x + 2y - 2z = 1$ meets the line joining the points $A(2, 1, 5)$ and $B(3, 4, 3)$ at the point C and C divides AB in the ratio $\lambda : 1$. Then the coordinates of C are $\left(\frac{3\lambda + 2}{\lambda + 1}, \frac{4\lambda + 1}{\lambda + 1}, \frac{3\lambda + 5}{\lambda + 1}\right)$. ...(1)

But the point C lies on the plane $2x + 2y - 2z = 1$. So its coordinates must satisfy the equation of this plane.

$$\therefore \qquad 2\left(\frac{3\lambda + 2}{\lambda + 1}\right) + 2\left(\frac{4\lambda + 1}{\lambda + 1}\right) - 2\left(\frac{3\lambda + 5}{\lambda + 1}\right) = 1$$

or
$$6\lambda + 4 + 8\lambda + 2 - 6\lambda - 10 = \lambda + 1 \quad \text{or} \quad 7\lambda = 5 \quad \text{or} \quad \lambda = \frac{5}{7}.$$

$\therefore$ the required ratio is $(5/7) : 1$ *i.e.*, $5 : 7$ *i.e.*, the plane divides AB internally in the ratio $5 : 7$.

Putting $\lambda = 5/7$ in (1), the coordinates of the point of division C are

$$\left(\frac{29}{12}, \frac{9}{4}, \frac{25}{6}\right).$$

Comprehensive Exercise 1

1. (i) Find the locus of a point P which moves in such a way that its distance from the point $A(u, v, w)$ is always equal to a.
   (ii) A, B, C are three points on the axes of x, y and z respectively at distances a, b, c from the origin O; find the co-ordinates of the point which is equidistant from A, B, C and O.
2. (i) Show that the points $A(0, 1, 2)$, $B(2, -1, 3)$ and $C(1, -3, 1)$ are the vertices of an isosceles right angled triangle.
   (ii) Show that the points $(1, 2, 3)$, $(2, 3, 1)$ and $(3, 1, 2)$ form an equilateral triangle.
3. (i) Find the co-ordinates of the point which divides the join of $(2, 3, 4)$ and $(3, -4, 7)$ in the ratio $2 : -4$. **(Meerut 2003)**
   (ii) Find the ratios in which the sphere $x^2 + y^2 + z^2 = 504$ divides the line joining the points $(12, -4, 8)$ and $(27, -9, 18)$.

4. (i) Using distance formula show that the points $A(3, 2, -4)$, $B(5, 4, -6)$ and $C(9, 8, -10)$ are collinear. Find the ratio in which B divides AC.

(Kumaun 2007)

(ii) Find the ratio in which the line joining the points $A(2, 3, 4)$ and $B(-3, 5, -4)$ is divided by the yz-plane. Also, find the point of intersection.

5. Three vertices of a parallelogram $ABCD$ are $A(3, 4, -1)$, $B(7, 10, -3)$ and $C(8, 1, 0)$. Find the fourth vertex D.

6. What are the perpendicular distances of the point (x, y, z) from the coordinate axes ?

7. Find the ratio in which the xy-plane divides the join of $A(-3, 4, -8)$ and $B(5, -6, 4)$. Also find the point of intersection of the line with the plane.

8. The mid-points of the sides of a triangle are $(1, 5, -1)$, $(0, 4, -2)$ and $(2, 3, 4)$. Find its vertices.

Answers 1

1. (i) $x^2 + y^2 + z^2 - 2xu - 2yv - 2wz + u^2 + v^2 + w^2 - a^2 = 0$

(ii) $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$

3. (i) $(1, 10, 1)$ (ii) $2 : 3$ and $2 : -3$

4. (i) $1 : 2$ (ii) $2 : 3$; $\left(0, \frac{19}{5}, \frac{4}{5}\right)$

5. $(4, -5, 2)$ 6. $\sqrt{z^2 + y^2}$; $\sqrt{z^2 + x^2}$; $\sqrt{x^2 + y^2}$

7. $(7/3, -8/3, 0)$ 8. $A(1, 2, 3)$, $B(3, 4, 5)$ and $C(-1, 6, -7)$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If the vertices of a triangle ABC are the points $A(2, -1, 0)$, $B(3, 3, -3)$ and $C(0, 1, 4)$, then the coordinates of its centroid are

(a) $\left(\frac{5}{3}, 1, \frac{1}{3}\right)$ (b) $\left(-\frac{5}{3}, 2, -\frac{1}{3}\right)$

(c) $(5, 2, 1)$ (d) $\left(4, \frac{1}{3}, \frac{2}{3}\right)$

2. If A, B and C are the points $A\ (2\ ,3,4)$, $B\ (3,-2\ ,2)$ and $C\ (6,-17,-4)$, then the ratio in which C divides AB is

(a) $-4:3$	(b) $3:4$
(c) $4:3$	(d) $1:2$

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. If the vertices of a triangle are the points $(x_1 , y_1 , z_1), (x_2 , y_2 , z_2)$ and (x_3 , y_3 , z_3), then the coordinates of its centroid are ……

2. The coordinates of the point dividing the line joining the points $P\ (x_1 , y_1 , z_1)$ and $Q\ (x_2 , y_2 , z_2)$ externally in the ratio $m : n$ are …… .

3. The distance between the points $A\ (-2\ , 1, -3)$ and $B\ (4, 3, -6)$ is …… .

4. The ratio in which the yz-plane divides the join of the points $(-2\ , 4, 7)$ and $(3, -5, 8)$ is …… .

Answers

Multiple Choice Questions

1. (a) **2.** (a)

Fill in the Blank(s)

1. $\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}, \dfrac{z_1 + z_2 + z_3}{3}\right)$

2. $\left(\dfrac{m\ x_2 - n\ x_1}{m - n}, \dfrac{m\ y_2 - n\ y_1}{m - n}, \dfrac{m\ z_2 - n\ z_1}{m - n}\right)$

3. 7

4. 2 : 3

☐

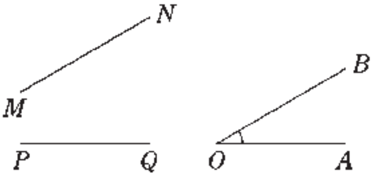
Chapter

3

Direction Cosines and Projections

1 Angle between Two Non-coplanar (i.e. Non-intersecting) Lines

Let PQ and MN be two non-coplanar lines. The angle between two non-coplanar lines PQ and MN is equal to the angle between two straight lines OA and OB drawn from any point O parallel to PQ and MN respectively. Thus the angle between the lines PQ and MN is equal to the angle AOB.



2 Direction Cosines of a Directed Line or a Vector

(Kumaun 2001; Kanpur 11)

Definition: *Suppose a directed line or a vector makes angles* α, β *and* γ *with the positive directions of x-axis, y-axis and z-axis respectively. Then*

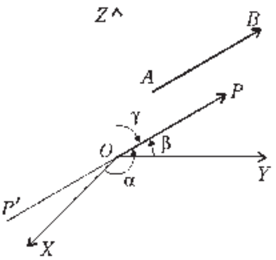
$$\cos\alpha,\ \cos\beta,\ \cos\gamma$$

are called the direction cosines (briefly written as d.c.'s) of that directed line or vector. These are usually denoted by l, m, n respectively.

Thus $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

The angles α, β, γ are known as *direction angles* of the line.

Let AB be a given directed line or a vector. Draw a line OP through the origin O in the direction of the line AB. Measure angles α, β, γ made by OP with OX, OY, OZ respectively as shown by arrows. Then $\cos\alpha, \cos\beta, \cos\gamma$ are d.c.'s of the line AB.



Clearly OP' *i.e.*, the line through O in the direction of the directed line BA makes angles $180° - \alpha, 180° - \beta, 180° - \gamma$ with OX, OY, OZ respectively. So d.c.'s of the directed line BA are

$$\cos(180° - \alpha), \cos(180° - \beta), \cos(180° - \gamma)$$

i.e., $$-\cos\alpha, -\cos\beta, -\cos\gamma.$$

Thus if the d.c.'s of a directed line AB are l, m, n, then the d.c.'s of the directed line BA whose direction is opposite to the direction of AB are $-l, -m, -n$.

Remark: Since the angles α, β, γ are not coplanar, therefore $\alpha + \beta + \gamma \neq 360°$.

Direction Cosines of the Coordinate Axes:

Since the axis of x makes angles 0°, 90°, 90° with the axes of x, y, z respectively, therefore by definition, its d.c.'s are cos 0°, cos 90°, cos 90° *i.e.*, 1, 0, 0. **(Kumaun 2008)**

Hence the **d.c.'s of the x-axis are 1, 0, 0.**

Similarly the **d.c.'s of the y-axis are 0, 1, 0 and the d.c.'s of the z-axis are 0, 0, 1.**

(Kumaun 2008)

3 Direction Ratios of a Line or a Vector

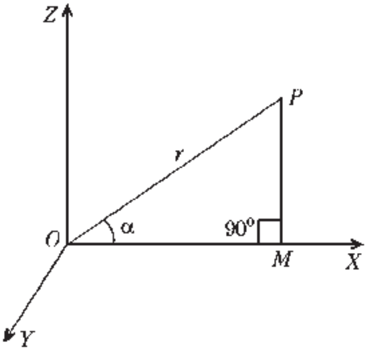
Definition: *Suppose l, m, n are the direction cosines of a line or a vector. Then any three numbers a, b, c which are proportional to l, m, n i.e., $l/a = m/b = n/c$ are called direction ratios (briefly written as d.r.'s) of that line or vector.*

Direction cosines of a line are unique. But the direction ratios of a line are by no means unique. If a, b, c are direction ratios of a line, then $\lambda a, \lambda b, \lambda c$ are also direction ratios of that line where λ is any non-zero real number.

4 Position of a Point by Radius Vector and Direction Cosines

If l, m, n are the direction cosines of a line OP, and OP = r, then the co-ordinates of P are (lr, mr, nr).

Let $P(x, y, z)$ be a point in the space and O the origin. Then length $OP = r$ is the radius vector of the point P. Draw PM perpendicular from P to OX meeting it at M. Then $x = OM$. From the right angled ΔOMP, we have



$$\frac{OM}{OP} = \cos\alpha = l$$

or $\frac{x}{r} = l$ or $x = lr$.

Similarly, $y = mr$ and $z = nr$.

Hence, co-ordinates of P are $(\boldsymbol{lr, mr, nr})$.

Corollary: If (x, y, z) be the co-ordinates of a point P, such that $OP = r$, then the direction cosines of OP are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$.

5 Relation between the Direction Cosines

If l, m, n are the direction cosines of a line, then $\boldsymbol{l^2 + m^2 + n^2 = 1}$.

(Kumaun 2001; Kanpur 05; Agra 14; Bundelkhand 12; Kashi 12)

Through O draw a line OP parallel to the given line. Then the direction cosines of OP are l, m, n and let $OP = 1$.

$\therefore$ the co-ordinates of P are (l, m, n) and the co-ordinates of O are $(0, 0, 0)$.

$\therefore$ $$OP^2 = (l - 0)^2 + (m - 0)^2 + (n - 0)^2$$

or $$(1)^2 = l^2 + m^2 + n^2 \qquad [\because OP = 1]$$

or $$\boldsymbol{l^2 + m^2 + n^2 = 1}.$$

Corollary: *If* α, β, γ *are the angles which a line makes with the axes, then*

$$\boldsymbol{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}.$$

(Kanpur 2004; Kashi 13)

6 Direction Cosines of the Vector $\vec{r} = a\vec{i} + b\vec{j} + c\vec{k}$, where $r = |\vec{r}|$

(Avadh 2014)

Theorem 1: *If* $\vec{r} = a\,\hat{i} + b\,\hat{j} + c\,\hat{k}$, *then prove that*

(i) *a, b, c are direction ratios of* $\vec{r}$

(ii) *direction cosines l, m, n of* $\vec{r}$ *are given by*

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}, n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

(iii) $l^2 + m^2 + n^2 = 1$

(iv) *unit vector in the direction of* $\vec{r}$ *is* $l\,\hat{i} + m\,\hat{j} + n\,\hat{k}$ *and*

$$\vec{r} = |\vec{r}|(l\,\hat{i} + m\,\hat{j} + n\,\hat{k}).$$

Proof: Suppose the vector $\vec{r} = a\,\hat{i} + b\,\hat{j} + c\,\hat{k}$ which is in the direction of a given line, makes angles α, β, γ with the positive directions of x-axis, y-axis and z-axis respectively. If l, m, n are the direction cosines of $\vec{r}$, then $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

We have $\vec{r} \bullet \hat{i} = (a\,\hat{i} + b\,\hat{j} + c\,\hat{k}) \bullet \hat{i} = a$, $\vec{r} \bullet \hat{j} = b$ and $\vec{r} \bullet \hat{k} = c$.

Now $\vec{r} \bullet \hat{i} = a \Rightarrow |\vec{r}||\hat{i}|\cos\alpha = a \Rightarrow |\vec{r}|\cos\alpha = a \Rightarrow \cos\alpha = \dfrac{a}{|\vec{r}|}$;

$\vec{r} \bullet \hat{j} = b \Rightarrow |\vec{r}||\hat{j}|\cos\beta = b \Rightarrow |\vec{r}|\cos\beta = b \Rightarrow \cos\beta = \dfrac{b}{|\vec{r}|}$;

and $\vec{r} \bullet \hat{k} = c \Rightarrow |\vec{r}||\hat{k}|\cos\gamma = c \Rightarrow |\vec{r}|\cos\gamma = c \Rightarrow \cos\gamma = \dfrac{c}{|\vec{r}|}$.

$\therefore$ $l = \cos\alpha = \dfrac{a}{|\vec{r}|} = \dfrac{a}{\sqrt{a^2 + b^2 + c^2}}$

$[\because |\vec{r}| = |a\,\hat{i} + b\,\hat{j} + c\,\hat{k}| = \sqrt{a^2 + b^2 + c^2}]$

$m = \cos\beta = \dfrac{b}{|\vec{r}|} = \dfrac{b}{\sqrt{a^2 + b^2 + c^2}}$

and $n = \cos\gamma = \dfrac{c}{|\vec{r}|} = \dfrac{c}{\sqrt{a^2 + b^2 + c^2}}$.

We have

$$l^2 + m^2 + n^2 = \frac{a^2}{a^2 + b^2 + c^2} + \frac{b^2}{a^2 + b^2 + c^2} + \frac{c^2}{a^2 + b^2 + c^2}$$

$$= \frac{a^2 + b^2 + c^2}{a^2 + b^2 + c^2} = 1.$$

Also $\dfrac{l}{a} = \dfrac{m}{b} = \dfrac{n}{c} = \dfrac{1}{\sqrt{a^2 + b^2 + c^2}}$

i.e., a, b, c are proportional to d.c.'s l, m, n of $\vec{r}$.

$\therefore$ a, b, c are direction ratios of $\vec{r} = a\,\hat{i} + b\,\hat{j} + c\,\hat{k}$.

Finally unit vector in the direction of $\vec{r}$

$$= \frac{1}{|\vec{r}|}\vec{r} = \frac{1}{|\vec{r}|}(a\,\hat{i} + b\,\hat{j} + c\,\hat{k}) = \frac{a}{|\vec{r}|}\hat{i} + \frac{b}{|\vec{r}|}\hat{j} + \frac{c}{|\vec{r}|}\hat{k}$$

$$= l\,\hat{i} + m\,\hat{j} + n\,\hat{k}\,.$$

$$\therefore \qquad \vec{r} = |\,\vec{r}\,|\,(l\,\hat{i} + m\,\hat{j} + n\,\hat{k}\,).$$

Remember the following results established in theorem 1:

(i) If l, m, n are the direction cosines of a line or a vector, then $l^2 + m^2 + n^2 = 1$.

(ii) l, m, n are the direction cosines of a line or a vector if and only if $l\,\hat{i} + m\,\hat{j} + n\,\hat{k}$ is a unit vector in the direction of that line or vector.

(iii) If $\vec{r} = a\,\hat{i} + b\,\hat{j} + c\,\hat{k}$ is any vector in the direction of a given line, then a, b, c are direction ratios of that vector or line.

Remark: If l, m, n are direction cosines of a line, then l, m, n are also direction ratios of that line. But if a, b, c are direction ratios of a line, then a, b, c are direction cosines of that line if and only if $a^2 + b^2 + c^2 = 1$.

Theorem 2: *To show that the direction cosines of a line whose direction ratios are a, b, c are* $\dfrac{a}{\sqrt{(a^2 + b^2 + c^2)}}, \dfrac{b}{\sqrt{(a^2 + b^2 + c^2)}}, \dfrac{c}{\sqrt{(a^2 + b^2 + c^2)}}$.

Proof: Let l, m, n be the direction cosines of a line whose direction ratios are a, b, c. Then

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \lambda \text{ (say).}$$

$$\therefore \qquad l = a\lambda\,,\quad m = b\,\lambda\,,\quad n = c\,\lambda\,. \qquad \ldots(1)$$

But $\qquad l^2 + m^2 + n^2 = 1.$

$$\therefore \qquad \lambda^2\,(a^2 + b^2 + c^2) = 1 \quad \text{or} \quad \lambda^2 = \frac{1}{a^2 + b^2 + c^2}$$

or $$\lambda = \pm\frac{1}{\sqrt{(a^2 + b^2 + c^2)}}\,.$$

$$\therefore \text{ from (1), } l = \frac{a}{\sqrt{(a^2 + b^2 + c^2)}}, m = \frac{b}{\sqrt{(a^2 + b^2 + c^2)}}, \ n = \frac{c}{\sqrt{(a^2 + b^2 + c^2)}}$$

or $$l = -\frac{a}{\sqrt{(a^2 + b^2 + c^2)}}, m = -\frac{b}{\sqrt{(a^2 + b^2 + c^2)}}, n = -\frac{c}{\sqrt{(a^2 + b^2 + c^2)}}$$

depending upon the direction of that line.

Thus, if l, m, n are the direction cosines of a line whose direction ratios are a, b, c, then

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{(a^2 + b^2 + c^2)}} = \frac{1}{\sqrt{(a^2 + b^2 + c^2)}}\,.$$

$$\therefore \qquad l = \frac{a}{\sqrt{(a^2 + b^2 + c^2)}},\ m = \frac{b}{\sqrt{(a^2 + b^2 + c^2)}},\ n = \frac{c}{\sqrt{(a^2 + b^2 + c^2)}}.$$

Rule: Let a, b, c be the d.r.'s of a given line, then, to find actual direction cosines of this line, divide each of a, b, c by $\sqrt{(a^2 + b^2 + c^2)}$.

Theorem 3: *If the length of a line OP through the origin O is r, then the coordinates of P are (lr, mr, nr), where l, m, n are the direction cosines of the line OP.*

Proof: The unit vector in the direction of the line OP whose d.c.'s are l, m, n

$$= l\hat{i} + m\hat{j} + n\hat{k}.$$

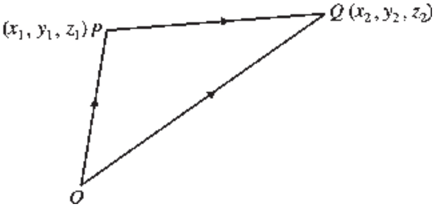
$$\therefore \qquad \overrightarrow{OP} = |\overrightarrow{OP}|\,(l\hat{i} + m\hat{j} + n\hat{k}) = r\,(l\hat{i} + m\hat{j} + n\hat{k}) = r\,l\,\hat{i} + r\,m\,\hat{j} + r\,n\,\hat{k}.$$

$\therefore$ the coordinates of P are $(r\,l, r\,m, r\,n)$.

Theorem 4: Direction cosines of the join of two points:

To show that the direction ratios of a line PQ joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1$ and its direction cosines are

$$\frac{x_2 - x_1}{PQ},\ \frac{y_2 - y_1}{PQ},\ \frac{z_2 - z_1}{PQ}.$$



Proof: Let O be the origin and (x_1, y_1, z_1) and (x_2, y_2, z_2) be the coordinates of the points P and Q respectively.

We have $\overrightarrow{PQ}$ = position vector of Q

− position vector of P

$$= (x_2\,\hat{i} + y_2\,\hat{j} + z_2\,\hat{k}) - (x_1\,\hat{i} + y_1\,\hat{j} + z_1\,\hat{k})$$
$$= (x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}.$$

Now direction ratios of PQ are the coefficients of $\hat{i},\ \hat{j},\ \hat{k}$ in the resolution of $\overrightarrow{PQ}$ as a linear combination of $\hat{i},\ \hat{j},\ \hat{k}$.

$\therefore$ direction ratios of PQ are $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

Also a unit vector in the direction of $\overrightarrow{PQ}$

$$= \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{\overrightarrow{PQ}}{PQ}$$

$$= \frac{1}{PQ}\{(x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}\}$$

$$= \frac{x_2 - x_1}{PQ}\,\hat{i} + \frac{y_2 - y_1}{PQ}\,\hat{j} + \frac{z_2 - z_1}{PQ}\,\hat{k}.$$

$\therefore$ the direction cosines of PQ are

$$\frac{x_2-x_1}{PQ}, \frac{y_2-y_1}{PQ}, \frac{z_2-z_1}{PQ},$$

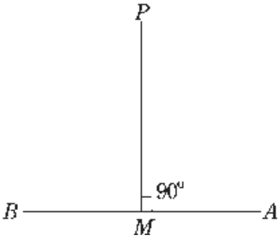
where $PQ = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$.

Remember: Direction ratios of a line PQ joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $x_2-x_1,\ y_2-y_1,\ z_2-z_1$.

7 Projections

Projection of a point on a given line:

Let P be a given point and AB the given straight line. Draw PM perpendicular from P to AB, meeting AB in M. Then the foot M of the perpendicular PM is called the projection of the given point P on the given line AB.



Projection of a given line segment on another given line:

To find the projection of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on another line whose direction cosines are l, m, n.

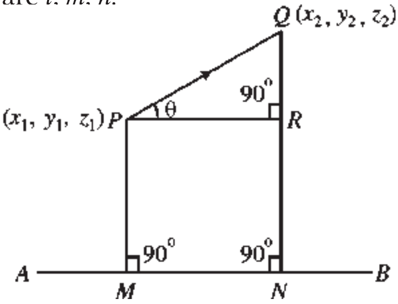
Let AB be a given line whose direction cosines are $l.\ m.\ n.$ If $\vec{a}$ is a unit vector along AB, then

$$\vec{a} = l\,\hat{i} + m\,\hat{j} + n\,\hat{k}\ .$$

Let P and Q be two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Draw PM and QN perpendiculars to AB. Then MN is the projection of PQ on AB.

Draw PR parallel to AB to meet QN at R.

Then $PR = MN$.



If θ is the angle between the lines PQ and AB, then $\angle QPR = \theta$.

The projection of PQ on $AB = MN = PR$

$$= PQ\cos\theta = |\overrightarrow{PQ}|\cos\theta = \overrightarrow{PQ}\bullet\vec{a},$$

where $\vec{a}$ is unit vector in the direction PR *i.e.*, in the direction of the given line AB

$$= \{(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})\}\bullet(l\hat{i} + m\hat{j} + n\hat{k})$$

$$= \{(x_2-x_1)\hat{i} + (y_2-y_1)\hat{j} + (z_2-z_1)\hat{k}\}\bullet(l\hat{i} + m\hat{j} + n\hat{k})$$

$$= l(x_2-x_1) + m(y_2-y_1) + n(z_2-z_1).$$

Hence the projection of the line joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ on a line whose direction cosines are l, m, n

$$= l(x_2-x_1) + m(y_2-y_1) + n(z_2-z_1).$$

Remark: If the projection of PQ on AB is zero, then PQ is perpendicular to AB.

Corollary 1: *If O and P are two points* $(0, 0, 0)$ *and* (x_1, y_1, z_1)*, then the projection of OP on a line whose direction cosines are l, m, n is* $lx_1 + my_1 + nz_1$. **(Kumaun 2007)**

Corollary 2: *The projection of the line joining the points* $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$ *on a line whose direction ratios are a, b, c is*

$$= \frac{(x_2 - x_1)a + (y_2 - y_1)b + (z_2 - z_1)c}{\sqrt{(a^2 + b^2 + c^2)}}.$$

Illustrative Examples

Example 1: *Find the direction cosines of a line whose direction ratios are* $2, 3, -6$.

Solution: We have $\sqrt{2^2 + 3^2 + (-6)^2} = \sqrt{4 + 9 + 36} = 7$.

Hence the direction cosines of the given line are

$$\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}.$$

Example 2: *Find the direction cosines of the line segment joining the points* $P(-2, 1, -8)$ *and* $Q(4, 3, -5)$. **(Kashi 2011)**

Solution: The direction ratios of the line segment PQ are

$$4 - (-2),\ 3 - 1,\ -5 - (-8) \quad i.e., \quad 6, 2, 3.$$

We have $\sqrt{(6)^2 + (2)^2 + (3)^2} = \sqrt{36 + 4 + 9} = \sqrt{49} = 7$.

$\therefore$ the direction cosines of PQ are $\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$.

Example 3: *If* $P(6, 3, 2)$, $Q(5, 1, 4)$, $R(3, -4, 7)$, $S(0, 2, 5)$ *are four points, find the projection of PQ on RS.*

Solution: The direction ratios of RS are

$$0 - 3, 2 + 4, 5 - 7 \quad i.e., \quad -3, 6, -2.$$

Hence the direction cosines of RS are

$$\frac{-3}{\sqrt{(9 + 36 + 4)}}, \frac{6}{\sqrt{(9 + 36 + 4)}}, \frac{-2}{\sqrt{(9 + 36 + 4)}}$$

or $\frac{-3}{7}, \frac{6}{7}, \frac{-2}{7}$.

Now the projection of PQ upon RS

$$= (5 - 6)\left(-\frac{3}{7}\right) + (1 - 3)\frac{6}{7} + (4 - 2)\left(-\frac{2}{7}\right)$$

$$= \frac{3}{7} - \frac{12}{7} - \frac{4}{7} = -\frac{13}{7} = \frac{13}{7} \text{ numerically.}$$

Comprehensive Exercise 1

1. Find the d.c.'s of a line whose direction ratios are $-1, 2, -1$.
2. Find the direction cosines of the line which is equally inclined to the axes.
   (Gorakhpur 2005; Kanpur 11; Bundelkhand 13; Purvanchal 13)
3. Find the direction cosines l, m, n of two lines which are connected by the relations $l - 5m + 3n = 0$ and $7l^2 + 5m^2 - 3n^2 = 0$.
   (Meerut 2010, 12; Purvanchal 13)
4. Find the direction cosines l, m, n of the two lines which are connected by the relations $l + m + n = 0$ and $mn - 2nl - 2lm = 0$.
   (Gorakhpur 2005; Kanpur 07; Purvanchal 08; Kumaun 13)
5. If P, Q, R, S are four points with co-ordinates $(3, 4, 5), (4, 6, 3), (-1, 2, 4), (1, 0, 5)$ respectively, then find the projection of PQ on RS. Also find the projection of RS on PQ. **(Agra 2001)**
6. Prove that $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$, where α, β, γ are the angles which the given line makes with the positive directions of the axes. **(Agra 2001)**

Answers 1

1. $\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}$
2. $\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}$
3. $-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$ and $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$
4. $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}; \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$
5. $(-\frac{4}{3}, -\frac{4}{3})$

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8 Angle between Two Lines or Vectors

(Kumaun 2002)

Theorem 1: *If θ is the angle between two lines or vectors whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2, then prove that*

(i) $\cos \theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

(ii) *the lines are perpendicular if and only if* $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$.
(Avadh 2013)

(iii) $$\sin \theta = \sqrt{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2}$$
$$= \sqrt{\Sigma\, (m_1 n_2 - m_2 n_1)^2}\ .$$

(iv) *the lines are parallel if and only if* $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$. **(Avadh 2013)**

Proof: Let $\vec{a}$ and $\vec{b}$ be unit vectors along the lines or vectors whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 respectively. Then

$$\vec{a} = l_1\,\hat{i} + m_1\,\hat{j} + n_1\,\hat{k} \text{ and } \vec{b} = l_2\,\hat{i} + m_2\,\hat{j} + n_2\,\hat{k}.$$

Since θ is the angle between the given lines, therefore the angle between the vectors $\vec{a}$ and $\vec{b}$ is also θ.

(i) We have $\vec{a} \bullet \vec{b} = (l_1\hat{i} + m_1\hat{j} + n_1\hat{k}) \bullet (l_2\hat{i} + m_2\hat{j} + n_2\hat{k})$

$\Rightarrow$ $|\vec{a}|\,|\vec{b}|\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

$\Rightarrow$ $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

[$\because\ |\vec{b}| = 1 = |\vec{b}|, \vec{a}$ and $\vec{b}$ being unit vectors]

Hence $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$.

(ii) The given lines or vectors are perpendicular if and only if $\theta = 90°$

i.e., if and only if $\cos\theta = 0$ *i.e.,* if and only if $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$.

(iii) To prove this result we shall make use of Lagrange's identity stated below.

Lagrange's Identity: If l_1, m_1, n_1 and l_2, m_2, n_2 are two sets of real numbers, then

$$(l_1^2 + m_1^2 + n_1^2)(l_2^2 + n_2^2 + n_2^2) - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2$$
$$= (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2.$$

Now, we have $\sin^2\theta = 1 - \cos^2\theta$

$$= (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2$$
$$= (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2$$

[By Lagrange's identity]

$$= \Sigma\,(m_1 n_2 - m_2 n_1)^2 \quad \therefore \quad \sin\theta = \sqrt{\Sigma\,(m_1 n_2 - m_2 n_1)^2}.$$

Alternative proof for the value of sin θ:

We have $|\vec{a} \times \vec{b}| = |(l_1\hat{i} + m_1\hat{j} + n_1\hat{k}) \times (l_2\hat{i} + m_2\hat{j} + n_2\hat{k})|$

$\Rightarrow$ $|\vec{b}|\,|\vec{b}|\sin\theta = |(m_1 n_2 - m_2 n_1)\,\hat{i} + (n_1 l_2 - n_2 l_1)\,\hat{j} + (l_1 m_2 - l_2 m_1)\,\hat{k}|$

$\Rightarrow$ $\sin\theta = \sqrt{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2}$.

[$\because\ |\vec{a}| = 1 = |\vec{b}|, \vec{a}$ and $\vec{b}$ being unit vectors]

(iv) The given lines or vectors are parallel

$\Leftrightarrow$ the vectors $\vec{a}$ and $\vec{b}$ are parallel $\Leftrightarrow$ $\vec{a} = \lambda\,\vec{b}$, where λ is some scalar

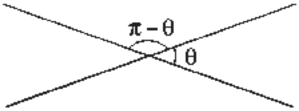
$\Leftrightarrow$ $l_1\hat{i} + m_1\hat{j} + n_1\hat{k} = \lambda\,(l_2\hat{i} + m_2\hat{j} + n_2\hat{k})$

$\Leftrightarrow \quad l_1 = \lambda\, l_2\ , m_1 = \lambda m_2\ , n_1 = \lambda n_2$

[Equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$ on both sides]

$\Leftrightarrow \quad \dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$, each ratio being equal to λ .

Remark 1: If θ is the angle between two lines, then $\pi - \theta$ is also the angle between those lines. We have $\cos(\pi - \theta) = -\cos\theta$ so that $|\cos(\pi - \theta)| = |\cos\theta|$.



If θ is an acute angle, then $\cos\theta$ is positive and if θ is an obtuse angle, then $\cos\theta$ is negative.

If on applying the formula $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$,

we get $\cos\theta > 0$, then θ is the acute angle between the lines and if we get $\cos\theta < 0$, then θ is the obtuse angle between the lines. If we wish to get the acute angle between the given lines, we can modify the formula giving $\cos\theta$ as

$$\cos\theta = |\, l_1 l_2 + m_1 m_2 + n_1 n_2 \,|.$$

Remark 2: Rule to remember the formula for sin θ:

Direction cosines of one line are l_1, m_1, n_1

and direction cosines of the other line are l_2, m_2, n_2.

We have $$\sin^2\theta = \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}^2 + \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2$$

Theorem 2: *If θ is the angle between two lines or vectors whose direction ratios are a_1, a_2, a_3 and b_1, b_2, b_3, then prove that*

(i) *cos* $\theta = \dfrac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{b_1^2 + b_2^2 + b_3^2}}$

(ii) *the lines are perpendicular if and only if* $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$

(Avadh 2013)

(iii) *sin* $\theta = \dfrac{\sqrt{(a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2 + (a_1 b_2 - a_2 b_1)^2}}{\sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{b_1^2 + b_2^2 + b_3^2}}$

(iv) *the lines are parallel if and only if* $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$ · **(Meerut 2013B; Avadh 13)**

Proof: Let $\vec{a}$ and $\vec{b}$ be vectors along the lines whose direction ratios are a_1, a_2, a_3 and b_1, b_2, b_3 respectively.

Then $\vec{b} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

Since θ is the angle between the given lines, therefore the angle between the vectors $\vec{a}$ and $\vec{b}$ is also θ.

(i) We have $\vec{a} \bullet \vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \bullet (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$$\Rightarrow \qquad |\vec{a}|\ |\vec{b}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$$

$$\Rightarrow \qquad \cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\vec{a}|\,|\vec{b}|}$$

$$= \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{b_1^2 + b_2^2 + b_3^2}}.$$

(ii) The given lines or vectors are perpendicular

$$\Leftrightarrow \ \theta = 90° \Leftrightarrow \cos\theta = 0 \Leftrightarrow a_1b_1 + a_2b_2 + a_3b_3 = 0.$$

(iii) We have

$$|\vec{a} \times \vec{b}| = |(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})|$$

$$\Rightarrow \qquad |\vec{a}|\,|\vec{b}|\sin\theta = |(a_2b_3 - a_3b_2)\,\hat{i} + (a_3b_1 - a_1b_3)\,\hat{j} + (a_1b_2 - a_2b_1)\,\hat{k}|$$

$$\Rightarrow \qquad \sin\theta = \frac{\sqrt{(a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2}}{\sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{b_1^2 + b_2^2 + b_3^2}}.$$

(iv) The given lines or vectors are parallel

$\Leftrightarrow$ the vectors $\vec{a}$ and $\vec{b}$ are parallel

$\Leftrightarrow$ $\vec{a} = \lambda\,\vec{b}$, where λ is some scalar

$\Leftrightarrow$ $a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \lambda\,(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$\Leftrightarrow$ $a_1 = \lambda b_1, a_2 = \lambda b_2, a_3 = \lambda b_3$ [Equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$]

$\Leftrightarrow$ $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$, each ratio being equal to λ.

Remark 1: The condition of perpendicularity of two lines is the same whether we use their direction cosines or direction ratios.

Remark 2: Two lines are parallel if and only if their direction ratios are proportional.

Remark 3: How to show that three given points are collinear ?

If we are to show that the three given points P, Q and R are collinear, we should find direction ratios of PQ and PR. If these direction ratios are proportional, then PQ and PR are parallel. Since both PQ and PR pass through P, so they are in the same straight line. Hence the points P, Q and R are collinear.

Remark 4: If θ is the angle between the straight lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2, then **(Bundelkhand 2014; Kumaun 14)**

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2 \quad \text{and} \quad \sin\theta = \sqrt{\Sigma\,(m_1n_2 - m_2n_1)^2}.$$

$$\therefore \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{\Sigma\,(m_1n_2 - m_2n_1)^2}}{l_1l_2 + m_1m_2 + n_1n_2}.$$

Again if θ is the angle between the straight lines whose direction ratios are a_1, b_1, c_1 and a_2, b_2, c_2, then

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

and
$$\sin\theta = \frac{\sqrt{\Sigma\,(b_1 c_2 - b_2 c_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

$\therefore$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{\Sigma\,(b_1 c_2 - b_2 c_1)^2}}{a_1 a_2 + b_1 b_2 + c_1 c_2}.$$

We observe that the formula for $\tan\theta$ is the same whether we are given direction cosines or direction ratios.

9 Perpendicular Distance of a Point from a Line

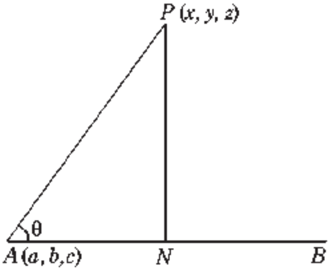
To find the perpendicular distance of a point $P\,(x', y', z')$ from a line through $A\,(a, b, c)$ and whose direction cosines are l, m, n.

Let AB be a line through $A\,(a, b, c)$ and whose d.c.'s are l, m, n.

Let PN be the perpendicular from P to AB.

Now AN = projection of the line segment joining $A\,(a, b, c)$ and $P\,(x', y', z')$ on the line AB

$$= (x' - a)\,l + (y' - b)\,m + (z' - c)\,n,$$



and AP = distance between the points A and P

$$= \sqrt{[(x' - a)^2 + (y' - b)^2 + (z' - c)^2]}.$$

We have, $PN^2 = AP^2 - AN^2$

or
$$PN^2 = \{(x' - a)^2 + (y' - b)^2 + (z' - c)^2\} - \{(x' - a)\,l + (y' - b)\,m + (z' - c)\,n\}^2$$

or
$$PN^2 = \{(x' - a)^2 + (y' - b)^2 + (z' - c)^2\}\{l^2 + m^2 + n^2\} - \{(x' - a)\,l + (y' - b)\,m + (z' - c)\,n\}^2$$

$[\because\ l^2 + m^2 + n^2 = 1]$

$$= \Sigma\,\{(y' - b)\,n - (z' - c)\,m\}^2$$ [By Lagrange's identity]

$\therefore$
$$PN = \sqrt{[\Sigma\,\{(y' - b)\,n - (z' - c)\,m\}^2]}.$$

Aliter: Let $\angle PAN = \theta$.

We have, $PN^2 = AP^2 \sin^2\theta$.

Now θ is the angle between the lines AP and AB. Here the d.c.'s of AP are

$$(x'-a)/AP, (y'-b)/AP, (z'-c)/AP$$

and the d.c.'s of AB are l, m, n

$$\therefore \quad \sin^2\theta = \begin{vmatrix} (y'-b)/AP & (z'-c)/AP \\ m & n \end{vmatrix}^2 + \begin{vmatrix} (x'-a)/AP & (z'-c)/AP \\ l & n \end{vmatrix}^2 + \begin{vmatrix} (x'-a)/AP & (y'-b)/AP \\ l & m \end{vmatrix}^2$$

$$= \frac{1}{AP^2}\left[\begin{vmatrix} y'-b & z'-c \\ m & n \end{vmatrix}^2 + \begin{vmatrix} x'-a & z'-c \\ l & n \end{vmatrix}^2 + \begin{vmatrix} x'-a & y'-b \\ l & m \end{vmatrix}^2\right].$$

$$\therefore \quad PN^2 = AP^2\sin^2\theta = \begin{vmatrix} y'-b & z'-c \\ m & n \end{vmatrix}^2 + \begin{vmatrix} x'-a & z'-c \\ l & n \end{vmatrix}^2 + \begin{vmatrix} x'-a & y'-b \\ l & m \end{vmatrix}^2. \quad \ldots(1)$$

Remark: In the formula (1), l, m, n are the d.c.'s of the line AB. If however, α, β, γ are the d.r.'s of the line AB, then to get PN^2 we should divide the R.H.S. of (1) by $\alpha^2 + \beta^2 + \gamma^2$.

Illustrative Examples

Example 4: *Show that the three points* $A(2, -1, 3)$, $B(4, 3, 1)$ *and* $C(3, 1, 2)$ *are collinear.* **(Meerut 2010B)**

Solution: The direction ratios of the line AB are

$$4-2, 3-(-1), 1-3 \quad i.e., \quad 2, 4, -2.$$

The direction ratios of the line AC are $3-2, 1-(-1), 2-3$ *i.e.,* $1, 2, -1$.

We see that the direction ratios of the two lines AB and AC are proportional because we have $\frac{2}{1} = \frac{4}{2} = \frac{-2}{-1}$, each ratio being equal to 2.

$\therefore$ the lines AB and AC are parallel.

But both the lines AB and AC pass through the point A. So AB and AC are in the same straight line. Hence the points A, B and C are collinear.

Example 5: *Prove that the straight lines whose direction cosines are given by the relations* $al + mb + cn = 0$ *and* $fmn + gnl + hlm = 0$ *are perpendicular if* $f/a + g/b + h/c = 0$ *and parallel if* $\sqrt{(af)} \pm \sqrt{(bg)} \pm \sqrt{(ch)} = 0$. **(Meerut 2007B, 10; Purvanchal 09, 10; Kumaun 15)**

Solution: As given, $n = -(al + bm)/c$.

Substituting this value of n in the second relation, we get

$$fm\left(-\frac{al+bm}{c}\right)+gl\left(-\frac{al+bm}{c}\right)+hlm=0$$

or $$afml+bfm^2+agl^2+bglm-chlm=0$$

or $$ag\frac{l^2}{m^2}+\frac{l}{m}(af+bg-ch)+bf=0. \qquad \text{...(1)}$$

Now if l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of the two lines, then the roots of (1) are l_1/m_1 and l_2/m_2.

$\therefore$ product of the roots $= \dfrac{l_1}{m_1}\cdot\dfrac{l_2}{m_2}=\dfrac{bf}{ag}$

or $$\frac{l_1 l_2}{f/a}=\frac{m_1 m_2}{g/b}.$$

$\therefore$ $$\frac{l_1 l_2}{(f/a)}=\frac{m_1 m_2}{(g/b)}=\frac{n_1 n_2}{(h/c)},\quad \text{by symmetry.}$$

Now the lines are perpendicular if $l_1 l_2+m_1 m_2+n_1 n_2=0$

i.e., if $f/a+g/b+h/c=0$.

Again, if the lines are parallel then the direction cosines are same *i.e.,* the roots of (1) are equal

i.e., $(af+bg-ch)^2=4ag\,.\,bf$. $\qquad [\because\ B^2=4AC]$

Taking square root, we get

$$af+bg-ch=\pm 2\sqrt{(afbg)}$$

or $$af\pm 2\sqrt{(afbg)}+bg=ch$$

or $$\{\sqrt{(af)}\pm\sqrt{(bg)}\}^2=(ch).$$

Taking square root,

$$\sqrt{(af)}\pm\sqrt{(bg)}=\pm\sqrt{(ch)}$$

or $$\sqrt{(af)}\pm\sqrt{(bg)}\pm\sqrt{(ch)}=0,$$

which proves the second result.

Example 6: *Show that the straight lines whose direction cosines are given by the equations*

$$al+bm+cn=0 \text{ and } ul^2+vm^2+wn^2=0$$

are perpendicular, if $a^2(v+w)+b^2(u+w)+c^2(u+v)=0$ *and parallel, if*

$$\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0.$$

(Meerut 2001, 12; Kanpur 09, 11, 14; Kumaun 12, 14)

Solution: The given relations are

$$al+bm+cn=0 \qquad \text{...(1)}$$

and $$ul^2+vm^2+wn^2=0 \qquad \text{...(2)}$$

From (1), we have $n=-(al+bm)/c$

Putting this value of n in (2), we have

$$ul^2 + vm^2 + w\left\{-\frac{(al+bm)}{c}\right\}^2 = 0$$

or $$(c^2u + a^2w)\,l^2 + 2abwlm + (b^2w + c^2v)\,m^2 = 0$$

or $$(c^2u + a^2w)\,(l/m)^2 + 2abw\,(l/m) + (b^2w + c^2v) = 0 \qquad \ldots(3)$$

Let l_1, m_1, n_1 and l_2, m_2, n_2 be the d.c.'s of the two lines. Then the roots of the quadratic equation (3) in (l/m) are l_1/m_1 and l_2/m_2.

$\therefore$ product of the roots $= \dfrac{l_1}{m_1}\cdot\dfrac{l_2}{m_2} = \dfrac{b^2w + c^2v}{c^2u + a^2w}$.

$$\therefore \quad \frac{l_1l_2}{b^2w + c^2v} = \frac{m_1m_2}{c^2u + a^2w} = \frac{n_1n_2}{a^2v + b^2u}, \text{ by symmetry} \qquad \ldots(4)$$

The lines will be perpendicular if

$$l_1l_2 + m_1m_2 + n_1n_2 = 0 \qquad \ldots(5)$$

i.e., $(b^2w + c^2v) + (c^2u + a^2w) + (a^2v + b^2u) = 0,$ putting the proportionate values of l_1l_2, m_1m_2 and n_1n_2 from (4) in (5)

i.e., $a^2(v + w) + b^2(u + w) + c^2(v + u) = 0.$

Now the roots of the equation (3) will be equal *i.e.,* we shall have

$$l_1/m_1 = l_2/m_2 \text{ if } B^2 = 4AC$$

i.e., $$4a^2b^2w^2 = 4(c^2u + a^2w)(b^2w + c^2v)$$

or $$c^2(a^2wv + b^2uw + c^2vu) = 0$$

or $$a^2wv + b^2uw + c^2vu = 0$$

or $$\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0. \qquad \ldots(6)$$

[Dividing by uvw]

If instead of eliminating n between (1) and (2), we eliminate l, then by symmetry of the result (6), (6) is also the condition for $m_1/n_1 = m_2/n_2$.

$\therefore$ if (6) is satisfied, we have

$$\frac{l_1}{m_1} = \frac{l_2}{m_2} \quad \text{and} \quad \frac{m_1}{n_1} = \frac{m_2}{n_2}$$

i.e., $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$ which is the condition for the lines to be parallel.

Hence the lines will be parallel if

$$\frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0.$$

Remark: The right hand side of (5) is 0. So instead of putting actual values of l_1l_2, m_1m_2, n_1n_2 in (5), we can put their proportionate values also.

Example 7: *Show that the lines whose d.c.'s are given by* $l + m + n = 0$ *and* $2\,mn + 3ln - 5lm = 0$ *are at right angles.*

(Meerut 2000, 02, 04, 05, 08, 13B; Kumaun 08, 11; Purvanchal 07, 11)

Solution: From first relation, we get

$$l = -m - n. \qquad \ldots(1)$$

Substituting this value of l in the second relation, we get

$$2\,mn + 3\,(-m - n).\,n - 5\,(-m - n)\,m = 0$$

or $$5m^2 + 4mn - 3n^2 = 0$$

or $$5\,(m/n)^2 + 4\,(m/n) - 3 = 0. \qquad \ldots(2)$$

If l_1, m_1, n_1 and l_2, m_2, n_2 be the d.c.'s of the two lines, then the roots of (2) are m_1/n_1 and m_2/n_2.

∴ product of the roots

$$\frac{m_1}{n_1} \cdot \frac{m_2}{n_2} = -\frac{3}{5}$$

or $$\frac{m_1\ m_2}{3} = \frac{n_1\ n_2}{-5} \qquad \ldots(3)$$

Also from (1), $n = -l - m$.

Putting this value of n in the second given relation, we get

$$2\,m\,(-l - m) + 3l\,(-l - m) - 5lm = 0$$

or $$3\,(l/m)^2 + 10\,(l/m) + 2 = 0.$$

∴ $$\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{2}{3} \quad \text{or} \quad \frac{l_1\ l_2}{2} = \frac{m_1\ m_2}{3} \qquad \ldots(4)$$

From (3) and (4), we have

$$\frac{l_1 l_2}{2} = \frac{m_1 m_2}{3} = \frac{n_1 n_2}{-5} = k \text{ (say).}$$

∴ $$l_1 l_2 + m_1 m_2 + n_1 n_2 = (2 + 3 - 5)\,k = 0.\,k = 0.$$

∴ the given lines are at right angles.

Example 8: *If* l_1, m_1, n_1 *and* l_2, m_2, n_2 *are direction cosines of the two lines, show that the direction cosines of the line perpendicular to both are proportional to*

$$m_1 n_2 - m_2 n_1,\ n_1 l_2 - n_2 l_1,\ l_1 m_2 - l_2 m_1.$$

Prove further if the given lines are at right angles to each other then these direction ratios are the actual direction cosines. **(Avadh 2012)**

Solution: Suppose that the required direction cosines of the line are $l, m, n.$ Since the line is perpendicular to the given lines, we have

$$ll_1 + mm_1 + nn_1 = 0 \qquad \ldots(1)$$

and $$ll_2 + mm_2 + nn_2 = 0. \qquad \ldots(2)$$

Solving (1) and (2), we have

$$\frac{l}{m_1 n_2 - m_2 n_1} = \frac{m}{n_1 l_2 - n_2 l_1} = \frac{n}{l_1 m_2 - l_2 m_1}. \qquad \ldots(3)$$

This shows that the required d.r.'s are

$$m_1 n_2 - m_2 n_1,\ n_1 l_2 - n_2 l_1,\ l_1 m_2 - l_2 m_1.$$

Now suppose θ is the angle between the two given lines whose d.c.'s are l_1, m_1, n_1 and l_2, m_2, n_2.

Then $\sin \theta = \sqrt{\{\Sigma (m_1 n_2 - m_2 n_1)^2\}}$ ...(4)

[See article 8, Theorem 1, part (iii)]

If $\theta = 90°$ *i.e.*, the lines are perpendicular, then (4) gives

$$\sqrt{\{\Sigma (m_1 n_2 - m_2 n_1)^2\}} = 1. \qquad \ldots(5)$$

$\therefore$ in this case from (3) the d.c.'s l, m, n of the line are given by

$$\frac{l}{m_1 n_2 - m_2 n_1} = \frac{m}{n_1 l_2 - n_2 l_1} = \frac{n}{l_1 m_2 - l_2 m_1}$$

$$= \frac{\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{\{\Sigma (m_1 n_2 - m_2 n_1)^2\}}} = \frac{1}{1} = 1.$$

[Using (5) and $l^2 + m^2 + n^2 = 1$]

Hence in this case the actual direction cosines l, m, n are

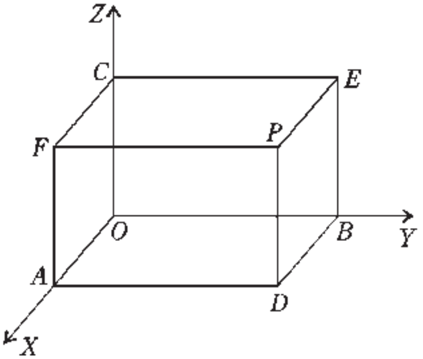
$$m_1 n_2 - m_2 n_1,\ n_1 l_2 - n_2 l_1,\ l_1 m_2 - l_2 m_1.$$ **Proved.**

Example 9: *If the edges of a rectangular parallelopiped be a, b, c, show that the angles between the four diagonals are given by*

$$\cos^{-1}\left\{\frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right\}.$$

(Meerut 2002; Agra 05; Avadh 07)

Solution: Let the vertex O of the rectangular parallelopiped be taken as origin and the coterminous edges OA, OB, OC as the coordinate axes. We have $OA = a$, $OB = b$, $OC = c$. The coordinates of the eight vertices of the parallelopiped are given by $O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$, $D(a, b, 0)$, $E(0, b, c)$, $F(a, 0, c)$ and $P(a, b, c)$.



The four diagonals of the parallelopiped are OP, AE, BF and CD whose direction ratios are respectively a, b, c; $-a, b, c$; $a, -b, c$; and $a, b, -c$.

∴ direction cosines of *OP*, *AE*, *BF* and *CD* are respectively

$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}};$$

$$\frac{-a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}};$$

$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{-b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}};$$

and
$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{-c}{\sqrt{a^2+b^2+c^2}}.$$

∴ the angle θ between the diagonals *OP* and *AE* is given by

$$\cos\theta = \frac{a(-a)+b\,.\,b+c\,.\,c}{\sqrt{a^2+b^2+c^2}\,\sqrt{a^2+b^2+c^2}}$$

$$= \frac{-a^2+b^2+c^2}{a^2+b^2+c^2}$$

or
$$\theta = \cos^{-1}\left(\frac{-a^2+b^2+c^2}{a^2+b^2+c^2}\right).$$

The total number of pairs of the diagonals is ${}^4C_2$ *i.e.*, 6. In a similar way the angles between the remaining five pairs of the diagonals are determined and all of these six angles are given by

$$\cos^{-1}\left\{\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right\}.$$

The above expression will give only six valid values because the ambiguous signs cannot be either all +ive or all –ive for in that case

$$\theta = \cos^{-1} 1 \quad \text{or} \quad \cos^{-1}(-1)$$

i.e.,
$$\theta = 0 \quad \text{or} \quad 180^\circ$$

which is impossible as no two of the diagonals are parallel.

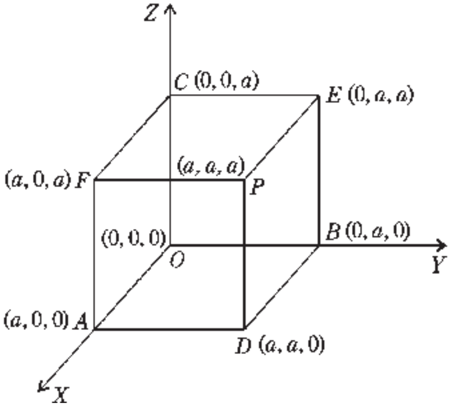
Example 10: *A line makes angles* $\alpha, \beta, \gamma, \delta$ *with the four diagonals of a cube; prove that*

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}.$$

(Meerut 2013B; Avadh 09)

Solution: Take the coordinate axes *OX*, *OY*, *OZ* along the coterminous edges *OA*, *OB*, *OC* respectively of a cube of edge of length *a*. Then the coordinates of the eight vertices of the cube are :

$O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, a, 0)$,
$C(0, 0, a)$, $D(a, a, 0)$, $E(0, a, a)$,
$F(a, 0, a)$ and $P(a, a, a)$.



The four diagonals of the cube are *OP*, *AE*, *BF* and *CD* whose direction ratios are respectively

$$a-0, a-0, a-0;$$
$$0-a, a-0, a-0;$$
$$a-0, 0-a, a-0;$$
$$\text{and } a-0, a-0, 0-a$$

i.e., $a, a, a;\ -a, a, a;\ a, -a, a;$ and $a, a, -a$

or $1, 1, 1;\ -1, 1, 1;\ 1, -1, 1;$ and $1, 1, -1$.

$\therefore$ the direction cosines of *OP*, *AE*, *BF* and *CD* are

$$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}; -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}};$$

and $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}$ respectively.

Let l, m, n be the direction cosines of a line which makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of the cube. Then

$$\cos\alpha = l \cdot \frac{1}{\sqrt{3}} + m \cdot \frac{1}{\sqrt{3}} + n \cdot \frac{1}{\sqrt{3}} = \frac{l+m+n}{\sqrt{3}},$$

$$\cos\beta = l \cdot \left(\frac{-1}{\sqrt{3}}\right) + m \cdot \frac{1}{\sqrt{3}} + n \cdot \frac{1}{\sqrt{3}} = \frac{-l+m+n}{\sqrt{3}},$$

$$\cos\gamma = l \cdot \frac{1}{\sqrt{3}} + m \cdot \left(\frac{-1}{\sqrt{3}}\right) + n \cdot \frac{1}{\sqrt{3}} = \frac{l-m+n}{\sqrt{3}}$$

and
$$\cos\delta = l \cdot \frac{1}{\sqrt{3}} + m \cdot \frac{1}{\sqrt{3}} + n \cdot \left(\frac{-1}{\sqrt{3}}\right) = \frac{l+m-n}{\sqrt{3}}.$$

On squaring and adding, we get $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$$= \frac{1}{3}\{(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2\}$$

$$= \frac{1}{3}\{4(l^2+m^2+n^2)\}$$

$$= \frac{4}{3}. \qquad [\because l^2+m^2+n^2=1]$$

Comprehensive Exercise 2

1. If points P, Q are $(2, 3, -6), (3, -4, 5)$, then find the angle between *OP* and *OQ*, where *O* is the origin.

2. Prove that the line joining the points $(1, 2, 3)$ and $(-1, -2, -3)$ is perpendicular to the line joining the points $(-2, 1, 5)$ and $(3, 3, 2)$.
3. Show that the three points $A(6, -7, -1)$, $B(2, -3, 1)$ and $C(4, -5, 0)$ are collinear.
4. Prove that the three lines drawn from a point with direction cosines proportional to $1, -1, 1$; $2, -3, 0$ and $1, 0, 3$ are coplanar.
5. Show that the lines whose direction cosines are given by the equations $2l + 2m - n = 0$, and $mn + nl + lm = 0$ are at right angles. **(Kanpur 2008)**
6. Prove that the acute angle between the lines whose direction cosines are given by the relations $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$ is $\pi / 3$. **(Meerut 2005B, 13)**
7. If l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both of them are $m_1 n_2 - m_2 n_1, n_1 l_2 - n_2 l_1, l_1 m_2 - l_2 m_1$. **(Kumaun 2007)**
8. If a variable line in two adjacent positions has direction cosines l, m, n and $l + \delta l, m + \delta m, n + \delta n$, show that the small angle $\delta\theta$ between the two positions is given by $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$. **(Avadh 2010)**
9. Show that the angle between any two diagonals of a cube is $\cos^{-1} (1/3)$. **(Garhwal 2001)**
10. If $(l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3)$ are the direction cosines of three mutually perpendicular lines, then find the direction cosines of a line whose direction cosines are proportional to $l_1 + l_2 + l_3$, $m_1 + m_2 + m_3$, $n_1 + n_2 + n_3$ and prove that this line is equally inclined to the given lines. **(Kanpur 2002)**
11. The direction cosines of two straight lines, inclined at an angle θ are l_1, m_1, n_1 and l_2, m_2, n_2. Show that direction cosines of the bisector of the angle between them are $\dfrac{l_1 + l_2}{2 \cos (\theta / 2)}, \dfrac{m_1 + m_2}{2 \cos (\theta / 2)}, \dfrac{n_1 + n_2}{2 \cos (\theta / 2)}$.
12. If two pairs of opposite edges of a tetrahedron are perpendicular, then prove that the third pair is also perpendicular. **(Garhwal 2003)**

Answers 2

1. $\cos^{-1}\left(-\dfrac{18\sqrt{2}}{35}\right)$

10. $\cos^{-1} (1/\sqrt{3})$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of a straight line, then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is equal to

(a) 1 (b) 2

(c) 0 (d) 3

2. Direction cosines of the line joining the point (0, 0, 0) and (1, 1, 1) are

(a) $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ (b) $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

(c) $(\sqrt{3}, \sqrt{3}, \sqrt{3})$ (d) (3, 3, 3)

3. The equation of x-axis are

(a) $x = 0, y = 0$ (b) $y = 0, z = 0$

(c) $z = 0, x = 0$ (d) none of these

(Kumaun 2015)

4. The direction cosines of x-axis are

(a) 1, 0, 0 (b) 1, 1, 0

(c) 0, 0, 1 (d) 0, 1, 1 **(Kumaun 2015)**

5. If (l, m, n) are direction cosines of OP and $OP = r$, then the co-ordinates of P are

(a) $\frac{l}{r}, \frac{m}{r}, \frac{n}{r}$ (b) lr, mr, nr

(c) $\frac{r}{l}, \frac{r}{m}, \frac{r}{n}$ (d) none of these

(Kumaun 2007)

6. The direction cosines of z-axis are

(a) 1, 0, 0 (b) 0, 1, 0

(c) 0, 0, 1 (d) 0, 1, 1 **(Kumaun 2008)**

7. The point (1, 1, 0) lies on

(a) xy-plane (b) yz-plane

(c) xz-plane (d) none of these

(Kumaun 2009, 14)

8. If the coordinates of the point A and B are $(1, -1, 0)$ $(0, 0, 1)$ respectively the direction cosines of AB are
   (a) $\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$
   (c) $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ (d) none of these
   (Kumaun 2009)

9. Direction cosines of the line joining the points $(0, 0, 0)$ and $(1, 1, 1)$ are
   (a) $(3, 3, 3)$ (b) $(\sqrt{3}, \sqrt{3}, \sqrt{3})$
   (c) $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ (d) $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$
   (Kumaun 2011, 13)

10. If $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of a straight line then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is equal to
    (a) 1 (b) 2
    (c) 0 (d) 3 **(Kumaun 2014)**

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. If α, β, γ are the angles which a given directed line makes with the positive directions of the axes of x, y and z respectively then the direction cosines of the line are …… .
2. D.C.'s of a line which makes equal angles with the positive directions of the coordinate axes are …… . **(Agra 2006; Bundelkhand 05)**
3. If l, m, n are the direction cosines of any line then $l^2 + m^2 + n^2 = \ldots\ldots$.
4. The direction cosines of a line whose direction ratios are $2, 3, -6$ are …… .
5. Projection of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) on a line whose direction ratios are a, b, c is …… .
6. The angle θ between any two lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 is given by $\cos\theta = \ldots\ldots$.
7. If θ is the angle between the lines whose d.c.'s are l_1, m_1, n_1 and l_2, m_2, n_2 then $\sin^2\theta = \ldots\ldots$.

True or False

Write 'T' for true and 'F' for false statement.

1. Two lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 will be perpendicular if $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$. **(Agra 2005)**
2. The direction cosines of intersecting lines are 0, 0, 0.
3. If l, m, n are the direction cosines of a line the $l^2 + m^2 + n^2 = 1$.

4. The direction cosines are also the direction ratios.

Answers

Multiple Choice Questions

1. (b) **2.** (a) **3.** (b) **4.** (a) **5.** (b)
6. (c) **7.** (a) **8.** (c) **9.** (d) **10.** (b)

Fill in the Blank(s)

1. $\cos\alpha, \cos\beta, \cos\gamma$ **2.** $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ **3.** 1

4. $\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}$ **5.** $\frac{a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1)}{\sqrt{(a^2 + b^2 + c^2)}}$

6. $l_1 l_2 + m_1 m_2 + n_1 n_2$

7. $(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2$

True or False

1. T **2.** F **3.** T **4.** T

□

Chapter

4

The Plane

1 Plane

Definition: *A plane is a surface such that every straight line joining any two points on it lies wholly on it.*

Normal to a plane: *A straight line which is perpendicular to every line lying in a plane is called **a normal to that plane.*** It is also called a line perpendicular to that plane. All the normals to a plane are parallel lines.

2 Normal Form of the Equation of a Plane

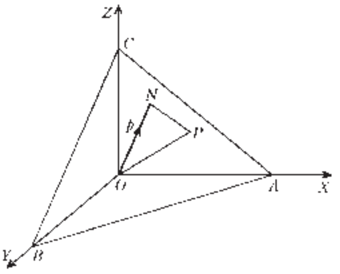
To find the equation of a plane whose perpendicular distance from the origin is p and cos α, cos β, cos γ are direction-cosines of this perpendicular.

Let $P(x, y, z)$ be any point on the plane. Let ON be the perpendicular drawn from origin to the plane. Since $ON = p$ and the direction cosines of ON are $\cos\alpha, \cos\beta, \cos\gamma$, therefore the co-ordinates of N are $(p\cos\alpha, p\cos\beta, p\cos\gamma)$. Therefore the direction cosines of PN will be proportional to

$$x - p\cos\alpha,\ y - p\cos\beta,\ z - p\cos\gamma.$$

Since ON and PN are at right angles to each other, therefore

$$\cos\alpha\,(x - p\cos\alpha) + \cos\beta\,(y - p\cos\beta) + \cos\gamma\,(z - p\cos\gamma) = 0$$



or $$x\cos\alpha + y\cos\beta + z\cos\gamma = p\,(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$$

or $$x\cos\alpha + y\cos\beta + z\cos\gamma = p$$

Hence if l, m, n be the direction cosines of the normal to a plane directed from the origin to the plane and p be the length of the perpendicular from the origin to the plane, then the equation of the plane is $lx + my + nz = p$.

This is known as the **equation of a plane in normal form.**

3 General Equation of a Plane

Theorem: *To prove that every equation* $ax + by + cz + d = 0$ *of first degree in* x, y *and* z *always represents a plane and the coefficients* a, b, c *of* x, y, z *in this equation are direction ratios of normal to this plane.* **(Garhwal 2000)**

Proof: The general equation of first degree in x, y, z is given by

$$ax + by + cz + d = 0. \qquad \ldots(1)$$

Let $A\,(x_1, y_1, z_1)$ and $B\,(x_2, y_2, z_2)$ be any two points on the surface represented by (1), so that we have

$$ax_1 + by_1 + cz_1 + d = 0 \qquad \ldots(2)$$

and $$ax_2 + by_2 + cz_2 + d = 0. \qquad \ldots(3)$$

Multiplying (3) by λ and adding to (2), we get

$$a\,(x_1 + \lambda\,x_2) + b\,(y_1 + \lambda\,y_2) + c\,(z_1 + \lambda\,z_2) + d\,(1 + \lambda) = 0.$$

Dividing both sides by $(1 + \lambda)$, we get

$$a\left(\frac{x_1 + \lambda\,x_2}{1 + \lambda}\right) + b\left(\frac{y_1 + \lambda\,y_2}{1 + \lambda}\right) + c\left(\frac{z_1 + \lambda\,z_2}{1 + \lambda}\right) + d = 0. \qquad \ldots(4)$$

The relation (4) shows that for every value of $\lambda \neq -1$, the point

$$\left(\frac{x_1 + \lambda x_2}{1 + \lambda}, \frac{y_1 + \lambda y_2}{1 + \lambda}, \frac{z_1 + \lambda z_2}{1 + \lambda}\right)$$

lies on the surface (1). But these are the general coordinates of a point which divides the join of $A\,(x_1, y_1, z_1)$ and $B\,(x_2, y_2, z_2)$ in the ratio $\lambda : 1$. Since λ may take any real value other than -1, every point of the straight line AB lies on the surface (1). Hence the equation (1) represents a plane.

Subtracting (2) from (3), we get

$$a\,(x_2 - x_1) + b\,(y_2 - y_1) + c\,(z_2 - z_1) = 0. \qquad \ldots(5)$$

The relation (5) shows that the two lines whose direction ratios are a, b, c and $x_2 - x_1, y_2 - y_1, z_2 - z_1$ are perpendicular. But $x_2 - x_1, y_2 - y_1, z_2 - z_1$ are direction ratios of the line AB which is any line lying in the plane (1). Therefore a line whose direction ratios are a, b, c is perpendicular to every line lying in the plane (1) and so it is perpendicular to the plane (1). Hence a, b, c are direction ratios of the normal to the plane (1).

Note: **The number of arbitrary constants in the general equation of the plane.**

The general equation of the plane is

$$ax + by + cz + d = 0 \quad \text{or} \quad (a/d)\,x + (b/d)\,y + (c/d)\,z = -1.$$

This equation shows that there are three arbitrary constants namely $a/d, b/d, c/d$ in the equation of a plane. Therefore the equation of a plane can be determined to satisfy the three conditions, each condition giving us the value of a constant.

An Important Remark: The equation of any plane passing through the origin is

$$ax + by + cz = 0.$$

4 To Reduce the General Equation of the Plane to the Normal Form

The general equation of the plane is

$$ax + by + cz + d = 0. \qquad \text{...(1)}$$

If l, m, n are the d.c.'s of the normal to the plane, then the equation of the plane in the normal form is

$$lx + my + nz = p. \qquad \text{...(2)}$$

If (1) and (2) represent the same plane, then

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{p}{-d} = \pm\frac{\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{(a^2 + b^2 + c^2)}} = \pm\frac{1}{\sqrt{(a^2 + b^2 + c^2)}},$$

where the same sign either +ive or –ive is to be chosen throughout.

$$\therefore \quad l = \pm a/\sqrt{(a^2 + b^2 + c^2)}, m = \pm b/\sqrt{(a^2 + b^2 + c^2)},$$

$$n = \pm c/\sqrt{(a^2 + b^2 + c^2)}, \text{and } p = \pm d/\sqrt{(a^2 + b^2 + c^2)}.$$

Substituting these values in (2), the normal form of the plane (1) is given by

$$\pm\frac{ax}{\sqrt{(a^2 + b^2 + c^2)}} \pm \frac{by}{\sqrt{(a^2 + b^2 + c^2)}} \pm \frac{cz}{\sqrt{(a^2 + b^2 + c^2)}} = \pm\frac{d}{\sqrt{(a^2 + b^2 + c^2)}} \qquad \text{...(3)}$$

The sign in equation (3) is so chosen that p *i.e.*, $\pm d/\sqrt{(a^2 + b^2 + c^2)}$ is always positive.

Working rule to reduce the general cartesian equation of a plane to normal form. Suppose the general cartesian equation of a plane is

$$ax + by + cz + d = 0. \qquad \ldots(1)$$

Transpose the constant term d in the equation (1) to the R.H.S. and adjust the equation in such a way that this constant term on the R.H.S. is positive. Now divide the equation by $\sqrt{a^2 + b^2 + c^2}$, where a, b, c are the coefficients of x, y, z in the equation of the plane. The resulting equation will be the equation of the plane in the normal form $lx + my + nz = p$.

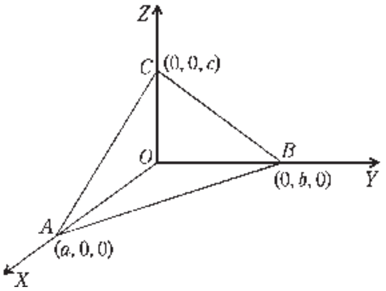
Here p will be the length of the perpendicular from the origin to the plane and l, m, n will be the direction cosines of the normal to the plane directed from origin to the plane.

5 Equation of a Plane in Intercepts Form

Theorem: *If a plane makes intercepts a, b and c on the axes of x, y and z respectively, then the equation of the plane is* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

(Kumaun 2007)

Proof: Let O be the origin and let the plane meet the coordinate axes at the points A, B, C respectively such that $OA = a, OB = b, OC = c$ with proper signs. Then the coordinates of the points A, B, C are $A\,(a, 0, 0)$, $B\,(0, b, 0)$ and $C\,(0, 0, c)$.



Let the equation of the plane be

$$Ax + By + Cz + D = 0, \qquad \ldots(1)$$

where d$D \neq 0$ because the plane does not pass through the origin $(0, 0, 0)$.

Since (1) passes through the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, therefore

$$Aa + D = 0 \quad \Rightarrow \quad A = -\frac{D}{a}$$

$$Bb + D = 0 \quad \Rightarrow \quad B = -\frac{D}{b}$$

and $$Cc + D = 0 \quad \Rightarrow \quad C = -\frac{D}{c}\cdot$$

Putting these values of A, B and C in (1), the required equation of the plane is

$$-\frac{D}{a}x - \frac{D}{b}y - \frac{D}{c}z + D = 0$$

or $$D\left[-\frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1\right] = 0$$

or $$-\frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0 \qquad [\because D \neq 0]$$

or $$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

This is called **equation of a plane in intercepts form.**

The intercept made by this plane on x-axis is a, that on y-axis is b and that on z-axis is c.

Working rule to reduce the equation of a plane $Ax + By + Cz + D = 0$ to intercepts form:

Transpose the constant term to the R.H.S. and then divide both sides of the equation by this transposed constant term to make the R.H.S. 1. Then put the resulting equation in the form $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

Equations of coordinate planes: The equation of xy-plane is $z=0$, the equation of yz-plane is $x=0$ and the equation of $z\ x$-plane is $y=0$.

6 General Equation of a Plane through a given Point and Perpendicular to a given Line

To find the equation of a plane through a given point $A\ (x_1, y_1, z_1)$ and perpendicular to a line whose direction ratios are a, b, c .

Let (x, y, z) be the coordinates of any current point P on the plane. Since the plane passes through the point $A\ (x_1, y_1, z_1)$, the line AP lies in the plane.

The d.r.'s of the line AP are $x - x_1, y - y_1, z - z_1$. Also the d.r.'s of the normal to the plane *i.e.*, of a line perpendicular to the plane are a, b, c .

Now the normal to the plane is perpendicular to every line lying in the plane and therefore the lines whose d.r.'s are a, b, c and $x - x_1, y - y_1, z - z_1$ are perpendicular.

$$\therefore \qquad a\ (x - x_1) + b\ (y - y_1) + c\ (z - z_1) = 0,$$

which is the equation of the required plane.

Remark: The equation of any plane passing through the point (x_1, y_1, z_1) is

$$a\ (x - x_1) + b\ (y - y_1) + c\ (z - z_1) = 0.$$

In this equation a, b, c are d.r.'s of normal to the plane.

As a particular case, the equation of any plane passing through the origin is $ax + by + cz = 0$, in which the coefficients of x, y, z *i.e.*, a, b, c are d.r.'s of the normal to the plane.

7 Equation of a Plane through Three Points

To find the equation of a plane which passes through three points whose co-ordinates are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3). **(Kumaun 2000)**

Let the general equation of the plane be

$$ax + by + cz + d = 0. \qquad ...(1)$$

If the equation (1) of the plane passes through the given points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) the coordinates of these points will satisfy the equation (1), so that we have

$$ax_1 + by_1 + cz_1 + d = 0, \qquad ...(2)$$

$$ax_2 + by_2 + cz_2 + d = 0, \qquad ...(3)$$

and

$$ax_3 + by_3 + cz_3 + d = 0. \qquad ...(4)$$

Eliminating a, b, c and d from the above equations (1), (2), (3) and (4) the equation of the required plane is given by

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \qquad ...(5)$$

Corollary: *Condition for four points* (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) *and* (x_4, y_4, z_4) *to be coplanar.*

The equation of the plane passing through first three points is given by equation (5). If the fourth point namely (x_4, y_4, z_4) also lies on this plane, then the co-ordinates of this point will satisfy the equation (5), so that we have

$$\begin{vmatrix} x_4 & y_4 & z_4 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0, \text{ i.e., } \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0. \qquad ...(6)$$

The condition (6) is the required condition for four given points to be coplanar.

8 Equations of The Co-ordinate Planes

(i) The equation to *yz*-plane: The x-coordinate of each point lying on the yz-plane is zero, and hence the equation to yz-plane is given by $x = 0$.

(ii) The equation to *zx*-plane: It is given by $y = 0$.

(iii) The equation to *xy*-plane: It is given by $z = 0$.

9 The Equations to the Planes Parallel to the Co-ordinate Planes

*The equation of the plane parallel to the yz-plane and at a distance `a' from it.* The $x$-coordinate of every point on this plane is equal to `a'. Hence the equation of the required plane is given by $x = a$.

Similarly, *the equation of the plane parallel to the xz-plane and at a distance 'b' from it is given by* $y = b$.

Also the equation of the plane parallel to the *xy*-plane and at a distance `c' from it is given by $z = c$.

10 The Equations of the Planes Perpendicular to the Co-ordinate Axes

The equation of the plane perpendicular to the x-axis. This plane is obviously parallel to the *yz*-plane and hence its equation is given by $x = a$. [See article 9]

Similarly the equations of the planes perpendicular to y and z axes are respectively given by $y = b$ and $z = c$.

11 Planes Parallel to Axes

If the plane $ax + by + cz + d = 0$ (where a, b, c are proportional to direction cosines of the normal to the plane) is parallel to the x-axis, then the normal to the plane will be at right angles to the x-axis, hence

Therefore the equation of the plane will be

$$by + cz + d = 0.$$

Similarly equations to the planes parallel to y and z axes can be written as

$$ax + cz + d = 0 \text{ and } ax + by + d = 0, \text{respectively.}$$

Remember: *Equation of the plane parallel to x-axis does not contain x , and equation of the plane parallel to the plane YOZ does not contain y and z.*

12 Angle between two Planes

Definition: *The angle between two planes is defined as the angle between their normals drawn from any point to the planes.*

Let the equations of the two planes be

$$a_1 x + b_1 y + c_1 z + d_1 = 0 \qquad \dots(1)$$

and

$$a_2 x + b_2 y + c_2 z + d_2 = 0. \qquad \dots(2)$$

The d.r.'s of the normal to the plane (1) are a_1, b_1, c_1 and the d.r.'s of the normal to the plane (2) are a_2, b_2, c_2 .

If θ is the angle between the planes (1) and (2), then θ is the angle between the lines whose d.r.'s are a_1, b_1, c_1 and a_2, b_2, c_2 .

$$\therefore \qquad \cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}} \qquad \dots(3)$$

and $$\tan\theta = \frac{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}{a_1a_2+b_1b_2+c_1c_2}.$$

For the acute angle between the two planes, cos θ is positive and for the obtuse angle it is negative. The numerical value of cos θ in both these cases is the same because cos (π – θ) = – cos θ.

Condition of perpendicularity of two planes:

Two planes are perpendicular if their normals are perpendicular. Therefore the planes (1) and (2) are perpendicular if the lines whose d.r.'s are a_1, b_1, c_1 and a_2, b_2, c_2 are perpendicular the condition for which is

$$a_1a_2+b_1b_2+c_1c_2=0. \qquad \ldots(4)$$

Condition of parallelism of two planes:

Two planes are parallel if their normals are parallel. Therefore the planes (1) and (2) are parallel if the lines whose d.r.'s are a_1, b_1, c_1 and a_2, b_2, c_2 are parallel the condition for which is

$$a_1/a_2=b_1/b_2=c_1/c_2 \qquad \ldots(5)$$

i.e., the coefficients of x, y, z in the equations of the two planes should be proportional.

Remember: The equation of any plane parallel to the plane

$ax+by+cz+d=0$ is $ax+by+cz+\lambda=0$.

13 The Two Sides of a Plane

To find the condition that the two points should lie on the same or opposite sides of a plane.

Let the equation of the plane be $ax+by+cz+d=0$

and $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points.

Suppose the line AB meets the plane in P where P divides AB in the ratio $\lambda : 1$. Now λ is positive or negative according as P divides AB internally or externally *i.e.*, according as A and B lie on opposite sides or on the same side of the plane.

The co-ordinates of the point P are

$$\left(\frac{\lambda x_2+x_1}{\lambda+1}, \frac{\lambda y_2+y_1}{\lambda+1}, \frac{\lambda z_2+z_1}{\lambda+1}\right).$$

It lies upon the plane, therefore

$$\frac{a(\lambda x_2+x_1)}{\lambda+1}+\frac{b(\lambda y_2+y_1)}{\lambda+1}+\frac{c(\lambda z_2+z_1)}{\lambda+1}+d=0$$

or $$ax_1+by_1+cz_1+d+\lambda(ax_2+by_2+cz_2+d)=0$$

or $$\lambda = -\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}.$$

If the points A and B lie on the same side of the plane, λ is negative, therefore

$$ax_1 + by_1 + cz_1 + d \quad \text{and} \quad ax_2 + by_2 + cz_2 + d$$

have the same sign (both positive or both negative).

If the points A and B lie on opposite sides of the plane, λ is positive, therefore

$$ax_1 + by_1 + cz_1 + d \quad \text{and} \quad ax_2 + by_2 + cz_2 + d$$

have opposite signs.

Thus we see that the points A and B lie on the same or on opposite sides of the plane according as the expressions

$$ax_1 + bx_1 + cz_1 + d \quad \text{and} \quad ax_2 + by_2 + cz_2 + d$$

have the same or opposite signs.

14 Perpendicular Distance of a Point from the Plane

To find the length of the perpendicular from the point (x_1, y_1, z_1) to a given plane:

Let the equation of the given plane be

$$ax + by + cz + d = 0 \qquad \ldots(1)$$

To find the length of the perpendicular from the point (x_1, y_1, z_1) to the plane (1).

Shifting the origin to the point (x_1, y_1, z_1) the equation (1) becomes

$$a(x + x_1) + b(y + y_1) + c(z + z_1) + d = 0$$

or $$ax + by + cz + ax_1 + by_1 + cz_1 + d = 0. \qquad \ldots(2)$$

Dividing both sides of (2) by $\sqrt{(a^2 + b^2 + c^2)}$, we get

$$\frac{a}{\sqrt{(a^2 + b^2 + c^2)}} x + \frac{b}{\sqrt{(a^2 + b^2 + c^2)}} y + \frac{c}{\sqrt{(a^2 + b^2 + c^2)}} z + \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{(a^2 + b^2 + c^2)}} = 0. \qquad \ldots(3)$$

The equation (3) of the plane is in the normal form with a proper adjustment of sign throughout the equation.

$\therefore$ The length of the perpendicular from the new origin to the plane (3)

$$= \pm \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{(a^2 + b^2 + c^2)}}.$$

Hence the length of the perpendicular from the point (x_1, y_1, z_1) to the plane (1)

$$= \pm \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{(a^2 + b^2 + c^2)}}.$$

Since the perpendicular distance of a point from the plane is always positive, therefore a positive or negative sign is to be attached before the radical according as $ax_1 + by_1 + cz_1 + d$ is positive or negative *i.e.*, according as (x_1, y_1, z_1) lies on the same side or on opposite side of the plane as the origin, provided d is positive.

Working Rule: To find the length p of the perpendicular from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$, we substitute the co-ordinates of the given point in the left hand side of the equation of the plane and then divide this expression by $\sqrt{[(\text{coeff. of } x)^2 + (\text{coeff. of } y)^2 + (\text{coeff. of } z)^2]}$.

Thus, $$p = \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{(a^2 + b^2 + c^2)}}.$$

If the value of p obtained from this formula is negative, we can ignore the sign and give the positive value in answer, unless there is some special reason.

Remark: To avoid the negative value of p, we can take

$$p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{(a^2 + b^2 + c^2)}}.$$

If the equation of the plane is in the normal form $lx + my + nz - p = 0$, the length p_1 of the perpendicular from the point (x_1, y_1, z_1) to the plane is given by

$$p_1 = lx_1 + my_1 + nz_1 - p, \text{ for in this case } \sqrt{(l^2 + m^2 + n^2)} = 1.$$

15 Distance between Two Parallel Planes

Find the lengths of perpendicular distances of each plane from the origin and retain their signs. The algebraic difference of these two perpendicular distances is the distance between the given parallel planes. But while applying this method we should be careful that the coefficients of x in the two equations of the planes are of the same sign.

Alternate method: Take a point on one of the two given planes, then the required distance is the length of the perpendicular drawn from this point to the other plane.

Illustrative Examples

Example 1: *Find the equation of the plane which cuts off intercepts* 6, 3, – 4 *from the axes of co-ordinates. Reduce it to normal form and find the perpendicular distance of the plane from the origin.*

Solution: The equation of the plane which cuts off intercepts 6, 3, – 4 from the co-ordinate axes is

$$\frac{x}{6} + \frac{y}{3} + \frac{z}{-4} = 1 \quad \text{or} \quad 2x + 4y - 3z = 12. \qquad \ldots(1)$$

Dividing both sides of (1) by $\sqrt{2^2 + 4^2 + (-3)^2}$ *i.e.*, by $\sqrt{29}$, we get

$$\frac{2}{\sqrt{29}}x + \frac{4}{\sqrt{29}}y - \frac{3}{\sqrt{29}}z = \frac{12}{\sqrt{29}},$$

which is the equation of the plane in normal form $lx + my + nz = p$.

The length of the perpendicular from the origin to this plane

$$= p = \frac{12}{\sqrt{29}} \cdot$$

Example 2: *A plane meets the co-ordinate axes in A, B, C such that the centroid of triangle ABC is the point (p, q, r). Show that the equation of the plane is* $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3$.

(Meerut 2000, 05, 10, 12; Avadh 09, 10; Kanpur 10; Kashi 13; Kumaun 11, 13)

Solution: Let the equation of the plane in intercepts form be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \qquad \ldots(1)$$

Then the co-ordinates of the points A, B and C are

$$A(a, 0, 0),\ B(0, b, 0) \text{ and } C(0, 0, c).$$

So the centroid of the triangle ABC is the point $\left(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c\right)$.

But it is given that the centroid of the Δ ABC is the point (p, q, r).

$$\therefore \qquad p = \frac{1}{3}a, q = \frac{1}{3}b, r = \frac{1}{3}c$$

$$\Rightarrow \qquad a = 3p, b = 3q, c = 3r.$$

Putting the values of a, b, c in (1), the equation of the required plane is

$$\frac{x}{3p} + \frac{y}{3q} + \frac{z}{3r} = 1$$

or $$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3.$$

Example 3: *Find the equation of the plane passing through the point* (1, 2, 1) *and perpendicular to the line joining the points* (1, 4, 2) *and* (2, 3, 5). *Find also the perpendicular distance of the origin from the plane.*

Solution: The direction ratios of the line joining the points $A(1, 4, 2)$ and $B(2, 3, 5)$ are $2 - 1, 3 - 4, 5 - 2$ *i.e.*, $1, -1, 3$. These are the direction ratios of the normal to the required plane. Also the required plane passes through the point $(1, 2, 1)$. Hence its equation is

$$1(x - 1) - 1(y - 2) + 3(z - 1) = 0$$

or $$x - y + 3z = 2.$$

The perpendicular distance of the origin from the plane (1)

$$= \frac{2}{\sqrt{1^2 + (-1)^2 + 3^2}} = \frac{2}{\sqrt{11}}.$$

Example 4: *Find the angle between the planes* $2x - y + z = 7$ *and* $x + y + 2z = 9$.

(Bundelkhand 2007)

Solution: The angle θ between the given planes is the angle between their normals whose direction ratios are $2, -1, 1$ and $1, 1, 2$.

$$\therefore \qquad \cos\theta = \frac{(2)(1) + (-1)(1) + (1)(2)}{\sqrt{2^2 + (-1)^2 + 1^2}\ \sqrt{1^2 + 1^2 + 2^2}} = \frac{3}{\sqrt{6}\sqrt{6}} = \frac{3}{6} = \frac{1}{2}$$

or $\qquad \theta = \pi / 3.$

Hence the acute angle between the given planes is 60°.

Example 5: *Find the equation of the plane passing through the points* $(1, -1, 2)$ *and* $(2, -2, 2)$ *and which is perpendicular to the plane* $6x - 2y + 2z = 9$.

(Meerut 2005; 12)

Solution: The equation of any plane passing through the point $(1, -1, 2)$ is

$$a(x - 1) + b(y + 1) + c(z - 2) = 0. \qquad \ldots(1)$$

If the plane (1) passes through the point $(2, -2, 2)$, then

$$a(2 - 1) + b(-2 + 1) + c(2 - 2) = 0$$

i.e., $\qquad a - b + 0c = 0 \qquad \ldots(2)$

Now we know that two planes are perpendicular if their normals are perpendicular. Direction ratios of normal to plane (1) are a, b, c and direction ratios of normal to the plane $6x - 2y + 2z = 9$ are $6, -2, 2$. So if the plane (1) is perpendicular to the plane $6x - 2y + 2z = 9$, then $6a - 2b + 2c = 0$. $\qquad \ldots(3)$

Solving (2) and (3) by cross-multiplication, we have

$$\frac{a}{-2 - 0} = \frac{b}{0 - 2} = \frac{c}{-2 + 6}$$

or $$\frac{a}{-2} = \frac{b}{-2} = \frac{c}{4}$$

or $$\frac{a}{1} = \frac{b}{1} = \frac{c}{-2}.$$

Putting the proportionate values of a, b, c in (1), the equation of the required plane is

$$1(x - 1) + 1(y + 1) - 2(z - 2) = 0$$

or $\qquad x + y - 2z + 4 = 0.$

Example 6: *Find the equation of the plane through the point* $(1, 3, 2)$ *and parallel to the plane* $3x - 2y + 2z + 33 = 0$. *Find the perpendicular distance of the point* $(3, 3, 2)$ *from this plane.*

Solution: The equation of the plane through the point $(1, 3, 2)$ and parallel to the plane $3x - 2y + 2z + 33 = 0$ is

$$3(x-1) - 2(y-3) + 2(z-2) = 0$$

or $$3x - 2y + 2z - 1 = 0. \qquad \ldots(1)$$

The perpendicular distance of the point (3, 3, 2) from the plane (1)

$$= \frac{|3(3) - 2(3) + 2(2) - 1|}{\sqrt{3^2 + (-2)^2 + 2^2}} = \frac{|9 - 6 + 4 - 1|}{\sqrt{17}} = \frac{6}{\sqrt{17}}.$$

Example 7: *Find the distance between the parallel planes*

$$2x - y + 3z - 4 = 0 \text{ and } 6x - 3y + 9z + 13 = 0.$$

(Bundelkhand 2007; Avadh 14)

Solution: Let $P(x_1, y_1, z_1)$ be any point on the plane $2x - y + 3z - 4 = 0$. Then

$$2x_1 - y_1 + 3z_1 - 4 = 0 \quad \text{i.e.,} \quad 2x_1 - y_1 + 3z_1 = 4. \qquad \ldots(1)$$

The distance between the given parallel planes = the length of the perpendicular from P to the plane $6x - 3y + 9z + 13 = 0$

$$= \frac{|6x_1 - 3y_1 + 9z_1 + 13|}{\sqrt{6^2 + (-3)^2 + 9^2}} = \frac{|3(2x_1 - y_1 + 3z_1) + 13|}{\sqrt{126}}$$

$$= \frac{|3(4) + 13|}{\sqrt{126}} \qquad \text{[Using (1)]}$$

$$= \frac{25}{3\sqrt{14}}.$$

Example 8: *A variable plane which remains at a constant distance $3p$ from the origin cuts the coordinate axes at A, B, C. Show that the locus of the centroid of triangle ABC is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.*

(Kumaun 2001; Meerut 01, 12)

Solution: Let the equation of the variable plane be $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$, $\qquad \ldots(1)$

where a, b, c are parameters *i.e.,* a, b, c are variables.

The plane (1) is at a constant distance $3p$ from the origin *i.e.,* the length of the perpendicular drawn from the origin to the plane (1) is always $3p$, whatever a, b, c may be.

$$\therefore \qquad \frac{1}{(1/a)^2 + (1/b)^2 + (1/c)^2} = 9p^2$$

or $$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{9p^2}. \qquad \ldots(2)$$

The plane (1) meets the coordinate axes at the points $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$. Let (x_1, y_1, z_1) be the coordinates of the centroid of $\Delta\ ABC$.

Then $$x_1 = \frac{a + 0 + 0}{3} = \frac{a}{3},\ y_1 = \frac{0 + b + 0}{3} = \frac{b}{3},\ z_1 = \frac{0 + 0 + c}{3} = \frac{c}{3}.$$

$$\therefore \qquad a = 3x_1,\ b = 3y_1,\ c = 3z_1.$$

Putting these values of a, b, c in (2), we get

$$\frac{1}{9x_1^2} + \frac{1}{9y_1^2} + \frac{1}{9z_1^2} = \frac{1}{9p^2} \quad \text{or} \quad x_1^{-2} + y_1^{-2} + z_1^{-2} = p^{-2}.$$

Hence the locus of the point (x_1, y_1, z_1) is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

Comprehensive Exercise 1

1. (i) Reduce the equation of the plane $x + 2y - 2z - 9 = 0$ to the normal form and hence find the length of the perpendicular drawn from the origin to the given plane.

(ii) Find the perpendicular distance from the origin to the plane $2x + y + 2z = 3$. Find also the direction-cosines of the normal to the plane.

2. (i) Find the equation to the plane through P $(2, 3, -1)$ at right angles to OP. **(Meerut 2007B)**

(ii) Find the equation of the plane passing through the points $(2, 2, -1)$, $(3, 4, 2)$ and $(7, 0, 6)$.

3. (i) The foot of the perpendicular from the origin to a plane is $(4, -2, -5)$. Find the equation of the plane.

(ii) O is the origin and A is the point (a, b, c). Find the equation of the plane through A and at right angles to OA.

4. A plane makes intercepts $9, 9/2, -9/2$ upon the co-ordinate axes. Find the length of the perpendicular from origin on it.

5. A plane meets the coordinate axes at A, B and C such that the centroid of the triangle ABC is $(1, -2, 3)$. Find the equation of the plane.

6. Find the equation of the plane perpendicular to the line segment from $(-3, 3, 2)$ to $(9, 5, 4)$ at the middle point of the segment.

7. Find the equation to the plane through the points $(1, 1, 0)$, $(1, 2, 1)$, $(-2, 2, -1)$. **(Agra 2006)**

8. Show that the four points $(0, -1, -1)$, $(4, 5, 1)$, $(3, 9, 4)$ and $(-4, 4, 4)$ are coplanar. **(Purvanchal 2011; Avadh 12)**

9. Show that the four points $(0, -1, 0)$, $(2, 1, -1)$, $(1, 1, 1)$ and $(3, 3, 0)$ are co-planar and hence show that the equation of the plane passing through these points is $4x - 3y + 2z = 3$. **(Kanpur 2011)**

10. Find the equation of the plane through $(1, 0, -2)$ and perpendicular to each of the planes $2x + y - z - 2 = 0$ and $x - y - z - 3 = 0$. **(Rohilkhand 2009B)**

11. (i) Find the equation of the plane through the points $(1, -2, 2), (-3, 1, -2)$ and perpendicular to the plane $x + 2y - 3z = 5$.

(Meerut 2009; Rohilkhand 13)

(ii) Find the equation to the plane through the points $(-1, 1, 1)$ and $(1, -1, 1)$ and perpendicular to the plane $x + 2y + 2z = 5$. **(Kumaun 2009)**

(iii) Find the equation of the plane through the points $(1, -2, 4)$ and $(3, -4, 5)$ and parallel to the x-axis (*i.e.*, perpendicular to the yz-plane).

(Kumaun 2007, 13)

(iv) Find the equation of the plane passing through $(2, 3, -4)$ and $(1, -1, 3)$ and parallel to the x-axis. **(Kumaun 2014)**

12. (i) Find the angle between the planes $2x - y + z = 11$ and $x + y + 2z = 3$.

(Rohilkhand 2012)

(ii) Find the angle between the planes $3x + 4y - 5z = 3$ and $2x + 6y + 6z = 7$. **(Kumaun 2007)**

13. Find the distance of the point $P\ (2, 1, -1)$ from the plane $x - 2y + 4z = 9$.

(Rohilkhand 2010)

14. (i) Find the distance between the parallel planes $2x - 2y + z + 3 = 0$ and $4x - 4y + 2z + 7 = 0$. **(Rohilkhand 2010)**

(ii) Find the distance between the parallel planes $2x - 2y + z + 1 = 0$ and $4x - 4y + 2z + 3 = 0$. **(Kumaun 2014)**

15. (i) Find the equations of the planes parallel to the plane $x + 2y - 2z + 8 = 0$ which are at a distance of 2 units from the point $(2, 1, 1)$.

(ii) Find the equations of the planes parallel to the plane which are at a unit distance from the point $(1, 2, 3)$.

(iii) Find the equation of the plane that passes through the point $(2, 3, 4)$ and is parallel to the plane $5x - 6y + 7z = 3$. **(Kumaun 2015)**

(iv) Find the equation of the plane through $(0, 1, -2)$ and parallel to the plane $2x - 3y + 4z = 0$. **(Kumaun 2008)**

16. A variable plane is at a constant distance p from the origin and meets the axes in A, B and C. Show that the locus of the centroid of the triangle ABC is $x^{-2} + y^{-2} + z^{-2} = 9p^{-2}$. **(Meerut 2005B, 07B, 13; Kanpur 2009,10)**

17. A variable plane is at a constant distance p from the origin and meets the coordinate axes in A, B, C. Show that the locus of the centroid of the tetrahedron $OABC$ is $x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$.

(Kanpur 2005; Agra 06; Meerut 06, 07; Avadh 13)

Answers 1

1. (i) $\frac{1}{3}x+\frac{2}{3}y-\frac{2}{3}z=3;\ 3$ (ii) $2/3, 1/3, 2/3$

2. (i) $2x+3y-z=14$ (ii) $5x+2y-3z=17$

3. (i) $4x-2y-5z=45$ (ii) $ax+by+cz=a^2+b^2+c^2$

4. 3 **5.** $6x-3y+2z=18$

6. $6x+y+z=25$ **7.** $2x+3y-3z-5=0$

10. $2x-y+3z+4=0$

11. (i) $x+16y+11z+9=0$ (ii) $4x+4y-6z+6=0$

(iii) $y+2z-6=0$ (iv) $7y+4z-5=0$

12. (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{2}$ **13.** $\frac{13}{\sqrt{21}}$

14. (i) $\frac{1}{6}$ (ii) $\frac{1}{6}$

15. (i) $x+2y-2z+8=0$; $x+2y-2z+4=0$

(ii) $x-2y+2z=0$; $x-2y+2z-6=0$

(iii) $5x-6y+7z-20=0$; $x-2y+2z-6=0$

(iv) $2x-3y+4z+11=0$; $x-2y+2z-6=0$

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16 A Plane through the Intersection of Two given Planes

Theorem : *The equation of any plane passing through the line of intersection of two given planes* $P \equiv a_1x+b_1y+c_1z+d_1=0$ and $Q \equiv a_2x+b_2y+c_2z+d_2=0$ *is* $P+\lambda Q=0$, *where* λ *is a parameter i.e.,* λ *is any real number.*

Proof: The equation

$$P+\lambda Q=0 \qquad \ldots(1)$$

is

$$a_1x+b_1y+c_1z+d_1+\lambda(a_2x+b_2y+c_2z+d_2)=0$$

or

$$(a_1+\lambda a_2)x+(b_1+\lambda b_2)y+(c_1+\lambda c_2)z+d_1+\lambda d_2=0 \qquad \ldots(2)$$

The equation (2) is the equation of a plane because it is an equation of first degree in x, y and z. Moreover all the points which satisfy both the equations $P=0$ and $Q=0$ *i.e.*, which lie on the line of intersection of the planes $P=0$ and $Q=0$ also satisfy the equation (1) because $0+\lambda(0)=0$.

Hence (1) is the equation of any plane passing through the line of intersection of the planes $P=0$ and $Q=0$.

Remark: The axis of x is the line of intersection of the planes $y = 0$ and $z = 0$. So the equation of any plane passing through the axis of x is $y + \lambda z = 0$. The axis of y is the line of intersection of the planes $z = 0$ and $x = 0$. So the equation of any plane passing through the axis of y is $z + \lambda x = 0$. Similarly the equation of any plane passing through the axis of z is $x + \lambda y = 0$.

17 Condition for a Line to be Parallel or Perpendicular to a given Plane

To find the condition that a line whose d.r.'s are l, m, n may be parallel or be perpendicular to a given plane.

Let the equation of the given plane be

$$ax + by + cz + d = 0. \qquad \ldots(1)$$

Then the d.r.'s of the normal to the plane (1) are a, b, c . The d.r.'s of the given line are l, m, n.

The line is parallel to the plane: If the given line is parallel to the plane (1), it is perpendicular to the normal to the plane (1), the condition for which is

$$al + bm + cn = 0. \qquad \ldots(2)$$

The line is perpendicular to the plane: If the given line is perpendicular to the plane (1), it is parallel to the normal to the plane (1), the condition for which is $a / l = b / m = c / n$.

18 The Angle between a Line and a Plane

Definition: *The angle between a line and a plane is defined to be the complement of the angle between the line and the normal to the plane.*

Clearly this angle can be determined by the methods explained earlier.

Illustrative Examples

Example 9: *Find the equation of the plane through the line of intersection of the planes* $x + 2y + 3z + 5 = 0, x - 3y + z + 6 = 0$ *and passing through the origin.*

Solution: The equation of any plane passing through the line of intersection of the planes $x + 2y + 3z + 5 = 0$ and $x - 3y + z + 6 = 0$ is

$$x + 2y + 3z + 5 + \lambda (x - 3y + z + 6) = 0. \qquad \ldots(1)$$

If the plane (1) passes through the origin $(0, 0, 0)$, then $5 + 6\lambda = 0$ or $\lambda = -5/6$.

Putting $\lambda = -5/6$ in (1), the equation of the required plane is

$$x + 2y + 3z + 5 - (5/6)(x - 3y + z + 6) = 0$$

or $\quad 6x + 12y + 18z + 30 - 5x + 15y - 5z - 30 = 0$

or $\quad x + 27y + 13z = 0.$

Example 10: *Find the equation of the plane which contains the line of intersection of the planes* $x + 2y + 3z - 4 = 0$ *and* $2x + y - z + 5 = 0$ *and which is perpendicular to the plane* $5x + 3y - 6z + 8 = 0$.

Solution: The equation of any plane passing through the line of intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$ is

$$(x + 2y + 3z - 4) + \lambda\,(2x + y - z + 5) = 0 \qquad \ldots(1)$$

or $$(1 + 2\lambda)\,x + (2 + \lambda)\,y + (3 - \lambda)\,z - 4 + 5\lambda = 0. \qquad \ldots(2)$$

If the plane (2) is perpendicular to the plane

$$5x + 3y - 6z + 8 = 0, \text{ then } 5\,(1 + 2\lambda) + 3\,(2 + \lambda) - 6\,(3 - \lambda) = 0$$

or $\quad -7 + 19\lambda = 0 \quad$ or $\quad \lambda = 7/19.$

Putting $\lambda = 7/19$ in (1), the required plane is

$$(x + 2y + 3z - 4) + (7/19)\,(2x + y - z + 5) = 0$$

or $\quad 19x + 38y + 57z - 76 + 14x + 7y - 7z + 35 = 0$

or $\quad 33x + 45y + 50\,z - 41 = 0.$

19 Planes Bisecting the Angles between Two Planes

Theorem: *The equations of the planes bisecting the angles between the planes* $a_1 x + b_1 y + c_1 z + d_1 = 0$ *and* $a_2 x + b_2 y + c_2 z + d_2 = 0$ *are*

$$\frac{a_1 x + b_1 y + c_1 z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2 x + b_2 y + c_2 z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}\cdot$$ **(Garhwal 2001)**

Proof: Let $P\,(x, y, z)$ be any point on either of the planes bisecting the angles between the given planes. Then the perpendicular distances of these planes from P must be equal.

$$\therefore \quad \frac{|\,a_1 x + b_1 y + c_1 z + d_1\,|}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{|\,a_2 x + b_2 y + c_2 z + d_2\,|}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

or $$\frac{a_1 x + b_1 y + c_1 z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2 x + b_2 y + c_2 z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}},$$

which are the required equations of the bisector planes.

Remark 1: Equation of the plane bisecting the angle in which the origin lies.

Let the given planes be

$$a_1 x + b_1 y + c_1 z + d_1 = 0 \qquad \ldots(1)$$

and $$a_2 x + b_2 y + c_2 z + d_2 = 0. \qquad \ldots(2)$$

First write the equations (1) and (2) in such a way that the constant terms d_1 and d_2 are of the same sign *i.e.*, either both d_1 and d_2 are positive or both are negative. Then the equation of the plane bisecting the angle in which the origin lies is

$$\frac{a_1 x + b_1 y + c_1 z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = + \frac{a_2 x + b_2 y + c_2 z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

and the equation of the plane bisecting the angle in which the origin does not lie is

$$\frac{a_1 x + b_1 y + c_1 z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = - \frac{a_2 x + b_2 y + c_2 z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}} .$$

Remark 2: How to find that the angle between the given planes in which the origin lies is acute or obtuse ?

First write the equations (1) and (2) in such a way that the constant terms d_1 and d_2 are of the same sign.

If $a_1 a_2 + b_1 b_2 + c_1 c_2 < 0$, then the angle between the planes in which the origin lies is acute.

If $a_1 a_2 + b_1 b_2 + c_1 c_2 > 0$, then the angle between the planes in which the origin lies is obtuse.

Note that the angle between the two planes in which the origin lies is acute or obtuse according as the angle between the normals to the two planes drawn from the origin is obtuse or acute.

Remark 3: If the angle between the bisecting plane and one of the given planes is less than 45°, then the bisecting plane will be the plane bisecting the acute angle.

Illustrative Examples

Example 11: *Show that the origin lies in the acute angle between the planes* $x + 2y + 2z = 9$ *and* $4x - 3y + 12z + 13 = 0$. *Find the equation of the plane which bisects the obtuse angle between them.*

Solution: The equations of the given planes in the normal form are

$$\frac{1}{3} x + \frac{2}{3} y + \frac{2}{3} z = 3 \text{ and } -\frac{4}{13} x + \frac{3}{13} y - \frac{12}{13} z = 1.$$

Therefore the direction cosines of the normals to the planes from the origin are $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$ and $-\frac{4}{13}, \frac{3}{13}, -\frac{12}{13}$ respectively.

If θ be the angle between these normals, then

$$\cos\theta = \frac{1}{3}\left(-\frac{4}{13}\right) + \frac{2}{3}\left(\frac{3}{13}\right) + \frac{2}{3}\left(-\frac{12}{13}\right) = -\frac{4}{39} + \frac{6}{39} - \frac{24}{39} = -\frac{22}{39} .$$

Therefore θ is an obtuse angle and hence the angle between the planes, in which the origin lies, is acute.

The equation of the plane which bisects the obtuse angle *i.e.*, the angle in which the origin does not lie is

$$\frac{x+2y+2z-9}{\sqrt{(1+4+4)}}=-\frac{-4x+3y-12z-13}{\sqrt{(16+9+144)}}$$

or $$\frac{x+2y+2z-9}{3}=\frac{4x-3y+12z+13}{13}$$

or $$13x+26y+26z-117=12x-9y+36z+39$$

or $$x+35y-10z=156.$$

20 Condition for the Equation to Represent a Pair of Planes and the Angle between the Two Planes

Prove that the equation

$$\boldsymbol{ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0}$$

represents a pair of planes if $\boldsymbol{abc+2fgh-af^2-bg^2-ch^2=0.}$

Prove also that the angle between the planes is

$$\tan^{-1}\left[\frac{2\,(f^2+g^2+h^2-bc-ca-ab)^{1/2}}{a+b+c}\right].$$

(Agra 2001, 02; Bundelkhand 09)

Proof: The given general homogeneous equation of second degree in x, y, z is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \qquad \ldots(1)$$

Let the equations of the two planes represented by (1) be

$$l_1x+m_1y+n_1z=0 \text{ and } l_2x+m_2y+n_2z=0.$$

These equations will not contain the constant terms for otherwise their product will not be homogeneous. Thus we have

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy\equiv(l_1x+m_1y+n_1z)(l_2x+m_2y+n_2z).$$

Comparing the coefficients of like terms on either side, we have

$$\left.\begin{aligned} l_1l_2=a,\ m_1m_2=b,\ n_1n_2=c,\ m_1n_2+m_2n_1=2f,\\ n_1l_2+n_2l_1=2g,\ l_1m_2+l_2m_1=2h\end{aligned}\right\}. \qquad \ldots(2)$$

The required condition is obtained by eliminating l_1, m_1, n_1 and l_2, m_2, n_2 from the relations (2). This is conveniently done by considering the following product of two zero-valued determinants :

$$\begin{vmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} \times \begin{vmatrix} l_2 & l_1 & 0 \\ m_2 & m_1 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} = 0.$$ **[Remember]**

Multiplying the two determinants by row-by-row multiplication rule, we have

$$\begin{vmatrix} 2\, l_1 l_2 & l_1 m_2 + l_2 m_1 & l_1 n_2 + l_2 n_1 \\ m_1 l_2 + m_2 l_1 & 2\, m_1 m_2 & m_1 n_2 + m_2 n_1 \\ n_1 l_2 + n_2 l_1 & n_1 m_2 + n_2 m_1 & 2 n_1 n_2 \end{vmatrix} = 0.$$

On putting the values of $l_1 l_2$, $l_1 m_2 + l_2 m_1$ etc. from (2), we have

$$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

or $$abc + 2fgh - af^2 - bg^2 - ch^2 = 0. \qquad \ldots(3)$$

This is the required condition that the equation (1) represents a pair of planes passing through the origin.

To find the angle between the planes: Let θ be the angle between the two planes represented by the equation (1).

Then θ is the angle between the planes $l_1 x + m_1 y + n_1 z = 0$ and $l_2 x + m_2 y + n_2 z = 0$ and so is given by

$$\tan \theta = \frac{[\Sigma\, (m_1 n_2 - m_2 n_1)^2]^{1/2}}{l_1 l_2 + m_1 m_2 + n_1 n_2},$$

where $$l_1 l_2 + m_1 m_2 + n_1 n_2 = a + b + c$$

and $$\Sigma\, (m_1 n_2 - m_2 n_1)^2 = \Sigma\, [(m_1 n_2 + m_2 n_1)^2 - 4 m_1 m_2 n_1 n_2]$$
$$= \Sigma\, (4f^2 - 4bc)$$
$$= 4\,(f^2 - bc) + 4\,(g^2 - ca) + 4\,(h^2 - ab),$$

so that $$[\Sigma\, (m_1 n_2 - m_2 n_1)^2]^{1/2} = 2\sqrt{(f^2 + g^2 + h^2 - bc - ca - ab)}.$$

∴ $$\boldsymbol{\tan \theta = \frac{2\sqrt{(f^2 + g^2 + h^2 - bc - ca - ab)}}{a + b + c}} \qquad \ldots(4)$$

or $$\theta = \tan^{-1}\left[\frac{2\sqrt{(f^2 + g^2 + h^2 - bc - ca - ab)}}{a + b + c}\right].$$

Condition of perpendicularity: The two planes given by (1) will be perpendicular if $\theta = \frac{1}{2}\pi$ *i.e.*, $\tan \theta = \tan \frac{1}{2}\pi = \infty$. Hence the relation (4) gives

$$\boldsymbol{a + b + c} = 0.$$

Thus, the two planes given by (1) will be **perpendicular if the coefficient of x^2 + the coefficient of y^2 + the coefficient of z^2 = 0.**

Illustrative Examples

Example 12: *Prove that the equation* $2x^2 - 6y^2 - 12z^2 + 18yz + 2zx + xy = 0$ *represents a pair of planes and find the angle between them.*

(Garhwal 2000; Agra 01; Rohilkhand 05, 07; Avadh 10)

Solution: Comparing the given equation with the equation

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, we get

$$a = 2, b = -6, c = -12, f = \frac{1}{2}\cdot 18 = 9, g = \frac{1}{2}\cdot 2 = 1, h = \frac{1}{2}\cdot$$

$$\therefore \quad abc + 2fgh - af^2 - bg^2 - ch^2$$

$$= 2(-6).(-12) + 2.9.1.\frac{1}{2} - 2.81 + 6.1 + 12.\frac{1}{4}$$

$$= 144 + 9 - 162 + 6 + 3 = 162 - 162 = 0.$$

Hence the given equation represents a pair of planes.

If θ be the angle between the planes, then

$$\tan\theta = \frac{2\sqrt{(f^2 + g^2 + h^2 - bc - ca - ab)}}{a + b + c}$$

$$= \frac{2\sqrt{(81 + 1 + \frac{1}{4} - 72 + 24 + 12)}}{2 - 6 - 12} = \frac{2\sqrt{[\frac{1}{4}.(185)]}}{-16} = -\frac{\sqrt{185}}{16}.$$

$$\therefore \quad \sec^2\theta = 1 + \tan^2\theta = 1 + \frac{185}{256} = \frac{441}{256}$$

$$\therefore \quad \sec\theta = \frac{21}{16} \quad \text{or} \quad \cos\theta = \frac{16}{21} \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{16}{21}\right).$$

Alternative method: Arranging the given equation as a quadratic in x, we have

$$2x^2 + x(2z + y) - (6y^2 + 12z^2 - 18yz) = 0.$$

$$\therefore \quad x = \frac{-(2z + y) \pm \sqrt{[(2z + y)^2 + 4.2.(6y^2 + 12z^2 - 18yz)]}}{2.2}$$

or $$4x = -2z - y \pm \sqrt{[4z^2 + 4zy + y^2 + 48y^2 + 96z^2 - 144yz]}$$

$$= -2z - y \pm \sqrt{(49y^2 - 140yz + 100z^2)}$$

$$= -2z - y \pm \sqrt{(7y - 10z)^2}$$

$$= -2z - y \pm (7y - 10z).$$

$\therefore$ $4x = -2z - y + 7y - 10z$ and $4x = -2z + y - 7y + 10z$

or $4x - 6y + 12z = 0$ and $4x + 8y - 8z = 0$

or $2x - 3y + 6z = 0$ and $x + 2y - 2z = 0$.

These being linear equations in x, y and z prepresent the two planes. If θ is the angle between these planes, then using

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}, \text{ we have}$$

$$\cos\theta = \frac{2\,.\,1 + (-3)\,.\,(2) + 6\,.\,(-2)}{\sqrt{(4+9+36)}\sqrt{(1+4+4)}} = \frac{-16}{21},$$

giving the obtuse angle between the planes.

If θ is the acute angle between the planes, then $\cos\theta = 16/21$.

$\therefore \qquad \theta = \cos^{-1}(16/21)$.

21 Projection on a Plane

Recall the definitions of the projection of a point and the projection of the segment of a line on a plane (see article 7 of chapter 3).

Similarly the projection of an area A on a given plane is defined. Let A be an area enclosed by the curve $PQR\ldots$. Let $P', Q', R', \ldots$ be the feet of the perpendiculars drawn from $P, Q, R, \ldots$ to the given plane. Then the projection of the area A enclosed by the curve $PQR\ldots$ on the given plane is the area A' enclosed by the curve $P'Q'R'\ldots$. If θ is the angle between the plane of the area A and the plane of projection, then $A' = A\cos\theta$.

Now we shall discuss two theorems on the projections.

Theorem 1: *Let the projections of an area A on the co-ordinate planes yz, zx and xy be* A_x, A_y *and* A_z *respectively, then* $A^2 = A_x^2 + A_y^2 + A_z^2$.

Proof: Let the direction cosines of the normal to the plane of area A be l, m, n.

Also the normal to the yz-plane is x-axis whose d.c.'s are 1, 0, 0. If α be the angle between the plane of area A and the yz-plane, then α is the angle between the normals to these planes and so $\cos\alpha = l\,.\,1 + m\,.\,0 + n\,.\,0 = l$.

Now the projection A_x of the area A on the yz-plane is given by

$$A_x = A\cos\alpha = Al.$$

Similarly we have $A_y = Am$, $A_z = An$.

Squaring and adding, we have

$$A_x^2 + A_y^2 + A_z^2 = A^2(l^2 + m^2 + n^2) = A^2\,.\,1 = A^2.$$

Theorem 2: *The projection of a given plane area A on a given plane* ξ *is equal to the sum of the projections of* A_x, A_y *and* A_z *on the given plane* ξ*, where* A_x, A_y *and* A_z *are the projections of the area A on the co-ordinate planes viz., yz, zx and xy-planes respectively.*

Proof: Let l, m, n be the d.c.'s of the normal to the plane A, and let l', m', n' be the d.c.'s of the normal to the plane ξ. Now if θ is the angle between these two planes, then $\qquad \cos\theta = ll' + mm' + nn'$. $\qquad \ldots(1)$

Now let the projection of the area A on the plane ξ be A' ; then we have

$$A' = A \cos \theta \quad \text{or} \quad A' = A\,(ll' + mm' + nn'). \qquad \ldots(2)$$

Also by definition and in view of theorem 1, we have

$$A_x = Al,\ A_y = Am,\ A_z = An. \qquad \ldots(3)$$

From (2), we have $A' = (Al)\,l' + (Am)\,m' + (An)\,n'$

$= A_x\, l' + A_y\, m' + A_z\, n'$ [using the relations (3)]

= (the projection of the area A_x on the plane ξ)

+ (the projection of the area A_y on the plane ξ)

+ (the projection of the area A_z on the plane ξ).

Proved.

22 Area of a Triangle

To find the area of a triangle ABC the co-ordinates of whose vertices are $A\,(x_1, y_1, z_1)$, $B\,(x_2, y_2, z_2)$ *and* $C\,(x_3, y_3, z_3)$.

Let l, m, n be the d.c.'s of the normal to the plane of the triangle ABC and let Δ denote the area of this triangle.

Let A_x, B_x and C_x be the projections of the three vertices A, B and C respectively on the yz-plane. Clearly the co-ordinates of these points are given by $A_x\ (0, y_1, z_1)$, $B_x\ (0, y_2, z_2)$ and $C_x\ (0, y_3, z_3)$.

Let Δ_x denote the area of the triangle $A_x B_x C_x$ *i.e.*, Δ_x is the area of projection of the area Δ on the yz-plane, so that we have

$$\Delta_x = \Delta\,.\,l. \qquad \ldots(1)$$

Also by the co-ordinate geometry of two dimensions,

$$\Delta_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_1 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} \qquad \ldots(2)$$

Similarly if Δ_y and Δ_z are the areas of the projections of the area Δ on zx and xy-planes, then

$$\Delta_y = \Delta \cdot m \qquad \ldots(3)$$

and

$$\Delta_z = \Delta \cdot n \qquad \ldots(4)$$

where

$$\Delta_y = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}, \quad \Delta_z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Squaring (1), (3) and (4) and adding, we get

$$\Delta_x^2 + \Delta_y^2 + \Delta_z^2 = \Delta^2\,(l^2 + m^2 + n^2) = \Delta^2\,.\,1$$

or

$$\Delta^2 = \Delta_x^2 + \Delta_y^2 + \Delta_z^2\,. \qquad \ldots(5)$$

This gives area Δ of the triangle ABC.

Illustrative Examples

Example 13: *A plane makes intercepts $OA = a, OB = b$ and $OC = c$ respectively on the co-ordinate axes. Show that the area of the triangle ABC is* $\frac{1}{2}\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}$.

(Meerut 2008, 09)

Solution: The points A, B and C lie on the axes of x, y and z respectively, so that their co-ordinates are $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$.

Let Δ denote the area of the triangle ABC. The projection of the triangle ABC on the yz-plane is the triangle OBC and if Δ_x denotes its area, then

$$\Delta_x = \frac{1}{2} . OB . OC = \frac{1}{2} bc . \qquad \ldots(1)$$

The projection of the triangle ABC on the zx-plane is the triangle OCA and its area Δ_y is given by $\Delta_y = \frac{1}{2} . OC . OA = \frac{1}{2} c\, a$. ...(2)

Also the projection of the triangle ABC on the xy-plane is the triangle OAB and its area Δ_z is given by

$$\Delta_z = \frac{1}{2} . OA . OB = \frac{1}{2} ab . \qquad \ldots(3)$$

$\therefore$ the area Δ of the triangle ABC is given by

$$\Delta^2 = \Delta_x^2 + \Delta_y^2 + \Delta_z^2 = \frac{1}{4}(b^2c^2 + c^2a^2 + a^2b^2)$$

or

$$\Delta = \frac{1}{2}\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}.$$

23 Volume of a Tetrahedron

If V is the volume of the tetrahedron A, BCD whose vertices are the points $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$, then

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

Particular case: If V is the volume of the tetrahedron O, ABC whose vertices are the points $O(0, 0, 0)$, $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$, then

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Illustrative Examples

Example 14: *Prove that the four planes* $my + nz = 0$, $nz + lx = 0$, $lx + my = 0$, $lx + my + nz = p$ *form a tetrahedron whose volume is* $\dfrac{2p^3}{3\,lmn}\cdot$

(Kanpur 2009; Kumaun 11)

Solution: The equations of the given planes are

$$my + nz = 0 \qquad \ldots(1)$$

$$nz + lx = 0 \qquad \ldots(2)$$

$$lx + my = 0 \qquad \ldots(3)$$

and

$$lx + my + nz = p. \qquad \ldots(4)$$

Solving (1), (2) and (3), we get $x = 0,\ y = 0,\ z = 0.$

Solving (2), (3) and (4), we get $x = -\dfrac{p}{l},\ y = \dfrac{p}{m},\ z = \dfrac{p}{n}\cdot$

Solving (1), (3) and (4), we get $x = \dfrac{p}{l},\ y = -\dfrac{p}{m},\ z = \dfrac{p}{n}\cdot$

Solving (1), (2) and (4), we get $x = \dfrac{p}{l},\ y = \dfrac{p}{m},\ z = -\dfrac{p}{n}\cdot$

Hence, the coordinates of the vertices of the tetrahedron are

$$(0, 0, 0)\ ;\ \left(-\frac{p}{l}, \frac{p}{m}, \frac{p}{n}\right);\ \left(\frac{p}{l}, -\frac{p}{m}, \frac{p}{n}\right)\ \text{and}\ \left(\frac{p}{l}, \frac{p}{m}, -\frac{p}{n}\right)\cdot$$

Therefore, the volume V of the tetrahedron

$$= \frac{1}{6}\begin{vmatrix} -\dfrac{p}{l} & \dfrac{p}{m} & \dfrac{p}{n} \\ \dfrac{p}{l} & -\dfrac{p}{m} & \dfrac{p}{n} \\ \dfrac{p}{l} & \dfrac{p}{m} & -\dfrac{p}{n} \end{vmatrix} = \frac{p^3}{6\,lmn}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \frac{p^3}{6\,lmn}\begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix}\ C_2 + C_1,\ C_3 + C_1$$

$$= \frac{p^3}{6\,lmn} \times 4 = \frac{2\,p^3}{3\,lmn}\cdot$$

Comprehensive Exercise 2

1. Find the equation of the plane passing through the line of intersection of the planes $2x - 7y + 4z = 3, 3x - 5y + 4z + 11 = 0$, and the point $(-2, 1, 3)$.

(Bundelkhand 2005)

2. Find the equation of the plane through the line of intersection of the planes $ax + by + cz + d = 0$ and $\alpha x + \beta y + \gamma z + \delta = 0$ and parallel to x-axis.

(Bundelkhand 2006, 14)

3. Find the equation of the plane which is perpendicular to the plane $5x + 3y + 6z + 8 = 0$ and which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0, 2x + y - z + 5 = 0$. **(Rohilkhand 2009)**

4. A variable plane at a constant distance p from the origin meets the axes in A, B and C. Through A, B, C planes are drawn parallel to the co-ordinate planes. Show that the locus of their point of intersection is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$. **(Rohilkhand 2008B)**

5. A variable plane passes through a fixed point (α, β, γ) and meets the axes of reference in A, B, C. Show that the locus of the point of intersection of the planes through A, B, C parallel to the co-ordinate planes is
$$\alpha x^{-1} + \beta y^{-1} + \gamma z^{-1} = 1.$$
(Meerut 2009B; Kumaun 12)

6. A point P moves on the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ which is fixed. The plane through P perpendicular to OP meets the co-ordinate axes in A, B and C. The planes through A, B and C parallel to the yz, zx and xy-planes intersect in Q. Prove that if the axes be rectangular, the locus of Q is
$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{ax} + \frac{1}{by} + \frac{1}{cz}.$$

7. Find the equations of the bisector planes of the angles between the planes $3x - 2y + 6z + 8 = 0$ and $2x - y + 2z + 3 = 0$ **(Purvanchal 2010)**

8. Find the equation of the plane that bisects the angle between the planes $3x - 6y + 2z + 5 = 0$ and $4x - 12y + 3z = 3$ which contains the origin. Is this the plane that bisects the obtuse angle ? **(Avadh 2011)**

9. Find the equations of the bisectors of the angles between the planes $2x - y - 2z - 6 = 0$ and $3x + 2y - 6z - 12 = 0$ and distinguish them. **(Purvanchal 2010)**

10. Prove that the equation $x^2 + 4y^2 - z^2 + 4xy = 0$ represents a pair of planes and find the angle between them. **(Rohilkhand 2008)**

11. Show that the equation $\frac{a}{y - z} + \frac{b}{z - x} + \frac{c}{x - y} = 0$ represents a pair of planes. **(Gorakhpur 2006; Kanpur 06; Rohilkhand 10)**

12. Find the area of the triangle whose vertices are $A(1, 2, 3)$, $B(2, -1, 1)$ and $C(1, 2, -4)$. **(Meerut 2013)**

13. Find the area of the triangle included between the plane $3x - 4y + z = 12$ and the co-ordinate planes. **(Meerut 2000; Rohilkhand 08)**

14. From a point $P(x', y', z')$ a plane is drawn at right angles to OP to meet the co-ordinate axes at A, B and C. Prove that the area of the triangle ABC is $r^5 / (2x'y'z')$, where r is the measure of OP. **(Meerut 2011)**

Answers 2

1. $15x - 47y + 28z = 7$

2. $(b\alpha - a\beta)\ y + (c\ \alpha - a\gamma)\ z + (d\alpha - a\delta) = 0$

3. $33x + 45y + 50z - 41 = 0$

7. $5x - y - 4z - 3 = 0, 23x - 13y + 32z + 45 = 0$

8. $67x - 162y + 47z + 44 = 0$

9. $5x - 13y + 4z - 6 = 0;\ 23x - y - 32z - 78 = 0$

10. $\tan^{-1}\left(\frac{1}{2}\sqrt{5}\right)$

12. $\frac{1}{2}\sqrt{(490)}$ square units

13. $3\sqrt{(26)}$ square units

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Equation of a plane through the point $P\ (\alpha, \beta, \gamma)$ and perpendicular to OP is

(a) $\alpha x + \beta y + \gamma z = \sqrt{\alpha^2 + \beta^2 + \gamma^2}$ (b) $\alpha x + \beta y + \gamma z = 0$

(c) $\alpha x + \beta y + \gamma z = \alpha^2 + \beta^2 + \gamma^2$ (d) $\alpha x + \beta y + \gamma z = 1$

2. The angle between the planes $3x - 4y + 5z = 0$ and $2x - y - 2z = 5$ is

(a) $\pi/3$ (b) $\pi/2$

(c) $\pi/6$ (d) $\pi/4$

(Meerut 2009B)

3. The ratio in which the xy-plane meets the line joining the points $(-3, 4, -8)$ and $(5, -6, 4)$ is

(a) 2 : 3 (b) 2 : 1

(c) 4 : 5 (d) none of these

(Agra 2007)

4. A plane meets the coordinate axes at A, B, C such that the centroid of the triangle ABC is the point (a, b, c). Then the equation of the plane ABC is

(a) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ (b) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$

(c) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$ (d) none of these

5. If d_1 and d_2 are both positive and the origin lies in the acute angle between the planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ then the value of $a_1\ a_2+b_1\ b_2+c_1\ c_2$ is

(a) negative (b) positive

(c) 0 (d) none of these

6. The intercept form of plane is

(a) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ (b) $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}=1$

(c) $ax+by+cz=1$ (d) none of these

(Bundelkhand 2005)

7. The equation of the plane parallel to x-axis is

(a) $y=0$ (b) $z=0$

(c) $ax+d=0$ (d) $by+cz+d=0$

(Kumaun 2008)

8. The angle between the plane $z=0$ and $x+y=0$ is

(a) 60° (b) 90°

(c) 180° (d) none of these

9. The angle between the planes $3x-4y+5z=0$ and $2x-y-z=5$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$ **(Kumaun 2011, 14)**

10. The equation of the XOY plane is

(a) $x=0$ (b) $=0$

(c) $z=0$ (d) none of these

(Kumaun 2014)

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. The equation $ax+by+cz+d=0$ represents a …… .

2. The direction cosines of normal to the plane $4x-y-8z+7=0$ directed from the origin to the plane are …… .

3. $2x+3y+4z=0$ is the equation of a plane which passes through the .... **(Meerut 2001)**

4. $\alpha(x-l)+\beta(y-m)+\gamma(z-n)=0$ represents a plane passing through the point .... **(Meerut 2001)**

5. Equation of the plane through a given point (x_1, y_1, z_1) and perpendicular to a line whose direction ratios are a, b, c is ....

6. The equation of the plane which cuts off intercepts a, b, c from the axes is ....

7. The intercept made by the plane $3x+4y+8z=2$ on y-axis is ....

8. The equation of the plane $x-2y+2z-9=0$ in normal form is ....

9. The planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ are perpendicular if and only if ....

10. The equation of the plane through the point $(1, 4, -2)$ and parallel to the plane $-2x+y-3z=7$ is ....

11. The equation of the plane parallel to the zx-plane and at a distance 'b' from it is given by ....

12. The length of the perpendicular drawn from the point (x_1, y_1, z_1) to the plane $ax+by+cz+d=0$ is ....

13. The length of the perpendicular drawn from the origin to the plane $x+4y-8z+18=0$ is ....

14. The perpendicular distance of the point $(2, -1, -4)$ from the plane $3x-4y+12z=9$ is ....

15. The distance between the parallel planes $x+y-z+4=0$ and $x+y-z+5=0$ is ....

16. The length of the perpendicular from the origin to a plane is 5 units and the direction ratios of a normal to the plane are $2, 3, -6$. The equation of the plane is .... **(Meerut 2002)**

17. The equation of the plane passing through the origin and parallel to the plane $4x-9y+7z+3=0$ is ....

18. The foot of the perpendicular drawn from the origin to a plane is $(12, -4, 3)$. The equation of the plane is ....

19. The angle between the normals to the planes $2x-y+z=13$ and $x+y+2z=9$ is ....

20. Two points (x_1, y_1, z_1) and (x_2, y_2, z_2) will lie on the same side of the plane $ax+by+cz+d=0$ if $ax_1+by_1+cz_1+d$ and $ax_2+by_2+cz_2+d$ are of the ...... signs.

21. The bisector of the acute angle between the planes $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$ is …… .

22. Two planes represented by $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ will be perpendicular if …… .

23. The areas of projections of ΔABC on the coordinate planes yz, zx, xy are respectively A_x, A_y, A_z. If A is the area of ΔABC, then $A^2 = \ldots\ldots$.

True or False

Write 'T' for true and 'F' for false statement.

1. The number of arbitrary constants in the general equation of a plane is 3.

2. The planes $x - y + z = 7$ and $3x + 2y - z + 9 = 0$ are perpendicular to each other.

3. The planes $3x - 4y + 8z + 7 = 0$ and $6x + 8y + 16z + 9 = 0$ are parallel to each other.

4. The planes $3x + 4y + 9z = 8$ and $6x + 8y + 18z - 7 = 0$ are parallel to each other.

5. The angle between the planes $2x - y - 2z - 6 = 0$ and $3x + 2y - 6z - 12 = 0$ in which the origin lies is acute.

Answers

Multiple Choice Questions

1. (c) **2.** (b) **3.** (b) **4.** (c) **5.** (a)
6. (a) **7.** (d) **8.** (b) **9.** (c) **10.** (c)

Fill in the Blank(s)

1. plane **2.** $-\frac{4}{9}, \frac{1}{9}, \frac{8}{9}$ **3.** origin

4. (l, m, n) **5.** $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

6. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ **7.** $\frac{1}{2}$

8. $\frac{1}{3}x - \frac{2}{3}y + \frac{2}{3}z = 3$ **9.** $a_1a_2 + b_1b_2 + c_1c_2 = 0$

10. $2x - y + 3z + 8 = 0$ **11.** $y = b$

12. $\dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

13. 2

14. $\dfrac{47}{13}$

15. $\dfrac{1}{\sqrt{3}}$

16. $2x + 3y - 6z = 35$

17. $4x - 9y + 7z = 0$

18. $12x - 4y + 3z = 169$

19. $\pi / 3$

20. same

21. $23x - 13y + 32z + 45 = 0$

22. $a + b + c = 0$

23. $A_x^2 + A_y^2 + A_z^2$

True or False

1. *T* **2.** *T* **3.** *F* **4.** *T* **5.** *F*

Chapter

5

The Straight Line

1 The Equations of a Straight Line

Every equation of the first degree in x, y, z represents a plane. Also as two planes intersect in a line, therefore the two equations together represent that line. Thus $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ represent a straight line.

These are called the general equations of a straight line.

2 Equations of the Straight Line in the Symmetrical Form

To find the equations of a straight line passing through a given point $A(\alpha, \beta, \gamma)$ and having direction cosines l, m, n. **(Kumaun 2001)**

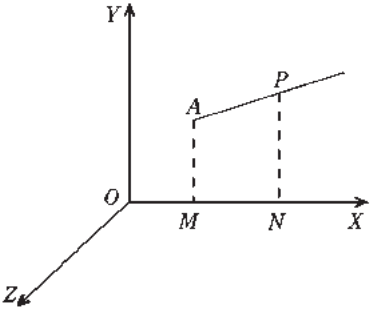
Let $P(x, y, z)$ be any point on the line such that $AP = r$.

Now projecting AP on the x-axis, we have

$$x - \alpha = lr \quad \text{or} \quad \frac{x - \alpha}{l} = r.$$

Similarly projecting AP on the y and z-axes, we have

$$\frac{y-\beta}{m}=r,\ \frac{z-\gamma}{n}=r\ .$$



$$\therefore \qquad \frac{\boldsymbol{x}-\alpha}{\boldsymbol{l}}=\frac{\boldsymbol{y}-\beta}{\boldsymbol{m}}=\frac{\boldsymbol{z}-\gamma}{\boldsymbol{n}}$$

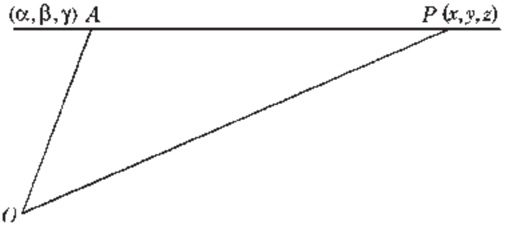
are the equations of a straight line in the symmetrical form.

Alternative method: *(By vectors).*

Let $P(x, y, z)$ be any point on the line AP, where $A(\alpha, \beta, \gamma)$ is the given point.

$$\therefore \qquad \vec{AP}=\vec{OP}-\vec{OA}=\vec{r}-\vec{a}=(xi+yj+zk)-(\alpha i+\beta j+\gamma k)$$

$$=(x-\alpha)\,i+(y-\beta)\,j+(z-\gamma)\,k\ . \qquad \ldots(1)$$



Since l, m, n are the direction cosines of the line AP, therefore $\vec{AP}=r\,\hat{t}$, where $AP=r$ and $\hat{t}$ is the unit vector along AP given by

$$\hat{t}=li+mj+nk\ .$$

$$\therefore \qquad \vec{AP}=r\,\hat{t}=r\,(li+mj+nk). \qquad \ldots(2)$$

From (1) and (2),

$$\vec{r}-\vec{a}=r\,\hat{t}$$

or $\qquad (x-\alpha)\,i+(y-\beta)\,j+(z-\gamma)\,k=r\,(li+mj+nk).$

$\therefore \qquad x-\alpha=lr\ ,\ y-\beta=mr\ ,\ z-\gamma=nr\ .$

Hence $\dfrac{\boldsymbol{x}-\alpha}{\boldsymbol{a}}=\dfrac{\boldsymbol{y}-\beta}{\boldsymbol{b}}=\dfrac{\boldsymbol{z}-\gamma}{\boldsymbol{c}}\cdot$ are the required equations of the line.

Corollary 1: The equations of the straight line passing through (α, β, γ) and having direction cosines proportional to a, b, c are $\dfrac{\boldsymbol{x}-\alpha}{\boldsymbol{a}}=\dfrac{\boldsymbol{y}-\beta}{\boldsymbol{b}}=\dfrac{\boldsymbol{z}-\gamma}{\boldsymbol{c}}\cdot$

Corollary 2: Any point on the line. Any point $P(x, y, z)$ on the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \text{ (say) is } \boldsymbol{(\alpha+lr,\ \beta+mr,\ \gamma+nr).}$$

Here, l, m, n are the d.c.'s of the line and r is the distance of any point $P(x, y, z)$ on the line from the given point (α, β, γ).

Similarly any point on the line

$$\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\gamma}{c}=r \text{ (say) is } (\alpha+ar,\ \ \beta+br,\ \ \gamma+cr).$$

It should be noted here that 'r' is **not the actual distance** of any point $P(x, y, z)$ on the line from the given point (α, β, γ).

3 Line through Two given Points

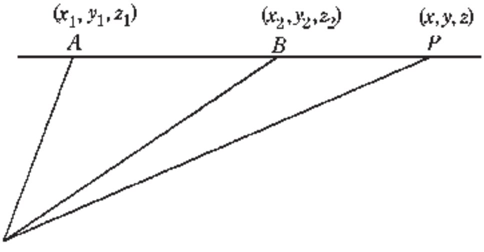
(Meerut 2010B)

To find the equations of a straight line through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2).

The direction cosines of the line will be proportional to $x_2 - x_1, y_2 - y_1, z_2 - z_1$ and it passes through (x_1, y_1, z_1).

Therefore the equations of the line will be $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$.

Alternate method: *(By vectors).*



Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ be the given points and $P(x, y, z)$ be any point on the line AB.

Since the vectors $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are collinear, therefore $\overrightarrow{AP} = t\,\overrightarrow{AB}$, where t is some scalar

or $$(x - x_1)\,i + (y - y_1)\,j + (z - z_1)\,k = t\,\{(x_2 - x_1)\,i + (y_2 - y_1)\,j + (z_2 - z_1)\,k\}.$$

$\therefore$ $x - x_1 = t\,(x_2 - x_1)$, $y - y_1 = t\,(y_2 - y_1)$

and $z - z_1 = t\,(z_2 - z_1)$.

Hence, $$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

are the required equations of a line through two given points.

Illustrative Examples

Example 1: *Find the co-ordinates of the point of intersection of the line*

$$\frac{x + 1}{1} = \frac{y + 3}{3} = \frac{z - 2}{2} \text{ with the plane } 3x + 4y + 5z = 20.$$

(Kumaun 2015)

Solution: Suppose $\dfrac{x + 1}{1} = \dfrac{y + 3}{3} = \dfrac{z - 2}{2} = r$, (say).

$\therefore$ $(-1 + r, -3 + 3r, 2 + 2r)$ are the co-ordinates of any point on the line.

If this point lies on the plane $3x + 4y + 5z = 20$, then

$$3(r - 1) + 4(3r - 3) + 5(2r + 2) = 20.$$

$\therefore$ $25r = 25$ or $r = 1$.

Putting the value of r we get the required co-ordinates of the point as $(0, 0, 4)$.

Example 2: *Find the distance of the point* (2 , 3, 4) *from the point where the line*

$$\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2} \quad \text{meets the plane } x+y+z=22.$$

Solution: Any point on the line is $(3+r\ , 4+2r\ , 5+2r)$.

If it also lies on the plane $x+y+z=22$, then

$$3+r+4+2r+5+2r=22 \qquad \text{or} \qquad 5r=10.$$

$$\therefore \qquad r=2\ .$$

Putting the value of r we get the required co-ordinates of the point as (5, 8, 9).

$\therefore$ the required distance = the distance between (2 , 3, 4) and (5, 8, 9)

$$=\sqrt{(5-2)^2+(8-3)^2+(9-4)^2}$$

$$=\sqrt{9+25+25}=\sqrt{59}\ .$$

Example 3: *Find the distance of the point* (1, 3, 4) *from the plane* $2x-y+z=3$ *measured parallel to the line* $\frac{x}{2}=\frac{y}{-1}=\frac{z}{-1}$. **(Meerut 2007B, 08, 12)**

Solution: Line through (1, 3, 4) parallel to the given line is

$$\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{-1}=r\ , \text{(say)}.$$

Any point on this line is $(1+2r\ , 3-r\ , 4-r)$.

If it also lies on the plane $2x-y+z=3$, then

$$2\,(1+2\,r)-(3-r)+4-r=3 \quad \text{or} \quad 2+4r-3+r+4-r=3$$

$$\text{or} \qquad 4r=0 \quad \text{or} \quad r=0.$$

Putting the value of r we get the co-ordinates of the point as (1, 3, 4).

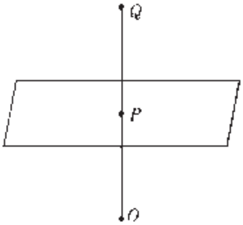
Distance required = the distance between the points (1, 3, 4) and (1, 3, 4) which is obviously zero.

Example 4: *Find the co-ordinates of the foot of the perpendicular drawn from the origin to the plane* $3x+4y-6z+1=0$. *Find also the co-ordinates of the point on the line which is at the same distance from the foot of the perpendicular as the origin is.*

(Bundelkhand 2005; Meerut 13B)

Solution: The equation of the given plane is

$$3x+4y-6z+1=0. \qquad \ldots(1)$$



The d.r.'s of the normal to the plane (1) are 3, 4, – 6.

$\therefore$ d.r.'s of the line perpendicular to the plane (1) are 3, 4, – 6.

Hence the equations of the line passing through (0, 0, 0) and perpendicular to the plane (1) are

$$\frac{x}{3}=\frac{y}{4}=\frac{z}{-6}=r \text{ (say)}. \qquad \ldots(2)$$

$\therefore$ any point P on the line (2) is $(3r, 4r, -6r)$. ...(3)

If this point lies on the plane (1), then

$$3(3r) + 4(4r) - 6(-6r) + 1 = 0$$

or $$r = -1/61 \cdot$$

Putting this value of r *i.e.*, $r = -1/61$ in (3), the coordinates of the foot of the perpendicular P are $\left(\frac{-3}{61}, \frac{-4}{61}, \frac{6}{61}\right)$.

Now if $Q(x_1, y_1, z_1)$ be the point on the line which is at the same distance from the foot of the perpendicular as the origin is, then P is the middle point of OQ.

$$\therefore \quad \frac{x_1 + 0}{2} = \frac{-3}{61}, \quad \frac{y_1 + 0}{2} = \frac{-4}{61} \quad \text{and} \quad \frac{z_1 + 0}{2} = \frac{6}{61}$$

giving $$x_1 = \frac{-6}{61}, \quad y_1 = \frac{-8}{61}, \quad z_1 = \frac{12}{61}.$$

$\therefore$ the co-ordinates of Q are $\left(\frac{-6}{61}, \frac{-8}{61}, \frac{12}{61}\right)$.

Comprehensive Exercise 1

1. (i) Find the point in which the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ meets the plane $x - 2y + z = 20$. **(Rohilkhand 2013)**

   (ii) Find the distance from the point $(3, 4, 5)$ to the pont where the line $\frac{x-3}{1} = \frac{y-4}{2} = \frac{z-5}{2}$ meets the plane $x + y + z = 2$. **(Kumaun 2009)**

2. Find the coordinates of the point where the line joining the points $(2, -3, 1)$ and $(3, -4, -5)$ meets the plane $2x + y + z = 7$. **(Kumaun 2007; Rohilkhand 08)**

3. Show that the distance of the point of intersection of the line $\frac{x-3}{1} = \frac{y-4}{2} = \frac{z-5}{2}$ and the plane $x + y + z = 17$ from the point $(3, 4, 5)$ is 3.

4. Find the points in which the line $\frac{x+1}{-1} = \frac{y-12}{5} = \frac{z-7}{2}$ cuts the surface $11x^2 - 5y^2 + z^2 = 0$. **(Meerut 2004, 05, 07, 09B; Kanpur 06)**

5. Find the distance of the point $(1, -2, 3)$ from the plane $x - y + z = 5$ measured along a line parallel to $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$. **(Kanpur 2010)**

6. Find the image of the point $(1, 3, 4)$ in the plane $2x - y + z + 3 = 0$. **(Meerut 2004, 05, 06B, 07, 10; Rohilkhand 08)**

7. A variable plane makes intercepts on the co-ordinate axes the sum of whose squares is constant and equal to k^2. Show that the locus of the foot of the perpendicular from the origin to the plane is

$$(x^{-2}+y^{-2}+z^{-2})(x^2+y^2+z^2)^2=k^2.$$ **(Garhwal 2001)**

Answers 1

1. (i) (8, 7, 26) (ii) 6 2. (1, – 2, 7)

4. (1, 2 , 3); (2 , – 3, 1) 5. 1 6. (– 3, 5, 2)

4 Transformation of the General Form of the Equations of a Straight Line to Symmetrical Form

To transform the equations $a_1x+b_1y+c_1z+d_1=0,\ a_2x+b_2y+c_2z+d_2=0$ *of a straight line to the symmetrical form.*

Let the general form of the equations of the straight line be given by the equations

$$\left.\begin{aligned}a_1x+b_1y+c_1z+d_1&=0\\a_2x+b_2y+c_2z+d_2&=0\end{aligned}\right\}. \qquad \ldots(1)$$

Now we are required to write down the symmetrical form of the straight line given by the equations (1). For this we must know (i) the direction cosines or direction ratios of the line and (ii) the co-ordinates of a point on the line. To find these two we proceed as follows :

Step 1: To find the direction cosines or the direction ratios of the line given by the equations (1).

Let l, m, n be the direction cosines or direction ratios of the line. Since the line is common to both the planes, therefore it is perpendicular to the normals of both the planes. The direction ratios of the normals to the planes given by equations (1) are a_1, b_1, c_1 and a_2, b_2, c_2 respectively. Hence we have

$$la_1+mb_1+nc_1=0 \quad\text{and}\quad la_2+mb_2+nc_2=0.$$

Solving these equations for $l, m, n,$ we have

$$\frac{l}{b_1c_2-b_2c_1}=\frac{m}{c_1a_2-c_2a_1}=\frac{n}{a_1b_2-a_2b_1}.$$

$\therefore$ the direction ratios of the line are

$$b_1c_2-b_2c_1,\ c_1a_2-c_2a_1,\ a_1b_2-a_2b_1. \qquad \ldots(2)$$

Step 2: To find the co-ordinates of a point on the line given by the equations (1).

The co-ordinates of a point on a line can be chosen in many ways. One of these ways is that we choose the point as the one where the line cuts the *xy*-plane (*i.e.*, the plane $z = 0$), provided the line is not parallel to the plane $z = 0$ *i.e.*, provided $a_1 b_2 - a_2 b_1 \neq 0$. Putting $z = 0$ in both the equations given by (1), we get

$$a_1 x + b_1 y + d_1 = 0, \; a_2 x + b_2 y + d_2 = 0.$$

Solving these equations for x, y, we get

$$\frac{x}{b_1 d_2 - b_2 d_1} = \frac{y}{d_1 a_2 - d_2 a_1} = \frac{1}{a_1 b_2 - a_2 b_1}.$$

Hence the co-ordinates of a point on the line (1), where it cuts the plane $z = 0$ are

$$\left(\frac{b_1 d_2 - b_2 d_1}{a_1 b_2 - a_2 b_1}, \frac{d_1 a_2 - d_2 a_1}{a_1 b_2 - a_2 b_1}, 0 \right). \qquad \ldots(3)$$

Hence the equations of the line in symmetrical form are

$$\frac{x - \left(\dfrac{b_1 d_2 - b_2 d_1}{a_1 b_2 - a_2 b_1} \right)}{b_1 c_2 - b_2 c_1} = \frac{y - \left(\dfrac{d_1 a_2 - d_2 a_1}{a_1 b_2 - a_2 b_1} \right)}{c_1 a_2 - c_2 a_1} = \frac{z - 0}{a_1 b_2 - a_2 b_1}.$$

Note: If $a_1 b_2 - a_2 b_1 = 0$, then instead of taking $z = 0$ we should take the point where the line cuts $x = 0$ plane or $y = 0$ plane.

Illustrative Examples

Example 5: *Find in symmetrical form the equations of the line*

$3x + 2y - z - 4 = 0 = 4x + y - 2z + 3$ *and find its direction cosines.*

(Meerut 2005B)

Solution: The equations of the given line in general form are

$$3x + 2y - z - 4 = 0, \; 4x + y - 2z + 3 = 0. \qquad \ldots(1)$$

Let l, m, n be the d.c.'s of the line (1). Since the line is common to both the planes, it is perpendicular to the normals to both the planes.

Hence we have $3l + 2m - n = 0, \; 4l + m - 2n = 0$.

Solving these, we get

$$\frac{l}{-4+1} = \frac{m}{-4+6} = \frac{n}{3-8} \quad \text{or} \quad \frac{l}{-3} = \frac{m}{2} = \frac{n}{-5}.$$

$\therefore$ the d.r.'s of the line (1) are $-3, 2, -5$.

We have, $\sqrt{(-3)^2 + 2^2 + (-5)^2} = \sqrt{38}$.

$\therefore$ The d.c.'s l, m, n of the line (1) are given by

$$l = -\frac{3}{\sqrt{38}}, \; m = \frac{2}{\sqrt{38}}, \; n = -\frac{5}{\sqrt{38}}.$$

Now to find the co-ordinates of a point on the line given by (1), let us find the point where it meets the plane $z = 0$. Putting $z = 0$ in the equations given by (1), we have

$$3x + 2y - 4 = 0, \quad 4x + y + 3 = 0.$$

Solving these, we get

$$\frac{x}{6+4} = \frac{y}{-16-9} = \frac{1}{3-8} \quad \text{or} \quad \frac{x}{10} = \frac{y}{-25} = \frac{1}{-5}$$

giving $x = -2, y = 5$.

$\therefore$ The line meets the plane $z = 0$ in the point $(-2, 5, 0)$ and has direction ratios as $-3, 2, -5$. Therefore the equations of the given line in symmetrical form are

$$\frac{x+2}{-3} = \frac{y-5}{2} = \frac{z-0}{-5}.$$

Example 6: *The planes* $3x - y + z + 1 = 0, 5x + y + 3z = 0$ *intersect in the line* PQ. *Find the equation to the plane through the point* $(2, 1, 4)$ *and perpendicular to* PQ.

Solution: Let l, m, n be the direction cosines of PQ. Since the normal to each of the given planes is perpendicular to the line PQ, therefore

$$3l - m + n = 0 \quad \text{and} \quad 5l + m + 3n = 0.$$

$$\therefore \quad \frac{l}{-3-1} = \frac{m}{5-9} = \frac{n}{3+5}$$

or $$\frac{l}{1} = \frac{m}{1} = \frac{n}{-2}.$$

Since we have to find the equation of the plane which is perpendicular to PQ, therefore the coefficients of x, y and z in its equation must be quantities proportional to l, m, n so let the equation of this plane be $x + y - 2z + \lambda = 0$.

If it passes through the point $(2, 1, 4)$, then

$$2 + 1 - 8 + \lambda = 0 \quad \text{or} \quad \lambda = 5.$$

Hence the required equation is

$$x + y - 2z + 5 = 0.$$

Comprehensive Exercise 2

1. Find in symmetrical form the equations of the line $x + y + z + 1 = 0 = 4x + y - 2z + 2$ and find its direction cosines.
2. Find the angle between the lines $x + 2y - 2z = 11, x - 2y + z = 9$ and $\frac{x-3}{1} = \frac{y+5}{-3} = \frac{z-1}{2}$.
3. Prove that the lines $x = ay + b, z = cy + d$ and $x = a'y + b', z = c'y + d'$ are perpendicular if $aa' + cc' + 1 = 0$.

(Kanpur 2006, 09; Rohilkhand 07; Bundelkhand 09; Kumaun 09)

4. Find the equations to the line through the point (1, 2 , 3) parallel to the line $x - y + 2z - 5 = 0 ; 3x + y + z - 6 = 0$. **(Avadh 2013)**

5. (i) Show that the lines $2x + 3y - 4z = 0, 3x - 4y + z = 7$ and $5x - y - 3z + 12 = 0, x - 7y + 5z - 6 = 0$ are parallel.

   (ii) Show that the lines $x + y - z = 5, 9x - 5y + z = 4$ and $6x - 8y + 4z = 3, x + 8y - 6z + 7 = 0$ are parallel. **(Kumaun 2008)**

6. Show that the lines $3x + 2y + z = 5, x + y - 2z = 3$ and $2x - y - z = 0, 7x + 10y - 8z = 15$ are mutually perpendicular.

Answers 2

1. $\dfrac{x + 1/3}{-1} = \dfrac{y + 2/3}{2} = \dfrac{z + 0}{-1}; \dfrac{-1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}, \dfrac{-1}{\sqrt{6}}$

2. $\cos^{-1}\left(\dfrac{1}{\sqrt{406}}\right)$

4. $\dfrac{x - 1}{-3} = \dfrac{y - 2}{5} = \dfrac{z - 3}{4}$

5 Angle between a Straight Line and a Plane

(Rohilkhand 2005)

To find the co-ordinates of the point of intersection of a given line and a given plane and to deduce the conditions that :

(i) the line may be parallel to the plane,

(ii) the line may be perpendicular to the plane, and

(iii) the line may be lying in the plane.

Let the equations of the given straight line in symmetrical form be

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} = r \text{ (say)}. \qquad \ldots(1)$$

and the equation of the given plane be

$$ax + by + cz + d = 0. \qquad \ldots(2)$$

The co-ordinates of any point on the line (1) are

$$(\alpha + lr, \beta + mr, \gamma + nr). \qquad \ldots(3)$$

If this point lines on the plane (2), then

$$a(\alpha + lr) + b(\beta + mr) + c(\gamma + nr) + d = 0$$

or $$r(al + bm + cn) + a\alpha + b\beta + c\gamma + d = 0$$

or $$r = -\frac{a\alpha + b\beta + c\gamma + d}{al + bm + cn}.$$

Substituting this value of r in (3), we get the point of intersection of a given line and a given plane.

Corollary 1: Conditions of parallelism of a line and a plane:

(Bundelkhand 2005; Agra 06)

To deduce the conditions that the line $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ *may be parallel to the plane*

$$ax+by+cz+d=0.$$

If the line (1) is parallel to the plane (2), then this line must be perpendicular to the normal to the plane (2).

$$\therefore \qquad al+bm+cn=0.$$

Again the point (α,β,γ) should not lie on the plane *i.e.*, we must have $a\alpha+b\beta+c\gamma+d\neq 0$, for otherwise the line (1) will not be simply parallel to the plane (2) but it will lie in the plane (2).

$$\therefore \qquad al+bm+cn=0 \quad \text{and} \quad a\alpha+b\beta+c\gamma+d\neq 0$$

are the required conditions.

Corollary 2: Condition of perpendicularity: If the line (1) is perpendicular to the plane (2), then it must be parallel to the normal of the plane (2), so that

$$\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$$

is the required condition of perpendicularity.

Corollary 3: Conditions for a line to lie in a plane: If the line (1) lies in the plane (2), then for all values of r the point $(\alpha+lr,\beta+mr,\gamma+nr)$ will lie on the plane (2) *i.e.*, $a(\alpha+lr)+b(\beta+mr)+c(\gamma+nr)+d=0$

or $\qquad r(al+bm+cn)+(a\alpha+b\beta+c\gamma+d)=0$ is true for all values of r.

$\therefore$ The coefficient of $r=0$ and the constant term $=0$

i.e., $\qquad al+bm+cn=0$ and $a\alpha+b\beta+c\gamma+d=0$,

which are the required conditions.

Illustrative Examples

Example 7: *Find the equation of the plane through the line* $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ *and parallel to the line* $\dfrac{x-\alpha'}{l'}=\dfrac{y-\beta'}{m'}=\dfrac{z-\gamma'}{n'}$.

Solution: Any plane through the line $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ is

$$a(x-\alpha)+b(y-\beta))+c(z-\gamma)=0 \qquad \ldots(1)$$

where $\qquad al+bm+cn=0. \qquad \ldots(2)$

Since the plane (1) is parallel to the line $\dfrac{x-\alpha'}{l'}=\dfrac{y-\beta'}{m'}=\dfrac{z-\gamma'}{n'}$, therefore

$$al'+bm'+cn'=0. \qquad \ldots(3)$$

From (2) and (3), we have

$$\frac{a}{mn'-m'n}=\frac{b}{nl'-n'l}=\frac{c}{lm'-l'm}.$$

Putting the proportionate values of a, b, c in (1), the required equation of the plane is

$$(mn'-m'n)(x-\alpha)+(nl'-n'l)(y-\beta)+(lm'-l'm)(z-\gamma)=0.$$

Example 8: *Find the equation of the plane through the line*

$$P\equiv ax+by+cz+d=0,\ Q\equiv a'x+b'y+c'z+d'=0$$

and parallel to the line $x/l=y/m=z/n$.

Solution: The equation of any plane through the line $P=0$, $Q=0$ *i.e.*, through the line of intersection of the planes $P=0$ and $Q=0$ is

$$P+\lambda Q=0 \qquad \ldots(1)$$

or $(ax+by+cz+d)+\lambda(a'x+b'y+c'z+d')=0$

or $(a+\lambda a')x+(b+\lambda b')y+(c+\lambda c')z+d+\lambda d'=0. \qquad \ldots(2)$

The d.c.'s of the normal to the plane (2) are proportional to

$$a+\lambda a', b+\lambda b', c+\lambda c'.$$

The plane (2) [or (1)] will be parallel to the line $x/l=y/m=z/n$ if the normal to the plane (2) is perpendicular to the line $x/l=y/m=z/n$. Hence we have

$$(a+\lambda a')l+(b+\lambda b')m+(c+\lambda c')n=0$$

or $\lambda(a'l+b'm+c'n)=-(al+bm+cn)$

or $\lambda=-(al+bm+cn)/(a'l+b'm+c'n)$.

Putting this value of λ in (1), the required equation of the plane is given by

$$P-\{(al+bm+cn)/(a'l+b'm+c'n)\}Q=0$$

or $P(a'l+b'm+c'n)=Q(al+bm+cn)$.

Comprehensive Exercise 3

1. Find the equation of the plane which passes through the line of intersection of the planes $u_1\equiv a_1x+b_1y+c_1z+d_1=0$ and $u_2\equiv a_2x+b_2y+c_2z+d_2=0$ and is parallel to x-axis.
2. Find the equation of the plane through the line $3x-4y+5z=10, 2x+2y-3z=4$ and parallel to the line $x=2y=3z$.
3. Find the direction cosines of the line whose equations are $x+y=3$ and $x+y+z=0$ and show that it makes an angle of 30° with the plane $y-z+2=0$.

4. Find the equation of the plane through the points $(2, -1, 0)$, $(3, -4, 5)$ and parallel to the line $3x = 2y = z$.
5. Find the equation of the plane through $(2, 1, 4)$ perpendicular to the line of intersection of the planes $3x + 4y + 7z + 4 = 0$ and $x - y + 2z + 3 = 0$.

Answers 3

1. $u_1 a_2 = u_2 a_1$ 2. $x - 20y + 27z = 14$ 3. $0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$

4. $33x - 4y - 9z - 70 = 0$ 5. $15x + y - 7z - 3 = 0$

---

6 Plane through a given Line (Equations of the given Line in the Symmetrical Form)

The equation of any plane through the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ *is*

$$a(x-\alpha) + b(y-\beta) + c(z-\gamma) = 0 \text{ where } al + bm + cn = 0.$$

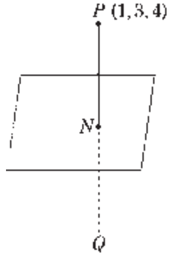
The equations of the given line in symmetrical form are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}. \quad \ldots(1)$$

The equation of any plane through (α, β, γ) is

$$a(x-\alpha) + b(y-\beta) + c(z-\gamma) = 0. \quad \ldots(2)$$

If it passes through the given line, its normal is perpendicular to the given line



i.e., $al + bm + cn = 0.$ ...(3)

Form (2) and (3), the equation of *any* plane through the given line is

$$\boldsymbol{a(x-\alpha) + b(y-\beta) + c(z-\gamma) = 0, \text{ where } al + bm + cn = 0.}$$

Corollary: Plane through one line and parallel to another line.

The equation of the plane through the line $\frac{x-\alpha}{l_1} = \frac{y-\beta}{m_1} = \frac{z-\gamma}{n_1}$, *and parallel to the line* $\frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2}$ *is*

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

The equation of any plane through the line $\frac{x-\alpha}{l_1} = \frac{y-\beta}{m_1} = \frac{z-\gamma}{n_1}$ is

$$a\,(x-\alpha)+b\,(y-\beta)+c\,(z-\gamma)=0, \qquad \ldots(1)$$

where $al_1+bm_1+cn_1=0. \qquad \ldots(2)$

If it is parallel to the line $\dfrac{x}{l_2}=\dfrac{y}{m_2}=\dfrac{z}{n_2}$, its normal is perpendicular to this line

i.e., $al_2+bm_2+cn_2=0. \qquad \ldots(3)$

Eliminating a, b, c from (1), (2), (3), we get

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0,$$ which is the required equation.

Illustrative Examples

Example 9: *Find the equation of the plane through the point* $(\alpha', \beta', \gamma')$ *and through the line whose equations are* $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$.

Solution: The equations of the given line are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}. \qquad \ldots(1)$$

The equation of any plane through the line (1) is

$$a\,(x-\alpha)+b\,(y-\beta)+c\,(z-\gamma)=0 \qquad \ldots(2)$$

where $al+bm+cn=0. \qquad \ldots(3)$

If the plane (2) passes through the point $(\alpha', \beta', \gamma')$, then from (2)

$$a\,(\alpha'-\alpha)+b\,(\beta'-\beta)+c\,(\gamma'-\gamma)=0 \qquad \ldots(4)$$

Eliminating a, b, c from(2), (4), (3), we get

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ l & m & n \end{vmatrix}=0,$$ which is the required equation.

Example 10: *Find the equation of the plane which contains the two parallel lines*

$$\frac{x+1}{3}=\frac{y-2}{2}=\frac{z}{1} \quad \textit{and} \quad \frac{x-3}{3}=\frac{y+4}{2}=\frac{z-1}{1}.$$

Solution: The equations of the two parallel lines are

$$\frac{x+1}{3}=\frac{y-2}{2}=\frac{z-0}{1} \qquad \ldots(1)$$

and $$\frac{x-3}{3}=\frac{y+4}{2}=\frac{z-1}{1}. \qquad \ldots(2)$$

The equation of any plane through the line (1) is

$$a(x+1)+b(y-2)+cz=0, \qquad \text{...(3)}$$

where $\quad 3a+2b+c=0. \qquad$ ...(4)

The line (2) will also lie on the plane (3) if the point $(3,-4,1)$ lying on the line (2) also lies on the plane (3). Hence

$$a(3+1)+b(-4-2)+c\,.\,1=0$$

or $\quad 4a-6b+c=0 \qquad$ ...(5)

Solving (4) and (5), we get

$$\frac{a}{8}=\frac{b}{1}=\frac{c}{-26}\cdot$$

Putting these proportional values of a, b, c in (3) the required equation of the plane is

$$8(x+1)+1\,.\,(y-2)-26z=0$$

or $\quad 8x+y-26z+6=0.$

Example 11: *Show that the plane through the point* (α,β,γ) *and the line* $x=py+q=rz+s$ *is given by* $\begin{vmatrix} x & py+q & rz+s \\ \alpha & p\beta+q & r\gamma+s \\ 1 & 1 & 1 \end{vmatrix}=0.$

(Meerut 2001; Kumaun 11)

Solution: The equations of the given line are $x=py+q=rz+s$,

or in symmetrical form are

$$\frac{x-0}{1}=\frac{y+(q/p)}{1/p}=\frac{z+(s/r)}{1/r} \qquad \text{...(1)}$$

The equation of any plane through the line (1) is

$$a(x-0)+b(y+q/p)+c(z+s/r)=0 \qquad \text{...(2)}$$

where $\quad 1\,.\,a+(1/p)\,.\,b+(1/r)\,.\,c=0. \qquad$ ...(3)

The plane (2) will also pass through the point (α,β,γ) if

$$a\alpha+b(\beta+q/p)+c(\gamma+s/r)=0. \qquad \text{...(4)}$$

Eliminating a, b, c from the equations (2), (4) and (3), the equation of the required plane is

$$\begin{vmatrix} x & y+q/p & z+s/r \\ \alpha & \beta+q/p & \gamma+s/r \\ 1 & 1/p & 1/r \end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix} x & py+q & rz+s \\ \alpha & p\beta+q & r\gamma+s \\ 1 & 1 & 1 \end{vmatrix}=0.$$

[Multiplying the second and third columns by p and r respectively.]

Example 12: *Find the equation of the plane through the point* $(2,-1,1)$ *and the line*

$$4x-3y+5=0=y-2z-5.$$

Solution: The given equations of the line are

$$4x - 3y + 5 = 0, \quad y - 2z - 5 = 0.$$

The equation of any plane through the given line is

$$(4x - 3y + 5) + \lambda\,(y - 2z - 5) = 0 \qquad \ldots(1)$$

If the plane (1) passes through the point $(2, -1, 1)$, we have

$$4(2) - 3(-1) + 5 + \lambda(-1 - 2.1 - 5) = 0$$

or $\quad 16 - 8\lambda = 0 \quad$ or $\quad \lambda = 2$.

Putting $\lambda = 2$ in (1), the equation of the required plane is

$$4x - 3y + 5 + 2(y - 2z - 5) = 0 \qquad \text{or} \qquad 4x - y - 4z = 5.$$

Comprehensive Exercise 4

1. Find the equations of the planes through the line $\dfrac{x-2}{2} = \dfrac{y-3}{3} = \dfrac{z-4}{5}$ and parallel to the co-ordinate axes.

2. Prove that the equation of the plane through the line $\dfrac{x-1}{3} = \dfrac{y+6}{4} = \dfrac{z+1}{2}$ and parallel to $\dfrac{x-2}{2} = \dfrac{y-1}{-3} = \dfrac{z+4}{5}$ is $26x - 11y - 17z - 109 = 0$ and show that the point $(2, 1, -4)$ lies on it.

3. Find the equation of the plane determined by the parallel lines $\dfrac{x-4}{1} = \dfrac{y-3}{-4} = \dfrac{z-2}{5}$ and $\dfrac{x-3}{1} = \dfrac{y+2}{-4} = \dfrac{z}{5}$.

4. Find the equation of the plane which contains the line $x = (y-3)/2 = (z-5)/3$ and which is perpendicular to the plane $2x + 7y - 3z = 1$.

5. Show that the equation to the plane containing the line $\dfrac{x+1}{-3} = \dfrac{y-3}{2} = \dfrac{z+2}{1}$ and the point $(0, 7, -7)$ is $x + y + z = 0$. Hence show that the line $\dfrac{x}{1} = \dfrac{y-7}{2} = \dfrac{z+7}{-3}$ also lies in the same plane. **(Garhwal 2001)**

6. Find the equation of the plane containing the line $\dfrac{y}{b} + \dfrac{z}{c} = 1, x = 0$ and parallel to the line $\dfrac{x}{a} - \dfrac{z}{c} = 1, y = 0$.

7. The plane $lx + my = 0$ is rotated about its line of intersection with the plane $z = 0$, through an angle α. Prove that the equation of the plane in its new position is $lx + my \pm z\sqrt{(l^2 + m^2)}\tan\alpha = 0$. **(Meerut 2000, 06B)**

Answers 4

1. $5y - 3z - 3 = 0;\ 5x - 2z - 2 = 0;\ 3x - 2y = 0$

3. $11x - y - 3z - 35 = 0$ **4.** $9x - 3y - z + 14 = 0$

6. $\frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0$

---

7 Foot and Length of Perpendicular from a Point to a Line

(A) Line in symmetrical form:

To find the perpendicular distance of a point $P\ (x_1, y_1, z_1)$, *from a given line when its equations are given in the symmetrical form.*

Let the equations of the given line in symmetrical form be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{ (say)}. \qquad \ldots(1)$$

The co-ordinates of any point N on the line (1) are

$$(\alpha + lr, \beta + mr, \gamma + nr). \qquad \ldots(2)$$

If N be the foot of the perpendicular from $P\ (x_1, y_1, z_1)$ to (1), then the line PN is perpendicular to (1). The direction ratios of the line PN are

$$\alpha + lr - x_1,\ \beta + mr - y_1,\ \gamma + nr - z_1\ . \qquad \ldots(3)$$

As the line (1) is perpendicular to PN, using the condition

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0, \text{we get}$$

$$l\,(\alpha + lr - x_1) + m\,(\beta + mr - y_1) + n\,(\gamma + nr - z_1) = 0$$

or $$r\,(l^2 + m^2 + n^2) = l\,(x_1 - \alpha) + m\,(y_1 - \beta) + n\,(z_1 - \gamma)$$

or $$r = \frac{l\,(x_1 - \alpha) + m\,(y_1 - \beta) + n\,(z_1 - \gamma)}{l^2 + m^2 + n^2}\ . \qquad \ldots(4)$$

Substituting this value of r from (4) in (2), we can determine the co-ordinates of N, the foot of perpendicular, and then PN can be easily calculated.

Corollary: Equations of the perpendicular line. The equations of the perpendicular from the point $P\ (x_1, y_1, z_1)$ to the line (1) are given by

$$\frac{x - x_1}{\alpha + lr - x_1} = \frac{y - y_1}{\beta + mr - y_1} = \frac{z - z_1}{\gamma + nr - z_1}\ .$$

(B) Line in general form:

To find the equations of the perpendicular line from the point $P(x_1, y_1, z_1)$ *to a line whose equations are given by*

$$ax + by + cz + d = 0 = a'x + b'y + c'z + d'.$$

The equations of the given line in general form are

$$ax + by + cz + d = 0, \quad a'x + b'y + c'z + d' = 0 \qquad \ldots(1)$$

The perpendicular from a given point P to a given line is the intersection of the two planes namely (i) the plane through the given point $P(x_1, y_1, z_1)$ and also through the given line ; and (ii) the plane through the point P perpendicular to the given line. Now the equation of any plane through the line (1) is given by

$$(ax + by + cz + d) + \lambda (a'x + b'y + c'z + d') = 0 \qquad \ldots(2)$$

If this plane (2) passes through $P(x_1, y_1, z_1)$, then

$$(ax_1 + by_1 + cz_1 + d) + \lambda (a'x_1 + b'y_1 + c'z_1 + d') = 0$$

or $$\lambda = -\frac{ax_1 + by_1 + cz_1 + d}{a'x_1 + b'y_1 + c'z_1 + d'}.$$

Substituting this value of λ in (2), the equation of the plane through the point P and the line (1) is given by

$$\frac{ax + by + cz + d}{ax_1 + by_1 + cz_1 + d} = \frac{a'x + b'y + c'z + d'}{a'x_1 + b'y_1 + c'z_1 + d'}. \qquad \ldots(3)$$

Also if l, m, n be the d.c.'s of the given line (1), then we have

$$al + bm + cn = 0 \quad \text{and} \quad a'l + b'm + c'n = 0.$$

Solving these, we get

$$\frac{l}{bc' - b'c} = \frac{m}{ca' - c'a} = \frac{n}{ab' - a'b}. \qquad \ldots(4)$$

Now we are to find the equation of the second plane which passes through P and is perpendicular to the line (1).

Since the plane is perpendicular to the line (1), therefore d.r.'s of its normal are proportional to l, m, n given by (4).

$\therefore$ the equation of the plane perpendicular to the line (1) and passing through $P(x_1, y_1, z_1)$ is

$$l(x - x_1) + m(y - y_1) + n(z - z_1) = 0 \qquad \ldots(5)$$

Therefore the equations of the perpendicular line from the point $P(x_1, y_1, z_1)$ to the line (1) are given by the equations (4) and (5).

Illustrative Examples

Example 13: *Find the equations of the perpendicular from the point* (3, – 1, 11) *to the line* $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$. *Find also the co-ordinates of the foot of the perpendicular. Hence find the length of the perpendicular.* **(Kanpur 2006; Kumaun 12)**

Solution: The given point is $P\,(3, -1, 11)$ and the equations of the given line are

$$\frac{x-0}{2}=\frac{y-2}{3}=\frac{z-3}{4}=r \text{ (say)}. \qquad \ldots(1)$$

The co-ordinates of any point N on the line (1) are

$$(2r, 3r+2, 4r+3). \qquad \ldots(2)$$

Let this point N be the foot of the perpendicular from the point $P\,(3, -1, 11)$ to the line (1). Then the d.r.'s of the perpendicular PN are

$$2r-3, (3r+2)-(-1), (4r+3)-11$$

or $$2r-3, 3r+3, 4r-8. \qquad \ldots(3)$$

The d.r.'s of the given line (1) are 2, 3, 4.

Now PN is perpendicular to the line (1). The condition of perpendicularity gives

$$(2r-3).2+(3r+3).3+(4r-8).4=0,$$

or $$29r-29=0, \quad \text{or} \quad r=1.$$

Putting the value of r in (2), the foot N of the perpendicular is the point $(2, 5, 7)$.

Putting the value of r in (3), the d.r.'s of PN are $-1, 6, -4$.

Hence the equations of the perpendicular PN from the point $P\,(3, -1, 11)$ to the line (1) are $\dfrac{x-3}{-1}=\dfrac{y+1}{6}=\dfrac{z-11}{-4}$.

The length of the perpendicular PN

$$= \text{the distance between the points } P\,(3, -1, 11) \text{ and } N\,(2, 5, 7)$$

$$= \sqrt{(3-2)^2+(-1-5)^2+(11-7)^2}$$

$$= \sqrt{1+36+16}=\sqrt{53}.$$

Example 14: *Find the distance of the point* $P\,(3, 8, 2)$ *from the line* $\dfrac{x-1}{2}=\dfrac{y-3}{4}=\dfrac{z-2}{3}$ *measured parallel to the plane* $3x+2y-2z+17=0$.

Solution: The equations of the given line are

$$\frac{x-1}{2}=\frac{y-3}{4}=\frac{z-2}{3}=r \text{ (say)}. \qquad \ldots(1)$$

Any point Q on the line (1) is $(2r+1, 4r+3, 3r+2)$.

Now P is the point $(3, 8, 2)$ and hence d.r.'s of PQ are

$$(2r+1)-3, (4r+3)-8, (3r+2)-2 \quad \textit{i.e.,} \quad 2r-2, 4r-5, 3r.$$

It is required to find the distance PQ measured parallel to the plane

$$3x+2y-2z+17=0. \qquad \ldots(2)$$

Now PQ is parallel to the plane (2) and therefore PQ will be perpendicular to the normal to the plane (2). Hence, we have

$$(2r-2)(3)+(4r-5)(2)+(3r)(-2)=0$$

or $\qquad 8r - 16 = 0, \quad$ or $\quad r = 2$.

Putting the value of r, the point Q is $(5, 11, 8)$.

$\therefore$ Required distance = The distance between $P\ (3, 8, 2)$ and $Q\ (5, 11, 8)$

$$= \sqrt{(3-5)^2 + (8-11)^2 + (2-8)^2} = \sqrt{4+9+36} = 7.$$

8 Projection of a Line on a given Plane

Definition 1: *The projection of a line on a given plane is the line of intersection of the two planes namely (i) the given plane and (ii) the plane through the given line and perpendicular to the given plane.*

Definition 2: *If P be the point of intersection of the given line with the given plane and Q be the foot of the perpendicular from any point on the line to the plane, then the line PQ is called the projection of the given line on the given plane.*

Illustrative Examples

Example 15: *Find the equations of the projection of the line* $\dfrac{x-1}{2} = \dfrac{y+1}{-1} = \dfrac{z-3}{4}$ *on the plane* $x + 2y + z = 6$. **(Rohilkhand 2012)**

Solution: The equations of the given line are

$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4} \qquad \ldots(1)$$

and the given plane is $\quad x + 2y + z = 6. \qquad \ldots(2)$

The equation of any plane through the given line (1) is

$$a\,(x-1) + b\,(y+1) + c\,(z-3) = 0 \qquad \ldots(3)$$

where $\qquad 2a - b + 4c = 0 \qquad \ldots(4)$

The plane (3) will be perpendicular to the plane (2), if $a + 2b + c = 0 \qquad \ldots(5)$

Solving (4) and (5), we get $\dfrac{a}{-9} = \dfrac{b}{2} = \dfrac{c}{5}$.

Putting these proportionate values of a, b, c in (3), we have

$$-9\,(x-1) + 2\,(y+1) + 5\,(z-3) = 0$$

or $\qquad 9x - 2y - 5z + 4 = 0 \qquad \ldots(6)$

The equations (2) and (6) together are the equations of the line of projection.

Alternate Method: (Use of definition 2)

The given line is $\dfrac{x-1}{2} = \dfrac{y+1}{-1} = \dfrac{z-3}{4} = r$ (say). $\qquad \ldots(1)$

Any point P on this line (1) is $(2r+1, -r-1, 4r+3)$.

If P lies on the given plane $x + 2y + z = 6$, then

$$2r + 1 + 2\,(-r-1) + 4r + 3 = 6, \quad \text{or} \quad r = 1.$$

Putting this value of r, the point of intersection P of the line (1) and the given plane is $(3, -2, 7)$.

Now evidently the line (1) passes through the point $(1, -1, 3)$ and hence this is a point on the given line (1). We are to find the foot of the perpendicular Q from $(1, -1, 3)$ to the given plane.

The d.r.'s of the normal to the given plane are $1, 2, 1$ and hence these are d.r.'s of the line through $(1, -1, 3)$ and perpendicular to the given plane, and therefore the equations of this perpendicular line are $\dfrac{x-1}{1} = \dfrac{y+1}{2} = \dfrac{z-3}{1} = r_1$ (say).

Any point on it is $(r_1 + 1, 2r_1 - 1, r_1 + 3)$.

If this be the point Q, then it will lie on the given plane $x + 2y + z = 6$. So we have

$$r_1 + 1 + 2(2r_1 - 1) + r_1 + 3 = 6 \quad \text{or} \quad r_1 = 2/3.$$

Putting this value of r_1, the foot of the perpendicular Q is $(5/3, 1/3, 11/3)$.

$\therefore$ the required equations of the projection *i.e.*, the equations of the line PQ joining the points $P(3, -2, 7)$ and $Q\left(\dfrac{5}{3}, \dfrac{1}{3}, \dfrac{11}{3}\right)$ are

$$\frac{x-3}{3-\frac{5}{3}} = \frac{y+2}{-2-\frac{1}{3}} = \frac{z-7}{7-\frac{11}{3}} \quad \text{or} \quad \frac{x-3}{4} = \frac{y+2}{-7} = \frac{z-7}{10}.$$

Comprehensive Exercise 5

1. Find the equations of the perpendicular from the point (1, 6, 3) to the line $\dfrac{x}{1} = \dfrac{y-1}{2} = \dfrac{z-2}{3}$. Find also the co-ordinates of the foot of the perpendicular.

(Kanpur 2011)

2. Find the equations of the perpendicular from the origin to the line $ax + by + cz + d = 0 = a'x + b'y + c'z + d' = 0$.
3. The equations to AB referred to rectangular axes are $\dfrac{x}{2} = \dfrac{y}{-3} = \dfrac{z}{6}$.

   Through a point $P(1, 2, 5)$, PN is drawn perpendicular to AB and PQ is drawn parallel to the plane $3x + 4y + 5z = 0$ to meet AB in Q. Find the equations to PN and PQ and the co-ordinates of N and Q.

Answers 5

1. $x - 1 = 0,\ 2y + 3z = 21;\ (1, 3, 5)$
2. $(ad' - a'd)\,x + (bd' - b'd)\,y + (cd' - c'd)\,z = 0$ and

   $(bc' - b'dc)\,x + (ca' - c'a)\,y + (ab' - a'b)\,z = 0$

3. $\frac{x-1}{3}=\frac{y-2}{-176}=\frac{z-5}{-89};\frac{x-1}{4}=\frac{y-2}{-13}=\frac{z-5}{-8};$

$N\left(\frac{52}{49},\frac{-78}{49},\frac{156}{49}\right)$; $Q(3,-9/2,9)$

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9 Coplanar Lines

To find the condition that two given lines may intersect and to obtain the equation of the plane containing them. **(Kumaun 2000)**

Let the equations to the given lines be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(1)$$

and $$\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}. \qquad \ldots(2)$$

If the lines intersect, they lie in a plane. Any plane through the line (1) is

$$a(x-\alpha)+b(y-\beta)+c(z-\gamma)=0 \qquad \ldots(3)$$

where $al+bm+cn=0.$ ...(4)

If the plane (3) contains the line (2), then

$$a(\alpha'-\alpha)+b(\beta'-\beta)+c(\gamma'-\gamma)=0 \qquad \ldots(5)$$

and $al'+bm'+cn'=0$ ...(6)

Eliminating *a*, *b* and *c* between (4), (5) and (6), the condition for lines (1) and (2) to be coplanar is given by $\begin{vmatrix}\alpha'-\alpha & \beta'-\beta & \gamma'-\gamma\\ l & m & n\\ l' & m' & n'\end{vmatrix}=0.$

Also eliminating *a*, *b* and *c* between (3), (4) and (6), we get the equation to the plane containing the given lines as $\begin{vmatrix}x-\alpha & y-\beta & z-\gamma\\ \lambda & m & n\\ l' & m' & n'\end{vmatrix}=0.$

Another Method: Any point on the line (1) is $(\alpha+lr,\beta+mr,\gamma+nr)$ and any point on the line (2) is $(\alpha'+l'r',\beta'+m'r',\gamma'+n'r')$.

If the lines are coplanar *i.e.*, they intersect, then they must have a common point.

Therefore $\alpha+lr=\alpha'+l'r'$ or $\alpha-\alpha'+lr-l'r'=0$

$\beta-\beta'+mr-m'r'=0,$ and $\gamma-\gamma'+nr-n'r'=0.$

Eliminating r and r', we have

$$\begin{vmatrix}\alpha-\alpha' & l & l'\\ \beta-\beta' & m & m'\\ \gamma-\gamma' & n & n'\end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix}\alpha'-\alpha & \beta'-\beta & \gamma'-\gamma\\ l & m & n\\ l' & m' & n'\end{vmatrix}=0.$$

10 Condition for the Two Lines to Intersect

To find the condition that two given lines **(one in symmetrical form and the other in general form)** *may intersect and to obtain the equation of the plane containing them.*

Let the equations of the given lines be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(1)$$

and $$a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2 \qquad \ldots(2)$$

The equation of any plane through the line (2) is

$$(a_1x+b_1y+c_1z+d_1)+\lambda\,(a_2x+b_2y+c_2z+d_2)=0 \qquad \ldots(3)$$

or $$(a_1+\lambda a_2)\,x+(b_1+\lambda b_2)\,y+(c_1+\lambda c_2)\,z+(d_1+\lambda d_2)=0.$$

If this plane is parallel to the line (1), then we have

$$l\,(a_1+\lambda a_2)+m\,(b_1+\lambda b_2)+n\,(c_1+\lambda c_2)=0$$

or $$\lambda\,(a_2l+b_2m+c_2n)=-\,(a_1l+b_1m+c_1n)$$

or $$\lambda=-\frac{a_1l+b_1m+c_1n}{a_2l+b_2m+c_2n}. \qquad \ldots(4)$$

Putting this value of λ in (3) the equation of the plane through the line (2) and parallel to the line (1) is given by

$$\frac{a_1x+b_1y+c_1z+d_1}{a_1l+b_1m+c_1n}=\frac{a_2x+b_2y+c_2z+d_2}{a_2l+b_2m+c_2n}. \qquad \ldots(5)$$

If the line (1) lies in this plane, then the point (α,β,γ) on the line (1) must satisfy (5) and so the condition for the lines (1) and (2) to be coplanar is

$$\frac{a_1\alpha+b_1\beta+c_1\gamma+d_1}{a_1l+b_1m+c_1n}=\frac{a_2\alpha+b_2\beta+c_2\gamma+d_2}{a_2l+b_2m+c_2n}. \qquad \ldots(6)$$

If the condition (6) is satisfied, the lines (1) and (2) are intersecting (or coplanar) and the plane containing both the lines is given by the equation (5).

11 The Condition that Two Lines whose Equations are given in General Form may Intersect

To find the condition that two lines whose equations are given in **general form** *may intersect.*

Let the equations of the two lines be

$$a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2 \qquad \ldots(1)$$

and $$a_3x+b_3y+c_3z+d_3=0=a_4x+b_4y+c_4z+d_4 \qquad \ldots(2)$$

If these two lines are coplanar, then they intersect and let (α,β,γ) be their point of intersection.

The co-ordinates of this point must satisfy the equations of the four planes representing the two lines.

∴ we have

$$a_1\alpha + b_1\beta + c_1\gamma + d_1 = 0,\ \ a_2\alpha + b_2\beta + c_2\gamma + d_2 = 0,$$
$$a_3\alpha + b_3\beta + c_3\gamma + d_3 = 0 \quad \text{and} \quad a_4\alpha + b_4\beta + c_4\gamma + d_4 = 0.$$

Eliminating α, β, γ from these we have the required condition as

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0.$$

Note: In numerical examples it is convenient to solve after transforming the given equations (general form) into the symmetrical form.

Illustrative Examples

Example 16: *Show that the lines* $\dfrac{x+1}{-3} = \dfrac{y-3}{2} = \dfrac{z+2}{1}$ *and* $\dfrac{x}{1} = \dfrac{y-7}{-3} = \dfrac{z+7}{2}$ *intersect.Find the co-ordinates of the point of intersection and the equation to the plane containing them.* **(Kanpur 2009; Kumaun 09, 13; Rohilkhand 09, 11)**

Solution: Any point on the line $\dfrac{x+1}{-3} = \dfrac{y-3}{2} = \dfrac{z+2}{1} = r_1$ (say)

is $$(-1-3r_1\ , 3+2\,r_1\ , -2+r_1) \qquad \text{...(1)}$$

and any point on the line $\dfrac{x}{1} = \dfrac{y-7}{-3} = \dfrac{z+7}{2} = r_2$ (say)

is $$(r_2\ , 7-3r_2\ , -7+2\,r_2) \qquad \text{...(2)}$$

If the two given lines intersect (*i.e.*, are coplanar) then for some values of r_1 and r_2 the above two points (1) and (2) must coincide *i.e.*,

$$-3r_1 - 1 = r_2\ , \quad \text{or} \quad 3r_1 + r_2 = -1 \qquad \text{...(3)}$$

$$2\,r_1 + 3 = -3r_2 + 7 \quad \text{or} \quad 2\,r_1 + 3r_2 = 4 \qquad \text{...(4)}$$

and $$r_1 - 2 = 2\,r_2 - 7 \quad \text{or} \quad r_1 - 2\,r_2 = -5 \qquad \text{...(5)}$$

Solving (3) and (4), we get

$$r_1 = -1, r_2 = 2\ .$$

These values of r_1 and r_2 also satisfy the third equation (5).

Hence the given lines intersect.

Substituting these values of r_1 and r_2 in (1) or (2), we get the required co-ordinates of the point of intersection as $(2\ , 1, -3)$.

Also the equation of the plane containing the given lines is

$$\begin{vmatrix} x+1 & y-3 & z+2 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix} = 0$$

or $\quad (x+1)(4+3) - (y-3)(-6-1) + (z+2)(9-2) = 0$

or $\quad x + y + z = 0.$

Example 17: *Prove that the lines*

$\dfrac{x}{\alpha} = \dfrac{y}{\beta} = \dfrac{z}{\gamma}, \ \dfrac{x}{a\alpha} = \dfrac{y}{b\beta} = \dfrac{z}{c\gamma}, \ \dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$ *will lie in one plane if*

$$\frac{l}{\alpha}(b-c) + \frac{m}{\beta}(c-a) + \frac{n}{\gamma}(a-b) = 0.$$

(Kanpur 2006; Rohilkhand 12; Kumaun 12)

Solution: We observe that all the three given lines pass through the origin O and hence they will be co-planar if they are perpendicular to a line through the origin O. Let d.c.'s of this line through the origin O be l_1, m_1, n_1.

Hence if this line is perpendicular to the given lines, we have

$$l_1\alpha + m_1\beta + n_1\gamma = 0 \qquad \ldots(1)$$

$$l_1 l + m_1 m + n_1 n = 0 \qquad \ldots(2)$$

$$l_1 a\alpha + m_1 b\beta + n_1 c\gamma = 0 \qquad \ldots(3)$$

Eliminating l_1, m_1, n_1 from (1), (2) and (3) we have the required condition as

$$\begin{vmatrix} \alpha & \beta & \gamma \\ l & m & n \\ a\alpha & b\beta & c\gamma \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 1 & 1 & 1 \\ l/\alpha & m/\beta & n/\gamma \\ a & b & c \end{vmatrix} = 0$$

or $\quad -(l/\alpha)(c-b) + (m/\beta)(c-a) - (n/\gamma)(b-a) = 0,$

expanding the determinant with respect to the second row

or $\quad \dfrac{l}{\alpha}(b-c) + \dfrac{m}{\beta}(c-a) + \dfrac{n}{\gamma}(a-b) = 0.$

Example 18: *Show that the lines* $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$ *and*

$$4x - 3y + 1 = 0 = 5x - 3z + 2$$

are coplanar. Also find their point of intersection.

(Bundelkhand 2014; Rohilkhand 10)

Solution: The equations of the given lines are

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = r \text{ (say)} \qquad \ldots(1)$$

and $\quad 4x - 3y + 1 = 0, \ 5x - 3z + 2 = 0. \qquad \ldots(2)$

The coordinates of any point on the line (1) are

$$(2r + 1,\ 3r + 2\ ,\ 4r + 3). \qquad \ldots(3)$$

The lines (1) and (2) will intersect (*i.e.*, will be coplanar) if the point (3) also lies on the line (2).

This point satisfies both the equations of the line (2), if we have

$$4\,(2r + 1) - 3\,(3r + 2) + 1 = 0 \quad \text{or} \quad r = -1$$

and $$5\,(2r + 1) - 3\,(4r + 3) + 2 = 0 \quad \text{or} \quad r = -1.$$

As both the values of r are the same, the two given lines intersect.

Putting this value of r in (3), the point of intersection is $(-1, -1, -1)$.

Comprehensive Exercise 6

1. (i) Show that the lines $(x + 3)/2 = (y + 5)/3 = -(z - 7)/3$ and $(x + 1)/4 = (y + 1)/5 = -(z + 1)$ are coplanar. Find the equation of the plane containing them. **(Purvanchal 2007)**

(ii) Show that the lines $7x - 4y + 7z + 16 = 0 = 4x + 3y - 2z + 3$ and $x - 3y + 4z + 6 = 0 = x - y + z + 1$ are coplanar.

2. Prove that the lines $(x - 1)/2 = (y - 2)/3 = (z - 3)/4$ and $(x - 2)/3 = (y - 3)/4 = (z - 4)/5$ are coplanar; find their point of intersection. Also find the equation of the plane in which they lie.

(Meerut 2003; Rohilkhand 08; Kanpur 10; Kashi 13; Kumaun 15)

3. (i) Show that the two given lines are coplanar :

$x = \frac{1}{2}(y - 2) = \frac{1}{3}(z + 3)\,;\ \frac{1}{2}(x - 2) = \frac{1}{3}(y - 6) = \frac{1}{4}(z - 3).$ Also find the point of intersection and the equation of the plane in which they lie.

(ii) Show that the lines $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3}$ and $\frac{x-4}{2} = \frac{y-6}{3} = \frac{z-8}{4}$ are coplanar. **(Kumaun 2009, 14)**

4. Prove that the lines

$$\frac{x-a}{a'} = \frac{y-b}{b'} = \frac{z-c}{c'} \quad \text{and} \quad \frac{x-a'}{a} = \frac{y-b'}{b} = \frac{z-c'}{c}$$

intersect and find the co-ordinates of the point of intersection and the equation of the plane in which they lie. **(Meerut 2000)**

5. Prove that the lines $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ and $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$ intersect, and find the co-ordinates of their point of intersection.

(Kanpur 2007)

6. Prove that the lines $\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1}$; $\frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2}$

and $\frac{x}{l_3} = \frac{y}{m_3} = \frac{z}{n_3}$ will be co-planar if $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$

7. Prove that the lines $x = ay + b = cz + d$ and $x = \alpha y + \beta = \gamma z + \delta$ are coplanar if $(a\beta - b\alpha)(\gamma - c) - (c\delta - d\gamma)(\alpha - a) = 0$.

8. Show that the following pairs of lines are coplanar :

(i) $\frac{1}{3}(x + 4) = \frac{1}{5}(y + 6) = -\frac{1}{2}(z - 1)$ and

$3x - 2y + z + 5 = 0 = 2x + 3y + 4z - 4$. Also find their point of intersection and the equation of the plane in which they lie.

(ii) $x - 4 = -(y + 1)/2 = z$ and $4x - y + 5z - 7 = 0 = 2x - 5y - z - 3$. Also find the equation of the plane containing them. **(Kumaun 2008)**

(iii) $(x - 3)/3 = -(y - 2)/4 = z + 1$ and $x + 2y + 3z = 0 = 2x + 4y + 3z + 3$. Also find the point of intersection.

9. Find the equation of the plane through the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

and perpendicular to the plane containing the lines

$$\frac{x}{m} = \frac{y}{n} = \frac{z}{l} \quad \text{and} \quad \frac{x}{n} = \frac{y}{l} = \frac{z}{m}.$$

10. Find the foot and hence the length of the perpendicular from the point $(5, 7, 3)$ to the line $\frac{x - 15}{3} = \frac{y - 29}{8} = \frac{z - 5}{-5}$. Find the equations of the perpendicular. Also find the equation of the plane in which the perpendicular and the given straight line lie. **(Purvanchal 2011)**

Answers 6

1. $6x - 5y - z = 0$

2. $(-1, -1, -1)$; $x - 2y + z = 0$

3. $x - 2y + z + 7 = 0$; $(2, 6, 3)$

4. $(a + a', b + b', c + c')$; $\begin{vmatrix} x & y & z \\ a & b & c \\ a' & b' & c' \end{vmatrix} = 0$

5. $(5, -7, 6)$

8. (i) $45(-4) - 17(-6) + 25(1) + 53 = 0$; $(2, 4, -3)$

(ii) $x + 2y + 3z = 2$

(iii) $(9, -6, 1)$

9. $(m-n)\,x+(n-l)\,y+(l-m)\,z=0$

10. Foot N (9, 13, 15); Length of the perpendicular from (5, 7, 3) is 14.

Equation of the perpendicular are $\frac{x-5}{2}=\frac{y-7}{3}=\frac{z-3}{6}$;

Equation of the required plane $9x-4y-z-14=0$.

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12 Equations of a Straight Line Intersecting Two given Lines

Case I: The equations of the two lines are given in symmetrical form.

Let the given lines be

$$\frac{x-\alpha_1}{l_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1}=r_1 \text{ (say)}, \qquad \ldots(1)$$

and

$$\frac{x-\alpha_2}{l_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2}=r_2 \text{ (say)}. \qquad \ldots(2)$$

Any point on the line (1) is $P\,(l_1r_1+\alpha_1\,,m_1r_1+\beta_1\,,n_1r_1+\gamma_1)$

and any point on the line (2) is $Q\,(l_2r_2+\alpha_2\,,m_2r_2+\beta_2\,,n_2r_2+\gamma_2)$.

We are required to find the equations of a line which intersects the lines (1) and (2). Let the required line intersect the lines (1) and (2) in the points P and Q respectively.

The required line is one which joins the points P and Q. The values of r_1 and r_2 will be determined by some additional given conditions.

Case II: The equations of the two lines are given in general form.

Let the equations of the given lines be

$$u_1=0=v_1 \quad\text{and}\quad u_2=0=v_2\,.$$

Then the equations of the required line intersecting both the given lines are

$$u_1+\mu_1v_1=0 \quad\text{and}\quad u_2+\mu_2v_2=0,$$

where the values of μ_1 and μ_2 are determined by some additional given conditions.

Illustrative Examples

Example 19: *A line with direction cosines proportional to* $2, 7, -5$ *is drawn to intersect the lines* $\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1}$ *and* $\frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4}$.

Find the co-ordinates of the points of intersection and the length intercepted on it.

Solution: The given lines are

$$\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1}=r_1 \text{ (say)}, \qquad \ldots(1)$$

and $$\frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4}=r_2 \text{ (say)}. \qquad \ldots(2)$$

Any point P on (1) is $(3r_1+5, -r_1+7, r_1-2)$,

and any point Q on (2) is $(-3r_2-3, 2r_2+3, 4r_2+6)$.

The direction ratios of QP are

$$(3r_1+3r_2+8, -r_1-2r_2+4, r_1-4r_2-8). \qquad \ldots(3)$$

Let the line with d.r.'s $2, 7, -5$ meet the lines (1) and (2) in the points P and Q respectively. Then the d.r.'s $(2, 7, -5)$ will be proportional to the d.r.'s given by (3).

$$\therefore \qquad \frac{3r_1+3r_2+8}{2}=\frac{-r_1-2r_2+4}{7}=\frac{r_1-4r_2-8}{-5}. \qquad \ldots(4)$$

From the first two of (4), we get $7(3r_1+3r_2+8)=2(-r_1-2r_2+4)$

or $$23r_1+25r_2+48=0. \qquad \ldots(5)$$

And from the 1st and 3rd of (4), we get $2(r_1-4r_2-8)=-5(3r_1+3r_2+8)$

or $$17r_1+7r_2+24=0. \qquad \ldots(6)$$

Solving (5) and (6), we get $r_1=r_2=-1$.

Putting these values of r_1 and r_2 the co-ordinates of the points of intersection are $P(2, 8, -3)$ and $Q(0, 1, 2)$.

The required length intercepted by the lines (1) and (2) on the line with d.r.'s $2, 7, -5$

$$=PQ=\sqrt{(2-0)^2+(8-1)^2+(-3-2)^2}$$

$$=\sqrt{4+49+25}=\sqrt{78}.$$

Note: The equations of the line PQ are given by $\dfrac{x-2}{2}=\dfrac{y-8}{7}=\dfrac{z+3}{-5}$.

Example 20: *Find the equations of the straight line through the origin and cutting each of the lines* $\dfrac{x-x_1}{l_1}=\dfrac{y-y_1}{m_1}=\dfrac{z-z_1}{n_1}$ *and* $\dfrac{x-x_2}{l_2}=\dfrac{y-y_2}{m_2}=\dfrac{z-z_2}{n_2}$.

(Agra 2007)

Solution: Equation of any plane through the first line $\dfrac{x-x_1}{l_1}=\dfrac{y-y_1}{m_1}=\dfrac{z-z_1}{n_1}$

is $$a(x-x_1)+b(y-y_1)+c(z-z_1)=0, \qquad \ldots(1)$$

where $$al_1+bm_1+cn_1=0. \qquad \ldots(2)$$

If the plane (1) passes through the origin (0, 0, 0), then from (1)

$$ax_1+by_1+cz_1=0. \qquad \ldots(3)$$

Eliminating a, b, c from (1), (3) and (2), the equation of the plane through the origin and through the first line is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_1 & y_1 & z_1 \\ l_1 & m_1 & n_1 \end{vmatrix} = 0.$$

Adding the second row to the first row, we get

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$$

or $$(n_1 y_1 - m_1 z_1) x + (l_1 z_1 - n_1 x_1) y + (m_1 x_1 - l_1 y_1) z = 0. \qquad \ldots(4)$$

Similarly the plane through the origin and through the second line is

$$\begin{vmatrix} x & y & z \\ x_2 & y_2 & z_2 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

or $$(n_2 y_2 - m_2 z_2) x + (l_2 z_2 - n_2 x_2) y + (m_2 x_2 - l_2 y_2) z = 0. \qquad \ldots(5)$$

The planes (4) and (5) together give the required line.

Comprehensive Exercise 7

1. Find the equations to the planes through the point $(1, 0, -1)$ and the lines $4x - y - 13 = 0 = 3y - 4z - 1$ and $y - 2z + 2 = 0 = x - 5$ and show that the equations to the line through the given point which intersects the two given lines can be written as $x = y + 1 = z + 2$.
2. Find the equations to the straight line drawn from the origin to intersect the lines $2x + 5y + 3z - 4 = 0 = x - y - 5z - 6$ and $3x - y + 2z - 1 = 0 = x + 2y - z - 2$.
3. Find the equations to the line drawn parallel to $\frac{1}{4} x = y = z$, so as to meet the lines $5x - 6 = 4y + 3 = z$ and $2x - 4 = 3y + 5 = z$.
4. A line with direction cosines proportional to $2, 1, 2$ meets each of the lines given by the equations $x = y + a = z$, $x + a = 2y = 2z$. Find the co-ordinates of each of the points of intersection.
5. Find the equations to the line intersecting the lines
$$x - 1 = y = z - 1, 2x + 2 = 2y = z + 1$$
and parallel to the line $\frac{1}{2}(x - 1) = (y - 1) = \frac{1}{3}(z - 2)$.
6. Find the equations to the straight line drawn through the origin which will intersect both the lines
$$\frac{x - 1}{1} = \frac{y + 3}{4} = \frac{z - 5}{3} \quad \text{and} \quad \frac{x - 4}{2} = \frac{y + 3}{3} = \frac{z - 14}{4}.$$

Answers 7

1. $x + 2y - 3z - 4 = 0$ and $x + y - 2z - 3 = 0$
2. $4x + 17y + 19z = 0$ and $5x - 4y + 5z = 0$
3. $15x - 76y + 16z - 75 = 0$ and $4x - 21y + 5z - 43 = 0$
4. $(3a, 2a, 3a)$; (a, a, a)
5. $\frac{1}{2}(x - 1) = y = \frac{1}{3}(z - 1)$
6. $9x - 2y - 3z = 0$

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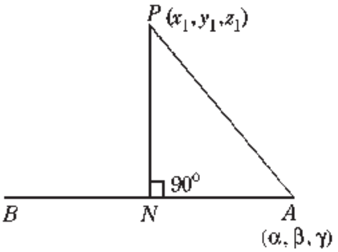
13 Perpendicular Distance Formula for the Line

To find the perpendicular distance of a point from a line and the co-ordinates of the foot of the perpendicular.



Let $P(x_1, y_1, z_1)$ be a given point and let AB be a given line. Let the equations of the line AB in symmetrical form be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}, \qquad \ldots(1)$$

where l, m, n are the d.c.'s of (1). The line (1) is passing through the point $A(\alpha, \beta, \gamma)$ and has direction cosines l, m, n. From P draw PN perpendicular to AB. Now it is required to find PN. From the right angled $\Delta\ APN$, we have

$$PN^2 = AP^2 - AN^2. \qquad \ldots(2)$$

Now AP = the distance between (α, β, γ) and $P(x_1, y_1, z_1)$

$$= \sqrt{(x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-\gamma)^2}, \qquad \ldots(3)$$

and AN = projection of AP on AB *i.e.*, the projection of AP on a line whose d.c.'s are l, m, n

$$= (x_1-\alpha)\,l + (y_1-\beta)\,m + (z_1-\gamma)\,n. \qquad \ldots(4)$$

Putting the values from (3) and (4) in (2), we get

$$PN^2 = \{(x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-\gamma)^2\} - \{(x_1-\alpha)\,l + (y_1-\beta)\,m + (z_1-\gamma)\,n\}^2$$

$$= \{(x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-\gamma)^2\}(l^2+m^2+n^2) - \{(x_1-\alpha)\,l + (y_1-\beta)\,m + (z_1-\gamma)\,n\}^2$$

$$[\because\ l^2 + m^2 + n^2 = 1]$$

$$= \{m(z_1-\gamma)-n(y_1-\beta)\}^2+\{n(x_1-\alpha)-l(z_1-\gamma)\}^2$$
$$+\{l(y_1-\beta)-m(x_1-\alpha)\}^2$$

[By using Lagrange's identity]

$$=\begin{vmatrix} m & n \\ y_1-\alpha & z_1-\gamma \end{vmatrix}^2+\begin{vmatrix} n & l \\ z_1-\gamma & x_1-\alpha \end{vmatrix}^2+\begin{vmatrix} l & m \\ x_1-\alpha & y_1-\beta \end{vmatrix}^2. \quad \ldots(5)$$

Note: In the equations (1) of the line AB, l, m, n have been taken as the actual direction cosines of the line. In case direction ratios a, b, c of AB are given, we should either first find the direction cosines of AB or we should divide the R.H.S. of (5) by $(a^2+b^2+c^2)$.

To find the co-ordinates of the foot of the perpendicular N.

Since N, the foot of the perpendicular, is a point on the line AB given by (1), its co-ordinates may be written as

$$(lr+\alpha,\ mr+\beta,\ nr+\gamma). \quad \ldots(6)$$

The d.r.'s of PN are $lr+\alpha-x_1,\ mr+\beta-y_1,\ nr+\gamma-z_1$.

Also PN is perpendicular to AB.

$\therefore \quad (lr+\alpha-x_1).l+(mr+\beta-y_1).m+(nr+\gamma-z_1).n=0$

or $\quad r(l^2+m^2+n^2)=l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)$

or $\quad r=l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma) \qquad [\because l^2+m^2+n^2=1]$

Putting this value of r in (6) the co-ordinates of N are obtained.

Illustrative Examples

Example 21: *From the point $P(1,2,3)$, PN is drawn perpendicular to the straight line*

$$1/3\,(x-2)=1/4\,(y-3)=1/5\,(z-4).$$

Find the distance PN, the equations to PN and co-ordinates of N.

Solution: The equations of the given line AB (say) are

$$(x-2)/3=(y-3)/4=(z-4)/5=r \text{ (say)} \quad \ldots(1)$$

The line (1) is passing through the point $A(2,3,4)$. Since N, the foot of the perpendicular, is a point on the line (1) [*i.e.*, AB], the co-ordinates of N may be written as $(3r+2,\ 4r+3,\ 5r+4)$ ...(2)

$\therefore$ d.r.'s of PN are

$$3r+2-1,\ 4r+3-2,\ 5r+4-3 \text{ i.e., are } 3r+1,\ 4r+1,\ 5r+1. \quad \ldots(3)$$

The d.r.'s of the line AB whose equations are given by (1), are 3, 4, 5.

Since PN is perpendicular to AB, we have

$$3 \, . \, (3r + 1) + 4 \, . \, (4r + 1) + 5 \, . \, (5r + 1) = 0, \quad \text{or} \quad r = -6/25.$$

Putting the value of r in (2), we get $N \equiv (32/25, 51/25, 14/5)$.

$\therefore$ PN = the distance between the points P and N

$$= \sqrt{\left\{\left(\frac{32}{25} - 1\right)^2 + \left(\frac{51}{25} - 2\right)^2 + \left(\frac{14}{5} - 3\right)^2\right\}} = \frac{\sqrt{3}}{5}.$$

Putting the value of r in (3), the d.r.'s of PN are $7/25, 1/25, -5/25$ *i.e.*, are $7, 1, -5$.

$\therefore$ the equations to PN *i.e.*, of a line passing through P (1, 2, 3) and having d.r.'s 7, 1, –5 are $\dfrac{x-1}{7} = \dfrac{y-2}{1} = \dfrac{z-3}{-5}$.

Example 22: *Find the locus of a point which moves so that its distance from the line $x = y = -z$ is twice its distance from the plane $x - y + z = 1$.*

Solution: Let the given point be $P(x_1, y_1, z_1)$ whose locus is required to be found.

The equations of the given line AB are

$$x/1 = y/1 = z/(-1). \qquad \ldots(1)$$

The line (1) is clearly passing through A (0, 0, 0) and has d.r.'s 1, 1, – 1. Hence d.c.'s l, m, n of (1) are $1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}$.

Let p_1 be the perpendicular distance of P from the line (1), then by article 4.13, we have

$$p_1^2 = \begin{vmatrix} 1/\sqrt{3} & -1/\sqrt{3} \\ y_1 - 0 & z_1 - 0 \end{vmatrix}^2 + \begin{vmatrix} -1/\sqrt{3} & 1/\sqrt{3} \\ z_1 - 0 & x_1 - 0 \end{vmatrix}^2 + \begin{vmatrix} 1/\sqrt{3} & 1/\sqrt{3} \\ x_1 - 0 & y_1 - 0 \end{vmatrix}^2$$

or

$$p_1^2 = \frac{1}{3}\{(z_1 + y_1)^2 + (-x_1 - z_1)^2 + (y_1 - x_1)^2\}$$

$$= \frac{2}{3}(x_1^2 + y_1^2 + z_1^2 + y_1 z_1 + z_1 x_1 - x_1 y_1) \qquad \ldots(2)$$

Let p_2 be the perpendicular distance of $P(x_1, y_1, z_1)$ from the plane $x - y + z = 1$. Then

$$p_2 = \frac{x_1 - y_1 + z_1 - 1}{\sqrt{1+1+1}}. \qquad \ldots(3)$$

According to the condition given in the question, $p_1 = 2p_2$.

Squaring, we get $p_1^2 = 4p_2^2$

or $$\frac{2}{3}(x_1^2 + y_1^2 + z_1^2 + y_1 z_1 + z_1 x_1 - x_1 y_1) = \frac{4}{3}(x_1 - y_1 + z_1 - 1)^2$$

or $$x_1^2 + y_1^2 + z_1^2 + y_1 z_1 + z_1 x_1 - x_1 y_1 = 2(x_1^2 + y_1^2 + z_1^2 + 1 - 2x_1 y_1 + 2x_1 z_1 - 2x_1 - 2y_1 z_1 + 2y_1 - 2z_1)$$

or $$x_1^2 + y_1^2 + z_1^2 - 5y_1 z_1 + 3z_1 x_1 - 3x_1 y_1 - 4x_1 + 4y_1 - 4z_1 + 2 = 0.$$

$\therefore$ the required locus of $P(x_1, y_1, z_1)$ is given by

$$x^2 + y^2 + z^2 - 5yz + 3zx - 3xy - 4x + 4y - 4z + 2 = 0.$$

Comprehensive Exercise 8

1. Find the distance of $(-2, 1, 5)$ from the line through $(2, 3, 5)$ whose direction cosines are proportional to $2, -3, 6$.

2. Prove that the equations of the perpendicular from the point $(1, 6, 3)$ to the line $x = \frac{y-1}{2} = \frac{z-2}{3}$ *are* $\frac{x-1}{0} = \frac{y-6}{-3} = \frac{z-3}{2}$ and the co-ordinates of the foot of the perpendicular are $(1, 3, 5)$.
3. How far is the point $(4, 1, 1)$ from the line of intersection of $x + y + z - 4 = 0 = x - 2y - z - 4$?
4. Find the length of the perpendicular drawn from origin to the line $x + 2y + 3z + 4 = 0 = 2x + 3y + 4z + 5$. Also find the equations of this perpendicular and the co-ordinates of the foot of the perpendicular.

Answers 8

1. $\frac{4\sqrt{61}}{7}$ 3. $\frac{3\sqrt{(42)}}{14}$

4. $(2/3, -1/3, -4/3)$; $\frac{x}{2} = \frac{y}{-1} = \frac{z}{-4}$; $\frac{\sqrt{(21)}}{3}$

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14 Intersection of Three Planes

Let the equations of three planes be given by

$$u_1 \equiv a_1 x + b_1 y + c_1 z + d_1 = 0, \quad \ldots(1)$$

$$u_2 \equiv a_2 x + b_2 y + c_2 z + d_2 = 0, \quad \ldots(2)$$

and $$u_3 \equiv a_3 x + b_3 y + c_3 z + d_3 = 0, \quad \ldots(3)$$

where u_1, u_2 and u_3 denote respectively the left hand members in the equations (1), (2) and (3). No two of these three planes are parallel.

We know that two non-parallel planes intersect in a straight line and hence we get three lines of intersection by taking two planes at a time out of the three planes given by (1), (2) and (3). There arise the following three cases :

Case I: The three lines of intersection explained above may coincide *i.e.*, the three given planes have a common line of intersection.

Case II: The three lines of intersection explained above may be parallel to each other and no two of them coincide. In this case the three given planes form a triangular prism.

Case III: The three lines of intersection explained above may intersect in a common point. In this case the three planes intersect in a point.

Before proceeding to prove the actual theorem, for convenience, we make use of some **notations** given as follows :

Consider the matrix (or a rectangular array)

$$D = \begin{Vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{Vmatrix} \qquad \ldots(\text{i})$$

Let the determinant obtained by omitting the first column in (i) be denoted by Δ_1

i.e., we put $$\Delta_1 = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}.$$

Similarly, the determinants obtained by omitting the second, third and fourth columns will be denoted respectively by Δ_2 , Δ_3 and Δ_4. Thus we put

$$\Delta_2 = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}, \ \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}, \ \Delta_4 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The symmetrical form of the line of intersection of the planes (1) and (2) is

$$\frac{x - \left(\dfrac{b_1 d_2 - b_2 d_1}{a_1 b_2 - a_2 b_1}\right)}{b_1 c_2 - b_2 c_1} = \frac{y - \left(\dfrac{d_1 a_2 - d_2 a_1}{a_1 b_2 - a_2 b_1}\right)}{c_1 a_2 - c_2 a_1} = \frac{z - 0}{a_1 b_2 - a_2 b_1}, \qquad \ldots(4)$$

where $a_1 \, b_2 - a_2 \, b_1 \neq 0$.

Now we shall discuss the three cases given above in detail as follows :

Case I: The three planes intersect in a common line:

The equation of any plane through the line of intersection of the planes (1) and (2) is given by $u_1 + \lambda u_2 = 0$

or $$(a_1 x + b_1 y + c_1 z + d_1) + \lambda\,(a_2 x + b_2 \, y + c_2 z + d_2) = 0$$

or $$(a_1 + \lambda a_2)\, x + (b_1 + \lambda b_2)\, y + (c_1 + \lambda c_2)\, z + (d_1 + \lambda d_2) = 0. \quad \ldots(5)$$

If the three given planes intersect in a common line, then for some value of λ the plane (5) should represent the plane (3). Thus comparing the coefficients in the equations (5) and (3), we have

$$\frac{a_1 + \lambda a_2}{a_3} = \frac{b_1 + \lambda b_2}{b_3} = \frac{c_1 + \lambda c_2}{c_3} = \frac{d_1 + \lambda d_2}{d_3} = \mu \text{ (say)}.$$

$\therefore$ $$a_1 + \lambda \, a_2 - \mu a_3 = 0, b_1 + \lambda b_2 - \mu b_3 = 0, c_1 + \lambda c_2 - \mu c_3 = 0,$$

and $$d_1 + \lambda d_2 - \mu d_3 = 0.$$

Now we are to eliminate two arbitrary constants λ and μ and this can be done from any three out of the four equations given above. Hence eliminating λ and μ from any three equations taken at a time out of these four equations, we have the conditions as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ *i.e.*, $\Delta_4 = 0$, $\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = 0$ *i.e.*, $\Delta_3 = 0$

$$\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} = 0 \ \ i.e., \ \Delta_2 = 0, \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} = 0 \ \ i.e., \ \Delta_1 = 0.$$

Hence the three planes (1), (2) and (3) will have a common line of intersection if $\Delta_4 = 0,\ \Delta_3 = 0,\ \Delta_2 = 0$ and $\Delta_1 = 0$.

Case II: The three planes form a triangular prism:

The three planes will form a triangular prism if the line of intersection of any two planes is parallel to the third plane and does not lie in it.

The line of intersection of the planes (1) and (2) is given by (4). The line (4) will be parallel to the plane (3) if

$$a_3\ (b_1c_2 - b_2c_1) + b_3\ (c_1a_2 - c_2a_1) + c_3\ (a_1b_2 - a_2b_1) = 0$$

i.e.,
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \ \ i.e., \ \Delta_4 = 0.$$

The line (1) will not lie in the plane (3) if

$$a_3\left(\frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}\right) + c_3\ .0 + d_3 \neq 0$$

i.e.,
$$a_3\ (b_1d_2 - b_2d_1) + b_3\ (d_1a_2 - d_2a_1) + d_3\ (a_1b_2 - a_2b_1) \neq 0$$

i.e.,
$$\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \neq 0 \ \ i.e., \ \Delta_3 \neq 0.$$

Hence the three planes will form a triangular prism if

$$\mathbf{\Delta_4 = 0 \text{ and } \Delta_3 \neq 0 \text{ or } \Delta_2 \neq 0 \text{ or } \Delta_1 \neq 0.}$$

Case III: The three planes intersect in a point:

The three planes will intersect in a point if the line of intersection of the planes (1) and (2) given by (4), is neither parallel to nor lie in the plane (3). Rather than the line (4) must meet the plane (3) in a point.

Thus the condition that the three planes meet in a point is that $\Delta_4 \neq 0$.

Alternative method: Solving the equations (1), (2) and (3) by the method of determinants [This method is called Cramer's Rule], we have

$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

or $$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta_4} \text{ or } x = -\frac{\Delta_1}{\Delta_4}, \; y = \frac{\Delta_2}{\Delta_4}, \; z = -\frac{\Delta_3}{\Delta_4} \cdot \quad \ldots(6)$$

Hence the three planes will intersect in the point whose coordinates are given by (6) if $\Delta_4 \neq 0$.

Working Rule: Let the three planes be given by the equations (1), (2) and (3). Now proceed as follows :

(1) First evaluate Δ_4. If $\Delta_4 \neq 0$, then the three planes intersect in a point whose co-ordinates are given by the relations (6) above.

(2) If $\Delta_4 = 0$, then evaluate Δ_3, Δ_2 and Δ_1.

(i) If $\Delta_3 \neq 0$ (or $\Delta_1 \neq 0$ or $\Delta_2 \neq 0$), then the three planes form a triangular prism.

(ii) If $\Delta_3 = 0, \Delta_2 = 0$ and $\Delta_1 = 0$, then the three planes intersect in a common line.

Remark: If $\Delta_4 = 0$ and $\Delta_3 = 0$ and at least one of the three common minors $a_1 b_2 - a_2 b_1$, $a_2 b_3 - a_3 b_2$ and $a_1 b_3 - a_3 b_1$ of Δ_4 and Δ_3 is not zero, then it can be proved algebraically that $\Delta_2 = 0$ and $\Delta_1 = 0$. Consequently in this case the three planes will have a common line of intersection.

Illustrative Examples

Example 23: *Find the nature of the intersection of the sets of planes :*

(i) $x - 2y + 2z = 3, 2x + 3y - z = 5, 3x - 4y + 5z = 10$;

(ii) $2x + 4y + 2z = 7, 5x + y - z = 9, x - y - z = 6$;

(iii) $x + 2y + z = 0, 3x + y - 2z = 1, 3x - 4y - 7z = 2$.

Solution: (i) The equations of the given planes are

$$x - 2y + 2z - 3 = 0 \quad \ldots(1)$$

$$2x + 3y - z - 5 = 0 \quad \ldots(2)$$

$$3x - 4y + 5z - 10 = 0. \quad \ldots(3)$$

The rectangular array of coefficients is

$$D = \left\| \begin{matrix} 1 & -2 & 2 & -3 \\ 2 & 3 & -1 & -5 \\ 3 & -4 & 5 & -10 \end{matrix} \right\|. \quad \ldots(4)$$

Omitting the fourth column from (4), we have

$$\Delta_4 = \begin{vmatrix} 1 & -2 & 2 \\ 2 & 3 & -1 \\ 3 & -4 & 5 \end{vmatrix}.$$

Expanding this determinant along the first row, we have

$$\Delta_4 = 1\,(15-4) - (-2)\,[10-(-3)] + 2\,(-8-9)$$
$$= 1\,.\,11 + 2\,.\,13 + 2\,.\,(-17) = 3, \text{which is} \neq 0.$$

Hence the given planes intersect in a point.

(ii) The equations of the given planes are

$$2\,x + 4y + 2\,z - 7 = 0 \qquad \ldots(1)$$
$$5x + y - z - 9 = 0 \qquad \ldots(2)$$
$$x - y - z - 6 = 0. \qquad \ldots(3)$$

The rectangular array of coefficients is

$$D = \left\| \begin{matrix} 2 & 4 & 2 & -7 \\ 5 & 1 & -1 & -9 \\ 1 & -1 & -1 & -6 \end{matrix} \right\| \qquad \ldots(4)$$

Omitting the fourth column from (4), we have

$$\Delta_4 = \begin{vmatrix} 2 & 4 & 2 \\ 5 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix}.$$

Expanding by means of the first row, we have

$$\Delta_4 = 2\,(-1-1) - 4\,[-5-(-1)] + 2\,(-5-1) = -4 + 16 - 12 = 0.$$

Since $\Delta_4 = 0$, therefore the three planes either intersect in a line or form a triangular prism. Now omitting the third column from (4), we have

$$\Delta_3 = \begin{vmatrix} 2 & 4 & -7 \\ 5 & 1 & -9 \\ 1 & -1 & -6 \end{vmatrix}$$
$$= 2\,(-6-9) - 4\,[-30-(-9)] + (-7)\,(-5-1)$$
$$= -30 + 84 + 42 = 96, \text{ which is} \neq 0.$$

Hence the given three planes form a triangular prism, as no two of the three planes are parallel.

(iii) The equations of the given planes are

$$x + 2\,y + z = 0 \qquad \ldots(1)$$
$$3x + y - 2\,z - 1 = 0 \qquad \ldots(2)$$
$$3x - 4y - 7z - 2 = 0. \qquad \ldots(3)$$

The rectangular array of coefficients is

$$D = \left\| \begin{matrix} 1 & 2 & 1 & 0 \\ 3 & 1 & -2 & -1 \\ 3 & -4 & -7 & -2 \end{matrix} \right\| \qquad \ldots(4)$$

Omitting the fourth column from (4), we have

$$\Delta_4 = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 3 & -4 & -7 \end{vmatrix}.$$

Expanding this determinant along the first row, we have

$$\Delta_4 = 1(-7-8) - 2[-21-(-6)] + 1(-12-3)$$
$$= -15 + 30 - 15 = 0.$$

Since $\Delta_4 = 0$, therefore the three planes either intersect in a line or form a triangular prism.

Now omitting the third column from (4), we have

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 0 \\ 3 & 1 & -1 \\ 3 & -4 & -2 \end{vmatrix}$$

$$= 1(-2-4) - 2[-6-(-3)] + 0 = -6 + 6 = 0.$$

Similarly, we find that

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 0 \\ 3 & -2 & -1 \\ 3 & -7 & -2 \end{vmatrix} = 0 \quad \text{and} \quad \Delta_1 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & -2 & -1 \\ -4 & -7 & -2 \end{vmatrix} = 0.$$

Hence the given three planes intersect in a line.

Comprehensive Exercise 9

1. Examine the nature of the intersection of the sets of the planes :

(i) $x - y + z = 3, 2x + 5y + 3z = 0, 3x - 2y - 6z + 1 = 0$;

(ii) $3x + 2y + z = 6, 5x + 4y + 3z = 4, 3x + 4y + 5z + 12 = 0$;

(iii) $x - y + z - 4 = 0, 2x - y - z + 4 = 0, x + y - 5z + 14 = 0$;

(iv) $2x - 3y - z = -3, x + 2y + 3z = 2$ and $-x + 2y + z = 2$.

(Kumaun 2010, 15)

2. Examine the nature of the intersection of the sets of planes :

(i) $x + 2y - 5z = 1, 4x + y + z = 2, 6x + y + 3z = 3$;

(ii) $x + 4y + 6z = 5, 2x + 5y + 9z = 10, x + 3y + 5z = 5$;

(iii) $x - y + z = 2, 2x - 3y + 4z = 8, x + y + z = 2$;

(iv) $x + 3y - z = 6, x + 2y + 4z + 5 = 0, 2x + 6y - 2z + 7 = 0$.

3. Show that the planes
   $2x - 3y - 7z = 0, 3x - 14y - 13z = 0, 8x - 31y - 33z = 0$
   pass through one line and find its equations. **(Meerut 2011)**
4. Prove that the planes $x + ay + (b + c)z + d = 0$, $x + by + (c + a)z + d = 0$, $x + cy + (a + b)z + d = 0$, pass through one line. **(Rohilkhand 2008)**
5. Prove that the planes $x = y \sin \psi + z \sin \phi$, $y = z \sin \theta + x \sin \psi$, and $z = x \sin \phi + y \sin \theta$ will intersect in the line $\dfrac{x}{\cos \theta} = \dfrac{y}{\cos \phi} = \dfrac{z}{\cos \psi}$ if $\theta + \phi + \psi = \dfrac{1}{2}\pi$.

Answers 9

1. (i) The planes intersect at a point
   (ii) The planes have a common line of intersection
   (iii) The planes form a triangular prism
   (iv) The planes have a common line of intersection
2. (i) The planes form a triangular prism
   (ii) The planes have a common line of intersection
   (iii) The planes intersect at a point
   (iv) The plane (3) and (1) are parallel, and the plane (2) intersects them
3. $\dfrac{x}{-59} = \dfrac{y}{5} = \dfrac{z}{-19}$

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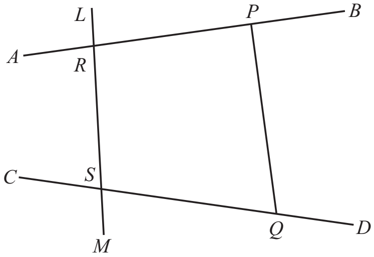
15 Shortest Distance between Two Lines

Skew lines: *Skew lines are those lines which do not intersect or the lines which do not lie in a plane.*

Shortest distance: *The length of the line intercepted between two lines which is perpendicular to both is the shortest distance between them.* The straight line which is perpendicular to each of the two skew lines is called the *line of shortest distance.* The shortest distance is sometimes abbreviated as **S.D.**

16 The Shortest Distance between any Two Non-intersecting Lines is Perpendicular to Both

To prove that the shortest distance between any two non-intersecting lines is perpendicular to both.



Let AB and CD be two non-intersecting lines and LM a line perpendicular to both of them. RS is the portion of LM intercepted between AB and CD. Now we have to prove that RS is the shortest distance between AB and CD.

Let P and Q be any points on AB and CD respectively. RS is the projection of PQ on LM. If θ is the angle between PQ and LM, then

$$RS = PQ \cos\theta \text{ or } \frac{RS}{PQ} = \cos\theta.$$

Since $\cos\theta < 1$, therefore $\dfrac{RS}{PQ} < 1$

or $\quad RS < PQ$ *i.e.*, RS is the shortest distance.

17 Length and Equations of the Line of Shortest Distance

To find the length and equations of the shortest distance between lines whose equations are given.

1st Method: Projection Method: The equations of the skew lines being given in symmetrical form. **(Bundelkhand 2006)**

Let the equations of the given lines be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad \ldots(1)$$

and

$$\frac{x-\alpha'}{l'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'} \qquad \ldots(2)$$

Let λ, μ, ν be the direction cosines of the S.D. Since S.D. is perpendicular to each of the given lines, therefore $l\lambda + m\mu + n\nu = 0$ and $l'\lambda + m'\mu + n'\nu = 0$.

$$\therefore \quad \frac{\lambda}{mn' - m'n} = \frac{\mu}{nl' - n'l} = \frac{\nu}{lm' - l'm}$$

$$= \frac{1}{\sqrt{(mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2}}.$$

$$\therefore \quad \lambda = \frac{mn' - m'n}{\sqrt{\{\Sigma (mn' - m'n)^2\}}}, \quad \mu = \frac{nl' - n'l}{\sqrt{\{\Sigma (mn' - m'n)^2\}}},$$

$$\nu = \frac{lm' - l'm}{\sqrt{\{\Sigma (mn' - m'n)^2\}}}.$$

If $P(\alpha, \beta, \gamma)$ is any point on the line (1) and $Q(\alpha', \beta', \gamma')$ is any point on the line (2), then the S.D. will be the projection of the line PQ joining these points on the line whose d.c.'s are λ, μ, ν.

$$\therefore \qquad \text{S.D.} = (\alpha - \alpha')\,\lambda + (\beta - \beta')\,\mu + (\gamma - \gamma')\,\nu$$

$$= \frac{(\alpha - \alpha')(mn' - m'n) + (\beta - \beta')(nl' - n'l) + (\gamma - \gamma')(lm' - l'm)}{\sqrt{\{\Sigma (mn' - m'n)^2\}}}$$

$$= \begin{vmatrix} \alpha - \alpha' & \beta - \beta' & \gamma - \gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix} \sqrt{\{\Sigma (mn' - m'n)^2\}}.$$

Equation of the plane containing the line (1) and the S.D. is

$$\begin{vmatrix} x - \alpha & y - \beta & z - \gamma \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix} = 0 \qquad \ldots(3)$$

Equation of the plane containing the line (2) and the S.D. is

$$\begin{vmatrix} x - \alpha' & y - \beta' & z - \gamma' \\ l' & m' & n' \\ \lambda & \mu & \nu \end{vmatrix} = 0 \qquad \ldots(4)$$

Equations (3) and (4) taken together will represent the equations of the line of shortest distance.

Note: If the lines are coplanar, the S.D. between them is zero.

Then
$$\begin{vmatrix} \alpha - \alpha' & \beta - \beta' & \gamma - \gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0.$$

Or **Two lines are coplanar if the shortest distance between them is zero.**

Another method ***(By vectors):***

Let the equations of the given lines be

$$\vec{r} = \vec{a} + t\,\vec{b} \quad \text{and} \quad \vec{r} = \vec{a}' + t\,\vec{b}',$$

where $\vec{a}$ and $\vec{a}'$ are the position vectors of the points $P(\alpha, \beta, \gamma)$ and $Q(\alpha', \beta', \gamma')$ and $\vec{b}, \vec{b}'$ are the unit vectors along AB and CD. (Refer fig. of article 16.)

Thus $\vec{b} = li + mj + nk$ and $\vec{b}' = l'i + m'j + n'k$ where (l, m, n) and (l', m', n') are the d.c.'s of AB and CD respectively.

RS, the S.D. between the lines, is perpendicular to both and hence parallel to $\vec{b} \times \vec{b}'$. Also it is the projection of PQ upon RS.

$\therefore$ $RS = PQ \cos\theta$, where θ is the angle between PQ and RS

$$= \frac{\overrightarrow{PQ} \cdot \overrightarrow{RS}}{|\overrightarrow{RS}|} = \frac{(\vec{a}' - \vec{a}) \cdot (\vec{b} \times \vec{b}')}{|(\vec{b} \times \vec{b}')|}$$

$$= \frac{\{(\alpha'-\alpha)\,i+(\beta'-\beta)\,j+(\gamma'-\gamma)\,k\}\bullet[(li+mj+nk)\times(l'i+m'j+n'k)]}{|(li+mj+nk)\times(l'i+m'j+n'k)|}$$

$$= \begin{vmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix} \div \sqrt{\{\Sigma\,(mn'-m'n)^2\}}.$$

Equation of the plane containing AB and RS is

$$[\vec{r}-\vec{a},\ \vec{b},\ \vec{b}\times\vec{b}\,'] = 0 \qquad \ldots(1)$$

[$\because$ it contains the vectors $\vec{r}-\vec{a}$, $\vec{b}$ and $\vec{b}\times\vec{b}\,'$]

Similarly the equation of the plane containing CD and RS is

$$[\vec{r}-\vec{a}\,',\ \vec{b}\,',\vec{b}\times\vec{b}\,'] = 0 \qquad \ldots(2)$$

The line of intersection of the planes (1) and (2) will be the required line of the S.D.

2nd Method: General co-ordinates: The equations of the two lines being given in symmetrical form :

Let the equations of the two lines be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{ (say)} \qquad \ldots(1)$$

and $$\frac{x-\alpha'}{l'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'} = r' \text{ (say)}. \qquad \ldots(2)$$

Any point on (1) is $P\,(lr+\alpha\,,\,mr+\beta,\,nr+\gamma)$

and any point on (2) is $Q\,(l'r'+\alpha',\,m'r'+\beta',\,n'r'+\gamma')$.

Let the line joining the points P and Q be the shortest distance. Then PQ is perpendicular to (1) and (2) both. The d.c.'s of this line PQ are proportional to

$$lr+\alpha-l'r'-\alpha',\ mr+\beta-m'r'-\beta',\ nr+\gamma-n'r'-\gamma'.$$

Since PQ is perpendicular to (1) and (2) both, therefore

$$l\,(lr+\alpha-l'r'-\alpha')+m\,(mr+\beta-m'r'-\beta') + n\,(nr+\gamma-n'r'-\gamma') = 0$$

and $$l'\,(lr+\alpha-l'r'-\alpha')+m'\,(mr+\beta-m'r'-\beta') + n'\,(nr+\gamma-n'r'-\gamma') = 0.$$

Solve these two equations to determine the values of r and r'. Then putting these values in the coordinates of the points P and Q, we get the two points at which the S.D. meets the two lines.

$\therefore$ S.D. = the distance between the points P and Q,

and equations of S.D. are the equations of the line joining the points P and Q.

Note: This method is useful when the coordinates of P and Q are also required.

3rd Method: (One line in general form and the other in symmetrical form):

This method is generally used when the equations of one line are given in general form while those of the other line in the symmetrical form.

Let the equations of one line be

$$u_1 = 0 = v_1 \qquad \ldots(1)$$

and the equations of the second line be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}. \qquad \ldots(2)$$

Equation of any plane through the line (1) is

$$u_1 + \lambda v_1 = 0. \qquad \ldots(3)$$

Find λ such that the plane (3) is parallel to the line (2) and then put this value of λ in (3).

Then the length of shortest distance is equal to the length of perpendicular from any point, say, (α, β, γ) on the line (2) to the plane through the line (1) and parallel to the line (2).

4th Method: (Both the lines in general form):

Let the equations of the lines be

$$u_1 = 0 = v_1 \;\ldots(1) \qquad \text{and} \qquad u_2 = 0 = v_2 \qquad \ldots(2)$$

Then planes containing respectively the lines (1) and (2) are

$$u_1 + \lambda v_1 = 0 \;\ldots(3) \qquad \text{and} \qquad u_2 + \mu v_2 = 0. \qquad \ldots(4)$$

Choose λ and μ such that the planes(3) and (4) are parallel.

Then the required shortest distance is the distance between these parallel planes.

Note: For convenience we generally reduce the given equations of the straight lines to symmetrical form and then use the method I or II as explained above.

Illustrative Examples

Example 24: *Find the shortest distance between the lines*

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}; \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}.$$

Show also that the equations of the shortest distance are

$$11x + 2y - 7z + 6 = 0 = 7x + y - 5z + 7.$$

(Rohilkhand 2008, 13; Kanpur 2005; Kumaun 11)

Solution: The given lines are

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = r_1 \text{ (say)} \qquad \ldots(1)$$

and

$$\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5} = r_2 \text{ (say)}. \qquad \ldots(2)$$

Method 1: (Projection method): Let l, m, n be the d.c.'s of the line of S.D. Since it is perpendicular to both the given lines (1) and (2), therefore we have

$$2l + 3m + 4n = 0\ ;\ 3l + 4m + 5n = 0.$$

Solving these, we get

$$\frac{l}{15-16} = \frac{m}{12-10} = \frac{n}{8-9}$$

or $$\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1} = \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{(-1)^2 + (2)^2 + (-1)^2}} = \frac{1}{\sqrt{6}}.$$

$\therefore$ The d.c.'s of S.D. are $\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}$.

Now A (1, 2, 3) is a point on the line (1) and B (2, 4, 5) is a point on the line (2).

The length of S.D.

= the projection of join of A and B on the line whose d.c.'s are $\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}$

$$= -\frac{1}{\sqrt{6}}(2-1) + \frac{2}{\sqrt{6}}(4-2) - \frac{1}{\sqrt{6}}(5-3) = \frac{1}{\sqrt{6}}.$$

The equations of S.D.

The equation of the plane through the line (1) and S.D. is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ -1 & 2 & -1 \end{vmatrix} = 0 \quad \text{or} \quad 11x + 2y - 7z + 6 = 0 \qquad \ldots(3)$$

And the equation of the plane through the line (2) and the S.D. is

$$\begin{vmatrix} x-2 & y-4 & z-5 \\ 3 & 4 & 5 \\ -1 & 2 & -1 \end{vmatrix} = 0 \quad \text{or} \quad 7x + y - 5z + 7 = 0 \qquad \ldots(4)$$

$\therefore$ From equations (3) and (4) the equations of the S.D. are

$$11x + 2y - 7z + 6 = 0,\ 7x + y - 5z + 7 = 0.$$

Method 2: Any point P on the line (1) is $(2r_1 + 1, 3r_1 + 2, 4r_1 + 3)$, ...(3)

and any point Q on the line (2) is $(3r_2 + 2, 4r_2 + 4, 5r_2 + 5)$. ...(4)

The d.r.'s of the line PQ are

$$(3r_2 + 2) - (2r_1 + 1), (4r_2 + 4) - (3r_1 + 2), (5r_2 + 5) - (4r_1 + 3)$$

or $$3r_2 - 2r_1 + 1, 4r_2 - 3r_1 + 2, 5r_2 - 4r_1 + 2. \qquad \ldots(5)$$

If PQ is the line of shortest distance, then PQ is perpendicular to both the given lines (1) and (2) and, therefore, we have

$$2(3r_2 - 2r_1 + 1) + 3(4r_2 - 3r_1 + 2) + 4(5r_2 - 4r_1 + 2) = 0$$

and $$3(3r_2 - 2r_1 + 1) + 4(4r_2 - 3r_1 + 2) + 5(5r_2 - 4r_1 + 2) = 0$$

or $$38r_2 - 29r_1 + 16 = 0 \text{ and } 50r_2 - 38r_1 + 21 = 0.$$

Solving these equations, we get $r_1 = 1/3, r_2 = -1/6$.

Substituting the values of r_1 and r_2 in (3), (4) and (5), we have the co-ordinates of P and Q as $\quad P\left(\dfrac{5}{3}, 3, \dfrac{13}{3}\right),\ Q\left(\dfrac{3}{2}, \dfrac{10}{3}, \dfrac{25}{6}\right)$

and the d.r.'s of the line of shortest distance PQ as $-\dfrac{1}{6}, \dfrac{1}{3}, -\dfrac{1}{6}$ *i.e.*, $-1, 2, -1$.

$\therefore$ the length of S.D. = the distance between the points P and Q

$$= \sqrt{\left\{\left(\frac{3}{2}-\frac{5}{3}\right)^2 + \left(\frac{10}{3}-3\right)^2 + \left(\frac{25}{6}-\frac{13}{3}\right)^2\right\}}$$

$$= \sqrt{\left\{\left(-\frac{1}{6}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(-\frac{1}{6}\right)^2\right\}} = \frac{1}{\sqrt{6}}.$$

The equations of S.D. are either given by equations (3) and (4) of method 1 above or we can write the equations of a line passing through the point P and having d.r.'s $-1, 2, -1$.

Example 25: *Find the shortest distance between the lines*

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1},\ \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}.$$

Find also its equations and the points in which it meets the given lines.

(Meerut 2002, 05B, 07B, 12; Bundelkhand 2006; Kanpur 2009, 11; Purvanchal 10, 11; Kumaun 12, 14)

Solution: The equations of the given lines are

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} = r_1 \text{ (say)} \qquad \ldots(1)$$

and
$$\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} = r_2 \text{ (say)}. \qquad \ldots(2)$$

Any point P on the line (1) is $(3r_1 + 3, -r_1 + 8, r_1 + 3)$, ...(3)

and any point Q on the line (2) is $(-3r_2 - 3, 2r_2 - 7, 4r_2 + 6)$ ...(4)

The d.r.'s of the line PQ are

$$(-3r_2 - 3) - (3r_1 + 3), (2\,r_2 - 7) - (-r_1 + 8), (4r_2 + 6) - (r_1 + 3)$$

or
$$-3r_2 - 3r_1 - 6, 2r_2 + r_1 - 15, 4r_2 - r_1 + 3. \qquad \ldots(5)$$

If PQ be the line of S.D., then PQ is perpendicular to both the given lines (1) and (2), and so we have

$$3\,(-3r_2 - 3r_1 - 6) - 1\,.\,(2\,r_2 + r_1 - 15) + 1\,.\,(4r_2 - r_1 + 3) = 0$$

and
$$-3\,(-3r_2 - 3r_1 - 6) + 2\,.\,(2\,r_2 + r_1 - 15) + 4\,(4r_2 - r_1 + 3) = 0$$

or
$$-7r_2 - 11r_1 = 0 \quad \text{and} \quad 29r_2 + 7r_1 = 0.$$

Solving these equations, we get $r_1 = r_2 = 0$.

Substituting these values of r_1 and r_2 in (3), (4) and (5), we have the co-ordinates of P and Q as $P\,(3, 8, 3)$ and $Q\,(-3, -7, 6)$

and the d.r.'s of the line of shortest distance PQ are $-6, -15, 3$ or $-2, -5, 1$.

The length of S.D. = the distance between the points P and Q

$$= \sqrt{(-3-3)^2 + (-7-8)^2 + (6-3)^2} = 3\sqrt{30}.$$

Also the line of shortest distance PQ is the line passing through $P(3, 8, 3)$ and having d.r.'s $-2, -5, 1$.

So its equations are given by

$$\frac{x-3}{-2} = \frac{y-8}{-5} = \frac{z-3}{1} \quad \text{or} \quad \frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}.$$

Example 26: *Find the length and position of the shortest distance between the lines*

$$\frac{x}{4} = \frac{y+1}{3} = \frac{z-2}{2}, 5x - 2y - 3z + 6 = 0 = x - 3y + 2z - 3.$$

(Meerut 2005, 09B; Rohilkhand 13)

Solution: The equations of the given lines are

$$\frac{x}{4} = \frac{y+1}{3} = \frac{z-2}{2} \qquad \text{...(1)}$$

and $$5x - 2y - 3z + 6 = 0,\ x - 3y + 2z - 3 = 0. \qquad \text{...(2)}$$

The equation of any plane through the line (2) is

$$(5x - 2y - 3z + 6) + \lambda(x - 3y + 2z - 3) = 0$$

or $$(5 + \lambda)x + (-2 - 3\lambda)y + (-3 + 2)z + (6 - 3\lambda) = 0. \qquad \text{...(3)}$$

If the plane (3) is parallel to the line (1), then the normal to the plane (3) will be perpendicular to the line (1) and so we have

$$4(5 + \lambda) + 3(-2 - 3\lambda) + 2(-3 + 2\lambda) = 0 \quad \text{or} \quad \lambda = 8.$$

Putting this value of λ in (3), the equation of the plane through the line (2) and parallel to the line (1) is given by

$$13x - 26y + 13z - 18 = 0. \qquad \text{...(4)}$$

Clearly, $A(0, -1, 2)$ is a point on the line (1).

$\therefore$ Length of the shortest distance

= The length of perpendicular from the point $A(0, -1, 2)$ to the plane (4)

$$= \frac{13 \cdot 0 - 26 \cdot (-1) + 13(2) - 18}{\sqrt{\{(13)^2 + (-26)^2 + (13)^2\}}} = \frac{34}{13\sqrt{6}} = \frac{17\sqrt{6}}{39}.$$

The position of S.D. *i.e.,* the equations of shortest distance.

The equation of the plane through the line (1) and perpendicular to the plane (4) is given by

$$\begin{vmatrix} x & y+1 & z-2 \\ 4 & 3 & 2 \\ 13 & -26 & 13 \end{vmatrix} = 0 \quad \text{or} \quad 13\begin{vmatrix} x & y+1 & z-2 \\ 4 & 3 & 2 \\ 1 & -2 & 1 \end{vmatrix} = 0$$

or $$x(3 + 4) - (y + 1)(4 - 2) + (z - 2)(-8 - 3) = 0$$

or $$7x - 2y - 11z + 20 = 0. \quad \ldots(5)$$

Again if the plane (3) which is any plane through the line (2) is perpendicular to the plane (4), we have

$$13(5+\lambda) - 26(-2-3\lambda) + 13(-3+2\lambda) = 0 \quad \text{or} \quad \lambda = -\frac{2}{3}.$$

Putting this value of λ *i.e.*, $\lambda = -2/3$ in (3), the equation of the plane through the line (2) and perpendicular to the plane (4) is given by

$$13x - 13z + 24 = 0 \quad \ldots(6)$$

$\therefore$ The equations (5) and (6) together are the required equations of the S.D.

Example 27: *Find the length and equations of the shortest distance between*

$$3x - 9y + 5z = 0 = x + y - z$$

and $$6x + 8y + 3z - 13 = 0 = x + 2y + z - 3.$$

Solution: Here we shall use method IV of article 17.

The equations of the planes through the given lines are

$$(3x - 9y + 5z) + \lambda_1 (x + y - z) = 0$$

and $$(6x + 8y + 3z - 13) + \lambda_2 (x + 2y + z - 3) = 0$$

or $$x(3+\lambda_1) + y(-9+\lambda_1) + z(5-\lambda_1) = 0 \quad \ldots(1)$$

and $$x(6+\lambda_2) + y(8+2\lambda_2) + z(3+\lambda_2) - (13+3\lambda_2) = 0. \quad \ldots(2)$$

If the planes (1) and (2) are parallel, then their coefficients are proportional and so we have

$$\frac{3+\lambda_1}{6+\lambda_2} = \frac{-9+\lambda_1}{8+2\lambda_2} = \frac{5-\lambda_1}{3+\lambda_2} = k \text{ (say)}. \quad \ldots(3)$$

Taking the ratios 1st, 2nd and 3rd with k respectively in (3), we get

$$(3+\lambda_1) = k(6+\lambda_2) \quad \text{or} \quad 3 + \lambda_1 - 6k - k\lambda_2 = 0 \quad \ldots(4)$$

$$(-9+\lambda_1) = k(8+2\lambda_2) \quad \text{or} \quad -9 + \lambda_1 - 8k - 2k\lambda_2 = 0 \quad \ldots(5)$$

$$(5-\lambda_1) = k(3+\lambda_2) \quad \text{or} \quad 5 - \lambda_1 - 3k - k\lambda_2 = 0. \quad \ldots(6)$$

Subtracting (6) from (4), we have

$$-2 + 2\lambda_1 - 3k = 0. \quad \ldots(7)$$

Subtracting 2 times (6) from (5), we have

$$-19 + 3\lambda_1 - 2k = 0. \quad \ldots(8)$$

Solving (7) and (8), we have

$$\lambda_1 = 53/5, k = 32/5.$$

Putting the values of λ_1 and k in (4), we get

$$\lambda_2 = -31/8.$$

Substituting the values of λ_1 and λ_2 in (1) and (2), the equations of the parallel planes through the given lines are

$$17x + 2y - 7z = 0 \quad \ldots(9)$$

and $17x + 2y - 7z - 11 = 0.$ ...(10)

The required S.D. is the distance between the parallel planes (9) and (10).

Any point on the plane (9) is (0, 0, 0).

$\therefore$ The length of S.D.

$$= \text{the length of perpendicular from } (0, 0, 0) \text{ to the plane (10)}$$

$$= \frac{0 + 0 - 0 - 11}{\sqrt{\{(17)^2 + (2)^2 + (-7)^2\}}} = \frac{11}{\sqrt{(342)}} \quad \text{[Numerically]}$$

The equations of S.D.

The equation of any plane through the first given line is

$$x(3 + \lambda_1) + y(-9 + \lambda_1) + z(5 - \lambda_1) = 0 \qquad \text{...(11)}$$

[See equation (1)]

If the plane (11) is perpendicular to (9) or (10), we have

$$17(3 + \lambda_1) + 2(-9 + \lambda_1) - 7(5 - \lambda_1) = 0 \text{ or } \lambda_1 = 1/13.$$

Putting the value of λ_1 in (11) the equation of the plane through the 1st given line and perpendicular to the plane (9) or (10) is given by

$$10x - 29y + 16z = 0. \qquad \text{...(12)}$$

Again the equation of any plane through the 2nd given line is

$$x(6 + \lambda_2) + y(8 + 2\lambda_2) + z(3 + \lambda_2) - (13 + 3\lambda_2) = 0. \qquad \text{...(13)}$$

[See equation (2)]

If the plane (13) is perpendicular to (9) or (10), we have

$$17(6 + \lambda_2) + 2(8 + 2\lambda_2) - 7(3 + \lambda_2) = 0$$

or $\lambda_2 = -58/7.$

Putting the value of λ_2 in (13), the equation of the plane through the 2nd given line and perpendicular to the plane (9) or (10) is given by

$$13x + 82y + 55z - 109 = 0. \qquad \text{...(14)}$$

The equations (12) and (14) are the required equations of the shortest distance.

Note: We can solve the above problem by reducing both the lines to symmetrical form and then using method I or II. The problem can also be solved by reducing only one line to symmetrical form and then using method III.

Comprehensive Problems 10

1. Find the length of the shortest distance between the lines
$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-2}{1}; \frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}.$$

(Meerut 2000, 06B; Rohilkhand 05, 09, 09B; Kanpur 06)

2. Find the equations of the shortest distance and its length between the lines $\dfrac{x-3}{1}=\dfrac{y-5}{-2}=\dfrac{z-7}{1}; \dfrac{x+1}{7}=\dfrac{y+1}{-6}=\dfrac{z+1}{1}$.

(Rohilkhand 2005; Kanpur 08; Lucknow 11; Meerut 12; Kumaun 09, 15)

Show also that its equations are given by $\dfrac{x-1}{2}=\dfrac{y-2}{3}=\dfrac{z-3}{4}$

3. **Find the length of the shortest distance between the lines** $\dfrac{x-3}{1}=\dfrac{y-5}{-2}=\dfrac{z-7}{1}; \dfrac{x+1}{7}=\dfrac{y+1}{-6}=\dfrac{z+1}{-1}$.

Find also its equations. **(Meerut 2006B; Kanpur 06, 10)**

4. Find the shortest distance between the lines $\dfrac{x-1}{2}=\dfrac{y-2}{3}=\dfrac{z-3}{4}; \dfrac{x-2}{3}=\dfrac{y-3}{4}=\dfrac{z-4}{5}$.

Hence show that the lines are coplanar. **(Bundelkhand 2005; Meerut 07B)**

5. Find the points on the lines

$\dfrac{x-6}{3}=-(y-7)=z-4$ and $\dfrac{-x}{3}=\dfrac{y+9}{2}=\dfrac{z-2}{4}$

which are nearest to each other. Hence find the shortest distance between the lines and also its equations.

6. Find the equations of the straight line perpendicular to both the lines $\dfrac{x-1}{1}=\dfrac{y-1}{2}=\dfrac{z+2}{2}; \dfrac{x+2}{2}=\dfrac{y-5}{-1}=\dfrac{z+3}{2}$.

7. Find the length and equations of the common perpendicular to the two lines $\dfrac{x+3}{-4}=\dfrac{y-6}{3}=\dfrac{z}{2}; \dfrac{x+2}{-4}=\dfrac{y}{1}=\dfrac{z-7}{1}$. **(Meerut 2010B)**

8. Show that the shortest distance between any two opposite edges of the tetrahedron formed by the planes

$y+z=0, z+x=0, x+y=0, x+y+z=a$ is $2a/\sqrt{6}$

and that the three lines of shortest distance intersect at the point $x=y=z=-a$. **(Garhwal 2002)**

9. Show that the shortest distance between the lines $x+a=2y=-12z$ and $x=y+2a=6z-6a$ is $2a$. **(Meerut 2011)**

10. Find the length of the shortest distance between the z-axis and the line $x+y+2z-3=0=2x+3y+4z-4$. **(Kumaun 2010)**

11. Find the shortest distance between the z-axis and the line $ax+by+cz+d=0=a'x+b'y+c'z+d'$. **(Kumaun 2008)**

Show also that it meets the z-axis at a point whose distance from the origin is

$$\frac{(ab'-d'b)(bc'-b'c)+(ca'-c'a)(ad'-a'd)}{(bc'-b'c)^2+(ca'-c'a)^2}.$$

12. Show that the equation of the plane containing the line $y/b + z/c = 1, x = 0$ and parallel to the line $x/a - z/c = 1, y = 0$ is $x/a - y/b - z/c + 1 = 0$ and if $2d$ is the shortest distance, then show that $d^{-2} = a^{-2} + b^{-2} + c^{-2}$. **(Meerut 2008; Rohilkhand 06; Avadh 09)**

13. Show that the shortest distance between the diagonals of a rectangular parallelopiped and the edges not meeting it are

$$\frac{bc}{\sqrt{(b^2 + c^2)}}, \frac{ca}{\sqrt{(c^2 + a^2)}}, \frac{ab}{\sqrt{(a^2 + b^2)}}$$

where a, b, c are the lengths of the edges.

Answers 10

1. $\frac{34}{\sqrt{29}}; -11x - 2y + 7z + 29 = 0, 27x + 26y - 33z - 34 = 0$

2. $2\sqrt{29}; \frac{x-3}{2} = \frac{y-5}{3} = \frac{z-7}{4}$ **3.** $6\sqrt{3}$ **4.** 0

5. $P(3, 8, 3), Q(-3, -7, 6); 3\sqrt{(30)}; \frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$

6. $\frac{x-2}{7} = \frac{y-3}{4} = \frac{z-1}{-5}$

7. $32x + 34y + 13z = 108, 4x + 11y + 5z = 27; 9$

10. 2 **11.** $\frac{(db' - d'b)(bc' - b'c) + (ca' - c'a)(ad' - a'd)}{(bc' - b'c)^2 + (ca' - c'a)^2}$

18 The Equations of Two Non-intersecting (Skew) Lines

To show that by a proper choice of axes the equations of two skew lines can be given by the equations $y = x \tan\alpha,\ z = c;\ y = -x \tan\alpha,\ z = -c$.

In the adjoining figure let AB and $A'B'$ be the two given skew (non-intersecting) lines, and let CD of length $2c$ be the shortest distance between them.

Take the axis of z along DC and O, the middle point of DC, as the origin. Draw OK and OL parallel to AB and $A'B'$ respectively and take the plane KOL as the plane $z = 0$. Take the internal and external bisectors of the angle KOL as the axes of x and y respectively. In the figure, OX and OY represent the axes of x and y respectively. Let the angle between the lines OK and OL (*i.e.*, between the given lines AB and $A'B'$) be 2α.

As explained above the line OK (which is parallel to AB) is inclined at angles $\alpha, \frac{1}{2}\pi - \alpha, \frac{1}{2}\pi$ with x, y, z axes respectively and therefore, the d.c.'s of OK (*i.e.*, of AB) are $\cos\alpha, \cos(\frac{1}{2}\pi - \alpha), \cos\frac{1}{2}\pi$ *i.e.*, $\cos\alpha, \sin\alpha, 0$.

Again the line OL (which is parallel to $A'B'$) is inclined at angles $-\alpha, \frac{1}{2}\pi + \alpha, \frac{1}{2}\pi$ with the co-ordinate axes respectively and, therefore, the d.c.'s of OL (*i.e.*, of $A'B'$) are $\cos(-\alpha), \cos(\frac{1}{2}\pi + \alpha), \cos\frac{1}{2}\pi$ or $\cos\alpha, -\sin\alpha, 0$.

Now it is required to find the equations of the given skew lines AB and $A'B'$.

Since $CD = 2c$ and O is the middle point of CD, therefore $OC = OD = c$. Hence the co-ordinates of the points C and D (on the z-axis) are $(0, 0, c)$ and $(0, 0, -c)$ respectively.

Thus we see that the given line AB has d.c.'s $\cos\alpha, \sin\alpha, 0$ and passes through the point $C(0, 0, c)$ and hence its equations are

$$\frac{x-0}{\cos\alpha} = \frac{y-0}{\sin\alpha} = \frac{z-c}{0} \quad \text{or} \quad y = x\tan\alpha, z = c. \qquad \ldots(1)$$

Also, the line $A'B'$ has d.c.'s $\cos\alpha, -\sin\alpha, 0$ and passes through the point $D(0, 0, -c)$ and hence its equations are

$$\frac{x-0}{\cos\alpha} = \frac{y-0}{-\sin\alpha} = \frac{z+c}{0} \quad \text{or} \quad y = -x\tan\alpha, z = -c. \qquad \ldots(2)$$

The equations (1) and (2) are the required equations of the given skew lines AB and $A'B'$.

If we put $\tan\alpha = m$, the equations (1) and (2) become

$$y = mx, z = c \text{ and } y = -mx, z = -c. \qquad \ldots(3)$$

The equations (3) may be written as

$$\frac{x}{1} = \frac{y}{m} = \frac{z-c}{0} \quad \text{and} \quad \frac{x}{1} = \frac{y}{-m} = \frac{z+c}{0}. \qquad \ldots(4)$$

Illustrative Examples

Example 28: *Prove that the locus of a variable line which intersects the three given lines*

$$y = mx, z = c;\ y = -mx, z = -c;\ y = z, mx = -c$$

is the surface $y^2 - m^2x^2 = z^2 - c^2$. **(Meerut 2006, 07, 10; Kanpur 08)**

Solution: The equations of the given lines are

$$y = mx, z = c; \qquad \ldots(1)$$

$$y = -mx, z = -c; \qquad \ldots(2)$$

and $$y = z, mx = -c. \qquad \ldots(3)$$

We know that any line intersecting the lines (1) and (2) is given by two planes, one through each line.

The equation of any plane through the line (1) is

$$(y - mx) + \lambda (z - c) = 0. \qquad \ldots(4)$$

Also, the equation of any plane through the line (2) is

$$(y + mx) + \mu (z + c) = 0. \qquad \ldots(5)$$

The planes (4) and (5) intersect in a line and if this line meets the line (3), then putting $mx = -c$ and $z = y$ in (4) and (5), we have

$$(y + c) + \lambda (y - c) = 0 \quad \text{and} \quad (y - c) + \mu (y + c) = 0$$

or $$\lambda = -\frac{y + c}{y - c} \quad \text{and} \quad \mu = -\frac{y - c}{y + c}.$$

Multiplying these relations, we get $\lambda\mu = 1$. ...(6)

The required locus is obtained by eliminating λ and μ between (4), (5) and (6) and so eliminating λ and μ between these equations, the required locus is given by

$$\left[-\frac{(y - mx)}{z + c}\right] \times \left[-\frac{y + mx}{z - c}\right] = 1$$

or $$y^2 - m^2 x^2 = z^2 - c^2.$$

Example 29: *A variable line intersects the x-axis and the curve $x = y$, $y^2 = cz$ and is parallel to the plane $x = 0$. Prove that it generates the paraboloid $xy = cz$.*

Solution: The equation of any plane through x-axis *i.e.*, $y = 0 = z$ is

$$y = \lambda z. \qquad \ldots(1)$$

The equation of any plane parallel to the plane $x = 0$ is

$$x = \mu. \qquad \ldots(2)$$

The planes (1) and (2) intersect in a line which intersects the x-axis and is parallel to the plane $x = 0$. If this line meets the curve $x = y$, $y^2 = cz$, we have by putting $x = y$ in (2),

$$y = \mu. \qquad \ldots(3)$$

From (1), $$\lambda = \frac{y}{z} = \frac{cy}{cz} = \frac{cy}{y^2} \qquad [\because cz = y^2]$$

or $$\lambda = c / y \quad \text{or} \quad \lambda = c / \mu, \qquad \text{[Using (3)]}$$

or $$\lambda\mu = c. \qquad \ldots(4)$$

The required locus is given by eliminating λ and μ between (1), (2) and (4), and is

$$(y / z)\, x = c$$

or $$xy = cz.$$

Comprehensive Exercise 11

1. Find the surface generated by the lines which intersect the lines $y = mx, z = c\ ;\ y = -mx, z = -c$ and x-axis.

2. Show that the straight lines which intersect the three lines $y - z = 1, x = 0\ ;\ z - x = 1, y = 0$ and $x - y = 1, z = 0$ lie on the surface whose equation is $x^2 + y^2 + z^2 - 2yz - 2zx - 2xy - 1 = 0$.

3. Prove that the locus of a line which meets two lines $y = \pm mx, z = \pm c$ and the circle $x^2 + y^2 = a^2, z = 0$ is
$$c^2 m^2 (cy - mzx)^2 + c^2 (yz - cmx)^2 = a^2 m^2 (z^2 - c^2).$$

4. Find the surface generated by a straight line which meets two lines $y = mx, z = c\ ;\ y = -mx, z = -c$ at the same angle.

5. Show that the locus of lines which meet the lines
$$\frac{x + a}{0} = \frac{y}{\sin\alpha} = \frac{z}{-\cos\alpha}\ ;\ \frac{x - a}{0} = \frac{y}{\sin\alpha} = \frac{z}{\cos\alpha}$$
at the same angle is $(xy\cos\alpha - az\sin\alpha)(zx\sin\alpha - ay\cos\alpha) = 0$.

6. P, P' are two variable points on two given non-intersecting lines and PP' is of constant length $2k$. Find the surface generated by PP'.

7. Find the equation to the surface generated by a straight line which is parallel to the line $y = mx, z = nx$ and intersects the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0.$$

8. How many lines can be drawn from a point to intersect two non-coplanar lines neither of which passes through the point? Find the equations of the lines or line which can be drawn from the point $(2, -1, 3)$ to intersect the lines
$$\frac{x - 1}{2} = \frac{y - 2}{3} = \frac{z - 3}{4}\ ;\ \frac{x - 4}{4} = \frac{y}{5} = \frac{z + 3}{3}.$$
(Bundelkhand 2005)

Answers 11

1. $cy = mzx$ 4. $(mcx - yz)(cy - mzx) = 0$

6. $c^2 (mzx - cy)^2 + c^2 m^2 (yz - mcx)^2 = m^2 (\lambda^2 - c^2)(z^2 - c^2)^2$

7. $\frac{1}{a^2}(xn - z)^2 + \frac{1}{b^2}(yn - mz)^2 = n^2$

8. 1; $12x + 4y - 9z + 7 = 0$ and $11x - 10y + 2z - 28 = 0$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The direction cosines of any straight line perpendicular to z-axis are

(a) $0, \cos\alpha, \sin\alpha$ (b) $\cos\alpha, 0, \sin\alpha$

(c) $\cos\alpha, \sin\alpha, 0$ (d) $\cos\alpha, \sin\alpha, 1$ **(Agra 2007)**

2. The equations of the straight line passing through the point $(2, 5, 7)$ and parallel to y-axis are

(a) $\frac{x-2}{0}=\frac{y-5}{1}=\frac{z-7}{0}$ (b) $\frac{x-2}{1}=\frac{y-5}{0}=\frac{z-7}{1}$

(c) $\frac{x+2}{0}=\frac{y+5}{1}=\frac{z+7}{0}$ (d) $\frac{x-2}{1}=\frac{y-5}{1}=\frac{z-7}{0}$

3. Perpendicular distance of the point (x_1, y_1, z_1) from the z-axis is

(a) $\sqrt{x_1^2+y_1^2}$ (b) $\sqrt{y_1^2+z_1^2}$

(c) $\sqrt{z_1^2+x_1^2}$ (d) none of these

4. Image of the point $(-5, 3, 2)$ in the yz-plane is

(a) $(5, 3, 2)$ (b) $(2, 3, -5)$

(c) $(-5, -3, 2)$ (d) $(5, 3, -2)$ **(Agra 2007)**

5. The shortest distance between the lines $x+a=2y=-12z$ and $x=y+2a=6z-6a$ is

(a) 0 (b) a

(c) $2a$ (d) none of these

6. The equation of the plane through $(2, 1, 4)$ perpendicular to the line of intersection of the planes $3x+4y+7z+4=0$ and $x-y+2z+3=0$ is

(a) $15x+y-7z-3=0$ (b) $15x-y-7z-3=0$

(c) $15x+y+7z-3=0$ (d) $15x+y+7z+3=0$

7. The symmetrical form of the line $x+2y+3z+4=0$,

$2x+3y+4z+5=0$ is

(a) $\frac{x+2}{1}=\frac{y-3}{2}=\frac{z-0}{-1}$ (b) $\frac{x-2}{-2}=\frac{y+3}{1}=\frac{z-0}{-1}$

(c) $\frac{x-2}{-1}=\frac{y+3}{2}=\frac{z-0}{-1}$ (d) none of these

(Kumaun 2008)

8. The equation of z-axis in symmetrical form is

(a) $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$ (b) $\frac{x}{0}=\frac{y}{1}=\frac{z}{0}$

(c) $\frac{x}{0}=\frac{y}{0}=\frac{z}{1}$ (d) none of these **(Kumaun 2010)**

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. If two planes are not parallel, they intersect in a …… .
2. The equations of the straight line passing through a given point (x_1, y_1, z_1) and having direction cosines l, m, n are …… .
3. The equations of the straight line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are …… . **(Meerut 2001)**
4. The equations of the straight line passing through the point $(2, -3, 5)$ and having its direction cosines proportional to $2, -1, 3$ are …… .
5. The direction cosines of the straight line $\frac{x+1}{2}=\frac{y-3}{1}=\frac{z-5}{-2}$ are …… .
6. The coordinates of any point P on the straight line
$$\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{-2} \text{ are …… .}$$
7. The coordinates of the two points on the straight line
$$\frac{x-2}{6}=\frac{y+3}{2}=\frac{z+1}{3}$$
which are at a distance 14 from the point $(2, -3, -1)$ are …… .
8. The equations of the straight line passing through the point $(2, -5, -3)$ and parallel to the straight line $\frac{x-5}{-1}=\frac{y+2}{3}=\frac{z-4}{5}$ are …… .
9. The equations of the straight line parallel to
$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z}{4} \text{ are } \frac{x-2}{\ldots}=\frac{y-2}{\ldots}=\frac{z}{8}.$$ **(Meerut 2001)**
10. The direction cosines of the straight line $y=0, z=0$ are …… .
11. The equations of the straight line $x-y=0, z=1$ in symmetrical form are …… .
12. The direction cosines l, m, n of the straight line
$$3x+2y-5z=0=4x+2y-3z+3 \text{ satisfy the equations …… .}$$
13. The straight line $3x+2y-z-4=0, 4x+y-2z+3=0$ meets the xy-plane at the point …… .
14. The equations of the straight line passing through the point $(2, 3, -7)$ and perpendicular to the plane $2x-3y+4z=7$ are …… .

15. The line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is perpendicular to the plane $ax+by+cz+d=0$, if ......

(Meerut 2001)

16. The conditions for the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ to be parallel to the plane $ax+by+cz+d=0$ but not lying in it are ...... **(Bundelkhand 2005)**

17. The angle between the straight line $\frac{x-3}{1}=\frac{y+4}{-1}=\frac{z}{0}$ and the plane $y-z+2=0$ is ....

18. The foot of perpendicular from $(2, 3, 4)$ to the plane $x+y-z+4=0$ is ......

19. The equation of the plane containing the straight line

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$$

and parallel to the straight line $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ is ....

20. The straight lines

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \text{ and } \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

are coplanar if ....

21. The shortest distance between two intersecting straight lines is ....

22. Two lines are ...... if the shortest distance between them vanishes.

23. The shortest distance between two skew lines is ...... to both the lines.

24. The shortest distance between two non-intersecting lines is the projection of the straight line joining any two points on these lines on a straight line perpendicular to ....

True or False

Write 'T' for true and 'F' for false statement.

1. The straight line $\frac{x+1}{1}=\frac{y-2}{3}=\frac{z+4}{-5}$ is parallel to the plane $6x+8y+6z=7$.

2. The straight line $\frac{x-2}{3}=\frac{y+9}{5}=\frac{z-6}{1}$ is perpendicular to the plane $6x+10y-2z=9$.

3. The straight line $\frac{x-2}{5}=\frac{y-1}{6}=\frac{z+1}{4}$ lies in the plane $2x+3y-7z=5$.

4. The lines $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}$ and $\frac{x-2}{1}=\frac{y-4}{4}=\frac{z-6}{7}$ are coplanar.

5. Two parallel straight lines are always coplanar.

6. If two straight lines intersect, they are always coplanar.
7. Two non-coplanar lines are always intersecting lines.
8. The lines $x = ay + b, z = cy + d$ and $x = a'y + b', z = c'y + d'$ are perpendicular if $aa' + cc' + 1 = 0$.
9. The conditions that the line $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$ is parallel to the plane $ax + by + cz + d = 0$ but does not lie in it are $al + bm + cn = 0, ax_1 + by_1 + cz_1 + d = 0$.
10. Straight line which is perpendicular to each of the two skew lines is called the line of shortest distance.
11. The symmetrical form of straight line is $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$.

Answers

Multiple Choice Questions

1. (c) 2. (a) 3. (a) 4. (a) 5. (c)
6. (a) 7. (c) 8. (c)

Fill in the Blank(s)

1. straight line
2. $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$
3. $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$
4. $\mathbf{\dfrac{x - 2}{2} = \dfrac{y + 3}{-1} = \dfrac{z - 5}{3}}$
5. $\dfrac{2}{3}, \dfrac{1}{3}, -\dfrac{2}{3}$
6. $(r + 3, 2r + 4, -2r + 5)$
7. $(14, 1, 5)$ and $(-10, -7, -7)$
8. $\dfrac{x - 2}{-1} = \dfrac{y + 5}{3} = \dfrac{z + 3}{5}$
9. 4 ; 6
10. 1, 0, 0
11. $\dfrac{x - 0}{1} = \dfrac{y - 0}{1} = \dfrac{z - 1}{0}$
12. $3l + 2m - 5n = 0, 4l + 2m - 3n = 0$
13. $(-2, 5, 0)$
14. $\dfrac{x - 2}{2} = \dfrac{y - 3}{-3} = \dfrac{z + 7}{4}$
15. $\dfrac{a}{l} = \dfrac{b}{m} = \dfrac{c}{n}$
16. $al + bm + cn = 0, ax_1 + by_1 + cz_1 + d \neq 0$
17. $\pi / 6$
18. $\left(\dfrac{1}{3}, \dfrac{4}{3}, \dfrac{17}{3}\right)$

19. $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

20. $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

21. 0

22. coplanar

23. perpendicular

24. both these lines

True or False

1. *T*

2. *F*

3. *F*

4. *T*

5. *T*

6. *T*

7. *F*

8. *T*

9. *F*

10. *T*

11. *T*

□

VECTOR ANALYSIS

Chapters

Chapter

1

Multiple Products

1 Triple Products

We know that the vector product $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ is itself a vector quantity. Therefore we can multiply it by another vector $\mathbf{c}$ both scalarly and vectorially. The product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is called **scalar triple product,** which is a pure number. On the other hand the product $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is called **vector triple product,** which is again a vector quantity.

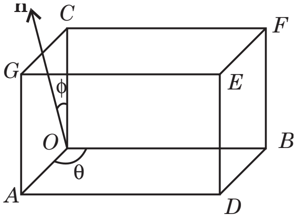
Note. Since $\mathbf{a} \cdot \mathbf{b}$ is a scalar quantity, therefore the products $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ and $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$ are meaningless. Moreover in the product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ we can omit the parentheses and we can simply write it as $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$**.** Obviously the product $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$ has meaning only if we regard it as the product $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.

2 Scalar Triple Product

The scalar product of two vectors one of which is itself the vector product of two vectors is a scalar quantity called "Scalar Triple Product". Thus if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors, then $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is called the scalar triple product of these three vectors.

Since the scalar triple product involves both the signs of 'cross' and 'dot' therefore it is sometimes also called the **mixed product.**

Geometrical Interpretation of Scalar Triple Product.
Let us consider a parallelopiped whose coterminous edges OA, OB, OC have the lengths and directions of the vectors **a, b, c** respectively. Let V be the volume of this parallelopiped. We shall regard V, as necessarily positive.



Let $\mathbf{a} \times \mathbf{b} = \mathbf{n}$. Then from our definition of vector product, the vector **n** is perpendicular to the face $OADB$, and its modulus n is the measure of the area of the parallelogram $OADB$. Also, by definition, the vectors **a, b** and **n** form a right handed triad.

Let ϕ denote the angle between the directions of the vectors $\overrightarrow{OC}$ and **n**. Then the vectors **a, b** and **c** will form a right handed or a left handed triad according as ϕ is acute or obtuse.

Now $$(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c} = |(\mathbf{a} \times \mathbf{b})||\mathbf{c}| \cos \phi = |\mathbf{n}||\mathbf{c}| \cos \phi$$
$$= (\text{area of the parallelogram } OADB).(OC \cos \phi)$$
$$[\because |\mathbf{c}| = OC]$$

Now $OC \cos \phi$ will be positive or negative according as ϕ is acute or obtuse. Its absolute value will give us the length of the perpendicular from C to the plane of the parallelogram $OADB$.

Now the volume V of the parallelopiped = (Area of the parallelogram $OADB$) × length of the perpendicular from C on this parallelogram. Therefore $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c} = +V$, if ϕ is acute *i.e.*, if **a, b, c** form a right handed triad and $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c} = -V$, if ϕ is obtuse *i.e.*, if **a, b, c** form a left handed triad.

Now we know that if the vectors **a, b, c** form a right handed triad, then the vector triads **b, c, a** and **c, a, b** are also right handed. Hence each of the products $(\mathbf{b} \times \mathbf{c}) \bullet \mathbf{a}$ and $(\mathbf{c} \times \mathbf{a}) \bullet \mathbf{b}$ will have the same value $+V$ or $-V$ according as **a, b, c** form a right handed or a left handed triad. Thus we conclude that in all the cases

$$(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \bullet \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \bullet \mathbf{b}.$$

Now $$\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a} \text{ and } \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}.$$

$\therefore$ $$(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c} = \mathbf{c} \bullet (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \times \mathbf{c}) \bullet \mathbf{a}$$
$$= \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \times \mathbf{a}) \bullet \mathbf{b} = \mathbf{b} \bullet (\mathbf{c} \times \mathbf{a})$$
$$= -(\mathbf{b} \times \mathbf{a}) \bullet \mathbf{c} = -\mathbf{c} \bullet (\mathbf{b} \times \mathbf{a}) = -(\mathbf{c} \times \mathbf{b}) \bullet \mathbf{a}$$
$$= -\mathbf{a} \bullet (\mathbf{c} \times \mathbf{b}) = -(\mathbf{a} \times \mathbf{c}) \bullet \mathbf{b} = -\mathbf{b} \bullet (\mathbf{a} \times \mathbf{c}).$$

From this we conclude that *the value of a scalar triple product depends on the cyclic order of the factors and is independent of the position of the dot and cross. These may be interchanged at pleasure. However, an anticyclic permutation of the three factors changes the value of the product in sign but not in magnitude.* **(Important)**

Notation: In view of the properties discussed above, the scalar triple product is usually written as $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c} = [\mathbf{abc}]$ or $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$. *This notation takes into consideration only the cyclic order of the three vectors and disregards the unimportant positions of dot and cross*. Thus $[\mathbf{abc}] = [\mathbf{bca}] = [\mathbf{cab}] = -[\mathbf{cba}]$ etc.

The signs of dot and cross can be inserted at pleasure *i.e.*,

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) \quad \text{or} \quad = (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}\,.$$

Note 1: If $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ constitute an orthogonal right handed triad of unit vectors,

then $\quad [\mathbf{i}, \mathbf{j}, \mathbf{k}] = (\mathbf{i} \times \mathbf{j}) \bullet \mathbf{k} = \mathbf{k} \bullet \mathbf{k} = 1.$

Note 2: The scalar triple product $[\mathbf{abc}]$ is positive or negative according as $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ form a right handed or a left handed triad of vectors.

3 Distributive Law for Vector Product

To prove that $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$, *where* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *are any three vectors.*

Let $\quad \mathbf{r} \equiv \mathbf{a} \times (\mathbf{b} + \mathbf{c}) - \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c} \quad$ ...(1)

Now forming the scalar product of both sides of (1) with an arbitrary vector $\mathbf{d}$, we get

$$\mathbf{d} \bullet \mathbf{r} = \mathbf{d} \bullet [\mathbf{a} \times (\mathbf{b} + \mathbf{c}) - \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c}] \quad \text{...(2)}$$

or $\quad \mathbf{d} \bullet \mathbf{r} = \mathbf{d} \bullet [\mathbf{a} \times (\mathbf{b} + \mathbf{c})] - \mathbf{d} \bullet (\mathbf{a} \times \mathbf{b}) - \mathbf{d} \bullet (\mathbf{a} \times \mathbf{c})$

[Since scalar product is distributive]

Now in a scalar triple product the positions of dot and cross can be interchanged without affecting its value. Therefore from (2), we get

$$\begin{aligned}\mathbf{d} \bullet \mathbf{r} &= (\mathbf{d} \times \mathbf{a}) \bullet (\mathbf{b} + \mathbf{c}) - (\mathbf{d} \times \mathbf{a}) \bullet \mathbf{b} - (\mathbf{d} \times \mathbf{a}) \bullet \mathbf{c} \\ &= (\mathbf{d} \times \mathbf{a}) \bullet \mathbf{b} + (\mathbf{d} \times \mathbf{a}) \bullet \mathbf{c} - (\mathbf{d} \times \mathbf{a}) \bullet \mathbf{b} - (\mathbf{d} \times \mathbf{a}) \bullet \mathbf{c}\end{aligned}$$

[Since scalar product is distributive]

$= 0.$

Therefore either $\mathbf{d} = \mathbf{0}$, or $\mathbf{r} = \mathbf{0}$ or $\mathbf{d}$ is perpendicular to $\mathbf{r}$. But the vector $\mathbf{d}$ is arbitrary. Therefore we can take it to be non-zero and not perpendicular to $\mathbf{r}$.

Hence $\quad \mathbf{r} = \mathbf{0}$ *i.e.*, $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) - \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c} = \mathbf{0}$

i.e., $\quad \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.$

4 Properties of Scalar Triple Product

(i) *The value of a scalar triple product, if two of its vectors are equal, is zero.*

We have $\quad [\mathbf{aab}] = \mathbf{a} \bullet (\mathbf{a} \times \mathbf{b}).$

Now $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$.

Therefore $\quad \mathbf{a} \bullet (\mathbf{a} \times \mathbf{b}) = 0.$

(ii) *The value of a scalar triple product, if two of its vectors are parallel, is zero.*

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be three vectors such that $\mathbf{a}$ and $\mathbf{b}$ are parallel *i.e.*, $\mathbf{b} = t\mathbf{a}$, where t is some scalar.

Now $[\mathbf{abc}] = (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c} = (\mathbf{a} \times t\mathbf{a}) \bullet \mathbf{c} = t\,(\mathbf{a} \times \mathbf{a}) \bullet \mathbf{c} = t\,(\mathbf{0} \bullet \mathbf{c})$

$[\because\ \mathbf{a} \times \mathbf{a} = \mathbf{0}]$

$= 0.$

(iii) The necessary and sufficient condition that three non-parallel and non-zero vectors **a, b, c** *be coplanar is that* $[\mathbf{abc}] = 0$.

Let **a, b, c** be three coplanar vectors. Now $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to the plane of **a** and **b.** Since **a, b, c** are coplanar, therefore $\mathbf{a} \times \mathbf{b}$ is also perpendicular to **c.** Now the dot product of two perpendicular vectors is equal to zero. Hence $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c} = 0$ *i.e.,* $[\mathbf{abc}] = \mathbf{0}$. **Therefore the condition is necessary.**

The condition is also sufficient: Because if $[\mathbf{abc}] = 0$ *i.e.,* $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c} = 0$, then **c** is perpendicular to $\mathbf{a} \times \mathbf{b}$. But $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to the plane of **a** and **b**. Since **c** is perpendicular to $\mathbf{a} \times \mathbf{b}$, therefore **c** is parallel to the plane of **a** and **b.**

Hence **a**, **b**, **c** are coplanar.

(iv) Since the distributive law holds for both scalar and vector products, it holds also for the scalar triple product.

Thus $[\mathbf{a}, \mathbf{b} + \mathbf{d}, \mathbf{c} + \mathbf{r}] = [\mathbf{abc}] + [\mathbf{abr}] + [\mathbf{adc}] + [\mathbf{adr}]$, the cyclic order of the factors being maintained in each term.

5 To Express The Value of The Scalar Triple Product [abc] In Terms of Rectangular Components of The Vectors

Let $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$, $\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$,

$\mathbf{c} = c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$.

Now
$$\mathbf{b} \times \mathbf{c} = (b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}) \times (c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k})$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$= (b_2\,c_3 - b_3\,c_2)\,\mathbf{i} - (b_1\,c_3 - b_3\,c_1)\,\mathbf{j} + (b_1\,c_2 - b_2\,c_1)\,\mathbf{k}.$$

$\therefore$
$$\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = (a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}) \bullet [(b_2\,c_3 - b_3\,c_2)\,\mathbf{i} - (b_1\,c_3 - b_3\,c_1)\,\mathbf{j} + (b_1\,c_2 - b_2\,c_1)\,\mathbf{k}]$$
$$= a_1\,(b_2\,c_3 - b_3\,c_2) - a_2\,(b_1\,c_3 - b_3\,c_1) + a_3\,(b_1\,c_2 - b_2\,c_1)$$

$[\because\ \mathbf{i} \bullet \mathbf{i} = \mathbf{j} \bullet \mathbf{j} = \mathbf{k} \bullet \mathbf{k} = 1$ and $\mathbf{i} \bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{k} = \mathbf{k} \bullet \mathbf{i} = 0]$

$\therefore$
$$[\mathbf{abc}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad \ldots(1)$$

Also $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c} = \mathbf{c} \bullet (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

showing that the value of a scalar triple product is independent of the positions of dot and cross.

Note: If OA, OB, OC be three concurrent edges of a parallelopiped and if $(a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3)$ be the rectangular coordinates of A, B, C referred to O as origin, then the determinant (1) gives the volume of that parallelopiped.

6 To Express the Scalar Triple Product [a, b, c] in Terms of any Three non-Coplanar Vectors l, m, n

Let $\mathbf{a} = a_1 \mathbf{l} + a_2 \mathbf{m} + a_3 \mathbf{n},\ \mathbf{b} = b_1 \mathbf{l} + b_2 \mathbf{m} + b_3 \mathbf{n},$

and $\mathbf{c} = c_1 \mathbf{l} + c_2 \mathbf{m} + c_3 \mathbf{n}.$

Now
$$\begin{aligned}\mathbf{b} \times \mathbf{c} &= (b_1 \mathbf{l} + b_2 \mathbf{m} + b_3 \mathbf{n}) \times (c_1 \mathbf{l} + c_2 \mathbf{m} + c_3 \mathbf{n}) \\ &= b_1 c_1 \mathbf{l} \times \mathbf{l} + b_1 c_2 \mathbf{l} \times \mathbf{m} + b_1 c_3 \mathbf{l} \times \mathbf{n} + b_2 c_1 \mathbf{m} \times \mathbf{l} \\ &\quad + b_2 c_2 \mathbf{m} \times \mathbf{m} + b_2 c_3 \mathbf{m} \times \mathbf{n} + b_3 c_1 \mathbf{n} \times \mathbf{l} \\ &\quad + b_3 c_2 \mathbf{n} \times \mathbf{m} + b_3 c_3 \mathbf{n} \times \mathbf{n} \\ &= (b_2 c_3 - b_3 c_2) \mathbf{m} \times \mathbf{n} - (b_1 c_3 - b_3 c_1) \mathbf{n} \times \mathbf{l} \\ &\quad + (b_1 c_2 - b_2 c_1) \mathbf{l} \times \mathbf{m}\end{aligned}$$

$[\because \mathbf{l} \times \mathbf{l} = \mathbf{0}$ and $\mathbf{l} \times \mathbf{m} = -\mathbf{m} \times \mathbf{l}$ etc.$]$

$$\begin{aligned}\therefore \quad \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) &= (a_1 \mathbf{l} + a_2 \mathbf{m} + a_3 \mathbf{n}) \bullet [(b_2 c_3 - b_3 c_2) \mathbf{m} \times \mathbf{n} \\ &\quad - (b_1 c_3 - b_3 c_1) \mathbf{n} \times \mathbf{l} + (b_1 c_2 - b_2 c_1) \mathbf{l} \times \mathbf{m}] \\ &= a_1 (b_2 c_3 - b_3 c_2) [\mathbf{lmn}] - a_2 (b_1 c_3 - b_3 c_1) [\mathbf{lmn}] \\ &\quad + a_3 (b_1 c_2 - b_2 c_1) [\mathbf{lmn}].\end{aligned}$$

$[\because$ $[\mathbf{lmn}] = [\mathbf{mnl}] = [\mathbf{nlm}]$ and all the scalar triple products of the type $[\mathbf{lml}]$ in which two vectors are equal vanish$]$

Hence, $$[\mathbf{abc}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\mathbf{lmn}].$$

Note: Since $[\mathbf{i}, \mathbf{j}, \mathbf{k}] = 1$, therefore article 5 is particular case of article 6.

Illustrative Examples

Example 1: *Find the volume of the parallelopiped whose edges are represented by*

$$\mathbf{a} = 2\,\mathbf{i} - 4\,\mathbf{j} + 5\,\mathbf{k},\ \mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k},\ \mathbf{c} = 3\,\mathbf{i} - 5\,\mathbf{j} + 2\,\mathbf{k}.$$

Solution: The required volume of the parallelopiped is equal to the absolute value of $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

Now $$[\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}] = \begin{vmatrix} 2 & -4 & 5 \\ 1 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix}$$

$$= 2\,(-2+5) + 4\,(2-3) + 5\,(-5+3) = 6 - 4 - 10 = -8.$$

Neglecting the negative sign, we get the volume of the parallelopiped = 8 cubic units.

Example 2: *Find the constant p such that the vectors* $\mathbf{a} = 2\,\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k}$, $\mathbf{c} = 3\,\mathbf{i} + p\,\mathbf{j} + 5\,\mathbf{k}$ *are coplanar.* **(Kumaun 2013)**

Solution: If the vectors **a, b, c** are coplanar, then we should have $[\mathbf{abc}] = 0$.

Now $$[\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}] = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & p & 5 \end{vmatrix}$$

$$= 2\,(10+3p) + 1\,(5+9) + 1\,(p-6) = 7p + 28.$$

$\therefore$ $[\mathbf{abc}]$ will be zero if $7p + 28 = 0$ or $p = -4$.

Hence for the given vectors to be coplanar, we should have $p = -4$.

Example 3: *Prove that the four points* $4\,\mathbf{i} + 5\,\mathbf{j} + \mathbf{k}$, $-(\mathbf{j} + \mathbf{k})$, $(3\,\mathbf{i} + 9\,\mathbf{j} + 4\,\mathbf{k})$ *and* $4\,(-\mathbf{i} + \mathbf{j} + \mathbf{k})$ *are coplanar.* **(Kumaun 2012)**

Solution: Let A, B, C, D be the four given points whose position vectors referred to some origin O are

$$4\,\mathbf{i} + 5\,\mathbf{j} + \mathbf{k},\ -(\mathbf{j} + \mathbf{k}), (3\,\mathbf{i} + 9\,\mathbf{j} + 4\,\mathbf{k}) \text{ and } 4\,(-\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

If the four points A, B, C, D are coplanar, then the vectors $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ should also be coplanar.

We have $\overrightarrow{AB}$ = position vector of B − position vector of A

$$= -(\mathbf{j} + \mathbf{k}) - (4\,\mathbf{i} + 5\,\mathbf{j} + \mathbf{k}) = -4\,\mathbf{i} - 6\,\mathbf{j} - 2\,\mathbf{k} = \mathbf{a} \quad \text{(say)}.$$

Similarly $$\overrightarrow{AC} = (3\,\mathbf{i} + 9\,\mathbf{j} + 4\,\mathbf{k}) - (4\,\mathbf{i} + 5\,\mathbf{j} + \mathbf{k}) = -\mathbf{i} + 4\,\mathbf{j} + 3\,\mathbf{k} = \mathbf{b} \quad \text{(say)},$$

and $$\overrightarrow{AD} = 4\,(-\mathbf{i} + \mathbf{j} + \mathbf{k}) - (4\,\mathbf{i} + 5\,\mathbf{j} + \mathbf{k}) = -8\,\mathbf{i} - \mathbf{j} + 3\,\mathbf{k} = \mathbf{c} \quad \text{(say)}.$$

Now the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ will be coplanar if $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$.

Now $$[\mathbf{a}\ \ \mathbf{b}\ \ \mathbf{c}] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

$$= -4\,(12+3) + 6\,(-3+24) - 2\,(1+32)$$
$$= -60 + 126 - 66 = 0.$$

∴ The points A, B, C, D are coplanar.

Example 4: *Show that the four points* $-\mathbf{a} + 4\mathbf{b} - 3\mathbf{c}, 3\mathbf{a} + 2\mathbf{b} - 5\mathbf{c}, -3\mathbf{a} + 8\mathbf{b} - 5\mathbf{c}$ *and* $-3\mathbf{a} + 2\mathbf{b} + \mathbf{c}$ *are coplanar.*

Solution: Let A, B, C and D be the points whose position vectors are respectively

$$-\mathbf{a} + 4\mathbf{b} - 3\mathbf{c}, 3\mathbf{a} + 2\mathbf{b} - 5\mathbf{c}, -3\mathbf{a} + 8\mathbf{b} - 5\mathbf{c} \text{ and } -3\mathbf{a} + 2\mathbf{b} + \mathbf{c}.$$

We have $\overrightarrow{AB}$ = position vector of B − position vector of A

$$= 3\mathbf{a} + 2\mathbf{b} - 5\mathbf{c} - (-\mathbf{a} + 4\mathbf{b} - 3\mathbf{c}) = 4\mathbf{a} - 2\mathbf{b} - 2\mathbf{c},$$

$\overrightarrow{AC}$ = position vector of C − position vector of A

$$= (-3\mathbf{a} + 8\mathbf{b} - 5\mathbf{c}) - (-\mathbf{a} + 4\mathbf{b} - 3\mathbf{c}) = -2\mathbf{a} + 4\mathbf{b} - 2\mathbf{c},$$

and $$\overrightarrow{AD} = (-3\mathbf{a} + 2\mathbf{b} + \mathbf{c}) - (-\mathbf{a} + 4\mathbf{b} - 3\mathbf{c}) = -2\mathbf{a} - 2\mathbf{b} + 4\mathbf{c}.$$

Now the scalar triple product of the vectors $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$

$$= [\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}] = \begin{vmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{vmatrix} [\mathbf{a}\ \mathbf{b}\mathbf{c}] \qquad \text{[Refer article 6]}$$

$$= \{4\,(16-4) + 2\,(-8-4) - 2\,(4+8)\}\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$
$$= (48 - 24 - 24)\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0.$$

Since the scalar triple product of the vectors $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ is zero, therefore these vectors are coplanar. Hence the points A, B, C and D are coplanar.

Example 5: *If* **a, b, c** *are the position vectors of* A, B, C *prove that* $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$ *is a vector perpendicular to the plane of* ABC.

Solution: We have $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{BC} = \mathbf{c} - \mathbf{b}$ and $\overrightarrow{CA} = \mathbf{a} - \mathbf{c}$.

Let $$\mathbf{d} = \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}.$$

Now $$\mathbf{d} \bullet \overrightarrow{AB} = \mathbf{d} \bullet (\mathbf{b} - \mathbf{a}) = (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}) \bullet (\mathbf{b} - \mathbf{a})$$

$$= (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{b} - (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{a} + (\mathbf{b} \times \mathbf{c}) \bullet \mathbf{b} - (\mathbf{b} \times \mathbf{c}) \bullet \mathbf{a}$$
$$+ (\mathbf{c} \times \mathbf{a}) \bullet \mathbf{b} - (\mathbf{c} \times \mathbf{a}) \bullet \mathbf{a}$$

$$= [\mathbf{abb}] - [\mathbf{aba}] + [\mathbf{bcb}] - [\mathbf{bca}] + [\mathbf{cab}] - [\mathbf{caa}]$$

$$= -[\mathbf{bca}] + [\mathbf{cab}], \quad \text{since } [\mathbf{abb}] = 0 \text{ etc.}$$

$= -[\mathbf{bca}] + [\mathbf{bca}]$, since $[\mathbf{cab}] = [\mathbf{bca}]$

$= 0.$

Therefore vector **d** is perpendicular to $\overrightarrow{AB}$. Similarly, we can show that **d** is perpendicular to $\overrightarrow{BC}$.

Now since **d** is perpendicular to two lines in the plane ABC, hence it is perpendicular to the plane ABC.

Example 6: *Prove that* $[\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}, \mathbf{c}+\mathbf{a}] = 2\,[\mathbf{abc}]$. **(Kumaun 2010)**

Solution: L.H.S.

$$= (\mathbf{a}+\mathbf{b}) \bullet [(\mathbf{b}+\mathbf{c}) \times (\mathbf{c}+\mathbf{a})]$$

$$= (\mathbf{a}+\mathbf{b}) \bullet [\mathbf{b}\times\mathbf{c} + \mathbf{b}\times\mathbf{a} + \mathbf{c}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}]$$

$$= (\mathbf{a}+\mathbf{b}) \bullet [\mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a} + \mathbf{b}\times\mathbf{a}], \text{ since } \mathbf{c}\times\mathbf{c} = \mathbf{0}$$

$$= \mathbf{a}\bullet(\mathbf{b}\times\mathbf{c}) + \mathbf{a}\bullet(\mathbf{c}\times\mathbf{a}) + \mathbf{a}\bullet(\mathbf{b}\times\mathbf{a}) + \mathbf{b}\bullet(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\bullet(\mathbf{c}\times\mathbf{a}) + \mathbf{b}\bullet(\mathbf{b}\times\mathbf{a})$$

$$= [\mathbf{abc}] + [\mathbf{aca}] + [\mathbf{aba}] + [\mathbf{bbc}] + [\mathbf{bca}] + [\mathbf{bba}] = [\mathbf{abc}] + [\mathbf{bca}],$$

since all the scalar triple products in which two vectors are equal vanish.

But $[\mathbf{abc}] = [\mathbf{bca}]$.

Hence the L.H.S. $= 2\,[\mathbf{abc}]$.

Example 7: *Prove that*

$$[\mathbf{lmn}]\,[\mathbf{abc}] = \begin{vmatrix} \mathbf{l}\bullet\mathbf{a} & \mathbf{l}\bullet\mathbf{b} & \mathbf{l}\bullet\mathbf{c} \\ \mathbf{m}\bullet\mathbf{a} & \mathbf{m}\bullet\mathbf{b} & \mathbf{m}\bullet\mathbf{c} \\ \mathbf{n}\bullet\mathbf{a} & \mathbf{n}\bullet\mathbf{b} & \mathbf{n}\bullet\mathbf{c} \end{vmatrix}.$$

Solution: Let $\mathbf{l} = l_1\mathbf{i} + l_2\mathbf{j} + l_3\mathbf{k}$, $\mathbf{m} = m_1\mathbf{i} + m_2\mathbf{j} + m_3\mathbf{k}$,

$\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$; $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$,

$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$.

Now L.H.S. $= [\mathbf{lmn}]\,[\mathbf{abc}]$

$$= \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1 a_1 + l_2 a_2 + l_3 a_3 & l_1 b_1 + l_2 b_2 + l_3 b_3 & l_1 c_1 + l_2 c_2 + l_3 c_3 \\ m_1 a_1 + m_2 a_2 + m_3 a_3 & m_1 b_1 + m_2 b_2 + m_3 b_3 & m_1 c_1 + m_2 c_2 + m_3 c_3 \\ n_1 a_1 + n_2 a_2 + n_3 a_3 & n_1 b_1 + n_2 b_2 + n_3 b_3 & n_1 c_1 + n_2 c_2 + n_3 c_3 \end{vmatrix}$$

by the rule for the multiplication of determinants of the same order.

Now $\mathbf{l} \bullet \mathbf{a} = (l_1 \ \mathbf{i} + l_2 \ \mathbf{j} + l_3 \ \mathbf{k}) \bullet (a_1 \ \mathbf{i} + a_2 \ \mathbf{j} + a_3 \ \mathbf{k})$

$= l_1 \ a_1 + l_2 \ a_2 + l_3 \ a_3$, etc.

Hence the $$\text{L.H.S.} = \begin{vmatrix} \mathbf{l} \bullet \mathbf{a} & \mathbf{l} \bullet \mathbf{b} & \mathbf{l} \bullet \mathbf{c} \\ \mathbf{m} \bullet \mathbf{a} & \mathbf{m} \bullet \mathbf{b} & \mathbf{m} \bullet \mathbf{c} \\ \mathbf{n} \bullet \mathbf{a} & \mathbf{n} \bullet \mathbf{b} & \mathbf{n} \bullet \mathbf{c} \end{vmatrix} \cdot$$

Example 8: *Prove that if* $\mathbf{l}, \mathbf{m}, \mathbf{n}$ *be three non-coplanar vectors, then*

$$[\mathbf{lmn}] (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} \mathbf{l} \bullet \mathbf{a} & \mathbf{l} \bullet \mathbf{b} & \mathbf{l} \\ \mathbf{m} \bullet \mathbf{a} & \mathbf{m} \bullet \mathbf{b} & \mathbf{m} \\ \mathbf{n} \bullet \mathbf{a} & \mathbf{n} \bullet \mathbf{b} & \mathbf{n} \end{vmatrix} \cdot$$

Solution: Let

$\mathbf{l} = l_1 \mathbf{i} + l_2 \mathbf{j} + l_3 \mathbf{k}, \ \mathbf{m} = m_1 \mathbf{i} + m_2 \mathbf{j} + m_3 \mathbf{k}, \ \mathbf{n} = n_1 \mathbf{i} + n_2 \mathbf{j} + n_3 \mathbf{k},$

and $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}, \ \mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}.$

Now $$[\mathbf{lmn}] = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \text{ and } (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot$$

$$\therefore \quad [\mathbf{lmn}] (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1 \mathbf{i} + l_2 \mathbf{j} + l_3 \mathbf{k} & l_1 a_1 + l_2 a_2 + l_3 a_3 & l_1 b_1 + l_2 b_2 + l_3 b_3 \\ m_1 \mathbf{i} + m_2 \mathbf{j} + m_3 \mathbf{k} & m_1 a_1 + m_2 a_2 + m_3 a_3 & m_1 b_1 + m_2 b_2 + m_3 b_3 \\ n_1 \mathbf{i} + n_2 \mathbf{j} + n_3 \mathbf{k} & n_1 a_1 + n_2 a_2 + n_3 a_3 & n_1 b_1 + n_2 b_2 + n_3 b_3 \end{vmatrix}.$$

Now $\mathbf{l} \bullet \mathbf{a} = (l_1 \ \mathbf{i} + l_2 \ \mathbf{j} + l_3 \ \mathbf{k}) \bullet (a_1 \ \mathbf{i} + a_2 \ \mathbf{j} + a_3 \ \mathbf{k})$

$= l_1 \ a_1 + l_2 \ a_2 + l_3 \ a_3$ etc.

$$\therefore \quad [\mathbf{lmn}] (\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} \mathbf{l} & \mathbf{l} \bullet \mathbf{a} & \mathbf{l} \bullet \mathbf{b} \\ \mathbf{m} & \mathbf{m} \bullet \mathbf{a} & \mathbf{m} \bullet \mathbf{b} \\ \mathbf{n} & \mathbf{n} \bullet \mathbf{a} & \mathbf{n} \bullet \mathbf{b} \end{vmatrix} = \begin{vmatrix} \mathbf{l} \bullet \mathbf{a} & \mathbf{l} \bullet \mathbf{b} & \mathbf{l} \\ \mathbf{m} \bullet \mathbf{a} & \mathbf{m} \bullet \mathbf{b} & \mathbf{m} \\ \mathbf{n} \bullet \mathbf{a} & \mathbf{n} \bullet \mathbf{b} & \mathbf{n} \end{vmatrix} \cdot$$

Comprehensive Exercise 1

1. Define scalar triple product of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and interpret the same geometrically.

2. Define scalar triple product of $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Prove that the value of the scalar triple product of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ remains unchanged if the cyclic order of the vectors is maintained.

3. If **a, b, c** are any three vectors, prove that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

4. Prove that $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$.
5. Show that $\mathbf{i} \cdot \mathbf{j} \times \mathbf{k} = 1$.
6. Show that $[\lambda\mathbf{a} + \mu\mathbf{b}, \mathbf{c}, \mathbf{d}] = \lambda\,[\mathbf{a}, \mathbf{c}, \mathbf{d}] + \mu\,[\mathbf{b}, \mathbf{c}, \mathbf{d}]$.
7. Prove that $[\mathbf{i} - \mathbf{j},\ \mathbf{j} - \mathbf{k},\ \mathbf{k} - \mathbf{i}] = 0$.
8. Find the volume of the parallelopiped whose edges are represented by
   (i) $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k},\ \mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k},\ \mathbf{c} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.
   (ii) $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k},\ \mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k},\ \mathbf{c} = \mathbf{j} + \mathbf{k}$.
9. Show that the vectors $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}, -2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}, \mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ are coplanar.
10. Show that the vectors $2\mathbf{a} - \mathbf{b} + 3\mathbf{c}, \mathbf{a} + \mathbf{b} - 2\mathbf{c}$ and $\mathbf{a} + \mathbf{b} - 3\mathbf{c}$ are non-coplanar where **a**, **b**, **c** are non-coplanar vectors.
    [**Hint:** Show that the scalar triple product of the three given vectors is not zero.]
11. Prove that the four points $6\mathbf{a} - 4\mathbf{b} + 10\mathbf{c}, -5\mathbf{a} + 3\mathbf{b} - 10\mathbf{c}, 4\mathbf{a} - 6\mathbf{b} - 10\mathbf{c}$ and $2\mathbf{b} + 10\mathbf{c}$ are coplanar.
12. Show that the vectors **a**, **b**, **c** are coplanar if $\mathbf{b} + \mathbf{c}, \mathbf{c} + \mathbf{a}, \mathbf{a} + \mathbf{b}$ are coplanar.
13. Prove that four points with position vectors **a**, **b**, **c**, **d** are coplanar if and only if $[\mathbf{b}, \mathbf{c}, \mathbf{d}] + [\mathbf{c}, \mathbf{a}, \mathbf{d}] + [\mathbf{a}, \mathbf{b}, \mathbf{d}] = [\mathbf{a}, \mathbf{b}, \mathbf{c}]$.

Answers 1

8. (i) 7 cubic units, (ii) 12 cubic units.

7 Vector Triple Product

The vector product of two vectors one of which is itself the vector product of two vectors is a vector quantity called a "Vector Triple Product". Thus if **a, b** and **c** be three vectors, the products of the form $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ etc. are called **"Vector Triple Products"**.

Theorem: *To prove that* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$.

Let $\mathbf{r} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and $\mathbf{b} \times \mathbf{c} = \mathbf{d}$.

Since $\mathbf{b} \times \mathbf{c} = \mathbf{d}$, therefore **d** is a vector perpendicular to the plane containing **b** and **c**. Also $\mathbf{r} = \mathbf{a} \times \mathbf{d}$. Therefore **r** is a vector perpendicular to both **a** and **d**. Now the vector **r** is perpendicular to the vector **d**, whereas the vector **d** is perpendicular to the plane containing **b** and **c**. Therefore the vector **r** must lie in the plane containing

b and **c**. Hence the vector **r** can be expressed linearly in terms of **b** and **c** in the form

$$\mathbf{r} = l\mathbf{b} + m\mathbf{c}, \qquad \ldots(1)$$

where l and m are scalars.

Since **r** is perpendicular to **a**, therefore $\mathbf{r} \bullet \mathbf{a} = 0$.

$$\therefore \quad (l\,\mathbf{b} + m\mathbf{c}) \bullet \mathbf{a} = 0 \quad \text{or} \quad l\,(\mathbf{b} \bullet \mathbf{a}) + m\,(\mathbf{c} \bullet \mathbf{a}) = 0.$$

$$\therefore \quad \frac{l}{\mathbf{c} \bullet \mathbf{a}} = \frac{-m}{\mathbf{b} \bullet \mathbf{a}} = \lambda \text{ (say).}$$

Putting the values of l and m in (1), we get

$$\mathbf{r} = \lambda\,(\mathbf{c} \bullet \mathbf{a})\,\mathbf{b} - \lambda\,(\mathbf{b} \bullet \mathbf{a})\,\mathbf{c} = \lambda\,[(\mathbf{c} \bullet \mathbf{a})\,\mathbf{b} - (\mathbf{b} \bullet \mathbf{a})\,\mathbf{c}] \qquad \ldots(2)$$

Now we are to find the value of λ.

Consider unit vectors **j** and **k**, the first parallel to **b** and the second perpendicular to it in the plane containing **b** and **c**. Then we may write $\mathbf{b} = b_2\,\mathbf{j}$ and $\mathbf{c} = c_2\,\mathbf{j} + c_3\,\mathbf{k}$.

In terms of **j** and **k** and the other unit vector **i** of the right handed system, the remaining vector **a** may be written as

$$\mathbf{a} = a_1\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}.$$

Now $\quad \mathbf{b} \times \mathbf{c} = b_2\,\mathbf{j} \times (c_2\,\mathbf{j} + c_3\ \mathbf{k}) = b_2\ c_2\ \ \mathbf{j} \times \mathbf{j} + b_2 c_3\ \ \mathbf{j} \times \mathbf{k} = b_2 c_3\ \ \mathbf{i}$

$$[\because\ \mathbf{j} \times \mathbf{j} = \mathbf{0} \text{ and } \mathbf{j} \times \mathbf{k} = \mathbf{i}]$$

$$\begin{aligned}
\therefore \quad \mathbf{r} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (a_1\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}) \times (b_2 c_3\,\mathbf{i}) \\
&= a_1 b_2 c_3\ \ \mathbf{i} \times \mathbf{i} + a_2 b_2 c_3\ \ \mathbf{j} \times \mathbf{i} + a_3 b_2 c_3\ \ \mathbf{k} \times \mathbf{i} \\
&= a_3 b_2 c_3\ \ \mathbf{j} - a_2 b_2 c_3\ \ \mathbf{k} \qquad \ldots(3)
\end{aligned}$$

$$[\because\ \mathbf{i} \times \mathbf{i} = \mathbf{0},\ \ \mathbf{j} \times \mathbf{i} = -\,\mathbf{k} \text{ and } \mathbf{k} \times \mathbf{i} = \mathbf{j}]$$

$$\begin{aligned}
\text{Also} \quad \mathbf{r} &= \lambda\,[(\mathbf{c} \bullet \mathbf{a})\,\mathbf{b} - (\mathbf{b} \bullet \mathbf{a})\,\mathbf{c}] \\
&= \lambda\,[(c_2\,\mathbf{j} + c_3\mathbf{k}) \bullet (a_1\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k})\,b_2\,\mathbf{j} \\
&\qquad - (b_2\,\mathbf{j}) \bullet (a_1\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k})\,(c_2\,\mathbf{j} + c_3\,\mathbf{k})] \\
&= \lambda\,[c_2 a_2 b_2\,\mathbf{j} + c_3 a_3 b_2\,\mathbf{j} - b_2 a_2 c_2\,\mathbf{j} - b_2 a_2 c_3\,\mathbf{k}] \\
&\qquad [\because\ \mathbf{i} \bullet \mathbf{i} = 1, \mathbf{i} \bullet \mathbf{j} = 0 \text{ etc.}] \\
&= \lambda\,[a_3 b_2 c_3\,\mathbf{j} - a_2 b_2 c_3\,\mathbf{k}]. \qquad \ldots(4)
\end{aligned}$$

Now from (3) and (4) we conclude that $\lambda = 1$.

Hence $\quad \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \bullet \mathbf{a})\,\mathbf{b} - (\mathbf{b} \bullet \mathbf{a})\,\mathbf{c} = (\mathbf{a} \bullet \mathbf{c})\,\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\,\mathbf{c}$

$$[\because\ \mathbf{c} \bullet \mathbf{a} = \mathbf{a} \bullet \mathbf{c}]$$

Corollary: $\quad (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\,[\mathbf{c} \times (\mathbf{a} \times \mathbf{b})]$

$$= -\,[(\mathbf{c} \bullet \mathbf{b})\,\mathbf{a} - (\mathbf{c} \bullet \mathbf{a})\,\mathbf{b}] = (\mathbf{c} \bullet \mathbf{a})\,\mathbf{b} - (\mathbf{c} \bullet \mathbf{b})\,\mathbf{a}.$$

Rule to remember a × (b × c). It is a vector to be expressed linearly in terms of **b** and **c** which are the vectors within the brackets. Also

a × (**b** × **c**) = [Dot product of **a** and **c**] **b**

– [Dot product of **a** and **b**] **c**.

Similarly we may remember (**a** × **b**) × **c**.

8 Vector Triple Product is not Associative

If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be three vectors, then $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ gives a vector which lies in the plane of $\mathbf{b}$ and $\mathbf{c}$ and which is perpendicular to $\mathbf{a}$. Moreover $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ gives a vector which lies in the plane of $\mathbf{a}$ and $\mathbf{b}$ and which is perpendicular to $\mathbf{c}$. Hence, in general, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. Thus in the case of vector triple product the position of brackets cannot be, in general, changed without altering the value of the product.

Illustrative Examples

Example 9: *Prove that* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$.

Solution: We have

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\,\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\,\mathbf{c},\quad \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = (\mathbf{b} \bullet \mathbf{a})\,\mathbf{c} - (\mathbf{b} \bullet \mathbf{c})\,\mathbf{a},$$

and $$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{c} \bullet \mathbf{b})\,\mathbf{a} - (\mathbf{c} \bullet \mathbf{a})\,\mathbf{b}.$$

Adding these three expressions, we get

$$\begin{aligned} &\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \\ &= (\mathbf{a} \bullet \mathbf{c})\,\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\,\mathbf{c} + (\mathbf{b} \bullet \mathbf{a})\,\mathbf{c} - (\mathbf{b} \bullet \mathbf{c})\,\mathbf{a} + (\mathbf{c} \bullet \mathbf{b})\,\mathbf{a} - (\mathbf{c} \bullet \mathbf{a})\,\mathbf{b} \\ &= \mathbf{0}. \qquad [\because\ \mathbf{a} \bullet \mathbf{c} = \mathbf{c} \bullet \mathbf{a},\ \mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a},\ \mathbf{b} \bullet \mathbf{c} = \mathbf{c} \bullet \mathbf{b}] \end{aligned}$$

Example 10: *Show that the vectors* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}),\ \mathbf{b} \times (\mathbf{c} \times \mathbf{a}),\ \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$ *are coplanar.*

Solution: Let $\mathbf{r}_1 = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}),\ \mathbf{r}_2 = \mathbf{b} \times (\mathbf{c} \times \mathbf{a}),\ \mathbf{r}_3 = \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$.

Now first prove that $\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3 = \mathbf{0}$, as we have done in the previous exercise. Since there exists a linear relation between the vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ therefore any of these vectors can be expressed as a linear combination of the other two. Hence these three vectors are coplanar.

Example 11: *If* $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k},\ \mathbf{b} = 2\mathbf{i} + \mathbf{j} + \mathbf{k},\ \mathbf{c} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, *find* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

Solution: We have

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \bullet \mathbf{c})\,\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\,\mathbf{c} \\ &= [(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \bullet (\mathbf{i} + 2\mathbf{j} - \mathbf{k})]\,(2\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &\qquad - [(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \bullet (2\mathbf{i} + \mathbf{j} + \mathbf{k})]\,(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \\ &= (1 - 4 - 1)\,(2\mathbf{i} + \mathbf{j} + \mathbf{k}) - (2 - 2 + 1)\,(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \\ &= (-8\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = -9\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}. \end{aligned}$$

Example 12: *Show that* $\mathbf{i} \times (\mathbf{a} \times \mathbf{i}) + \mathbf{j} \times (\mathbf{a} \times \mathbf{j}) + \mathbf{k} \times (\mathbf{a} \times \mathbf{k}) = 2\mathbf{a}$.

(Kumaun 2009)

Solution: We have

$$\mathbf{i} \times (\mathbf{a} \times \mathbf{i}) = (\mathbf{i} \bullet \mathbf{i})\,\mathbf{a} - (\mathbf{i} \bullet \mathbf{a})\,\mathbf{i} = \mathbf{a} - (\mathbf{i} \bullet \mathbf{a})\,\mathbf{i} \qquad [\because\ \mathbf{i} \bullet \mathbf{i} = 1]$$

$$\mathbf{j} \times (\mathbf{a} \times \mathbf{j}) = (\mathbf{j} \bullet \mathbf{j})\,\mathbf{a} - (\mathbf{j} \bullet \mathbf{a})\,\mathbf{j} = \mathbf{a} - (\mathbf{j} \bullet \mathbf{a})\,\mathbf{j} \qquad [\because\ \mathbf{j} \bullet \mathbf{j} = 1]$$

and $\mathbf{k} \times (\mathbf{a} \times \mathbf{k}) = (\mathbf{k} \bullet \mathbf{k})\,\mathbf{a} - (\mathbf{k} \bullet \mathbf{a})\,\mathbf{k} = \mathbf{a} - (\mathbf{k} \bullet \mathbf{a})\,\mathbf{k}$. $[\because\ \mathbf{k} \bullet \mathbf{k} = 1]$

Adding these three expressions, we get

$$\mathbf{i} \times (\mathbf{a} \times \mathbf{i}) + \mathbf{j} \times (\mathbf{a} \times \mathbf{j}) + \mathbf{k} \times (\mathbf{a} \times \mathbf{k})$$
$$= 3\mathbf{a} - (\mathbf{i} \bullet \mathbf{a})\,\mathbf{i} - (\mathbf{j} \bullet \mathbf{a})\,\mathbf{j} - (\mathbf{k} \bullet \mathbf{a})\,\mathbf{k}$$
$$= 3\mathbf{a} - [(\mathbf{a} \bullet \mathbf{i})\,\mathbf{i} + (\mathbf{a} \bullet \mathbf{j})\,\mathbf{j} + (\mathbf{a} \bullet \mathbf{k})\,\mathbf{k}]. \qquad [\because\ \mathbf{a} \bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{a} \text{ etc.}]$$

Now we shall show that

$$\mathbf{a} = (\mathbf{a} \bullet \mathbf{i})\,\mathbf{i} + (\mathbf{a} \bullet \mathbf{j})\,\mathbf{j} + (\mathbf{a} \bullet \mathbf{k})\,\mathbf{k}.$$

Let $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Taking dot product of both sides with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ successively, we get

$$x = \mathbf{a} \bullet \mathbf{i}, \quad y = \mathbf{a} \bullet \mathbf{j}, \quad z = \mathbf{a} \bullet \mathbf{k}.$$

$\therefore$ $\mathbf{a} = (\mathbf{a} \bullet \mathbf{i})\,\mathbf{i} + (\mathbf{a} \bullet \mathbf{j})\,\mathbf{j} + (\mathbf{a} \bullet \mathbf{k})\,\mathbf{k}$.

Hence $\mathbf{i} \times (\mathbf{a} \times \mathbf{i}) + \mathbf{j} \times (\mathbf{a} \times \mathbf{j}) + \mathbf{k} \times (\mathbf{a} \times \mathbf{k}) = 3\mathbf{a} - \mathbf{a} = 2\mathbf{a}$.

Example 13: *Show that* $\mathbf{i} \times (\mathbf{j} \times \mathbf{k}) = \mathbf{0}$.

Solution: We have $\mathbf{i} \times (\mathbf{j} \times \mathbf{k}) = \mathbf{i} \times \mathbf{i}$ $[\because\ \mathbf{j} \times \mathbf{k} = \mathbf{i}]$

$= \mathbf{0}$. $[\because\ \mathbf{i} \times \mathbf{i} = \mathbf{0}]$

Example 14: *Show that* $[\mathbf{a} \times \mathbf{b},\ \mathbf{b} \times \mathbf{c},\ \mathbf{c} \times \mathbf{a}] = [\mathbf{abc}]^2$, *and express the result by means of determinants.* **(Kumaun 2011)**

Solution: We have

$$[\mathbf{a} \times \mathbf{b},\ \mathbf{b} \times \mathbf{c},\ \mathbf{c} \times \mathbf{a}] = (\mathbf{a} \times \mathbf{b}) \bullet [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})].$$

Let us first find the value of $(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})$.

Let $\mathbf{b} \times \mathbf{c} = \mathbf{d}$.

Then $(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a}) = \mathbf{d} \times (\mathbf{c} \times \mathbf{a}) = (\mathbf{d} \bullet \mathbf{a})\,\mathbf{c} - (\mathbf{d} \bullet \mathbf{c})\,\mathbf{a}$

$$= [(\mathbf{b} \times \mathbf{c}) \bullet \mathbf{a}] \bullet \mathbf{c} - [(\mathbf{b} \times \mathbf{c}) \bullet \mathbf{c}]\,\mathbf{a}$$
$$= [\mathbf{bca}]\,\mathbf{c} - [\mathbf{bcc}]\,\mathbf{a} = [\mathbf{abc}]\,\mathbf{c},$$

since $[\mathbf{bcc}] = 0$ and $[\mathbf{bca}] = [\mathbf{abc}]$.

$\therefore$ $[\mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}] = (\mathbf{a} \times \mathbf{b}) \bullet [\mathbf{abc}]\,\mathbf{c}$

$$= [\mathbf{abc}]\,(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c} = [\mathbf{abc}]\,[\mathbf{abc}] = [\mathbf{abc}]^2.$$

Second part: Let

$$\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k},\quad \mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k},$$
$$\mathbf{c} = c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}.$$

We have $$[\mathbf{abc}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Again $$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2 b_3 - b_2 a_3)\,\mathbf{i} + (b_1 a_3 - a_1 b_3)\,\mathbf{j} + (a_1 b_2 - a_2 b_1)\,\mathbf{k}.$$

Similarly $\quad \mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= (b_2 c_3 - b_3 c_2)\,\mathbf{i} + (c_1 b_3 - b_1 c_3)\,\mathbf{j} + (b_1 c_2 - c_1 b_2)\,\mathbf{k}$$

and $\quad \mathbf{c} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$

$$= (c_2 a_3 - a_2 c_3)\,\mathbf{i} + (a_1 c_3 - a_3 c_1)\,\mathbf{j} + (c_1 a_2 - a_1 c_2)\,\mathbf{k}.$$

$\therefore \quad [\mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}]$

$$= \begin{vmatrix} a_2 b_3 - b_2 a_3 & b_1 a_3 - a_1 b_3 & a_1 b_2 - a_2 b_1 \\ b_2 c_3 - b_3 c_2 & c_1 b_3 - b_1 c_3 & b_1 c_2 - b_2 c_1 \\ c_2 a_3 - c_3 a_2 & a_1 c_3 - a_3 c_1 & c_1 a_2 - c_2 a_1 \end{vmatrix}$$

$$= \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix},$$

where the capital letters A_1, A_2, A_3 etc. denote the cofactors of the corresponding small letters a_1, a_2, a_3 etc. in the determinant $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Since $\quad [\mathbf{abc}]^2 = [\mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}]$,

$$\therefore \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}.$$

Example 15: *Prove that* $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{c}) \bullet \mathbf{d} = (\mathbf{a} \bullet \mathbf{d})\,[\mathbf{abc}]$.

Solution: Let $\mathbf{a} \times \mathbf{b} = \mathbf{r}$.

Then $\quad (\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{c}) = \mathbf{r} \times (\mathbf{a} \times \mathbf{c}) = (\mathbf{r} \bullet \mathbf{c})\,\mathbf{a} - (\mathbf{r} \bullet \mathbf{a})\,\mathbf{c}$

$$= [(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}]\,\mathbf{a} - [(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{a}]\,\mathbf{c}$$

$$= [\mathbf{abc}]\,\mathbf{a} - [\mathbf{aba}]\,\mathbf{c} = [\mathbf{abc}]\,\mathbf{a}, \text{ since } [\mathbf{aba}] = 0.$$

Therefore $\quad (\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{c}) \bullet \mathbf{d} = [\mathbf{abc}]\,\mathbf{a} \bullet \mathbf{d} = (\mathbf{a} \bullet \mathbf{d})\,[\mathbf{abc}]$.

Example 16: *If* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *be three unit vectors such that* $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \frac{1}{2}\mathbf{b}$, *find the angles which* $\mathbf{a}$ *makes with* $\mathbf{b}$ *and* $\mathbf{c}$, $\mathbf{b}$ *and* $\mathbf{c}$ *being non-parallel.* **(Kumaun 2009)**

Solution: It is given that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \frac{1}{2}\mathbf{b}$.

$$\therefore \qquad (\mathbf{a} \bullet \mathbf{c})\,\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\,\mathbf{c} = \frac{1}{2}\,\mathbf{b},$$

$$\text{or} \qquad \left(\mathbf{a} \bullet \mathbf{c} - \frac{1}{2}\right)\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\,\mathbf{c} = \mathbf{0}. \qquad \ldots(1)$$

Since **b** and **c** are non-parallel, therefore for the existence of the relation (1) the coefficients of **b** and **c** should vanish separately. Therefore, we get

$$\mathbf{a} \bullet \mathbf{c} - \frac{1}{2} = 0, \quad \textit{i.e.,} \quad \mathbf{a} \bullet \mathbf{c} = \frac{1}{2} \quad \text{and} \quad \mathbf{a} \bullet \mathbf{b} = 0.$$

Let θ and ϕ be the angles which **a** makes with **b** and **c** respectively. Since **a**, **b**, **c** are unit vectors, we have

$$\mathbf{a} \bullet \mathbf{b} = \cos\theta = 0 \quad \Rightarrow \quad \theta = 90^\circ,$$

$$\text{and} \qquad \mathbf{a} \bullet \mathbf{c} = \cos\phi = \frac{1}{2} \quad \Rightarrow \quad \phi = 60^\circ.$$

Example 17: *Prove that* $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, *if and only if* $(\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = \mathbf{0}$.

Solution: We have $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

if and only if $\quad (\mathbf{c} \bullet \mathbf{a})\,\mathbf{b} - (\mathbf{c} \bullet \mathbf{b})\,\mathbf{a} = (\mathbf{a} \bullet \mathbf{c})\,\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\,\mathbf{c}$

i.e., if and only if $\quad -(\mathbf{c} \bullet \mathbf{b})\,\mathbf{a} = -(\mathbf{a} \bullet \mathbf{b})\,\mathbf{c}$, since $\mathbf{c} \bullet \mathbf{a} = \mathbf{a} \bullet \mathbf{c}$

i.e., if and only if $\quad (\mathbf{c} \bullet \mathbf{b})\,\mathbf{a} - (\mathbf{a} \bullet \mathbf{b})\,\mathbf{c} = \mathbf{0}$

i.e., if and only if $\quad (\mathbf{b} \bullet \mathbf{c})\,\mathbf{a} - (\mathbf{b} \bullet \mathbf{a})\,\mathbf{c} = \mathbf{0}$, since $\mathbf{c} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{c}$ and $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$

i.e., if and only if $\quad (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = \mathbf{0}.$

Note: $(\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = \mathbf{0}$ is possible when (i) **a** and **c** are collinear because then $\mathbf{c} \times \mathbf{a} = \mathbf{0}$ or (ii) **b** is parallel to $\mathbf{c} \times \mathbf{a}$ *i.e.,* **b** is perpendicular to both **c** and **a** or (iii) at least one of the vectors **a, b, c** is a null vector.

Comprehensive Exercise 2

1. Evaluate $(\mathbf{b} \times \mathbf{c}) \times \mathbf{a}$,
   where $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$, and $\mathbf{c} = 4\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$.
2. (i) Verify the formula for vector triple product
   $$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\,\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\,\mathbf{c}$$
   by taking $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{k}$, $\mathbf{c} = \mathbf{j} + \mathbf{k}$.
   (ii) Verify $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\,\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\,\mathbf{c}$ for $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. **(Kumaun 2014)**
3. Prove that $(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a}) = [\mathbf{abc}]\,\mathbf{c}$.
4. Prove that for any three vectors **A**, **B** and **C**,
   $$(\mathbf{A} \times \mathbf{B}) \bullet (\mathbf{B} \times \mathbf{C}) \times (\mathbf{C} \times \mathbf{A}) = (\mathbf{A} \bullet \mathbf{B} \times \mathbf{C})^2.$$

5. Prove that for any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, $[\mathbf{a}\times\mathbf{b}, \mathbf{b}\times\mathbf{c}, \mathbf{c}\times\mathbf{a}] = [\mathbf{abc}]^2$.

Hence show that the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar if and only if the vectors $\mathbf{a}\times\mathbf{b}, \mathbf{b}\times\mathbf{c}, \mathbf{c}\times\mathbf{a}$ are non-coplanar.

6. Show that $[\mathbf{a}\times\mathbf{b}, \mathbf{b}\times\mathbf{c}, \mathbf{c}\times\mathbf{a}] = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} = [\mathbf{abc}]^2$.

[**Hint:** First prove that $[\mathbf{a}\times\mathbf{b}, \mathbf{b}\times\mathbf{c}, \mathbf{c}\times\mathbf{a}] = [\mathbf{abc}]^2$.

For proof see Ex. 6. For the next part proceed as in Ex. 7.]

7. Prove that $\mathbf{a}\times(\mathbf{b}\times\mathbf{a}) = (\mathbf{a}\times\mathbf{b})\times\mathbf{a}$.

Answers 2

1. $8(-4\mathbf{i} + \mathbf{j} - \mathbf{k})$.

9 Scalar Product of Four Vectors

If $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are four vectors, the products $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$, $(\mathbf{a}\times\mathbf{d})\cdot(\mathbf{b}\times\mathbf{c})$ etc. are called scalar products of four vectors.

Theorem: To prove that

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}.$$

Let $\mathbf{a}\times\mathbf{b} = \mathbf{r}$.

Then $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = \mathbf{r}\cdot(\mathbf{c}\times\mathbf{d})$.

Now in a scalar triple product the position of dot and cross may be interchanged without altering the value of the product.

Therefore $\mathbf{r}\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{r}\times\mathbf{c})\cdot\mathbf{d}$.

$$\therefore \quad (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = [(\mathbf{a}\times\mathbf{b})\times\mathbf{c}]\cdot\mathbf{d} = [(\mathbf{c}\cdot\mathbf{a})\,\mathbf{b} - (\mathbf{c}\cdot\mathbf{b})\,\mathbf{a}]\cdot\mathbf{d}$$

$$= (\mathbf{c}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{c}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{d})$$

$$= (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}.$$

This relation is known as **Lagrange's Identity.**

10 Vector Product of Four Vectors

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be four vectors. Consider the vector product of the vectors $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$. This product can be written as $(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})$ and is called the vector product of four vectors. It is a vector perpendicular to $\mathbf{a} \times \mathbf{b}$ and, therefore coplanar with $\mathbf{a}$ and $\mathbf{b}$. Similarly it is a vector coplanar with $\mathbf{c}$ and $\mathbf{d}$. Hence this vector must be parallel to the line of intersection of a plane parallel to $\mathbf{a}$ and $\mathbf{b}$ with another plane parallel to $\mathbf{c}$ and $\mathbf{d}$.

Theorem: *To prove that*

(*i*) $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{abd}]\,\mathbf{c} - [\mathbf{abc}]\,\mathbf{d}$

(*ii*) $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{acd}]\,\mathbf{b} - [\mathbf{bcd}]\,\mathbf{a}$.

Proof:

(i) $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$ is a vector which can be either expressed in terms of $\mathbf{c}$ and $\mathbf{d}$ or in terms of $\mathbf{a}$ and $\mathbf{b}$. To express it in terms of $\mathbf{c}$ and $\mathbf{d}$, let us put $\mathbf{a} \times \mathbf{b} = \mathbf{l}$. Then

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= \mathbf{l} \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{l} \bullet \mathbf{d})\,\mathbf{c} - (\mathbf{l} \bullet \mathbf{c})\,\mathbf{d} \\ &= [(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{d}]\,\mathbf{c} - [(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}]\,\mathbf{d} \\ &= [\mathbf{abd}]\,\mathbf{c} - [\mathbf{abc}]\,\mathbf{d}.\end{aligned}$$

(ii) Again to express $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$ in terms of $\mathbf{a}$ and $\mathbf{b}$, let us put $\mathbf{c} \times \mathbf{d} = \mathbf{m}$.

Then
$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= (\mathbf{a} \times \mathbf{b}) \times \mathbf{m} = -\,\mathbf{m} \times (\mathbf{a} \times \mathbf{b}) \\ &= -\,[(\mathbf{m} \bullet \mathbf{b})\,\mathbf{a} - (\mathbf{m} \bullet \mathbf{a})\,\mathbf{b}] = (\mathbf{m} \bullet \mathbf{a})\,\mathbf{b} - (\mathbf{m} \bullet \mathbf{b})\,\mathbf{a} \\ &= [(\mathbf{c} \times \mathbf{d}) \bullet \mathbf{a}]\,\mathbf{b} - [(\mathbf{c} \times \mathbf{d}) \bullet \mathbf{b}]\,\mathbf{a} \\ &= [\mathbf{cda}]\,\mathbf{b} - [\mathbf{cdb}]\,\mathbf{a} = [\mathbf{acd}]\,\mathbf{b} - [\mathbf{bcd}]\,\mathbf{a}.\end{aligned}$$

Linear Relation connecting four vectors: Equating the above two expressions for the value of $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$, we get

$$[\mathbf{abd}]\,\mathbf{c} - [\mathbf{abc}]\,\mathbf{d} = [\mathbf{acd}]\,\mathbf{b} - [\mathbf{bcd}]\,\mathbf{a}$$

or
$$[\mathbf{bcd}]\,\mathbf{a} - [\mathbf{acd}]\,\mathbf{b} + [\mathbf{abd}]\,\mathbf{c} - [\mathbf{abc}]\,\mathbf{d} = 0, \qquad \ldots(1)$$

which is the required linear relation connecting the four vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$.

To find an expression for any vector r, in space, as a linear combination of three non-coplanar vectors a, b, c.

Replacing $\mathbf{d}$ by $\mathbf{r}$ in the relation (1) just established, we get

$$[\mathbf{bcr}]\,\mathbf{a} - [\mathbf{acr}]\,\mathbf{b} + [\mathbf{abr}]\,\mathbf{c} - [\mathbf{abc}]\,\mathbf{r} = 0$$

or
$$[\mathbf{abc}]\,\mathbf{r} = [\mathbf{bcr}]\,\mathbf{a} - [\mathbf{acr}]\,\mathbf{b} + [\mathbf{abr}]\,\mathbf{c}. \qquad \ldots(2)$$

Since $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar, therefore $[\mathbf{abc}] \neq 0$.
Therefore dividing both sides of (2) by $[\mathbf{abc}]$, we get

$$\mathbf{r} = \frac{[\mathbf{bcr}]\,\mathbf{a} - [\mathbf{acr}]\,\mathbf{b} + [\mathbf{abr}]\,\mathbf{c}}{[\mathbf{abc}]},$$

or
$$\mathbf{r} = \frac{[\mathbf{bcr}]\,\mathbf{a} + [\mathbf{car}]\,\mathbf{b} + [\mathbf{abr}]\,\mathbf{c}}{[\mathbf{abc}]}, \quad \text{since } [\mathbf{acr}] = -\,[\mathbf{car}]$$

or
$$\mathbf{r} = \frac{[\mathbf{rbc}]\,\mathbf{a} + [\mathbf{rca}]\,\mathbf{b} + [\mathbf{rab}]\,\mathbf{c}}{[\mathbf{abc}]}, \qquad \ldots(3)$$

which is the required expression for $\mathbf{r}$.

11 Reciprocal System of Vectors

If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *be any three non-coplanar vectors so that* $[\mathbf{abc}] \neq 0$, *then the three vectors* $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ *defined by the equations*

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{abc}]},\quad \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{abc}]},\quad \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{abc}]}$$

are called reciprocal system of vectors to the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

(i) *To show that* $\mathbf{a} \bullet \mathbf{a}' = \mathbf{b} \bullet \mathbf{b}' = \mathbf{c} \bullet \mathbf{c}' = 1$.

We have $$\mathbf{a} \bullet \mathbf{a}' = \mathbf{a} \bullet \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{abc}]} = \frac{\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})}{[\mathbf{abc}]} = \frac{[\mathbf{abc}]}{[\mathbf{abc}]} = 1.$$

Similarly $$\mathbf{b} \bullet \mathbf{b}' = \mathbf{b} \bullet \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{abc}]} = \frac{\mathbf{b} \bullet (\mathbf{c} \times \mathbf{a})}{[\mathbf{abc}]} = \frac{[\mathbf{bca}]}{[\mathbf{abc}]} = \frac{[\mathbf{abc}]}{[\mathbf{abc}]} = 1$$

and $$\mathbf{c} \bullet \mathbf{c}' = \mathbf{c} \bullet \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{abc}]} = \frac{\mathbf{c} \bullet (\mathbf{a} \times \mathbf{b})}{[\mathbf{abc}]} = \frac{[\mathbf{cab}]}{[\mathbf{abc}]} = \frac{[\mathbf{abc}]}{[\mathbf{abc}]} = 1.$$

Note: The reason for the name reciprocal lies in the relations

$$\mathbf{a} \bullet \mathbf{a}' = \mathbf{b} \bullet \mathbf{b}' = \mathbf{c} \bullet \mathbf{c}' = 1.$$

(ii) *The scalar product of any other pair of vectors, one from each system, is zero i.e.,*

$$\mathbf{a} \bullet \mathbf{b}' = \mathbf{a} \bullet \mathbf{c}' = \mathbf{b} \bullet \mathbf{a}' = \mathbf{b} \bullet \mathbf{c}' = \mathbf{c} \bullet \mathbf{a}' = \mathbf{c} \bullet \mathbf{b}' = 0.$$

We have

$$\mathbf{a} \bullet \mathbf{b}' = \mathbf{a} \bullet \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{abc}]} = \frac{\mathbf{a} \bullet (\mathbf{c} \times \mathbf{a})}{[\mathbf{abc}]} = \frac{[\mathbf{aca}]}{[\mathbf{abc}]} = 0, \quad \text{since } [\mathbf{aca}] = 0.$$

Similarly we can prove the other results.

(iii) *The scalar triple product* $[\mathbf{abc}]$ *formed from three non-coplanar vectors* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *is the reciprocal of the scalar triple product* $[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']$ *formed from the reciprocal system* $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ *i.e.,* $[\mathbf{abc}]\,[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'] = 1$.

We have $$[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'] = \mathbf{a}' \bullet (\mathbf{b}' \times \mathbf{c}') = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{abc}]} \bullet \left\{ \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{abc}]} \times \frac{(\mathbf{a} \times \mathbf{b})}{[\mathbf{abc}]} \right\}$$

$$= \frac{(\mathbf{b} \times \mathbf{c}) \bullet [(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b})]}{[\mathbf{abc}]^3}.$$

Now expanding $(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b})$ by vector triple product treating $\mathbf{c} \times \mathbf{a}$ as one vector, we get

$$(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b}) = [(\mathbf{c} \times \mathbf{a}) \bullet \mathbf{b}]\, \mathbf{a} - [(\mathbf{c} \times \mathbf{a}) \bullet \mathbf{a}]\, \mathbf{b} = [\mathbf{cab}]\, \mathbf{a} - [\mathbf{caa}]\, \mathbf{b}$$

$$= [\mathbf{abc}]\, \mathbf{a}, \quad \text{since } [\mathbf{caa}] = 0$$

and $$[\mathbf{cab}] = [\mathbf{abc}].$$

$\therefore$ $$[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'] = \frac{(\mathbf{b} \times \mathbf{c}) \bullet [\mathbf{abc}]\ \mathbf{a}}{[\mathbf{abc}]^3} = \frac{[(\mathbf{b} \times \mathbf{c}) \bullet \mathbf{a}]\ [\mathbf{abc}]}{[\mathbf{abc}]^3}$$

$$= \frac{[\mathbf{bca}]\,[\mathbf{abc}]}{[\mathbf{abc}]^3} = \frac{[\mathbf{abc}]^2}{[\mathbf{abc}]^3} = \frac{1}{[\mathbf{abc}]}.$$

$\therefore \qquad [\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']\,[\mathbf{abc}] = 1.$

Note 1: Since $[\mathbf{abc}] \neq 0$, therefore from the relation, $[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']\,[\mathbf{abc}] = 1$, we conclude that $[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'] \neq 0$.

Hence the vectors $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are also non-coplanar.

Note 2: The symmetry of results proved in properties (i), (ii) and (iii) suggest that if $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ is the reciprocal system to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ then $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is also the reciprocal system to $\mathbf{a}', \mathbf{b}', \mathbf{c}'$.

Note 3: The relation $[\mathbf{abc}]\,[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}'] = 1$ shows that the scalar triple products $[\mathbf{abc}]$ and $[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']$ are either both positive or both negative. Hence the two systems of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are either both right handed or both left handed.

(iv) The orthonormal vector triads $\mathbf{i}, \mathbf{j}, \mathbf{k}$ *form a self reciprocal system.*

Let $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ be the system of vectors reciprocal to the system $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

Then by definition $\mathbf{i}' = \dfrac{\mathbf{j} \times \mathbf{k}}{[\mathbf{i\ j\ k}]} = \dfrac{\mathbf{i}}{1} = \mathbf{i}$.

Similarly, $\qquad \mathbf{j}' = \mathbf{j}$ and $\mathbf{k}' = \mathbf{k}$.

Hence the result.

Theorem: *If* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *be three non-coplanar vectors and* $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ *constitute the reciprocal system of vectors, then any vector* $\mathbf{r}$ *can be expressed as*

$$\mathbf{r} = (\mathbf{r} \bullet \mathbf{a}')\,\mathbf{a} + (\mathbf{r} \bullet \mathbf{b}')\,\mathbf{b} + (\mathbf{r} \bullet \mathbf{c}')\,\mathbf{c}.$$

Proof: Let $\mathbf{r}$ be expressed as a linear combination of the non-coplanar vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in the form

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c} \qquad \ldots(1)$$

where x, y, z are some scalars.

Multiplying both sides of (1) scalarly with $\mathbf{b} \times \mathbf{c}$, we get

$$\mathbf{r} \bullet (\mathbf{b} \times \mathbf{c}) = x\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) + y\mathbf{b} \bullet (\mathbf{b} \times \mathbf{c}) + z\mathbf{c} \bullet (\mathbf{b} \times \mathbf{c})$$

$$= x\,[\mathbf{abc}] + y\,[\mathbf{bbc}] + z\,[\mathbf{cbc}] = x\,[\mathbf{abc}],$$

since $[\mathbf{bbc}] = 0 = [\mathbf{cbc}]$.

$$\therefore \qquad x = \frac{\mathbf{r} \bullet (\mathbf{b} \times \mathbf{c})}{[\mathbf{abc}]} = \mathbf{r} \bullet \frac{(\mathbf{b} \times \mathbf{c})}{[\mathbf{abc}]} = \mathbf{r} \bullet \mathbf{a}', \text{ since } \mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{abc}]}.$$

Similarly multiplying both sides of (1) scalarly with $\mathbf{c} \times \mathbf{a}$ and $\mathbf{a} \times \mathbf{b}$, we can show that

$$y = \mathbf{r} \bullet \mathbf{b}' \text{ and } z = \mathbf{r} \bullet \mathbf{c}'.$$

Putting the values of x, y and z in (1), we get

$$\mathbf{r} = (\mathbf{r} \bullet \mathbf{a}')\,\mathbf{a} + (\mathbf{r} \bullet \mathbf{b}')\,\mathbf{b} + (\mathbf{r} \bullet \mathbf{c}')\,\mathbf{c}. \qquad \ldots(2)$$

Note 1: In a similar manner, we can prove that

$$\mathbf{r} = (\mathbf{r} \cdot \mathbf{a})\,\mathbf{a}' + (\mathbf{r} \cdot \mathbf{b})\,\mathbf{b}' + (\mathbf{r} \cdot \mathbf{c})\,\mathbf{c}'.$$

Note 2: Since the system of vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ is self-reciprocal, therefore from (2) we conclude that $\mathbf{r} = (\mathbf{r} \cdot \mathbf{i})\,\mathbf{i} + (\mathbf{r} \cdot \mathbf{j})\,\mathbf{j} + (\mathbf{r} \cdot \mathbf{k})\,\mathbf{k}$.

Illustrative Examples

Example 16: *Find a set of vectors reciprocal to the set* $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, $-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

Solution: Let $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$, $\mathbf{c} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

Let $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ be the set of vectors reciprocal to the set $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Then by definition

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{abc}]},\ \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{abc}]},\ \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{abc}]}.$$

Now $$[\mathbf{abc}] = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix} = 2\,(2) - 3\,(0) - 1\,(1) = 3$$

and $$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix} = 2\mathbf{i} + 0\mathbf{j} + \mathbf{k} = 2\,\mathbf{i} + \mathbf{k}.$$

$\therefore$ $$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{abc}]} = \frac{2\mathbf{i} + \mathbf{k}}{3} = \left(\frac{2}{3}\,\mathbf{i} + \frac{1}{3}\,\mathbf{k}\right).$$

Similarly $$\mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{abc}]} = \frac{1}{3}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 2 \\ 2 & 3 & -1 \end{vmatrix} = \frac{-8\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}}{3}$$

and $$\mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{abc}]} = \frac{1}{3}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ 1 & -1 & -2 \end{vmatrix} = \frac{-7\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}}{3}.$$

Example 17: *Prove that* $\mathbf{a} \times \{\mathbf{b} \times (\mathbf{c} \times \mathbf{d})\} = (\mathbf{b} \cdot \mathbf{d})\,(\mathbf{a} \times \mathbf{c}) - (\mathbf{b} \cdot \mathbf{c})\,(\mathbf{a} \times \mathbf{d})$.

Hence expand $\mathbf{a} \times [\mathbf{b} \times \{\mathbf{c} \times (\mathbf{d} \times \mathbf{e})\}]$.

Solution: **First part:** We have

$$\begin{aligned} \mathbf{a} \times \{\mathbf{b} \times (\mathbf{c} \times \mathbf{d})\} &= \mathbf{a} \times \{(\mathbf{b} \cdot \mathbf{d})\,\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\,\mathbf{d}\} \\ &= (\mathbf{b} \cdot \mathbf{d})\,(\mathbf{a} \times \mathbf{c}) - (\mathbf{b} \cdot \mathbf{c})\,(\mathbf{a} \times \mathbf{d}). \end{aligned}$$

Second part: We have

$$\begin{aligned} \mathbf{b} \times \{\mathbf{c} \times (\mathbf{d} \times \mathbf{e})\} &= \mathbf{b} \times \{(\mathbf{c} \cdot \mathbf{e})\,\mathbf{d} - (\mathbf{c} \cdot \mathbf{d})\,\mathbf{e}\} \\ &= (\mathbf{c} \cdot \mathbf{e})\,(\mathbf{b} \times \mathbf{d}) - (\mathbf{c} \cdot \mathbf{d})\,(\mathbf{b} \times \mathbf{e}). \end{aligned}$$

$$\therefore \quad \mathbf{a} \times [\mathbf{b} \times \{\mathbf{c} \times (\mathbf{d} \times \mathbf{e})\}] = \mathbf{a} \times [(\mathbf{c} \bullet \mathbf{e})(\mathbf{b} \times \mathbf{d}) - (\mathbf{c} \bullet \mathbf{d})(\mathbf{b} \times \mathbf{e})]$$
$$= (\mathbf{c} \bullet \mathbf{e})[\mathbf{a} \times (\mathbf{b} \times \mathbf{d})] - (\mathbf{c} \bullet \mathbf{d})[\mathbf{a} \times (\mathbf{b} \times \mathbf{e})]$$
$$= (\mathbf{c} \bullet \mathbf{e})[(\mathbf{a} \bullet \mathbf{d})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{d}]$$
$$- (\mathbf{c} \bullet \mathbf{d})[(\mathbf{a} \bullet \mathbf{e})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{e}].$$

Example 18: *Prove that* $\mathbf{d} \bullet [\mathbf{a} \times \{\mathbf{b} \times (\mathbf{c} \times \mathbf{d})\}] = (\mathbf{b} \bullet \mathbf{d})[\mathbf{acd}]$.

Solution: $\mathbf{a} \times \{\mathbf{b} \times (\mathbf{c} \times \mathbf{d})\} = \mathbf{a} \times \{(\mathbf{b} \bullet \mathbf{d})\mathbf{c} - (\mathbf{b} \bullet \mathbf{c})\mathbf{d}\}$

$$= (\mathbf{b} \bullet \mathbf{d})(\mathbf{a} \times \mathbf{c}) - (\mathbf{b} \bullet \mathbf{c})(\mathbf{a} \times \mathbf{d}).$$

$$\therefore \quad \mathbf{d} \bullet [\mathbf{a} \times \{\mathbf{b} \times (\mathbf{c} \times \mathbf{d})\}] = \mathbf{d} \bullet [(\mathbf{b} \bullet \mathbf{d})(\mathbf{a} \times \mathbf{c}) - (\mathbf{b} \bullet \mathbf{c})(\mathbf{a} \times \mathbf{d})]$$
$$= (\mathbf{b} \bullet \mathbf{d})[\mathbf{d} \bullet (\mathbf{a} \times \mathbf{c})] - (\mathbf{b} \bullet \mathbf{c})[\mathbf{d} \bullet (\mathbf{a} \times \mathbf{d})]$$
$$= (\mathbf{b} \bullet \mathbf{d})[\mathbf{dac}] - (\mathbf{b} \bullet \mathbf{c})[\mathbf{dad}]$$
$$= (\mathbf{b} \bullet \mathbf{d})[\mathbf{acd}],$$

since $[\mathbf{dad}] = 0$ and $[\mathbf{dac}] = [\mathbf{acd}]$.

Example 19: *If the four vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ *are coplanar, show that* $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{0}$.

Solution: $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$. Similarly $\mathbf{c} \times \mathbf{d}$ is a vector perpendicular to the plane containing $\mathbf{c}$ and $\mathbf{d}$.

Since $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are all coplanar, therefore the vectors $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c} \times \mathbf{d}$ are perpendicular to the same plane. Therefore $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c} \times \mathbf{d}$ are parallel.

Now we know that the vector product of two parallel vectors is equal to a zero vector, therefore $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{0}$.

Example 20: *Prove that*

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) + (\mathbf{a} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{d}) \times (\mathbf{b} \times \mathbf{c}) = -2[\mathbf{bcd}]\mathbf{a}.$$

Solution: $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{l} \times (\mathbf{c} \times \mathbf{d})$, where $\mathbf{l} = \mathbf{a} \times \mathbf{b}$

$$= (\mathbf{l} \bullet \mathbf{d})\mathbf{c} - (\mathbf{l} \bullet \mathbf{c})\mathbf{d}$$
$$= [(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{d})]\mathbf{c} - [(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}]\mathbf{d}$$
$$= [\mathbf{abd}]\mathbf{c} - [\mathbf{abc}]\mathbf{d}. \qquad \ldots(1)$$

Again $(\mathbf{a} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{c}) \times \mathbf{m}$, where $\mathbf{m} = \mathbf{d} \times \mathbf{b}$

$$= (\mathbf{m} \bullet \mathbf{a})\mathbf{c} - (\mathbf{m} \bullet \mathbf{c})\mathbf{a}$$
$$= [(\mathbf{d} \times \mathbf{b}) \bullet \mathbf{a}]\mathbf{c} - [(\mathbf{d} \times \mathbf{b}) \bullet \mathbf{c}]\mathbf{a}$$
$$= [\mathbf{dba}]\mathbf{c} - [\mathbf{dbc}]\mathbf{a} = -[\mathbf{abd}]\mathbf{c} - [\mathbf{bcd}]\mathbf{a}, \qquad \ldots(2)$$

since $[\mathbf{dba}] = -[\mathbf{abd}]$, as we have changed the cyclic order of the vectors and $[\mathbf{dbc}] = [\mathbf{bcd}]$, as the cyclic order has been maintained.

Also $(\mathbf{a} \times \mathbf{d}) \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{d}) \times \mathbf{n}$, where $\mathbf{n} = \mathbf{b} \times \mathbf{c}$

$$= (\mathbf{n} \bullet \mathbf{a})\mathbf{d} - (\mathbf{n} \bullet \mathbf{d})\mathbf{a}$$
$$= [(\mathbf{b} \times \mathbf{c}) \bullet \mathbf{a}]\mathbf{d} - [(\mathbf{b} \times \mathbf{c}) \bullet \mathbf{d}]\mathbf{a}$$
$$= [\mathbf{bca}]\mathbf{d} - [\mathbf{bcd}]\mathbf{a} = [\mathbf{abc}]\mathbf{d} - [\mathbf{bcd}]\mathbf{a}. \qquad \ldots(3)$$

Adding (1), (2) and (3), we get

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})+(\mathbf{a}\times\mathbf{c})\times(\mathbf{d}\times\mathbf{b})+(\mathbf{a}\times\mathbf{d})\times(\mathbf{b}\times\mathbf{c})=-2\,[\mathbf{bcd}]\,\mathbf{a}.$$

Example 21: *Prove that*

$$[\mathbf{a}\times\mathbf{p},\mathbf{b}\times\mathbf{q},\mathbf{c}\times\mathbf{r}]+[\mathbf{a}\times\mathbf{q},\mathbf{b}\times\mathbf{r},\mathbf{c}\times\mathbf{p}]+[\mathbf{a}\times\mathbf{r},\mathbf{b}\times\mathbf{p},\mathbf{c}\times\mathbf{q}]=0.$$

(Kumaun 2012)

Solution: We have

$$\begin{aligned}[\mathbf{a}\times\mathbf{p},\mathbf{b}\times\mathbf{q},\mathbf{c}\times\mathbf{r}]&=(\mathbf{a}\times\mathbf{p})\bullet[(\mathbf{b}\times\mathbf{q})\times(\mathbf{c}\times\mathbf{r})]\\&=(\mathbf{a}\times\mathbf{p})\bullet[\{(\mathbf{b}\times\mathbf{q})\bullet\mathbf{r}\}\,\mathbf{c}-\{(\mathbf{b}\times\mathbf{q})\bullet\mathbf{c}\}\,\mathbf{r}]\\&=(\mathbf{a}\times\mathbf{p})\bullet\{[\mathbf{bqr}]\,\mathbf{c}-[\mathbf{bqc}]\,\mathbf{r}\}\\&=[\mathbf{apc}]\,[\mathbf{bqr}]-[\mathbf{apr}]\,[\mathbf{bqc}].\qquad\ldots(1)\end{aligned}$$

Again

$$\begin{aligned}[\mathbf{a}\times\mathbf{q},\mathbf{b}\times\mathbf{r},\mathbf{c}\times\mathbf{p}]&=[\mathbf{b}\times\mathbf{r},\mathbf{c}\times\mathbf{p},\mathbf{a}\times\mathbf{q}]\\&=(\mathbf{b}\times\mathbf{r})\bullet[(\mathbf{c}\times\mathbf{p})\times(\mathbf{a}\times\mathbf{q})]\\&=(\mathbf{b}\times\mathbf{r})\bullet[\{(\mathbf{c}\times\mathbf{p})\bullet\mathbf{q}\}\,\mathbf{a}-\{(\mathbf{c}\times\mathbf{p})\bullet\mathbf{a}\}\,\mathbf{q}]\\&=[\mathbf{bra}]\,[\mathbf{cpq}]-[\mathbf{brq}]\,[\mathbf{cpa}].\qquad\ldots(2)\end{aligned}$$

and

$$\begin{aligned}[\mathbf{a}\times\mathbf{r},\mathbf{b}\times\mathbf{p},\mathbf{c}\times\mathbf{q}]&=[\mathbf{c}\times\mathbf{q},\mathbf{a}\times\mathbf{r},\mathbf{b}\times\mathbf{p}]\\&=(\mathbf{c}\times\mathbf{q})\bullet[(\mathbf{a}\times\mathbf{r})\times(\mathbf{b}\times\mathbf{p})]\\&=(\mathbf{c}\times\mathbf{q})\bullet[\{(\mathbf{a}\times\mathbf{r})\bullet\mathbf{p}\}\,\mathbf{b}-\{(\mathbf{a}\times\mathbf{r})\bullet\mathbf{b}\}\,\mathbf{p}]\\&=[\mathbf{cqb}]\,[\mathbf{arp}]-[\mathbf{cqp}]\,[\mathbf{arb}].\qquad\ldots(3)\end{aligned}$$

Adding (1), (2) and (3) we get

$$\begin{aligned}&[\mathbf{a}\times\mathbf{p},\mathbf{b}\times\mathbf{q},\mathbf{c}\times\mathbf{r}]+[\mathbf{a}\times\mathbf{q},\mathbf{b}\times\mathbf{r},\mathbf{c}\times\mathbf{p}]+[\mathbf{a}\times\mathbf{r},\mathbf{b}\times\mathbf{p},\mathbf{c}\times\mathbf{q}]\\&=[\mathbf{apc}]\,[\mathbf{bqr}]-[\mathbf{apr}]\,[\mathbf{bqc}]+[\mathbf{bra}]\,[\mathbf{cpq}]\\&\qquad-[\mathbf{brq}]\,[\mathbf{cpa}]+[\mathbf{cqb}]\,[\mathbf{arp}]-[\mathbf{cqp}]\,[\mathbf{arb}]\\&=[\mathbf{apc}]\,[\mathbf{bqr}]-[\mathbf{apr}]\,[\mathbf{bqc}]+[\mathbf{bra}]\,[\mathbf{cpq}]\\&\qquad-[\mathbf{bqr}]\,[\mathbf{apc}]+[\mathbf{bqc}]\,[\mathbf{apr}]-[\mathbf{cpq}]\,[\mathbf{bra}]\\&=0,\text{ since }[\mathbf{brq}]=-[\mathbf{bqr}],\ [\mathbf{cpa}]=-[\mathbf{apc}]\text{ etc.}\end{aligned}$$

Example 22: *Prove that* $[\mathbf{a}\times\mathbf{b},\mathbf{c}\times\mathbf{d},\mathbf{e}\times\mathbf{f}]=[\mathbf{abd}]\,[\mathbf{cef}]-[\mathbf{abc}]\,[\mathbf{def}]$

$$=[\mathbf{abe}]\,[\mathbf{fcd}]-[\mathbf{abf}]\,[\mathbf{ecd}]=[\mathbf{cda}]\,[\mathbf{bef}]-[\mathbf{cdb}]\,[\mathbf{aef}].$$

(Kumaun 2012, 15)

Solution: We have $[\mathbf{a}\times\mathbf{b},\mathbf{c}\times\mathbf{d},\mathbf{e}\times\mathbf{f}]=(\mathbf{a}\times\mathbf{b})\bullet[(\mathbf{c}\times\mathbf{d})\times(\mathbf{e}\times\mathbf{f})]$

$$\begin{aligned}&=(\mathbf{a}\times\mathbf{b})\bullet[\mathbf{l}\times(\mathbf{e}\times\mathbf{f})],\text{ where }\mathbf{l}=\mathbf{c}\times\mathbf{d}\\&=(\mathbf{a}\times\mathbf{b})\bullet[(\mathbf{l}\bullet\mathbf{f})\,\mathbf{e}-(\mathbf{l}\bullet\mathbf{e})\,\mathbf{f}]\\&=(\mathbf{a}\times\mathbf{b})\bullet[\{(\mathbf{c}\times\mathbf{d})\bullet\mathbf{f}\}\,\mathbf{e}-\{(\mathbf{c}\times\mathbf{d})\bullet\mathbf{e}\}\,\mathbf{f}]\\&=[\mathbf{cdf}]\,[\mathbf{abe}]-[\mathbf{cde}]\,[\mathbf{abf}]\\&=[\mathbf{abe}]\,[\mathbf{fcd}]-[\mathbf{abf}]\,[\mathbf{ecd}],\text{ since }[\mathbf{cdf}]=[\mathbf{fcd}]\text{ etc.}\end{aligned}$$

Again $$[\mathbf{a}\times\mathbf{b},\mathbf{c}\times\mathbf{d},\mathbf{e}\times\mathbf{f}]=[\mathbf{c}\times\mathbf{d},\mathbf{e}\times\mathbf{f},\mathbf{a}\times\mathbf{b}]$$
$$=(\mathbf{c}\times\mathbf{d})\bullet[(\mathbf{e}\times\mathbf{f})\times(\mathbf{a}\times\mathbf{b})]$$
$$=(\mathbf{c}\times\mathbf{d})\bullet[\{(\mathbf{e}\times\mathbf{f})\bullet\mathbf{b}\}\,\mathbf{a}-\{(\mathbf{e}\times\mathbf{f})\bullet\mathbf{a}\}\,\mathbf{b}]$$
$$=[\mathbf{cda}]\,[\mathbf{efb}]-[\mathbf{cd\ b}]\,[\mathbf{efa}]$$
$$=[\mathbf{cda}]\,[\mathbf{bef}]-[\mathbf{cdb}]\,[\mathbf{aef}].$$

and $$[\mathbf{a}\times\mathbf{b},\mathbf{c}\times\mathbf{d},\mathbf{e}\times\mathbf{f}]=[\mathbf{e}\times\mathbf{f},\mathbf{a}\times\mathbf{b},\mathbf{c}\times\mathbf{d}]$$
$$=(\mathbf{e}\times\mathbf{f})\bullet[(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d})]$$
$$=(\mathbf{e}\times\mathbf{f})\bullet[\{(\mathbf{a}\times\mathbf{b})\bullet\mathbf{d}\}\,\mathbf{c}-\{(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}\}\,\mathbf{d}]$$
$$=[\mathbf{efc}]\,[\mathbf{abd}]-[\mathbf{efd}]\,[\mathbf{abc}]$$
$$=[\mathbf{abd}]\,[\mathbf{cef}]-[\mathbf{abc}]\,[\mathbf{def}].$$

Example 23: *Prove that* $(\mathbf{b}\times\mathbf{c})\bullet(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\bullet(\mathbf{b}\times\mathbf{d})+(\mathbf{a}\times\mathbf{b})\bullet(\mathbf{c}\times\mathbf{d})=0.$

Solution: We have

$$(\mathbf{b}\times\mathbf{c})\bullet(\mathbf{a}\times\mathbf{d})=\begin{vmatrix}\mathbf{b}\bullet\mathbf{a} & \mathbf{b}\bullet\mathbf{d}\\ \mathbf{c}\bullet\mathbf{a} & \mathbf{c}\bullet\mathbf{d}\end{vmatrix}$$
$$=(\mathbf{b}\bullet\mathbf{a})\,(\mathbf{c}\bullet\mathbf{d})-(\mathbf{c}\bullet\mathbf{a})\,(\mathbf{b}\bullet\mathbf{d}) \qquad \ldots(1)$$

Similarly, $$(\mathbf{c}\times\mathbf{a})\bullet(\mathbf{b}\times\mathbf{d})=\begin{vmatrix}\mathbf{c}\bullet\mathbf{b} & \mathbf{c}\bullet\mathbf{d}\\ \mathbf{a}\bullet\mathbf{b} & \mathbf{a}\bullet\mathbf{d}\end{vmatrix}$$
$$=(\mathbf{c}\bullet\mathbf{b})\,(\mathbf{a}\bullet\mathbf{d})-(\mathbf{a}\bullet\mathbf{b})\,(\mathbf{c}\bullet\mathbf{d}) \qquad \ldots(2)$$

and $$(\mathbf{a}\times\mathbf{b})\bullet(\mathbf{c}\times\mathbf{d})=\begin{vmatrix}\mathbf{a}\bullet\mathbf{c} & \mathbf{a}\bullet\mathbf{d}\\ \mathbf{b}\bullet\mathbf{c} & \mathbf{b}\bullet\mathbf{d}\end{vmatrix}$$
$$=(\mathbf{a}\bullet\mathbf{c})\,(\mathbf{b}\bullet\mathbf{d})-(\mathbf{b}\bullet\mathbf{c})\,(\mathbf{a}\bullet\mathbf{d}) \qquad \ldots(3)$$

Adding (1), (2) and (3), we get

$$(\mathbf{b}\times\mathbf{c})\bullet(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\bullet(\mathbf{b}\times\mathbf{d})+(\mathbf{a}\times\mathbf{b})\bullet(\mathbf{c}\times\mathbf{d})=0,$$

since $\mathbf{a}\bullet\mathbf{b}=\mathbf{b}\bullet\mathbf{a}$ etc.

Comprehensive Exercise 3

1. Prove the identity $\mathbf{a}\times[\mathbf{a}\times(\mathbf{a}\times\mathbf{b})]=(\mathbf{a}\bullet\mathbf{a})\,(\mathbf{b}\times\mathbf{a})$. **(Kumaun 2011)**

2. Establish the identity $[\mathbf{a\ b\ c}]\,\mathbf{d}=[\mathbf{b\ c\ d}]\,\mathbf{a}+[\mathbf{c\ a\ d}]\,\mathbf{b}+[\mathbf{a\ b\ d}]\,\mathbf{c}$ for any four vectors $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}$. Hence show that any vector $\mathbf{r}$ can always be expressed as a linear combination of three given non-coplanar vectors.

3. Obtain a set of vectors reciprocal to the three vectors
$$-\mathbf{i}+\mathbf{j}+\mathbf{k},\ \mathbf{i}-\mathbf{j}+\mathbf{k},\ \mathbf{i}+\mathbf{j}+\mathbf{k}.$$

4. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be a set of non-coplanar vectors and

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{abc}]},\ \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{abc}]},\ \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{c}}{[\mathbf{abc}]},$$

then prove that

$$\mathbf{a} = \frac{\mathbf{b}' \times \mathbf{c}'}{[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']},\ \mathbf{b} = \frac{\mathbf{c}' \times \mathbf{a}'}{[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']} \text{ and } \mathbf{c} = \frac{\mathbf{a}' \times \mathbf{b}'}{[\mathbf{a}'\ \mathbf{b}'\ \mathbf{c}']}.$$

5. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$ are reciprocal system of vectors, prove that

(i) $\mathbf{a} \times \mathbf{a}' + \mathbf{b} \times \mathbf{b}' + \mathbf{c} \times \mathbf{c}' = \mathbf{0}$

(ii) $\mathbf{a}' \times \mathbf{b}' + \mathbf{b}' \times \mathbf{c}' + \mathbf{c}' \times \mathbf{a}' = \dfrac{\mathbf{a} + \mathbf{b} + \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$

(iii) $\mathbf{a} \bullet \mathbf{a}' + \mathbf{b} \bullet \mathbf{b}' + \mathbf{c} \bullet \mathbf{c}' = 3.$

Answers 3

3. $-\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{k},\ -\frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k},\ \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}.$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The value of $\mathbf{i} \bullet (\mathbf{j} \times \mathbf{k}) + \mathbf{j} \bullet (\mathbf{k} \times \mathbf{i}) + \mathbf{k} \bullet (\mathbf{i} \times \mathbf{j})$ is

(a) 0
(b) 1
(c) 2
(d) 3

2. The volume of the parallelopiped whose edges are given by

$\vec{OA} = 2\mathbf{i} - 3\mathbf{j},\ \vec{OB} = \mathbf{i} + \mathbf{j} - \mathbf{k},\ \vec{OC} = 3\mathbf{i} - \mathbf{k}$ is

(a) 1
(b) 4
(c) 2/7
(d) None of these

3. If $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ is the scalar triple product of three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ is equal to

(a) $[\mathbf{b}\ \mathbf{a}\ \mathbf{c}]$
(b) $[\mathbf{c}\ \mathbf{b}\ \mathbf{a}]$
(c) $[\mathbf{b}\ \mathbf{c}\ \mathbf{a}]$
(d) $[\mathbf{a}\ \mathbf{c}\ \mathbf{b}]$

4. If $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ constitute an orthogonal right handed triad of unit vectors and $\mathbf{a}$ is any vector, then $\mathbf{i} \times (\mathbf{a} \times \mathbf{i}) + \mathbf{j} \times (\mathbf{a} \times \mathbf{j}) + \mathbf{k} \times (\mathbf{a} \times \mathbf{k})$ is equal to

(a) $\mathbf{a}$ (b) $2\mathbf{a}$

(c) $3\mathbf{a}$ (d) $\mathbf{0}$

5. $[\bar{a}', \bar{b}', \bar{c}']$ is qual to

(a) $[\bar{a}, \bar{b}, \bar{c}]$ (b) $[\bar{a}, \bar{b}, \bar{c}]^2$

(c) $\dfrac{1}{[\bar{a}, \bar{b}, \bar{c}]}$ (d) None of these

(Kumaun 2009)

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. If $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be a set of orthonormal unit vectors, then $[\mathbf{i}\ \mathbf{j}\ \mathbf{k}] =$ …… .

2. If $\vec{A}, \vec{B}, \vec{C}$ be three non-coplanar vectors, then

$$\frac{\vec{A} \bullet \vec{B} \times \vec{C}}{\vec{C} \times \vec{A} \bullet \vec{B}} + \frac{\vec{B} \bullet \vec{A} \times \vec{C}}{\vec{C} \bullet \vec{A} \times \vec{B}} = \ldots\ldots$$

3. For any three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) =$ … .

4. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are reciprocal system of vectors, then

$$\mathbf{a} \bullet \mathbf{a}' + \mathbf{b} \bullet \mathbf{b}' + \mathbf{c} \bullet \mathbf{c}' = \ldots\ldots$$

5. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are any three coplanar vectors, then $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c} =$ …… . **(Kumaun 2008)**

6. $(\bar{a} \times \bar{b}) \times \bar{c} =$ ……… **(Kumaun 2015)**

7. If $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors, then $(\mathbf{i} \times \mathbf{j}) \bullet \mathbf{k} =$ ……… **(Kumaun 2009)**

8. Value of vector $[(\mathbf{b} + \mathbf{c}) \times (\mathbf{c} + \mathbf{a})] =$ ……….. **(Kumaun 2013)**

True or False

Write 'T' for true and 'F' for false statement.

1. If $\mathbf{x} \bullet \mathbf{a} = \mathbf{x} \bullet \mathbf{b} = \mathbf{x} \bullet \mathbf{c} = 0$, for some non-zero vector $\mathbf{x}$, then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$.

2. If $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be a set of orthonormal unit vectors, then $\mathbf{i} \times (\mathbf{j} \times \mathbf{k}) \neq \mathbf{0}$.

3. The orthonormal vector triads $\mathbf{i}, \mathbf{j}, \mathbf{k}$ form a self reciprocal system.

Answers

Multiple Choice Questions

1. (d) **2.** (b) **3.** (c) **4.** (b) **5.** (c)

Fill in the Blank(s)

1. 1 **2.** 0 **3.** 0 **4.** 3 **5.** 0
6. $(\mathbf{c} \bullet \mathbf{a})\,\mathbf{b} - (\mathbf{c} \bullet \mathbf{b})\,\mathbf{a}$ **7.** 1 **8.** $\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{b} \times \mathbf{a}$

True or False

1. *T* **2.** *F* **3.** *T*

□

Chapter 2

Differentiation of Vectors

1 Vector Function

We know that a scalar quantity possesses only magnitude and has no concern with direction. A single real number gives us a complete representation of a scalar quantity. Thus a scalar quantity is nothing but a real number.

Let D be any subset of the set of all real numbers. If to each element t of D, we associate by some rule a unique real number $f(t)$, then this rule defines a **scalar function** of the scalar variable t. Here $f(t)$ is a scalar quantity and thus f is a scalar function.

In a similar manner we define a vector function.

Let D be any subset of the set of all real numbers. If to each element t of D, we associate by some rule a unique vector $\mathbf{f}(t)$, *then this rule defines a* **vector function** *of the scalar variable t. Here* $\mathbf{f}(t)$ *is a vector quantity and thus* $\mathbf{f}$ *is a vector function.*

We know that every vector can be uniquely expressed as a linear combination of three fixed non-coplanar vectors. Therefore we may write

$$\mathbf{f}(t) = f_1(t)\,\mathbf{i} + f_2(t)\,\mathbf{j} + f_3(t)\,\mathbf{k}$$

where **i , j , k** denote a fixed right handed triad of three mutually perpendicular non-coplanar unit vectors.

2 Scalar Fields and Vector Fields

If to each point $P(x, y, z)$ of a region R in space there corresponds a unique scalar $f(P)$, then f is called a **scalar point function** and we say that a **scalar field** f has been defined in R.

Examples: (1) The temperature at any point within or on the surface of earth at a certain time defines a scalar field.

(2) $f(x, y, z) = x^2 - y^3 - 3z^2$ defines a scalar field.

If to each point $P(x, y, z)$ of a region R in space there corresponds a unique vector $\mathbf{f}(P)$, then $\mathbf{f}$ is called a **vector point function** and we say that a **vector field f** has been defined in R.

Examples: (1) If the velocity at any point (x, y, z) of a particle moving in a curve is known at a certain time, then a vector field is defined.

(2) $\mathbf{f}(x, y, z) = xy^2\mathbf{i} + 3yz^3\mathbf{j} - 2x^2z\mathbf{k}$ defines a vector field.

3 Limit and Continuity of a Vector Function

Definition 1: *A vector function $\mathbf{f}(t)$ is said to tend to a limit $\mathbf{l}$, when t tends to t_0, if for any given positive number ε, however small, there corresponds a positive number δ such that*

$$|\mathbf{f}(t) - \mathbf{l}| < \varepsilon \quad \textit{whenever} \quad 0 < |t - t_0| < \delta.$$

If $\mathbf{f}(t)$ tends to a limit $\mathbf{l}$ as t tends to t_0, we write $\lim_{t \to t_0} \mathbf{f}(t) = \mathbf{l}$.

Definition 2: *A vector function $\mathbf{f}(t)$ is said to be continuous for a value t_0 of t if*

(i) $\mathbf{f}(t_0)$ *is defined and*

(ii) for any given positive number ε, however small, there corresponds a positive number δ such that $|\mathbf{f}(t) - \mathbf{f}(t_0)| < \varepsilon$, *whenever* $|t - t_0| < \delta$.

Further a vector function $\mathbf{f}(t)$ is said to be continuous if it is continuous for every value of t for which it has been defined.

We shall give here (without proof) some important results about the limits and continuity of a vector function.

Theorem 1: *The necessary and sufficient condition for a vector function $\mathbf{f}(t)$ to be* continuous at $t = t_0$ is that $\lim_{t \to t_0} \mathbf{f}(t) = \mathbf{f}(t_0)$.

(Purvanchal 2014)

Theorem 2: *If* $\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ *, then* $\mathbf{f}(t)$ *is continuous if and only if* $f_1(t),\ f_2(t),\ f_3(t)$ *are continuous.*

Theorem 3: *Let* $\mathbf{f}(t) = f_1(t)\mathbf{i} + f_2(t)\mathbf{j} + f_3(t)\mathbf{k}$ *and* $\mathbf{l} = l_1\mathbf{i} + l_2\mathbf{j} + l_3\mathbf{k}$.

Then the necessary and sufficient conditions that $\lim\limits_{t \to t_0} \mathbf{f}(t) = \mathbf{l}$ *are*

$$\lim_{t \to t_0} f_1(t) = l_1 , \lim_{t \to t_0} f_2(t) = l_2 \text{ and } \lim_{t \to t_0} f_3(t) = l_3 .$$

Theorem 4: *If* $\mathbf{f}(t), \mathbf{g}(t)$ *are vector functions of scalar variable t and* $\phi(t)$ *is a scalar function of scalar variable t , then*

(i) $\lim\limits_{t \to t_0} [\mathbf{f}(t) \pm \mathbf{g}(t)] = \lim\limits_{t \to t_0} \mathbf{f}(t) \pm \lim\limits_{t \to t_0} \mathbf{g}(t)$

(ii) $\lim\limits_{t \to t_0} [\mathbf{f}(t) \bullet \mathbf{g}(t)] = \left[\lim\limits_{t \to t_0} \mathbf{f}(t)\right] \bullet \left[\lim\limits_{t \to t_0} \mathbf{g}(t)\right]$

(iii) $\lim\limits_{t \to t_0} [\mathbf{f}(t) \times \mathbf{g}(t)] = \left[\lim\limits_{t \to t_0} \mathbf{f}(t)\right] \times \left[\lim\limits_{t \to t_0} \mathbf{g}(t)\right]$

(iv) $\lim\limits_{t \to t_0} [\phi(t)\, \mathbf{f}(t)] = \left[\lim\limits_{t \to t_0} \phi(t)\right]\left[\lim\limits_{t \to t_0} \mathbf{f}(t)\right]$

(v) $\lim\limits_{t \to t_0} |\mathbf{f}(t)| = \left|\lim\limits_{t \to t_0} \mathbf{f}(t)\right|$.

4 Derivative of a Vector Function with Respect to a Scalar

Definition: *Let* $\mathbf{r} = \mathbf{f}(t)$ *be a vector function of the scalar variable t. We define*

$$\mathbf{r} + \delta\mathbf{r} = \mathbf{f}(t + \delta t).$$

$$\therefore \quad \delta\mathbf{r} = \mathbf{f}(t + \delta t) - \mathbf{f}(t).$$

Consider the vector $\dfrac{\delta\mathbf{r}}{\delta t} = \dfrac{\mathbf{f}(t + \delta t) - \mathbf{f}(t)}{\delta t}$.

If $\lim\limits_{\delta t \to 0} \dfrac{\delta\mathbf{r}}{\delta t} = \lim\limits_{\delta t \to 0} \dfrac{\mathbf{f}(t + \delta t) - \mathbf{f}(t)}{\delta t}$ *exists, then the value of this limit, which we shall denote by* $\dfrac{d\mathbf{r}}{dt}$ *, is called the derivative of the vector function* **r** *with respect to the scalar t. Symbolically*

$$\frac{d\mathbf{r}}{dt} = \lim_{\delta t \to 0} \frac{(\mathbf{r} + \delta\mathbf{r}) - \mathbf{r}}{\delta t} = \lim_{\delta t \to 0} \frac{\mathbf{f}(t + \delta t) - \mathbf{f}(t)}{\delta t} .$$

If $\dfrac{d\mathbf{r}}{dt}$ exists, then **r** is said to be differentiable. Since $\dfrac{\delta\mathbf{r}}{\delta t}$ is a vector quantity, therefore $\dfrac{d\mathbf{r}}{dt}$ is also a vector quantity.

Successive Derivatives: If we differentiate $\frac{d\mathbf{r}}{dt}$ again, we get $\frac{d^2\mathbf{r}}{dt^2}$ which is called the second derivative of $\mathbf{r}$ w.r.t. t , and so on.

Thus, we continue differentiating successively upto n times and get

$$\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}, \frac{d^3\mathbf{r}}{dt^3}, \ldots\ldots, \frac{d^n\mathbf{r}}{dt^n},$$

where $\frac{d^n\mathbf{r}}{dt^n}$ is called the n^{th} diff. coeff. (or derivative) of $\mathbf{r}$ w.r.t. t. (Here diff. coeff. means differential coefficient.)

We often represent $\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}, \ldots\ldots$ by $\dot{\mathbf{r}}$, $\ddot{\mathbf{r}}$, …… respectively.

A scalar or vector function of t is called differentiable of order n if its n^{th} order derivative exists.

5 Differentiation Formulae

Theorem: *If* **a**, **b** *and* **c** *are differentiable vector functions of a scalar t and* ϕ *is a differentiable scalar function of the same variable t, then*

1. $\frac{d}{dt}(\mathbf{a}+\mathbf{b}) = \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{b}}{dt}$
2. $\frac{d}{dt}(\mathbf{a}\bullet\mathbf{b}) = \mathbf{a}\bullet\frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt}\bullet\mathbf{b}$
3. $\frac{d}{dt}(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times\frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt}\times\mathbf{b}$
4. $\frac{d}{dt}(\phi\mathbf{a}) = \phi\frac{d\mathbf{a}}{dt} + \frac{d\phi}{dt}\mathbf{a}$
5. $\frac{d}{dt}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \left[\frac{d\mathbf{a}}{dt}\ \mathbf{b}\ \mathbf{c}\right] + \left[\mathbf{a}\ \frac{d\mathbf{b}}{dt}\ \mathbf{c}\right] + \left[\mathbf{a}\ \mathbf{b}\ \frac{d\mathbf{c}}{dt}\right]$ **(Meerut 2013B; Purvanchal 08)**
6. $\frac{d}{dt}\{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\} = \frac{d\mathbf{a}}{dt}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{a}\times\left(\frac{d\mathbf{b}}{dt}\times\mathbf{c}\right) + \mathbf{a}\times\left(\mathbf{b}\times\frac{d\mathbf{c}}{dt}\right)\cdot$

(Meerut 2000; Purvanchal 08)

Proof: 1. $\frac{d}{dt}(\mathbf{a}+\mathbf{b}) = \lim_{\delta t\to 0}\frac{\{(\mathbf{a}+\delta\mathbf{a})+(\mathbf{b}+\delta\mathbf{b})\}-(\mathbf{a}+\mathbf{b})}{\delta t}$

$$= \lim_{\delta t\to 0}\frac{\delta\mathbf{a}+\delta\mathbf{b}}{\delta t} = \lim_{\delta t\to 0}\left(\frac{\delta\mathbf{a}}{\delta t}+\frac{\delta\mathbf{b}}{dt}\right)$$

$$= \lim_{\delta t\to 0}\frac{\delta\mathbf{a}}{\delta t} + \lim_{\delta t\to 0}\frac{\delta\mathbf{b}}{dt} = \frac{d\,\mathbf{a}}{dt} + \frac{d\,\mathbf{b}}{dt}\cdot$$

Thus the derivative of the sum of two vectors is equal to the sum of their derivatives, as it is also in Scalar Calculus.

Similarly we can prove that $\frac{d}{dt}(\mathbf{a}-\mathbf{b})=\frac{d\mathbf{a}}{dt}-\frac{d\mathbf{b}}{dt}\cdot$

In general if $\mathbf{r}_1$, $\mathbf{r}_2$,……, $\mathbf{r}_n$ are vector functions of a scalar t, then

$$\frac{d}{dt}(\mathbf{r}_1+\mathbf{r}_2+\ldots\ldots+\mathbf{r}_n)=\frac{d\mathbf{r}_1}{dt}+\frac{d\mathbf{r}_2}{dt}+\ldots\ldots+\frac{d\mathbf{r}_n}{dt}\cdot$$

2. $$\frac{d}{dt}(\mathbf{a}\bullet\mathbf{b})=\lim_{\delta t\to 0}\frac{(\mathbf{a}+\delta\mathbf{a})\bullet(\mathbf{b}+\delta\mathbf{b})-\mathbf{a}\bullet\mathbf{b}}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{\mathbf{a}\bullet\mathbf{b}+\mathbf{a}\bullet\delta\mathbf{b}+\delta\mathbf{a}\bullet\mathbf{b}+\delta\mathbf{a}\bullet\delta\mathbf{b}-\mathbf{a}\bullet\mathbf{b}}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{\mathbf{a}\bullet\delta\mathbf{b}+\delta\mathbf{a}\bullet\mathbf{b}+\delta\mathbf{a}\bullet\delta\mathbf{b}}{\delta t}$$

$$=\lim_{\delta t\to 0}\left\{\mathbf{a}\bullet\frac{\delta\mathbf{b}}{\delta t}+\frac{\delta\mathbf{a}}{\delta t}\bullet\mathbf{b}+\frac{\delta\mathbf{a}}{\delta t}\bullet\delta\mathbf{b}\right\}$$

$$=\lim_{\delta t\to 0}\mathbf{a}\bullet\frac{\delta\mathbf{b}}{\delta t}+\lim_{\delta t\to 0}\frac{\delta\mathbf{a}}{\delta t}\bullet\mathbf{b}+\lim_{\delta t\to 0}\frac{\delta\mathbf{a}}{\delta t}\bullet\delta\mathbf{b}$$

$$=\mathbf{a}\bullet\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\bullet\mathbf{b}+\frac{d\mathbf{a}}{dt}\bullet\mathbf{0},\qquad\text{since}\,\delta\mathbf{b}\to\text{ zero vector as}\,\delta t\to 0$$

$$=\mathbf{a}\bullet\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\bullet\mathbf{b}$$

Note: We know that $\mathbf{a}\bullet\mathbf{b}=\mathbf{b}\bullet\mathbf{a}$. Therefore while evaluating $\frac{d}{dt}(\mathbf{a}\bullet\mathbf{b})$, we should not bother about the order of the factors.

3. $$\frac{d}{dt}(\mathbf{a}\times\mathbf{b})=\lim_{\delta t\to 0}\frac{(\mathbf{a}+\delta\mathbf{a})\times(\mathbf{b}+\delta\mathbf{b})-\mathbf{a}\times\mathbf{b}}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\delta\mathbf{b}+\delta\mathbf{a}\times\mathbf{b}+\delta\mathbf{a}\times\delta\mathbf{b}-\mathbf{a}\times\mathbf{b}}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{\mathbf{a}\times\delta\mathbf{b}+\delta\mathbf{a}\times\mathbf{b}+\delta\mathbf{a}\times\delta\mathbf{b}}{\delta t}$$

$$=\lim_{\delta t\to 0}\left\{\mathbf{a}\times\frac{\delta\mathbf{b}}{\delta t}+\frac{\delta\mathbf{a}}{\delta t}\times\mathbf{b}+\frac{\delta\mathbf{a}}{\delta t}\times\delta\mathbf{b}\right\}$$

$$=\lim_{\delta t\to 0}\mathbf{a}\times\frac{\delta\mathbf{b}}{\delta t}+\lim_{\delta t\to 0}\frac{\delta\mathbf{a}}{\delta t}\times\mathbf{b}+\lim_{\delta t\to 0}\frac{\delta\mathbf{a}}{\delta t}\times\delta\mathbf{b}$$

$$=\mathbf{a}\times\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\times\mathbf{b}+\frac{d\mathbf{a}}{dt}\times\mathbf{0}\,,\ \text{since }\delta\mathbf{b}\to\text{ zero vector as }\delta t\to 0$$

$$=\mathbf{a}\times\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\times\mathbf{b}+\mathbf{0}=\mathbf{a}\times\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\times\mathbf{b}\,.$$

Note: We know that cross product of two vectors is not commutative because $\mathbf{a} \times \mathbf{b} = -\, \mathbf{b} \times \mathbf{a}$. Therefore while evaluating $\frac{d}{dt}(\mathbf{a} \times \mathbf{b})$, we must maintain the order of the factors **a** and **b**.

4. $$\frac{d}{dt}(\phi\mathbf{a}) = \lim_{\delta t \to 0} \frac{(\phi + \delta\phi)(\mathbf{a} + \delta\mathbf{a}) - \phi\mathbf{a}}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\phi\mathbf{a} + \phi\delta\mathbf{a} + \delta\phi\mathbf{a} + \delta\phi\delta\mathbf{a} - \phi\mathbf{a}}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\phi\delta\mathbf{a} + \delta\phi\mathbf{a} + \delta\phi\delta\mathbf{a}}{\delta t}$$

$$= \lim_{\delta t \to 0} \left\{\phi \frac{\delta\mathbf{a}}{\delta t} + \frac{\delta\phi}{\delta t}\mathbf{a} + \frac{\delta\phi}{\delta t}\delta\mathbf{a}\right\}$$

$$= \lim_{\delta t \to 0} \phi \frac{\delta\mathbf{a}}{\delta t} + \lim_{\delta t \to 0} \frac{\delta\phi}{\delta t}\mathbf{a} + \lim_{\delta t \to 0} \frac{\delta\phi}{\delta t}\delta\mathbf{a}$$

$$= \phi \frac{d\mathbf{a}}{dt} + \frac{d\phi}{dt}\mathbf{a} + \frac{d\phi}{dt}\mathbf{0}, \text{ since } \delta\mathbf{a} \to \text{zero vector as } \delta t \to 0$$

$$= \phi \frac{d\mathbf{a}}{dt} + \frac{d\phi}{dt}\mathbf{a} + \mathbf{0} = \phi\frac{d\mathbf{a}}{dt} + \frac{d\phi}{dt}\mathbf{a}.$$

Note: $\phi\mathbf{a}$ is the multiplication of a vector by a scalar. In the case of such multiplication we usually write the scalar in the first position and the vector in the second position.

5. $$\frac{d}{dt}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \frac{d}{dt}\{\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})\} = \mathbf{a} \bullet \frac{d}{dt}(\mathbf{b} \times \mathbf{c}) + \frac{d\mathbf{a}}{dt} \bullet (\mathbf{b} \times \mathbf{c}) \qquad \text{[by rule (2)]}$$

$$= \mathbf{a} \bullet \left(\mathbf{b} \times \frac{d\mathbf{c}}{dt} + \frac{d\mathbf{b}}{dt} \times \mathbf{c}\right) + \frac{d\mathbf{a}}{dt} \bullet (\mathbf{b} \times \mathbf{c}) \qquad \text{[by rule (3)]}$$

$$= \mathbf{a} \bullet \left(\mathbf{b} \times \frac{d\mathbf{c}}{dt}\right) + \mathbf{a} \bullet \left(\frac{d\mathbf{b}}{dt} \times \mathbf{c}\right) + \frac{d\mathbf{a}}{dt} \bullet (\mathbf{b} \times \mathbf{c})$$

$$= \left[\mathbf{a}\ \mathbf{b} \frac{d\mathbf{c}}{dt}\right] + \left[\mathbf{a} \frac{d\mathbf{b}}{dt}\ \mathbf{c}\right] + \left[\frac{d\mathbf{a}}{dt}\ \mathbf{b}\ \mathbf{c}\right]$$

$$= \left[\frac{d\mathbf{a}}{dt}\ \mathbf{b}\ \mathbf{c}\right] + \left[\mathbf{a} \frac{d\mathbf{b}}{dt}\ \mathbf{c}\right] + \left[\mathbf{a}\ \mathbf{b} \frac{d\mathbf{c}}{dt}\right].$$

Note: Here **[abc]** is the scalar triple product of three vectors **a** , **b**and **c**. Therefore while evaluating $\frac{d}{dt}$ **[abc]** we must maintain the cyclic order of each factor.

6. $$\frac{d}{dt}\{\mathbf{a} \times (\mathbf{b} \times \mathbf{c})\} = \mathbf{a} \times \frac{d}{dt}(\mathbf{b} \times \mathbf{c}) + \frac{d\mathbf{a}}{dt} \times (\mathbf{b} \times \mathbf{c}) \qquad \text{[by rule (3)]}$$

$$= \mathbf{a} \times \left(\frac{d\mathbf{b}}{dt} \times \mathbf{c} + \mathbf{b} \times \frac{d\mathbf{c}}{dt}\right) + \frac{d\mathbf{a}}{dt} \times (\mathbf{b} \times \mathbf{c})$$

$$= \mathbf{a} \times \left(\frac{d\mathbf{b}}{dt} \times \mathbf{c} \right) + \mathbf{a} \times \left(\mathbf{b} \times \frac{d\mathbf{c}}{dt} \right) + \frac{d\mathbf{a}}{dt} \times (\mathbf{b} \times \mathbf{c})$$

$$= \frac{d\mathbf{a}}{dt} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{a} \times \left(\frac{d\mathbf{b}}{dt} \times \mathbf{c} \right) + \mathbf{a} \times \left(\mathbf{b} \times \frac{d\mathbf{c}}{dt} \right).$$

6 Derivative of a Function of a Function

Suppose **r** is a differentiable vector function of a scalar variable s and s is a differentiable scalar function of another scalar variable t. Then **r** is a function of t.

An increment δt in t produces an increment $\delta \mathbf{r}$ in **r** and an increment δs in s. When $\delta t \to 0$, $\delta \mathbf{r} \to \mathbf{0}$ and $\delta s \to 0$.

We have $$\frac{d\mathbf{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \lim_{\delta t \to 0} \left(\frac{\delta s}{\delta t} \frac{\delta \mathbf{r}}{\delta s} \right)$$

$$= \left(\lim_{\delta t \to 0} \frac{\delta s}{\delta t} \right) \left(\lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta s} \right) = \frac{ds}{dt} \frac{d\mathbf{r}}{ds}.$$

7 Derivative of a Constant Vector

A vector is said to be constant only if both its magnitude and direction are fixed. If either of these changes then the vector will change and thus it will not be constant.

Let **r** be a constant vector function of the scalar variable t. Let $\mathbf{r} = \mathbf{c}$, where **c** is a constant vector. Then $\mathbf{r} + \delta \mathbf{r} = \mathbf{c}$.

$\therefore$ $\delta \mathbf{r} = \mathbf{0}$ (zero vector). $\qquad \therefore$ $\dfrac{\delta \mathbf{r}}{\delta t} = \dfrac{\mathbf{0}}{\delta t} = \mathbf{0}$.

$\therefore$ $\lim\limits_{\delta t \to 0} \dfrac{\delta \mathbf{r}}{\delta t} = \lim\limits_{\delta t \to 0} \mathbf{0} = \mathbf{0}$. $\qquad \therefore$ $\dfrac{d\mathbf{r}}{dt} = \mathbf{0}$ (zero vector).

Thus the derivative of a constant vector is equal to the null vector.

8 Derivative of a Vector Function in Terms of its Components

Let **r** be a vector function of the scalar variable t. **(Avadh 2014)**

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where the components x, y, z are scalar functions of the scalar variable t and **i, j, k** are fixed unit vectors.

We have $\mathbf{r} + \delta \mathbf{r} = (x + \delta x)\, \mathbf{i} + (y + \delta y)\, \mathbf{j} + (z + \delta z)\, \mathbf{k}$.

$\therefore$ $\delta \mathbf{r} = (\mathbf{r} + \delta \mathbf{r}) - \mathbf{r} = \delta x\, \mathbf{i} + \delta y \mathbf{j} + \delta z \mathbf{k}$.

$\therefore$ $$\frac{\delta \mathbf{r}}{\delta t} = \frac{\delta x}{\delta t} \mathbf{i} + \frac{\delta y}{\delta t} \mathbf{j} + \frac{\delta z}{\delta t} \mathbf{k}.$$

$\therefore$ $$\lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \lim_{\delta t \to 0} \left\{ \frac{\delta x}{\delta t} \mathbf{i} + \frac{\delta y}{\delta t} \mathbf{j} + \frac{\delta z}{\delta t} \mathbf{k} \right\}.$$

$$\therefore \qquad \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}.$$

Thus in order to differentiate a vector we should differentiate its components.

Note: If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then sometimes we also write it as $\mathbf{r} = (x, y, z)$. In this notation $\dfrac{d\mathbf{r}}{dt} = \left(\dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt}\right), \dfrac{d^2\mathbf{r}}{dt^2} = \left(\dfrac{d^2x}{dt^2}, \dfrac{d^2y}{dt^2}, \dfrac{d^2z}{dt^2}\right)$, and so on.

Alternative Method:

We have $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are constant vectors and so their derivatives will be zero.

Now,
$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{d}{dt}(x\mathbf{i}) + \frac{d}{dt}(y\mathbf{j}) + \frac{d}{dt}(z\mathbf{k})$$
$$= \frac{dx}{dt}\,\mathbf{i} + x\frac{d\mathbf{i}}{dt} + \frac{dy}{dt}\,\mathbf{j} + y\frac{d\mathbf{j}}{dt} + \frac{dz}{dt}\,\mathbf{k} + z\frac{d\mathbf{k}}{dt}$$
$$= \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}\text{, since } \frac{d\mathbf{i}}{dt}\text{, etc. vanish.}$$

9 Some Important Results

Theorem 1: *The necessary and sufficient condition for the vector function* $\mathbf{a}(t)$ *to be constant is that* $\dfrac{d\mathbf{a}}{dt} = \mathbf{0}.$

(Purvanchal 2014)

Proof: **The condition is necessary.** Let $\mathbf{a}(t)$ be a constant vector function of the scalar variable t. Then $\mathbf{a}(t + \delta t) = \mathbf{a}(t)$. We have

$$\frac{d\mathbf{a}}{dt} = \lim_{\delta t \to 0} \frac{\mathbf{a}(t + \delta t) - \mathbf{a}(t)}{\delta t} = \lim_{\delta t \to 0} \frac{\mathbf{0}}{\delta t} = \mathbf{0}.$$

Therefore the condition is necessary.

The condition is sufficient. Let $\dfrac{d\mathbf{a}}{dt} = \mathbf{0}.$ Then to prove that $\mathbf{a}$ is a constant vector. Let

$$\mathbf{a}(t) = a_1(t)\,\mathbf{i} + a_2(t)\,\mathbf{j} + a_3(t)\,\mathbf{k}.$$

Then
$$\frac{d\mathbf{a}}{dt} = \frac{da_1}{dt}\,\mathbf{i} + \frac{da_2}{dt}\,\mathbf{j} + \frac{da_3}{dt}\,\mathbf{k}.$$

Therefore $\dfrac{d\mathbf{a}}{dt} = \mathbf{0}$ gives, $\dfrac{da_1}{dt}\,\mathbf{i} + \dfrac{da_2}{dt}\,\mathbf{j} + \dfrac{da_3}{dt}\,\mathbf{k} = \mathbf{0}.$

Equating to zero the coefficients of **i, j** and **k**, we get

$$\frac{da_1}{dt} = 0, \frac{da_2}{dt} = 0, \frac{da_3}{dt} = 0.$$

Hence a_1, a_2, a_3 are constant scalars *i.e.*, they are independent of t. Therefore $\mathbf{a}(t)$ is a constant vector function.

Theorem 2: *If* **a** *is a differentiable vector function of the scalar variable t and if* $|\mathbf{a}| = a$, *then*

(i) $\dfrac{d}{dt}(\mathbf{a}^2) = 2a\dfrac{da}{dt}$; *(ii)* $\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = a\dfrac{da}{dt}\cdot$

Proof: (i) We have $\mathbf{a}^2 = \mathbf{a} \bullet \mathbf{a} = (a)(a)\cos 0 = a^2$.

Therefore $\dfrac{d}{dt}(\mathbf{a}^2) = \dfrac{d}{dt}(a^2) = 2a\dfrac{da}{dt}\cdot$

(ii) We have $\dfrac{d}{dt}(\mathbf{a}^2) = \dfrac{d}{dt}(\mathbf{a} \bullet \mathbf{a}) = \dfrac{d\mathbf{a}}{dt} \bullet \mathbf{a} + \mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = 2\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt}\cdot$

Also $\dfrac{d}{dt}(\mathbf{a}^2) = \dfrac{d}{dt}(a^2) = 2a\dfrac{da}{dt}\cdot$

$\therefore$ $2\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = 2a\dfrac{da}{dt}$ or $\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = a\dfrac{da}{dt}\cdot$

Theorem 3: *If* **a** *has constant length (fixed magnitude), then* **a** *and* $\dfrac{d\mathbf{a}}{dt}$ *are perpendicular provided* $\left|\dfrac{d\mathbf{a}}{dt}\right| \neq 0.$

Proof: Let $|\mathbf{a}| = a$ = constant. Then $\mathbf{a} \bullet \mathbf{a} = a^2$ = constant.

$\therefore$ $\dfrac{d}{dt}(\mathbf{a} \bullet \mathbf{a}) = 0$ or $\dfrac{d\mathbf{a}}{dt} \bullet \mathbf{a} + \mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = 0$

or $2\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = 0$ or $\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = 0.$

Thus the scalar product of two vectors **a** and $\dfrac{d\mathbf{a}}{dt}$ is zero.

Therefore **a** is perpendicular to $\dfrac{d\mathbf{a}}{dt}$ provided $\dfrac{d\mathbf{a}}{dt}$ is not null vector *i.e.*, provided

$$\left|\frac{d\mathbf{a}}{dt}\right| \neq 0.$$

Thus the derivative of a vector of constant length is perpendicular to the vector provided the vector itself is not constant.

Theorem 4: *The necessary and sufficient condition for the vector* **a** (t) *to have constant magnitude is* $\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = 0.$ **(Meerut 2001, 04B, 06, 07, 11, 13; Avadh 13; Kashi 10, 12; Purvanchal 11)**

Proof: Let **a** be a vector function of the scalar variable t. Let $|\mathbf{a}| = a$ = constant. Then $\mathbf{a} \bullet \mathbf{a} = a^2$ = constant.

$\therefore$ $\dfrac{d}{dt}(\mathbf{a} \bullet \mathbf{a}) = 0$ or $\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} + \dfrac{d\mathbf{a}}{dt} \bullet \mathbf{a} = 0$

or $2\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = 0$ or $\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = 0.$

Therefore the condition is necessary.

Condition is sufficient. If $\mathbf{a} \bullet \dfrac{d\mathbf{a}}{dt} = 0$, then

$$\mathbf{a} \bullet \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{a}}{dt} \bullet \mathbf{a} = 0 \quad \text{or} \quad \frac{d}{dt}(\mathbf{a} \bullet \mathbf{a}) = 0$$

or $\mathbf{a} \bullet \mathbf{a} = \text{constant}$ or $\mathbf{a}^2 = \text{constant}$

or $a^2 = \text{constant}$ or $|\mathbf{a}| = \text{constant}$

Theorem 5: *If* **a** *is a differentiable vector function of the scalar variable t, then*

$$\frac{d}{dt}\left(\mathbf{a} \times \frac{d\mathbf{a}}{dt}\right) = \mathbf{a} \times \frac{d^2\mathbf{a}}{dt^2}.$$

Proof: We have

$$\frac{d}{dt}\left(\mathbf{a} \times \frac{d\mathbf{a}}{dt}\right) = \frac{d\mathbf{a}}{dt} \times \frac{d\mathbf{a}}{dt} + \mathbf{a} \times \frac{d^2\mathbf{a}}{dt^2} = \mathbf{0} + \mathbf{a} \times \frac{d^2\mathbf{a}}{dt^2},$$

since the cross product of two equal vectors $\dfrac{d\mathbf{a}}{dt}$ is zero $= \mathbf{a} \times \dfrac{d^2\mathbf{a}}{dt^2}$.

Theorem 6: *The necessary and sufficient condition for the vector* **a** (t) *to have constant direction is* $\mathbf{a} \times \dfrac{d\mathbf{a}}{dt} = \mathbf{0}$.

(Meerut 2004B, 06, 07, 10, 11, 13B; Avadh 14)

Proof: Let **a** be a vector function of the scalar variable t. Let **A** be a unit vector in the direction of **a**. If a be the magnitude of **a**, then $\mathbf{a} = a\mathbf{A}$.

$$\therefore \quad \frac{d\mathbf{a}}{dt} = a\frac{d\mathbf{A}}{dt} + \frac{da}{dt}\mathbf{A}.$$

Hence

$$\mathbf{a} \times \frac{d\mathbf{a}}{dt} = (a\,\mathbf{A}) \times \left(a\frac{d\mathbf{A}}{dt} + \frac{da}{dt}\mathbf{A}\right)$$

$$= a^2\mathbf{A} \times \frac{d\mathbf{A}}{dt} + a\frac{da}{dt}\mathbf{A} \times \mathbf{A}$$

$$= a^2\mathbf{A} \times \frac{d\mathbf{A}}{dt} \qquad [\because \ \mathbf{A} \times \mathbf{A} = \mathbf{0}] \qquad \ldots(1)$$

The condition is necessary. Suppose **a** has a constant direction. Then **A** is a constant vector because it has constant direction as well as constant magnitude. Therefore $\dfrac{d\mathbf{A}}{dt} = \mathbf{0}$.

$\therefore$ From (1), we get $\mathbf{a} \times \dfrac{d\mathbf{a}}{dt} = a^2\mathbf{A} \times \mathbf{0} = \mathbf{0}$.

Therefore the condition is necessary.

The condition is sufficient. Suppose that $\mathbf{a} \times \dfrac{d\mathbf{a}}{dt} = \mathbf{0}$.

Then from (1), we get

$$a^2\mathbf{A} \times \frac{d\mathbf{A}}{dt} = \mathbf{0} \quad \text{or} \quad \mathbf{A} \times \frac{d\mathbf{A}}{dt} = \mathbf{0}. \qquad \ldots(2)$$

Since **A** is of constant length, therefore $\mathbf{A} \bullet \dfrac{d\mathbf{A}}{dt} = 0.$ ...(3)

From (2) and (3), we get $\dfrac{d\mathbf{A}}{dt} = \mathbf{0}.$

Hence **A** is a constant vector *i.e.*, the direction of **a** is constant.

10 Curves in Space

A curve in a three dimensional Euclidean space may be regarded as the intersection of two surfaces represented by two equations of the form

$$F_1 \ (x, y, z) = 0, F_2 \ (x, y, z) = 0.$$

It can be easily seen that the parametric equations of the form

$$x = f_1 (t), y = f_2 (t), z = f_3 (t)$$

where x , y , z are scalar functions of the scalar t, also represent a curve in three dimensional space. Here (x, y, z) are the coordinates of a current point on the curve. The scalar variable t may range over a set of values $a \leq t \leq b$.

In vector notation an equation of the form $\mathbf{r} = \mathbf{f} \ (t)$, represents a curve in three dimensional space if **r** is the position vector of a current point on the curve. As t changes, **r** will give position vectors of different points on the curve. The vector $\mathbf{f} \ (t)$ can be expressed as

$$f_1 (t)\mathbf{i} + f_2 (t)\mathbf{j} + f_3 (t)\mathbf{k}.$$

Also if (x, y, z) are the coordinates of a current point on the curve whose position vector is **r**, then $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Therefore the single vector equation $\mathbf{r} = \mathbf{f} \ (t)$

i.e., $$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = f_1 (t)\mathbf{i} + f_2 (t)\mathbf{j} + f_3 (t)\mathbf{k}$$

is equivalent to the three parametric equations

$$x = f_1 (t), y = f_2 (t), z = f_3 (t).$$

Thus *a curve in a space may be defined as the locus of a point whose coordinates may be expressed as a function of a* **single** *parameter.*

For example, the two equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \ \ x = a \cosh \frac{z}{a}$$

specify a curve in three dimensional space. The parametric equations of this curve are $$x = a \cosh u, \ \ y = b \sinh u, \ \ z = au.$$

And its vectorial equation is

$$\mathbf{r} = a \cosh u \ \mathbf{i} + b \sinh u \ \mathbf{j} + au \ \mathbf{k}.$$

The vector equation $\mathbf{r} = a \cos t\mathbf{i} + b \sin t\mathbf{j} + 0\mathbf{k}$ represents an ellipse, as for different values of t, the end point of **r** describes an ellipse.

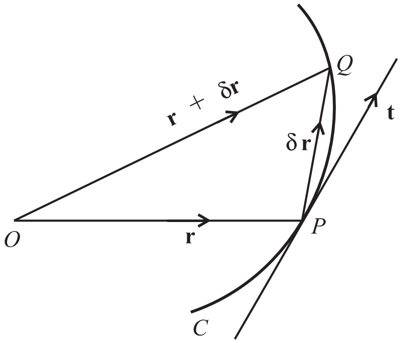
Similarly $\mathbf{r} = at^2 \mathbf{i} + 2at\mathbf{j} + 0\mathbf{k}$ is the vector equation of a parabola.

The terms **skew, twisted** or **tortuous** are often used for curves in a space.

11 Geometrical Significance of dr / dt and Unit Tangent Vector to a Curve

Geometrical Significance of dr/dt :

Let $\mathbf{r} = \mathbf{f}\ (t)$ be the vector equation of a curve in space. Let $\mathbf{r}$ and $\mathbf{r} + \delta\mathbf{r}$ be the position vectors of two neighbouring points P and Q on this curve. Thus we have



$$\overrightarrow{OP} = \mathbf{r} = \mathbf{f}(t)$$

and $$\overrightarrow{OQ} = \mathbf{r} + \delta\mathbf{r} = \mathbf{f}(t + \delta t).$$

$\therefore$ $$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= (\mathbf{r} + \delta\mathbf{r}) - \mathbf{r} = \delta\mathbf{r}.$$

Thus $\dfrac{\delta\mathbf{r}}{dt}$ is a vector parallel to the chord PQ.

As $Q \to P$ *i.e.*, as $\delta t \to 0$, chord $PQ \to$ tangent at P to the curve.

$\therefore$ $\lim\limits_{\delta t \to 0} \dfrac{\delta\mathbf{r}}{\delta t} = \dfrac{d\mathbf{r}}{dt}$ is a vector parallel to the tangent at P to the curve $\mathbf{r} = \mathbf{f}\ (t)$.

Unit tangent vector to a curve:

Suppose in place of the scalar parameter t, we take the parameter as s where s denotes the arc length measured along the curve from any convenient fixed point C on the curve. Thus arc $CP = s$ and arc $CQ = s + \delta s$.

In this case $\dfrac{d\mathbf{r}}{ds}$ will be a vector along the tangent at P to the curve and in the direction of s increasing. Also we have

$$\left|\frac{d\,\mathbf{r}}{ds}\right| = \lim_{\delta t \to 0}\left|\frac{\delta\,\mathbf{r}}{\delta s}\right| = \lim_{\delta t \to 0} \frac{|\delta\mathbf{r}\,|}{\text{arc } PQ} = \lim_{Q \to P} \frac{\text{chord } PQ}{\text{arc } PQ} = 1.$$

Thus $\dfrac{d\mathbf{r}}{ds}$ is a unit vector along the tangent at P in the direction of s increasing. We denote it by $\mathbf{t}$. Note that $\mathbf{t}$ always points in the direction of motion along the curve.

12 Application to Velocity and Acceleration

Velocity: If the scalar variable t be the time and $\mathbf{r}$ be the position vector of a moving particle P with respect to the origin O, then $\delta\mathbf{r}$ is the displacement of the particle in time δt.

The vector $\dfrac{\delta\mathbf{r}}{\delta t}$ is the average velocity of the particle during the interval δt. If $\mathbf{v}$ represents the velocity vector of the particle at P, then $\mathbf{v} = \lim\limits_{\delta t \to 0} \dfrac{\delta\mathbf{r}}{\delta t} = \dfrac{d\mathbf{r}}{dt}$.

Since $\frac{d\mathbf{r}}{dt}$ is a vector along the tangent at P to the curve in which the particle is moving, therefore the direction of velocity is along the tangent.

Acceleration: If $\delta\mathbf{v}$ be the change in the velocity $\mathbf{v}$ during the time δt, then $\frac{\delta\mathbf{v}}{\delta t}$ is the average acceleration during that interval. If $\mathbf{a}$ represents the acceleration of the particle at time t, then

$$\mathbf{a} = \lim_{\delta t \to 0} \frac{\delta\mathbf{v}}{\delta t} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d^2\mathbf{r}}{dt^2}.$$

Illustrative Examples

Example 1: *If* $\mathbf{r} = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + t\mathbf{k}$, *find*

(i) $\frac{d\mathbf{r}}{dt}$, **(Kashi 2013)** *(ii)* $\frac{d^2\mathbf{r}}{dt^2}$, **(Meerut 2010B)**

(iii) $\left|\frac{d\mathbf{r}}{dt}\right|$, **(Kashi 2013)** *(iv)* $\left|\frac{d^2\mathbf{r}}{dt^2}\right|$. **(Meerut 2009)**

Solution: Since $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are constant vectors, therefore $\frac{d\mathbf{i}}{dt} = \mathbf{0}$ etc. Therefore

(i) $$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(\sin t)\,\mathbf{i} + \frac{d}{dt}(\cos t)\,\mathbf{j} + \frac{d}{dt}(t)\,\mathbf{k} = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \mathbf{k}.$$

(ii) $$\frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d}{dt}(\cos t)\,\mathbf{i} - \frac{d}{dt}(\sin t)\,\mathbf{j} + \frac{d\mathbf{k}}{dt}$$

$$= -\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + \mathbf{0} = -\sin t\,\mathbf{i} - \cos t\,\mathbf{j}.$$

(iii) $$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{[(\cos t)^2 + (-\sin t)^2 + (1)^2]} = \sqrt{2}.$$

(iv) $$\left|\frac{d^2\mathbf{r}}{dt^2}\right| = \sqrt{[(-\sin t)^2 + (-\cos t)^2]} = 1.$$

Example 2: *If* $\mathbf{a}$, $\mathbf{b}$ *are constant vectors,* ω *is a constant, and* $\mathbf{r}$ *is a vector function of the scalar variable* t *given by* $\mathbf{r} = \cos \omega t\,\mathbf{a} + \sin \omega t\,\mathbf{b}$, *show that*

(i) $\frac{d^2\mathbf{r}}{dt^2} + \omega^2\mathbf{r} = \mathbf{0}$ **(Rohilkhand 2006)**

(ii) $\mathbf{r} \times \frac{d\mathbf{r}}{dt} = \omega\mathbf{a} \times \mathbf{b}$. **(Bundelkhand 2004; Kanpur 05)**

Solution: Since $\mathbf{a}$, $\mathbf{b}$ are constant vectors, therefore

$$\frac{d\mathbf{a}}{dt} = \mathbf{0},\ \frac{d\mathbf{b}}{dt} = \mathbf{0}.$$

(i) $$\frac{d\mathbf{r}}{dt}=\frac{d}{dt}(\cos \omega t)\,\mathbf{a}+\frac{d}{dt}(\sin \omega t)\,\mathbf{b}$$

$$=-\,\omega \sin \omega t\ \mathbf{a}+\omega \cos \omega t\ \mathbf{b}.$$

$\therefore$ $$\frac{d^2\mathbf{r}}{dt^2}=-\,\omega^2\cos \omega t\ \mathbf{a}-\omega^2\sin \omega t\ \mathbf{b}$$

$$=-\,\omega^2(\cos \omega t\mathbf{a}+\sin \omega t\mathbf{b})=-\,\omega^2\,\mathbf{r}.$$

$\therefore$ $$\frac{d^2\mathbf{r}}{dt^2}+\omega^2\,\mathbf{r}=\mathbf{0}.$$

(ii) $$\mathbf{r}\times\frac{d\mathbf{r}}{dt}=(\cos \omega t\mathbf{a}+\sin \omega t\ \mathbf{b})\times(-\,\omega \sin \omega t\mathbf{a}+\omega \cos \omega t\mathbf{b})$$

$$=\omega \cos^2 \omega t\ \mathbf{a}\times\mathbf{b}-\omega \sin^2 \omega t\ \mathbf{b}\times\mathbf{a}\qquad[\because\ \mathbf{a}\times\mathbf{a}=\mathbf{0},\ \mathbf{b}\times\mathbf{b}=\mathbf{0}]$$

$$=\omega \cos^2 \omega t\ \mathbf{a}\times\mathbf{b}+\omega \sin^2 \omega t\ \mathbf{a}\times\mathbf{b}$$

$$=\omega(\cos^2 \omega t+\sin^2 \omega t)\ \mathbf{a}\times\mathbf{b}=\omega\mathbf{a}\times\mathbf{b}.$$

Example 3: *If* $\mathbf{r}=a\cos t\ \mathbf{i}+a\sin t\ \mathbf{j}+at\tan\alpha\ \mathbf{k}$, *find*

$$\left|\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right|\ \text{and}\ \left[\frac{d\mathbf{r}}{dt},\frac{d^2\mathbf{r}}{dt^2},\frac{d^3\mathbf{r}}{dt^3}\right].$$

(Purvanchal 07, 10; Agra 14)

Solution: We have $$\frac{d\mathbf{r}}{dt}=-\,a\sin t\ \mathbf{i}+a\cos t\ \mathbf{j}+a\tan\alpha\ \mathbf{k}$$

$$\frac{d^2\mathbf{r}}{dt^2}=-\,a\cos t\ \mathbf{i}-a\sin t\ \mathbf{j},\qquad\left[\because\ \frac{d\,\mathbf{k}}{dt}=\mathbf{0}\right]$$

$$\frac{d^3\mathbf{r}}{dt^3}=a\sin t\ \mathbf{i}-a\cos t\ \mathbf{j}.$$

$\therefore$ $$\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-\,a\sin t&a\cos t&a\tan\alpha\\-\,a\cos t&-\,a\sin t&0\end{vmatrix}$$

$$=a^2\sin t\tan\alpha\ \mathbf{i}-a^2\cos t\tan\alpha\ \mathbf{j}+a^2\,\mathbf{k}.$$

$\therefore$ $$\left|\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right|=\sqrt{}(a^4\sin^2 t\tan^2\alpha+a^4\cos^2 t\tan^2\alpha+a^4)$$

$$=a^2\sec\alpha.$$

Also $$\left[\frac{d\mathbf{r}}{dt},\frac{d^2\mathbf{r}}{dt^2},\frac{d^3\mathbf{r}}{dt^3}\right]=\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)\bullet\frac{d^3\mathbf{r}}{dt^3}$$

$$=(a^2\sin t\tan\alpha\ \mathbf{i}-a^2\cos t\tan\alpha\ \mathbf{j}+a^2\ \mathbf{k})\bullet(a\sin t\ \mathbf{i}-a\cos t\ \mathbf{j})$$

$$=a^3\sin^2 t\tan\alpha\ \mathbf{i}\bullet\mathbf{i}+a^3\cos^2 t\tan\alpha\ \mathbf{j}\bullet\mathbf{j}\qquad[\because\ \mathbf{i}\bullet\mathbf{j}=0\ \text{etc.}]$$

$$=a^3\tan\alpha(\sin^2 t+\cos^2 t)\qquad[\because\ \mathbf{i}\bullet\mathbf{i}=1=\mathbf{j}\bullet\mathbf{j}]$$

$$=a^3\tan\alpha.$$

Example 4: *If* $\dfrac{d\mathbf{u}}{dt} = \mathbf{w} \times \mathbf{u}, \dfrac{d\mathbf{v}}{dt} = \mathbf{w} \times \mathbf{v}$, *show that* $\dfrac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \mathbf{w} \times (\mathbf{u} \times \mathbf{v})$.

(Garhwal 2003; Bundelkhand 09)

Solution: We have

$$\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \frac{d\mathbf{u}}{dt} \times \mathbf{v} + \mathbf{u} \times \frac{d\mathbf{v}}{dt} = (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} + \mathbf{u} \times (\mathbf{w} \times \mathbf{v})$$

$$= (\mathbf{v} \bullet \mathbf{w})\,\mathbf{u} - (\mathbf{v} \bullet \mathbf{u})\,\mathbf{w} + (\mathbf{u} \bullet \mathbf{v})\,\mathbf{w} - (\mathbf{u} \bullet \mathbf{w})\,\mathbf{v}$$

$$= (\mathbf{v} \bullet \mathbf{w})\,\mathbf{u} - (\mathbf{u} \bullet \mathbf{w})\,\mathbf{v} \qquad [\because \ \mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}]$$

$$= (\mathbf{w} \bullet \mathbf{v})\,\mathbf{u} - (\mathbf{w} \bullet \mathbf{u})\,\mathbf{v} = \mathbf{w} \times (\mathbf{u} \times \mathbf{v}).$$

Example 5: *If* $\mathbf{R}$ *be a unit vector in the direction of* $\mathbf{r}$, *prove that*

$$\mathbf{R} \times \frac{d\mathbf{R}}{dt} = \frac{1}{r^2}\,\mathbf{r} \times \frac{d\mathbf{r}}{dt}, \text{ where } r = |\mathbf{r}|.$$

(Bundelkhand 2008, 11)

Solution: We have $\mathbf{r} = r\mathbf{R}$; so that $\mathbf{R} = \dfrac{1}{r}\mathbf{r}$.

$$\therefore \qquad \frac{d\mathbf{R}}{dt} = \frac{1}{r}\frac{d\mathbf{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\mathbf{r}.$$

Hence
$$\mathbf{R} \times \frac{d\mathbf{R}}{dt} = \frac{1}{r}\mathbf{r} \times \left(\frac{1}{r}\frac{d\mathbf{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\mathbf{r}\right)$$

$$= \frac{1}{r^2}\mathbf{r} \times \frac{d\mathbf{r}}{dt} - \frac{1}{r^3}\frac{dr}{dt}\mathbf{r} \times \mathbf{r}$$

$$= \frac{1}{r^2}\mathbf{r} \times \frac{d\mathbf{r}}{dt}. \qquad [\because \ \mathbf{r} \times \mathbf{r} = \mathbf{0}]$$

Example 6: *If* $\mathbf{r}$ *is a vector function of a scalar t and* $\mathbf{a}$ *is a constant vector, m a constant, differentiate the following with respect to t :*

(i)	$\mathbf{r} \bullet \mathbf{a}$,	*(ii)*	$\mathbf{r} \times \mathbf{a}$,
(iii)	$\mathbf{r} \times \dfrac{d\mathbf{r}}{dt}$,	*(iv)*	$\mathbf{r} \bullet \dfrac{d\mathbf{r}}{dt}$,
(v)	$\mathbf{r}^2 + \dfrac{1}{\mathbf{r}^2}$,	*(vi)*	$m\left(\dfrac{d\mathbf{r}}{dt}\right)^2$,
(vii)	$\dfrac{\mathbf{r} + \mathbf{a}}{\mathbf{r}^2 + \mathbf{a}^2}$,	*(viii)*	$\dfrac{\mathbf{r} \times \mathbf{a}}{\mathbf{r} \bullet \mathbf{a}}$.

Solution: (i) Let $R = \mathbf{r} \bullet \mathbf{a}$. [Note that $\mathbf{r} \bullet \mathbf{a}$ is a scalar]

Then
$$\frac{dR}{dt} = \frac{d\mathbf{r}}{dt} \bullet \mathbf{a} + \mathbf{r} \bullet \frac{d\mathbf{a}}{dt}$$

$$= \frac{d\mathbf{r}}{dt} \bullet \mathbf{a} + \mathbf{r} \bullet \mathbf{0} \qquad \left[\because \ \frac{d\mathbf{a}}{dt} = \mathbf{0}, \text{ as } \mathbf{a} \text{ is constant}\right]$$

$$= \frac{d\mathbf{r}}{dt} \bullet \mathbf{a} + 0$$

$$= \frac{d\mathbf{r}}{dt} \bullet \mathbf{a}.$$

(ii) Let $\mathbf{R} = \mathbf{r} \times \mathbf{a}$.

Then $$\frac{d\mathbf{R}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{a} + \mathbf{r} \times \frac{d\mathbf{a}}{dt}$$

$$= \frac{d\mathbf{r}}{dt} \times \mathbf{a} + \mathbf{r} \times \mathbf{0} \qquad \left[\because \quad \frac{d\mathbf{a}}{dt} = \mathbf{0}\right]$$

$$= \frac{d\mathbf{r}}{dt} \times \mathbf{a} + \mathbf{0} = \frac{d\mathbf{r}}{dt} \times \mathbf{a}.$$

(iii) Let $\mathbf{R} = \mathbf{r} \times \dfrac{d\mathbf{r}}{dt}\cdot$

Then $$\frac{d\mathbf{R}}{dt} = \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} + \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}$$

$$= \mathbf{0} + \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} \qquad \left[\because \quad \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} = \mathbf{0}\right]$$

$$= \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\cdot$$

(iv) Let $R = \mathbf{r} \bullet \dfrac{d\mathbf{r}}{dt}\cdot$

Then $$\frac{dR}{dt} = \frac{d\mathbf{r}}{dt} \bullet \frac{d\mathbf{r}}{dt} + \mathbf{r} \bullet \frac{d^2\mathbf{r}}{dt^2}$$

$$= \left(\frac{d\mathbf{r}}{dt}\right)^2 + \mathbf{r} \bullet \frac{d^2\mathbf{r}}{dt^2}\cdot$$

(v) Let $R = \mathbf{r}^2 + \dfrac{1}{\mathbf{r}^2}\cdot$

Then $$\frac{dR}{dt} = \frac{d}{dt}(\mathbf{r}^2) + \frac{d}{dt}\left(\frac{1}{\mathbf{r}^2}\right)$$

$$= \frac{d}{dt}(r^2) + \frac{d}{dt}\left(\frac{1}{r^2}\right), \text{where } r = |\mathbf{r}|$$

$$= 2r\frac{dr}{dt} - \frac{2}{r^3}\frac{dr}{dt}\cdot$$

(vi) Let $R = m\left(\dfrac{d\mathbf{r}}{dt}\right)^2\cdot$

Then $$\frac{dR}{dt} = m\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)^2$$

$$= 2\,m\frac{d\mathbf{r}}{dt} \bullet \frac{d^2\mathbf{r}}{dt^2} \qquad \left[\text{Note:} \quad \frac{d\mathbf{r}^2}{dt} = 2\,\mathbf{r} \bullet \frac{d\mathbf{r}}{dt}\right]$$

$$= 2\,m\frac{d\mathbf{r}}{dt} \bullet \frac{d^2\mathbf{r}}{dt^2}\cdot$$

(vii) Let $\mathbf{R} = \dfrac{\mathbf{r}+\mathbf{a}}{\mathbf{r}^2+\mathbf{a}^2}$.

Then $$\frac{d\mathbf{R}}{dt} = \frac{1}{(\mathbf{r}^2+\mathbf{a}^2)}\frac{d}{dt}(\mathbf{r}+\mathbf{a}) + \left\{\frac{d}{dt}\left(\frac{1}{\mathbf{r}^2+\mathbf{a}^2}\right)\right\}(\mathbf{r}+\mathbf{a})$$

[Note that $\mathbf{r}^2+\mathbf{a}^2$ is a scalar]

$$= \frac{1}{\mathbf{r}^2+\mathbf{a}^2}\left(\frac{d\mathbf{r}}{dt}+\frac{d\mathbf{a}}{dt}\right) - \left\{\frac{1}{(\mathbf{r}^2+\mathbf{a}^2)^2}\frac{d}{dt}(\mathbf{r}^2+\mathbf{a}^2)\right\}(\mathbf{r}+\mathbf{a})$$

$$= \frac{1}{\mathbf{r}^2+\mathbf{a}^2}\frac{d\mathbf{r}}{dt} - \frac{2\,\mathbf{r}\bullet\dfrac{d\mathbf{r}}{dt}}{(\mathbf{r}^2+\mathbf{a}^2)^2}(\mathbf{r}+\mathbf{a})$$

$$\left[\because\ \frac{d\mathbf{a}}{dt}=\mathbf{0},\ \frac{d}{dt}\mathbf{r}^2 = 2\mathbf{r}\bullet\frac{d\mathbf{r}}{dt},\ \frac{d}{dt}\mathbf{a}^2 = 0\right]$$

(viii) Let $\mathbf{R} = \dfrac{\mathbf{r}\times\mathbf{a}}{\mathbf{r}\bullet\mathbf{a}}$.

Then $$\frac{d\mathbf{R}}{dt} = \frac{1}{\mathbf{r}\bullet\mathbf{a}}\frac{d}{dt}(\mathbf{r}\times\mathbf{a}) + \left\{\frac{d}{dt}\left(\frac{1}{\mathbf{r}\bullet\mathbf{a}}\right)\right\}(\mathbf{r}\times\mathbf{a})$$

[Note that $\mathbf{r}\bullet\mathbf{a}$ is a scalar quantity]

$$= \frac{1}{\mathbf{r}\bullet\mathbf{a}}\left(\frac{d\,\mathbf{r}}{dt}\times\mathbf{a}+\mathbf{r}\times\frac{d\,\mathbf{a}}{dt}\right) - \left\{\frac{1}{(\mathbf{r}\bullet\mathbf{a})^2}\frac{d}{dt}(\mathbf{r}\bullet\mathbf{a})\right\}(\mathbf{r}\times\mathbf{a})$$

$$= \frac{\dfrac{d\mathbf{r}}{dt}\times\mathbf{a}}{\mathbf{r}\bullet\mathbf{a}} - \left\{\frac{1}{(\mathbf{r}\bullet\mathbf{a})^2}\left(\frac{d\mathbf{r}}{dt}\bullet\mathbf{a}+\mathbf{r}\bullet\frac{d\mathbf{a}}{dt}\right)\right\}(\mathbf{r}\times\mathbf{a})$$

$$= \frac{\dfrac{d\mathbf{r}}{dt}\times\mathbf{a}}{\mathbf{r}\bullet\mathbf{a}} - \frac{\dfrac{d\mathbf{r}}{dt}\bullet\mathbf{a}}{(\mathbf{r}\bullet\mathbf{a})^2}(\mathbf{r}\times\mathbf{a}). \qquad \left[\because\ \frac{d\mathbf{a}}{dt}=\mathbf{0}\right]$$

Example 7: *Find*

(i) $\dfrac{d}{dt}\left[\mathbf{r}, \dfrac{d\mathbf{r}}{dt}, \dfrac{d^2\mathbf{r}}{dt^2}\right]$; *(ii)* $\dfrac{d^2}{dt^2}\left[\mathbf{r}, \dfrac{d\mathbf{r}}{dt}, \dfrac{d^2\mathbf{r}}{dt^2}\right]$;

(iii) $\dfrac{d}{dt}\left[\mathbf{r}\times\left(\dfrac{d\mathbf{r}}{dt}\times\dfrac{d^2\mathbf{r}}{dt^2}\right)\right]$.

(Kumaun 2010)

Solution: (i) Let $R = \left[\mathbf{r}, \dfrac{d\mathbf{r}}{dt}, \dfrac{d^2\mathbf{r}}{dt^2}\right]$. Then R is the scalar triple product of three vectors $\mathbf{r}$, $\dfrac{d\mathbf{r}}{dt}$ and $\dfrac{d^2\mathbf{r}}{dt^2}$. Therefore using the rule for finding the derivative of a scalar triple product, we have

$$\frac{dR}{dt}=\left[\frac{d\mathbf{r}}{dt},\frac{d\mathbf{r}}{dt},\frac{d^2\mathbf{r}}{dt^2}\right]+\left[\mathbf{r},\frac{d^2\mathbf{r}}{dt^2},\frac{d^2\mathbf{r}}{dt^2}\right]+\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^3\mathbf{r}}{dt^3}\right]$$

$$=\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^3\mathbf{r}}{dt^3}\right],$$

since scalar triple products having two equal vectors vanish.

(ii) Let $R=\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^2\mathbf{r}}{dt^2}\right]\cdot$ Then as in part (i)

$$\frac{dR}{dt}=\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^3\mathbf{r}}{dt^3}\right]\cdot$$

Differentiating again, we get

$$\frac{d^2R}{dt^2}=\left[\frac{d\mathbf{r}}{dt},\frac{d\mathbf{r}}{dt},\frac{d^3\mathbf{r}}{dt^3}\right]+\left[\mathbf{r},\frac{d^2\mathbf{r}}{dt^2},\frac{d^3\mathbf{r}}{dt^3}\right]+\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^4\mathbf{r}}{dt^4}\right]$$

$$=\left[\mathbf{r},\frac{d^2\mathbf{r}}{dt^2},\frac{d^3\mathbf{r}}{dt^3}\right]+\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^4\mathbf{r}}{dt^4}\right]\cdot$$

(iii) Let $\mathbf{R}=\mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)\cdot$

Then **R** is the vector triple product of three vectors. Therefore using the rule for finding the derivative of a vector triple product, we have

$$\frac{d\mathbf{R}}{dt}=\frac{d\mathbf{r}}{dt}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)+\mathbf{r}\times\left(\frac{d^2\mathbf{r}}{dt^2}\times\frac{d^2\mathbf{r}}{dt^2}\right)+\mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right)$$

$$=\frac{d\mathbf{r}}{dt}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)+\mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right),$$

since $\frac{d^2\mathbf{r}}{dt^2}\times\frac{d^2\mathbf{r}}{dt^2}=\mathbf{0}$, being vector product of two equal vectors.

Example 8: *If* $\mathbf{a}=\sin\theta\,\mathbf{i}+\cos\theta\,\mathbf{j}+\theta\mathbf{k}$, $\mathbf{b}=\cos\theta\,\mathbf{i}-\sin\theta\,\mathbf{j}-3\mathbf{k}$, *and* $\mathbf{c}=2\mathbf{i}+3\mathbf{j}-3\mathbf{k}$, *find* $\frac{d}{d\theta}\{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\}$ *at* $\theta=\frac{\pi}{2}\cdot$ **(Kumaun 2010)**

Solution: We have $\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \cos\theta&-\sin\theta&-3\\ 2&3&-3\end{vmatrix}$

$$=(3\sin\theta+9)\,\mathbf{i}+(3\cos\theta-6)\,\mathbf{j}+(3\cos\theta+2\sin\theta)\,\mathbf{k}\,.$$

$$\therefore\quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \sin\theta&\cos\theta&\theta\\ 3\sin\theta+9&3\cos\theta-6&3\cos\theta+2\sin\theta\end{vmatrix}$$

$$= (3\cos^2\theta + 2\sin\theta\cos\theta - 3\theta\cos\theta + 6\theta)\,\mathbf{i}$$
$$+ (3\theta\sin\theta + 9\theta - 3\sin\theta\cos\theta - 2\sin^2\theta)\,\mathbf{j}$$
$$+ (-6\sin\theta - 9\cos\theta)\,\mathbf{k}\,.$$

$$\therefore \qquad \frac{d}{d\theta}\{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\} = (-6\cos\theta\sin\theta + 2\cos^2\theta - 2\sin^2\theta$$
$$- 3\cos\theta + 3\theta\sin\theta + 6)\,\mathbf{i} + (3\sin\theta + 3\theta\cos\theta + 9$$
$$- 3\cos^2\theta + 3\sin^2\theta - 4\sin\theta\cos\theta)\,\mathbf{j}$$
$$+ (-6\cos\theta + 9\sin\theta)\,\mathbf{k}\,.$$

Putting $\theta = \pi/2$, we get the required derivative

$$= \left(4 + \frac{3}{2}\pi\right)\mathbf{i} + 15\mathbf{j} + 9\mathbf{k}\,.$$

Example 9: *A particle moves along the curve $x = 4\cos t$, $y = 4\sin t$, $z = 6t$. Find the velocity and acceleration at time $t = 0$ and $t = \frac{1}{2}\pi$. Find also the magnitudes of the velocity and acceleration at any time t.* **(Bundelkhand 2007, 14)**

Solution: Let $\mathbf{r}$ be the position vector of the particle at time t.

Then $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + 6t\,\mathbf{k}$. If $\mathbf{v}$ is the velocity of the particle at time t and $\mathbf{a}$ its acceleration at that time, then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -4\sin t\,\mathbf{i} + 4\cos t\,\mathbf{j} + 6\mathbf{k}$$

and $$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -4\cos t\,\mathbf{i} - 4\sin t\,\mathbf{j}.$$

Magnitude of the velocity at time $t = |\mathbf{v}|$

$$= \sqrt{(16\sin^2 t + 16\cos^2 t + 36)} = \sqrt{(52)} = 2\sqrt{(13)}.$$

Magnitude of the acceleration $= |\mathbf{a}| = \sqrt{(16\cos^2 t + 16\sin^2 t)} = 4.$

At $t = 0$, $\mathbf{v} = 4\,\mathbf{j} + 6\,\mathbf{k}$, $\mathbf{a} = -4\,\mathbf{i}$.

At $t = \frac{1}{2}\pi$, $\mathbf{v} = -4\,\mathbf{i} + 6\,\mathbf{k}$, $\mathbf{a} = -4\,\mathbf{j}$.

Example 10: *A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t + 5$, where t is the time. Find the components of its velocity and acceleration at $t = 1$ in the direction* $\mathbf{i} + \mathbf{j} + 3\mathbf{k}$. **(Purvanchal 2012)**

Solution: If $\mathbf{r}$ is the position vector of any point (x, y, z) on the given curve, then

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (t^3 + 1)\,\mathbf{i} + t^2\,\mathbf{j} + (2t + 5)\,\mathbf{k}\,.$$

Velocity $= \mathbf{v} = \dfrac{d\mathbf{r}}{dt} = 3t^2\mathbf{i} + 2t\mathbf{j} + 2\mathbf{k} = 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ at $t = 1$.

Acceleration $= \mathbf{a} = \dfrac{d^2\mathbf{r}}{dt^2} = \dfrac{d}{dt}\left(\dfrac{d\mathbf{r}}{dt}\right) = 6t\mathbf{i} + 2\mathbf{j} = 6\mathbf{i} + 2\mathbf{j}$ at $t = 1$.

Now the unit vector in the given direction $\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

$$= \frac{\mathbf{i}+\mathbf{j}+3\mathbf{k}}{|\mathbf{i}+\mathbf{j}+3\mathbf{k}|} = \frac{\mathbf{i}+\mathbf{j}+3\mathbf{k}}{\sqrt{(11)}} = \mathbf{b}, \text{ say.}$$

$\therefore$ the component of velocity in the given direction

$$= \mathbf{v} \bullet \mathbf{b} = \frac{(3\mathbf{i}+2\mathbf{j}+2\mathbf{k}) \bullet (\mathbf{i}+\mathbf{j}+3\mathbf{k})}{\sqrt{(11)}} = \frac{11}{\sqrt{(11)}} = \sqrt{(11)};$$

and the component of acceleration in the given direction

$$= \mathbf{a} \bullet \mathbf{b} = \frac{(6\mathbf{i}+2\mathbf{j}) \bullet (\mathbf{i}+\mathbf{j}+3\mathbf{k})}{\sqrt{(11)}} = \frac{8}{\sqrt{(11)}}.$$

Comprehensive Exercise 1

1. (i) If $\mathbf{r}$ is the position vector of a moving point and r is the modulus of $\mathbf{r}$, show that $\mathbf{r} \bullet \frac{d\mathbf{r}}{dt} = r\frac{dr}{dt}$. Interpret the relations $\mathbf{r} \bullet \frac{d\mathbf{r}}{dt} = 0$ and $\mathbf{r} \times \frac{d\mathbf{r}}{dt} = \mathbf{0}$.

   (ii) If $\mathbf{r} \times d\mathbf{r} = \mathbf{0}$, show that $\hat{\mathbf{r}}$ = constant. **(Garhwal 2001; Meerut 05B; Purvanchal 12; Bundelkhand 12; Avadh 13)**

2. (i) If $\mathbf{r} = (t+1)\,\mathbf{i} + (t^2+t+1)\,\mathbf{j} + (t^3+t^2+t+1)\,\mathbf{k}$, find $\frac{d\mathbf{r}}{dt}$ and $\frac{d^2\mathbf{r}}{dt^2}$.

   (ii) If $\mathbf{r} = t^2\,\mathbf{i} - t\,\mathbf{j} + (2t+1)\,\mathbf{k}$, find at $t = 0$, the values of

   $\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}, \left|\frac{d\mathbf{r}}{dt}\right|, \left|\frac{d^2\mathbf{r}}{dt^2}\right|.$ **(Kanpur 2002; Bundelkhand 2010, 13)**

3. Show that $\hat{\mathbf{r}} \times d\hat{\mathbf{r}} = (\mathbf{r} \times d\,\mathbf{r})/r^2$, where $\mathbf{r} = r\,\hat{\mathbf{r}}$.

4. (i) The position vector of a moving particle at time t is given by $\underline{\mathbf{r}} = (3t-4)\,\mathbf{i} + (t^2-2)\,\mathbf{j} + 4t^3\,\mathbf{k}$. Find its velocity and acceleration at time $t = 2$. **(Kanpur 2005; Bundelkhand 13)**

   (ii) If $\mathbf{r} = t^3\,\mathbf{i} + \left(2t^3 - \frac{1}{5t^2}\right)\mathbf{j}$, show that $\mathbf{r} \times \frac{d\mathbf{r}}{dt} = \mathbf{k}$. **(Agra 2007)**

   (iii) If $\mathbf{r} = (\cos nt)\,\mathbf{i} + (\sin nt)\,\mathbf{j}$, where n is a constant and t varies, show that $\mathbf{r} \times \frac{d\mathbf{r}}{dt} = n\mathbf{k}$. **(Garhwal 2002)**

5. (i) If $\mathbf{r} = (\sinh t)\,\mathbf{a} + (\cosh t)\,\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, then show that $\frac{d^2\mathbf{r}}{dt^2} = \mathbf{r}$.

   (ii) If $\mathbf{u} = t^2\mathbf{i} - t\mathbf{j} + (2t+1)\,\mathbf{k}$ and $\mathbf{v} = (2t-3)\,\mathbf{i} + \mathbf{j} - t\mathbf{k}$, find $\frac{d}{dt}(\mathbf{u} \bullet \mathbf{v})$, when $t = 1$. **(Bundelkhand 2007)**

6. (i) If $\mathbf{r} = e^{nt}\,\mathbf{a} + e^{-nt}\,\mathbf{b}$, where $\mathbf{a}$, $\mathbf{b}$ are constant vectors, show that

$$\frac{d^2\mathbf{r}}{dt^2} - n^2\mathbf{r} = \mathbf{0}.$$ **(Kumaun 2000; Rohilkhand 07; Agra 07; Kashi 14)**

(ii) Show that $\mathbf{r} = \mathbf{a}\,e^{mt} + \mathbf{b}\,e^{nt}$, where $\mathbf{a}$ and $\mathbf{b}$ are the constant vectors, is the solution of the differential equation $\dfrac{d^2\mathbf{r}}{dt^2} - (m+n)\dfrac{d\mathbf{r}}{dt} + mn\,\mathbf{r} = \mathbf{0}$.

Hence solve the equation

$$\frac{d^2\mathbf{r}}{dt^2} - \frac{d\mathbf{r}}{dt} - 2\mathbf{r} = \mathbf{0}, \quad \text{where } \mathbf{r} = \mathbf{i} \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{j} \text{ for } t = 0.$$

(iii) If $\mathbf{r} = \mathbf{a}\sin\omega t + \mathbf{b}\cos\omega t + \dfrac{\mathbf{c}\,t}{\omega^2}\sin\omega t$, prove that

$$\frac{d^2\mathbf{r}}{dt^2} + \omega^2\mathbf{r} = \frac{2\mathbf{c}}{\omega}\cos\omega t,$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are constant vectors and ω is a constant scalar.

7. (i) A particle moves along the curve $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$.

Determine the velocity and acceleration at any time t and their magnitudes at $t = 0$. **(Bundelkhand 2005; Rohilkhand 08)**

(ii) Show that if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are constant vectors, then $\mathbf{r} = \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c}$ is the path of a particle moving with constant acceleration.

8. A particle moves so that its position vector is given by $\mathbf{r} = \cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j}$ where ω is a constant; show that

(i) the velocity of the particle is perpendicular to $\mathbf{r}$,

(ii) the acceleration is directed towards the origin and has magnitude proportional to the distance from the origin,

(iii) $\mathbf{r} \times \dfrac{d\mathbf{r}}{dt}$ is a constant vector.

9. If $\mathbf{A} = 5t^2\,\mathbf{i} + t\,\mathbf{j} - t^3\,\mathbf{k}$ and $\mathbf{B} = \sin t\,\mathbf{i} - \cos t\,\mathbf{j}$, find

(i) $\dfrac{d}{dt}(\mathbf{A} \bullet \mathbf{B})$; (ii) $\dfrac{d}{dt}(\mathbf{A} \times \mathbf{B})$; (iii) $\dfrac{d}{dt}(\mathbf{A} \bullet \mathbf{A})$. **(Agra 2005)**

10. Prove the following :

(i) $$\frac{d}{dt}\left[\mathbf{a} \bullet \frac{d\mathbf{b}}{dt} - \frac{d\mathbf{a}}{dt} \bullet \mathbf{b}\right] = \mathbf{a} \bullet \frac{d^2\mathbf{b}}{dt^2} - \frac{d^2\mathbf{a}}{dt^2} \bullet \mathbf{b}$$

(ii) $$\frac{d}{dt}\left[\mathbf{a} \times \frac{d\mathbf{b}}{dt} - \frac{d\mathbf{a}}{dt} \times \mathbf{b}\right] = \mathbf{a} \times \frac{d^2\mathbf{b}}{dt^2} - \frac{d^2\mathbf{a}}{dt^2} \times \mathbf{b}$$

11. If $\mathbf{r}$ is a unit vector, then prove that

$$\left|\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right| = \left|\frac{d\mathbf{r}}{dt}\right|.$$

(Purvanchal 2006)

12. If $\mathbf{r}$ is a vector function of a scalar t, r its module, and $\mathbf{a}$, $\mathbf{b}$ are constant vectors, differentiate the following with respect to t :

(i) $r^3 \ \mathbf{r} + \mathbf{a} \times \dfrac{d\,\mathbf{r}}{dt}$, (ii) $r^2 \ \mathbf{r} + (\mathbf{a} \rceil \mathbf{r})\,\mathbf{b}$,

(iii) $r^n \ \mathbf{r}$, (iv) $(a\mathbf{r} + r\,\mathbf{b})^2$.

13. Find the unit tangent vector to any point on the curve

$x = a \cos t$, $y = a \sin t$, $z = bt$.

14. If the direction of a differentiable vector function $\mathbf{r}\ (t)$ is constant, show that

$$\mathbf{r} \times \left(\frac{d\,\mathbf{r}}{dt}\right) = \mathbf{0}. \qquad \textit{Or}$$

If $\mathbf{r}\ (t)$ is a vector of constant direction, show that its derivative is collinear with it.

15. If $\mathbf{e}$ is the unit vector making an angle θ with x-axis, show that $d\mathbf{e}/d\theta$ is a unit vector obtained by rotating $\mathbf{e}$ through a right angle in the direction of θ increasing.

Answers 1

2. (i) $\mathbf{i} + (2t+1)\,\mathbf{j} + (3t^2 + 2t + 1)\,\mathbf{k}$; $2\mathbf{j} + (6t+2)\,\mathbf{k}$

(ii) $-\mathbf{j} + 2\mathbf{k}$; $2i$; $\sqrt{5}$; 2

4. (i) $\mathbf{V} = 3\mathbf{i} + 4\mathbf{j} + 48\mathbf{k}$, $\mathbf{a} = 2\mathbf{j} + 48\mathbf{k}$

5. (i) $\mathbf{r}$ (ii) $6t^2 - 10t - 2$; -6

7. (i) $\sqrt{37}$; $\sqrt{(325)}$

9. (i) $(5t^2 - 1)\cos t + 11t \sin t$

(ii) $t^2\,(t \sin t - 3\cos t)\,\mathbf{i} - t^2\,(t\cos t + 3\sin t)\,\mathbf{j}$
$- (11t\cos t - 5t^2 \sin t + \sin t)\,\mathbf{k}$

(iii) $100\,t^3 + 2\,t + 6t^5$

12. (i) $3r^2 \dfrac{dr}{dt}\,\mathbf{r} + r^3 \dfrac{d\,\mathbf{r}}{dt} + \mathbf{a} \times \dfrac{d^2\,\mathbf{r}}{dt^2}$

(ii) $2\,r \dfrac{dr}{dt}\,\mathbf{r} + r^2 \dfrac{d\,\mathbf{r}}{dt} + \left(\mathbf{a} \bullet \dfrac{d\,\mathbf{r}}{dt}\right)\mathbf{b}$

(iii) $\left(nr^{n-1} \dfrac{dr}{dt}\right)\mathbf{r} + r^n \dfrac{d\,\mathbf{r}}{dt}$

(iv) $2\,(a\,\mathbf{r} + r\,\mathbf{b}) \bullet \left(a \dfrac{d\,\mathbf{r}}{dt} + \dfrac{dr}{dt}\,\mathbf{b}\right)$

13. $\dfrac{1}{\sqrt{(a^2 + b^2)}}\,(-a \sin t\,\mathbf{i} + a \cos t\,\mathbf{j} + b\,\mathbf{k})$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If $\mathbf{r} = \mathbf{a}\, e^{\omega t} + \mathbf{b}\, e^{-\omega t}$, where **a, b** are constant vectors then $\dfrac{d^2\mathbf{r}}{dt^2} - \omega^2 \mathbf{r}$ is equal to

(a) 1 (b) **0**
(c) 2 (d) none of these

(Bundelkhand 2001; Agra 06)

2. A particle moves along the curve $\mathbf{r} = e^{-t} \cos t\, \mathbf{i} + e^{-t} \sin t\, \mathbf{j} + e^{-t}\, \mathbf{k}$. The magnitude of its velocity at $t = 0$ is

(a) $2\sqrt{3}$ (b) $\sqrt{3}/2$
(c) $\sqrt{3}$ (d) none of these

3. If $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, then $\dfrac{d^2\mathbf{r}}{dt^2} = \dots\dots$

(a) $\mathbf{i}\, x + \mathbf{j}\, y + \mathbf{k}\, z$ (b) $\mathbf{i}\dfrac{dx}{dt} + \mathbf{j}\dfrac{dy}{dt} + \mathbf{k}\dfrac{dz}{dt}$
(c) $\mathbf{i}\dfrac{d^2x}{dt^2} + \mathbf{j}\dfrac{d^2y}{dt^2} + \mathbf{k}\dfrac{d^2z}{dt^2}$ (d) none of these

(Kumaun 2011)

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. If $\mathbf{r} = 3\mathbf{i} - 6t^2\, \mathbf{j} + 4t\, \mathbf{k}$, then $\dfrac{d\mathbf{r}}{dt} = \dots\dots;\ \dfrac{d^2\mathbf{r}}{dt^2} = \dots\dots$ **(Bundelkhand 2008)**

2. If $\mathbf{u} = t^2\, \mathbf{i} - t\, \mathbf{j} + (2t+1)\, \mathbf{k}$, $\mathbf{v} = (2t-3)\, \mathbf{i} + \mathbf{j} - t\, \mathbf{k}$, then $\dfrac{d}{dt}(\mathbf{u} \bullet \mathbf{v}) = \dots\dots$

3. If $\mathbf{r} = (\cos \omega t)\, \mathbf{i} + (\sin \omega t)\, \mathbf{j}$, then $\mathbf{r} \times \dfrac{d^2\mathbf{r}}{dt^2} = \dots\dots$

4. The necessary and sufficient condition for the vector **a** (t) to have constant direction is ……

5. If $\mathbf{r} = 5\mathbf{i} + 3t^2\, \mathbf{j} + 2t\, \mathbf{k}$, then $\dfrac{d\mathbf{r}}{dt} = \dots\dots;\ \dfrac{d^2\mathbf{r}}{dt^2} = \dots\dots$ **(Bundelkhand 2010)**

6. $\dfrac{d}{dt}(\mathbf{a} \bullet \mathbf{b}) = \dots\dots$

(Kumaun 2009)

True or False

Write 'T' for true and 'F' for false statement.

1. A vector is said to be constant only if its magnitude is fixed and direction changes.
2. The necessary and sufficient condition for the vector **a** (t) to have constant magnitude is $\mathbf{a} \bullet \frac{d\mathbf{a}}{dt} = 0$.

Answers

Multiple Choice Questions

1. (b)
2. (c)
3. (c)

Fill in the Blank(s)

1. $-12\,t\,\mathbf{j} + 4\,\mathbf{k}\,; -12\,\mathbf{j}$
2. $6t^2 - 10\,t - 2$
3. 0
4. $\mathbf{a} \times \frac{d\,\mathbf{a}}{dt} = \mathbf{0}$
5. $6\,t\,\mathbf{j} + 2\,\mathbf{k}\,; 6\,\mathbf{j}$
6. $\mathbf{a} \bullet \frac{d\mathbf{b}}{dt} + \frac{d\mathbf{a}}{dt} \bullet \mathbf{b}$

True or False

1. *F*
2. *T*

☐

Chapter

3

Gradient, Divergence and Curl

1 Partial Derivatives of Vectors

Suppose $\mathbf{r}$ is a vector depending on more than one scalar variable. Let $\mathbf{r} = \mathbf{f}(x, y, z)$ *i.e.*, let $\mathbf{r}$ be a function of three scalar variables x, y and z. The partial derivative of $\mathbf{r}$ with respect to x is defined as

$$\frac{\partial \mathbf{r}}{\partial x} = \lim_{\delta x \to 0} \frac{\mathbf{f}(x + \delta x, y, z) - \mathbf{f}(x, y, z)}{\delta x}$$

if this limit exists. Thus $\partial \mathbf{r} / \partial x$ is nothing but the ordinary derivative of $\mathbf{r}$ with respect to x provided the other variables y and z are regarded as constants. Similarly we may define the partial derivatives $\dfrac{\partial \mathbf{r}}{\partial y}$ and $\dfrac{\partial \mathbf{r}}{\partial z}$.

Higher partial derivatives can also be defined as in Scalar Calculus. Thus, for example,

$$\frac{\partial^2 \mathbf{r}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{r}}{\partial x}\right), \quad \frac{\partial^2 \mathbf{r}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \mathbf{r}}{\partial y}\right), \quad \frac{\partial^2 \mathbf{r}}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial \mathbf{r}}{\partial z}\right),$$

$$\frac{\partial^2 \mathbf{r}}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{r}}{\partial y}\right), \quad \frac{\partial^2 \mathbf{r}}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \mathbf{r}}{\partial x}\right).$$

If $\mathbf{r}$ has continuous partial derivatives of the second order at least, then, $\dfrac{\partial^2 \mathbf{r}}{\partial x\, \partial y} = \dfrac{\partial^2 \mathbf{r}}{\partial y\, \partial x}$ *i.e.*, the order of differentiation is immaterial. If $\mathbf{r} = \mathbf{f}\,(x, y, z)$, the total differential $d\mathbf{r}$ of $\mathbf{r}$ is given by $d\mathbf{r} = \dfrac{\partial \mathbf{r}}{\partial x} dx + \dfrac{\partial \mathbf{r}}{\partial y} dy + \dfrac{\partial \mathbf{r}}{\partial z} dz$.

2 The Vector Differential Operator Del (∇)

The vector differential operator ∇ (read as *del* or *nabla*) is defined as

$$\nabla \equiv \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \equiv \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$$

and operates distributively.

The vector operator ∇ can generally be treated to behave as an ordinary vector. It possesses properties like ordinary vectors. The symbols $\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}$ can be treated as its components along $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

3 Gradient of a Scalar Field

(Agra 2005)

Definition: *Let $f\,(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space (i.e., defines a differentiable scalar field). Then the gradient of f, written as ∇f or grad f, is defined as*

$$\nabla f = \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} .$$

It should be noted that ∇f is a vector whose three successive components are $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z} \cdot$ Thus the gradient of a scalar field defines a vector field. *If f is a scalar point function, then ∇f is a vector point function.*

4 Formulas Involving Gradient

Theorem 1: Gradient of the sum of two scalar point functions. *If f and g are two scalar point functions, then*

$$grad\,(f + g) = grad\, f + grad\, g \quad or \quad \nabla (f + g) = \nabla f + \nabla g .$$

Proof: We have

$$\text{grad}\,(f + g) = \nabla (f + g) = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) (f + g)$$

$$= \mathbf{i}\frac{\partial}{\partial x}(f+g) + \mathbf{j}\frac{\partial}{\partial y}(f+g) + \mathbf{k}\frac{\partial}{\partial z}(f+g)$$

$$= \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{i}\frac{\partial g}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{j}\frac{\partial g}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} + \mathbf{k}\frac{\partial g}{\partial z}$$

$$= \left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}\right) + \left(\mathbf{i}\frac{\partial g}{\partial x} + \mathbf{j}\frac{\partial g}{\partial y} + \mathbf{k}\frac{\partial g}{\partial z}\right)$$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) f + \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) g$$

$$= \nabla f + \nabla g = \text{grad } f + \text{grad } g .$$

Similarly, we can prove that $\nabla (f - g) = \nabla f - \nabla g.$

Theorem 2: Gradient of a constant. *The necessary and sufficient condition for a scalar point function to be constant is that* $\nabla f = \mathbf{0}.$

Proof: If $f(x, y, z)$ is constant, then $\dfrac{\partial f}{\partial x} = 0, \dfrac{\partial f}{\partial y} = 0, \dfrac{\partial f}{\partial z} = 0.$

Therefore, $\text{grad } f = \mathbf{i}\dfrac{\partial f}{\partial x} + \mathbf{j}\dfrac{\partial f}{\partial y} + \mathbf{k}\dfrac{\partial f}{\partial z} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$

Hence the condition is necessary.

Conversely, let $\text{grad } f = \mathbf{0}$. Then $\mathbf{i}\dfrac{\partial f}{\partial x} + \mathbf{j}\dfrac{\partial f}{\partial y} + \mathbf{k}\dfrac{\partial f}{\partial z} = \mathbf{0}$.

Therefore, $\dfrac{\partial f}{\partial x} = 0, \dfrac{\partial f}{\partial y} = 0, \dfrac{\partial f}{\partial z} = 0.$

$\therefore$ f must be independent of x, y and z .

$\therefore$ f must be a constant. Hence the condition is sufficient.

Theorem 3: Gradient of the product of two scalar point functions.

If f and g are scalar point functions, then $\text{grad}(fg) = f \text{ grad } g + g \text{ grad } f$

or $\nabla (fg) = f\,\nabla g + g\,\nabla f.$ **(Kumaun 2014)**

Proof: We have $\nabla (fg) = \left(\mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}\right)(fg)$

$$= \mathbf{i}\frac{\partial}{\partial x}(fg) + \mathbf{j}\frac{\partial}{\partial y}(fg) + \mathbf{k}\frac{\partial}{\partial z}(fg)$$

$$= \mathbf{i}\left(f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}\right) + \mathbf{j}\left(f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right) + \mathbf{k}\left(f\frac{\partial g}{\partial z} + g\frac{\partial f}{\partial z}\right)$$

$$= f\left(\mathbf{i}\frac{\partial g}{\partial x} + \mathbf{j}\frac{\partial g}{\partial y} + \mathbf{k}\frac{\partial g}{\partial z}\right) + g\left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z}\right)$$

$$= f\,\nabla g + g\,\nabla f = f \text{ grad } g + g \text{ grad } f .$$

In particular, if c is a constant, then

$$\nabla (c\,f) = c\,\nabla f + f\,\nabla c = c\,\nabla f + \mathbf{0} = c\,\nabla f.$$

Theorem 4: Gradient of the quotient of two scalar functions. *If f and g are two scalar point functions, then* $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\,\nabla f - f\,\nabla g}{g^2}$.

(Kumaun 2013)

Proof: We have

$$\nabla\left(\frac{f}{g}\right) = \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right)\left(\frac{f}{g}\right)$$

$$= \mathbf{i}\,\frac{\partial}{\partial x}\left(\frac{f}{g}\right) + \mathbf{j}\,\frac{\partial}{\partial y}\left(\frac{f}{g}\right) + \mathbf{k}\,\frac{\partial}{\partial z}\left(\frac{f}{g}\right).$$

But $$\frac{\partial}{\partial x}\left(\frac{f}{g}\right) = \frac{g\,\dfrac{\partial f}{\partial x} - f\,\dfrac{\partial g}{\partial x}}{g^2}, \quad \frac{\partial}{\partial y}\left(\frac{f}{g}\right) = \frac{g\,\dfrac{\partial f}{\partial y} - f\,\dfrac{\partial g}{\partial y}}{g^2},$$

and $$\frac{\partial}{\partial z}\left(\frac{f}{g}\right) = \frac{g\,\dfrac{\partial f}{\partial z} - f\,\dfrac{\partial g}{\partial z}}{g^2}.$$

$$\therefore \quad \nabla\left(\frac{f}{g}\right) = \frac{1}{g^2}\left\{\mathbf{i}\left(g\,\frac{\partial f}{\partial x} - f\,\frac{\partial g}{\partial x}\right) + \mathbf{j}\left(g\,\frac{\partial f}{\partial y} - f\,\frac{\partial g}{\partial y}\right) + \mathbf{k}\left(g\,\frac{\partial f}{\partial z} - f\,\frac{\partial g}{\partial z}\right)\right\}$$

$$= \frac{1}{g^2}\left\{g\left(\mathbf{i}\,\frac{\partial f}{\partial x} + \mathbf{j}\,\frac{\partial f}{\partial y} + \mathbf{k}\,\frac{\partial f}{\partial z}\right) - f\left(\mathbf{i}\,\frac{\partial g}{\partial x} + \mathbf{j}\,\frac{\partial g}{\partial y} + \mathbf{k}\,\frac{\partial g}{\partial z}\right)\right\}$$

$$= \frac{1}{g^2}\{g\,\nabla f - f\,\nabla g\}.$$

Illustrative Examples

Example 1: *If* $\mathbf{A} = x^2 yz\,\mathbf{i} - 2xz^3\,\mathbf{j} + xz^2\,\mathbf{k}$, $\mathbf{B} = 2z\mathbf{i} + y\mathbf{j} - x^2\mathbf{k}$, *find the value of* $\dfrac{\partial^2}{\partial x\,\partial y}(\mathbf{A}\times\mathbf{B})$ *at* $(1, 0, -2)$.

Solution: We have $$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x^2 yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$$

$$= (2x^3z^3 - xyz^2)\,\mathbf{i} + (2xz^3 + x^4 yz)\,\mathbf{j} + (x^2y^2z + 4xz^4)\,\mathbf{k}.$$

$$\therefore \quad \frac{\partial}{\partial y}(\mathbf{A}\times\mathbf{B}) = -xz^2\mathbf{i} + x^4 z\,\mathbf{j} + 2x^2 yz\mathbf{k}.$$

Again $$\frac{\partial^2}{\partial x\,\partial y}(\mathbf{A}\times\mathbf{B}) = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}(\mathbf{A}\times\mathbf{B})\right\} = -z^2\mathbf{i} + 4x^3z\mathbf{j} + 4xyz\,\mathbf{k}. \quad \ldots(1)$$

Putting $x = 1$, $y = 0$ and $z = -2$ in (1), we get the required derivative at the point $(1, 0, -2) = -4\mathbf{i} - 8\mathbf{j}$.

Example 2: *If* $f(x, y, z) = 3x^2 y - y^3 z^2$, *find grad f at the point* $(1, -2, -1)$.

(Rohilkhand 2009)

Solution: We have

$$\text{grad } f = \nabla f = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(3x^2 y - y^3 z^2)$$

$$= \mathbf{i}\frac{\partial}{\partial x}(3x^2 y - y^3 z^2) + \mathbf{j}\frac{\partial}{\partial y}(3x^2 y - y^3 z^2)$$

$$+ \mathbf{k}\frac{\partial}{\partial z}(3x^2 y - y^3 z^2)$$

$$= \mathbf{i}(6xy) + \mathbf{j}(3x^2 - 3y^2 z^2) + \mathbf{k}(-2y^3 z)$$

$$= 6xy\,\mathbf{i} + (3x^2 - 3y^2 z^2)\,\mathbf{j} - 2y^3 z\,\mathbf{k}.$$

Putting $x = 1$, $y = -2$, $z = -1$, we get

$$\nabla f = 6(1)(-2)\,\mathbf{i} + \{3(1)^2 - 3(-2)^2(-1)^2\}\,\mathbf{j} - 2(-2)^3(-1)\,\mathbf{k}$$

$$= -12\,\mathbf{i} - 9\mathbf{j} - 16\mathbf{k}.$$

Example 3: *If* $r = |\mathbf{r}|$ *where* $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, *prove that*

(i) $\nabla f(r) = f'(r)\nabla r$, **(Rohilkhand 2008; Meerut 11)**

(ii) $\nabla r = \dfrac{1}{r}\mathbf{r}$,

(iii) $\nabla f(r) \times \mathbf{r} = \mathbf{0}$,

(iv) $\nabla\left(\dfrac{1}{r}\right) = -\dfrac{\mathbf{r}}{r^3}$, **(Rohilkhand 2008, 09B)**

(v) $\nabla \log|\mathbf{r}| = \dfrac{\mathbf{r}}{r^2}$,

(Garhwal 2000; Kumaun 07; Rohilkhand 08; Bundelkhand 08; Meerut 11)

(vi) $\nabla r^n = nr^{n-2}\,\mathbf{r}$. **(Rohilkhand 2008, 11; Bundelkhand 11)**

Solution: If $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, then $r = |\mathbf{r}| = \sqrt{(x^2 + y^2 + z^2)}$.

$\therefore \qquad r^2 = x^2 + y^2 + z^2.$

(i) $$\nabla f(r) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) f(r) = \mathbf{i}\frac{\partial}{\partial x} f(r) + \mathbf{j}\frac{\partial}{\partial y} f(r) + \mathbf{k}\frac{\partial}{\partial z} f(r)$$

$$= \mathbf{i}\,f'(r)\frac{\partial r}{\partial x} + \mathbf{j}\,f'(r)\frac{\partial r}{\partial y} + \mathbf{k}\,f'(r)\frac{\partial r}{\partial z} = f'(r)\left(\mathbf{i}\frac{\partial r}{\partial x} + \mathbf{j}\frac{\partial r}{\partial y} + \mathbf{k}\frac{\partial r}{\partial z}\right)$$

$$= f'(r)\nabla r.$$

(ii) We have $\nabla r = \mathbf{i}\dfrac{\partial r}{\partial x} + \mathbf{j}\dfrac{\partial r}{\partial y} + \mathbf{k}\dfrac{\partial r}{\partial z}\cdot$

Now $\qquad r^2 = x^2 + y^2 + z^2.$

$\therefore \qquad 2r\dfrac{\partial r}{\partial x} = 2x \quad$ *i.e.*, $\quad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}\cdot$

Similarly, $\quad \dfrac{\partial r}{\partial y} = \dfrac{y}{r} \quad$ and $\quad \dfrac{\partial r}{\partial z} = \dfrac{z}{r}\cdot$

$$\therefore \qquad \nabla r = \frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k} = \frac{1}{r}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{1}{r}\mathbf{r} = \hat{\mathbf{r}}.$$

(iii) We have as in part (i), $\nabla f(r) = f'(r)\nabla r.$

But as in part (ii), $\nabla r = \dfrac{1}{r}\mathbf{r}$.

$$\therefore \qquad \nabla f(r) = f'(r)\frac{1}{r}\mathbf{r}.$$

$$\therefore \qquad \nabla f(r) \times \mathbf{r} = \left\{f'(r)\frac{1}{r}\mathbf{r}\right\} \times \mathbf{r} = \left\{\frac{1}{r}f'(r)\right\}(\mathbf{r} \times \mathbf{r}) = \mathbf{0}.$$

$[\because \ \mathbf{r} \times \mathbf{r} = \mathbf{0}]$

(iv) We have $\nabla\left(\dfrac{1}{r}\right) = \mathbf{i}\dfrac{\partial}{\partial x}\left(\dfrac{1}{r}\right) + \mathbf{j}\dfrac{\partial}{\partial y}\left(\dfrac{1}{r}\right) + \mathbf{k}\dfrac{\partial}{\partial z}\left(\dfrac{1}{r}\right)$

$$= \mathbf{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \mathbf{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + \mathbf{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= -\frac{1}{r^2}\left(\frac{\partial r}{\partial x}\mathbf{i} + \frac{\partial r}{\partial y}\mathbf{j} + \frac{\partial r}{\partial z}\mathbf{k}\right)$$

$$= -\frac{1}{r^2}\left(\frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k}\right) = -\frac{1}{r^3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = -\frac{1}{r^3}\mathbf{r}.$$

[see part (ii)]

(v) We have $\nabla \log |\mathbf{r}| = \nabla \log r$

$$= \mathbf{i}\frac{\partial}{\partial x}\log r + \mathbf{j}\frac{\partial}{\partial y}\log r + \mathbf{k}\frac{\partial}{\partial z}\log r$$

$$= \frac{1}{r}\frac{\partial r}{\partial x}\mathbf{i} + \frac{1}{r}\frac{\partial r}{\partial y}\mathbf{j} + \frac{1}{r}\frac{\partial r}{\partial z}\mathbf{k} = \frac{1}{r}\left(\frac{x}{r}\mathbf{i} + \frac{y}{r}\mathbf{j} + \frac{z}{r}\mathbf{k}\right)$$

$$= \frac{1}{r^2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{1}{r^2}\mathbf{r}.$$

(vi) We have $\nabla r^n = \mathbf{i}\dfrac{\partial}{\partial x}r^n + \mathbf{j}\dfrac{\partial}{\partial y}r^n + \mathbf{k}\dfrac{\partial}{\partial z}r^n$

$$= \mathbf{i}\,nr^{n-1}\frac{\partial r}{\partial x} + \mathbf{j}\,nr^{n-1}\frac{\partial r}{\partial y} + \mathbf{k}\,nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left(\mathbf{i}\frac{\partial r}{\partial x} + \mathbf{j}\frac{\partial r}{\partial y} + \mathbf{k}\frac{\partial r}{\partial z}\right) = nr^{n-1}\,\nabla r$$

$$= nr^{n-1}\,\frac{1}{r}\,\mathbf{r} \qquad \left[\because\ \nabla r = \frac{\mathbf{r}}{r} \text{ as in part (ii)}\right]$$

$$= nr^{n-2}\,\mathbf{r}.$$

Example 4: (i) *Interpret the symbol* $\mathbf{a}\bullet\nabla$ (ii) *Show that* $(\mathbf{a}\bullet\nabla)\,\phi = \mathbf{a}\bullet\nabla\,\phi$

(iii) *Show that* $(\mathbf{a}\bullet\nabla)\,\mathbf{r} = \mathbf{a}$. **(Kumaun 2008; Purvanchal 14)**

Solution: (i) Let $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. Then

$$\mathbf{a}\bullet\nabla = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\bullet\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)$$

$$= a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}.$$

(ii) $$(\mathbf{a}\bullet\nabla)\,\phi = \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)\phi.$$

Also $$\mathbf{a}\bullet\nabla\phi = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\bullet\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$= a_1\frac{\partial\phi}{\partial x} + a_2\frac{\partial\phi}{\partial y} + a_3\frac{\partial\phi}{\partial z}.$$

Hence $(\mathbf{a}\bullet\nabla)\,\phi = \mathbf{a}\bullet\nabla\phi.$

(iii) $$(\mathbf{a}\bullet\nabla)\,\mathbf{r} = \left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)\mathbf{r} = a_1\frac{\partial\mathbf{r}}{\partial x} + a_2\frac{\partial\mathbf{r}}{\partial y} + a_3\frac{\partial\mathbf{r}}{\partial z}.$$

But $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$

$$\therefore\quad \frac{\partial\mathbf{r}}{\partial x} = \mathbf{i},\ \frac{\partial\mathbf{r}}{\partial y} = \mathbf{j},\ \frac{\partial\mathbf{r}}{\partial z} = \mathbf{k}.$$

$$\therefore\quad (\mathbf{a}\bullet\nabla)\,\mathbf{r} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = \mathbf{a}.$$

Comprehensive Exercise 1

1. If $\mathbf{F} = e^{xy}\,\mathbf{i} + (x - 2y)\,\mathbf{j} + x\sin y\,\mathbf{k}$, calculate

   (i) $\dfrac{\partial\mathbf{F}}{\partial x}$, (ii) $\dfrac{\partial\mathbf{F}}{\partial y}$, (iii) $\dfrac{\partial^2\mathbf{F}}{\partial x^2}$,

   (iv) $\dfrac{\partial^2\mathbf{F}}{\partial x\,\partial y}$, (v) $\dfrac{\partial^2\mathbf{F}}{\partial y^2}$.

2. If $\mathbf{f} = (2x^2y - x^4)\,\mathbf{i} + (e^{xy} - y\sin x)\,\mathbf{j} + x^2\cos y\,\mathbf{k}$, verify that

$$\frac{\partial^2\mathbf{f}}{\partial y\,\partial x} = \frac{\partial^2\mathbf{f}}{\partial x\,\partial y}.$$

3. If $\mathbf{u} = xyz\,\mathbf{i} + xz^2\,\mathbf{j} - y^3\,\mathbf{k}$ and $\mathbf{v} = x^3\,\mathbf{i} - xyz\,\mathbf{j} + x^2 z\,\mathbf{k}$, calculate

$$\frac{\partial^2 \mathbf{u}}{\partial y^2} \times \frac{\partial^2 \mathbf{v}}{\partial x^2} \text{ at the point } (1, 1, 0).$$

4. If $\phi(x, y, z) = x^2 y + y^2 x + z^2$, find $\nabla\phi$ at the point $(1, 1, 1)$.

5. Find grad f, where f is given by $f = x^3 - y^3 + xz^2$, at the point $(1, -1, 2)$.

6. If $\phi(x, y, z) = xy^2 z$ and $\mathbf{f} = xz\mathbf{i} - xy\mathbf{j} + yz^2\mathbf{k}$, show that

$$\frac{\partial^3}{\partial x^2 \partial z}(\phi \mathbf{f}) \text{ at } (2, -1, 1) \text{ is } 4\mathbf{i} + 2\mathbf{j}.$$

7. If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$, prove that

$$(\text{grad } u) \bullet [(\text{grad } v) \times (\text{grad } w)] = 0.$$ **(Meerut 2007B)**

8. If $\mathbf{F} = \left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right)\mathbf{i} + \left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right)\mathbf{j} + \left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right)\mathbf{k}$, prove that

(i) $\mathbf{F} = \mathbf{r} \times \nabla f$, (ii) $\mathbf{F} \bullet \mathbf{r} = 0$, (iii) $\mathbf{F} \bullet \nabla f = 0$.

9. If $\phi = (3r^2 - 4r^{1/2} + 6r^{-1/3})$, show that $\nabla\phi = 2(3 - r^{-3/2} - r^{-7/3})\,\mathbf{r}$.

10. Prove that $\nabla\phi \bullet d\mathbf{r} = d\phi$. **(Meerut 2005, 06, 09B; Kumaun 08)**

11. Prove that $f(u)\nabla u = \nabla \int f(u)\, du$. **(Kumaun 2012, 13)**

12. ρ and p are two scalar point functions such that ρ is a function of p; show that $\nabla\rho = \frac{d\rho}{dp}\nabla p$.

13. Show that $\frac{d\phi}{ds} = \nabla\phi \bullet \frac{d\mathbf{r}}{ds}$, where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and ϕ is a function of x, y and z.

14. Prove that $\mathbf{A} \bullet \left(\nabla \frac{1}{r}\right) = -\frac{\mathbf{A} \bullet \mathbf{r}}{r^3}$. **(Meerut 2010)**

15. Prove that $\nabla r^{-3} = -3r^{-5}\,\mathbf{r}$. **(Meerut 2009, 12)**

16. Show that

(i) grad $(\mathbf{r} \bullet \mathbf{a}) = \mathbf{a}$, **(Avadh 2010)**

(ii) grad $[\mathbf{r}, \mathbf{a}, \mathbf{b}] = \mathbf{a} \times \mathbf{b}$,

where **a** and **b** are constant vectors.

Answers 1

1. (i) $ye^{xy}\,\mathbf{i} + \mathbf{j} + \sin y\,\mathbf{k}$ (ii) $xe^{xy}\,\mathbf{i} - 2\,\mathbf{j} + x\cos y\,\mathbf{k}$

(iii) $y^2 e^{xy}\,\mathbf{i}$ (iv) $e^{xy}(xy + 1)\,\mathbf{i} + \cos y\,\mathbf{k}$

(v) $x^2 e^{xy} \mathbf{i} - x \sin y \mathbf{k}$

3. $-36 \mathbf{j}$

4. $3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$

5. $7\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$

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5 Equipotential Surfaces or Level Surfaces

Let $f(x, y, z)$ be a scalar field over a region R. The points satisfying an equation of the type

$$f(x, y, z) = c, \text{ (arbitrary constant)}$$

constitute a family of surfaces in three dimensional space. The surfaces of this family are called ***level surfaces***. Any surface of this family is such that the value of the function f at any point of it is the same. Therefore these surfaces are also called ***iso-f-surfaces.***

Theorem 1: *Let $f(x, y, z)$ be a scalar field over a region R. Then through any point of R there passes one and only one level surface.*

Proof: Let (x_1, y_1, z_1) be any point of the region R.

Then the level surface $f(x, y, z) = f(x_1, y_1, z_1)$ passes through this point.

Now suppose the level surfaces $f(x, y, z) = c_1$ and $f(x, y, z) = c_2$ pass through the point (x_1, y_1, z_1). Then

$$f(x_1, y_1, z_1) = c_1 \quad \text{and} \quad f(x_1, y_1, z_1) = c_2.$$

Since $f(x, y, z)$ has a unique value at (x_1, y_1, z_1), therefore we have $c_1 = c_2$.

Hence only one level surface passes through the point (x_1, y_1, z_1).

Theorem 2: *∇f is a vector normal to the surface $f(x, y, z) = c$ where c is a constant.*

Proof: Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector of any point $P(x, y, z)$ on the level surface $f(x, y, z) = c$. Let $Q(x + \delta x, y + \delta y, z + \delta z)$ be a neighbouring point on this surface.

Then the position vector of $Q = \mathbf{r} + \delta\mathbf{r} = (x + \delta x)\,\mathbf{i} + (y + \delta y)\,\mathbf{j} + (z + \delta z)\,\mathbf{k}$.

$$\therefore \qquad \overrightarrow{PQ} = (\mathbf{r} + \delta\mathbf{r}) - \mathbf{r} = \delta\mathbf{r} = \delta x\,\mathbf{i} + \delta y\,\mathbf{j} + \delta z\,\mathbf{k}.$$

As $Q \to P$, the line PQ tends to tangent at P to the level surface.

Therefore $d\,\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ lies in the tangent plane to the surface at P.

From the differential calculus, we have

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$

$$= \left(\mathbf{i}\,\frac{\partial f}{\partial x} + \mathbf{j}\,\frac{\partial f}{\partial y} + \mathbf{k}\,\frac{\partial f}{\partial z}\right) \bullet (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}) = \nabla f \bullet d\,\mathbf{r}.$$

Since $f\ (x\ ,\ y\ ,z) =$ constant, therefore $df\ = 0$.

$\therefore\ \ \nabla f\ \bullet d\mathbf{r} = 0$ so that ∇f is a vector perpendicular to $d\ \mathbf{r}$ and therefore to the tangent plane at P to the surface $f\ (x, y, z) = c$.

Hence $\nabla\ f$ is a vector normal to the surface $f\ (x, y, z) = c$.

Thus if $f\ (x, y, z)$ is a scalar field defined over a region R, then ∇f at any point $(x\ ,\ y\ ,z)$ is a vector in the direction of normal at that point to the level surface $f\ (x, y, z) = c$ passing through that point.

6 Directional Derivative of a Scalar Point Function

Definition: *Let $f\ (x, y, z)$ define a scalar field in a region R and let P be any point in this region. Suppose Q is a point in this region in the neighbourhood of P in the direction of a given unit vector $\hat{\mathbf{a}}$.*

*Then $\underset{Q\to P}{lim}\ \dfrac{f\ (Q) - f\ (P\ )}{PQ}$, if it exists, is called the **directional derivative** of f at P in the direction of $\hat{\mathbf{a}}$.*

Interpretation of Directional Derivative: Let P be the point $(x\ ,\ \ y\ ,\ \ z)$ and let Q be the point $(x + \delta x,\ \ y + \delta y,\ \ z + \delta z)$. Suppose $PQ = \delta s$ .Then δs is a small element at P in the direction of $\hat{\mathbf{a}}$. If

$$\delta f = f\ (x + \delta x\ ,\ y + \delta y\ ,\ z + \delta z) - f\ (x\ ,\ y\ ,z) = f\ (Q) - f\ (P\ ),$$

then $\dfrac{\delta f}{\delta s}$ represents the average rate of change of f per unit distance in the direction of $\hat{\mathbf{a}}$. Now the directional derivative of f at P in the direction of $\hat{\mathbf{a}}$ is $\displaystyle\lim_{Q\to P}\ \frac{f\ (Q) - f\ (P\ )}{PQ} = \lim_{\delta s\ \to\ 0}\ \frac{\delta f}{\delta s} = \frac{df}{ds}\cdot$ It represents the rate of change of f with respect to distance s at the point P in the direction of unit vector $\hat{\mathbf{a}}$.

Theorem 1: *The directional derivative of a scalar field f at a point $P\ (x, y, z)$ in the direction of a unit vector $\hat{\mathbf{a}}$ is given by* $\dfrac{df}{ds} = \nabla\ f\ \bullet \hat{\mathbf{a}}$.

Proof: Let $f\ (x, y, z)$ define a scalar field in the region R. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ denote the position vector of any point $P\ (x, y, z)$ in this region. If s denotes the distance of P from some fixed point A in the direction of $\hat{\mathbf{a}}$, then δs denotes small element at P in the direction of $\hat{\mathbf{a}}$. Therefore $\dfrac{d\mathbf{r}}{ds}$ is a unit vector at P in this direction *i.e.*, $\dfrac{d\mathbf{r}}{ds} = \hat{\mathbf{a}}$.

But $\qquad\qquad \mathbf{r} = x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}$.

$$\therefore \qquad \frac{d\mathbf{r}}{ds} = \frac{dx}{ds}\,\mathbf{i} + \frac{dy}{ds}\,\mathbf{j} + \frac{dz}{ds}\,\mathbf{k} = \hat{\mathbf{a}}.$$

Now
$$\nabla f \bullet \hat{\mathbf{a}} = \left(\mathbf{i}\,\frac{\partial f}{\partial x} + \mathbf{j}\,\frac{\partial f}{\partial y} + \mathbf{k}\,\frac{\partial f}{\partial z}\right) \bullet \left(\frac{dx}{ds}\,\mathbf{i} + \frac{dy}{ds}\,\mathbf{j} + \frac{dz}{ds}\,\mathbf{k}\right)$$

$$= \frac{\partial f}{dx}\,\frac{dx}{ds} + \frac{\partial f}{dy}\,\frac{dy}{ds} + \frac{\partial f}{dz}\,\frac{dz}{ds} = \frac{d\,f}{ds}$$

= directional derivative of f at P in the direction of $\hat{\mathbf{a}}$.

Alternative Proof: Let Q be a point in the neighbourhood of P in the direction of the given unit vector $\hat{\mathbf{a}}$. If l, m, n are the direction cosines of the line PQ, then $l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k}$ = the unit vector in the direction of $PQ = \hat{\mathbf{a}}$. Further if $PQ = \delta s$, then the co-ordinates of Q are $(x + l\delta s, y + m\delta s, z + n\delta s)$. Now the directional derivative of f at P in the direction of $\hat{\mathbf{a}}$ is

$$= \lim_{Q \to P} \frac{f\,(Q) - f\,(P\,)}{PQ}$$

$$= \lim_{\delta s \to 0} \frac{f\,(x + l\delta s\,,\, y + m\delta s\,,\, z + n\delta s) - f\,(x\,,\, y\,,\, z)}{\delta s}$$

$$= \lim_{\delta s \to 0} \frac{f\,(x\,,\,y\,,\,z) + \left(l\delta s\,\frac{\partial f}{\partial x} + m\delta s\,\frac{\partial f}{\partial y} + n\delta s\,\frac{\partial f}{\partial z}\right) + \ldots - f\,(x\,,\,y\,,\,z)}{\delta s},$$

on expanding by Taylor's theorem

$$= l\,\frac{\partial f}{\partial x} + m\,\frac{\partial f}{\partial y} + n\,\frac{\partial f}{\partial z}$$

$$= \left(\frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}\right) \bullet (l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k}) = \nabla f \bullet \hat{\mathbf{a}}\,.$$

Theorem 2: *If $\hat{\mathbf{n}}$ be a unit vector normal to the level surface $f\,(x\,,\,y\,,\,z) = c$ at a point $P\,(x\,,\,y\,,\,z)$ and n be the distance of P from some fixed point A in the direction of $\hat{\mathbf{n}}$ so that δn represents element of normal at P in the direction of $\hat{\mathbf{n}}$, then* $\text{grad } f = \dfrac{df}{dn}\,\hat{\mathbf{n}}$.

Proof: We have $\text{grad } f = \nabla f = \dfrac{\partial f}{\partial x}\,\mathbf{i} + \dfrac{\partial f}{\partial y}\,\mathbf{j} + \dfrac{\partial f}{\partial z}\,\mathbf{k}$.

Also grad f is a vector normal to the surface $f\,(x\,,\,y\,,\,z) = c$. Since $\hat{\mathbf{n}}$ is a unit vector normal to the surface $f\,(x\,,\,y\,,\,z) = c$, therefore let $\text{grad } f = A\,\hat{\mathbf{n}}$, where A is some scalar to be determined.

Now $\dfrac{df}{dn}$ = directional derivative of f in the direction of $\hat{\mathbf{n}}$

$$= \nabla f \bullet \hat{\mathbf{n}} = A\,\hat{\mathbf{n}} \bullet \hat{\mathbf{n}} = A. \qquad [\because\ \nabla f = \text{grad } f = A\,\hat{\mathbf{n}}\ ]$$

$$\therefore \quad \text{grad } f = \nabla f = \frac{df}{dn}\,\hat{\mathbf{n}}.$$

Note: If the vector $\hat{\mathbf{n}}$ is in the direction of f increasing, then $\dfrac{df}{dn}$ is positive. Therefore ∇f is a vector normal to the surface $f(x, y, z) = c$ in the direction of f increasing.

Theorem 3: *Grad f is a vector in the direction of which the maximum value of the directional derivative of f i.e.,* $\dfrac{df}{ds}$ *occurs.*

Proof: The directional derivative of f in the direction of $\hat{\mathbf{a}}$ is given by

$$\frac{df}{ds} = \nabla f \bullet \hat{\mathbf{a}} = \left(\frac{df}{dn}\,\hat{\mathbf{n}}\right) \bullet \hat{\mathbf{a}} \qquad \left[\because\ \nabla f = \frac{df}{dn}\,\hat{\mathbf{n}}\right]$$

$$= \frac{df}{dn}(\hat{\mathbf{n}} \bullet \hat{\mathbf{a}}) = \frac{df}{dn}\cos\theta, \text{ where } \theta \text{ is the angle between } \hat{\mathbf{a}} \text{ and } \hat{\mathbf{n}}.$$

Now $\dfrac{df}{dn}$ is fixed. Therefore $\dfrac{df}{dn}\cos\theta$ is maximum when $\cos\theta$ is maximum *i.e.,* when $\cos\theta = 1$. But $\cos\theta$ will be 1 when the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{n}}$ is 0 *i.e.,* when $\hat{\mathbf{a}}$ is along the unit normal vector $\hat{\mathbf{n}}$.

Therefore the directional derivative is maximum along the normal to the surface. Its maximum value is $= \dfrac{df}{dn} = |\text{grad } f|$.

7 Tangent Plane and Normal to a Level Surface

Tangent plane to a surface: The tangent to any curve drawn on a surface is called a *tangent line* to the surface. All the tangent lines to a surface at the point P lie in a plane. This plane is called the *tangent plane* to the surface at the point P on it.

Normal to a surface: The normal to a surface at the point P is a straight line passing through P and perpendicular to the tangent plane at P.

Angle between two surfaces: The angle between the two surfaces at a point P is the angle between the normals to the two surfaces at that point.

Equations of the Tangent Plane and Normal to a Surface:

To find the equations of the tangent plane and normal to the surface $f(x, y, z) = c$.

Let $f(x, y, z) = c$ be the equation of a level surface. Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ be the position vector of any point $P(x, y, z)$ on this surface.

Then $\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$ is a vector along the normal to the surface at P *i.e.*, ∇f is perpendicular to the tangent plane at P.

Tangent plane at *P*: Let $\mathbf{R} = X\,\mathbf{i} + Y\,\mathbf{j} + Z\,\mathbf{k}$ be the position vector of any current point $Q\,(X, Y, Z)$ on the tangent plane at P to the surface. The vector

$$\overrightarrow{PQ} = \mathbf{R} - \mathbf{r} = (X - x)\,\mathbf{i} + (Y - y)\,\mathbf{j} + (Z - z)\,\mathbf{k}$$

lies in the tangent plane at P. Therefore it is perpendicular to the vector ∇f.

$$\therefore \qquad (\mathbf{R} - \mathbf{r}) \bullet \nabla f = 0$$

or
$$[(X - x)\,\mathbf{i} + (Y - y)\,\mathbf{j} + (Z - z)\,\mathbf{k}] \bullet \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right) = 0$$

or
$$(X - x)\frac{\partial f}{\partial x} + (Y - y)\frac{\partial f}{\partial y} + (Z - z)\frac{\partial f}{\partial z} = 0, \qquad \ldots(1)$$

is the equation of the tangent plane at P.

Normal at *P*: Let $\mathbf{R} = X\,\mathbf{i} + Y\,\mathbf{j} + Z\,\mathbf{k}$

be the position vector of any current point $Q\,(X, Y, Z)$ on the normal at P to the surface. The vector $\overrightarrow{PQ} = \mathbf{R} - \mathbf{r} = (X - x)\,\mathbf{i} + (Y - y)\,\mathbf{j} + (Z - z)\,\mathbf{k}$ lies along the normal at P to the surface. Therefore it is parallel to the vector ∇f.

$$\therefore \qquad (\mathbf{R} - \mathbf{r}) \times \nabla f = \mathbf{0} \qquad \ldots(2)$$

is the vector equation of the normal at P to the given surface.

Cartesian form: The vectors

$$(X - x)\,\mathbf{i} + (Y - y)\,\mathbf{j} + (Z - z)\,\mathbf{k} \text{ and } \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k},$$

will be parallel if

$$(X - x)\,\mathbf{i} + (Y - y)\,\mathbf{j} + (Z - z)\,\mathbf{k} = p\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}\right),$$

where p is some scalar.

Equating the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, we get

$$X - x = p\frac{\partial f}{\partial x},\ Y - y = p\frac{\partial f}{\partial y},\ Z - z = p\frac{\partial f}{\partial z}$$

or
$$\frac{X - x}{\dfrac{\partial f}{\partial x}} = \frac{Y - y}{\dfrac{\partial f}{\partial y}} = \frac{Z - z}{\dfrac{\partial f}{\partial z}} \qquad \ldots(3)$$

which are the equations of the normal at P.

Note: The vector $\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$ is along the normal to the surface $F\,(x, y, z) = 0$ at the point (x, y, z).

Illustrative Examples

Example 5: *Find a unit normal vector to the level surface* $x^2 y + 2xz = 4$ *at the point* $(2, -2, 3)$. **(Kashi 2014)**

Solution: The equation of the level surface is $f(x, y, z) \equiv x^2 y + 2xz = 4$.

The vector grad f is along the normal to the surface at the point (x, y, z).

We have grad $f = \nabla (x^2 y + 2xz) = (2xy + 2z)\,\mathbf{i} + x^2\,\mathbf{j} + 2x\,\mathbf{k}$.

$\therefore$ at the point $(2, -2, 3)$, grad $f = -2\,\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$.

$\therefore$ $-2\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}$ is a vector along the normal to the given surface at the point $(2, -2, 3)$.

Hence a unit normal vector to the surface at this point

$$= \frac{-2\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}}{|-2\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}|} = \frac{-2\,\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k}}{\sqrt{(4 + 16 + 16)}} = -\frac{1}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{2}{3}\,\mathbf{k}.$$

The vector $-\left(-\frac{1}{3}\,\mathbf{i} + \frac{2}{3}\,\mathbf{j} + \frac{2}{3}\,\mathbf{k}\right)$ *i.e.*, $\frac{1}{3}\,\mathbf{i} - \frac{2}{3}\,\mathbf{j} - \frac{2}{3}\,\mathbf{k}$ is also a unit normal vector to the given surface at the point $(2, -2, 3)$.

Alternate Solution: The given surface is

$$\phi(x, y, z) \equiv x^2 y + 2xz - 4 = 0.$$

We have $\frac{\partial \phi}{\partial x} = 2xy + 2z,\ \frac{\partial \phi}{\partial y} = x^2,\ \frac{\partial \phi}{\partial z} = 2z.$

$\therefore$ At the point $(2, -2, 3)$, we have

$$\frac{\partial \phi}{\partial x} = -2,\ \frac{\partial \phi}{\partial y} = 4,\ \frac{\partial \phi}{\partial z} = 4.$$

$\therefore$ A vector along the normal to the given surface at the point $(2, -2, 3)$

$$= \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) = (-2, 4, 4) \quad \text{or} \quad (-1, 2, 2).$$

We have $|(-1, 2, 2)| = \sqrt{(-1)^2 + 2^2 + 2^2} = 3.$

Hence, a unit normal vector to the given surface at the point $(2, -2, 3)$

$$= \frac{1}{3}(-1, 2, 2) = \left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right).$$

Example 6: *Find the directional derivative of* $f(x, y, z) = x^2 yz + 4xz^2$ *at the point* $(1, -2, -1)$ *in the direction of the vector* $2\,\mathbf{i} - \mathbf{j} - 2\,\mathbf{k}$.

(Bundelkhand 2007; Rohilkhand 11; Kashi 13; Agra 14)

Solution: We have $f(x, y, z) = x^2 yz + 4xz^2$.

$$\therefore \quad \text{grad } f = (2xyz + 4z^2)\,\mathbf{i} + x^2 z\mathbf{j} + (x^2 y + 8xz)\,\mathbf{k}$$

$$= 8\mathbf{i} - \mathbf{j} - 10\,\mathbf{k} \text{ at the point } (1, -2, -1).$$

If $\hat{\mathbf{a}}$ be the unit vector in the direction of the vector $2\mathbf{i}-\mathbf{j}-2\mathbf{k}$, then

$$\hat{\mathbf{a}}=\frac{2\,\mathbf{i}-\mathbf{j}-2\,\mathbf{k}}{\sqrt{(4+1+4)}}=\frac{2}{3}\,\mathbf{i}-\frac{1}{3}\,\mathbf{j}-\frac{2}{3}\,\mathbf{k}.$$

Therefore the required directional derivative is

$$\frac{df}{ds}=\text{grad } f\bullet\hat{\mathbf{a}}=(8\mathbf{i}-\mathbf{j}-10\mathbf{k})\bullet\left(\frac{2}{3}\,\mathbf{i}-\frac{1}{3}\,\mathbf{j}-\frac{2}{3}\,\mathbf{k}\right)$$

$$=\frac{16}{3}+\frac{1}{3}+\frac{20}{3}=\frac{37}{3}.$$

Since this is positive, f is increasing in this direction.

Example 7: *Find the directional derivative of the function* $f=x^2-y^2+2z^2$ *at the point* $P\,(1,2\,,3)$ *in the direction of the line* PQ *where* Q *is the point* $(5,0,4)$.

(Purvanchal 2007; Kashi 14)

Solution: Here

$$\text{grad } f=\frac{\partial f}{\partial x}\,\mathbf{i}+\frac{\partial f}{\partial y}\,\mathbf{j}+\frac{\partial f}{\partial z}\,\mathbf{k}$$

$$=2x\mathbf{i}-2y\mathbf{j}+4z\mathbf{k}=2\mathbf{i}-4\mathbf{j}+12\mathbf{k}\text{ at the point }(1,2\,,3).$$

Also $\overrightarrow{PQ}=$ position vector of Q − position vector of P

$$=(5\mathbf{i}+0\,\mathbf{j}+4\mathbf{k})-(\mathbf{i}+2\,\mathbf{j}+3\mathbf{k})=4\mathbf{i}-2\mathbf{j}+\mathbf{k}.$$

If $\hat{\mathbf{a}}$ be the unit vector in the direction of the vector $\overrightarrow{PQ}$, then

$$\hat{\mathbf{a}}=\frac{4\,\mathbf{i}-2\,\mathbf{j}+\mathbf{k}}{\sqrt{(16+4+1)}}=\frac{4\,\mathbf{i}-2\,\mathbf{j}+\mathbf{k}}{\sqrt{(21)}}.$$

∴ the required directional derivative

$$=(\text{grad } f\,)\bullet\hat{\mathbf{a}}=(2\mathbf{i}-4\,\mathbf{j}+12\,\mathbf{k})\bullet\left\{\frac{4\mathbf{i}-2\,\mathbf{j}+\mathbf{k}}{\sqrt{(21)}}\right\}$$

$$=\frac{28}{\sqrt{(21)}}=\frac{28}{21}\sqrt{(21)}=\frac{4}{3}\sqrt{(21)}.$$

Example 8: *In what direction from the point* $(1,1,-1)$ *is the directional derivative of* $f=x^2-2y^2+4z^2$ *a maximum ? Also find the value of this maximum directional derivative.*

(Kanpur 2008)

Solution: We have

$$\text{grad } f=2x\,\mathbf{i}-4y\,\mathbf{j}+8z\,\mathbf{k}$$

$$=2\,\mathbf{i}-4\,\mathbf{j}-8\,\mathbf{k}\text{ at the point }(1,1,-1).$$

The directional derivative of f is maximum in the direction of grad f

$$=2\,\mathbf{i}-4\,\mathbf{j}-8\,\mathbf{k}.$$

The maximum value of this directional derivative

$$=|\text{grad } f\,|=|2\,\mathbf{i}-4\,\mathbf{j}-8\,\mathbf{k}|=\sqrt{(4+16+64)}=\sqrt{(84)}=2\sqrt{(21)}.$$

Example 9: *What is the greatest rate of increase of* $u = xyz^2$ *at the point* $(1, 0, 3)$ *?*

Solution: We have $\nabla u = yz^2\ \mathbf{i} + xz^2\ \mathbf{j} + 2xyz\ \mathbf{k}$.

$\therefore$ at the point (1, 0, 3), we have $\nabla u = 0\ \mathbf{i} + 9\ \mathbf{j} + 0\ \mathbf{k} = 9\ \mathbf{j}$.

The greatest rate of increase of u at the point (1, 0, 3)

$$= \text{the maximum value of } \frac{du}{ds} \text{ at the point } (1, 0, 3)$$

$$= |\nabla u|, \text{ at the point } (1, 0, 3) \quad = |9\mathbf{j}| = 9.$$

Example 10: *Find the equations of the tangent plane and normal to the surface* $2xz^2 - 3xy - 4x = 7$ *at the point* $(1, -1, 2)$.

Solution: The equation of the surface is

$$f(x, y, z) \equiv 2xz^2 - 3xy - 4x = 7.$$

We have

$$\text{grad } f = (2z^2 - 3y - 4)\ \mathbf{i} - 3x\ \mathbf{j} + 4xz\ \mathbf{k}$$

$$= 7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}, \text{ at the point } (1, -1, 2).$$

$\therefore$ $7\ \mathbf{i} - 3\ \mathbf{j} + 8\ \mathbf{k}$ is a vector along the normal to the surface at the point (1, – 1, 2).

The position vector of the point (1, – 1, 2) is $= \mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

If $\mathbf{R} = X\ \mathbf{i} + Y\ \mathbf{j} + Z\ \mathbf{k}$ is the position vector of any current point (X, Y, Z) on the tangent plane at (1, – 1, 2), then the vector $\mathbf{R} - \mathbf{r}$ is perpendicular to the vector grad f.

$\therefore$ the equation of the tangent plane is $(\mathbf{R} - \mathbf{r}) \bullet \text{grad } f = 0$

i.e., $\{(X\ \mathbf{i} + Y\ \mathbf{j} + Z\ \mathbf{k}) - (\mathbf{i} - \mathbf{j} + 2\mathbf{k})\} \bullet (7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}) = 0$

i.e., $\{(X - 1)\ \mathbf{i} + (Y + 1)\ \mathbf{j} + (Z - 2)\ \mathbf{k}\} \bullet (7\mathbf{i} - 3\mathbf{j} + 8\mathbf{k}) = 0$

i.e., $7(X - 1) - 3(Y + 1) + 8(Z - 2) = 0.$

The equations of the normal to the surface at the point (1, – 1, 2) are

$$\frac{X - 1}{\dfrac{\partial f}{\partial x}} = \frac{Y + 1}{\dfrac{\partial f}{\partial y}} = \frac{Z - 2}{\dfrac{\partial f}{\partial z}} \quad \textit{i.e.,} \quad \frac{X - 1}{7} = \frac{Y + 1}{-3} = \frac{Z - 2}{8}.$$

Example 11: *Find the equations of the tangent plane and normal to the surface* $xyz = 4$ *at the point* $(1, 2, 2)$. **(Meerut 2000)**

Solution: The equation of the surface is $f(x, y, z) \equiv xyz - 4 = 0$.

We have grad $f = yz\ \mathbf{i} + xz\ \mathbf{j} + xy\ \mathbf{k} = 4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, at the point (1, 2, 2).

$\therefore$ $4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ is a vector along the normal to the surface at the point (1, 2, 2).

The position vector of the point (1, 2, 2) is $= \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

If $\mathbf{R} = X\ \mathbf{i} + Y\ \mathbf{j} + Z\ \mathbf{k}$ is the position vector of any current point (X, Y, Z) on the tangent plane at (1, 2, 2), the equation of the tangent plane is

$$(\mathbf{R} - \mathbf{r}) \bullet \text{grad } f = 0,$$

i.e., $\{(X\,\mathbf{i}+Y\,\mathbf{j}+Z\,\mathbf{k})-(\mathbf{i}+2\mathbf{j}+2\mathbf{k})\}\bullet(4\mathbf{i}+2\mathbf{j}+2\mathbf{k})=0$

i.e., $\{(X-1)\,\mathbf{i}+(Y-2)\,\mathbf{j}+(Z-2)\,\mathbf{k}\}\bullet(4\mathbf{i}+2\mathbf{j}+2\mathbf{k})=0$

i.e., $4(X-1)+2(Y-2)+2(Z-2)=0$

i.e., $4X+2Y+2Z=12$, *i.e.,* $2X+Y+Z=6$.

The equations of the normal to the surface at the point (1, 2 , 2) are

$$\frac{X-1}{\dfrac{\partial f}{\partial x}}=\frac{Y-2}{\dfrac{\partial f}{\partial y}}=\frac{Z-2}{\dfrac{\partial f}{\partial z}}$$

i.e., $$\frac{X-1}{4}=\frac{Y-2}{2}=\frac{Z-2}{2},\ \textit{i.e.,}\ \frac{X-1}{2}=\frac{Y-2}{1}=\frac{Z-2}{1}.$$

Example 12: *Find the angle between the surfaces* $x^2+y^2+z^2=9$, *and* $z=x^2+y^2-3$ *at the point* $(2,-1,2)$. **(Meerut 2001)**

Solution: Angle between two surfaces at a point is the angle between the normals to the surfaces at that point. Let $f_1=x^2+y^2+z^2$ and $f_2=x^2+y^2-z$.

Then grad $f_1=2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}$ and grad $f_2=2x\,\mathbf{i}+2y\,\mathbf{j}-\mathbf{k}$.

Let $\mathbf{n}_1=\text{grad } f_1$ at the point $(2,-1,2)$ and $\mathbf{n}_2=\text{grad } f_2$ at the point $(2,-1,2)$.

Then $\mathbf{n}_1=4\mathbf{i}-2\mathbf{j}+4\mathbf{k}$ and $\mathbf{n}_2=4\mathbf{i}-2\mathbf{j}-\mathbf{k}$.

The vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are along normals to the two surfaces at the point $(2,-1,2)$. If θ is the angle between these vectors, then

$$\mathbf{n}_1\bullet\mathbf{n}_2=|\mathbf{n}_1|\,|\mathbf{n}_2|\cos\theta$$

or $$16+4-4=\sqrt{(16+4+16)}\,\sqrt{(16+4+1)}\cos\theta.$$

$\therefore$ $$\cos\theta=\frac{16}{6\sqrt{(21)}}\quad\text{or}\quad\theta=\cos^{-1}\frac{8}{3\sqrt{(21)}}.$$

Comprehensive Exercise 2

1. (i) Find the gradient and the unit normal to the level surface $x^2+y-z=4$ at the point $(2,0,0)$.

   (ii) Find the unit normal to the surface $z=x^2+y^2$ at the point $(-1,-2,5)$.

2. (i) Find the unit vector normal to the surface $x^2-y^2+z=2$ at the point $(1,-1,2)$.

   (ii) Find the unit normal to the surface $x^4-3xyz+z^2+1=0$ at the point $(1,1,1)$.

   (iii) Find a unit normal vector to the surface $x^2y+2xz=4$ at the point $(2,-2,3)$.

3. (i) Find the directional derivatives of a scalar point function f in the direction of coordinate axes.

(ii) Find the directional derivative of $\phi = xy + yz + zx$ in the direction of the vector $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ at $(1, 2, 0)$. **(Kumaun 2015)**

(iii) Find the directional derivative of $\phi(x, y, z) = x^2 yz + 4xz^2$ at the point $(1, -2, 1)$ in the direction of $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

(iv) Find the directional derivative of $f(x, y, z) = x^2 - 2y^2 + 4z^2$ at the point $(1, 1, -1)$ in the direction of $2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

(v) Find the directional derivative of the function $f = xy + yz + zx$ in the direction of the vector $2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ at the point $(3, 1, 2)$.

(vi) Find the directional derivative of the function $xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t, y = t^2, z = t^3$ at the point $(1, 1, 1)$.

4. Find the directional derivatives of $\phi = xyz$ at the point $(2, 2, 2)$, in the directions (i) $\mathbf{i}$, (ii) $\mathbf{j}$, (iii) $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

5. For the function $f = y/(x^2 + y^2)$, find the value of the directional derivative making an angle $30°$ with the positive x-axis at the point $(0, 1)$.

6. (i) Find the greatest value of the directional derivative of the function $2x^2 - y - z^4$ at the point $(2, -1, 1)$.

(ii) In what direction the directional derivative of $\phi = x^2 y^2 z$ from $(1, 1, 2)$ will be maximum and what is its magnitude ? Also find a unit normal vector to the surface $x^2 y^2 z = 2$ at the point $(1, 1, 2)$.

(iii) Find the maximum value of the directional derivative of $\phi = x^2 yz$ at the point $(1, 4, 1)$.

(iv) Calculate the maximum rate of change and the corresponding direction for the function $\phi = x^2 y^3 z^4$ at the point $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

(v) Find the values of the constants a, b, c so that the directional derivative of $\phi = ax^2 + by^2 + cz^2$ at $(1, 1, 2)$ has a maximum magnitude 4 in the direction parallel to y-axis.

7. Find the equation of the tangent plane to the surface $yz - zx + xy + 5 = 0$, at the point $(1, -1, 2)$.

8. (i) Find the equations of the tangent plane and normal to the surface $x^2 + y^2 + z^2 = 25$ at the point $(4, 0, 3)$.

(ii) Given the curve $x^2 + y^2 + z^2 = 1, x + y + z = 1$ (intersection of two surfaces), find the equations of the tangent line at the point $(1, 0, 0)$.

(iii) Find the equations of the tangent plane and normal to the surface $z = x^2 + y^2$ at the point $(2, -1, 5)$.

9. (i) Find the equations of the tangent plane and the normal to the surface $x^2 + 2y^2 + 3z^2 = 12$ at the point $(1, 2, -1)$.

(ii) Find the equations of the tangent plane and the normal to the surface $xy + yz + zx = 1$, at the point $(2, 3, -1)$.

(iii) Find the equations of the tangent plane and the normal to the surface $z = x^2 - 2xy - y^2$ at the point $(1, 2, -7)$.

(iv) Find the equation of the tangent plane to the surface $x^2 + y^2 + z^2 = 9$ at $(2, -1, 2)$.

(v) Find the equation of the tangent plane to the surface $z = x^2 + y^2$ at the point $(1, -1, 2)$.

10. Show that the directional derivative of a scalar point function at any point along any tangent line to the level surface at the point is zero.

11. If **F** and f are point functions, show that the components of the former, tangential and normal to the level surface $f = 0$ are

$$\frac{\nabla f \times (\mathbf{F} \times \nabla f)}{(\nabla f)^2} \text{ and } \frac{(\mathbf{F} \bullet \nabla f)\, \nabla f}{(\nabla f)^2}.$$

12. Find the angle of intersection at $(4, -3, 2)$ of spheres $x^2 + y^2 + z^2 = 29$ and $x^2 + y^2 + z^2 + 4x - 6y - 8z - 47 = 0$.

13. Find the constants a and b so that the surface $ax^2 - byz = (a + 2)x$ will be orthogonal to the surface $4x^2 y + z^3 = 4$ at the point $(1, -1, 2)$.

14. Show that the sum of the squares of the intercepts on the coordinate axes made by the tangent plane to the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$ is constant.

Answers 2

1. (i) $4\mathbf{i} + \mathbf{j} - \mathbf{k}$; $\frac{1}{3\sqrt{2}}(4\mathbf{i} + \mathbf{j} - \mathbf{k})$ (ii) $\frac{-(2\mathbf{i} + 4\mathbf{j} + \mathbf{k})}{\sqrt{(21)}}$

2. (i) $\frac{1}{3}(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ (ii) $\frac{\mathbf{i} - 3\mathbf{j} - \mathbf{k}}{\sqrt{(11)}}$ (iii) $\left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

3. (i) $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ in the directions of **i**, **j** and **k**

(ii) $\frac{10}{3}$ (iii) $-\frac{13}{3}$ (iv) $\frac{8}{\sqrt{6}}$

(v) $\frac{45}{7}$ (vi) $\frac{18}{\sqrt{(14)}}$

4. (i) 4 (ii) 4 (iii) $4\sqrt{3}$

5. $-1/2$

6. (i) 9

(ii) $\sqrt{(33)}$ in the direction of the vector $4\,\mathbf{i} + 4\,\mathbf{j} + \mathbf{k}$; $\dfrac{4\,\mathbf{i} + 4\,\mathbf{j} + \mathbf{k}}{\sqrt{(33)}}$

(iii) 9

(iv) $324\sqrt{2}$ in the direction of the vector $108\,(\mathbf{i} + \mathbf{j} - 4\,\mathbf{k})$

(v) $a = 0, b = 2, c = 0$

7. $3X - 3Y + 2Z = 10$

8. (i) $4x + 3z = 25$; $\dfrac{x-4}{4} = \dfrac{y}{0} = \dfrac{z-3}{3}$

(ii) $X = 1, \dfrac{Y}{-1} = \dfrac{Z}{1}$ (iii) $4x - 2y - z = 5$; $\dfrac{x-2}{4} = \dfrac{y+1}{-2} = \dfrac{z-5}{-1}$

9. (i) $x + 4y - 3z = 12$; $\dfrac{x-1}{1} = \dfrac{y-2}{4} = \dfrac{z+1}{-3}$

(ii) $2x + y + 5z = 2$; $\dfrac{x-2}{2} = \dfrac{y-3}{1} = \dfrac{z+1}{5}$

(iii) $2x + 6y + z = 7$; $\dfrac{x-1}{2} = \dfrac{y-2}{6} = \dfrac{z+7}{1}$

(iv) $2x - y + 2z = 9$ (v) $2x - 2y - z = 2$

12. $\cos^{-1}\sqrt{(19/29)}$ **13.** $a = 5/2, b = 1$

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8 Divergence of a Vector Point Function

(Agra 2005)

Definition: *Let* **V** *be any given differentiable vector point function. Then the divergence of* **V**, *written as,* $\nabla \bullet \mathbf{V}$ or *div* **V**,

is defined as

$$\text{div } \mathbf{V} = \nabla \bullet \mathbf{V} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \bullet \mathbf{V}$$

$$= \mathbf{i} \bullet \frac{\partial \mathbf{V}}{\partial x} + \mathbf{j} \bullet \frac{\partial \mathbf{V}}{\partial y} + \mathbf{k} \bullet \frac{\partial \mathbf{V}}{\partial z} = \Sigma\, \mathbf{i} \bullet \frac{\partial \mathbf{V}}{\partial x}.$$

It should be noted that div **V** is a scalar quantity. Thus *the divergence of a vector point function is a scalar point function.*

Theorem: *If* $\mathbf{V} = V_1\,\mathbf{i} + V_2\,\mathbf{j} + V_3\,\mathbf{k}$ *is a differentiable vector point function, then*

$$\text{div } \mathbf{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

Proof: We have by definition

$$\text{div } \mathbf{V} = \nabla \bullet \mathbf{V} = \mathbf{i} \bullet \frac{\partial \mathbf{V}}{\partial x} + \mathbf{j} \bullet \frac{\partial \mathbf{V}}{\partial y} + \mathbf{k} \bullet \frac{\partial \mathbf{V}}{\partial z}.$$

Now $\mathbf{V} = V_1 \mathbf{i} + V_2 \mathbf{j} + V_3 \mathbf{k}$.

$\therefore$ $$\frac{\partial \mathbf{V}}{\partial x} = \frac{\partial V_1}{\partial x} \mathbf{i} + \frac{\partial V_2}{\partial x} \mathbf{j} + \frac{\partial V_3}{\partial x} \mathbf{k}.$$

$\therefore$ $$\mathbf{i} \bullet \frac{\partial \mathbf{V}}{\partial x} = \mathbf{i} \bullet \left(\frac{\partial V_1}{\partial x} \mathbf{i} + \frac{\partial V_2}{\partial x} \mathbf{j} + \frac{\partial V_3}{\partial x} \mathbf{k} \right) = \frac{\partial V_1}{\partial x}.$$

Similarly, $\mathbf{j} \bullet \dfrac{\partial \mathbf{V}}{\partial y} = \dfrac{\partial V_2}{\partial y}$ and $\mathbf{k} \bullet \dfrac{\partial \mathbf{V}}{\partial z} = \dfrac{\partial V_3}{\partial z}$.

Hence, $$\text{div } \mathbf{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

Solenoidal Vector: *A vector* $\mathbf{V}$ *is said to be solenoidal if div* $\mathbf{V} = 0$.

9 Curl of a Vector Point Function

(Agra 2005)

Definition: *Let* $\mathbf{f}$ *be any given differentiable vector point function. Then the curl or rotation of* $\mathbf{f}$, *written as* $\nabla \times \mathbf{f}$, *curl* $\mathbf{f}$ *or rot* $\mathbf{f}$ *is defined as*

$$\text{curl } \mathbf{f} = \nabla \times \mathbf{f} = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \times \mathbf{f}$$

$$= \mathbf{i} \times \frac{\partial \mathbf{f}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{f}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{f}}{\partial z} = \Sigma \, \mathbf{i} \times \frac{\partial \mathbf{f}}{\partial x}.$$

It should be noted that curl $\mathbf{f}$ is a vector quantity. *Thus the curl of a vector point function is a vector point function.*

Theorem: *If* $\mathbf{f} = f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k}$ *is a differentiable vector point function, then*

$$\text{curl } \mathbf{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \mathbf{k}.$$

Proof: We have by definition

$$\text{curl } \mathbf{f} = \nabla \times \mathbf{f} = \mathbf{i} \times \frac{\partial \mathbf{f}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{f}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{f}}{\partial z}$$

$$= \mathbf{i} \times \frac{\partial}{\partial x} (f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k}) + \mathbf{j} \times \frac{\partial}{\partial y} (f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k})$$
$$+ \mathbf{k} \times \frac{\partial}{\partial z} (f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k})$$

$$= \mathbf{i} \times \left(\frac{\partial f_1}{\partial x} \mathbf{i} + \frac{\partial f_2}{\partial x} \mathbf{j} + \frac{\partial f_3}{\partial x} \mathbf{k} \right) + \mathbf{j} \times \left(\frac{\partial f_1}{\partial y} \mathbf{i} + \frac{\partial f_2}{\partial y} \mathbf{j} + \frac{\partial f_3}{\partial y} \mathbf{k} \right)$$
$$+ \mathbf{k} \times \left(\frac{\partial f_1}{\partial z} \mathbf{i} + \frac{\partial f_2}{\partial z} \mathbf{j} + \frac{\partial f_3}{\partial z} \mathbf{k} \right)$$

$$= \left(\frac{\partial f_2}{\partial x} \mathbf{k} - \frac{\partial f_3}{\partial x} \mathbf{j} \right) + \left(- \frac{\partial f_1}{\partial y} \mathbf{k} + \frac{\partial f_3}{\partial y} \mathbf{i} \right) + \left(\frac{\partial f_1}{\partial z} \mathbf{j} - \frac{\partial f_2}{\partial z} \mathbf{i} \right)$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \mathbf{k}.$$

Note: It should be noted that the expression for curl **f** can be written immediately if we treat the operator ∇ as a vector quantity. Thus

$$\text{Curl } \mathbf{f} = \nabla \times \mathbf{f} = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \times (f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_2 & f_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ f_1 & f_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ f_1 & f_2 \end{vmatrix} \mathbf{k}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \mathbf{k}.$$

But we must take care that in the expansion of the determinant the operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ must *precede* the functions f_1, f_2, f_3.

Irrotational vector: *A vector* **f** *is said to be irrotational if* $\nabla \times \mathbf{f} = \mathbf{0}$.

10 The Laplacian Operator ∇^2

The Laplacian operator ∇^2 *is defined as* $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

If f is a scalar point function, then $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

It should be noted that $\nabla^2 f$ is also a scalar quantity.

If **f** is a vector point function, then $\nabla^2 \mathbf{f} = \frac{\partial^2 \mathbf{f}}{\partial x^2} + \frac{\partial^2 \mathbf{f}}{\partial y^2} + \frac{\partial^2 \mathbf{f}}{\partial z^2}$.

It should be noted that $\nabla^2 \mathbf{f}$ is also a vector quantity.

Laplace's equation: *The equation* $\nabla^2 f = 0$ *is called Laplace's equation.* A function which satisfies Laplace's equation is called a **harmonic function.**

Illustrative Examples

Example 13: *Prove that div* $\mathbf{r} = 3$. **(Kumaun 2000; Garhwal 01; Bundelkhand 01, 04; Meerut 03, 04, 07B, 08; Agra 08; Lucknow 05; Kanpur 13; Avadh 09; Purvanchal 11)**

Solution: We have $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

By definition, $\text{div}\,\mathbf{r} = \nabla \bullet \mathbf{r} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \bullet \mathbf{r} = \mathbf{i} \bullet \frac{\partial \mathbf{r}}{\partial x} + \mathbf{j} \bullet \frac{\partial \mathbf{r}}{\partial y} + \mathbf{k} \bullet \frac{\partial \mathbf{r}}{\partial z}$

$$= \mathbf{i}\bullet\mathbf{i} + \mathbf{j}\bullet\mathbf{j} + \mathbf{k}\bullet\mathbf{k} \qquad \left[\because \frac{\partial \mathbf{r}}{\partial x} = \mathbf{i}, \frac{\partial \mathbf{r}}{\partial y} = \mathbf{j}, \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k}\right]$$

$$= 1 + 1 + 1 = 3.$$

Example 14: *Prove that curl* $\mathbf{r} = \mathbf{0}$.

(Kumaun 2000; Garhwal 02; Meerut 11; Bundelkhand 06; Agra 08; Avadh 09; Kashi 13; Purvanchal 11; Rohilkhand 14)

Solution: We have by definition

$$\text{curl}\,\mathbf{r} = \nabla \times \mathbf{r} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times \mathbf{r}$$

$$= \mathbf{i} \times \frac{\partial \mathbf{r}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{r}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{r}}{\partial z}.$$

Now $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

$\therefore$ $\frac{\partial \mathbf{r}}{\partial x} = \mathbf{i}, \frac{\partial \mathbf{r}}{\partial y} = \mathbf{j}, \frac{\partial \mathbf{r}}{\partial z} = \mathbf{k}$.

$\therefore$ $\text{curl}\,\mathbf{r} = \mathbf{i} \times \mathbf{i} + \mathbf{j} \times \mathbf{j} + \mathbf{k} \times \mathbf{k}$

$$= \mathbf{0} + \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Example 15: *If* $\mathbf{f} = x^2 y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k}$, *find*

(i) *div* $\mathbf{f}$, **(Kumaun 2008)**

(ii) *curl* $\mathbf{f}$, **(Meerut 2012)**

(iii) *curl curl* $\mathbf{f}$. **(Meerut 2001; Kanpur 14)**

Solution: (i) We have

$$\text{div}\,\mathbf{f} = \nabla \bullet \mathbf{f} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \bullet (x^2 y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k})$$

$$= \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(-2xz) + \frac{\partial}{\partial z}(2yz)$$

$$= 2xy + 0 + 2y$$

$$= 2y(x + 1).$$

(ii) We have curl $\mathbf{f} = \nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & -2xz & 2yz \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(-2xz)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(x^2 y)\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}(x^2 y)\right]\mathbf{k}$$

$$= (2z + 2x)\,\mathbf{i} - 0\,\mathbf{j} + (-2z - x^2)\,\mathbf{k}$$

$$= (2x + 2z)\,\mathbf{i} - (x^2 + 2z)\,\mathbf{k}\,.$$

(iii) We have curl curl $\mathbf{f} = \nabla \times (\nabla \times \mathbf{f}) = \nabla \times [(2x + 2z)\,\mathbf{i} - (x^2 + 2z)\,\mathbf{k}]$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x + 2z & 0 & -x^2 - 2z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-x^2 - 2z)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(-x^2 - 2z) - \frac{\partial}{\partial z}(2x + 2z)\right]\mathbf{j}$$

$$+ \left[0 - \frac{\partial}{\partial y}(2x + 2z)\right]\mathbf{k}$$

$$= 0\,\mathbf{i} - (-2x - 2)\,\mathbf{j} + (0 - 0)\,\mathbf{k} = (2x + 2)\,\mathbf{j}.$$

Example 16: *Determine the constant a so that the vector* $\mathbf{V} = (x + 3y)\,\mathbf{i} + (y - 2z)\,\mathbf{j} + (x + az)\,\mathbf{k}$ *is solenoidal.*

(Rohilkhand 2009B)

Solution: A vector $\mathbf{V}$ is said to be solenoidal if div $\mathbf{V} = 0$.

We have div $\mathbf{V} = \nabla \bullet \mathbf{V} = \dfrac{\partial}{\partial x}(x + 3y) + \dfrac{\partial}{\partial y}(y - 2z) + \dfrac{\partial}{\partial z}(x + az)$

$$= 1 + 1 + a = 2 + a.$$

Now div $\mathbf{V} = 0$ if $2 + a = 0$ *i.e.*, if $a = -2$.

Example 17: *Show that the vector* $\mathbf{V} = (\sin y + z)\,\mathbf{i} + (x \cos y - z)\,\mathbf{j} + (x - y)\,\mathbf{k}$ *is irrotational.*

Solution: A vector $\mathbf{V}$ is said to be irrotational if curl $\mathbf{V} = \mathbf{0}$. We have

curl $\mathbf{V} = \nabla \times \mathbf{V}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \sin y + z & x\cos y - z & x - y \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(x - y) - \frac{\partial}{\partial z}(x \cos y - z)\right] \mathbf{i} - \left[\frac{\partial}{\partial x}(x - y) - \frac{\partial}{\partial z}(\sin y + z)\right] \mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}(x \cos y - z) - \frac{\partial}{\partial y}(\sin y + z)\right] \mathbf{k}$$

$$= (-1 + 1)\,\mathbf{i} - (1 - 1)\,\mathbf{j} + (\cos y - \cos y)\,\mathbf{k} = \mathbf{0}.$$

$\therefore$ **V** is irrotational.

Example 18: *Prove that* $\nabla \bullet (r^3 \mathbf{r}) = 6r^3$. **(Purvanchal 2006, 10)**

Solution: We have $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

$\therefore$ $r^3 \mathbf{r} = r^3\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = r^3 x\,\mathbf{i} + r^3 y\,\mathbf{j} + r^3 z\,\mathbf{k}$.

$\therefore$ $\nabla \bullet (r^3 \mathbf{r}) = \text{div}\,(r^3 \mathbf{r})$

$$= \frac{\partial}{\partial x}(r^3 x) + \frac{\partial}{\partial y}(r^3 y) + \frac{\partial}{\partial z}(r^3 z)$$

$$= r^3 + 3r^2\, x\frac{\partial r}{\partial x} + r^3 + 3r^2 y\frac{\partial r}{\partial y} + r^3 + 3r^2 z\frac{\partial r}{\partial z}$$

$$= 3r^3 + 3r^2\left(x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right) \quad \ldots(1)$$

Now $r^2 = x^2 + y^2 + z^2$.

$\therefore$ $2r\dfrac{\partial r}{\partial x} = 2x$ or $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}\cdot$

Similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}\cdot$

$\therefore$ from (1), $\nabla \bullet (r^3 \mathbf{r}) = 3r^3 + 3r^2\left(x\cdot\dfrac{x}{r} + y\cdot\dfrac{y}{r} + z\cdot\dfrac{z}{r}\right)$

$$= 3r^3 + 3r^2\left(\frac{x^2 + y^2 + z^2}{r}\right)$$

$$= 3r^3 + 3r^2\cdot\frac{r^2}{r} = 3r^3 + 3r^3 = 6r^3.$$

Example 19: *If* **V** *is a constant vector, show that*

(i) *div* $\mathbf{V} = 0$, **(Rohilkhand 2005; Purvanchal 14)**

(ii) *curl* $\mathbf{V} = \mathbf{0}$. **(Purvanchal 2014; Kumaun 15)**

Solution: (i) We have

$$\text{div}\,\mathbf{V} = \mathbf{i} \bullet \frac{\partial \mathbf{V}}{\partial x} + \mathbf{j} \bullet \frac{\partial \mathbf{V}}{\partial y} + \mathbf{k} \bullet \frac{\partial \mathbf{V}}{\partial z}$$

$$= \mathbf{i} \bullet \mathbf{0} + \mathbf{j} \bullet \mathbf{0} + \mathbf{k} \bullet \mathbf{0} = 0.$$

(ii) We have

$$\text{curl } \mathbf{V} = \mathbf{i} \times \frac{\partial \mathbf{V}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{V}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{V}}{\partial z}$$

$$= \mathbf{i} \times \mathbf{0} + \mathbf{j} \times \mathbf{0} + \mathbf{k} \times \mathbf{0} = \mathbf{0}.$$

Example 20: *If* **a** *is a constant vector, find*

(i) *div* **(r × a)** , **(Bundelkhand 2006; Kumaun 07, 11)**

(ii) *curl* **(r × a)**. **(Bundelkhand 2006; Kanpur 06; Kumaun 07; Agra 08; Rohilkhand 14)**

Solution: We have $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

Let $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$. Then the scalars a_1, a_2, a_3 are all constants.

We have

$$\mathbf{r} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= (a_3 y - a_2 z)\,\mathbf{i} + (a_1 z - a_3 x)\,\mathbf{j} + (a_2 x - a_1 y)\,\mathbf{k}.$$

(i) div **(r × a)** $= \dfrac{\partial}{\partial x}(a_3 y - a_2 z) + \dfrac{\partial}{\partial y}(a_1 z - a_3 x) + \dfrac{\partial}{\partial z}(a_3 x - a_1 y)$

$$= 0 + 0 + 0 = 0.$$

(ii) curl **(r × a)** $= \nabla \times (\mathbf{r} \times \mathbf{a})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_3 y - a_2 z & a_1 z - a_3 x & a_2 x - a_1 y \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(a_2 x - a_1 y) - \frac{\partial}{\partial z}(a_1 z - a_3 x)\right]\mathbf{i}$$

$$- \left[\frac{\partial}{\partial x}(a_2 x - a_1 y) - \frac{\partial}{\partial z}(a_3 y - a_2 z)\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}(a_1 z - a_3 x) - \frac{\partial}{\partial y}(a_3 y - a_2 z)\right]\mathbf{k}$$

$$= -2a_1\,\mathbf{i} - 2a_2\,\mathbf{j} - 2a_3\,\mathbf{k} = -2\,(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) = -2\mathbf{a}.$$

Example 21: *If* $\mathbf{V} = e^{xyz}$ **(i + j + k)**, *find curl* **V**.

Solution: We have curl $\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{xyz} & e^{xyz} & e^{xyz} \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(e^{xyz}) - \frac{\partial}{\partial z}(e^{xyz})\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(e^{xyz}) - \frac{\partial}{\partial z}(e^{xyz})\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}(e^{xyz}) - \frac{\partial}{\partial y}(e^{xyz})\right]\mathbf{k}$$

$$= e^{xyz}\,(xz - xy)\,\mathbf{i} + e^{xyz}\,(xy - yz)\,\mathbf{j} + e^{xyz}\,(yz - xz)\,\mathbf{k}.$$

Example 22: *Evaluate div* $\mathbf{f}$ *where* $\mathbf{f} = 2x^2 z\,\mathbf{i} - xy^2 z\,\mathbf{j} + 3y^2 x\,\mathbf{k}$.

Solution: We have

$$\text{div}\,\mathbf{f} = \nabla \bullet \mathbf{f} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \bullet (2x^2 z\,\mathbf{i} - xy^2 z\,\mathbf{j} + 3y^2 x\,\mathbf{k})$$

$$= \frac{\partial}{\partial x}(2x^2 z) + \frac{\partial}{\partial y}(-xy^2 z) + \frac{\partial}{\partial z}(3y^2 x)$$

$$= 4xz - 2xyz + 0 = 2xz\,(2 - y).$$

Example 23: *Show that* $\nabla^2\,(x/r^3) = 0.$ **(Kumaun 2008)**

Solution: $$\nabla^2\left(\frac{x}{r^3}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{x}{r^3}\right).$$

Now $$\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right) = \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)\right\} = \frac{\partial}{\partial x}\left\{\frac{1}{r^3} - \frac{3x}{r^4}\frac{\partial r}{\partial x}\right\}$$

$$= \frac{\partial}{\partial x}\left\{\frac{1}{r^3} - \frac{3x}{r^4}\frac{x}{r}\right\} \quad \left[\because\ r^2 = x^2 + y^2 + z^2 \text{ gives } \frac{\partial r}{\partial x} = \frac{x}{r}\right]$$

$$= \frac{\partial}{\partial x}\left\{\frac{1}{r^3} - \frac{3x^2}{r^5}\right\} = -\frac{3}{r^4}\frac{\partial r}{\partial x} - \frac{6x}{r^5} + \frac{15x^2}{r^6}\frac{\partial r}{\partial x}$$

$$= -\frac{3}{r^4}\frac{x}{r} - \frac{6x}{r^5} + \frac{15x^2}{r^6}\frac{x}{r} = -\frac{9x}{r^5} + \frac{15x^3}{r^7}.$$

Again $$\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right) = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)\right\} = \frac{\partial}{\partial y}\left\{-\frac{3x}{r^4}\frac{\partial r}{\partial y}\right\}$$

$$= \frac{\partial}{\partial y}\left\{-\frac{3x}{r^4}\frac{y}{r}\right\} \qquad \left[\because \frac{\partial r}{\partial y} = \frac{y}{r}\right]$$

$$= \frac{\partial}{\partial y}\left(-\frac{3xy}{r^5}\right) = -\frac{3x}{r^5} + \frac{15xy}{r^6}\frac{\partial r}{\partial y} = -\frac{3x}{r^5} + \frac{15xy^2}{r^7}.$$

Similarly $$\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right) = -\frac{3x}{r^5} + \frac{15xz^2}{r^7}.$$

Therefore, adding we get

$$\nabla^2\left(\frac{x}{r^3}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{x}{r^3}\right)$$

$$= -\frac{9x}{r^5} + \frac{15x^3}{r^7} - \frac{3x}{r^5} + \frac{15xy^2}{r^7} - \frac{3x}{r^5} + \frac{15xz^2}{r^7}$$

$$= -\frac{15x}{r^5} + \frac{15x}{r^7}(x^2 + y^2 + z^2) = -\frac{15x}{r^5} + \frac{15x}{r^7}r^2 = 0.$$

Comprehensive Exercise 3

1. (i) If $\mathbf{F} = x^2 z\,\mathbf{i} - 2y^3 z^2\,\mathbf{j} + xy^2 z\,\mathbf{k}$, find div **F**, curl **F** at (1, – 1, 1).

   (ii) If $\mathbf{f} = (y^2 + z^2 - x^2)\,\mathbf{i} + (z^2 + x^2 - y^2)\,\mathbf{j} + (x^2 + y^2 - z^2)\,\mathbf{k}$,

   find div **f** and curl **f**.

   (iii) If $\mathbf{f} = xy^2\,\mathbf{i} + 2x^2 yz\,\mathbf{j} - 3yz^2\,\mathbf{k}$, find div **f** and curl **f**. What are their values at the point (1, – 1, 1) ? **(Rohilkhand 2005)**

2. Find div **F** and curl **F** where **F** = grad $(x^3 + y^3 + z^3 - 3xyz)$. **(Rohilkhand 2007)**

3. Find the divergence and curl of the vector

   $\mathbf{f} = (x^2 - y^2)\,\mathbf{i} + 2xy\mathbf{j} + (y^2 - xy)\,\mathbf{k}$. **(Bundelkhand 2004)**

4. Given $\phi = 2x^3 y^2 z^4$, find div (grad ϕ).

5. If $u = x^2 - y^2 + 4z$, show that $\nabla^2 u = 0$.

6. If $u = 3x^2 y$ and $v = xz^2 - 2y$, then find grad [(grad u) • (grad v)].

7. If $\mathbf{f} = (x + y + 1)\,\mathbf{i} + \mathbf{j} + (-x - y)\,\mathbf{k}$, prove that **f** • curl **f** = 0.

8. (i) If $\mathbf{f} = f_1\,\mathbf{i} + f_2\,\mathbf{j} + f_3\,\mathbf{k}$, show that

   $\nabla \bullet \mathbf{f} = \nabla f_1 \bullet \mathbf{i} + \nabla f_2 \bullet \mathbf{j} + \nabla f_3 \bullet \mathbf{k}$. **(Bundelkhand 2001)**

   (ii) Prove that $\nabla \bullet (r^3\,\mathbf{r}) = 6\,r^3$.

9. Find the constants a, b, c so that the vector

   $\mathbf{F} = (x + 2y + az)\,\mathbf{i} + (bx - 3y - z)\,\mathbf{j} + (4x + cy + 2z)\,\mathbf{k}$ is irrotational.

   (Bundelkhand 2005; Rohilkhand 08B)

10. Show that the vector $\mathbf{F} = 3y^4 z^2\,\mathbf{i} + 4x^3 z^2\,\mathbf{j} - 3x^2 y^2\,\mathbf{k}$ is solenoid but not irrotational. **(Kashi 2014)**

Answers 3

1. (i) $-3; -6i$

   (ii) $-2(x + y + z)$; $2(y - z)\,\mathbf{i} + 2(z - x)\,\mathbf{j} + 2(x - y)\,\mathbf{k}$

(iii) $y^2 + 2x^2 z - 6yz; 9\,; -(3z^2 + 2x^2 y)\,\mathbf{i} + (4xyz - 2xy)\,\mathbf{k}; -\mathbf{i} - 2\mathbf{k}$

2. $6\,(x + y + z); 0$

3. $4x; (2y - x)\,\mathbf{i} + y\,\mathbf{j} + 4y\,\mathbf{k}$

4. $12xy^2 z^4 + 4x^3 z^4 + 24\,x^3 y^2 z^2$

6. $(6yz^2 - 12x)\,\mathbf{i} + (6xz^2)\,\mathbf{j} + (12xyz)\,\mathbf{k}$

9. $a = 4\,, b = 2\,, c = -1$

11 Vector Identities

1. *Prove that div* $(\mathbf{A} + \mathbf{B})$ = *div* $\mathbf{A}$ + *div* $\mathbf{B}$ *or* $\nabla \bullet (\mathbf{A} + \mathbf{B}) = \nabla \bullet \mathbf{A} + \nabla \bullet \mathbf{B}$.

Proof: We have

$$\text{div }(\mathbf{A} + \mathbf{B}) = \nabla \bullet (\mathbf{A} + \mathbf{B}) = \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right) \bullet (\mathbf{A} + \mathbf{B})$$

$$= \mathbf{i} \bullet \frac{\partial}{\partial x}(\mathbf{A} + \mathbf{B}) + \mathbf{j} \bullet \frac{\partial}{\partial y}(\mathbf{A} + \mathbf{B}) + \mathbf{k} \bullet \frac{\partial}{\partial z}(\mathbf{A} + \mathbf{B})$$

$$= \mathbf{i} \bullet \left(\frac{\partial \mathbf{A}}{\partial x} + \frac{\partial \mathbf{B}}{\partial x}\right) + \mathbf{j} \bullet \left(\frac{\partial \mathbf{A}}{\partial y} + \frac{\partial \mathbf{B}}{\partial y}\right) + \mathbf{k} \bullet \left(\frac{\partial \mathbf{A}}{\partial z} + \frac{\partial \mathbf{B}}{\partial z}\right)$$

$$= \left(\mathbf{i} \bullet \frac{\partial \mathbf{A}}{\partial x} + \mathbf{j} \bullet \frac{\partial \mathbf{A}}{\partial y} + \mathbf{k} \bullet \frac{\partial \mathbf{A}}{\partial z}\right) + \left(\mathbf{i} \bullet \frac{\partial \mathbf{B}}{\partial x} + \mathbf{j} \bullet \frac{\partial \mathbf{B}}{\partial y} + \mathbf{k} \bullet \frac{\partial \mathbf{B}}{\partial z}\right)$$

$$= \nabla \bullet \mathbf{A} + \nabla \bullet \mathbf{B} = \text{div }\mathbf{A} + \text{div }\mathbf{B}.$$

2. *Prove that curl* $(\mathbf{A} + \mathbf{B})$ = *curl* $\mathbf{A}$ + *curl* $\mathbf{B}$ *or* $\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$.

(Rohilkhand 2008)

Proof: We have curl $(\mathbf{A} + \mathbf{B}) = \nabla \times (\mathbf{A} + \mathbf{B})$

$$= \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right) \times (\mathbf{A} + \mathbf{B})$$

$$= \Sigma\,\mathbf{i} \times \frac{\partial}{\partial x}(\mathbf{A} + \mathbf{B}) = \Sigma\,\mathbf{i} \times \left(\frac{\partial \mathbf{A}}{\partial x} + \frac{\partial \mathbf{B}}{\partial x}\right)$$

$$= \Sigma\,\mathbf{i} \times \frac{\partial \mathbf{A}}{\partial x} + \Sigma\,\mathbf{i} \times \frac{\partial \mathbf{B}}{\partial x} = \text{curl }\mathbf{A} + \text{curl }\mathbf{B}.$$

3. *If* $\mathbf{A}$ *is a differentiable vector function and* ϕ *is a differentiable scalar function, then*

div $(\phi\mathbf{A})$ = (*grad* ϕ) $\bullet\, \mathbf{A} + \phi$ *div* $\mathbf{A}$ *or* $\nabla \bullet (\phi\mathbf{A}) = (\nabla\phi) \bullet \mathbf{A} + \phi\,(\nabla \bullet \mathbf{A})$.

(Garhwal 2001; Agra 05; Kashi 14)

Proof: We have

$$\text{div }(\phi\mathbf{A}) = \nabla \bullet (\phi\mathbf{A}) = \left(\mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z}\right) \bullet (\phi\mathbf{A})$$

$$= \mathbf{i} \bullet \frac{\partial}{\partial x}(\phi \mathbf{A}) + \mathbf{j} \bullet \frac{\partial}{\partial y}(\phi \mathbf{A}) + \mathbf{k} \bullet \frac{\partial}{\partial z}(\phi \mathbf{A})$$

$$= \Sigma \left\{ \mathbf{i} \bullet \left(\frac{\partial}{\partial x}(\phi \mathbf{A}) \right) \right\} = \Sigma \left\{ \mathbf{i} \bullet \left(\frac{\partial \phi}{\partial x} \mathbf{A} + \phi \frac{\partial \mathbf{A}}{\partial x} \right) \right\}$$

$$= \Sigma \left\{ \mathbf{i} \bullet \left(\frac{\partial \phi}{\partial x} \mathbf{A} \right) \right\} + \Sigma \left\{ \mathbf{i} \bullet \left(\phi \frac{\partial \mathbf{A}}{\partial x} \right) \right\}$$

$$= \Sigma \left\{ \left(\frac{\partial \phi}{\partial x} \mathbf{i} \right) \bullet \mathbf{A} \right\} + \Sigma \left\{ \phi \left(\mathbf{i} \bullet \frac{\partial \mathbf{A}}{\partial x} \right) \right\}$$

[∵ $\mathbf{a} \bullet (m\mathbf{b}) = (m\mathbf{a}) \bullet \mathbf{b} = m(\mathbf{a} \bullet \mathbf{b})$]

$$= \left\{ \Sigma \frac{\partial \phi}{\partial x} \mathbf{i} \right\} \bullet \mathbf{A} + \phi\, \Sigma \left(\mathbf{i} \bullet \frac{\partial \mathbf{A}}{\partial x} \right) = (\nabla \phi) \bullet \mathbf{A} + \phi (\nabla \bullet \mathbf{A}).$$

4. *Prove that curl* $(\phi \mathbf{A}) = $ (*grad* ϕ) $\times \mathbf{A} + \phi$ *curl* $\mathbf{A}$

or $\nabla \times (\phi \mathbf{A}) = (\nabla \phi) \times \mathbf{A} + \phi (\nabla \times \mathbf{A})$.

(Garhwal 2002, 03; Bundelkhand 06)

Proof: We have curl $(\phi \mathbf{A}) = \nabla \times (\phi \mathbf{A}) = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \times (\phi \mathbf{A})$

$$= \Sigma \left\{ \mathbf{i} \times \frac{\partial}{\partial x}(\phi \mathbf{A}) \right\} = \Sigma \left\{ \mathbf{i} \times \left(\frac{\partial \phi}{\partial x} \mathbf{A} + \phi \frac{\partial \mathbf{A}}{\partial x} \right) \right\}$$

$$= \Sigma \left\{ \mathbf{i} \times \left(\frac{\partial \phi}{\partial x} \mathbf{A} \right) \right\} + \Sigma \left\{ \mathbf{i} \times \left(\phi \frac{\partial \mathbf{A}}{\partial x} \right) \right\}$$

$$= \Sigma \left\{ \left(\frac{\partial \phi}{\partial x} \mathbf{i} \right) \times \mathbf{A} \right\} + \Sigma \left\{ \phi \left(\mathbf{i} \times \frac{\partial \mathbf{A}}{\partial x} \right) \right\}$$

[∵ $\mathbf{a} \times (m\mathbf{b}) = (m\mathbf{a}) \times \mathbf{b} = m(\mathbf{a} \times \mathbf{b})$]

$$= \left\{ \Sigma \left(\frac{\partial \phi}{\partial x} \mathbf{i} \right) \right\} \times \mathbf{A} + \phi\, \Sigma \left(\mathbf{i} \times \frac{\partial \mathbf{A}}{\partial x} \right) = (\nabla \phi) \times \mathbf{A} + \phi (\nabla \times \mathbf{A}).$$

5. *Prove that div* $(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \bullet$ *curl* $\mathbf{A} - \mathbf{A} \bullet$ *curl* $\mathbf{B}$

or $\nabla \bullet (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \bullet (\nabla \times \mathbf{A}) - \mathbf{A} \bullet (\nabla \times \mathbf{B})$.

(Garhwal 2003; Agra 05; Meerut 04, 05B, 08, 09; Bundelkhand 05, 07; Kashi 13; Avadh 09)

Proof: We have

$$\text{div } (\mathbf{A} \times \mathbf{B}) = \Sigma \left\{ \mathbf{i} \bullet \frac{\partial}{\partial x}(\mathbf{A} \times \mathbf{B}) \right\} = \Sigma \left\{ \mathbf{i} \bullet \left(\frac{\partial \mathbf{A}}{\partial x} \times \mathbf{B} + \mathbf{A} \times \frac{\partial \mathbf{B}}{\partial x} \right) \right\}$$

$$= \Sigma \left\{ \mathbf{i} \bullet \left(\frac{\partial \mathbf{A}}{\partial x} \times \mathbf{B} \right) \right\} + \Sigma \left\{ \mathbf{i} \bullet \left(\mathbf{A} \times \frac{\partial \mathbf{B}}{\partial x} \right) \right\}$$

$$= \Sigma \left\{ \left(\mathbf{i} \times \frac{\partial \mathbf{A}}{\partial x} \right) \bullet \mathbf{B} \right\} - \Sigma \left\{ \mathbf{i} \bullet \left(\frac{\partial \mathbf{B}}{\partial x} \times \mathbf{A} \right) \right\}$$

[∵ $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}$ and $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = -\, \mathbf{a} \bullet (\mathbf{c} \times \mathbf{b})$]

$$= \left\{ \Sigma \left(\mathbf{i} \times \frac{\partial \mathbf{A}}{\partial x} \right) \right\} \bullet \mathbf{B} - \Sigma \left\{ \left(\mathbf{i} \times \frac{\partial \mathbf{B}}{\partial x} \right) \bullet \mathbf{A} \right\}$$

$$= (\text{curl } \mathbf{A}) \bullet \mathbf{B} - \left\{ \Sigma \left(\mathbf{i} \times \frac{\partial \mathbf{B}}{\partial x} \right) \right\} \bullet \mathbf{A}$$

$$= (\text{curl } \mathbf{A}) \bullet \mathbf{B} - (\text{curl } \mathbf{B}) \bullet \mathbf{A} = \mathbf{B} \bullet \text{curl } \mathbf{A} - \mathbf{A} \bullet \text{curl } \mathbf{B}.$$

6. *Prove that* *curl* $(\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \bullet \nabla)\, \mathbf{A} - \mathbf{B}\, \textit{div}\, \mathbf{A} - (\mathbf{A} \bullet \nabla)\, \mathbf{B} + \mathbf{A}\, \textit{div}\, \mathbf{B}$.

(Garhwal 2001; Meerut 06B; Kumaun 09, 11, 12)

Proof: We have curl $(\mathbf{A} \times \mathbf{B}) = \nabla \times (\mathbf{A} \times \mathbf{B})$

$$= \Sigma \left\{ \mathbf{i} \times \frac{\partial}{\partial x} (\mathbf{A} \times \mathbf{B}) \right\} = \Sigma \left\{ \mathbf{i} \times \left(\mathbf{A} \times \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} \times \mathbf{B} \right) \right\}$$

$$= \Sigma \left\{ \mathbf{i} \times \left(\mathbf{A} \times \frac{\partial \mathbf{B}}{\partial x} \right) \right\} + \Sigma \left\{ \mathbf{i} \times \left(\frac{\partial \mathbf{A}}{\partial x} \times \mathbf{B} \right) \right\}$$

$$= \Sigma \left\{ \left(\mathbf{i} \bullet \frac{\partial \mathbf{B}}{\partial x} \right) \mathbf{A} - (\mathbf{i} \bullet \mathbf{A}) \frac{\partial \mathbf{B}}{\partial x} \right\} + \Sigma \left\{ (\mathbf{i} \bullet \mathbf{B}) \frac{\partial \mathbf{A}}{\partial x} - \left(\mathbf{i} \bullet \frac{\partial \mathbf{A}}{\partial x} \right) \mathbf{B} \right\}$$

$$= \Sigma \left\{ \left(\mathbf{i} \bullet \frac{\partial \mathbf{B}}{\partial x} \right) \mathbf{A} \right\} - \Sigma \left\{ (\mathbf{A} \bullet \mathbf{i}) \frac{\partial \mathbf{B}}{\partial x} \right\}$$
$$+ \Sigma \left\{ (\mathbf{B} \bullet \mathbf{i}) \frac{\partial \mathbf{A}}{\partial x} \right\} - \Sigma \left\{ \left(\mathbf{i} \bullet \frac{\partial \mathbf{A}}{\partial x} \right) \mathbf{B} \right\}$$

$$= \left\{ \Sigma \left(\mathbf{i} \bullet \frac{\partial \mathbf{B}}{\partial x} \right) \right\} \mathbf{A} - \left\{ \mathbf{A} \bullet \Sigma\, \mathbf{i} \frac{\partial}{\partial x} \right\} \mathbf{B} + \left\{ \mathbf{B} \bullet \Sigma\, \mathbf{i} \frac{\partial}{\partial x} \right\} \mathbf{A}$$
$$- \left\{ \Sigma \left(\mathbf{i} \bullet \frac{\partial \mathbf{A}}{\partial x} \right) \right\} \mathbf{B}$$

$$= (\text{div } \mathbf{B})\, \mathbf{A} - (\mathbf{A} \bullet \nabla)\, \mathbf{B} + (\mathbf{B} \bullet \nabla)\, \mathbf{A} - (\text{div } \mathbf{A})\, \mathbf{B}.$$

7. *Prove that* *grad* $(\mathbf{A} \bullet \mathbf{B}) = (\mathbf{B} \bullet \nabla)\, \mathbf{A} + (\mathbf{A} \bullet \nabla)\, \mathbf{B} + \mathbf{B} \times \textit{curl}\, \mathbf{A} + \mathbf{A} \times \textit{curl}\, \mathbf{B}$.

(Agra 2001; Kumaun 15)

Proof: We have

$$\text{grad } (\mathbf{A} \bullet \mathbf{B}) = \nabla (\mathbf{A} \bullet \mathbf{B}) = \Sigma\, \mathbf{i} \frac{\partial}{\partial x} (\mathbf{A} \bullet \mathbf{B}) = \Sigma\, \mathbf{i} \left(\mathbf{A} \bullet \frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x} \bullet \mathbf{B} \right)$$

$$= \Sigma \left\{ \left(\mathbf{A} \bullet \frac{\partial \mathbf{B}}{\partial x} \right) \mathbf{i} \right\} + \Sigma \left\{ \left(\mathbf{B} \bullet \frac{\partial \mathbf{A}}{\partial x} \right) \mathbf{i} \right\}. \qquad \ldots(1)$$

Now we know that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\, \mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\, \mathbf{c}$.

$\therefore$ $\quad (\mathbf{a} \bullet \mathbf{b})\, \mathbf{c} = (\mathbf{a} \bullet \mathbf{c})\, \mathbf{b} - \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

$\therefore$ $$\left(\mathbf{A} \bullet \frac{\partial \mathbf{B}}{\partial x} \right) \mathbf{i} = (\mathbf{A} \bullet \mathbf{i}) \frac{\partial \mathbf{B}}{\partial x} - \mathbf{A} \times \left(\frac{\partial \mathbf{B}}{\partial x} \times \mathbf{i} \right)$$

$$= (\mathbf{A} \bullet \mathbf{i}) \frac{\partial \mathbf{B}}{\partial x} + \mathbf{A} \times \left(\mathbf{i} \times \frac{\partial \mathbf{B}}{\partial x} \right).$$

Thus $$\Sigma\left\{\left(\mathbf{A}\bullet\frac{\partial\mathbf{B}}{\partial x}\right)\mathbf{i}\right\}=\Sigma\left\{(\mathbf{A}\bullet\mathbf{i})\frac{\partial\mathbf{B}}{\partial x}\right\}+\Sigma\left\{\mathbf{A}\times\left(\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}$$

$$=\left\{\mathbf{A}\bullet\Sigma\,\mathbf{i}\,\frac{\partial}{\partial x}\right\}\mathbf{B}+\mathbf{A}\times\Sigma\left(\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}\right)$$

$$=(\mathbf{A}\bullet\nabla)\,\mathbf{B}+\mathbf{A}\times(\nabla\times\mathbf{B}). \qquad \ldots(2)$$

Similarly $$\Sigma\left\{\left(\mathbf{B}\bullet\frac{\partial\mathbf{A}}{\partial x}\right)\mathbf{i}\right\}=(\mathbf{B}\bullet\nabla)\,\mathbf{A}+\mathbf{B}\times(\nabla\times\mathbf{A}). \qquad \ldots(3)$$

Putting the values from (2) and (3) in (1), we get

$$\text{grad}\,\mathbf{(A\bullet B)=(A\bullet\nabla)\,B+A\times(\nabla\times B)+(B\bullet\nabla)\,A+B\times(\nabla\times A).}$$

Note: If we put **A** in place of **B**, then

$$\text{grad}\,(\mathbf{A}\bullet\mathbf{A})=2\,(\mathbf{A}\bullet\nabla)\,\mathbf{A}+2\mathbf{A}\times(\nabla\times\mathbf{A})$$

or $$\frac{1}{2}\,\text{grad}\,\mathbf{A}^2=(\mathbf{A}\bullet\nabla)\,\mathbf{A}+\mathbf{A}\times\text{curl}\,\mathbf{A}.$$

8. *Prove that div grad* $\phi=\nabla^2\phi$ *i.e.,* $\nabla\bullet(\nabla\phi)=\nabla^2\phi$.

Proof: We have

$$\nabla\bullet(\nabla\phi)=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\bullet\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)$$

$$=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi=\nabla^2\phi.$$

9. *Prove that curl of the gradient of* ϕ *is zero i.e.,* $\nabla\times(\nabla\phi)=\mathbf{0}$, *i.e., curl grad* $\phi=\mathbf{0}$.

(Bundelkhand 2014)

Proof: We have grad $\phi=\dfrac{\partial\phi}{\partial x}\mathbf{i}+\dfrac{\partial\phi}{\partial y}\mathbf{j}+\dfrac{\partial\phi}{\partial z}\mathbf{k}$.

∴ curl grad $\phi=\nabla\times\text{grad}\,\phi$

$$=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} & \dfrac{\partial\phi}{\partial z}\end{vmatrix}$$

$$=\left(\frac{\partial^2\phi}{\partial y\,\partial z}-\frac{\partial^2\phi}{\partial z\,\partial y}\right)\mathbf{i}+\left(\frac{\partial^2\phi}{\partial z\,\partial x}-\frac{\partial^2\phi}{\partial x\,\partial z}\right)\mathbf{j}+\left(\frac{\partial^2\phi}{\partial x\,\partial y}-\frac{\partial^2\phi}{\partial y\,\partial x}\right)\mathbf{k}$$

$$= 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k} = \mathbf{0},$$

provided we suppose that ϕ has continuous second partial derivatives so that the order of differentiation is immaterial.

10. *Prove that div curl* $\mathbf{A} = 0$ *, i.e.,* $\nabla \bullet (\nabla \times \mathbf{A}) = 0$.

(Garhwal 2000; Meerut 02, 06B, 09; Agra 2000; Rohilkhand 05; Bundelkhand 08; Kumaun 14)

Proof: Let $\mathbf{A} = A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}$. Then

$$\text{curl } \mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial y} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \mathbf{k}.$$

Now $\quad \text{div curl } \mathbf{A} = \nabla \bullet (\nabla \times \mathbf{A})$

$$= \frac{\partial}{\partial x} \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right)$$

$$= \frac{\partial^2 A_3}{\partial x\, \partial y} - \frac{\partial^2 A_2}{\partial x\, \partial z} + \frac{\partial^2 A_1}{\partial y\, \partial z} - \frac{\partial^2 A_3}{\partial y\, \partial x} + \frac{\partial^2 A_2}{\partial z\, \partial x} - \frac{\partial^2 A_1}{\partial z\, \partial y}$$

$= 0$, assuming that $\mathbf{A}$ has continuous second partial derivatives.

11. *Prove that* $\nabla \times (\nabla \times \mathbf{A}) = \nabla\,(\nabla \bullet \mathbf{A}) - \nabla^2 \mathbf{A}$.

(Meerut 2000, 07B, 10B; Rohilkhand 14; Kumaun 10)

Proof: Let $\mathbf{A} = A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}$.

Then $\quad \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \mathbf{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \mathbf{k}.$$

$\therefore \quad \nabla \times (\nabla \times \mathbf{A}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial A_3}{\partial y} - \dfrac{\partial A_2}{\partial z} & \dfrac{\partial A_1}{\partial z} - \dfrac{\partial A_3}{\partial x} & \dfrac{\partial A_2}{\partial x} - \dfrac{\partial A_1}{\partial y} \end{vmatrix}$

$$= \Sigma\left[\left\{\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\right\}\mathbf{i}\right]$$

$$= \Sigma\left[\left\{\left(\frac{\partial^2 A_2}{\partial y\,\partial x}+\frac{\partial^2 A_3}{\partial z\,\partial x}\right)-\left(\frac{\partial^2 A_1}{\partial y^{\,2}}+\frac{\partial^2 A_1}{\partial z^{\,2}}\right)\right\}\mathbf{i}\right]$$

$$= \Sigma\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\left(\frac{\partial^2 A_1}{\partial y^{\,2}}+\frac{\partial^2 A_1}{\partial z^{\,2}}\right)\right\}\mathbf{i}\right]$$

$$= \Sigma\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\left(\frac{\partial^2 A_1}{\partial x^{\,2}}+\frac{\partial^2 A_1}{\partial y^{\,2}}+\frac{\partial^2 A_1}{\partial z^{\,2}}\right)\right\}\mathbf{i}\right]$$

$$= \Sigma\left[\left\{\frac{\partial}{\partial x}(\nabla \bullet \mathbf{A})-(\nabla^2 A_1)\right\}\mathbf{i}\right]$$

$$= \Sigma\left[\left\{\frac{\partial}{\partial x}(\nabla \bullet \mathbf{A})\right\}\mathbf{i}\right]-\nabla^2\ \Sigma\ A_1\mathbf{i} = \nabla(\nabla \bullet \mathbf{A})-\nabla^2\mathbf{A}.$$

Illustrative Examples

Example 24: *Taking* $\mathbf{F} = x^2 y\mathbf{i} + xz\mathbf{j} + 2yz\mathbf{k}$, *verify that div curl* $\mathbf{F} = 0$.

(Garhwal 2002; Bundelkhand 04; Kanpur 10)

Solution: We have curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & xz & 2yz \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(2yz)-\frac{\partial}{\partial z}(xz)\right]\mathbf{i}-\left[\frac{\partial}{\partial x}(2yz)-\frac{\partial}{\partial z}(x^2 y)\right]\mathbf{j}$$

$$+\left[\frac{\partial}{\partial x}(xz)-\frac{\partial}{\partial y}(x^2 y)\right]\mathbf{k}$$

$$= (2z - x)\,\mathbf{i} - 0\ \mathbf{j} + (z - x^2)\,\mathbf{k} = (2z - x)\,\mathbf{i} + (z - x^2)\,\mathbf{k}\,.$$

Now div curl $\mathbf{F}$ = div $[(2z - x)\,\mathbf{i} + (z - x^2)\,\mathbf{k}]$

$$= \frac{\partial}{\partial x}(2z - x) + \frac{\partial}{\partial z}(z - x^2) = -1 + 1 = 0.$$

Example 25: *Prove that div* $(r^n\ \mathbf{r}) = (n + 3)\,r^n$. **(Bundelkhand 2006)**

Solution: We have

$$\text{div}\,(\phi\mathbf{A}) = \phi\,(\text{div}\ \mathbf{A}) + \mathbf{A} \bullet \text{grad}\ \phi\,.$$

Putting $\mathbf{A} = \mathbf{r}$ and $\phi = r^n$ in this identity, we get

$$\begin{aligned}\operatorname{div}(r^n\,\mathbf{r}) &= r^n \operatorname{div}\mathbf{r} + \mathbf{r}\bullet\operatorname{grad} r^n\\ &= 3r^n + \mathbf{r}\bullet(n\,r^{n-1}\operatorname{grad} r)\\ &\qquad\qquad [\because\ \operatorname{div}\mathbf{r} = 3 \text{ and } \operatorname{grad} f(u) = f'(u)\operatorname{grad} u]\\ &= 3r^n + \mathbf{r}\bullet\left[n\,r^{n-1}\,\frac{1}{r}\,\mathbf{r}\right] \qquad \left[\because\ \operatorname{grad} r = \hat{\mathbf{r}} = \frac{1}{r}\,\mathbf{r}\right]\\ &= 3r^n + n\,r^{n-2}\,(\mathbf{r}\bullet\mathbf{r})\\ &= 3\,r^n + n\,r^{n-2}\,r^2\\ &= (n+3)\,r^n.\end{aligned}$$

Example 26: *Prove that div* $\left(\dfrac{\mathbf{r}}{r^3}\right) = 0.$ **(Bundelkhand 2010)**

Solution: We have

$$\begin{aligned}\operatorname{div}\left(\frac{1}{r^3}\,\mathbf{r}\right) &= \operatorname{div}(r^{-3}\,\mathbf{r}) = r^{-3}\operatorname{div}\mathbf{r} + \mathbf{r}\bullet\operatorname{grad} r^{-3}\\ &= 3r^{-3} + \mathbf{r}\bullet(-3r^{-4}\operatorname{grad} r) = 3r^{-3} + \mathbf{r}\bullet\left(-3r^{-4}\,\frac{1}{r}\,\mathbf{r}\right)\\ &= 3r^{-3} - 3r^{-5}\,(\mathbf{r}\bullet\mathbf{r}) = 3r^{-3} - 3r^{-5}r^2\\ &= 3r^{-3} - 3r^{-3} = 0.\end{aligned}$$

$\therefore$ the vector $r^{-3}\,\mathbf{r}$ is solenoidal.

Example 27: *Prove that div* $\hat{\mathbf{r}} = 2/r.$ **(Bundelkhand 2007; Kanpur 05)**

Solution: $\operatorname{div}(\hat{\mathbf{r}}) = \operatorname{div}\left(\dfrac{1}{r}\,\mathbf{r}\right)\cdot$ Now proceed as in Example 25.

Alternative Method:

$$\begin{aligned}\operatorname{div}\hat{\mathbf{r}} &= \operatorname{div}\left(\frac{1}{r}\,\mathbf{r}\right) = \operatorname{div}\left[\frac{1}{r}\,(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})\right]\\ &= \operatorname{div}\left(\frac{x}{r}\,\mathbf{i} + \frac{y}{r}\,\mathbf{j} + \frac{z}{r}\,\mathbf{k}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)\\ &= \left(\frac{1}{r} - \frac{x}{r^2}\,\frac{\partial r}{\partial x}\right) + \left(\frac{1}{r} - \frac{y}{r^2}\,\frac{\partial r}{\partial y}\right) + \left(\frac{1}{r} - \frac{z}{r^2}\,\frac{\partial r}{\partial z}\right).\end{aligned}$$

Now $\quad r^2 = x^2 + y^2 + z^2.$

$\therefore \quad 2r\,\dfrac{\partial r}{\partial x} = 2x \quad$ *i.e.,* $\quad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}\cdot$

Similarly $\quad \dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}\cdot$

$\therefore \quad \operatorname{div}\hat{\mathbf{r}} = \dfrac{3}{r} - \left(\dfrac{x}{r^2}\,\dfrac{x}{r} + \dfrac{y}{r^2}\,\dfrac{y}{r} + \dfrac{z}{r^2}\,\dfrac{z}{r}\right)$

$$= \frac{3}{r} - \frac{x^2 + y^2 + z^2}{r^3} = \frac{3}{r} - \frac{r^2}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}.$$

Example 28: *Prove that* $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$.

(Meerut 2003, 05, 06B)

Solution: We know that if ϕ is a scalar function, then $\nabla^2 \phi = \nabla \bullet (\nabla \phi)$.

$\therefore$
$$\nabla^2 f(r) = \nabla \bullet \{\nabla f(r)\} = \text{div}\,\{\text{grad}\, f(r)\}$$

$$= \text{div}\,\{f'(r)\,\text{grad}\, r\} = \text{div}\left\{\frac{1}{r} f'(r)\,\mathbf{r}\right\}$$

$$= \frac{1}{r} f'(r)\,\text{div}\,\mathbf{r} + \mathbf{r} \bullet \text{grad}\left\{\frac{1}{r} f'(r)\right\}$$

$$= \frac{3}{r} f'(r) + \mathbf{r} \bullet \left[\frac{d}{dr}\left\{\frac{1}{r} f'(r)\right\} \text{grad}\, r\right]$$

$$= \frac{3}{r} f'(r) + \mathbf{r} \bullet \left[\left\{-\frac{1}{r^2} f'(r) + \frac{1}{r} f''(r)\right\} \frac{1}{r}\,\mathbf{r}\right]$$

$$= \frac{3}{r} f'(r) + \left[\frac{1}{r}\left\{-\frac{1}{r^2} f'(r) + \frac{1}{r} f''(r)\right\}\right](\mathbf{r} \bullet \mathbf{r})$$

$$= \frac{3}{r} f'(r) + \left[\frac{1}{r}\left\{-\frac{1}{r^2} f'(r) + \frac{1}{r} f''(r)\right\}\right] r^2$$

$$= \frac{3}{r} f'(r) - \frac{1}{r} f'(r) + f''(r) = f''(r) + \frac{2}{r} f'(r).$$

Example 29: *Prove that* $\nabla^2 \left(\frac{1}{r}\right) = 0$ or $\text{div}\left(\textit{grad}\,\frac{1}{r}\right) = 0$.

(Agra 2002; Meerut 07, 13)

Solution: We have

$$\nabla^2 \left(\frac{1}{r}\right) = \nabla \bullet \left(\nabla \frac{1}{r}\right) = \text{div}\left(\text{grad}\,\frac{1}{r}\right)$$

$$= \text{div}\left(-\frac{1}{r^2}\,\text{grad}\, r\right) = \text{div}\left(-\frac{1}{r^2}\frac{1}{r}\,\mathbf{r}\right) = \text{div}\left(-\frac{1}{r^3}\,\mathbf{r}\right)$$

$$= \left(-\frac{1}{r^3}\right)\text{div}\,\mathbf{r} + \mathbf{r} \bullet \text{grad}\left(-\frac{1}{r^3}\right) = -\frac{3}{r^3} + \mathbf{r} \bullet \left[\frac{d}{dr}\left(-\frac{1}{r^3}\right)\text{grad}\, r\right]$$

$$= -\frac{3}{r^3} + \mathbf{r} \bullet \left(\frac{3}{r^4}\frac{1}{r}\,\mathbf{r}\right) = -\frac{3}{r^3} + \frac{3}{r^5}(\mathbf{r} \bullet \mathbf{r}) = -\frac{3}{r^3} + \frac{3}{r^5} r^2 = 0.$$

$\therefore$ $1/r$ is a solution of Laplace's equation.

Example 30: *Prove that div grad* $r^n = n(n+1)\,r^{n-2}$, *i.e.*, $\nabla^2 r^n = n(n+1)\,r^{n-2}$.

(Bundelkhand 2005, 10; Meerut 2000, 08; Avadh 10; Agra 14)

Solution: We have $\nabla^2 r^n = \nabla \bullet (\nabla r^n) = \text{div}\,(\text{grad}\, r^n)$

$$= \text{div}\,(nr^{n-1}\ \text{grad}\ r) = \text{div}\left(nr^{n-1}\ \frac{1}{r}\ \mathbf{r}\right) = \text{div}\,(nr^{n-2}\ \mathbf{r})$$

$$= (nr^{n-2})\ \text{div}\ \mathbf{r} + \mathbf{r} \bullet (\text{grad}\ nr^{n-2})$$

$$= 3nr^{n-2} + \mathbf{r} \bullet [n\,(n-2)\,r^{n-3}\ \text{grad}\ r]$$

$$= 3nr^{n-2} + \mathbf{r} \bullet \left[n\,(n-2)\,r^{n-3}\ \frac{1}{r}\ \mathbf{r}\right]$$

$$= 3nr^{n-2} + \mathbf{r} \bullet [n\,(n-2)\,r^{n-4}\ \mathbf{r}] = 3nr^{n-2} + n\,(n-2)\,r^{n-4}\ (\mathbf{r} \bullet \mathbf{r})$$

$$= 3nr^{n-2} + n\,(n-2)\,r^{n-4}\ r^2 = nr^{n-2}\ (3 + n - 2) = n\,(n+1)\,r^{n-2}.$$

Note: If $n = -1$, then $\nabla^2 (r^{-1}) = (-1)(-1+1)\,r^{-3} = 0$.

Example 31: *Prove that* $\nabla \bullet \left\{r\,\nabla\left(\frac{1}{r^3}\right)\right\} = \frac{3}{r^4}$ *or,* $\textit{div}\,[r\ \textit{grad}\ r^{-3}] = 3r^{-4}$.

(Meerut 2009B)

Solution: We have $\nabla\left(\frac{1}{r^3}\right) = \text{grad}\ r^{-3}$

$$= \frac{\partial}{\partial x}(r^{-3})\,\mathbf{i} + \frac{\partial}{\partial y}(r^{-3})\,\mathbf{j} + \frac{\partial}{\partial z}(r^{-3})\,\mathbf{k}.$$

Now $\frac{\partial}{\partial x}(r^{-3}) = -3r^{-4}\ \frac{\partial r}{\partial x}\cdot$

But $r^2 = x^2 + y^2 + z^2$.

Therefore $2r\,\frac{\partial r}{\partial x} = 2x$ or $\frac{\partial r}{\partial x} = \frac{x}{r}\cdot$

So $\frac{\partial}{\partial x}(r^{-3}) = -3r^{-4}\ \frac{x}{r} = -3r^{-5}\,x$.

Similarly $\frac{\partial}{\partial y}(r^{-3}) = -3r^{-5}\,y$ and $\frac{\partial}{\partial z}(r^{-3}) = -3r^{-5}\,z$.

Therefore $\nabla\left(\frac{1}{r^3}\right) = -3r^{-5}\ (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$.

$\therefore$ $r\,\nabla\left(\frac{1}{r^3}\right) = -3r^{-4}\ (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$.

$\therefore$ $\nabla \bullet \left(r\,\nabla\,\frac{1}{r^3}\right) = \frac{\partial}{\partial x}(-3r^{-4}x) + \frac{\partial}{\partial y}(-3r^{-4}y) + \frac{\partial}{\partial z}(-3r^{-4}z)$.

Now $\frac{\partial}{\partial x}(-3r^{-4}\ x) = 12r^{-5}\ \frac{\partial r}{\partial x}\ x - 3r^{-4}$

$$= 12r^{-5}\ \frac{x}{r}\ x - 3r^{-4} = 12r^{-6}x^2 - 3r^{-4}.$$

Similarly $\frac{\partial}{\partial y}(-3r^{-4}\,y) = 12r^{-6}y^2 - 3r^{-4}$

and $\frac{\partial}{\partial z}(-3r^{-4}\ z) = 12r^{-6}\ z^2 - 3r^{-4}$.

Hence $\nabla \bullet \left(r \nabla \frac{1}{r^3} \right) = 12\, r^{-6}\, (x^2 + y^2 + z^2) - 9r^{-4}$

$$= 12\, r^{-6} r^2 - 9r^{-4} = 12\, r^{-4} - 9r^{-4} = 3r^{-4}.$$

Example 32: *If* **a** *is a constant vector, prove that div* $\{r^n\, (\mathbf{a} \times \mathbf{r})\} = 0$ **(Kanpur 2008)**

Solution: We have div $(\phi \mathbf{A}) = \phi$ div $\mathbf{A} + \mathbf{A} \bullet$ grad ϕ.

$\therefore$ div $\{r^n\, (\mathbf{a} \times \mathbf{r})\} = r^n$ div $(\mathbf{a} \times \mathbf{r}) + (\mathbf{a} \times \mathbf{r}) \bullet$ grad r^n

$= r^n$ div $(\mathbf{a} \times \mathbf{r}) + (\mathbf{a} \times \mathbf{r}) \bullet (nr^{n-1}$ grad $r)$

$= r^n\, (\mathbf{r} \bullet \text{curl } \mathbf{a} - \mathbf{a} \bullet \text{curl } \mathbf{r}) + (\mathbf{a} \times \mathbf{r}) \bullet \left(nr^{n-1}\, \frac{1}{r}\, \mathbf{r} \right)$

$= r^n\, (\mathbf{r} \bullet \mathbf{0} - \mathbf{a} \bullet \mathbf{0}) + nr^{n-2}\, (\mathbf{a} \times \mathbf{r}) \bullet \mathbf{r}$

[$\because$ curl of a constant vector is zero and curl $\mathbf{r} = \mathbf{0}$]

$= nr^{n-2}\, [\mathbf{a}, \mathbf{r}, \mathbf{r}]$

$= 0$, since a scalar triple product having two equal vectors is zero.

Example 33: *If* **a** *and* **b** *are constant vectors, prove that*

(i) *div* $[(\mathbf{r} \times \mathbf{a}) \times \mathbf{b}] = -2\mathbf{b} \bullet \mathbf{a}$. **(Bundelkhand 2010)**

(ii) *curl* $[(\mathbf{r} \times \mathbf{a}) \times \mathbf{b}] = \mathbf{b} \times \mathbf{a}$. **(Bundelkhand 2010)**

Solution: (i) We have $(\mathbf{r} \times \mathbf{a}) \times \mathbf{b} = (\mathbf{b} \bullet \mathbf{r})\, \mathbf{a} - (\mathbf{b} \bullet \mathbf{a})\, \mathbf{r}$.

$\therefore$ div $[(\mathbf{r} \times \mathbf{a}) \times \mathbf{b}] =$ div $[(\mathbf{b} \bullet \mathbf{r})\, \mathbf{a} - (\mathbf{b} \bullet \mathbf{a})\, \mathbf{r}]$

$=$ div $[(\mathbf{b} \bullet \mathbf{r})\, \mathbf{a}] -$ div $[(\mathbf{b} \bullet \mathbf{a})\, \mathbf{r}]$ ...(1)

But div $(\phi \mathbf{A}) = \phi$ div $\mathbf{A} + \mathbf{A} \bullet$ grad ϕ.

Taking $\phi = \mathbf{b} \bullet \mathbf{r}$ and $\mathbf{A} = \mathbf{a}$, we get

div $[(\mathbf{b} \bullet \mathbf{r})\, \mathbf{a}] = (\mathbf{b} \bullet \mathbf{r})$ div $\mathbf{a} + \mathbf{a} \bullet$ grad $(\mathbf{b} \bullet \mathbf{r})$.

Since **a** is a constant vector, therefore div $\mathbf{a} = 0$.

Also let $\mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}$.

Then $\mathbf{b} \bullet \mathbf{r} = (b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}) \bullet (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

$= b_1 x + b_2 y + b_3 z$, where b_1, b_2, b_3 are constants.

$\therefore$ grad $(\mathbf{b} \bullet \mathbf{r}) = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k} = \mathbf{b}$.

$\therefore$ div $[(\mathbf{b} \bullet \mathbf{r})\, \mathbf{a}] = \mathbf{a} \bullet \mathbf{b}$. ...(2)

Again div $[(\mathbf{b} \bullet \mathbf{a})]\, \mathbf{r} = (\mathbf{b} \bullet \mathbf{a})$ div $\mathbf{r} + \mathbf{r} \bullet$ grad $(\mathbf{b} \bullet \mathbf{a})$.

But div $\mathbf{r} = 3$.

Also grad $(\mathbf{b} \bullet \mathbf{a}) = \mathbf{0}$ because $\mathbf{b} \bullet \mathbf{a}$ is constant.

$\therefore$ div $[(\mathbf{b} \bullet \mathbf{a})\, \mathbf{r}] = 3\, (\mathbf{b} \bullet \mathbf{a})$. ...(3)

Substituting the values from (2) and (3) in (1), we get

div $[(\mathbf{r} \times \mathbf{a}) \times \mathbf{b}] = (\mathbf{a} \bullet \mathbf{b}) - 3\, (\mathbf{b} \bullet \mathbf{a}) = -2\mathbf{b} \bullet \mathbf{a}$.

(ii) curl $[(\mathbf{r} \times \mathbf{a}) \times \mathbf{b}] =$ curl $[(\mathbf{b} \bullet \mathbf{r})\, \mathbf{a} - (\mathbf{b} \bullet \mathbf{a})\, \mathbf{r}]$

$=$ curl $[(\mathbf{b} \bullet \mathbf{r})\, \mathbf{a}] -$ curl $[(\mathbf{b} \bullet \mathbf{a})\, \mathbf{r}]$.

But curl $(\phi \mathbf{A}) = \text{grad } \phi \times \mathbf{A} + \phi \text{ curl } \mathbf{A}$.

$\therefore$ $\quad$ $\text{curl } [(\mathbf{b} \bullet \mathbf{r})\, \mathbf{a}] = [\text{grad } (\mathbf{b} \bullet \mathbf{r})] \times \mathbf{a} + (\mathbf{b} \bullet \mathbf{r}) \text{ curl } \mathbf{a} = \mathbf{b} \times \mathbf{a}$

$[\because \text{ curl } \mathbf{a} = \mathbf{0} \text{ and grad } (\mathbf{b} \bullet \mathbf{r}) = \mathbf{b}]$

Also $\quad$ $\text{curl } [(\mathbf{b} \bullet \mathbf{a})\, \mathbf{r}] = [\text{grad } (\mathbf{b} \bullet \mathbf{a})] \times \mathbf{r} + (\mathbf{b} \bullet \mathbf{a}) \text{ curl } \mathbf{r} = \mathbf{0}$

$[\because \text{ grad } (\mathbf{b} \bullet \mathbf{a}) = \mathbf{0} \text{ and curl } \mathbf{r} = \mathbf{0}]$

$\therefore$ $\quad$ $\text{curl } [(\mathbf{r} \times \mathbf{a}) \times \mathbf{b}] = \mathbf{b} \times \mathbf{a} - \mathbf{0} = \mathbf{b} \times \mathbf{a}.$

Example 34: *Prove that curl* $[\mathbf{r} \times (\mathbf{a} \times \mathbf{r})] = 3\mathbf{r} \times \mathbf{a}$, *where* **a** *is a constant vector.*

(Purvanchal 2009)

Solution: $\text{curl } [\mathbf{r} \times (\mathbf{a} \times \mathbf{r})]$

$= \nabla \times [(\mathbf{r} \bullet \mathbf{r})\, \mathbf{a} - (\mathbf{r} \bullet \mathbf{a})\, \mathbf{r}]$ $\quad [\because \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\, \mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\, \mathbf{c}]$

$= \nabla \times [r^2 \mathbf{a} - (\mathbf{r} \bullet \mathbf{a})\, \mathbf{r}]$ $\quad [\because \mathbf{r} \bullet \mathbf{r} = \mathbf{r}^2 = r^2]$

$= \nabla \times (r^2 \mathbf{a}) - \nabla \times [(\mathbf{r} \bullet \mathbf{a})\, \mathbf{r}]$ $\quad [\because \nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}]$

$= (\nabla r^2) \times \mathbf{a} + r^2\, (\nabla \times \mathbf{a}) - [\nabla\, (\mathbf{r} \bullet \mathbf{a})] \times \mathbf{r} - (\mathbf{r} \bullet \mathbf{a})\, (\nabla \times \mathbf{r})$

$[\because \nabla \times (\phi \mathbf{A}) = (\nabla \phi) \times \mathbf{A} + \phi\, (\nabla \times \mathbf{A})]$

$= (2r\, \nabla r) \times \mathbf{a} + r^2\, \mathbf{0} - [\nabla\, (\mathbf{r} \bullet \mathbf{a})] \times \mathbf{r} - (\mathbf{r} \bullet \mathbf{a})\, \mathbf{0}$

$[\because \nabla f(r) = f'(r)\, \nabla r\,; \nabla \times \mathbf{a} = \mathbf{0}$, **a** being a constant vector ; and $\nabla \times \mathbf{r} = \mathbf{0}]$

$= \left(2r\, \frac{1}{r}\, \mathbf{r}\right) \times \mathbf{a} - [\nabla\, (\mathbf{r} \bullet \mathbf{a})] \times \mathbf{r}$

$= 2\mathbf{r} \times \mathbf{a} - \mathbf{a} \times \mathbf{r}$ $\quad [\because \nabla\, (\mathbf{r} \bullet \mathbf{a}) = \mathbf{a}$, if **a** is a constant vector$]$

$= 2\mathbf{r} \times \mathbf{a} + \mathbf{r} \times \mathbf{a} = 3\mathbf{r} \times \mathbf{a}.$

Example 35: *If* **a** *is a constant vector, prove that*

$$\text{curl } \frac{\mathbf{a} \times \mathbf{r}}{r^3} = -\frac{\mathbf{a}}{r^3} + \frac{3\mathbf{r}}{r^5}\, (\mathbf{a} \bullet \mathbf{r}).$$

(Meerut 2009B; Bundelkhand 09, 11)

Solution: We have $\quad \text{curl } \frac{\mathbf{a} \times \mathbf{r}}{r^3} = \nabla \times \left(\frac{\mathbf{a} \times \mathbf{r}}{r^3}\right) = \Sigma \left\{\mathbf{i} \times \frac{\partial}{\partial x} \left(\frac{\mathbf{a} \times \mathbf{r}}{r^3}\right)\right\}.$

Now $\quad \frac{\partial}{\partial x} \left(\frac{\mathbf{a} \times \mathbf{r}}{r^3}\right) = -\frac{3}{r^4}\, \frac{\partial r}{\partial x}\, (\mathbf{a} \times \mathbf{r}) + \frac{1}{r^3} \left(\mathbf{a} \times \frac{\partial \mathbf{r}}{\partial x}\right) + \frac{1}{r^3} \left(\frac{\partial \mathbf{a}}{\partial x} \times \mathbf{r}\right) \quad \ldots(1)$

Now $\frac{\partial \mathbf{a}}{\partial x} = \mathbf{0}$ because **a** is a constant vector.

Also $\quad \mathbf{r} = x\, \mathbf{i} + y\, \mathbf{j} + z\, \mathbf{k}\,. \quad \therefore \quad \frac{\partial \mathbf{r}}{\partial x} = \mathbf{i}\,.$

Further $\quad \frac{\partial r}{\partial x} = \frac{x}{r} \cdot$

$\therefore$ (1) becomes $\quad \frac{\partial}{\partial x} \left(\frac{\mathbf{a} \times \mathbf{r}}{r^3}\right) = -\frac{3}{r^4}\, \frac{x}{r}\, (\mathbf{a} \times \mathbf{r}) + \frac{1}{r^3}\, (\mathbf{a} \times \mathbf{i})$

$$= -\frac{3x}{r^5}(\mathbf{a}\times\mathbf{r}) + \frac{1}{r^3}(\mathbf{a}\times\mathbf{i}).$$

$$\therefore \qquad \mathbf{i}\times\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right) = -\frac{3x}{r^5}\,\mathbf{i}\times(\mathbf{a}\times\mathbf{r}) + \frac{1}{r^3}\,\mathbf{i}\times(\mathbf{a}\times\mathbf{i})$$

$$= -\frac{3x}{r^5}\left[(\mathbf{i}\bullet\mathbf{r})\,\mathbf{a} - (\mathbf{i}\bullet\mathbf{a})\,\mathbf{r}\right] + \frac{1}{r^3}\left[(\mathbf{i}\bullet\mathbf{i})\,\mathbf{a} - (\mathbf{i}\bullet\mathbf{a})\,\mathbf{i}\right]$$

$$= -\frac{3x}{r^5}\,x\mathbf{a} + \frac{3x}{r^5}\,a_1\mathbf{r} + \frac{1}{r^3}\,\mathbf{a} - \frac{1}{r^3}\,a_1\,\mathbf{i}$$

$[\because\ \mathbf{i}\bullet\mathbf{r} = x$ and $\mathbf{i}\bullet\mathbf{a} = a_1$ if $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}]$

$$= -\frac{3x^2}{r^5}\,\mathbf{a} + \frac{3}{r^5}\,a_1 x\mathbf{r} + \frac{1}{r^3}\,\mathbf{a} - \frac{1}{r^3}\,a_1\mathbf{i}\,.$$

$$\therefore \qquad \Sigma\left\{\mathbf{i}\times\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)\right\}$$

$$= \left\{-\frac{3}{r^5}\,\Sigma\,x^2\right\}\mathbf{a} + \left\{\frac{3}{r^5}\,\Sigma\,a_1\,x\right\}\mathbf{r} + \frac{3}{r^3}\,\mathbf{a} - \frac{1}{r^3}\,\Sigma\,a_1\mathbf{i}$$

$$= -\frac{3}{r^5}\,r^2\,\mathbf{a} + \frac{3}{r^5}(\mathbf{r}\bullet\mathbf{a})\,\mathbf{r} + \frac{3}{r^3}\,\mathbf{a} - \frac{1}{r^3}\,\mathbf{a}$$

$[\because\ \Sigma\,x^2 = r^2, \Sigma\,a_1 x = \mathbf{r}\bullet\mathbf{a}, \Sigma\,a_1\mathbf{i} = \mathbf{a}]$

$$= -\frac{\mathbf{a}}{r^3} + \frac{3}{r^5}(\mathbf{a}\bullet\mathbf{r})\,\mathbf{r}.$$

Comprehensive Exercise 4

1. (i) Verify that curl grad $f = \mathbf{0}$, where $f = x^2 y + 2xy + z^2$.

(ii) Prove that grad $f(u) = f'(u)$ grad u.

(iii) Find $\nabla\phi$ and $|\nabla\phi|$ when $\phi = (x^2 + y^2 + z^2)\,e^{-(x^2+y^2+z^2)^{1/2}}$.

(Kumaun 2012)

2. (i) Prove that curl $(\psi\nabla\phi) = \nabla\psi\times\nabla\phi = -$ curl $(\phi\nabla\psi)$.

(ii) Prove that $\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi\bullet\nabla\psi + \psi\nabla^2\phi$. **(Kumaun 2014)**

(iii) Prove that div $(\nabla\phi\times\nabla\psi) = 0$.

(iv) If **A** and **B** are irrotational, prove that **A** × **B** is solenoidal.

(v) Prove that curl $(\phi$ grad $\phi) = \mathbf{0}$.

3. (i) Prove that $\mathbf{a}\bullet\{\nabla(\mathbf{v}\bullet\mathbf{a}) - \nabla\times(\mathbf{v}\times\mathbf{a})\} =$ div **v**, where **a** is a constant unit vector.

(ii) Prove that vector $f(r)\,\mathbf{r}$ is irrotational.

(iii) If f and g are two scalar point functions, prove that

div $(f\,\nabla\,g) = f\,\nabla^2\,g + \nabla f\bullet\nabla g$.

(iv) Show that curl $(\mathbf{a} \bullet \mathbf{r})\, \mathbf{a} = \mathbf{0}$, where **a** is a constant vector.

(Kumaun 2007, 09)

[**Hint:** Use identity 4. Note that $\nabla (\mathbf{a} \bullet \mathbf{r}) = \mathbf{a}$, if **a** is a constant vector.]

4. If **a** is a constant vector, then prove that

(i) $\nabla (\mathbf{a} \bullet \mathbf{u}) = (\mathbf{a} \bullet \nabla)\, \mathbf{u} + \mathbf{a} \times \text{curl } \mathbf{u}$,

(ii) $\nabla \bullet (\mathbf{a} \times \mathbf{u}) = -\, \mathbf{a} \bullet \text{curl } \mathbf{u}$,

(iii) $\nabla \times (\mathbf{a} \times \mathbf{u}) = \mathbf{a} \text{ div } \mathbf{u} - (\mathbf{a} \bullet \nabla)\, \mathbf{u}$,

5. (i) Given that $\rho\, \mathbf{F} = \nabla p$, where ρ, p, **F** are point functions, prove that

$$\mathbf{F} \bullet \text{curl } \mathbf{F} = 0.$$

(ii) A vector function **f** is the product of a scalar function and the gradient of a scalar function. Show that $\mathbf{f} \bullet \text{curl } \mathbf{f} = 0$.

(iii) Show that curl $\mathbf{a}\, \phi (r) = \dfrac{1}{r}\, \phi' (r)\, \mathbf{r} \times \mathbf{a}$, where **a** is a constant vector.

6. (i) Prove that curl $[r^{n} (\mathbf{a} \times \mathbf{r})] = (n + 2)\, r^{n}\, \mathbf{a} - n r^{n-2} (\mathbf{r} \bullet \mathbf{a})\, \mathbf{r}$,

where **a** is a constant vector. **(Kumaun 2008)**

(ii) Prove that $\nabla^2 (r^n\, \mathbf{r}) = n\, (n + 3)\, r^{n-2}\, \mathbf{r}$.

7. (i) Prove that curl grad $r^n = \mathbf{0}$. **(Avadh 2010; Kanpur 11)**

(ii) If $\nabla^2 f\, (r) = 0$, show that $f\, (r) = \dfrac{c_1}{r} + c_2$,

where $r^2 = x^2 + y^2 + z^2$ and c_1, c_2 are arbitrary constants.

8. If **r** is the position vector of the point $(x\,,\, y\,,\, z)$ show that curl $(r^n\, \mathbf{r}) = \mathbf{0}$, where r is the module of **r**.

9. Prove that $r^n\, \mathbf{r}$ is an irrotational vector for any value of n but is solenoidal only if $n + 3 = 0$.

10. (i) If $\mathbf{u} = (1 / r)\, \mathbf{r}$, show that $\nabla \times \mathbf{u} = \mathbf{0}$. **(Avadh 2010)**

(ii) Prove that div $(\mathbf{A} \times \mathbf{r}) = \mathbf{r} \bullet \text{curl } \mathbf{A}$.

11. (i) If $\nabla^2 f\, (r) = 0$ show that $f (r) = c_1 \log r + c_2$ where $r^2 = x^2 + y^2$ and c_1, c_2 are arbitrary constants.

(ii) If **a** and **b** are constant vectors, then show that $\nabla \bullet (\mathbf{a} \bullet \mathbf{b r}) = 3 \mathbf{a} \bullet \mathbf{b}$.

12. If $\mathbf{u} = (1 / r)\, \mathbf{r}$ find grad (div **u**).

13. (i) Prove that $\nabla^2 \left[\nabla \bullet \left(\dfrac{\mathbf{r}}{r^2} \right) \right] = 2 r^{-4}$.

(Kumaun 2011, 13)

(ii) Prove that $\mathbf{a} \bullet \left(\nabla \dfrac{1}{r} \right) = -\, \dfrac{\mathbf{a} \bullet \mathbf{r}}{r^3}$.

(iii) Prove that $\nabla \bullet (U\, \nabla V - V\, \nabla U) = U\, \nabla^2 V - V\, \nabla^2 U$.

(iv) Prove that $\mathbf{b} \bullet \nabla \left(\mathbf{a} \bullet \nabla \frac{1}{r} \right) = \frac{3\,(\mathbf{a} \bullet \mathbf{r})\,(\mathbf{b} \bullet \mathbf{r})}{r^5} - \frac{\mathbf{a} \bullet \mathbf{b}}{r^3}$,

where **a** and **b** are constant vectors. **(Kumaun 2015)**

14. Evaluate div $\{\mathbf{a} \times (\mathbf{r} \times \mathbf{a})\}$, where **a** is a constant vector. **(Kanpur 2007)**

15. (i) Prove that div $\left\{ \frac{f(r)\,\mathbf{r}}{r} \right\} = \frac{1}{r^2} \frac{d}{dr} (r^2 f)$. **(Kumaun 2007, 14)**

(ii) Prove that $\frac{1}{2} \nabla \mathbf{a}^2 = (\mathbf{a} \bullet \nabla)\,\mathbf{a} + \mathbf{a} \times \text{curl } \mathbf{a}$.

16. Prove that curl $[\mathbf{r} \times (\mathbf{a} \times \mathbf{r})] = 3\mathbf{r} \times \mathbf{a}$, where **a** is a constant vector.

17. Prove that $\nabla \times (\mathbf{F} \times \mathbf{r}) = 2\mathbf{F} - (\nabla \bullet \mathbf{F})\,\mathbf{r} + (\mathbf{r} \bullet \nabla)\,\mathbf{F}$. **(Kumaun 2015)**

18. If **a** and **b** are constant vectors, prove that

$$\text{grad } [(\mathbf{r} \times \mathbf{a}) \bullet (\mathbf{r} \times \mathbf{b})] = (\mathbf{b} \times \mathbf{r}) \times \mathbf{a} + (\mathbf{a} \times \mathbf{r}) \times \mathbf{b}.$$ **(Kumaun 2015)**

Answers 4

1. (iii) $(2 - r)\,e^{-r}\,\mathbf{r}\,;\ (2 - r)\,e^{-r}\,r$

12. $-\frac{2}{r^3}\,\mathbf{r}$

14. $2\mathbf{a}^2$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If the vector $\mathbf{V} = (x + 3y)\,\mathbf{i} + (y - 2z)\,\mathbf{j} + (x + az)\,\mathbf{k}$ is solenoidal, then the constant a is

(a) 0 (b) 1

(c) –2 (d) 2

2. The directional derivative of $\phi(x, y, z) = x^2 yz + 4xz^2$ at $(1, -2, -1)$ in the direction of the vector $2\,\mathbf{i} - \mathbf{j} - 2\,\mathbf{k}$ is

(a) 37/3 (b) 3/37

(c) 3 (d) none of these

3. If $r = |\mathbf{r}|$ where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, then $\nabla^2 r^n =$

(a) $n(n+1)\,r^n$ (b) $n(n+1)\,r^{n-1}$

(c) $n(n+1)\,r^{n-2}$ (d) none of these **(Agra 2007)**

4. $\nabla^2 \left(\frac{1}{r}\right) =$

(a) $-2 / r^3$ (b) 0

(c) $2 / r^3$ (d) none of these

(Kumaun 2014)

5. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{a}$ is a constant vector, then curl $(\mathbf{r} \times \mathbf{a})$ is

(a) $-\mathbf{a}$ (b) $-3\mathbf{a}$

(c) $-2\mathbf{a}$ (d) none of these

6. The value of div $\hat{\mathbf{r}}$ is

(a) $2 / r$ (b) 0

(c) $1 / r$ (d) none of these **(Agra 2014)**

7. The $\mathbf{V}$ is a constant vector, then div $\mathbf{V}$ is :

(a) 3 (b) $3\mathbf{V}$

(c) 0 (d) none of these

(Kumaun 2007, 11)

8. If $\mathbf{f} = x^2 y\mathbf{i} - 2xz\mathbf{j} + 2yz\mathbf{k}$, then div $\mathbf{f}$ is equal to

(a) $2x(x+1)$ (b) $2y(x+1)$

(c) $y(y+1)$ (d) none of these

(Kumaun 2008)

9. $\nabla \times (\nabla f)$ is equal to

(a) $\nabla^2 f$ (b) $\mathbf{0}$

(c) 0 (d) none of these

(Kumaun 2010)

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. If $\mathbf{F} = (x^2 + y^2)\,\mathbf{i} - 2\,xy\,\mathbf{j}$, then $\mathbf{F} \bullet d\mathbf{r} = \ldots\ldots$

2. If $\mathbf{P} = e^{xy}\,\mathbf{i} + (x - 2y)\,\mathbf{j} + (x \sin y)\,\mathbf{k}$, then $\dfrac{\partial \mathbf{P}}{\partial x} = \ldots\ldots$

3. If $\mathbf{a}$ is a constant vector then grad $(\mathbf{a} \bullet \mathbf{r}) = \ldots\ldots$

(Rohilkhand 2007; Kumaun 12)

4. If $\mathbf{a}$ is a constant vector, then $\nabla \bullet (\mathbf{a} \times \mathbf{r}) = \ldots\ldots$

5. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then the value of div $\mathbf{r} = \ldots\ldots$

(Agra 2008; Kumaun 15)

6. If $\mathbf{A} = x^2 z\mathbf{i} - 2y^3 z^2\,\mathbf{j} + xy^2 z\mathbf{k}$, then div $\mathbf{A}$ at $(1, -1, 1) = \ldots\ldots$

7. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then the value of curl $\mathbf{r} = \ldots\ldots$

(Agra 2008; Rohilkhand 14)

8. For any vector $\mathbf{A}$, div curl $\mathbf{A} = \ldots\ldots$.

9. A vector $\mathbf{V}$ is said to be solenoidal if …… . **(Rohilkhand 2005; Agra 06)**

10. A vector $\mathbf{F}$ is said to be irrotational if …… .

11. If $\phi = x^2 y + 2xy + z^2$, then curl grad $\phi = \ldots\ldots$. **(Bundelkhand 2007)**

12. If $\mathbf{f} = x^2 y\mathbf{i} + 2xz\mathbf{j} + 2yz\mathbf{k}$, then div (curl $\mathbf{f}$) $= \ldots\ldots$. **(Kumaun 2010, 12)**

13. Value of div grad ϕ is ……. **(Kumaun 2011)**

14. If $\mathbf{a}$ is a constant vector, then curl $(\mathbf{r} \times \mathbf{a}) = \ldots\ldots$. **(Kumaun 2013)**

True or False

Write 'T' for true and 'F' for false statement.

1. If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\mathbf{r}$ is solenoidal.
2. If $\mathbf{V}$ is a constant vector, then div $\mathbf{V} = 0$. **(Rohilkhand 2005)**
3. If $\mathbf{F} = 2xyz\ \mathbf{i} + y^2 z\mathbf{j} - 2yz^2\mathbf{k}$, then $\mathbf{F}$ is irrotational.
4. If ϕ is a differentiable scalar function, then curl grad $\phi = \mathbf{0}$.
5. If ϕ is a differentiable scalar function then div grad $\phi = \nabla^2 \phi$.
6. $\nabla \bullet (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \bullet (\nabla \times \mathbf{B}) - \mathbf{B} \bullet (\nabla \times \mathbf{A})$.
7. A function which satisfies Laplace's equation is called a harmonic function.

Answers

Multiple Choice Questions

1.	(c)	2.	(a)	3.	(c)
4.	(b)	5.	(c)	6.	(a)
7.	(c)	8.	(b)	9.	(b)

Fill in the Blank(s)

1.	$(x^2 + y^2)\,dx - 2xy\,dy$	2.	$y\,e^{xy}\,\mathbf{i} + \mathbf{j} + \sin y\ \mathbf{k}$	3.	$\mathbf{a}$
4.	0	5.	3	6.	–3
7.	0	8.	0	9.	div $\mathbf{V} = 0$
10.	curl $\mathbf{F} = \mathbf{0}$	11.	0	12.	0
13.	$\nabla^2 \phi$	14.	$-2\mathbf{a}$		

True or False

1.	*F*	2.	*T*	3.	*F*
4.	*T*	5.	*T*	6.	*F*
7.	*T*				

Chapter

4

Integration of Vectors

1 Integration of Vector Functions

We shall define *integration as the reverse process of differentiation.* Let $\mathbf{f}\ (t)$ and $\mathbf{F}\ (t)$ be two vector functions of the scalar t such that

$$\frac{d}{dt}\ \mathbf{F}\ (t) = \mathbf{f}\ (t).$$

Then $\mathbf{F}\ (t)$ is called the *indefinite integral* of $\mathbf{f}\ (t)$ with respect to t and symbolically we write

$$\int \mathbf{f}\ (t)\ dt = \mathbf{F}\ (t). \qquad \ldots(1)$$

The function $\mathbf{f}\ (t)$ to be integrated is called the *integrand.*

If $\mathbf{c}$ is any *arbitrary constant vector* independent of t, then

$$\frac{d}{dt}\ \{\mathbf{F}\ (t) + \mathbf{c}\ \} = \mathbf{f}\ (t).$$

This is equivalent to

$$\int \mathbf{f}\ (t)\ dt = \mathbf{F}\ (t) + \mathbf{c}. \qquad \ldots(2)$$

From (2) it is obvious that the integral $\mathbf{F}\ (t)$ of $\mathbf{f}\ (t)$ is indefinite to the extent of an additive arbitrary constant $\mathbf{c}$. Therefore $\mathbf{F}\ (t)$ is called the indefinite integral of $\mathbf{f}\ (t)$.

The constant vector **c** is called the *constant of integration*. It can be determined if we are given some initial conditions.

If $\frac{d}{dt}\mathbf{F}(t)=\mathbf{f}(t)$ for all t in the interval $[a,b]$, then the *definite integral* between the limits $t=a$ and $t=b$ can, in such case, be written as

$$\int_a^b \mathbf{f}(t)\,dt=\int_a^b\left\{\frac{d}{dt}\mathbf{F}(t)\right\}dt=\left[\mathbf{F}(t)+\mathbf{c}\right]_a^b=\mathbf{F}(b)-\mathbf{F}(a).$$

Theorem: *If* $\mathbf{f}(t)=f_1(t)\,\mathbf{i}+f_2(t)\,\mathbf{j}+f_3(t)\,\mathbf{k}$, *then*

$$\int \mathbf{f}(t)\,dt=\mathbf{i}\int f_1(t)\,dt+\mathbf{j}\int f_2(t)\,dt+\mathbf{k}\int f_3(t)\,dt.$$

Proof: Let $\frac{d}{dt}\mathbf{F}(t)=\mathbf{f}(t)$. ...(1)

Then $\int \mathbf{f}(t)\,dt=\mathbf{F}(t)$. ...(2)

Let $\mathbf{F}(t)=F_1(t)\,\mathbf{i}+F_2(t)\,\mathbf{j}+F_3(t)\,\mathbf{k}$.

Then from (1), we have

$$\frac{d}{dt}\{F_1(t)\,\mathbf{i}+F_2(t)\,\mathbf{j}+F_3(t)\,\mathbf{k}\}=\mathbf{f}(t)$$

or
$$\left\{\frac{d}{dt}F_1(t)\right\}\mathbf{i}+\left\{\frac{d}{dt}F_2(t)\right\}\mathbf{j}+\left\{\frac{d}{dt}F_3(t)\right\}\mathbf{k}$$
$$=f_1(t)\,\mathbf{i}+f_2(t)\,\mathbf{j}+f_3(t)\,\mathbf{k}.$$

Equating the coefficients of **i**, **j**, **k**, we get

$$\frac{d}{dt}F_1(t)=f_1(t),\ \frac{d}{dt}F_2(t)=f_2(t),\ \frac{d}{dt}F_3(t)=f_3(t).$$

$\therefore$ $F_1(t)=\int f_1(t)\,dt,\ F_2(t)=\int f_2(t)\,dt,\ F_3(t)=\int f_3(t)\,dt.$

$\therefore$ $\mathbf{F}(t)=\left\{\int f_1(t)\,dt\right\}\mathbf{i}+\left\{\int f_2(t)\,dt\right\}\mathbf{j}+\left\{\int f_3(t)\,dt\right\}\mathbf{k}.$

So from (2), we get

$$\int \mathbf{f}(t)\,dt=\mathbf{i}\int f_1(t)\,dt+\mathbf{j}\int f_2(t)\,dt+\mathbf{k}\int f_3(t)\,dt.$$

Note: From this theorem we conclude that the definition of the integral of a vector function implies the definition of integrals of three scalar functions which are the components of that vector function. **Thus in order to integrate a vector function we should integrate its components.**

2 Some Standard Results

We have already obtained some standard results for differentiation. With the help of these results we can obtain some standard results for integration.

1. We have $\frac{d}{dt}(\mathbf{r}\bullet\mathbf{s})=\frac{d\mathbf{r}}{dt}\bullet\mathbf{s}+\mathbf{r}\bullet\frac{d\mathbf{s}}{dt}\cdot$

Therefore $\int \left(\frac{d\mathbf{r}}{dt} \bullet \mathbf{s} + \mathbf{r} \bullet \frac{d\mathbf{s}}{dt} \right) dt = \mathbf{r} \bullet \mathbf{s} + c,$

where c is the constant of integration. It should be noted that c is here a scalar quantity since the integrand is also scalar.

2. We have $\frac{d}{dt}(\mathbf{r}^2) = 2\mathbf{r} \bullet \frac{d\mathbf{r}}{dt}$. Therefore $\int \left(2\,\mathbf{r} \bullet \frac{d\mathbf{r}}{dt} \right) dt = \mathbf{r}^2 + c.$

   Here the constant of integration c is a scalar quantity.

3. We have $\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)^2 = 2\,\frac{d\mathbf{r}}{dt} \bullet \frac{d^2\mathbf{r}}{dt^2}$.

   Therefore we have $\int \left(2\,\frac{d\mathbf{r}}{dt} \bullet \frac{d^2\mathbf{r}}{dt^2} \right) dt = \left(\frac{d\mathbf{r}}{dt}\right)^2 + c.$

   Here the constant of integration c is a scalar quantity.

   Also $\left(\frac{d\mathbf{r}}{dt}\right)^2 = \frac{d\mathbf{r}}{dt} \bullet \frac{d\mathbf{r}}{dt}$.

4. We have $\frac{d}{dt}\left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right) = \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} + \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}$.

   $\therefore \quad \int \left(\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} \right) dt = \mathbf{r} \times \frac{d\mathbf{r}}{dt} + \mathbf{c}.$

   Here the constant of integration **c** is a vector quantity since the integrand $\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}$ is also a vector quantity.

5. If **a** is a constant vector, we have

$$\frac{d}{dt}(\mathbf{a} \times \mathbf{r}) = \frac{d\mathbf{a}}{dt} \times \mathbf{r} + \mathbf{a} \times \frac{d\mathbf{r}}{dt} = \mathbf{a} \times \frac{d\mathbf{r}}{dt}.$$

   Therefore $\int \left(\mathbf{a} \times \frac{d\mathbf{r}}{dt} \right) dt = \mathbf{a} \times \mathbf{r} + \mathbf{c}.$

   Here the constant of integration **c** is a vector quantity.

6. If $r = |\mathbf{r}|$ and $\hat{\mathbf{r}}$ is a unit vector in the direction of **r**, then

$$\frac{d}{dt}(\hat{\mathbf{r}}) = \frac{d}{dt}\left(\frac{1}{r}\mathbf{r}\right) = \frac{1}{r}\frac{d\mathbf{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\mathbf{r}.$$

   Therefore $\int \left(\frac{1}{r}\frac{d\mathbf{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\mathbf{r} \right) dt = \hat{\mathbf{r}} + \mathbf{c}.$

7. If c is a constant scalar and **r** a vector function of a scalar t, then obviously

$$\int c\mathbf{r}\, dt = c \int \mathbf{r}\, dt.$$

8. If **r** and **s** are two vector functions of the scalar t, then obviously

$$\int (\mathbf{r} + \mathbf{s})\, dt = \int \mathbf{r}\, dt + \int \mathbf{s}\, dt.$$

Illustrative Examples

Example 1: *Evaluate* $\int_1^2 \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\,dt$, *where* $\mathbf{r} = 2t^2\,\mathbf{i} + t\,\mathbf{j} - 3t^3\,\mathbf{k}$.

(Bundelkhand 2004; Kumaun 08; Kanpur 13)

Solution: Given $\mathbf{r} = 2t^2\mathbf{i} + t\,\mathbf{j} - 3t^3\,\mathbf{k}$.

$$\therefore \quad \frac{d\mathbf{r}}{dt} = 4t\,\mathbf{i} + \mathbf{j} - 9t^2\,\mathbf{k} \text{ and } \frac{d^2\mathbf{r}}{dt^2} = 4\mathbf{i} + 0\,\mathbf{j} - 18t\mathbf{k}.$$

$$\therefore \quad \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = (2t^2\mathbf{i} + t\mathbf{j} - 3t^3\mathbf{k}) \times (4\mathbf{i} + 0\,\mathbf{j} - 18t\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t^2 & t & -3t^3 \\ 4 & 0 & -18t \end{vmatrix}$$

$$= -18t^2\mathbf{i} - (-36t^3 + 12t^3)\,\mathbf{j} - 4t\,\mathbf{k}$$

$$= -18\,t^2\mathbf{i} + 24t^3\,\mathbf{j} - 4t\mathbf{k}.$$

$$\therefore \quad \int_1^2 \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\,dt = \int_1^2 (-18t^2\mathbf{i} + 24t^3\,\mathbf{j} - 4t\mathbf{k})\,dt$$

$$= -18\mathbf{i}\int_1^2 t^2\,dt + 24\mathbf{j}\int_1^2 t^3\,dt - 4\mathbf{k}\int_1^2 t\,dt$$

$$= -18\mathbf{i}\left[\frac{t^3}{3}\right]_1^2 + 24\mathbf{j}\left[\frac{t^4}{4}\right]_1^2 - 4\mathbf{k}\left[\frac{t^2}{2}\right]_1^2$$

$$= -6\,(8-1)\,\mathbf{i} + 6\,(16-1)\,\mathbf{j} - 2\,(4-1)\,\mathbf{k}$$

$$= -42\mathbf{i} + 90\,\mathbf{j} - 6\mathbf{k}.$$

Example 2: *Find the value of* $\mathbf{r}$ *satisfying the equation* $\frac{d^2\mathbf{r}}{dt^2} = t\mathbf{a} + \mathbf{b}$, *where* **a** *and* **b** *are constant vectors.*

Solution: Integrating the equation $\frac{d^2\mathbf{r}}{dt^2} = t\mathbf{a} + \mathbf{b}$, we get $\frac{d\mathbf{r}}{dt} = \frac{1}{2}t^2\,\mathbf{a} + t\mathbf{b} + \mathbf{c}$, where **c** is constant.

Again integrating, we get

$$\mathbf{r} = \frac{1}{6}t^3 + \frac{1}{2}t^2\,\mathbf{b} + t\mathbf{c} + \mathbf{d}\text{, where } \mathbf{d} \text{ is constant.}$$

Example 3: *If* $\mathbf{r}\,(t) = 5t^2\,\mathbf{i} + t\mathbf{j} - t^3\,\mathbf{k}$, *prove that*

$$\int_1^2 \left(\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\right) dt = -14\mathbf{i} + 75\,\mathbf{j} - 15\mathbf{k}.$$

(Kumaun 2000; Meerut 01, 04, 05, 07, 10, 10B; Kanpur 10; Rohilkhand 13)

Solution: We have $\int \left(\mathbf{r} \times \dfrac{d^2 \mathbf{r}}{dt^2} \right) dt = \mathbf{r} \times \dfrac{d\mathbf{r}}{dt} + \mathbf{c}$.

$$\therefore \qquad \int_1^2 \left(\mathbf{r} \times \frac{d^2 \mathbf{r}}{dt^2} \right) dt = \left[\mathbf{r} \times \frac{d\mathbf{r}}{dt} \right]_1^2 \cdot$$

Let us now find $\mathbf{r} \times \dfrac{d\mathbf{r}}{dt} \cdot$ We have $\dfrac{d\mathbf{r}}{dt} = 10\, t\, \mathbf{i} + \mathbf{j} - 3t^2\, \mathbf{k}$.

$$\therefore \qquad \mathbf{r} \times \frac{d\mathbf{r}}{dt} = (5t^2\, \mathbf{i} + t\, \mathbf{j} - t^3\, \mathbf{k}) \times (10t\, \mathbf{i} + \mathbf{j} - 3t^2\, \mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5t^2 & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix} = -\,2t^3\, \mathbf{i} + 5t^4\, \mathbf{j} - 5t^2\, \mathbf{k}.$$

$$\therefore \qquad \int_1^2 \left(\mathbf{r} \times \frac{d^2 \mathbf{r}}{dt^2} \right) dt = \left[-2t^3\, \mathbf{i} + 5t^4\, \mathbf{j} - 5t^2\, \mathbf{k} \right]_1^2$$

$$= \left[-\,2t^3 \right]_1^2 \mathbf{i} + \left[5t^4 \right]_1^2 \mathbf{j} - \left[5t^2 \right]_1^2 \mathbf{k}$$

$$= -\,14\mathbf{i} + 75\mathbf{j} - 15\mathbf{k}.$$

Example 4: *Given that* $\mathbf{r}\,(t) = \begin{cases} 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}, & \textit{when} \quad t = 2 \\ 4\mathbf{i} - 2\,\mathbf{j} + 3\mathbf{k}, & \textit{when} \quad t = 3, \end{cases}$

show that $\int_2^3 \left(\mathbf{r} \bullet \dfrac{d\mathbf{r}}{dt} \right) dt = 10.$

(Meerut 2003, 13B; Bundelkhand 08; Kanpur 09, 11; Agra 06; Avadh 10; Purvanchal 13)

Solution: We have $\int \left(\mathbf{r} \bullet \dfrac{d\mathbf{r}}{dt} \right) dt = \left[\dfrac{1}{2}\, \mathbf{r}^2 \right]_2^3 \cdot$

When $t = 3, \mathbf{r} = 4\mathbf{i} - 2\,\mathbf{j} + 3\mathbf{k}$.

$\therefore$ when $t = 3, \mathbf{r}^2 = (4\mathbf{i} - 2\,\mathbf{j} + 3\mathbf{k}) \bullet (4\mathbf{i} - 2\,\mathbf{j} + 3\mathbf{k}) = 16 + 4 + 9 = 29.$

When $t = 2\,, \mathbf{r} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

$\therefore$ when $t = 2\,, \mathbf{r}^2 = 4 + 1 + 4 = 9.$

$$\therefore \qquad \int_2^3 \left(\mathbf{r} \bullet \frac{d\mathbf{r}}{dt} \right) dt = \frac{1}{2}\, [29 - 9] = 10.$$

Example 5: *The acceleration of a particle at any time* $t \geq 0$ *is given by*

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = 12\, \mathit{cos}\; 2t\; \mathbf{i} - 8\, \mathit{sin}\; 2t\; \mathbf{j} + 16\, t\; \mathbf{k}.$$

If the velocity $\mathbf{v}$ *and displacement* $\mathbf{r}$ *are zero at* $t = 0$*, find* $\mathbf{v}$ *and* $\mathbf{r}$ *at any time.*

(Agra 2007)

Solution: We have $\dfrac{d\mathbf{v}}{dt} = 12 \cos 2t\; \mathbf{i} - 8 \sin 2t\; \mathbf{j} + 16t\; \mathbf{k}$.

Integrating, we get

$$\mathbf{v} = \mathbf{i}\int 12\cos 2t\,dt + \mathbf{j}\int -8\sin 2t\,dt + \mathbf{k}\int 16t\,dt$$

or $$\mathbf{v} = 6\sin 2t\,\mathbf{i} + 4\cos 2t\,\mathbf{j} + 8t^2\,\mathbf{k} + \mathbf{c}.$$

When $t = 0$, $\mathbf{v} = \mathbf{0}.$

$\therefore$ $$\mathbf{0} = 0\mathbf{i} + 4\mathbf{j} + 0\mathbf{k} + \mathbf{c} \quad \text{or} \quad \mathbf{c} = -4\mathbf{j}.$$

$\therefore$ $$\mathbf{v} = \frac{d\mathbf{r}}{dt} = 6\sin 2t\,\mathbf{i} + (4\cos 2t - 4)\,\mathbf{j} + 8t^2\,\mathbf{k}.$$

Integrating, we get

$$\mathbf{r} = \mathbf{i}\int 6\sin 2t\,dt + \mathbf{j}\int (4\cos 2t - 4)\,dt + \mathbf{k}\int 8t^2\,dt$$

$$= -3\cos 2t\,\mathbf{i} + (2\sin 2t - 4t)\,\mathbf{j} + \frac{8}{3}t^3\,\mathbf{k} + \mathbf{d},$$

where $\mathbf{d}$ is constant.

When $t = 0$, $\mathbf{r} = \mathbf{0}$.

$\therefore$ $$\mathbf{0} = -3\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} + \mathbf{d}. \qquad \therefore \qquad \mathbf{d} = 3\mathbf{i}.$$

$\therefore$ $$\mathbf{r} = -3\cos 2t\,\mathbf{i} + (2\sin 2t - 4t)\,\mathbf{j} + \frac{8}{3}t^3\,\mathbf{k} + 3\mathbf{i}$$

$$= (3 - 3\cos 2t)\,\mathbf{i} + (2\sin 2t - 4t)\,\mathbf{j} + \frac{8}{3}t^3\,\mathbf{k}.$$

Comprehensive Exercise 1

1. If $\mathbf{f}(t) = (t - t^2)\,\mathbf{i} + 2t^3\,\mathbf{j} - 3\mathbf{k}$, find (i) $\int \mathbf{f}(t)\,dt$ (ii) $\int_1^2 \mathbf{f}(t)\,dt.$

2. Evaluate $\int_0^1 e^t\,\mathbf{i} + e^{-2t}\,\mathbf{j} + t\,\mathbf{k})\,dt$. **(Garhwal 2001, 02)**

3. If $\mathbf{f}(t) = t\,\mathbf{i} + (t^2 - 2t)\,\mathbf{j} + (3t^2 + 3t^3)\,\mathbf{k}$, find $\int_0^1 \mathbf{f}(t)\,dt.$

   (Garhwal 2003; Bundelkhand 07)

4. If $\mathbf{r} = t\,\mathbf{i} - t^2\,\mathbf{j} + (t - 1)\,\mathbf{k}$ and $\mathbf{s} = 2t^2\,\mathbf{i} + 6t\,\mathbf{k}$, evaluate

   (i) $\int_0^2 \mathbf{r} \bullet \mathbf{s}\,dt,$ (ii) $\int_0^2 \mathbf{r} \times \mathbf{s}\,dt$

   (Rohilkhand 2008)

5. (i) Find the value of $\mathbf{r}$ satisfying the equation $\dfrac{d^2\mathbf{r}}{dt^2} = \mathbf{a}$, where $\mathbf{a}$ is a constant vector. Also it is given that when $t = 0$, $\mathbf{r} = \mathbf{0}$ and $\dfrac{d\mathbf{r}}{dt} = \mathbf{u}$.

   (ii) Solve the equation $\dfrac{d^2\mathbf{r}}{dt^2} = \mathbf{a}$ where $\mathbf{a}$ is a constant vector ; given that $\mathbf{r} = \mathbf{0}$ and $\dfrac{d\mathbf{r}}{dt} = \mathbf{0}$ when $t = 0$.

   (Bundelkhand 2008)

6. Find the value of $\mathbf{r}$ satisfying the equation $\dfrac{d^2\mathbf{r}}{dt^2} = 6t\mathbf{i} - 24t^2\,\mathbf{j} + 4\sin t\,\mathbf{k}$,

given that $\mathbf{r} = 2\mathbf{i} + \mathbf{j}$ and $d\mathbf{r}/dt = -\,\mathbf{i} - 3\,\mathbf{k}$ at $t = 0$.

(Agra 2001; Meerut 11)

7. The acceleration of a particle at any time t is $e^t\mathbf{i} + e^{2t}\mathbf{j} + \mathbf{k}$.

Find $\mathbf{v}$, given that $\mathbf{v} = \mathbf{i} + \mathbf{j}$ at $t = 0$.

8. Evaluate $\int_1^2 (\mathbf{a} \bullet \mathbf{b} \times \mathbf{c})\, dt$, where

$\mathbf{a} = t\mathbf{i} - 3\mathbf{j} + 2t\mathbf{k}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = 3\mathbf{i} + t\,\mathbf{j} - \mathbf{k}$. **(Meerut 2013)**

9. Integrate $\dfrac{d^2\mathbf{r}}{dt^2} = -\,n\mathbf{r}^2$.

(Kumaun 2009)

Answers 1

1. (i) $\left(\dfrac{t^2}{2} + \dfrac{t^3}{3}\right)\mathbf{i} + \dfrac{t^4}{2}\,\mathbf{j} - 3t\,\mathbf{k} + \mathbf{c}$ (ii) $-\dfrac{5}{6}\,\mathbf{i} + \dfrac{15}{2}\,\mathbf{j} - 3\,\mathbf{k}$

2. $(e - 1)\,\mathbf{i} - \dfrac{1}{2}(e^{-2} - 1)\,\mathbf{j} + \dfrac{1}{2}\,\mathbf{k}$ 3. $\dfrac{1}{2}\,\mathbf{i} - \dfrac{2}{3}\,\mathbf{j} + \dfrac{7}{4}\,\mathbf{k}$

4. (i) 12 (ii) $-24\,\mathbf{i} - \dfrac{40}{3}\,\mathbf{j} + \dfrac{64}{5}\,\mathbf{k}$

5. (i) $\dfrac{1}{2}t^2\,\mathbf{a} + t\mathbf{u}$ (ii) $\dfrac{1}{2}t^2\,\mathbf{a}$

6. $\mathbf{r} = (t^3 - t + 2)\,\mathbf{i} + (1 - 2t^4)\,\mathbf{j} + (t - 4\sin t)\,\mathbf{k}$

7. $e^t\,\mathbf{i} + \dfrac{1}{2}(e^{2t} + 1)\,\mathbf{j} + t\,\mathbf{k}$ 8. 0

9. $-\,n^2\mathbf{r}^2 + c$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If $\mathbf{F}(t) = t\,\mathbf{i} + (t^2 - 2t)\,\mathbf{j} + (3t^2 + 3t^3)\,\mathbf{k}$, then the value of $\int_0^1 \mathbf{F}(t)\,dt$ is

(a) $\dfrac{1}{2}\,\mathbf{i} + \dfrac{2}{3}\,\mathbf{j} + \dfrac{7}{4}\,\mathbf{k}$ (b) $\dfrac{1}{2}\,\mathbf{i} - \dfrac{2}{3}\,\mathbf{j} + \dfrac{7}{4}\,\mathbf{k}$

(c) $-\dfrac{1}{2}\,\mathbf{i} - \dfrac{2}{3}\,\mathbf{j} + \dfrac{7}{4}\,\mathbf{k}$ (d) None of these

(Bundelkhand 2001)

2. If $\mathbf{r} = t\,\mathbf{i} - t^2\,\mathbf{j} + (t-1)\,\mathbf{k}$, and $\mathbf{s} = 2t^2\,\mathbf{i} + 6t\,\mathbf{k}$, then the value of $\int_0^1 \mathbf{r} \bullet \mathbf{s}\,dt$ is

(a) 10	(b) 12
(c) 15	(d) None of these

(Kumaun 2007, 10)

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. $\int_0^1 [t\,\mathbf{i} + (t^2 - 2t)\,\mathbf{j}]\,dt = \ldots\ldots$.

2. If $\mathbf{F}(t) = 3t^2\,\mathbf{i} + t\,\mathbf{j} + 2\,\mathbf{k}$ and $\mathbf{G}(t) = 6t^2\,\mathbf{i} + (t-1)\,\mathbf{j} + 3t\,\mathbf{k}$, then

$$\int_0^1 \left(\frac{d\mathbf{F}}{dt} \bullet \mathbf{G} + \mathbf{F} \bullet \frac{d\mathbf{G}}{dt}\right) dt = \ldots\ldots.$$

(Meerut 2011)

3. If $\mathbf{r}(t) = 5t^2\,\mathbf{i} + t\,\mathbf{j} - t^3\,\mathbf{k}$, then $\int_1^2 \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\,dt = \ldots\ldots$. **(Kumaun 2013)**

4. If $\mathbf{r} = t^2\,\mathbf{i} + \mathbf{j} - t\,\mathbf{k}$, then $\int_0^1 \mathbf{r} \bullet \frac{d\mathbf{r}}{dt}\,dt = \ldots\ldots$. **(Kumaun 2014)**

True or False

Write 'T' for true and 'F' for false statement.

1. $\int \left(2\,\frac{d\mathbf{r}}{dt} \bullet \frac{d^2\mathbf{r}}{dt^2}\right) dt = \left(\frac{d\mathbf{r}}{dt}\right)^2 + c.$

2. The value of $\int_0^1 (e^t\,\mathbf{i} + e^{-2t}\,\mathbf{j} + t\,\mathbf{k})\,dt$ is $(e-1)\,\mathbf{i} - (e^{-2} - 1)\,\mathbf{j} + \mathbf{k}$.

Answers

Multiple Choice Questions

1. (b)	**2.** (d)

Fill in the Blank(s)

1. $\frac{1}{2}i - \frac{2}{3}j$	**2.** 24
3. $-14\,\mathbf{i} + 75\,\mathbf{j} - 15\,\mathbf{k}$	**4.** 1

True or False

1. *T*	**2.** *F*

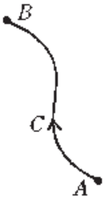
☐

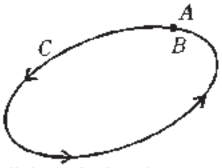
Chapter 5

Line Integrals

1 Some Preliminary Concepts

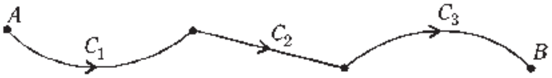
Oriented curve. Suppose C is a curve in space. Let us orient C by taking one of the two directions along C as the *positive direction*; the opposite direction along C is then called the *negative direction* . Suppose A is the initial point and B the terminal point of C under the chosen orientation. In case these two points coincide, the curve C is called a *closed curve.*





Oriented closed curve

Smooth curve. Let $\mathbf{r}\,(t) = x\,(t)\,\mathbf{i} + y\,(t)\,\mathbf{j} + z\,(t)\,\mathbf{k}$, where $\mathbf{r}\,(t)$ is the position vector of (x, y, z), be the parametric representation of a curve C joining the points A and B, where $t = t_1$ and $t = t_2$ respectively. We know that $d\mathbf{r} \,/\, dt$ is a tangent vector to this curve at the point **r**. Suppose the function **r** (t) is continuous and has a continuous first derivative not equal to zero vector for all values of t under consideration. Then the curve C possesses a unique tangent at each of its points. A curve satisfying these assumptions is called a **smooth curve.**



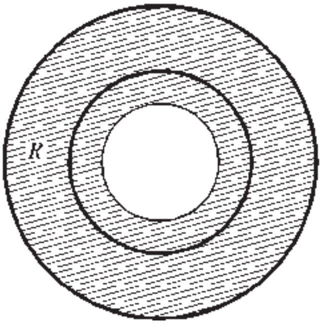
Piecewise Smooth Curve

A curve C is said to be **piecewise smooth** if it is composed of a finite number of smooth curves. The curve C in the adjoining figure is piecewise smooth as it is composed of three smooth curves C_1, C_2 and C_3. The circle is a smooth closed curve while the curve consisting of the four sides of a rectangle is a piecewise smooth closed curve.

Smooth surface. Suppose S is a surface which has a unique normal at each of its points and the direction of this normal depends continuously on the points of S. Then S is called a **smooth surface.**

If a surface S is not smooth but can be subdivided into a finite number of smooth surfaces, then it is called a **piecewise smooth surface.** The surface of a sphere is smooth while the surface of a cube is piecewise smooth.

Classification of regions. A region R in which every closed curve can be contracted to a point without passing out of the region is called a **simply connected region.** Otherwise the region R is **multiply-connected**. The region interior to a circle is a simply-connected plane region. The region interior to a sphere is a simply-connected region in space. The region between two concentric circles lying in the same plane is a multiply-connected plane region.



If we take a closed curve in this region surrounding the inner circle, then it cannot be contracted to a point without passing out of the region. Therefore the region is not simply-connected. However the region between two concentric spheres is a simply-connected region in space. The region between two infinitely long coaxial cylinders is a multiply-connected region in space.

2 Line Integrals

(Avadh 2014)

Any integral which is to be evaluated along a curve is called a line integral.

Suppose $\mathbf{r}\ (t) = x\ (t)\ \mathbf{i} + y\ (t)\ \mathbf{j} + z\ (t)\ \mathbf{k}$,where $\mathbf{r}\ (t)$ is the position vector of (x, y, z) *i.e.,* $\mathbf{r}\ (t) = x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}$,defines a piecewise smooth curve joining two points A and B. Let $t = t_1$ at A and $t = t_2$ at B. Suppose $\mathbf{F}(x, y, z) = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a vector point function defined and continuous along C. If s denotes the arc length of the curve C, then $\dfrac{d\mathbf{r}}{ds} = \mathbf{t}$ is a unit vector along the tangent to the curve C at the point $\mathbf{r}$. The component of the vector $\mathbf{F}$ along this tangent is $\mathbf{F} \bullet \dfrac{d\mathbf{r}}{ds}$. The integral of $\mathbf{F} \bullet \dfrac{d\mathbf{r}}{ds}$ along C from A to B written as

$$\int_A^B \left[\mathbf{F} \bullet \frac{d\mathbf{r}}{ds}\right] ds = \int_A^B \mathbf{F} \bullet d\mathbf{r} = \int_C \mathbf{F} \bullet d\mathbf{r}$$

is an example of a *line integral.* It is called *the tangent line integral* of $\mathbf{F}$ along C.

Since $\mathbf{r} = x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k}$, therefore, $d\mathbf{r} = dx\ \mathbf{i} + dy\ \mathbf{j} + dz\ \mathbf{k}$.

$$\therefore \qquad \mathbf{F} \bullet d\mathbf{r} = (F_1\mathbf{i} + F_2\ \mathbf{j} + F_3\mathbf{k}) \bullet (dx\ \mathbf{i} + dy\ \mathbf{j} + dz\ \mathbf{k})$$
$$= F_1\ dx + F_2\ dy + F_3\ dz.$$

Therefore in components form the above line integral is written as

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C (F_1\ dx + F_2\ dy + F_3\ dz).$$

The parametric equations of the curve C are $x = x(t)$, $y = y(t)$ and $z = z(t)$. Therefore we may write

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_{t_1}^{t_2} \left[F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right] dt.$$

Circulation: *If C is a simple closed curve (i.e. a curve which does not intersect itself anywhere), then the tangent line integral of* **F** *around C is called the circulation of* **F** *about C. It is often denoted by*

$$\oint \mathbf{F} \bullet d\mathbf{r} = \oint (F_1\ dx + F_2\ dy + F_3\ dz).$$

Work done by a Force. Suppose a force $\mathbf{F}$ acts upon a particle. Let the particle be displaced along a given path C in space. If $\mathbf{r}$ denotes the position vector of a point on C, then $\dfrac{d\mathbf{r}}{ds}$ is a unit vector along the tangent to C at the point $\mathbf{r}$ in the direction of s increasing. The component of force $\mathbf{F}$ along tangent to C is $\mathbf{F} \bullet \dfrac{d\mathbf{r}}{ds}$. Therefore the work done by $\mathbf{F}$ during a small displacement ds of the particle along C is $\left[\mathbf{F} \bullet \dfrac{d\mathbf{r}}{ds}\right] ds$ *i.e.,* $\mathbf{F} \bullet d\mathbf{r}$. The total work W done by $\mathbf{F}$ in this displacement along C, is given by the line integral

$$W = \int_C \mathbf{F} \bullet d\mathbf{r},$$

the integration being taken in the sense of the displacement.

Illustrative Examples

Example 1: *Evaluate* $\int_C \mathbf{F}\bullet d\mathbf{r}$, *where* $\mathbf{F} = x^2\mathbf{i} + y^3\mathbf{j}$ *and curve C is the arc of the parabola* $y = x^2$ *in the x-y plane from (0, 0) to (1, 1).*

Solution: We shall illustrate two methods for the solution of such a problem.

Method 1. The curve C is the parabola $y = x^2$ from (0, 0) to (1, 1).

Let $x = t$; then $y = t^2$. If $\mathbf{r}$ is the position vector of any point (x, y) on C, then

$$\mathbf{r}(t) = x\mathbf{i} + y\mathbf{j} = t\mathbf{i} + t^2\mathbf{j}. \qquad \therefore \qquad \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j}.$$

Also in terms of t, $\mathbf{F} = t^2\mathbf{i} + t^6\mathbf{j}$.

At the point (0, 0), $t = x = 0$. At the point (1, 1), $t = 1$.

$$\therefore \qquad \int_C \mathbf{F}\bullet d\mathbf{r} = \int_C \left(\mathbf{F}\bullet\frac{d\mathbf{r}}{dt}\right) dt = \int_0^1 (t^2\,\mathbf{i} + t^6\,\mathbf{j})\bullet(\mathbf{i} + 2t\,\mathbf{j})\,dt$$

$$= \int_0^1 t^2 + 2t^7)\,dt = \left[\frac{t^3}{3} + \frac{2t^8}{8}\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

Method 2: In the xy-plane we have $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.

$$\therefore \qquad d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}.$$

Therefore, $\mathbf{F}\bullet d\mathbf{r} = (x^2\mathbf{i} + y^3\mathbf{j})\bullet(dx\,\mathbf{i} + dy\,\mathbf{j}) = x^2\,dx + y^3\,dy.$

$$\therefore \qquad \int_C \mathbf{F}\bullet d\mathbf{r} = \int_C (x^2\,dx + y^3\,dy).$$

Now along the curve C, $y = x^2$. Therefore $dy = 2x\,dx$.

$$\therefore \qquad \int_C \mathbf{F}\bullet d\mathbf{r} = \int_{x=0}^1 [x^2\,dx + x^6\,(2x)\,dx]$$

$$= \int_0^1 (x^2 + 2x^7)\,dx = \left[\frac{x^3}{3} + \frac{2x^8}{8}\right]_0^1 = \frac{7}{12}.$$

Example 2: *If* $\mathbf{F} = 3xy\,\mathbf{i} - y^2\,\mathbf{j}$, *evaluate* $\int_C \mathbf{F}\bullet d\mathbf{r}$, *where C is the curve in the xy-plane,* $y = 2x^2$, *from* (0, 0) *to* (1, 2). **(Garhwal 2001, 02; Kumaun 07; Rohilkhand 12)**

Solution: The parametric equations of the parabola $y = 2x^2$ can be taken as

$$x = t,\ y = 2t^2.$$

At the point (0, 0), $x = 0$ and so $t = 0$. Again at the point (1, 2), $x = 1$ and so $t = 1$.

Now $$\int_C \mathbf{F}\bullet d\mathbf{r} = \int_C (3xy\mathbf{i} - y^2\mathbf{j})\bullet(dx\,\mathbf{i} + dy\,\mathbf{j})$$

[$\because$ $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, so that $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$]

$$= \int_C (3xy\ dx - y^2\ dy) = \int_{t=0}^{1} \left(3xy \frac{dx}{dt} - y^2 \frac{dy}{dt}\right) dt$$

$$= \int_0^1 (3 \,.\, t \,.\, 2\, t^2 \,.\, 1 - 4\, t^4 \,.\, 4t)\ dt$$

$$[\because \quad x = t, \quad y = 2t^2 \quad \text{so that} \quad dx / dt = 1 \quad \text{and} \quad dy / dt = 4t]$$

$$= \int_0^1 (6t^3 - 16t^5)\ dt = \left[6 \,.\, \frac{t^4}{4} - 16 \,.\, \frac{t^6}{6}\right]_0^1$$

$$= \frac{6}{4} - \frac{16}{6} = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}.$$

Example 3: *Evaluate* $\int \mathbf{F} \bullet d\mathbf{r}$ *along the curve* $x^2 + y^2 = 1, \ z = 1$ *in the positive direction from* $(0, 1, 1)$ *to* $(1, 0, 1)$ *where* $\mathbf{F} = (2x + yz)\ \mathbf{i} + xz\mathbf{j} + (xy + 2z)\ \mathbf{k}$.

Solution: Let the given curve be denoted by C and let A and B be points $(0, 1, 1)$ and $(1, 0, 1)$ respectively.

Along the given curve C, we have $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$$\therefore \qquad d\mathbf{r} = dx\ \mathbf{i} + dy\ \mathbf{j} + dz\ \mathbf{k}.$$

$$\therefore \qquad \int_C \mathbf{F} \bullet d\mathbf{r} = \int_A^B [(2x + yz)\ \mathbf{i} + xz\ \mathbf{j} + (xy + 2z)\ \mathbf{k}] \bullet (dx\ \mathbf{i} + dy\ \mathbf{j} + dz\ \mathbf{k})$$

$$= \int_A^B [(2x + yz)\ dx + xz\ dy + (xy + 2z)\ dz]. \qquad \text{...(1)}$$

In moving from A to B, x varies from 0 to 1 , y varies from 1 to 0 and z remains constant. We have $z = 1$ and so $dz = 0$.

Hence from (1)

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^1 (2x + y)\ dx + \int_1^0 x\ dy + 0$$

$$= \int_0^1 [2x + \sqrt{(1 - x^2)}]\ dx - \int_0^1 \sqrt{(1 - y^2)}\ dy = [x^2]_0^1 = 1,$$

the last two integrals cancel by a property of definite integrals.

Example 4: *Evaluate* $\int (x\ dy - y\ dx)$ *around the circle* $x^2 + y^2 = 1.$ **(Meerut 2002)**

Solution: Let C denote the circle $x^2 + y^2 = 1.$ The parametric equations of this circle are $x = \cos t, y = \sin t$.

To integrate around the circle C we should vary t from 0 to 2π.

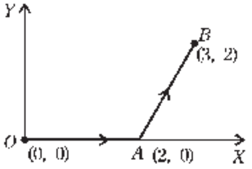
$$\therefore \qquad \oint_C (x\ dy - y\ dx) = \int_0^{2\pi} \left(x \frac{dy}{dt} - y \frac{dx}{dt}\right) dt$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t)\ dt = \int_0^{2\pi} dt = 2\pi.$$

Example 5: *If* $\mathbf{F} = (2\,x + y)\ \mathbf{i} + (3\,y - x)\ \mathbf{j}$, *evaluate* $\int_C \mathbf{F} \bullet d\mathbf{r}$ *where C is the curve in the xy-plane consisting of the straight lines from* $(0, 0)$ *to* $(2, 0)$ *and then to* $(3, 2)$.

(Agra 2007; Meerut 11; Purvanchal 14)

Solution: The path of integration C has been shown in the figure. It consists of the straight lines OA and AB.



We have $\int_C \mathbf{F} \bullet d\mathbf{r}$

$$= \int_C [(2x + y)\,\mathbf{i} + (3y - x)\,\mathbf{j}] \bullet (dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= \int_C [(2x + y)\,dx + (3y - x)\,dy].$$

Now along the straight line OA, $y = 0$, $dy = 0$ and x varies from 0 to 2. The equation of the straight line AB is

$$y - 0 = \frac{2 - 0}{3 - 2}(x - 2) \quad i.e., \quad y = 2x - 4.$$

$\therefore$ along AB, $y = 2x - 4$, $dy = 2\,dx$ and x varies from 2 to 3.

$$\therefore \quad \int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^2 [(2x + 0)\,dx + 0] + \int_2^3 [(2x + 2x - 4)\,dx + (6x - 12 - x)\,2\,dx]$$

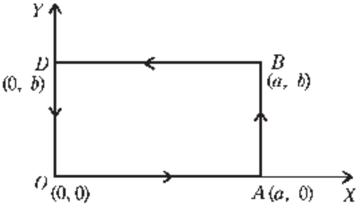
$$= \left[\,x^2\,\right]_0^2 + \int_2^3 (14x - 28)\,dx$$

$$= 4 + 14\int_2^3 (x - 2)\,dx = 4 + 14\left[\frac{(x - 2)^2}{2}\right]_2^3 = 4 + 7 = 11.$$

Example 6: *Evaluate* $\int_C \mathbf{F} \bullet d\mathbf{r}$ *where* $\mathbf{F} = (x^2 + y^2)\,\mathbf{i} - 2xy\mathbf{j}$, *curve C is the rectangle in the x y-plane bounded by* $y = 0,\ x = a,\ y = b,\ x = 0$.

(Meerut 2000, 06B, 07B, 09B, 12, 13; Kanpur 10; Bundelkhand 09; Purvanchal 09)

Solution: In the xy-plane $z = 0$.



Therefore

$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ and $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$.

The path of integration C has been shown in the figure. It consists of the straight lines OA, AB, BD and DO.

We have

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C [(x^2 + y^2)\,\mathbf{i} - 2xy\,\mathbf{j}] \bullet (dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= \int_C [(x^2 + y^2)\,dx - 2xy\,dy].$$

Now on OA, $y = 0$, $dy = 0$ and x varies from 0 to a ; on AB, $x = a$, $dx = 0$ and y varies from 0 to b ; on BD, $y = b$, $dy = 0$ and x varies from a to 0; on DO, $x = 0$, $dx = 0$ and y varies from b to 0.

$$\therefore \quad \int_C \mathbf{F} \bullet d\mathbf{r} = \int_0^a x^2\,dx - \int_0^b 2ay\,dy + \int_a^0 (x^2 + b^2)\,dx + \int_b^0 0\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^a - 2a\left[\frac{y^2}{2}\right]_0^b + \left[\frac{x^3}{3} + b^2 x\right]_a^0 + 0 = -2ab^2.$$

Example 7: *Find the total work done in moving a particle in a force field given by*

$$\mathbf{F} = 3xy\mathbf{i} - 5z\mathbf{j} + 10x\mathbf{k}$$

along the curve $x = t^2 + 1,\ \ y = 2t^2,\ \ z = t^3$ *from* $t = 1$ *to* $t = 2$.

Solution: Let C denote the arc of the given curve from $t = 1$ to $t = 2$. Then the total work done

$$= \int_C \mathbf{F} \bullet d\mathbf{r} = \int_C (3xy\mathbf{i} - 5z\ \mathbf{j} + 10x\ \mathbf{k}) \bullet (dx\ \mathbf{i} + dy\ \mathbf{j} + dz\ \mathbf{k})$$

$$= \int_C (3xy\ dx - 5z\ dy + 10x\ dz)$$

$$= \int_1^2 \left(3xy\frac{dx}{dt} - 5z\frac{dy}{dt} + 10x\frac{dz}{dt}\right) dt$$

$$= \int_1^2 [3(t^2 + 1)(2t)^2(2t) - (5t^3)(4t) + 10(t^2 + 1)(3t^2)]\ dt$$

$$= \int_1^2 (12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2)\ dt$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\ dt = 303.$$

Example 8: *Evaluate* $\int_C \mathbf{F} \bullet d\mathbf{r}$, *where* $\mathbf{F} = yz\ \mathbf{i} + zx\ \mathbf{j} + xy\ \mathbf{k}$ *and* C *is the portion of the curve* $\mathbf{r} = a\cos t\ \mathbf{i} + b\sin t\ \mathbf{j} + ct\ \mathbf{k}$, *from* $t = 0$ *to* $t = \pi/2$. **(Avadh 2010)**

Solution: Along the curve C,

$$\mathbf{r} = x\ \mathbf{i} + y\ \mathbf{j} + z\ \mathbf{k} = a\cos t\ \mathbf{i} + b\sin t\ \mathbf{j} + ct\ \mathbf{k}.$$

$\therefore$ $\quad x = a\cos t,\ \ y = b\sin t,\ \ z = ct.$

Now $\quad \int_C \mathbf{F} \bullet d\mathbf{r} = \int_C (yz\ \mathbf{i} + zx\ \mathbf{j} + xy\ \mathbf{k}) \bullet (dx\ \mathbf{i} + dy\ \mathbf{j} + dz\ \mathbf{k})$

$$= \int_C (yz\ dx + zx\ dy + xy\ dz) = \int_C d(xyz)$$

$$= [xyz]_{t=0}^{t=\pi/2} = [(a\cos t)\,.\,(b\sin t)\,.\,(ct)]_0^{\pi/2}$$

$$= abc\,[t\cos t\sin t]_0^{\pi/2} = abc\,(0 - 0) = 0.$$

Comprehensive Exercise 1

1. Find $\int_C \mathbf{t} \bullet d\mathbf{r}$ where t is the unit tangent vector and C is the unit circle, in xy-plane, with centre at the origin. **(Bundelkhand 2008)**

2. (i) Integrate the function $\mathbf{F} = x^2\mathbf{i} - xy\mathbf{j}$ from the point $(0, 0)$ to $(1, 1)$ along parabola $y^2 = x$.

(ii) Evaluate $\int_C xy^3 \, ds,$ where C is the segment of the line $y = 2x$ in the xy-plane from $(-1, -2)$ to $(1, 2)$.

3. Evaluate $\int_C \mathbf{F} \bullet d\mathbf{r},$ where $\mathbf{F} = \mathbf{i} \cos y - \mathbf{j}\, x \sin y$ and C is the curve $y = \sqrt{(1 - x^2)}$ in the xy-plane from $(1, 0)$ to $(0, 1)$. **(Agra 2001)**

4. (i) Evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F}$ is $x^2 y^2 \mathbf{i} + y\mathbf{j}$ and C is $y^2 = 4x$ in the xy-plane from $(0, 0)$ to $(4, 4)$.
(Garhwal 2003; Meerut 04, 06; Kanpur 09, 11, 12; Rohilkhand 11)

(ii) Evaluate $\int_C \mathbf{F} \bullet d\mathbf{r},$ where $\mathbf{F} = xy\, \mathbf{i} + yz\, \mathbf{j} + zx\, \mathbf{k}$ and C is the curve $\mathbf{r} = t\mathbf{i} + t^2 \mathbf{j} + t^3 \mathbf{k}, t$ varying from -1 to $+1$.

(iii) Evaluate $\int_C \mathbf{F} \bullet d\mathbf{r},$ where $\mathbf{F} = (2x + y)\, \mathbf{i} + (3y - x)\, \mathbf{j} + yzx\, \mathbf{k}$ and C is the curve $x = 2t^2, \, y = t, \, z = t^3$ from $t = 0$ to $t = 1$. **(Kumaun 2011)**

5. Evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$ where
$\mathbf{F} = c\, [-3a \sin^2 t \cos t\, \mathbf{i} + a\, (2 \sin t - 3 \sin^3 t)\, \mathbf{j} + b \sin 2t\, \mathbf{k}]$
and C is given by $\mathbf{r} = a \cos t\, \mathbf{i} + a \sin t\, \mathbf{j} + bt\, \mathbf{k}$ from $t = \pi / 4$ to $\pi / 2$.

6. Evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F} = z\, \mathbf{i} + x\, \mathbf{j} + y\, \mathbf{k}$ and C is the arc of the curve $\mathbf{r} = \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + t\, \mathbf{k}$ from $t = 0$ to $t = 2\pi$.

7. Evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$ where $\mathbf{F} = xy\, \mathbf{i} + (x^2 + y^2)\, \mathbf{j}$ and C is the x-axis from $x = 2$ to $x = 4$ and the straight line $x = 4$ from $y = 0$ to $y = 12$.

8. Find the work done in moving a particle in a force field
$\mathbf{F} = 3x^2\, \mathbf{i} + (2xz - y)\, \mathbf{j} + z\, \mathbf{k}$ along the line joining $(0, 0, 0)$ to $(2, 1, 3)$.

9. Calculate $\int_C [(x^2 + y^2)\, \mathbf{i} + (x^2 - y^2)\, \mathbf{j}] \bullet d\mathbf{r}$ where C is the curve :

(i) $y^2 = x$ joining $(0, 0)$ to $(1, 1)$. **(Meerut 2005B)**

(ii) $x^7 = y$ joining $(0, 0)$ to $(1, 1)$.

(iii) consisting of two straight lines joining $(0, 0)$ to $(1, 0)$ and $(1, 0)$ to $(1, 1)$.

(iv) consisting of three straight lines joining $(0, 0)$ to $(2, -2)$, $(2, -2)$ to $(0, -1)$ and $(0, -1)$ to $(1, 1)$.

10. Find the circulation of $\mathbf{F}$ round the curve C, where
$\mathbf{F} = e^x \sin y\, \mathbf{i} + e^x \cos y\, \mathbf{j}$ and C is the rectangle whose vertices are $(0, 0), (1, 0), (1, \frac{1}{2}\pi), (0, \frac{1}{2}\pi)$.

11. Find the circulation of $\mathbf{F}$ round the curve C, where $\mathbf{F} = (x - y)\, \mathbf{i} + (x + y)\, \mathbf{j}$ and C is the circle $x^2 + y^2 = 4, \; z = 0$.

12. (i) If $\mathbf{F} = (2x^2 + y^2)\,\mathbf{i} + (3y - 4x)\,\mathbf{j}$, evaluate $\int \mathbf{F} \bullet d\mathbf{r}$ around the triangle ABC whose vertices are $A\,(0, 0)$, $B\,(2, 0)$ and $C\,(2, 1)$.

(Kumaun 2008)

(ii) If $\mathbf{F} = (3x^2 + 6y)\,\mathbf{i} - 14\,yz\,\mathbf{j} + 20\,xz^2\,\mathbf{k}$, then evaluate $\int \mathbf{F} \bullet d\mathbf{r}$ from $(0, 0, 0)$ to $(1, 1, 1)$ along the curve $x = t,\ y = t^2,\ z = t^3$.

(Kumaun 2012)

Answers 1

1. 2π **2.** (i) $\dfrac{1}{12}$ (ii) $\dfrac{16}{\sqrt{5}}$ **3.** -1

4. (i) 264 (ii) $\dfrac{10}{7}$ **5.** $\dfrac{1}{2}c\,(a^2 + b^2)$ **6.** $2\pi + \pi = 3\pi$

7. 768 **8.** 16

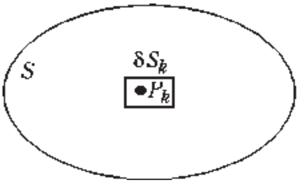
9. (i) $\dfrac{7}{10}$ (ii) $\dfrac{38}{45}$ (iii) 1 (iv) $-\dfrac{7}{3}$ **10.** 0

11. 8π **12.** (i) $-14/3$ (ii) 5

3 Surface Integrals

(Avadh 2014)

Any integral which is to be evaluated over a surface is called a surface integral.



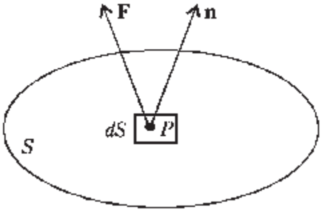
Suppose S is a surface of finite area. Suppose $f\,(x, y, z)$ is a single valued function of position defined over S. Subdivide the area S into n elements of areas $\delta S_1, \delta S_2, \ldots, \delta S_n$. In each part δS_k we choose an arbitrary point P_k whose coordinates are (x_k, y_k, z_k).

We define $f\,(P_k) = f\,(x_k, y_k, z_k)$.

Form the sum $\sum_{k=1}^{n} f\,(P_k)\,\delta S_k$.

Now take the limit of this sum as $n \to \infty$ in such a way that the largest of the areas δS_k approaches zero. This limit if it exists, is called the *surface integral* of $f\,(x, y, z)$ over S and is denoted by $\iint_S f\,(x, y, z)\,dS$.

It can be shown that if the surface S is piecewise smooth and the function $f\,(x, y, z)$ is continuous over S, then the above limit exists *i.e.,* is independent of the choice of sub-divisions and points P_k.



Flux: Suppose S is a piecewise smooth surface and $\mathbf{F}(x, y, z)$ is a vector function of position defined and continuous over S. Let P be any point on the surface S and let $\mathbf{n}$ be the unit vector at P in the direction of **outward drawn normal** to the surface S at P. Then $\mathbf{F} \bullet \mathbf{n}$ is the normal component of $\mathbf{F}$ at P. The integral of $\mathbf{F} \bullet \mathbf{n}$ over S is $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS$.

It is called the *flux* of $\mathbf{F}$ over S.

Let us associate with the differential of surface area dS a vector $d\mathbf{S}$ (called *vector area*) whose magnitude is dS and whose direction is that of $\mathbf{n}$. Then $d\mathbf{S} = \mathbf{n}\, dS$. Therefore we can write $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS = \iint_S \mathbf{F} \bullet d\mathbf{S}$.

Suppose the outward drawn normal to the surface S at P makes angles α, β, γ with the positive directions of x, y and z axes respectively. If l, m, n are the direction cosines of the outward drawn normal, then

$$l = \cos \alpha,\ m = \cos \beta,\ n = \cos \gamma.$$

Also $\mathbf{n} = \cos \alpha\, \mathbf{i} + \cos \beta\, \mathbf{j} + \cos \gamma\, \mathbf{k} = l\, \mathbf{i} + m\, \mathbf{j} + n\, \mathbf{k}$.

Let $\mathbf{F}(x, y, z) = F_1\, \mathbf{i} + F_2\, \mathbf{j} + F_3\, \mathbf{k}$. Then

$$\mathbf{F} \bullet \mathbf{n} = F_1 \cos \alpha + F_2 \cos \beta + F_3 \cos \gamma = F_1 l + F_2 m + F_3 n.$$

Therefore, we can write

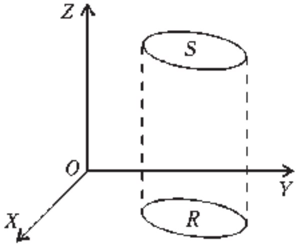
$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dS = \iint_S (F_1 \cos \alpha + F_2 \cos \beta + F_3 \cos \gamma)\, dS$$

$$= \iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy),$$

if we define $\iint_S F_1 \cos \alpha\, dS = \iint_S F_1\, dy\, dz$,

$$\iint_S F_2 \cos \beta\, dS = \iint_S F_2\, dz\, dx, \qquad \iint_S F_3 \cos \gamma\, dS = \iint_S F_3\, dx\, dy.$$

Note 1: Other examples of surface integrals are $\iint_S f\, \mathbf{n}\, dS, \iint_S \mathbf{F} \times d\mathbf{S}$ where $f(x, y, z)$ is a scalar function of position.



Note 2: **Important.** In order to evaluate surface integrals it is convenient to express them as double integrals taken over the orthogonal projection of the surface S on one of the coordinate planes. But this is possible only if any line perpendicular to the co-ordinate plane chosen meets the surface S in no more than one point. If the surface S does not satisfy this condition, then it can be sub-divided into surfaces which do satisfy this condition.

Suppose the surface S is such that any line perpendicular to the xy-plane meets S in no more than one point. Then the equation of the surface S can be written in the form $z = h(x, y)$.

Let R be the orthogonal projection of S on the $x\ y$-plane. If γ is the acute angle which the undirected normal $\mathbf{n}$ at $P(x, y, z)$ to the surface S makes with z-axis, then it can be shown that $\cos\gamma\, dS = dx\, dy$,
where dS is the small element of area of surface S at the point P.

Therefore $dS = \dfrac{dx\, dy}{\cos\gamma} = \dfrac{dx\, dy}{|\mathbf{n} \bullet \mathbf{k}|}$, where $\mathbf{k}$ is the unit vector along z-axis.

Hence $$\iint_S \mathbf{F} \bullet \mathbf{n}\, dS = \iint_R \mathbf{F} \bullet \mathbf{n} \frac{dx\, dy}{|\mathbf{n} \bullet \mathbf{k}|}.$$

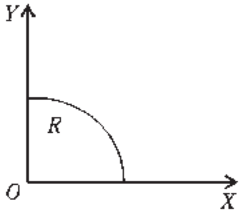
Thus the surface integral on S can be evaluated with the help of a double integral integrated over R.

Illustrative Examples

Example 9: *Evaluate* $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS$, *where* $\mathbf{F} = yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}$ *and* S *is that part of the surface of the sphere* $x^2 + y^2 + z^2 = 1$ *which lies in the first octant.*

(Garhwal 2003; kumaun 14))

Solution: A vector normal to the surface S is given by
$$\nabla(x^2 + y^2 + z^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}.$$



Therefore $\mathbf{n}$ = a unit normal to any point (x, y, z) of S
$$= \frac{2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}}{\sqrt{(4x^2 + 4y^2 + 4z^2)}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$
since $x^2 + y^2 + z^2 = 1$ on the surface S.

We have $$\iint_S \mathbf{F} \bullet \mathbf{n}\, dS = \iint_R \mathbf{F} \bullet \mathbf{n} \frac{dx\, dy}{|\mathbf{n} \bullet \mathbf{k}|},$$
where R is the projection of S on the xy-plane.

The region R is bounded by x-axis, y-axis and the circle $x^2 + y^2 = 1,\ z = 0$.

We have $\mathbf{F} \bullet \mathbf{n} = (yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}) \bullet (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = 3xyz.$

Also $\mathbf{n} \bullet \mathbf{k} = (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \bullet \mathbf{k} = z.\quad \therefore\quad |\mathbf{n} \bullet \mathbf{k}| = z.$

Hence $$\iint_S \mathbf{F} \bullet \mathbf{n}\, dS = \iint_R \frac{3xyz}{z}\, dx\, dy = 3\iint_R xy\, dx\, dy$$
$$= 3\int_{\theta=0}^{\pi/2} \int_{r=0}^{1} (r\cos\theta)(r\sin\theta)\, r\, d\theta\, dr, \text{ on changing to polars}$$
$$= 3\int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^1 \cos\theta \sin\theta\, d\theta = \frac{3}{4}\left(\frac{1}{2}\right) = \frac{3}{8}.$$

Example 10: *Evaluate* $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS$*, where* $\mathbf{F} = z\ \mathbf{i} + x\ \mathbf{j} - 3y^2 z\mathbf{k}$ *and* S *is the surface of the cylinder* $x^2 + y^2 = 16$ *included in the first octant between* $z = 0$ *and* $z = 5$.

Solution: A vector normal to the surface S is given by

$$\nabla (x^2 + y^2) = 2x\ \mathbf{i} + 2y\ \mathbf{j}.$$

Therefore $\mathbf{n}$ = a unit normal to any point of S

$$= \frac{2x\ \mathbf{i} + 2y\ \mathbf{j}}{\sqrt{(4x^2 + 4y^2)}} = \frac{x\ \mathbf{i} + y\ \mathbf{j}}{4},$$

since $x^2 + y^2 = 16$, on the surface S.

We have $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS = \iint_R \mathbf{F} \bullet \mathbf{n} \dfrac{dx\, dz}{|\mathbf{n} \bullet \mathbf{j}|}$, where R is the projection of S on the x-z plane. It should be noted that in this case we cannot take the projection of S on the x-y plane as the surface S is perpendicular to the x-y plane.

Now $$\mathbf{F} \bullet \mathbf{n} = (z\ \mathbf{i} + x\ \mathbf{j} - 3y^2 z\ \mathbf{k}) \bullet \left(\frac{x\ \mathbf{i} + y\ \mathbf{j}}{4}\right) = \frac{1}{4}(xz + xy),$$

$$\mathbf{n} \bullet \mathbf{j} = \left(\frac{x\ \mathbf{i} + y\ \mathbf{j}}{4}\right) \bullet \mathbf{j} = \frac{y}{4}.$$

Therefore the required surface integral is

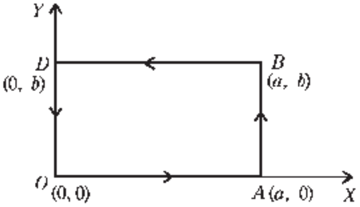
$$= \iint_R \frac{xz + xy}{4} \frac{dx\, dz}{y/4}$$

$$= \int_{z=0}^{5} \int_{x=0}^{4} \left(\frac{xz}{\sqrt{(16 - x^2)}} + x\right) dx\, dz, \text{ since } y = \sqrt{(16 - x^2)} \text{ on } S$$

$$= \int_0^5 (4z + 8)\, dz = 90.$$

4 Volume Integrals

(Avadh 2014)



Suppose V is a volume bounded by a surface S. Suppose $f(x, y, z)$ is a single valued function of position defined over V. Subdivide the volume V into n elements of volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$. In each part δV_k we choose an arbitrary point P_k whose co-ordinates are (x_k, y_k, z_k). We define

$$f(P_k) = f(x_k, y_k, z_k).$$

Form the sum

$$\sum_{k=1}^{n} f(P_k)\, \delta V_k.$$

Now take the limit of this sum as $n \to \infty$ in such a way that the largest of the volumes δV_k approaches zero. This limit, if it exists, is called the volume integral of $f(x, y, z)$ over V and is denoted by $\iiint_V f(x, y, z)\, dV$.

It can be shown that if the surface is piecewise smooth and the function $f(x, y, z)$ is continuous over V, then the above limit exists *i.e.*, is independent of the choice of sub-divisions and points P_k.

If we subdivide the volume V into small cuboids by drawing lines parallel to the three co-ordinates axes, then $dV = dx\, dy\, dz$ and the above volume integral becomes $\iiint_V f(x, y, z)\, dx\, dy\, dz$.

If $\mathbf{F}(x, y, z)$ is a vector function, then $\iiint_V \mathbf{F}\, dV$

is also an example of a volume integral.

Example 11: *If* $\mathbf{F} = (2x^2 - 3z)\,\mathbf{i} - 2xy\,\mathbf{j} - 4x\,\mathbf{k}$, *then evaluate* $\iiint_V \nabla \bullet \mathbf{F}\, dV$ *where* V *is the closed region bounded by the planes* $x = 0,\ y = 0,\ z = 0$ *and* $2x + 2y + z = 4$. *Also Evaluate* $\iiint_V \nabla \times \mathbf{F}\, dV$.

Solution: We have $\mathbf{F} = (2x^2 - 3z)\,\mathbf{i} - 2xy\,\mathbf{j} - 4x\,\mathbf{k}$.

$$\therefore \quad \nabla \bullet \mathbf{F} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \bullet [(2x^2 - 3z)\,\mathbf{i} - 2xy\,\mathbf{j} - 4x\,\mathbf{k}]$$

$$= \frac{\partial}{\partial x}(2x^2 - 3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x) = 4x - 2x = 2x.$$

$$\therefore \quad \iiint_V \nabla \bullet \mathbf{F}\, dV = \iiint_V 2x\, dx\, dy\, dz \qquad [\because \ dV = dx\, dy\, dz\ ]$$

$$= 2\int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} x\ dx\, dy\, dz.$$

[Note that we have taken a thin column parallel to z-axis as the elementary volume. It cuts the boundary at $z = 0$ and $z = 4 - 2x - 2y$. Also the projection of the plane $2x + 2y + z = 4$ on the xy-plane is bounded by the axes $y = 0,\ x = 0$ and the line $x + y = 2$. Hence the limits for y are from 0 to $2 - x$ and those for x are from 0 to 2]

$$\therefore \quad \iiint_V \nabla \bullet \mathbf{F}\, dV = 2\int_{x=0}^{2} \int_{y=0}^{2-x} x \Big[\, z \,\Big]_{z=0}^{4-2x-2y} dx\, dy$$

$$= 2\int_{x=0}^{2} \int_{y=0}^{2-x} x(4 - 2x - 2y)\, dx\, dy$$

$$= 2\int_{x=0}^{2} \Big[4xy - 2x^2 y - xy^2\Big]_{y=0}^{2-x} dx$$

$$= 2\int_0^2 [4x(2-x) - 2x^2(2-x) - x(2-x)^2]\, dx$$

$$= 2\int_0^2 [x^3 - 4x^2 + 4x]\, dx, \text{ on simplifying}$$

$$= 2\left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2\right]_0^2 = 2\left[4 - \frac{32}{3} + 8\right] = \frac{8}{3}.$$

Second part: We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-4x) - \frac{\partial}{\partial z}(-2xy) \right] \mathbf{i} - \left[\frac{\partial}{\partial x}(-4x) - \frac{\partial}{\partial z}(2x^2 - 3z) \right] \mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(2x^2 - 3z) \right] \mathbf{k}$$

$$= 0\,\mathbf{i} - (-4 + 3)\,\mathbf{j} + (-2y)\,\mathbf{k} = \mathbf{j} - 2y\,\mathbf{k}.$$

$$\therefore \qquad \iiint_V \nabla \times \mathbf{F}\, dV = \iiint_V (\mathbf{j} - 2y\,\mathbf{k})\, dx\, dy\, dz$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} (\mathbf{j} - 2y\,\mathbf{k})\, dx\, dy\, dz$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} (\mathbf{j} - 2y\,\mathbf{k})(4 - 2x - 2y)\, dx\, dy$$

$$= \int_{x=0}^{2} \left[\mathbf{j}(4y - 2xy - y^2) - 2\,\mathbf{k}\left(2y^2 - xy^2 - \frac{2}{3}y^3\right) \right]_{y=0}^{2-x} dx$$

$$= \int_{x=0}^{2} \Big[\mathbf{j}(2 - x)(4 - 2x - 2 + x)$$

$$- 2\,\mathbf{k}(2 - x)^2 \left\{ 2 - x - \frac{2}{3}(2 - x) \right\} \Big] dx$$

$$= \int_0^2 \left[(2 - x)^2\,\mathbf{j} - \frac{2}{3}(2 - x)^3\,\mathbf{k} \right] dx$$

$$= \int_0^2 \left[(x - 2)^2\,\mathbf{j} + \frac{2}{3}(x - 2)^3\,\mathbf{k} \right] dx$$

$$= \left[\frac{(x - 2)^3}{3} \right]_0^2 \mathbf{j} + \left[\frac{2}{3}\,\frac{(x - 2)^4}{4} \right]_0^2 \mathbf{k} = \frac{8}{3}\,\mathbf{j} - \frac{8}{3}\,\mathbf{k} = \frac{8}{3}(\mathbf{j} - \mathbf{k}).$$

Comprehensive Exercise 2

1. (i) Evaluate $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS$, where $\mathbf{F} = 18z\,\mathbf{i} - 12\,\mathbf{j} + 3y\,\mathbf{k}$ and S is the surface of the plane $2x + 3y + 6z = 12$ in the first octant.
(Bundelkhand 2005)

(ii) Evaluate $\iiint_V \phi\, dV$, where $\phi = 45\, x^2 y$ and V is the closed region bounded by the planes $4x + 2y + z = 8$, $x = 0$, $y = 0, z = 0$.

2. Evaluate $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS$, where $\mathbf{F} = (x + y^2)\,\mathbf{i} - 2x\,\mathbf{j} + 2yz\,\mathbf{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.
3. Evaluate $\iint_S \mathbf{A} \bullet \mathbf{n}\, dS$, where $\mathbf{A} = xy\,\mathbf{i} - x^2\,\mathbf{j} + (x + z)\,\mathbf{k}$, S is the portion of the plane $2x + 2y + z = 6$ included in the first octant and **n** is a unit normal to S.
4. If $\mathbf{F} = 2y\mathbf{i} - 3\,\mathbf{j} + x^2\,\mathbf{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$, then evaluate $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS$.
5. Evaluate $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS$, where $\mathbf{F} = y\mathbf{i} + 2x\mathbf{j} - z\,\mathbf{k}$ and S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$.
6. Evaluate $\iiint_V \mathbf{F}\, dV$ where $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and V is the region bounded by the surfaces $x = 0$, $x = 2$, $y = 0$, $y = 6$, $z = 4$ and $z = x^2$.

Answers 2

1. (i) 24 (ii) 128
2. 81
3. $\frac{27}{4}$
4. 132
5. 108
6. $24\,\mathbf{i} + 96\,\mathbf{j} + \frac{384}{5}\,\mathbf{k}$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If $\mathbf{F} = x^2\,\mathbf{i} + y^3\,\mathbf{j}$ and curve C is the arc of the parabola $y = x^2$ in the xy-plane from (0, 0) to (1, 1), then $\int_C \mathbf{F} \bullet d\,\mathbf{r}$ is

   (a) 7 / 12 (b) 5 / 12 (c) 11/12 (d) None of these
2. The work done in moving a particle in a force field

   $$\mathbf{F} = 3x^2\,\mathbf{i} + (2xz - y)\,\mathbf{j} + 3\mathbf{k}$$

   along the line joining (0, 0, 0) to (2 , 1, 3) is

   (a) 12 (b) 16 (c) 0 (d) 20

3. For a closed surface S, the value of $\iint_S \mathbf{r} \bullet \mathbf{n}\, dS$ is

(a) V (b) $2V$ (c) $3V$ (d) 0

where V is the volume enclosed by S. **(Kumaun 2013)**

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. Any integral which is to be evaluated along a curve is called a …… .
2. Any integral which is to be evaluated over a surface is called a …… .
3. If $\mathbf{r}$ denotes the position vector of a point on a curve C, then $\dfrac{d\mathbf{r}}{ds}$ is a unit vector along the …… to C at the point $\mathbf{r}$ in the direction of s increasing.
4. $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS$ is called the …… of $\mathbf{F}$ over S.
5. If $\mathbf{t}$ is the unit tangent vector and C is the unit circle in xy-plane, with centre at the origin, then $\int_C \mathbf{t} \bullet d\mathbf{r} =$ …… .
6. The value of $\int (x\, dy - y\, dx)$ around the circle $x^2 + y^2 = 1$ is …… .
7. If $\mathbf{F} = (3x^2 + 6y)\,\mathbf{i} - 14yz\,\mathbf{j} + 20\,xz^2\,\mathbf{k}$ and C is a straight line joining $(0, 0, 0)$ to $(1, 1, 1)$, then $\int_C \mathbf{F} \bullet d\,\mathbf{r} =$ …… .

True or False

Write 'T' for true and 'F' for false statement.

1. If C is a simple closed curve, then $\oint_C \mathbf{F} \bullet d\,\mathbf{r}$ is called the circulation of $\mathbf{F}$ about C.
2. If $\mathbf{F} = ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}$, a, b, c are constants, then $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS = \dfrac{2}{3}\pi\,(a + b + c)$, where S is the surface of a unit sphere.

Answers

Multiple Choice Questions

1. (a) 2. (b) 3. (c)

Fill in the Blank(s)

1. line integral
2. surface integral
3. tangent
4. flux
5. 2π
6. 2π
7. $13/3$

True or False

1. *T* 2. *F*

□

Chapter 6

Green's, Gauss's and Stoke's Theorems

1 Green's Theorem in the Plane

Let R be a closed bounded region in the x-y plane whose boundary C consists of finitely many smooth curves. Let M and N be continuous functions of x and y having continuous partial derivatives $\dfrac{\partial M}{\partial y}$ *and* $\dfrac{\partial N}{\partial x}$ *in R. Then*

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C (M\,dx + N\,dy),$$

the line integral being taken along the entire boundary C of R such that R is on the left as one advances in the direction of integration.

(Meerut 2004, 05, 07, 10; Avadh 10; Purvanchal 08; Rohilkhand 09B, 12; Kashi 13)

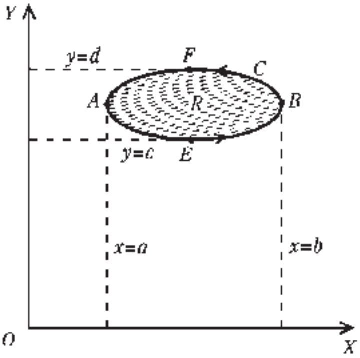
Proof: We shall first prove the theorem for a special region R bounded by a closed curve C and having the property that any straight line parallel to any one of the coordinate axes and intersecting R has only one segment (or a single point) in common with R. This means that R can be represented in both of the forms

$$a \le x \le b, \quad f(x) \le y \le g(x)$$

and

$$c \le y \le d, \quad p(y) \le x \le q(y).$$

In the adjoining figure, the equations of the curves AEB and BFA are $y = f(x)$ and $y = g(x)$ respectively. Similarly the equations of the curves FAE and EBF are $x = p(y)$ and $x = q(y)$ respectively.



We have

$$\iint_R \frac{\partial M}{\partial y}\, dx\, dy$$

$$= \int_{x=a}^{b} \left[\int_{y=f(x)}^{g(x)} \frac{\partial M}{\partial y}\, dy \right] dx$$

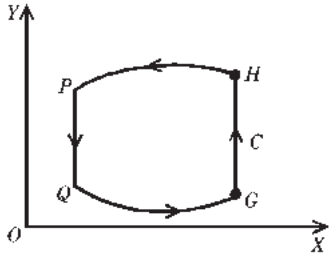
$$= \int_{x=a}^{b} \Big[M(x, y) \Big]_{y=f(x)}^{y=g(x)} dx$$

$$= \int_{x=a}^{b} \big[M[x, g(x)] - M[x, f(x)]\big]\, dx$$

$$= -\int_a^b M[x, f(x)]\, dx - \int_b^a M[x, g(x)]\, dx$$

$$= -\oint_C M(x, y)\, dx,$$ since $y = f(x)$ represents the curve AEB and $y = g(x)$ represents the curve BFA.

If portions of C are segments parallel to y-axis such as GH and PQ in the adjoining figure, then the above result is not affected. The line integral $\int M\, dx$ over GH is zero because on GH, we have x = constant implies $dx = 0$. Similarly the line integral over PQ is zero. The equations of QG and HP are $y = f(x)$ and $y = g(x)$ respectively. Hence we have



$$-\iint_R \frac{\partial M}{\partial y}\, dx\, dy = \oint_C M(x, y)\, dx. \qquad \ldots (1)$$

Similarly, $$\iint_R \frac{\partial N}{\partial x}\, dx\, dy = \int_{y=c}^{d} \left[\int_{x=p(y)}^{q(y)} \frac{\partial N}{\partial x}\, dx \right] dy$$

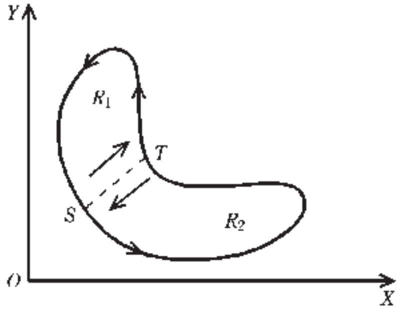
$$= \int_{y=c}^{d} \Big[N(x, y) \Big]_{x=p(y)}^{x=q(y)} dy$$

$$= \int_{y=c}^{d} \big[N[q(y), y] - N[p(y), y]\big]\, dy$$

$$= \int_c^d N[q(y), y]\, dy + \int_d^c N[p(y), y]\, dy$$

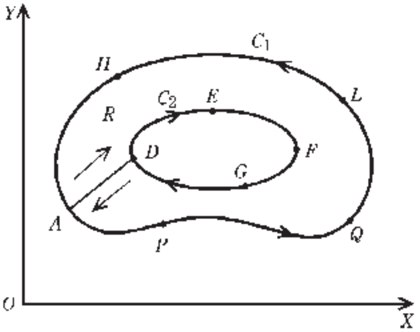
$$= \oint_C N(x, y)\, dy. \qquad \ldots (2)$$

From (1) and (2), we get on adding $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \oint_C (M\, dx + N\, dy)$.



The proof of the theorem can now be extended to a region R which can be subdivided into finitely many special regions of the above type by drawing lines (TS in the adjoining figure). In this case we apply the theorem to each subregion (R_1 and R_2 in the figure) and then add the results. The sum of the left hand members will be equal to the integral over R. The sum of the right hand members will be equal to the integral over C plus the line integrals over the curves introduced for subdividing R. Each of the latter integrals comes twice, taken once in each direction (as ST and TS in the figure). Therefore these two integrals cancel each other and thus the sum of the right hand members will be equal to the line integral over C.

Note: **Extension of Green's theorem in plane to multiply connected regions.**



Green's theorem in the plane is also valid for a multiply-connected region R such as shown in the adjoining figure. Here the boundary C of R consists of two parts; the exterior boundary C_1 is traversed in the anticlockwise sense so that R is on the left, while the interior boundary C_2 is traversed in the clockwise sense so that R is on the left.

In order to establish the theorem, we construct a line such as AD (called a *cross cut*) connecting the exterior and interior boundaries. The region bounded by $ADEFGDAPQLHA$ is simply-connected and so Green's theorem is valid for it. Therefore

$$\oint_{\substack{C \\ ADEFGDAPQLHA}} M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy.$$

The integral on the left hand side leaving out the integrand is equal to

$$\int_{AD} + \int_{C_2} + \int_{DA} + \int_{C_1} = \int_{C_2} + \int_{C_1}, \text{ since } \int_{AD} = -\int_{DA}$$

$$= \oint_C (M\, dx + N\, dy).$$

Hence the theorem.

2 Green's Theorem in the Plane in Vector Notation

We have $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ so that $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$.

Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$.

Then $M\,dx + N\,dy = (M\,\mathbf{i} + N\,\mathbf{j}) \bullet (dx\,\mathbf{i} + dy\,\mathbf{j}) = \mathbf{F} \bullet d\mathbf{r}$.

Also $\text{curl}\mathbf{F} = \nabla \times \mathbf{F}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = -\frac{\partial N}{\partial z}\,\mathbf{i} + \frac{\partial M}{\partial z}\,\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.$$

$$\therefore \qquad (\nabla \times \mathbf{F}) \bullet \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

Hence Green's theorem in plane can be written as

$$\iint_R (\nabla \times \mathbf{F}) \bullet \mathbf{k}\; dR = \oint_C \mathbf{F} \bullet d\mathbf{r}$$

where $dR = dx\,dy$ and $\mathbf{k}$ is unit vector perpendicular to the xy-plane.

If s denotes the arc length of C and $\mathbf{t}$ is the unit tangent vector to C, then $d\mathbf{r} = \dfrac{d\mathbf{r}}{ds}\,ds = \mathbf{t}\,ds$. Therefore the above result can also be written as

$$\iint_R (\nabla \times \mathbf{F}) \bullet \mathbf{k}\; dR = \oint_C \mathbf{F} \bullet \mathbf{t}\; ds.$$

Illustrative Examples

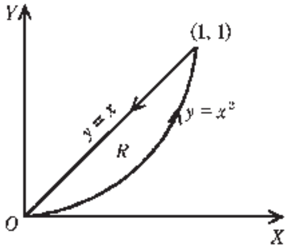
Example 1: *Verify Green's theorem in the plane for* $\oint_C (xy + y^2)\,dx + x^2\,dy$, *where* C *is the closed curve of the region bounded by* $y = x$ *and* $y = x^2$. **(Rohilkhand 2011)**

Solution: By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C (M\,dx + N\,dy).$$

Here $M = xy + y^2$, $N = x^2$.

The curves $y = x$ and $y = x^2$ intersect at $(0, 0)$ and $(1, 1)$. The positive direction in traversing C is as shown in the figure.



We have $\displaystyle\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$= \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dx\,dy$$

$$= \iint_R (2x - x - 2y)\,dx\,dy = \iint_R (x - 2y)\,dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)\, dy\, dx = \int_{x=0}^{1} \Big[xy - y^2 \Big]_{y=x^2}^{x} dx$$

$$= \int_0^1 [x^2 - x^2 - x^3 + x^4]\, dx = \int_0^1 (x^4 - x^3)\, dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4} \right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}.$$

Now let us evaluate the line integral along C. Along $y = x^2, dy = 2x\, dx$. Therefore along $y = x^2$, the line integral equals

$$\int_0^1 [\{(x)(x^2) + x^4\}\, dx + x^2 (2x)\, dx] = \int_0^1 (3x^3 + x^4)\, dx = \frac{19}{20}.$$

Along $y = x$, $dy = dx$. Therefore along $y = x$, the line integral equals

$$\int_1^0 [\{(x)\,(x) + x^2\}\, dx + x^2\; dx] = \int_1^0 3x^4\; dx = -1.$$

Therefore the required line integral $= \dfrac{19}{20} - 1 = -\dfrac{1}{20}\cdot$ Hence the theorem is verified.

Example 2: *Evaluate by Green's theorem* $\oint_C (x^2 - \cosh y)\, dx + (y + \sin x)\, dy$, *where* C *is the rectangle with vertices* $(0, 0)$, $(\pi, 0)$, $(\pi, 1)$, $(0, 1)$.

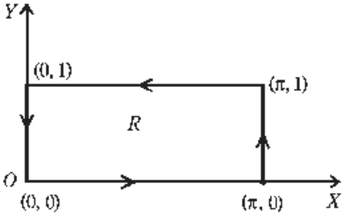
(Meerut 2002, 05B, 06, 13B; Rohilkhand 14)

Solution: By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy = \oint_C (M\, dx + N\, dy).$$

Here $M = x^2 - \cosh y$, $N = y + \sin x$.

$$\therefore \qquad \frac{\partial N}{\partial x} = \cos x, \quad \frac{\partial M}{\partial y} = -\sinh y.$$



Hence the given line integral is equal to

$$\iint_R (\cos x + \sinh y)\, dx\, dy$$

$$= \int_{x=0}^{\pi} \int_{y=0}^{1} (\cos x + \sinh y)\, dy\, dx$$

$$= \int_{x=0}^{\pi} \Big[y \cos x + \cosh y \Big]_{y=0}^{1} dx$$

$$= \int_{x=0}^{\pi} [\cos x + \cosh 1 - 1]\, dx = \Big[\sin x + x \cosh 1 - x \Big]_0^{\pi} = (\cosh 1 - 1).$$

Example 3: *Show that the area bounded by a simple closed curve* C *is given by* $\frac{1}{2} \oint_C (x\, dy - y\, dx)$. *Hence find the area of the ellipse* $x = a \cos \theta$, $y = b \sin \theta$.

Solution: By Green's theorem in plane, if R is a plane region bounded by a simple closed curve C, then

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \oint_C M\, dx + N\, dy.$$

Putting $M = -y,\ N = x$, we get

$$\oint_C (x\, dy - y\, dx) = \iint_R \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right] dx\, dy$$

$$= 2 \iint_R dx\, dy$$

$= 2\ A$, where A is the area bounded by C.

Hence $\quad A = \dfrac{1}{2} \oint_C (x\, dy - y\, dx).$

The area of the ellipse

$$= \frac{1}{2} \oint_C (x\, dy - y\, dx) = \frac{1}{2} \int_{\theta=0}^{2\pi} \left(a \cos\theta \frac{dy}{d\theta} - b \sin\theta \frac{dx}{d\theta}\right) d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} (ab \cos^2 \theta + ab \sin^2 \theta)\, d\theta = \frac{1}{2}\, ab \int_0^{2\pi} d\theta = \pi ab.$$

Comprehensive Exercise 1

1. Verify Green's theorem in the plane for
$\int_C [(2xy - x^2)\, dx + (x^2 + y^2)\, dy]$,
where C is the boundary of the region enclosed by $y = x^2$ and $y^2 = x$ described in the positive sense.

2. Verify Green's theorem in the plane for
$\oint_C [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy]$,
where C is the boundary of the region defined by $y = \sqrt{x},\ y = x^2$.

3. Apply Green's theorem in the plane to evaluate
$\int_C \{(y - \sin x)\, dx + \cos x\, dy\}$,
where C is the triangle enclosed by the lines $y = 0,\ x = 2\pi, \pi y = 2x$.
(Avadh 2010)

4. Evaluate by Green's theorem in plane $\int_C (e^{-x} \sin y\, dx + e^{-x} \cos y\, dy)$,
where C is the rectangle with vertices $(0, 0), (\pi, 0), \left(\pi, \frac{1}{2}\pi\right), \left(0, \frac{1}{2}\pi\right)$.

5. Evaluate by Green's theorem $\oint_C (\cos x \sin y - xy)\, dx + \sin x \cos y\, dy$,
where C is the circle $x^2 + y^2 = 1$. **(Kumaun 2012)**

6. If $\mathbf{F} = (x^2 - y^2)\,\mathbf{i} + 2xy\,\mathbf{j}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, find the value of $\int \mathbf{F} \bullet d\mathbf{r}$ around the rectangular boundary $x = 0,\ x = a,\ y = 0,\ y = b$.

7. Verify Green's theorem in the plane for

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy,$$

where C is the square with vertices $(0,0),\ (2,0),\ (2,2),\ (0,2)$.

(Meerut 2001)

8. Apply Green's theorem in the plane to evaluate

$\int_C [(2x^2 - y^2)\,dx + (x^2 + y^2)\,dy]$, where C is the boundary of the surface enclosed by the x-axis and the semi-circle $y = (1 - x^2)^{1/2}$.

9. If C is the simple closed curve in the xy-plane not enclosing the origin, show that $\int_C \mathbf{F} \bullet d\mathbf{r} = 0$, where $\mathbf{F} = \dfrac{-\mathbf{i}\,y + \mathbf{j}\,x}{x^2 + y^2}$.

Answers 1

3. $-\dfrac{\pi}{4} - \dfrac{2}{\pi}$

4. $2(e^{-\pi} - 1)$

5. 0

6. $2ab^2$

---

3 The Divergence Theorem of Gauss

Suppose V is the volume bounded by a closed piecewise smooth surface S. Suppose $\mathbf{F}(x, y, z)$ *is a vector function of position which is continuous and has continuous first partial derivatives in V. Then* $\iiint_V \nabla \bullet \mathbf{F}\,dV = \iint_S \mathbf{F} \bullet \mathbf{n}\,dS,$

where **n** *is the outward drawn unit normal vector to S.*

(Meerut 2000, 01, 06, 10B, 12, 12B; Bundelkhand 09, 11; Avadh 14; Kashi 14)

Since $\mathbf{F} \bullet \mathbf{n}$ is the normal component of vector **F**, therefore divergence theorem may also be stated as follows :

The surface integral of the normal component of a vector **F** *taken over a closed surface is equal to the integral of the divergence of* **F** *taken over the volume enclosed by the surface.*

Cartesian equivalent of Divergence Theorem:

Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$.

Then $\nabla \bullet \mathbf{F} = \text{div}\ \mathbf{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$.

If α, β, γ are the angles which outward drawn unit normal **n** makes with positive directions of x, y, z-axes, then $\cos\alpha, \cos\beta, \cos\gamma$ are direction cosines of **n** and we have $\mathbf{n} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$.

$$\therefore \quad \mathbf{F} \bullet \mathbf{n} = (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}) \bullet (\cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k})$$
$$= F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma.$$

Therefore the divergence theorem can be written as

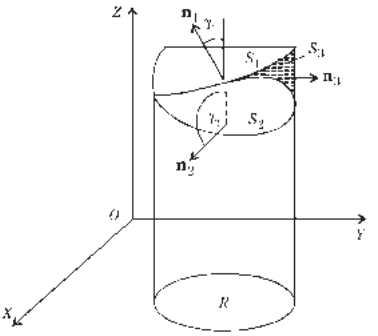
$$\iiint_V \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right) dx\, dy\, dz$$

$$=\iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, dS$$

$$=\iint F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy).$$

The significance of divergence theorem lies in the fact that a surface integral may be expressed as a volume integral and vice versa.

Proof of the divergence theorem:



We shall first prove the theorem for a special region V which is bounded by a piecewise smooth closed surface S and has the property that any straight line parallel to any one of the coordinate axes and intersecting V has only one segment (or a single point) in common with V. If R is the orthogonal projection of S on the xy-plane, then V can be represented in the form $f(x, y) \le z \le g(x, y)$ where (x, y) varies in R.

Obviously $z = g(x, y)$ represents the upper portion S_1 of S, $z = f(x, y)$ represents the lower portion S_2 of S and there may be a remaining vertical portion S_3 of S.

We have

$$\iiint_V \frac{\partial F_3}{\partial z}\, dV = \iiint_V \frac{\partial F_3}{\partial z}\, dx\, dy\, dz$$

$$=\iint_R \left[\int_{z=f(x,y)}^{g(x,y)} \frac{\partial F_3}{\partial z}\, dz\right] dx\, dy$$

$$=\iint_R \left[F_3(x, y, z)\right]_{z=f(x,y)}^{g(x,y)} dx\, dy$$

$$=\iint_R \left[F_3\,[x, y, g(x, y)] - F_3\,[x, y, f(x, y)]\right] dx\, dy$$

$$=\iint_R F_3\,[x, y, g(x, y)]\, dx\, dy$$

$$-\iint_R F_3\,[x, y, f(x, y)]\, dx\, dy \qquad \ldots(1)$$

Now for the vertical portion S_3 of S, the normal $\mathbf{n}_3$ to S_3 makes a right angle γ with $\mathbf{k}$. Therefore $\iint_{S_3} F_3\, \mathbf{k} \bullet \mathbf{n}_3\, dS_3 = 0$, since $\mathbf{k} \bullet \mathbf{n}_3 = 0$.

For the upper portion S_1 of S, the normal $\mathbf{n}_1$ to S_1 makes an acute angle γ_1 with $\mathbf{k}$.

Therefore $\mathbf{k} \bullet \mathbf{n}_1\, dS_1 = \cos\gamma_1\, dS_1 = dx\, dy.$

Hence $\iint_{S_1} F_3\, \mathbf{k} \bullet \mathbf{n}_1\, dS_1 = \iint_R F_3\,[x, y, g(x, y)]\, dx\, dy.$

For the lower portion S_2 of S, the normal $\mathbf{n}_2$ to S_2 makes an obtuse angle γ_2 with $\mathbf{k}$.

Therefore $\quad \mathbf{k} \bullet \mathbf{n}_2 \; dS_2 = \cos \gamma_2 \; dS_2 = -\, dx\, dy.$

Hence $\quad \iint_{S_1} F_3 \; \mathbf{k} \bullet \mathbf{n}_2 \; dS_2 = -\iint_R F_3 \; [x \;\; y, \; f\,(x, y)]\; dx\, dy.$

$$\therefore \quad \iint_S F_3 \; \mathbf{k} \bullet \mathbf{n}_3 \; dS_3 + \iint_{S_1} F_3 \; \mathbf{k} \bullet \mathbf{n}_1 \; dS_1 + \iint_{S_2} F_3 \; \mathbf{k} \bullet \mathbf{n}_2 \; dS_2$$

$$= 0 + \iint_R F_3 \; [\, x, \; y, \; g \; (x, \; y)]\; dx\, dy$$

$$- \iint_R F_3 \; [\, x, \; y, \; f \; (x, y)]\; dx\, dy$$

or with the help of (1), we get

$$\iint_S F_3 \; \mathbf{k} \bullet \mathbf{n} \, dS = \iiint_V \frac{\partial F_3}{\partial z} \, dV. \qquad \text{...(2)}$$

Similarly, by projecting S on the other co-ordinate planes, we get

$$\iint_S F_2 \; \mathbf{j} \bullet \mathbf{n} \, dS = \iiint_V \frac{\partial F_2}{\partial y} \, dV \qquad \text{...(3)}$$

and

$$\iint_S F_1 \; \mathbf{i} \bullet \mathbf{n} \, dS = \iiint_V \frac{\partial F_1}{\partial x} \, dV \qquad \text{...(4)}$$

Adding (2), (3) and (4), we get

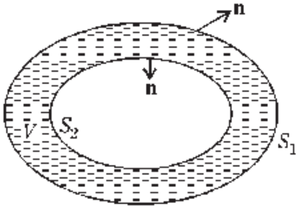
$$\iint_S (F_1 \; \mathbf{i} + F_2 \; \mathbf{j} + F_3 \; \mathbf{k}) \bullet \mathbf{n} \, dS = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dV$$

or

$$\iiint_V \nabla \bullet \mathbf{F} \, dV = \iint_S \mathbf{F} \bullet \mathbf{n} \, dS.$$

The proof of the theorem can now be extended to a region V which can be subdivided into finitely many special regions of the above type by drawing auxiliary surfaces. In this case we apply the theorem to each sub-region and then add the results. The sum of the volume integrals over parts of V will be equal to the volume integral over V. The surface integrals over auxiliary surfaces cancel in pairs, while the sum of the remaining surface integrals is equal to the surface integral over the whole boundary S of V.

Note: The divergence theorem is applicable for a region V if it is bounded by two closed surfaces S_1 and S_2 one of which lies within the other. Here outward drawn normals will have the directions as shown in the figure.



4 Some Deductions from Divergence Theorem

1. Green's theorem: *Let* ϕ *and* ψ *be scalar point functions which together with their derivatives in any direction are uniform and continuous within the region V bounded by a closed surface S, then*

$$\iiint_V (\phi\nabla^2 \psi - \psi\nabla^2 \phi)\, dV = \iint_S (\phi\nabla \psi - \psi\nabla \phi) \bullet \mathbf{n}\, dS.$$

Proof: By divergence theorem, we have $\iiint_V \nabla \bullet \mathbf{F}\, dV = \iint_S \mathbf{F} \bullet \mathbf{n}\, dS.$

Putting $\mathbf{F} = \phi\nabla\psi$, we get $\nabla \bullet \mathbf{F} = \nabla \bullet (\phi\nabla\psi)$

$$= \phi(\nabla \bullet \nabla\psi) + (\nabla\phi) \bullet (\nabla\psi) = \phi\nabla^2 \psi + (\nabla\phi) \bullet (\nabla\psi).$$

Also $\mathbf{F} \bullet \mathbf{n} = (\phi\nabla\psi) \bullet \mathbf{n}.$

$\therefore$ divergence theorem gives

$$\iiint_V [\phi\nabla^2 \psi + (\nabla\phi) \bullet (\nabla\psi)]\, dV = \iint_S (\phi\nabla\psi) \bullet \mathbf{n}\, dS \qquad ...(1)$$

This is called *Green's first identity* or *theorem.*

Interchanging ϕ and ψ in (1), we get

$$\iiint_V [\psi\nabla^2 \phi + (\nabla\psi) \bullet (\nabla\phi)]\, dV = \iint_S [\psi\nabla\phi] \bullet \mathbf{n}\, dS \qquad ...(2)$$

Subtracting (2) from (1), we get

$$\iiint_V (\phi\nabla^2 \psi - \psi\nabla^2 \phi)\, dV = \iint_S (\phi\nabla \psi - \psi\nabla\phi) \bullet \mathbf{n}\, dS \qquad ...(3)$$

This is called *Green's second identity* or *Green's theorem in symmetrical form.*

Since $\nabla\psi = \frac{\partial\psi}{\partial n}\mathbf{n}$ and $\nabla\phi = \frac{\partial\phi}{\partial n}\mathbf{n}$, therefore

$$(\phi\nabla\psi - \psi\nabla\phi) \bullet \mathbf{n} = \left(\phi\frac{\partial\psi}{\partial n}\mathbf{n} - \psi\frac{\partial\phi}{\partial n}\mathbf{n}\right) \bullet \mathbf{n}$$

$$= \phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}.$$

Hence (3) can also be written as

$$\iiint_V (\phi\nabla^2 \psi - \psi\nabla^2 \phi)\, dV = \iint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) dS.$$

Note: **Harmonic function:** *If a scalar point function* ϕ *satisfies Laplace's equation* $\nabla^2 \phi = 0$*, then* ϕ *is called harmonic function.* If ϕ and ψ are both harmonic functions, then $\nabla^2 \phi = 0,\ \nabla^2 \psi = 0.$

Hence from Green's second identity, we get $\iint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) dS = 0.$

2. *Prove that* $\iiint_V \nabla\phi\, dV = \iint_S \phi\mathbf{n}\, dS.$

Proof: By divergence theorem, we have $\iiint_V \nabla \bullet \mathbf{F}\, dV = \iint_S \mathbf{F} \bullet \mathbf{n}\, dS.$

Taking $\mathbf{F} = \phi\mathbf{C}$ where $\mathbf{C}$ is an arbitrary constant non-zero vector, we get

$$\iiint_V \nabla \bullet (\phi\mathbf{C})\, dV = \iint_S (\phi\mathbf{C}) \bullet \mathbf{n}\, dS. \qquad ...(1)$$

Now $\nabla \bullet (\phi\mathbf{C}) = (\nabla\phi) \bullet \mathbf{C} + \phi(\nabla \bullet \mathbf{C}) = (\nabla\phi) \bullet \mathbf{C}$, since $\nabla \bullet \mathbf{C} = 0$.

Also $(\phi\mathbf{C}) \bullet \mathbf{n} = \mathbf{C} \bullet (\phi\,\mathbf{n})$.

$\therefore$ (1) becomes

$$\iiint_V \mathbf{C} \bullet (\nabla\phi)\, dV = \iint_S \mathbf{C} \bullet (\phi\mathbf{n})\, dS$$

or $$C \bullet \iiint_V \nabla\phi\, dV = \mathbf{C} \bullet \int_S (\phi\mathbf{n})\, dS$$

or $$\mathbf{C} \bullet \left[\iiint_V \nabla\phi\, dV - \iint_S \phi\mathbf{n}\, dS\right] = 0.$$

Since $\mathbf{C}$ is an arbitrary vector, therefore we must have

$$\iiint_V \nabla\phi\, dV = \iint_S \phi\mathbf{n}\, dS.$$

3. *Prove that* $\iiint_V \nabla \times \mathbf{B}\, dV = \iint_S \mathbf{n} \times \mathbf{B}\, dS.$

Proof: In divergence theorem taking $\mathbf{F} = \mathbf{B} \times \mathbf{C}$, where $\mathbf{C}$ is an arbitrary constant vector, we get

$$\iiint_V \nabla \bullet (\mathbf{B} \times \mathbf{C})\, dV = \iint_S (\mathbf{B} \times \mathbf{C}) \bullet \mathbf{n}\, dS. \qquad ...(1)$$

Now $\nabla \bullet (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \bullet \text{curl } \mathbf{B} - \mathbf{B} \bullet \text{curl } \mathbf{C} = \mathbf{C} \bullet \text{curl } \mathbf{B}$, since curl $\mathbf{C} = \mathbf{0}$.

Also $(\mathbf{B} \times \mathbf{C}) \bullet \mathbf{n} = [\mathbf{B}, \mathbf{C}, \mathbf{n}] = [\mathbf{C}, \mathbf{n}, \mathbf{B}] = \mathbf{C} \bullet (\mathbf{n} \times \mathbf{B})$.

$\therefore$ (1) becomes

$$\iiint_V (\mathbf{C} \bullet \text{curl } \mathbf{B})\, dV = \iint_S \mathbf{C} \bullet (\mathbf{n} \times \mathbf{B})\, dS$$

or $$\mathbf{C} \bullet \iiint_V (\nabla \times \mathbf{B})\, dV = \mathbf{C} \bullet \iint_S (\mathbf{n} \times \mathbf{B})\, dS$$

or $$\mathbf{C} \bullet \left[\iiint_V (\nabla \times \mathbf{B})\, dV - \iint_S (\mathbf{n} \times \mathbf{B})\, dS\right] = 0.$$

Since $\mathbf{C}$ is an arbitrary vector therefore we can take $\mathbf{C}$ as a non-zero vector which is not perpendicular to the vector

$$\iiint_V (\nabla \times \mathbf{B})\, dV - \iint_S (\mathbf{n} \times \mathbf{B})\, dS.$$

Hence we must have

$$\iiint_V (\nabla \times \mathbf{B})\, dV - \iint_S (\mathbf{n} \times \mathbf{B})\, dS = \mathbf{0}$$

or $$\iiint_V (\nabla \times \mathbf{B})\, dV = \iint_S (\mathbf{n} \times \mathbf{B})\, dS.$$

Illustrative Examples

Example 4: *(i) For any closed surface S, prove that* $\iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\, dS = 0.$

(Meerut 2009B; Purvanchal 14; Avadh 14)

(ii) Evaluate $\iint_S \mathbf{r} \bullet \mathbf{n}\, dS$, *where S is a closed surface.*

(Agra 2006; Kumaun 07; Bundelkhand 09; Purvanchal 14)

(iii) If $\mathbf{F} = ax\ \mathbf{i} + by\ \mathbf{j} + cz\ \mathbf{k},\ a, b, c$ *are constants show that*

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, d\mathbf{S} = \frac{4}{3} \pi\,(a + b + c),$$

where S is the surface of a unit sphere.

(Bundelkhand 2001, 07, 08; Rohilkhand 07; Kashi 13)

Solution: (i) By divergence theorem, we have

$$\iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\, dS = \iiint_V (\text{div curl } \mathbf{F})\, dV,$$

where V is the volume enclosed by S

$$= 0, \text{ since div curl } \mathbf{F} = 0.$$

(ii) By the divergence theorem, we have

$$\iint_S \mathbf{r} \bullet \mathbf{n}\, dS = \iiint_V \nabla \bullet \mathbf{r}\, dV = \iiint_V 3\, dV,$$

since $\nabla \bullet \mathbf{r} = \text{div } \mathbf{r} = 3$

$$= 3V, \text{ where } V \text{ is the volume enclosed by } S.$$

(iii) By the divergence theorem, we have

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dS = \iiint_V (\nabla \bullet \mathbf{F})\, dV,$$

where V is the volume enclosed by S

$$= \iiint_V [\nabla \bullet (a\ x\ \mathbf{i} + by\ \mathbf{j} + cz\ \mathbf{k})]\, dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(a\ x) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)\right] dV$$

$$= \iiint_V (a + b + c)\, dV$$

$$= (a + b + c)\, V = (a + b + c)\, \frac{4}{3} \pi,$$

since the volume V enclosed by a sphere of unit radius is equal to $\frac{4}{3} \pi (1)^3$ *i.e.*, $\frac{4}{3} \pi$.

Example 5(i): *Show that* $\iint_S \mathbf{n}\, dS = \mathbf{0}$ *for any closed surface S.*

(Purvanchal 14)

(ii) Prove that $\iint_S \mathbf{r} \times \mathbf{n}\, dS = \mathbf{0}$ *for any closed surface S.*

(Agra 2007)

(iii) Prove that $\iint_S \mathbf{n} \times (\mathbf{a} \times \mathbf{r})\, dS = 2V\mathbf{a}$, *where* **a** *is a constant vector and V is the volume enclosed by the closed surface S.*

(Avadh 2011, 12)

Solution: (i) Let **C** be any arbitrary constant vector.

Then $$\mathbf{C} \bullet \iint_S \mathbf{n}\, dS = \iint_S \mathbf{C} \bullet \mathbf{n}\, dS$$

$$= \iiint_V (\nabla \bullet \mathbf{C})\, dV, \text{ by divergence theorem}$$

$$= 0, \quad \text{since div } \mathbf{C} = 0.$$

Thus $\mathbf{C}\bullet\iint_S \mathbf{n}\,dS = 0$, where $\mathbf{C}$ is an arbitrary vector.

Therefore we must have $\iint_S \mathbf{n}\,dS = \mathbf{0}$.

(ii) Let $\mathbf{C}$ be any arbitrary constant vector. Then

$$\mathbf{C}\bullet\iint_S \mathbf{r}\times\mathbf{n}\,dS = \iint_S \mathbf{C}\bullet[(\mathbf{r}\times\mathbf{n})]\,dS = \iint_S (\mathbf{C}\times\mathbf{r})\bullet\mathbf{n}\,dS$$

$$= \iiint_V [\nabla\bullet(\mathbf{C}\times\mathbf{r})]\,dV, \text{ by divergence theorem}$$

$$= \iiint_V [\mathbf{r}\bullet\text{curl }\mathbf{C} - \mathbf{C}\bullet\text{curl }\mathbf{r}]\,dV = 0,$$

since curl $\mathbf{C} = \mathbf{0}$ and curl $\mathbf{r} = \mathbf{0}$.

Thus $\mathbf{C}\bullet\iint_S \mathbf{r}\times\mathbf{n}\,dS = 0$, where $\mathbf{C}$ is an arbitrary vector.

Therefore, we must have $\iint_S \mathbf{r}\times\mathbf{n}\,dS = \mathbf{0}$.

(iii) We know that

$$\iiint_V \nabla\times\mathbf{B}\,dV = \iint_S \mathbf{n}\times\mathbf{B}\,dS. \qquad \text{[See article 4, part 3]}$$

Putting $\mathbf{B} = \mathbf{a}\times\mathbf{r}$, we get

$$\iint_S \mathbf{n}\times(\mathbf{a}\times\mathbf{r})\,dS = \iiint_V \nabla\times(\mathbf{a}\times\mathbf{r})\,dV$$

$$= \iiint_V \text{curl }(\mathbf{a}\times\mathbf{r})\,dV$$

$$= \iiint_V 2\mathbf{a}\,dV, \text{ since curl }(\mathbf{a}\times\mathbf{r}) = 2\mathbf{a}$$

$$= 2\mathbf{a}\iiint_V dV = 2V\mathbf{a}.$$

Example 6: *Using the divergence theorem, show that the volume V of a region T bounded by a surface S is*

$$V = \iint_S x\,dy\,dz = \iint_S y\,dz\,dx = \iint_S z\,dx\,dy$$

$$= \frac{1}{3}\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy).$$

(Meerut 2010)

Solution: By divergence theorem, we have

$$\iint_S x\,dy\,dz = \iiint_V \left(\frac{\partial}{\partial x}(x)\right)dV = \iiint_V dV = V$$

$$\iint_S y\,dz\,dx = \iiint_V \left(\frac{\partial}{\partial y}(y)\right)dV = \iiint_V dV = V$$

$$\iint_S z\,dx\,dy = \iiint_V \left(\frac{\partial}{\partial z}(z)\right)dV = \iiint_V dV = V.$$

Adding these results, we get

$$3V = \iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$$

or $$V = \frac{1}{3}\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy).$$

Example 7: *Verify divergence theorem for*

$$\mathbf{F} = (x^2 - yz) \mathbf{i} + (y^2 - zx) \mathbf{j} + (z^2 - xy) \mathbf{k}$$

taken over the rectangular parallelopiped $0 \le x \le a, 0 \le y \le b, \ 0 \le z \le c$.

(Meerut 2006B; Avadh 09; Rohilkhand 13)

Solution: We have

$$\begin{aligned} \text{div } \mathbf{F} &= \nabla \bullet \mathbf{F} \\ &= \frac{\partial}{\partial x} (x^2 - yz) + \frac{\partial}{\partial y} (y^2 - zx) + \frac{\partial}{\partial z} (z^2 - xy) \\ &= 2x + 2y + 2z. \end{aligned}$$

$$\therefore \quad \text{volume integer} = \iiint_V \nabla \bullet \mathbf{F} \, dV = \iiint_V 2(x + y + z) \, dV$$

$$= 2\int_{z=0}^{c} \int_{y=0}^{b} \int_{x=0}^{a} (x + y + z) \, dx \, dy \, dz$$

$$= 2 \int_{z=0}^{c} \int_{y=0}^{b} \left[\frac{x^2}{2} + yx + zx \right]_{x=0}^{a} dy \, dz$$

$$= 2 \int_{z=0}^{c} \int_{y=0}^{b} \left[\frac{a^2}{2} + ay + az \right] dy \, dz$$

$$= 2\int_{z=0}^{c} \left[\frac{a^2}{2} y + a \frac{y^2}{2} + azy \right]_{y=0}^{b} dz$$

$$= 2 \int_{z=0}^{c} \left[\frac{a^2 b}{2} + \frac{ab^2}{2} + abz \right] dz$$

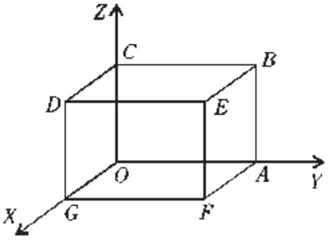
$$= 2 \left[\frac{a^2 b}{2} z + \frac{ab^2}{2} z + ab \frac{z^2}{2} \right]_0^c$$

$$= [a^2 bc + ab^2 c + abc^2] = abc \, (a + b + c).$$

Surface Integral: We shall now calculate $\iint_S \mathbf{F} \bullet \mathbf{n} \, dS$ over the six faces of the rectangular parallelopiped.

Over the face *DEFG*, $\mathbf{n} = \mathbf{i}, \ x = a$.

Therefore, $\iint_{DEFG} \mathbf{F} \bullet \mathbf{n} \, dS$



$$= \int_{z=0}^{c} \int_{y=0}^{b} [(a^2 - y\,z) \,\mathbf{i} + (y^2 - za) \,\mathbf{j} + (z^2 - ay) \,\mathbf{k}] \bullet \mathbf{i} \, dy \, dz$$

$$= \int_{z=0}^{c} \int_{y=0}^{b} (a^2 - yz) \, dy \, dz$$

$$= \int_{z=0}^{c} \left[a^2 y - z \frac{y^2}{2} \right]_{y=0}^{b} dz$$

$$= \int_{z=0}^{c} \left[a^2 b - \frac{zb^2}{2} \right] dz = \left[a^2 bz - \frac{z^2}{4} b^2 \right]_0^c = a^2 bc - \frac{c^2 b^2}{4}.$$

Over the face $ABCO$, $\mathbf{n} = -\mathbf{i}$, $x = 0$. Therefore

$$\iint_{ABCO} \mathbf{F} \bullet \mathbf{n}\, dS = \iint [(0 - yz)\,\mathbf{i} + \ldots + \ldots] \bullet (-\mathbf{i})\, dy\, dz$$

$$= \int_{z=0}^{c} \int_{y=0}^{b} y\, z\, dy\, dz = \int_{z=0}^{c} \left[\frac{y^2}{2} z \right]_{y=0}^{b} dz$$

$$= \int_{z=0}^{c} \frac{b^2}{2} z\, dz = \frac{b^2 c^2}{4}.$$

Over the face $ABEF$, $\mathbf{n} = \mathbf{j}$, $y = b$. Therefore

$$\iint_{ABEF} \mathbf{F} \bullet \mathbf{n}\, dS = \int_{z=0}^{c} \int_{x=0}^{a} [(x^2 - bz)\,\mathbf{i} + (b^2 - zx)\,\mathbf{j}$$

$$+ (z^2 - bx)\,\mathbf{k}] \bullet \mathbf{j}\, dx\, dz$$

$$= \int_{z=0}^{c} \int_{x=0}^{a} (b^2 - zx)\, dx\, dz = b^2\, ca - \frac{a^2 c^2}{4}.$$

Over the face $OGDC$, $\mathbf{n} = -\mathbf{j}$, $y = 0$. Therefore

$$\iint_{OGDC} \mathbf{F} \bullet \mathbf{n}\, dS = \int_{z=0}^{c} \int_{x=0}^{a} zx\, dx\, dz = \frac{c^2 a^2}{4}.$$

Over the face $BCDE$, $\mathbf{n} = \mathbf{k}$, $z = c$. Therefore

$$\iint_{BCDE} \mathbf{F} \bullet \mathbf{n}\, dS = \int_{y=0}^{b} \int_{x=0}^{a} (c^2 - xy)\, dx\, dy = c^2 ab - \frac{a^2 b^2}{4}.$$

Over the face $AFGO$, $\mathbf{n} = -\mathbf{k}$, $z = 0$. Therefore

$$\iint_{AFGO} \mathbf{F} \bullet \mathbf{n}\, dS = \int_{y=0}^{b} \int_{x=0}^{a} x\, y\, dx\, dy = \frac{a^2\, b^2}{4}.$$

Adding the six surface integrals, we get

$$\iint_S \mathbf{F} \bullet \mathbf{n}\, dS = \left(a^2 bc - \frac{c^2 b^2}{4} + \frac{c^2 b^2}{4} \right) + \left(b^2 ca - \frac{a^2 c^2}{4} + \frac{a^2 c^2}{4} \right)$$

$$+ \left(c^2 ab - \frac{a^2 b^2}{4} + \frac{a^2 b^2}{4} \right)$$

$$= abc\, (a + b + c).$$

Hence the theorem is verified.

Example 8: *If* $\mathbf{F} = x\,\mathbf{i} - y\,\mathbf{j} + (z^2 - 1)\mathbf{k}$, *find the value of* $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS$ *where* S *is the closed surface bounded by the planes* $z = 0$, $z = 1$ *and the cylinder* $x^2 + y^2 = 4$.

(Garhwal 2000; Kanpur 05; Avadh 13; Kumaun 15)

Solution: By divergence theorem, we have $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS = \iiint_V \text{div}\, \mathbf{F}\, dV$.

Here $$\operatorname{div}\mathbf{F}=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(-y)+\frac{\partial}{\partial z}(z^2-1)=1-1+2z=2z.$$

$\therefore$ $$\iiint_V \operatorname{div}\mathbf{F}\,dV=\int_{z=0}^{1}\int_{y=-2}^{2}\int_{x=-\sqrt{(4-y^2)}}^{\sqrt{(4-y^2)}}2z\,dx\,dy\,dz$$

$$=\int_{z=0}^{1}\int_{y=-2}^{2}[2zx]_{x=-\sqrt{(4-y^2)}}^{\sqrt{(4-y^2)}}\,dy\,dz$$

$$=\int_{z=0}^{1}\int_{y=-2}^{2}4z\sqrt{(4-y^2)}\,dy\,dz$$

$$=\int_{y=-2}^{2}\left[4\frac{z^2}{2}\sqrt{(4-y^2)}\right]_{z=0}^{1}dy$$

$$=2\int_{y=-2}^{2}\sqrt{(4-y^2)}\,dy=4\int_0^2\sqrt{(4-y^2)}\,dy$$

$$=4\left[\frac{y}{2}\sqrt{(4-y^2)}+2\sin^{-1}\frac{y}{2}\right]_0^2$$

$$=4\,[2\sin^{-1}1]=4\,(2)\,\frac{\pi}{2}=4\pi.$$

Comprehensive Exercise 2

1. (i) Verify divergence theorem for $\mathbf{F}=(2x-z)\mathbf{i}+x^2y\mathbf{j}-xz^2\mathbf{k}$ taken over the region bounded by $x=0, x=2, y=0, y=2, z=0, z=2$.

   (ii) Verify divergence theorem for $\mathbf{F}=(2x-z)\,\mathbf{i}+x^2y\,\mathbf{j}-xz^2\mathbf{k}$ taken over the region bounded by $x=0,\ x=1,\ y=0, y=1, z=0, z=1$.
   (Garhwal 2001; Kumaun 14)

2. (i) If $\mathbf{F}=4xz\,\mathbf{i}-y^2\mathbf{j}+yz\,\mathbf{k}$ and S is the surface bounded by $x=0,\ y=0,\ z=0,\ x=1,\ y=1,\ z=1,$ evaluate $\iint_S \mathbf{F}\bullet\mathbf{n}\,dS$.

   (ii) Evaluate $\iint_S x^2dy\,dz+y^2dz\,dx+2z\,(xy-x-y)\,dx\,dy$

   where S is the surface of the cube $0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 1$.
   (Kumaun 2015)

3. (i) Evaluate $\iint_S [4xz\,dy\,dz-y^2\,dz\,dx+yz\,dx\,dy]$ where S is the surface of the cube bounded by the planes $x=0, y=0, z=0, x=1,$ $y=1$ and $z=1$. **(Meerut 2005, 06B, 10B, 11)**

(ii) Apply Gauss's divergence theorem to evaluate

$$\iint_S [(x^3 - yz)\, dy\, dz - 2x^2 y\, dz\, dx + z\, dx\, dy]$$

over the surface of a cube bounded by the coordinate planes and the planes $x = y = z = a$. **(Rohilkhand 2011)**

4. (i) State divergence theorem of Gauss.

(ii) Use Gauss divergence theorem to show that

$$\iint_S \{(x^3 - yz)\,\mathbf{i} - 2x^2 y\,\mathbf{j} + 2\,\mathbf{k}\} \bullet \mathbf{n}\, dS = \frac{1}{3} a^5,$$

where S denotes the surface of the cube bounded by the planes $x = 0,\ x = a,\ y = 0,\ y = a,\ z = 0,\ z = a.$ **(Bundelkhand 2005, 06)**

5. Evaluate $\iint_S (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \bullet \mathbf{n}\, dS$ where S denotes the surface of the cube bounded by the planes $x = 0,\ y = 0,\ z = 0,\ x = a,\ y = a,\ z = a$ by the application of Gauss divergence theorem. Verify your answer by evaluating the integral directly. **(Garhwal 2003)**

6. (i) Evaluate by divergence theorem the integral

$$\iint_S xz^2\, dy\, dz + (x^2 y - z^3)\, dz\, dx + (2xy + y^2 z)\, dx\, dy,$$

where S is the entire surface of the hemispherical region bounded by $z = \sqrt{(a^2 - x^2 - y^2)}$ and $z = 0$.

(ii) Evaluate $\iint_S (y^2 z^2\,\mathbf{i} + z^2 x^2\,\mathbf{j} + z^2 y^2\,\mathbf{k}) \bullet \mathbf{n}\, dS$

where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ above the xy-plane and bounded by this plane. **(Bundelkhand 2006)**

7. (i) If $\mathbf{F} = ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}$, where a, b, c are constants, show that

$$\iint_S (\mathbf{n} \bullet \mathbf{F})\, dS = \frac{4\pi}{3}(a + b + c),$$

S being the surface of the sphere $(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 1.$

(ii) If S is any closed surface enclosing a volume V and $\mathbf{F} = x\,\mathbf{i} + 2\,y\,\mathbf{j} + 3\,z\,\mathbf{k}$, prove that $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS = 6V.$ **(Rohilkhand 2009B)**

8. Verify the divergence theorem for $\mathbf{F} = 4x\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k}$ taken over the region bounded by the surfaces $x^2 + y^2 = 4,\ z = 0,\ z = 3.$ **(Garhwal 2002; Bundelkhand 08)**

9. Use Gauss divergence theorem to find $\iint_S \mathbf{F} \bullet \mathbf{n}\, dS$, where $\mathbf{F} = 2x^2 y\,\mathbf{i} - y^2\,\mathbf{j} + 4xz^2\,\mathbf{k}$ and S is the closed surface in the first octant bounded by $y^2 + z^2 = 9$ and $x = 2$.

10. If $\mathbf{F} = y\,\mathbf{i} + (x - 2xz)\,\mathbf{j} - xy\,\mathbf{k}$, evaluate $\iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, dS$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane.

11. Evaluate $\iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, dS$,

where $\mathbf{F} = (x^2 + y - 4)\,\mathbf{i} + 3xy\,\mathbf{j} + (2xz + z^2)\,\mathbf{k}$ and S is the surface of the paraboloid $z = 4 - (x^2 + y^2)$ above the xy-plane.

12. Compute

(i) $\iint_S (a^2x^2 + b^2y^2 + c^2z^2)^{1/2}\, dS$, and

(ii) $\iint_S (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}\, dS$

over the ellipsoid $ax^2 + by^2 + cz^2 = 1$.

13. Evaluate $\iint_S (x^2 + y^2)\, dS$, where S is the surface of the cone $z^2 = 3(x^2 + y^2)$ bounded by $z = 0$ and $z = 3$.

14. Show that $\iint_S (x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}) \bullet \mathbf{n}\, dS$ vanishes where S denotes the surface of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

(Meerut 2005, 07; Kumaun 11, 13)

15. If $\mathbf{n}$ is the unit outward drawn normal to any closed surface S, show that

$$\iiint_V \text{div}\, \mathbf{n}\, dV = S.$$

Answers 2

2. (i) $\dfrac{3}{2}$ (ii) $\dfrac{1}{2}$ **3.** (i) $\dfrac{3}{2}$ (ii) $a^2\left(\dfrac{a^3}{3} + a\right)$

5. $3a^3$ **6.** (i) $\dfrac{2\pi a^5}{5}$ (ii) $\dfrac{\pi}{12}$

9. 180 **10.** 0

11. -4π **12.** (i) $\dfrac{4}{3}\pi abc.$ (ii) $\dfrac{4\pi}{\sqrt{(abc)}}$

13. 9π

5 Stoke's Theorem

Let S be a piecewise smooth open surface bounded by a piecewise smooth simple closed curve C. Let $\mathbf{F}(x, y, z)$ *be a continuous vector function which has continuous first partial derivatives in a region of space which contains S in its interior. Then*

$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, dS = \iint_S (curl\ \mathbf{F}) \bullet d\mathbf{S}$$

where C is traversed in the positive direction. The direction of C is called positive if an observer, walking on the boundary of S in this direction, with his head pointing in the direction of outward drawn normal **n** *to S, has the surface on the left.*

(Meerut 2009; Bundelkhand 10)

Note: $\oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C \left(\mathbf{F} \bullet \dfrac{d\mathbf{r}}{ds}\right) ds = \oint_C (\mathbf{F} \bullet \mathbf{t})\, ds,$ where **t** is unit tangent vector to C. Therefore $\mathbf{F} \bullet \mathbf{t}$ is the component of **F** in the direction of the tangent vector of C. Also $(\nabla \times \mathbf{F}) \bullet \mathbf{n}$ is the component of curl **F** in the direction of outward drawn normal vector **n** of S. Therefore in words Stoke's theorem may be stated as follows:

The line integral of the tangential component of vector **F** *taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of* **F** *taken over any surface S having C as its boundary.*

Cartesian equivalent of Stoke's theorem:

Let $\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$. Let outward drawn normal vector **n** of S make angles α, β, γ with positive directions of x, y, z axes.

Then $\mathbf{n} = \cos\alpha\, \mathbf{i} + \cos\beta\, \mathbf{j} + \cos\gamma\, \mathbf{k}$.

Also
$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}.$$

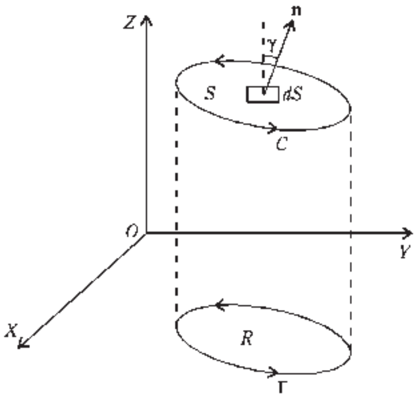
$\therefore$
$$(\nabla \times \mathbf{F}) \bullet \mathbf{n} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma.$$

Also
$$\mathbf{F} \bullet d\mathbf{r} = (F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}) \bullet (dx\, \mathbf{i} + dy\, \mathbf{j} + dz\, \mathbf{k})$$
$$= F_1\, dx + F_2\, dy + F_3\, dz.$$

$\therefore$ Stoke's theorem can be written as

$$\oint_C F_1\, dx + F_2\, dy + F_3\, dz$$
$$= \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right] dS.$$

Proof of Stoke's theorem: Let S be a surface which is such that its projections on the xy, yz and zx planes are regions bounded by simple closed curves. Suppose S can be represented simultaneously in the forms $z = f(x, y)$, $y = g(x, z)$, $x = h(z, y)$ where f, g, h are continuous functions and have continuous first partial derivatives.



Consider the integral

$$\iint_S [\nabla \times (F_1 \, \mathbf{i})] \bullet \mathbf{n} \, dS.$$

We have $$\nabla \times (F_1 \, \mathbf{i}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & 0 & 0 \end{vmatrix} = \frac{\partial F_1}{\partial z} \mathbf{j} - \frac{\partial F_1}{\partial y} \mathbf{k}.$$

$$\therefore \quad [\nabla \times (F_1 \, \mathbf{i})] \bullet \mathbf{n} = \left(\frac{\partial F_1}{\partial z} \mathbf{j} \bullet \mathbf{n} - \frac{\partial F_1}{\partial y} \mathbf{k} \bullet \mathbf{n} \right) = \frac{\partial F_1}{\partial z} \cos \beta - \frac{\partial F_1}{\partial y} \cos \gamma.$$

$$\therefore \quad \iint_S [\nabla \times (F_1 \, \mathbf{i})] \bullet \mathbf{n} \, dS = \iint_S \left(\frac{\partial F_1}{\partial z} \cos \beta - \frac{\partial F_1}{\partial y} \cos \gamma \right) dS.$$

We shall prove that

$$\iint_S \left(\frac{\partial F_1}{\partial z} \cos \beta - \frac{\partial F_1}{\partial y} \cos \gamma \right) dS = \oint_C F_1 \, dx.$$

Let R be the orthogonal projection of S on the xy-plane and let Γ be its boundary which is oriented as shown in the figure. Using the representation $z = f(x, y)$ of S, we may write the line integral over C as a line integral over Γ. Thus

$$\oint_{\mathbf{C}} F_1 (x, y, z) \, dx = \oint_\Gamma F_1 [x, y, f(x, y)] \, dx$$

$$= \oint_\Gamma \{ F_1 [x, y, f(x, y)] \, dx + 0 \, dy \}$$

$$= - \iint_R \frac{\partial F_1}{\partial y} \, dx \, dy,$$

by Green's theorem in plane for the region R.

But $$\frac{\partial F_1 [x, y, f(x, y)]}{\partial y} = \frac{\partial F_1 (x, y, z)}{\partial y} + \frac{\partial F_1 (x, y, z)}{\partial z} \frac{\partial f}{\partial y}.$$

$[\because \ z = f(x, y)]$

$$\therefore \quad \oint_C F_1 (x, y, z) \, dx = - \iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z} \frac{\partial f}{\partial y} \right) dx \, dy \qquad \ldots(1)$$

Now the equation $z = f(x, y)$ of the surface S can be written as

$$\phi (x, y, z) \equiv z - f(x, y) = 0.$$

We have grad $\phi = -\dfrac{\partial f}{\partial x}\,\mathbf{i} - \dfrac{\partial f}{\partial y}\,\mathbf{j} + \mathbf{k}$.

Let $|\text{grad } \phi| = a$.

Since grad ϕ is normal to S, therefore, we get $\mathbf{n} = \pm \dfrac{\text{grad } \phi}{a} \cdot$

But the components of both **n** and grad ϕ in positive direction of z-axis are positive. Therefore

$$\mathbf{n} = +\frac{\text{grad } \phi}{a}$$

or
$$\cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k} = -\frac{1}{a}\frac{\partial f}{\partial x}\,\mathbf{i} - \frac{1}{a}\frac{\partial f}{\partial y}\,\mathbf{j} + \frac{1}{a}\,\mathbf{k}.$$

$\therefore$
$$\cos\alpha = -\frac{1}{a}\frac{\partial f}{\partial x},\ \cos\beta = -\frac{1}{a}\frac{\partial f}{\partial y},\ \cos\gamma = \frac{1}{a}\cdot$$

Now
$$dS = \frac{dx\,dy}{\cos\gamma} = a\,dx\,dy.$$

$\therefore$
$$\iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) dS$$
$$= \iint_R \left[\frac{\partial F_1}{\partial z}\left(-\frac{1}{a}\frac{\partial f}{\partial y}\right) - \frac{\partial F_1}{\partial y}\frac{1}{a}\right] a\,dx\,dy$$
$$= -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y}\right) dx\,dy. \qquad \text{...(2)}$$

From (1) and (2), we get

$$\oint_C F_1\,dx = \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) dS$$
$$= \iint_S [\nabla \times (F_1\,\mathbf{i})] \bullet \mathbf{n}\,dS \qquad \text{...(3)}$$

Similarly, by projections on the other coordinate planes, we get

$$\oint_C F_2\,dy = \iint_S [\nabla \times (F_2\,\mathbf{j})] \bullet \mathbf{n}\,dS \qquad \text{...(4)}$$
$$\oint_C F_3\,dz = \iint_S [\nabla \times (F_3\,\mathbf{k})] \bullet \mathbf{n}\,dS \qquad \text{...(5)}$$

Adding (3), (4), (5), we get

$$\oint_C (F_1\,dx + F_2\,dy + F_3\,dz) = \iint_S [\nabla \times (F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k})] \bullet \mathbf{n}\,dS$$

or
$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\,dS.$$

If the surface S does not satisfy the restrictions imposed above, even then Stoke's theorem will be true provided S can be subdivided into surfaces $S_1, S_2, \ldots, S_k$ with boundaries $C_1, C_2, \ldots, C_k$ which do satisfy the restrictions. Stoke's theorem holds for each such surface. The sum of surface integrals over $S_1, S_2, \ldots, S_k$ will give us surface integral over S while the sum of the integrals over $C_1, C_2, \ldots, C_k$ will give us line integral over C.

Note: **Green's theorem in plane is a special case of Stoke's theorem.** If R is a region in the xy-plane bounded by a closed curve C, then in vector form Green's theorem in plane can be written as

$$\iint_R (\nabla \times \mathbf{F}) \bullet \mathbf{k}\, dR = \int_C \mathbf{F} \bullet d\mathbf{r}.$$

This is nothing but a special case of Stoke's theorem because here $\mathbf{k} = \mathbf{n}$ = outward drawn unit normal to the surface of region R.

Illustrative Examples

Example 9: *Prove that* $\oint_C \mathbf{r} \bullet d\mathbf{r} = 0.$ **(Meerut 2010)**

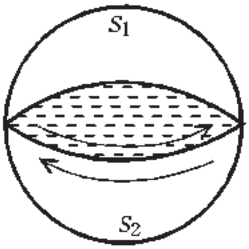
Solution: By Stoke's theorem $\oint_C \mathbf{r} \bullet d\mathbf{r} = \iint_S (\text{curl } \mathbf{r}) \bullet \mathbf{n}\, dS = 0$, since curl $\mathbf{r} = \mathbf{0}$.

Example 10: *By Stoke's theorem prove that div curl* $\mathbf{F} = 0$.

Solution: Let V be any volume enclosed by a closed surface.

Then by divergence theorem

$$\iiint_V \nabla \bullet (\text{curl } \mathbf{F})\, dV = \iint_S (\text{curl } \mathbf{F}) \bullet \mathbf{n}\, dS.$$



Divide the surface S into two portions S_1 and S_2 by a closed curve C. Then

$$\iint_S (\text{curl } \mathbf{F}) \bullet \mathbf{n}\, dS = \iint_{S_1} (\text{curl } \mathbf{F}) \bullet \mathbf{n}\, dS_1 + \iint_{S_2} (\text{curl } \mathbf{F}) \bullet \mathbf{n}\, dS_2 . \qquad \ldots (1)$$

By Stoke's theorem right hand side of (1) is $= \oint_C \mathbf{F} \bullet d\mathbf{r} - \oint_C \mathbf{F} \bullet d\mathbf{r} = 0.$

Negative sign has been taken in the second integral because the positive directions about the boundaries of the two surfaces are opposite.

$$\therefore \qquad \iiint_V \nabla \bullet (\text{curl } \mathbf{F})\, dV = 0.$$

Now this equation is true for all volume elements V. Therefore we have

$$\nabla \bullet (\text{curl } \mathbf{F}) = 0 \quad \text{or} \quad \text{div curl } \mathbf{F} = 0.$$

Example 11: *Verify Stoke's theorem for* $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$ *where S is the upper half surface of the sphere* $x^2 + y^2 + z^2 = 1$ *and C is its boundary.*

(Agra 2000, 06; Kanpur 09; Kumaun 07, 10, 13)

Solution: The boundary C of S is a circle in the xy-plane of radius unity and centre origin. The equations of the curve C are $x^2 + y^2 = 1$, $z = 0$. Suppose $x = \cos t$, $y = \sin t$, $z = 0, 0 \le t < 2\pi$ are parametric equation of C. Then

$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C (y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}) \bullet (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \oint_C (y\,dx + z\,dy + x\,dz)$$

$$= \oint_C y\,dx, \text{ since on } C,\ z = 0 \text{ and } dz = 0$$

$$= \int_0^{2\pi} \sin t \frac{dx}{dt}\,dt = \int_0^{2\pi} -\sin^2 t\,dt$$

$$= -\frac{1}{2}\int_0^{2\pi} (1 - \cos 2t)\,dt = -\frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= -\pi. \qquad \ldots(1)$$

Now let us evaluate $\iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\,dS$. We have

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\mathbf{i} - \mathbf{j} - \mathbf{k}.$$

If S_1 is the plane region bounded by the circle C, then by an application of divergence theorem, we have

$$\iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\,dS = \iint_{S_1} \text{curl } \mathbf{F} \bullet \mathbf{k}\,dS$$

[See example 4 after article 4]

$$= \iint_{S_1} (-\mathbf{i} - \mathbf{j} - \mathbf{k}) \bullet \mathbf{k}\,dS$$

$$= \iint_{S_1} (-1)\,dS = -\iint_{S_1} dS = -S_1.$$

But S_1 = area of a circle of radius $1 = \pi\,(1)^2 = \pi$.

$$\therefore \qquad \iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\,dS = -\pi. \qquad \ldots(2)$$

Hence from (1) and (2), the theorem is verified.

Example 12: *Verify Stoke's theorem for* $\mathbf{F} = (2x - y)\,\mathbf{i} - yz^2\,\mathbf{j} - y^2 z\,\mathbf{k}$, *where S is the upper half surface of the sphere* $x^2 + y^2 + z^2 = 1$ *and C is its boundary.*

(Kumaun 2003; Kanpur 10, 14; Avadh 09)

Solution: The boundary C of S is a circle in the xy-plane of radius unity and centre origin. Suppose $x = \cos t,\ y = \sin t,\ z = 0, 0 \le t < 2\pi$ are parametric equations of C. Then

$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C [(2x - y)\mathbf{i} - yz^2\,\mathbf{j} - y^2 z\,\mathbf{k}] \bullet (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \oint_C [(2x - y)\,dx - yz^2\,dy - y^2 z\,dz]$$

$$= \oint_C (2x - y)\,dx, \text{ since } z = 0 \text{ and } dz = 0$$

$$= \int_0^{2\pi} (2\cos t - \sin t)\frac{dx}{dt}\,dt$$

$$= -\int_0^{2\pi} (2\cos t - \sin t)\sin t\, dt$$

$$= -\int_0^{2\pi} [\sin 2t - \frac{1}{2}(1 - \cos 2t)]\, dt$$

$$= -\left[-\frac{\cos 2t}{2} - \frac{1}{2}t + \frac{1}{2}\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= -[(-\frac{1}{2} + \frac{1}{2}) - \frac{1}{2}(\pi - 0) + \frac{1}{4}(0 - 0)] = \pi. \quad \text{...(1)}$$

And $$(\nabla \times \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= (-2yz + 2yz)\,\mathbf{i} - (0 - 0)\,\mathbf{j} + (0 + 1)\,\mathbf{k} = \mathbf{k}.$$

Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface.

∴ by an application of Gauss divergence theorem, we have

$$\iint_{S'} \text{curl } \mathbf{F} \bullet \mathbf{n}\, dS = 0 \qquad \text{[See example 4(i) after article 4]}$$

or $$\iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\, dS + \iint_{S_1} \text{curl } \mathbf{F} \bullet \mathbf{n}\, dS = 0$$

[∵ S' consists of S and S_1]

or $$\iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\, dS - \iint_{S_1} \text{curl } \mathbf{F} \bullet \mathbf{k}\, dS = 0 \qquad [\because \text{ on } S_1,\ \mathbf{n} = -\mathbf{k}]$$

or $$\iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\, dS = \iint_{S_1} \text{curl } \mathbf{F} \bullet \mathbf{k}\, dS.$$

∴ $$\iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\, dS = \iint_{S_1} \text{curl } \mathbf{F} \bullet \mathbf{k}\, dS$$

$$= \iint_{S_1} \mathbf{k} \bullet \mathbf{k}\, dS = \iint_{S_1} dS = S_1 = \pi. \quad \text{...(2)}$$

Note that S_1 = area of a circle of radius $1 = \pi(1)^2 = \pi$.

Hence from (1) and (2) Stoke's theorem is verified.

Example 13: *Verify Stoke's theorem for* $\mathbf{F} = (x^2 + y^2)\,\mathbf{i} - 2xy\mathbf{j}$ *taken round the rectangle bounded by* $x = \pm a,\ y = 0,\ y = b.$ **(Bundelkhand 2007; Agra 08; Kumaun 15)**

Solution: We have $$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$$

$$= (-2y - 2y)\,\mathbf{k} = -4y\,\mathbf{k}.$$

Also $\mathbf{n} = \mathbf{k}$.

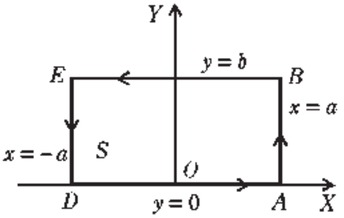
∴ $$\iint_S (\text{curl } \mathbf{F}) \bullet \mathbf{n}\, dS = \int_{y=0}^{b} \int_{x=-a}^{a} (-4y\,\mathbf{k}) \bullet \mathbf{k}\, dx\, dy$$

$$= -4\int_{y=0}^{b}\int_{x=-a}^{a} y\,dx$$

$$= -4\int_{y=0}^{b} [xy]_{x=-a}^{a}\,dy$$

$$= -4\int_{y=0}^{b} 2ay\,dy$$

$$= -4\,[ay^2]_0^b = -4ab^2.$$



Also $\oint_C \mathbf{F}\bullet d\mathbf{r} = \oint_C [(x^2+y^2)\,\mathbf{i} - 2xy\,\mathbf{j}]\bullet(dx\,\mathbf{i} + dy\,\mathbf{j})$

$$= \oint_C [(x^2+y^2)\,dx - 2xy\,dy\,]$$

$$= \int_{DA} [(x^2+y^2)\,dx - 2xy\,dy\,] + \int_{AB} + \int_{BE} + \int_{ED}.$$

Along DA, $y = 0$ and $dy = 0$. Along AB, $x = a$ and $dx = 0$.
Along BE, $y = b$ and $dy = 0$. Along ED, $x = -a$ and $dx = 0$.

$\therefore$
$$\oint_C \mathbf{F}\bullet d\mathbf{r} = \int_{x=-a}^{a} x^2\,dx + \int_{y=0}^{b} -2ay\,dy$$
$$+ \int_{x=a}^{-a} (x^2+b^2)\,dx + \int_{y=b}^{0} 2ay\,dy$$

$$= \int_{-a}^{a} x^2\,dx - \int_{-a}^{a} (x^2+b^2)\,dx - 4a\int_0^b y\,dy$$

$$= -\int_{-a}^{a} x^2\,dx - 4a\int_0^b y\,dy = -2ab^2 - 4a\left[\frac{y^2}{2}\right]_0^b = -4ab^2.$$

Thus $\oint_C \mathbf{F}\bullet d\mathbf{r} = \iint_S (\text{curl }\mathbf{F})\bullet\mathbf{n}\,dS.$

Hence the theorem is verified.

Example 14: *Evaluate* $\oint_C \mathbf{F}\bullet d\mathbf{r}$ *by Stoke's theorem where*

$$\mathbf{F} = y^2\,\mathbf{i} + x^2\,\mathbf{j} - (x+z)\,\mathbf{k}$$

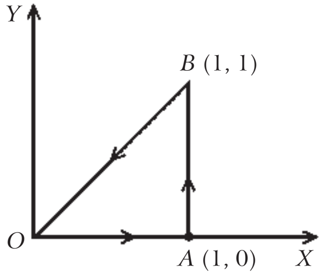
and C is the boundary of the triangle with vertices at (0, 0, 0), (1, 0, 0), (1, 1, 0).

(Avadh 2013)

Solution: We have

$$\text{Curl }\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 & x^2 & (x+z) \end{vmatrix}$$

$$= 0\,\mathbf{i} + \mathbf{j} + 2\,(x-y)\,\mathbf{k}.$$



Also we note that z co-ordinate of each vertex of the triangle is zero. Therefore the triangle lies in the xy-plane. So $\mathbf{n} = \mathbf{k}$.

$\therefore$ $\text{Curl }\mathbf{F}\bullet\mathbf{n} = [\,\mathbf{j} + 2\,(x-y)\mathbf{k}]\bullet\mathbf{k} = 2\,(x-y).$

In the figure, we have only considered the xy plane.

The equation of the line OB is $y = x$.

By Stoke's theorem

$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \bullet \mathbf{n}\, dS$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} 2(x - y)\, dx\, dy$$

$$= 2\int_{x=0}^{1} \left[x\, y - \frac{y^2}{2} \right]_{y=0}^{x} dx$$

$$= 2\int_0^1 \left[x^2 - \frac{x^2}{2} \right] dx = 2\int_0^1 \frac{x^2}{2}\, dx$$

$$= \int_0^1 x^2\, dx = \frac{1}{3}.$$

Comprehensive Exercise 3

1. (i) State Stoke's theorem.
   (ii) By Stoke's theorem prove that curl grad $\phi = \mathbf{0}$. **(Kumaun 2014)**
   (iii) Verify Stoke's theorem for the function $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\,\mathbf{k}$ where curve is the unit circle in the xy-plane bounding the hemisphere $z = \sqrt{(1 - x^2 - y^2)}$. **(Garhwal 2003; Agra 07)**
2. Verify Stoke's theorem for the vector $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ taken over the half of the sphere $x^2 + y^2 + z^2 = a^2$ lying above the xy-plane.
3. (i) Verify Stoke's theorem for the function $\mathbf{F} = x^2\mathbf{i} + xy\,\mathbf{j}$ integrated along the rectangle, in the plane $z = 0$, whose sides are along the lines $x = 0,\ y = 0,\ x = a$ and $y = b$.
   (ii) Verify Stoke's theorem for the function $\mathbf{F} = (x^2 - y^2)\,\mathbf{i} + 2xy\,\mathbf{j}$ in the rectangular region in the xy-plane bounded by the lines $x = 0,\ x = a,\ y = 0$ and $y = b$. **(Kanpur 2008)**
   (iii) Verify Stoke's theorem for the function $\mathbf{F} = x^2\mathbf{i} + xy\,\mathbf{j}$, integrated round the square, in the plane $z = 0$, whose sides are along the lines $x = 0,\ y = 0,\ x = a,\ y = a$. **(Agra 2002)**
4. Verify Stoke's theorem for the vector $\mathbf{A} = 3y\,\mathbf{i} - xz\,\mathbf{j} + yz^2\,\mathbf{k}$, where S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$ and C is its boundary.

5. (i) By converting into a line integral evaluate
$\iint_S (\nabla \times \mathbf{A}) \bullet \mathbf{n}\, dS$, where $\mathbf{A} = (x - z)\,\mathbf{i} + (x^3 + yz)\,\mathbf{j} - 3xy^2\,\mathbf{k}$
and S is the surface of the cone $z = 2 - \sqrt{(x^2 + y^2)}$ above the xy-plane.

(ii) By converting into a line integral evaluate $\iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, dS$
where $\mathbf{F} = (x^2 + y - 4)\,\mathbf{i} + 3\,xy\,\mathbf{j} + (2xy + z^2)\,\mathbf{k}$ and S is the surface of the paraboloid $z = 4 - (x^2 + y^2)$ above the xy-plane.

6. (i) Evaluate by Stoke's theorem $\oint_C (e^x\, dx + 2y\, dy - dz)$
where C is the curve $x^2 + y^2 = 4, z = 2.$ **(Garhwal 2001; Meerut 09B)**

(ii) Evaluate by Stoke's theorem $\oint_C (yz\, dx + xz\, dy + xy\, dz)$
where C is the curve $x^2 + y^2 = 1,\ z = y^2$. **(Meerut 2006B)**

7. (i) Evaluate $\iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, dS$, where $\mathbf{F} = (y - z + 2)\,\mathbf{i} +$
$(yz + 4)\,\mathbf{j} - xz\,\mathbf{k}$ and S is the surface of the cube $x = y = z = 0$, $x = y = z = 2$ above the xy-plane.

(ii) Evaluate by Stoke's theorem $\oint_C (\sin z\, dx - \cos x\, dy + \sin y\, dz)$ where
C is the boundary of the rectangle $0 \le x \le \pi, 0 \le y \le 1, z = 3$.
(Meerut 2009)

8. If $\mathbf{f} = \nabla\phi$ and $g = \nabla\psi$ are two vector point functions, such that $\nabla^2\phi = 0$, $\nabla^2\psi = 0$, show that

$$\iint_S (\mathbf{g} \bullet \nabla)\,\mathbf{f} \bullet d\mathbf{S} = \int_C (\mathbf{f} \times \mathbf{g}) \bullet d\mathbf{r} + \iint_S (\mathbf{f} \bullet \nabla)\mathbf{g} \bullet d\mathbf{S}.$$

9. Prove that a necessary and sufficient condition that $\oint_C \mathbf{F} \bullet d\mathbf{r} = 0$ for every closed curve C lying in a simply connected region R is that $\nabla \times \mathbf{F} = \mathbf{0}$ identically.

10. Apply Stoke's theorem to prove that

$$\int_C (y\, dx + z\, dy + x\, dz) = -2\sqrt{2}\,\pi a^2$$

where C is the curve given by $x^2 + y^2 + z^2 - 2ax - 2ay = 0, x + y = 2a$ and begins at the point $(2a, 0, 0)$ and goes at first below the z-plane.
(Meerut 2005, 06B)

11. Use Stoke's theorem to evaluate $\iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, dS$,
where $\mathbf{F} = y\,\mathbf{i} + (x - 2xz)\,\mathbf{j} - xy\,\mathbf{k}$ and S is the surface of sphere $x^2 + y^2 + z^2 = a^2$, above the xy-plane. **(Kumaun 2012)**

Answers 3

5. (i) 12π (ii) -4π **7.** (i) -4 (ii) 2

11. 0

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6 Line Integrals Independent of Path

Let $\mathbf{F}(x, y, z) = f(x, y, z)\,\mathbf{i} + g(x, y, z)\,\mathbf{j} + h(x, y, z)\,\mathbf{k}$ be a vector point function defined and continuous in a region R of space. Let P and Q be two points in R and let C be a path joining P to Q. Then

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int (f\,dx + g\,dy + h\,dz) \qquad \ldots (1)$$

is called the line integral of **F** along C. In general the value of this line integral depends not only on the end points P and Q of the path C but also on C.

In other words, if we integrate from P to Q along different paths, we shall, in general, get different values of the integral. *The line integral* (1) *is said to be* **independent of path** *in R, if for every pair of end points P and Q in R the value of the integral is the same for all paths C in R starting from P and ending at Q.*

In this case the value of this line integral will depend on the choice of P and Q and not on the choice of the path joining P to Q.

Definition: *The expression $f\,dx + g\,dy + h\,dz$ is said to be an exact differential if there exists a single valued scalar point function $\phi(x, y, z)$, having continuous first partial derivatives such that* $d\phi = f\,dx + g\,dy + h\,dz$.

It can be easily seen that $f\,dx + g\,dy + h\,dz$ is an exact differential if and only if the vector function $\mathbf{F} = f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}$ is the gradient of a single valued scalar function $\phi(x, y, z)$.

Because $\mathbf{F} = \text{grad}\ \phi$

if, and only if $f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k} = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}$

if, and only if $f = \dfrac{\partial\phi}{\partial x},\ g = \dfrac{\partial\phi}{\partial y},\ h = \dfrac{\partial\phi}{\partial z}$

if, and only if $f\,dx + g\,dy + h\,dz = \dfrac{\partial\phi}{\partial x}dx + \dfrac{\partial\phi}{\partial y}dy + \dfrac{\partial\phi}{\partial z}dz$

if, and only if $f\,dx + g\,dy + h\,dz = d\,\phi$.

Thus $\mathbf{F} = \text{grad}\ \phi$ if, and only if $f\,dx + g\,dy + h\,dz$ is an exact differential $d\phi$.

Theorem 1: *Let $f(x, y, z)$, $g(x, y, z)$ and $h(x, y, z)$ be continuous in a region R of space. Then the line integral*

$$\int (f\,dx + g\,dy + h\,dz)$$

is independent of path in R if and only if the differential form under the integral sign is exact in R.

Or

Let $\mathbf{F}(x, y, z)$ *be continuous in region R of space. Then the line integral* $\int_C \mathbf{F} \bullet d\mathbf{r}$ *is independent of the path C in R joining P and Q if and only if* $\mathbf{F} = \text{grad}\,\phi$ *where* $\phi(x, y, z)$ *is a single-valued scalar function having continuous first partial derivatives in R.*

Proof: Suppose $\mathbf{F} = \text{grad}\,\phi$ in R. Let P and Q be any two points in R and let C be any path from P to Q in R.

Then
$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \nabla\phi \bullet d\mathbf{r}$$
$$= \int_C \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right) \bullet (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$
$$= \int_C \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$
$$= \int_C d\phi$$
$$= \int_P^Q d\phi = [\phi]_P^Q$$
$$= \phi(Q) - \phi(P).$$

Thus the line integral depends only on points P and Q and not on the path joining them. This is true, of course, only if $\phi(x, y, z)$ is single valued at all points P and Q.

Conversely, suppose the line integral $\int_C \mathbf{F} \bullet d\mathbf{r}$ is independent of the path C joining any two points P and Q in R. Let P be a fixed point (x_0, y_0, z_0) in R and let Q be any point (x, y, z) in R.

Let
$$\phi(x, y, z) = \int_{(x_0, y_0, z_0)}^{(x, y, z)} \mathbf{F} \bullet d\mathbf{r}$$
$$= \int_{x_0, y_0, z_0}^{(x, y, z)} \left(\mathbf{F} \bullet \frac{d\mathbf{r}}{ds}\right) ds.$$

Differentiating both sides with respect to s, we get
$$\frac{d\phi}{ds} = \mathbf{F} \bullet \frac{d\mathbf{r}}{ds}.$$

But
$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds}$$
$$= \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right) \bullet \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}\right)$$

$$= \nabla \phi \bullet \frac{d\mathbf{r}}{ds}.$$

$$\therefore \qquad \mathbf{F} \bullet \frac{d\mathbf{r}}{ds} = \nabla \phi \bullet \frac{d\mathbf{r}}{ds}$$

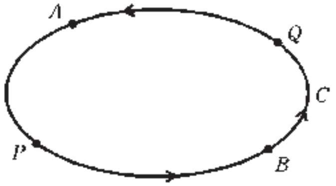
or
$$(\nabla \phi - \mathbf{F}) \bullet \frac{d\mathbf{r}}{ds} = 0.$$

Now this result is true irrespective of the path joining P to Q *i.e.* this result is true irrespective of the direction of $\frac{d\mathbf{r}}{ds}$ which is tangent vector to C. Therefore we must have $\nabla\phi - \mathbf{F} = \mathbf{0}$ i.e., $\nabla \phi = \mathbf{F}$.
This completes the proof of the theorem.

Definition: *A vector field* $\mathbf{F}(x, y, z)$ *defined and continuous in a region R of space is said to be a conservative vector field if the line integral* $\int_C \mathbf{F} \bullet d\mathbf{r}$ *is independent of the path C in R joining P and Q where P and Q are any two points in R.*

By theorem 1, vector field $\mathbf{F}(x, y, z)$ is conservative if and only if $\mathbf{F} = \nabla \phi$ where $\phi(x, y, z)$ is a single valued scalar function having continuous first partial derivatives in R. The function $\phi(x, y, z)$ is called the **scalar potential** *of the vector field* $\mathbf{F}$.

Theorem 2: *Let* $\mathbf{F}(x, y, z)$ *be a vector function defined and continuous in a region R of space. Then the line integral* $\int_P^Q \mathbf{F} \bullet d\mathbf{r}$ *is independent of the path joining any two points P and Q in R if and only if* $\oint_{\mathbf{C}} \mathbf{F} \bullet d\mathbf{r} = 0$ *for every simple closed path in R.*



Proof: Let C be any simple closed path in R and let the line integral be independent of path in R. Take two points P and Q on C and subdivide C into two arcs PBQ and QAP. Then

$$\oint_{\mathbf{C}} \mathbf{F} \bullet d\mathbf{r} = \int_{PBQAP} \mathbf{F} \bullet d\mathbf{r}$$
$$= \int_{PBQ} \mathbf{F} \bullet d\mathbf{r} + \int_{QAP} \mathbf{F} \bullet d\mathbf{r}$$
$$= \int_{PBQ} \mathbf{F} \bullet d\mathbf{r} - \int_{PAQ} \mathbf{F} \bullet d\mathbf{r}$$
$$= 0,$$

since the integral from P to Q along a path through B is equal to the integral from P to Q along a path through A.

Conversely, suppose that the integral under consideration is zero on every simple closed path in R. Let P and Q be any two points in R which join P to Q and do not cross. Then

$$\oint_{PBQAP} \mathbf{F} \bullet d\mathbf{r} = \int_{PBQ} \mathbf{F} \bullet d\mathbf{r} + \int_{QAP} \mathbf{F} \bullet d\mathbf{r}$$

$$= \int_{PBQ} \mathbf{F} \bullet d\mathbf{r} - \int_{PAQ} \mathbf{F} \bullet d\mathbf{r}.$$

But as given, we have

$$\oint_{PBQAP} \mathbf{F} \bullet d\mathbf{r} = 0.$$

$\therefore$ $$\int_{PBQ} \mathbf{F} \bullet d\mathbf{r} - \int_{PAQ} \mathbf{F} \bullet d\mathbf{r} = 0$$

or $$\int_{PBQ} \mathbf{F} \bullet d\mathbf{r} = \int_{PAQ} \mathbf{F} \bullet d\mathbf{r}.$$

This completes the proof of the theorem.

Theorem 3: *Let* $\mathbf{F}(x, y, z) = f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}$ *be a continuous vector function having continuous first partial derivatives in a region* R *of space. If* $\int f\,dx + g\,dy + h\,dz$ *is independent of path in* R *and consequently* $f\,dx + g\,dy + h\,dz$ *is an exact differential in* R*, then curl* $\mathbf{F} = \mathbf{0}$ *everywhere in* R*. Conversely, if* R *is simply connected and curl* $\mathbf{F} = \mathbf{0}$ *everywhere in* R*, then* $f\,dx + g\,dy + h\,dz$ *is an exact differential in* R *or* $\int f\,dx + g\,dy + h\,dz$ *is independent of path in* R*.*

Proof: Suppose $\int (f\,dx + g\,dy + h\,dz)$ is independent of path in R. Then $f\,dx + g\,dy + h\,dz$ is an exact differential in R. Therefore

$$\mathbf{F} = f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k} = \text{grad}\,\phi.$$

$\therefore$ $$\text{curl}\,\mathbf{F} = \text{curl}\,(\text{grad}\,\phi) = \mathbf{0}.$$

Converse. Suppose R is simply connected and curl $\mathbf{F} = \mathbf{0}$ everywhere in R. Let C be any simple closed path in R. Since R is simply connected, therefore we can find a surface S in R having C as its boundary. Therefore by Stoke's theorem

$$\oint_{\mathbf{C}} \mathbf{F} \bullet d\mathbf{r} = \iint_S (\text{curl}\,\mathbf{F}) \bullet \mathbf{n}\,dS = 0.$$

Thus $\oint_{\mathbf{C}} \mathbf{F} \bullet d\mathbf{r}$ is zero for every simple closed path C in R.

Therefore $\int \mathbf{F} \bullet d\mathbf{r}$ is independent of path in R.

Therefore $\mathbf{F} = \nabla\,\phi$ and consequently $f\,dx + g\,dy + h\,dz$ is an exact differential $d\phi$.

Note: The assumption that R be simply connected is essential and cannot be omitted. It is obvious from the following illustration.

Illustration: Let $$\mathbf{F} = -\frac{y}{x^2 + y^2}\,\mathbf{i} + \frac{x}{x^2 + y^2}\,\mathbf{j}.$$

Here $\mathbf{F}$ is not defined at origin. In every region R of the xy-plane not containing the origin, we have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0 \end{vmatrix}$$

$$= 0\mathbf{i} + 0\mathbf{j} + \left\{ \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) \right\} \mathbf{k}$$

$$= \left\{ \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} \right\} \mathbf{k}$$

$$= 0\,\mathbf{k}$$

$$= \mathbf{0}.$$

Suppose R is simply connected. For example let R be the region enclosed by a simple closed curve C not enclosing the origin. Then

$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \oint_C \left(-\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy \right)$$

$$= \int\int_R \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) \right] dx\,dy,$$

by Green's theorem in plane

$$= 0.$$

Suppose R is not simply connected. Let R be the region of the xy-plane contained between concentric circles of radii $\frac{1}{2}$ and $\frac{3}{2}$ and having centre at origin. Obviously R is not simply connected. We have $z = 0$, everywhere in R. Let C be a closed curve in R. The parametric equations of C can be taken as $x = \cos t$, $y = \sin t$, $z = 0$, $0 \leq t < 2\pi$.

We have $$\oint_C \mathbf{F} \bullet d\mathbf{r} = \int_C \left(-\frac{y}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy \right)$$

$$= \int_{t=0}^{2\pi} \left[-\frac{\sin t}{\cos^2 t + \sin^2 t}\frac{dx}{dt} + \frac{\cos t}{\cos^2 t + \sin^2 t}\frac{dy}{dt} \right] dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt$$

$$= 2\,\pi.$$

Thus we see that $\oint_C \mathbf{F} \bullet d\mathbf{r} \neq 0$.

Definition: Irrotational vector field: *A vector field* **F** *is said to be irrotational if curl* **F = 0.**

We see that an irrotational field **F** is characterized by any one of the three conditions :

(i) $\mathbf{F} = \nabla \phi$,

(ii) $\nabla \times \mathbf{F} = \mathbf{0}$,

(iii) $\oint_C \mathbf{F} \bullet d\mathbf{r} = 0$ for every closed path.

Any one of these conditions implies the other two.

Illustrative Examples

Example 15: *Are the following forms exact ?*

(i) $e^y\ dx + e^x\ dy + e^z\ dz$.

(ii) $yz\ dx + xz\ dy + xy\ dz$.

Solution:

(i) Here $\mathbf{F} = e^y\ \mathbf{i} + e^x\ \mathbf{j} + e^z\ \mathbf{k}$.

We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^y & e^x & e^z \end{vmatrix}$$

$$= 0\ \mathbf{i} + 0\ \mathbf{j} + (e^x - e^y)\ \mathbf{k}.$$

Since curl $\mathbf{F} \neq \mathbf{0}$, therefore the given form is not exact.

(ii) Here $\mathbf{F} = yz\ \mathbf{i} + xz\ \mathbf{j} + xy\ \mathbf{k}$.

We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= (x - x)\mathbf{i} - (y - y)\mathbf{j} + (z - z)\mathbf{k}$$

$$= \mathbf{0}.$$

Since curl $\mathbf{F} = \mathbf{0}$, therefore the given form is exact.

Example 16: *In each of following cases show that the given differential form is exact and find a function ϕ such that the form equals $d\phi$:*

(i) $\cos x\ dx - 2yz\ dy - y^2\ dz$.

(ii) $(z^2 - 2xy)\ dx - x^2\ dy + 2xz\ dz$.

Solution: (i) Here $\mathbf{F} = \cos x\ \mathbf{i} - 2yz\ \mathbf{j} - y^2\ \mathbf{k}$.

We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x & -2yz & -y^2 \end{vmatrix}$$

$$= (-2y + 2y)\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k} = \mathbf{0}.$$

∴ the given form is exact.

Let $\mathbf{F} = \nabla\phi$,

or $\cos x\,\mathbf{i} - 2yz\,\mathbf{j} - y^2\,\mathbf{k} = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$.

Then

$$\frac{\partial\phi}{\partial x} = \cos x \text{ whence } \phi = \sin x + f_1(y, z) \qquad \ldots(1)$$

$$\frac{\partial\phi}{\partial y} = -2yz \text{ whence } \phi = -y^2 z + f_2(x, z) \qquad \ldots(2)$$

$$\frac{\partial\phi}{\partial z} = -y^2 \text{ whence } \phi = -y^2 z + f_3(x, y). \qquad \ldots(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y, z) = -y^2 z,\ f_2(x, z) = \sin x,\ f_3(x, y) = \sin x.$$

∴ $\phi = \sin x - y^2 z$ to which may be added any constant.

∴ $\phi = \sin x - y^2 z + C$.

(ii) Here $\mathbf{F} = (z^2 - 2xy)\,\mathbf{i} - x^2\,\mathbf{j} + 2xz\,\mathbf{k}$. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 - 2xy & -x^2 & 2xz \end{vmatrix}$$

$$= 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k} = \mathbf{0}.$$

∴ the given form is exact.

Let $\mathbf{F} = \nabla\phi$

or $(z^2 - 2xy)\,\mathbf{i} - x^2\,\mathbf{j} + 2xz\,\mathbf{k} = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$.

Then

$$\frac{\partial\phi}{\partial x} = z^2 - 2xy \text{ whence } \phi = z^2 x - x^2 y + f_1(y, z) \qquad \ldots(1)$$

$$\frac{\partial\phi}{\partial y} = -x^2 \text{ whence } \phi = -x^2 y + f_2(x, z) \qquad \ldots(2)$$

$$\frac{\partial\phi}{\partial z} = 2xz \text{ whence } \phi = xz^2 + f_3(x, y). \qquad \ldots(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y, z) = 0, \quad f_2(x, z) = xz^2, \quad f_3(x, y) = -x^2 y.$$

$\therefore$ $\phi = z^2 x - x^2 y$ to which may be added any constant.

$$\therefore \quad \phi = z^2 x - x^2 y + C.$$

Example 17: *Show that the vector field* **F** *given by*

$$\mathbf{F} = (x^2 - yz)\,\mathbf{i} + (y^2 - zx)\,\mathbf{j} + (z^2 - xy)\,\mathbf{k}$$

is irrotational. Find a scalar ϕ *such that* $\mathbf{F} = \nabla\phi$.

Solution: We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$$= (-x + x)\,\mathbf{i} - (-y + y)\,\mathbf{j} + (-z + z)\,\mathbf{k} = \mathbf{0}.$$

$\therefore$ The vector field **F** is irrotational.

Let $\mathbf{F} = \nabla\phi$

or $(x^2 - yz)\,\mathbf{i} + (y^2 - zx)\,\mathbf{j} + (z^2 - xy)\,\mathbf{k} = \dfrac{\partial\phi}{\partial x}\,\mathbf{i} + \dfrac{\partial\phi}{\partial y}\,\mathbf{j} + \dfrac{\partial\phi}{\partial z}\,\mathbf{k}.$

Then

$$\frac{\partial\phi}{\partial x} = x^2 - yz \text{ whence } \phi = \frac{x^3}{3} - xyz + f_1(y, z) \qquad \ldots (1)$$

$$\frac{\partial\phi}{\partial y} = y^2 - zx \text{ whence } \phi = \frac{y^3}{3} - xyz + f_2(x, z) \qquad \ldots (2)$$

$$\frac{\partial\phi}{\partial z} = z^2 - xy \text{ whence } \phi = \frac{z^3}{3} - xyz + f_3(x, y). \qquad \ldots (3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y, z) = \frac{y^3}{3} + \frac{z^3}{3}, \quad f_2(x, z) = \frac{x^3 + z^3}{3}, \quad f_3(x, y) = \frac{x^3 + y^3}{3}.$$

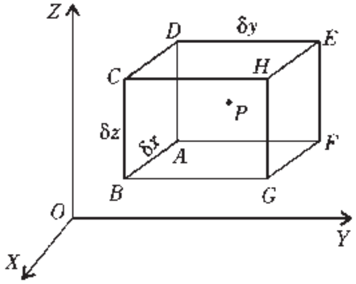
Therefore

$$\phi = \frac{x^3 + y^3 + z^3}{3} - xyz + C.$$

7 Physical Interpretation of Divergence and Curl

Physical interpretation of divergence: *Suppose that there is a fluid motion whose velocity at any point is* **v** (x, y, z). *Then the loss of fluid per unit volume per unit time in a small parallelopiped having centre at* $P(x, y, z)$ *and edges parallel to the co-ordinate axes and having lengths* $\delta x, \delta y, \delta z$ *respectively, is given approximately by*

$$div\ \mathbf{v} = \nabla \bullet \mathbf{v}.$$



Let $\mathbf{v} = v_1\ \mathbf{i} + v_2\ \mathbf{j} + v_3\ \mathbf{k}$.

x-component of velocity $\mathbf{v}$ at $P = v_1\ (x, y, z)$.

$\therefore$ x-component of $\mathbf{v}$ at centre of face $AFED$ which is perpendicular to x-axis and is nearer to origin

$$= v_1 \left(x - \frac{\delta x}{2}, y, z \right)$$

$$= v_1\ (x, y, z) - \frac{\delta x}{2} \frac{\partial v_1}{\partial x} + \ldots$$

by Taylor's theorem

$$= v_1\ (x, y, z) - \frac{\delta x}{2} \frac{\partial v_1}{\partial x} \text{ approximately.}$$

Similarly x-component of $\mathbf{v}$ at centre of opposite face $GHCB$

$$= v_1 + \frac{\delta x}{2} \frac{\partial v_1}{\partial x} \text{ approximately.}$$

$\therefore$ volume of fluid entering the parallelopiped across $AFED$ per unit time

$$= \left(v_1 - \frac{\delta x}{2} \frac{\partial v_1}{\partial x} \right) \delta y\ \delta z.$$

Also volume of fluid going out the parallelopiped across $GHCB$ per unit time

$$= \left(v_1 + \frac{\delta x}{2} \frac{\partial v_1}{\partial x} \right) \delta y\ \delta z.$$

$\therefore$ loss in volume per unit time in the direction of x-axis

$$= \left(v_1 + \frac{\delta x}{2} \frac{\partial v_1}{\partial x} \right) \delta y\ \delta z - \left(v_1 - \frac{\delta x}{2} \frac{\partial v_1}{\partial x} \right) \delta y\ \delta z$$

$$= \frac{\partial v_1}{\partial x}\ \delta x\ \delta y\ \delta z.$$

Similarly, loss in volume per unit time in y direction

$$= \frac{\partial v_2}{\partial y}\ \delta x\ \delta y\ \delta z,$$

and loss in volume per unit time in z direction

$$= \frac{\partial v_3}{\partial z}\ \delta x\ \delta y\ \delta z.$$

$\therefore$ total loss of the fluid per unit volume per unit time symbol ·

$$= \frac{\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right) \delta x\ \delta y\ \delta z}{\delta x\ \delta y\ \delta z}$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \nabla \bullet \mathbf{v} = \text{div } \mathbf{v}.$$

Physical interpretation of curl: Let S be a circular disc of small radius r and centre P bounded by the circle C. Let $\mathbf{F}(x, y, z)$ be a continuously differentiable vector function in S. Then by Stoke's theorem

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \bullet \mathbf{n}\, dS = \overline{(\text{curl } \mathbf{F}) \bullet \mathbf{n}} \iint_S dS,$$

by mean value theorem of integral calculus where $\overline{(\text{curl } \mathbf{F}) \bullet \mathbf{n}}$ is some value intermediate between the maximum and minimum values of $(\text{curl } \mathbf{F}) \bullet \mathbf{n}$ over S.

$$\therefore \qquad \int_C \mathbf{F} \bullet d\mathbf{r} = \overline{(\text{curl } \mathbf{F}) \bullet \mathbf{n}}\ S.$$

$$\therefore \qquad \overline{(\text{curl } \mathbf{F}) \bullet \mathbf{n}} = \frac{\left[\oint_C \mathbf{F} \bullet d\mathbf{r}\right]}{S}.$$

Taking limit as $r \to 0$, we get at P,

$$(\text{curl } \mathbf{F}) \bullet \mathbf{n} = \lim_{r \to 0} \frac{\oint_C \mathbf{F} \bullet d\mathbf{r}}{S}.$$

Now $(\text{curl } \mathbf{F}) \bullet \mathbf{n}$ is normal component of curl $\mathbf{F}$ at P and $\oint_C \mathbf{F} \bullet d\mathbf{r}$ is circulation of $\mathbf{F}$ about C. Therefore the normal component of the curl can be interpreted physically as the limit of the circulation per unit area.

Comprehensive Exercise 4

1. Are the following forms exact ?
   (i) $x\, dx - y\, dy + z\, dz$.
   (ii) $y^2 z^3\, dx + 2xyz^3\, dy + 3xy^2 z^2\, dz$.
2. In each of following cases show that the given differential form is exact and find a function ϕ such that the form equals $d\phi$:
   (i) $x\, dx - y\, dy - z\, dz$.
   (ii) $dx + z\, dy + y\, dz$.
3. (i) Show that
   $(y^2 z^3 \cos x - 4x^3 z)\, dx + 2z^3 y \sin x\, dy + (3y^2 z^2 \sin x - x^4)\, dz$
   is an exact differential of some function ϕ and find this function.
   (ii) Show that $\mathbf{F} = (2xy + z^3)\,\mathbf{i} + x^2\,\mathbf{j} + 3xz^2\,\mathbf{k}$ is a conservative force field. Find the scalar potential. Find also the work done in moving an object in this field from (1, – 2, 1) to (3, 1, 4).

4. (i) Show that the vector field

$$\mathbf{F} = (2xy^2 + yz)\,\mathbf{i} + (2x^2y + xz + 2yz^2)\,\mathbf{j} + (2y^2z + xy)\,\mathbf{k}$$

is conservative.

(ii) Show that $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is conservative and find ϕ such that $\mathbf{F} = \nabla\phi$.

5. (i) Show that $\mathbf{F} = (\sin y + z)\,\mathbf{i} + (x \cos y - z)\,\mathbf{j} + (x - y)\,\mathbf{k}$ is a conservative vector field and find a function ϕ such that $\mathbf{F} = \nabla\phi$.

(ii) Evaluate $\int_C 2xyz^2\, dx + (x^2z^2 + z\cos yz)\, dy + (2x^2yz + y\cos yz)\, dz$ where C is any path from $(0, 0, 1)$ to $(1, \pi/4, 2)$.

6. Show that the following vector functions $\mathbf{F}$ are irrotational and find the corresponding scalar ϕ such that $\mathbf{F} = \nabla\phi$.

(i) $\mathbf{F} = (\sin y + z\cos x)\,\mathbf{i} + (x\cos y + \sin z)\,\mathbf{j} + (y\cos z + \sin x)\,\mathbf{k}$.

(ii) $\mathbf{F} = (y\sin z - \sin x)\,\mathbf{i} + (x\sin z + 2yz)\,\mathbf{j} + (xy\cos z + y^2)\,\mathbf{k}$.

(iii) $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$.

7. Find a, b, c if $\mathbf{F} = (3x - 3y + az)\,\mathbf{i} + (bx + 2y - 4z)\,\mathbf{j} + (2x + cy + z)\,\mathbf{k}$ is irrotational.

8. Evaluate $\int_C yz\, dx + (xz + 1)\, dy + xy\, dz$, where C is any path from $(1, 0, 0)$ to $(2, 1, 4)$.

9. Show that the form under the integral sign is exact and evaluate

$$\int_{(0,2,1)}^{(2,0,1)} [ze^x\, dx + 3yz\, dy + (e^x + y^2)\, dz].$$

Answers 4

1. (i) Exact (ii) Exact

2. (i) Exact; $\phi = \dfrac{x^2 - y^2 - z^2}{2} + C$

(ii) Exact; $\phi = x + yz + C$

3. (i) $\phi = y^2z^3 \sin x - x^4z + C$

(ii) $\phi = x^2y - xz^3 + C; 202$

4. (ii) $\phi = \dfrac{1}{2}(x^2 + y^2 + z^2) + C.$

5. (i) $\phi = x\sin y + xz - yz + C.$

(ii) $\pi + 1$

6. (i) $\phi = x \sin y + z \sin x + y \sin z + C$

(ii) $\phi = xy \sin z + \cos x + y^2 z + C$

(iii) $\phi = \frac{1}{4}(x^4 + y^4 + z^4) + C$

7. $a = 2, \ b = -3, c = -4.$

8. 9 **9.** $e^2 - 5$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If C is the curve $x^2 + y^2 = 1, z = y^2$, then by Stoke's theorem

$$\oint_C (yz\, dx + zx\, dy + xy\, dz) \text{ is}$$

(a) 0 (b) 3

(c) 5 (d) None of these

2. If S denotes the surface of the cube bounded by the planes $x = 0$, $x = a$, $y = 0$, $y = a$, $z = 0, z = a$ then by the application of Gauss divergence theorem the value of $\iint_S (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \bullet \mathbf{n}\, dS$ is

(a) a^3 (b) $2a^3$

(c) $3a^3$ (d) 0

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. For any closed surface $S, \iint_S \text{curl } \mathbf{F} \bullet \mathbf{n}\, dS =$ …….

2. If $\mathbf{n}$ is the unit outward drawn normal to any closed surface S, then

$$\iiint_V \text{div } \mathbf{n}\, dV = \ldots\ldots.$$

(Bundelkhand 2008)

3. The value of $\oint_C \mathbf{r} \bullet d\,\mathbf{r} =$ …….

(Agra 2008)

4. A necessary and sufficient condition that $\oint_C \mathbf{F} \bullet d\,\mathbf{r} = 0$ for every closed curve C lying in a simply connected region R is that $\nabla \times \mathbf{F} =$ …… identically.

5. By Stoke's theorem, $\oint_C \mathbf{F} \bullet d\,\mathbf{r} = \ldots\ldots$

6. $\iint_S \mathbf{r} \bullet \mathbf{n}\, dS = \ldots\ldots$

(Kumaun 2009)

True or False

Write 'T' for true and 'F' for false statement.

1. $\iint_S \mathbf{n}\, dS = \mathbf{0}$ for any surface S.
2. Green's theorem in plane is a special case of Stoke's theorem.
3. Green's theorem states that "the surface integral of the normal component of a vector **F** taken over a closed surface is equal to the integral of the divergence of **F** taken over the volume enclosed by the surface".

Answers

Multiple Choice Questions

1. (a)   2. (c)

Fill in the Blank(s)

1. 0   2. S   3. 0   4. 0

5. $\iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, dS$ 6. $3V$

True or False

1. *F*   2. *T*   3. *F*

☐