Krishna's

TEXT BOOK on

Functions of Several Variables and Partial Differential Equations

(For B.A. and B.Sc. Vth Semester students of Kumaun University)

Kumaun University Semester Syllabus *w.e.f.* 2018-19

By

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Dedicated

to

Lord

Krishna

Authors & Publishers

Preface

This book on **Functions of Several Variables and Partial Differential Equations** has been specially written according to the latest **Syllabus** to meet the requirements of **B.A. and B.Sc. Semester-V Students** of all colleges affiliated to Kumaun University.

The subject matter has been discussed in such a simple way that the students will find no difficulty to understand it. The proofs of various theorems and examples have been given with minute details. Each chapter of this book contains complete theory and a fairly large number of solved examples. Sufficient problems have also been selected from various university examination papers. At the end of each chapter an exercise containing objective questions has been given.

We have tried our best to keep the book free from misprints. The authors shall be grateful to the readers who point out errors and omissions which, inspite of all care, might have been there.

The authors, in general, hope that the present book will be warmly received by the students and teachers. We shall indeed be very thankful to our colleagues for their recommending this book to their students.

The authors wish to express their thanks to Mr. S.K. Rastogi, *M.D.*, Mr. Sugam Rastogi, *Executive Director*, Mrs. Kanupriya Rastogi, *Director* and entire team of **KRISHNA Prakashan Media (P) Ltd., Meerut** for bringing out this book in the present nice form.

The authors will feel amply rewarded if the book serves the purpose for which it is meant. Suggestions for the improvement of the book are always welcome.

— Authors

Syllabus

Functions of Several Variables and Partial Differential Equations

B.A./B.Sc. V Semester

Kumaun University

Fifth Semester – Third Paper

B.A./B.Sc. Paper-III M.M.-60

Functions of several variables: Limit, continuity and differentiability of functions of several variables.

Partial Derivatives: Partial derivatives and their geometrical interpretation, differentials, derivatives of composite and implicit functions, Jacobians, Chain rule, Euler's theorem on homogeneous functions, harmonic functions, Taylor's expansion of functions of several variables.

Maxima and Minima: Maxima and minima of functions of several variables – Lagrange's method of multipliers.

Partial differential equations: Partial differential equations of first order, Charpit's method, Linear partial differential equations with constant coefficients. First-order linear, quasi-linear PDE's using the method of characteristics.

Partial differential equations of 2nd-order: Classification of 2nd-order linear equations in two independent variables: hyperbolic, parabolic and elliptic types (with examples).

Brief Contents

Krishna's

FUNCTIONS OF SEVERAL VARIABLES AND PARTIAL DIFFERENTIAL EQUATIONS

Chapters

4. Limit and Continuity of Functions of Several Variables

5. Partial Derivation and Differentiability of Functions of Several Variables

6. Partial Differential Equations of the First Order

7. Linear Partial Differential Equations with Constant Coefficients

Chapter

1

Partial Differentiation

1 Functions of Several Independent Variables

So far we have considered functions of one independent variable only. However, in practice, we often come across functions of more than one independent variable. For example, the area of a rectangle depends upon two independent variables, namely the length and the breadth. Similarly, the volume of a rectangular parallelopiped depends upon three independent variables, namely the length, the breadth and the height.

There are a number of differences between the calculus of one and of two variables. Fortunately the calculus of functions of three or more variable differs only slightly from that of functions of two variables. The study here will be limited largely to functions of two variables.

Definition: *Let z be a symbol which has one definite value for every pair of values of x and y. Then z is called a function of the two independent variables x and y, and is usually written as $z = f(x, y)$. A function of x and y is also written as $\phi(x, y)$ or $\psi(x, y)$ etc.*

(Kanpur 2014)

A similar definition can be given for functions of more than two independent variables.

If to each point (x, y), of a part of the xy-plane is assigned a unique real number z, even then z is said to be given as a function, $z = f(x, y)$, of the independent variables x and y. The locus of all points (x, y, z) satisfying $z = f(x, y)$ is a surface in ordinary space.

2 Continuity of a Function of Two Variables

A function $f(x, y)$ is said to have a limit A as $x \to a$ and $y \to b$ if for any arbitrarily chosen positive number ε, however small (but not zero), there exists a corresponding number $\delta > 0$ such that

$$|f(x, y) - A| < \varepsilon,$$

for all values of x and y satisfying $0 < \sqrt{\{(x-a)^2 + (y-b)^2\}} < \delta$.

Here $0 < \sqrt{\{(x-a)^2 + (y-b)^2\}} < \delta$ defines a deleted neighbourhood of (a, b), namely all points except (a, b) lying within a circle of radius δ and centre (a, b).

A function $f(x, y)$ is said to be continuous at (a, b) provided $f(a, b)$ is defined and

$$\lim_{x \to a,\, y \to b} f(x, y) = f(a, b).$$

3 Partial Differential Coefficients

Suppose $z = f(x, y)$ is a function of two independent variables x and y. Since x and y are independent, we may (i) allow x to vary while y is kept fixed, (ii) allow y to vary while x is kept fixed, (*iii*) allow x and y to vary simultaneously. In the first two cases, z is practically a function of a single variable and we can differentiate it in accordance with the usual rules.

The partial differential coefficient of $z = f(x, y)$ with respect to x is the ordinary differential coefficient of $f(x, y)$ with respect to x when y is regarded as a constant. It is usually written as

$$\frac{\partial f}{\partial x} \text{ or } \frac{\partial z}{\partial x} \text{ or } f_x.$$

Thus, $$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x},$$ provided the limit exists.

Similarly, the partial differential coefficient of $z = f(x, y)$ with respect to y is the ordinary differential coefficient of $f(x, y)$ with respect to y when x is kept as constant. It is written as

$$\frac{\partial f}{\partial y} \text{ or } \frac{\partial z}{\partial y} \text{ or } f_y.$$

In a similar manner, if $z = f(x_1, x_2, \ldots, x_n)$ be a function of n independent variables $x_1, x_2, \ldots, x_n$, then the partial differential coefficient of z with respect to x_1, is the ordinary differential coefficient of z with respect to x_1, when all the variables except x_1 are regarded as constants.

We shall write it as $\dfrac{\partial z}{\partial x_1}$ or $\dfrac{\partial f}{\partial x_1}$.

4 Partial Differential Coefficients of Second Order

The partial differential coefficient $\dfrac{\partial z}{\partial x}$ of $z = f(x, y)$ may again be differentiated partially with respect to x and to y, thus giving the second partial differential coefficients

$$\frac{\partial^2 z}{\partial x^2} = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \quad \text{and} \quad \frac{\partial^2 z}{\partial y\, \partial x} = f_{yx} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right).$$

Similarly, from $\dfrac{\partial z}{\partial y}$ may be obtained

$$\frac{\partial^2 z}{\partial x\, \partial y} = f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right).$$

If $z = f(x, y)$ and its partial derivatives are continuous (as is true in all ordinary cases), the order of differentiation is immaterial, that is,

$$\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}.$$

Illustrative Examples

Example 1: *If* $u = ax^2 + 2hxy + by^2$, *find* $\dfrac{\partial^2 u}{\partial y\, \partial x}$ *and* $\dfrac{\partial^2 u}{\partial x\, \partial y}$.

Solution: We have, $u = ax^2 + 2hxy + by^2$.

$\therefore$ $$\frac{\partial u}{\partial x} = 2ax + 2hy. \qquad [\text{Treating } y \text{ as constant}]$$

Hence, $$\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(2ax + 2hy) = 2h.$$

[Treating x as constant]

Again $$\frac{\partial u}{\partial y} = 2hx + 2by. \qquad [\text{Treating } x \text{ as constant}]$$

$\therefore$ $$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(2hx + 2by) = 2h.$$

[Treating y as constant)]

Here we note that $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{\partial^2 u}{\partial y\, \partial x}$.

Example 2: *If* $u = f(y/x)$, *show that* $x(\partial u/\partial x) + y(\partial u/\partial y) = 0$.

(Agra 2003; Garhwal 07)

Solution: We have $u = f(y/x)$.

$\partial u / \partial x = [f'(y/x)](-y/x^2),$ [Treating y as constant]

$\therefore$ $x(\partial u/\partial x) = -(y/x) f'(y/x).$ ...(1)

Again $\partial u / \partial y = [f'(y/x)] . (1/x),$ [Treating x as constant]

$\therefore$ $y(\partial u/\partial y) = (y/x) f'(y/x).$ ...(2)

Adding (1) and (2), we get

$$x(\partial u/\partial x) + y(\partial u/\partial y) = 0.$$

Example 3: *If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that*

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z} \quad \textit{and} \quad \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}.$$

(Kanpur 2007; Purvanchal 07; Garhwal 08, 11; Rohilkhand 12; Avadh 13; Kashi 14)

Solution: We have $u = \log(x^3 + y^3 + z^3 - 3xyz)$.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}, \quad \frac{\partial u}{\partial y} = \frac{3y^2 - 3zx}{x^3 + y^3 + z^3 - 3xyz}$$

and
$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}.$$

$$\therefore \quad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - yz - zx - xy)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - yz - zx - xy)}{(x+y+z)(x^2 + y^2 + z^2 - yz - zx - xy)} = \frac{3}{x+y+z}. \quad \text{...(1)}$$

Now ,
$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right), \quad \text{[From (1)]}$$

$$= 3\left[\frac{\partial}{\partial x}\left(\frac{1}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{1}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{1}{x+y+z}\right)\right]$$

$$= 3\left[\frac{-1}{(x+y+z)^2} + \frac{-1}{(x+y+z)^2} + \frac{-1}{(x+y+z)^2}\right] = \frac{-9}{(x+y+z)^2}.$$

Example 4: *If $x = r\cos\theta$, $y = r\sin\theta$, show that*

$$\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}, \quad \frac{\partial x}{r\,\partial\theta} = r\frac{\partial\theta}{\partial x}, \quad \textit{and} \quad \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0.$$

Also find the value of $\dfrac{\partial\theta}{\partial x}$.

(Garhwal 2002)

Solution : We have $x = r\cos\theta$.

$$\therefore \quad \frac{\partial x}{\partial r} = \cos\theta. \quad \text{[Regarding } \theta \text{ as constant]}$$

Also we have, $r^2 = x^2 + y^2$.

$\therefore$ $$2r\frac{\partial r}{\partial x} = 2x \qquad \text{[Regarding } y \text{ as constant]}$$

or $$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta.$$

Thus $$\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}.$$

Again, $$\frac{\partial x}{\partial \theta} = -r\sin\theta. \qquad \text{[Regarding } r \text{ as constant]}$$

$\therefore$ $$\frac{1}{r}\frac{\partial x}{\partial \theta} = -\sin\theta.$$

Also we have, $\theta = \tan^{-1}(y/x)$.

$\therefore$ $$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}.$$

$\therefore$ $$r\frac{\partial \theta}{\partial x} = -\sin\theta.$$

Hence $$\frac{\partial x}{r\,\partial \theta} = r\frac{\partial \theta}{\partial x}.$$

Finally, we have $\theta = \tan^{-1}(y/x)$.

$\therefore$ $$\frac{\partial \theta}{\partial x} = -\frac{y}{(x^2+y^2)}$$

and $$\frac{\partial^2 \theta}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2}. \qquad \ldots(1)$$

Also $$\frac{\partial \theta}{\partial y} = \frac{1}{\left(1+\frac{y^2}{x^2}\right)}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

and $$\frac{\partial^2 \theta}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2}. \qquad \ldots(2)$$

Adding (1) and (2), we get
$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0.$$

Example 5: *If* $u = (1-2xy+y^2)^{-1/2}$, *prove that*
$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\} = 0.$$

Solution: Here $u = (1-2xy+y^2)^{-1/2}$.

$\therefore$ $$\frac{\partial u}{\partial x} = -\frac{1}{2}(1-2xy+y^2)^{-3/2}(-2y) = yu^3,$$

and $$\frac{\partial u}{\partial y}=-\frac{1}{2}(1-2xy+y^2)^{-3/2}\,.(-2x+2y)=(x-y)u^3.$$

Now, $$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\}=\frac{\partial}{\partial x}\{(1-x^2)\,.\,yu^3\}$$

$$=y(-2x)u^3+y(1-x^2)\,.\,3u^2\frac{\partial u}{\partial x}$$

$$=-2xy\,u^3+3y(1-x^2)u^2\,.\,yu^3$$

$$=-2xyu^3+3y^2u^5(1-x^2). \qquad \text{...(1)}$$

Also $$\frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\}=\frac{\partial}{\partial y}\{y^2(x-y)u^3\}=\frac{\partial}{\partial y}\{(y^2x-y^3)u^3\}$$

$$=(2xy-3y^2)u^3+(y^2x-y^3)\,.\,3u^2\frac{\partial u}{\partial y}$$

$$=(2xy-3y^2)u^3+y^2(x-y)\,.\,3u^2\,.\,(x-y)u^3$$

$$=(2xy-3y^2)u^3+y^2(x-y)^2\,.\,3u^5$$

$$=2xyu^3+3y^2u^5[(x-y)^2-u^{-2}]$$

$$=2xyu^3+3y^2u^5[(x-y)^2-(1-2xy+y^2)], \qquad [\because u^{-2}=1-2xy+y^2]$$

$$=2xyu^3+3y^2u^5[x^2-1]=2xyu^3-3y^2u^5(1-x^2). \qquad \text{...(2)}$$

Adding (1) and (2), we have

$$\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial u}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{y^2\frac{\partial u}{\partial y}\right\}=0.$$

Example 6: *If* $\theta=t^n e^{-r^2/4t}$, *what value of n will make* $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}$?

(Garhwal 2003; Lucknow 11; Meerut12B)

Solution: We have $$\frac{\partial\theta}{\partial r}=t^n\,.\,e^{-r^2/4t}\cdot\left(-\frac{2r}{4t}\right)=-\frac{r}{2}t^{n-1}e^{-r^2/4t}.$$

$\therefore$ $$r^2\frac{\partial\theta}{\partial r}=-\frac{1}{2}r^3\,t^{n-1}e^{-r^2/4t}.$$

$\therefore$ $$\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=-\frac{3r^2}{2}t^{n-1}e^{-r^2/4t}-\frac{1}{2}r^3\,t^{n-1}e^{-r^2/4t}\cdot\left(-\frac{2r}{4t}\right)$$

$$=-\frac{3}{2}r^2\,t^{n-1}e^{-r^2/4t}+\frac{1}{4}r^4\,t^{n-2}e^{-r^2/4t}.$$

$\therefore$ $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=-\frac{3}{2}t^{n-1}e^{-r^2/4t}+\frac{1}{4}r^2\,t^{n-2}e^{-r^2/4t}.$$

Also $$\frac{\partial\theta}{\partial t}=nt^{n-1}e^{-r^2/4t}+t^n e^{-r^2/4t}\cdot\frac{r^2}{4t^2}$$

$$=nt^{n-1}e^{-r^2/4t}+\frac{1}{4}r^2\,t^{n-2}e^{-r^2/4t}.$$

Now $$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}$$

$$\Rightarrow \quad -\frac{3}{2}t^{n-1}e^{-r^2/4t}+\frac{1}{4}r^2t^{n-2}e^{-r^2/4t}=n\,t^{n-1}e^{-r^2/4t}+\frac{1}{4}r^2t^{n-2}e^{-r^2/4t}$$

$$\Rightarrow \quad -\frac{3}{2}t^{n-1}e^{-r^2/4t}=n\,t^{n-1}e^{-r^2/4t},$$ for all possible values of r and t

$$\Rightarrow \quad n=-\frac{3}{2}\cdot$$

Comprehensive Exercise 1

1. Find $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ when $u=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1.$

2. Prove that $\dfrac{\partial^2 u}{\partial x\,\partial y}=\dfrac{\partial^2 u}{\partial y\,\partial x}$ in each of the following cases :

   (i) $u=x^4+x^2y^2+y^4,$ (ii) $u=\log\tan(y/x),$

   (iii) $u=\log\left\{\dfrac{x^2+y^2}{x+y}\right\},$ (iv) $u=x^y.$

3. If $u=\sin^{-1}\dfrac{x}{y}+\tan^{-1}\dfrac{y}{x}$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=0.$

4. If $u=xyf\left(\dfrac{y}{x}\right)$, then write the value of the expression $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}\cdot$

   (Meerut 2001; Kanpur 07)

5. If $z=f(x+ay)+\phi(x-ay)$, prove that $\partial^2 z/\partial y^2=a^2(\partial^2 z/\partial x^2)$.

   (Bundelkhand 2001; Kanpur 05; Meerut 2013B)

6. If $u=f(r)$, where $r^2=x^2+y^2$, show that
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r).$$
   (Meerut 2001; Agra 01; Avadh 04)

7. If $z=x^2\tan^{-1}\dfrac{y}{x}-y^2\tan^{-1}\dfrac{x}{y}$, prove that $\dfrac{\partial^2 z}{\partial x\,\partial y}=\dfrac{x^2-y^2}{x^2+y^2}\cdot$

8. (i) If $u=2(ax+by)^2-(x^2+y^2)$ and $a^2+b^2=1$, prove that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0.$

   (ii) If $u=e^x(x\cos y-y\sin y)$, then show that $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0.$

   (Garhwal 2003)

9. If $z = (x^2 + y^2) / (x + y)$, show that $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$.

(Kumaun 2000; Bundelkhand 11; Kashi 12, 13; Kanpur 11; Avadh 14)

10. If $u = e^{xyz}$, show that $\partial^3 u / \partial x\, \partial y\, \partial z = (1 + 3xyz + x^2 y^2 z^2)\, e^{xyz}$.

(Kumaun 2001; Kashi 12; Kanpur 11; Rohilkhand 13)

11. If $x^x y^y z^z = c$, show that at $x = y = z$, $\partial^2 z / \partial x \partial y = -\{x \log (ex)\}^{-1}$.

(Garhwal 2009; Rohilkhand 13; Bundelkhand 14; Purvanchal 14)

12. Show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ when

(i) $u = e^{my} \cos mx$. **(Agra 2014)**

(ii) $u = \log (x^2 + y^2)$ **(Agra 2014)**

(iii) $u = \tan^{-1} (y/x)$

13. If $V = (x^2 + y^2 + z^2)^{-1/2}$, show that

(i) $x \dfrac{\partial V}{\partial x} + y \dfrac{\partial V}{\partial y} + z \dfrac{\partial V}{\partial z} = -V$

(ii) $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$.

(Kumaun 2008)

14. If $u = \tan^{-1} \dfrac{xy}{\sqrt{(1 + x^2 + y^2)}}$, show that $\dfrac{\partial^2 u}{\partial x\, \partial y} = \dfrac{1}{(1 + x^2 + y^2)^{3/2}}$.

15. If $\dfrac{x^2}{a^2 + u} + \dfrac{y^2}{b^2 + u} + \dfrac{z^2}{c^2 + u} = 1$, prove that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}\right).$$

(Garhwal 2004; Rohilkhand 11B, 12B)

16. If $x = r \cos \theta$, $y = r \sin \theta$, prove that

(i) $(\partial r / \partial x)^2 + (\partial r / \partial y)^2 = 1$

(ii) $\dfrac{\partial^2 r}{\partial x^2} + \dfrac{\partial^2 r}{\partial y^2} = \dfrac{1}{r}\left[\left(\dfrac{\partial r}{\partial x}\right)^2 + \left(\dfrac{\partial r}{\partial y}\right)^2\right]$.

(Lucknow 2007, 11; Garhwal 11)

17. (i) If $u = x^2 y + y^2 z + z^2 x$, show that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = (x + y + z)^2$.

(ii) If $u = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$, show that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

(Rohilkhand 2013B)

Answers 1

1. $\dfrac{2x}{a^2}, \dfrac{2y}{b^2}$

4. $2u$

5 Homogeneous Functions

(Gorakhpur 2006)

An expression in x and y in which every term is of the same degree is called a *homogeneous function* of x and y. Consider the function defined by

$$f(x, y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots.. + a_{n-1} xy^{n-1} + a_n y^n. \quad \ldots(1)$$

In this function every term is of degree n. Therefore it is a homogeneous function of x and y of degree n. Moreover, (1) may be written as

$$f(x, y) = x^n \left\{a_0 + a_1 \left(\frac{y}{x}\right) + a_2 \left(\frac{y}{x}\right)^2 + \ldots + a_n \left(\frac{y}{x}\right)^n\right\}$$

or $$f(x, y) = x^n F(y/x),$$

where $F(y/x)$ is some function of y/x. Thus a homogeneous function of x and y of degree n may be put in the form $x^n F(y/x)$. Therefore we give the general definition of a homogeneous function as follows :

$x^n F(y/x)$ is called a *homogeneous function of* x and y of degree n, whatever the function F may be. Similarly, $y^n F(x/y)$ is also a *homogeneous function* of x and y of degree n.

Thus $x^3 \sin(y/x)$ is a homogeneous function of x and y of degree 3. Similarly, $y^2 \cos(x/y)$ is a homogeneous function of x and y of degree 2.

In general, if the function $f(x_1, x_2, \ldots, x_p)$ of the p variables $x_1, x_2, \ldots, x_p$ can be put in the form

$$x_r^n F\left(\frac{x_1}{x_r}, \frac{x_2}{x_r}, \ldots, \frac{x_p}{x_r}\right),$$

then $f(x_1, x_2, \ldots, x_p)$ is called a homogeneous function of $x_1, x_2, \ldots, x_p$ of degree n.

Note 1: **To test whether a given function $f(x, y)$ is homogeneous or not we put tx for x and ty for y in it.**

If we get $$f(tx, ty) = t^n f(x, y),$$

the function $f(x, y)$ is homogeneous of degree n; otherwise $f(x, y)$ is not a homogeneous function.

Note 2: **If u is a homogeneous function of x and y of degree n then $\partial u / \partial x$ and $\partial u / \partial y$ are also homogeneous functions of x and y each being of degree $n - 1$.**

Let $\quad u = x^n f\,(y/x).$

[$\because$ u is a homogeneous function of x and y of degree n]

Then $$\frac{\partial u}{\partial x} = nx^{n-1} f\,(y/x) + x^n\{f'(y/x)\}\,.\,(-y/x^2)$$
$$= x^{n-1}[n\,f\,(y/x) - (y/x)\,f'(y/x)]$$
$$= x^{n-1}.\ \text{(some function of } y/x)$$
$$= \text{a homogeneous function of } x \text{ and } y \text{ of degree } (n-1).$$

Similarly, $$\frac{\partial u}{\partial y} = x^n\{f'(y/x)\}\cdot\frac{1}{x} = x^{n-1} f'(y/x)$$
$$= x^{n-1}.\ \text{(some function of } y/x)$$
$$= \text{a homogeneous function of } x \text{ and } y \text{ of degree } (n-1).$$

6 Euler's Theorem on Homogeneous Functions

If u is a homogeneous function of x and y of degree n , then $x\,\dfrac{\partial u}{\partial x} + y\,\dfrac{\partial u}{\partial y} = nu.$

(Meerut 2000; Garhwal 06; Gorakhpur 06; Kashi 11, 13, 14)

Proof: Since u is a homogeneous function of x and y of degree n, therefore u may be put in the form

$$u = x^n f\,(y/x). \qquad ...(1)$$

Differentiating (1) partially w.r.t. 'x', we have

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[x^n f\,(y/x)]$$
$$= [f\,(y/x)]\,nx^{n-1} + x^n[f'(y/x)](-y/x^2).$$

$\therefore$ $$x\,\frac{\partial u}{\partial x} = nx^n f\,(y/x) - x^{n-1} y\,.\,f'(y/x). \qquad ...(2)$$

Again differentiating (1) partially w.r.t. 'y', we have

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[x^n f\,(y/x)] = x^n[f'(y/x)]\cdot\frac{1}{x} = x^{n-1} f'(y/x).$$

$\therefore$ $$y\,\frac{\partial u}{\partial y} = y\,.\,x^{n-1} f'(y/x). \qquad ...(3)$$

Adding (2) and (3), we have

$$x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} = nx^n f\,(y/x) = nu. \qquad \text{[From (1)]}$$

Hence $$\boldsymbol{x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} = nu}\,.$$

This proves the theorem.

Note: **Euler's theorem can be extended to a homogeneous function of any number of variables.** *Thus if $f(x_1, x_2, \ldots, x_n)$ be a homogeneous function of $x_1, x_2, \ldots, x_n$ of degree n, then*

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \ldots + x_n \frac{\partial f}{\partial x_n} = nf.$$

This proof is similar to that of two variables.

Illustrative Examples

Example 7: *Verify Euler's theorem for the function* $u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$.

(Meerut 2012B; Rohilkhand 14)

Solution: Here u is a homogeneous function of x and y of degree

$$\frac{1}{4} - \frac{1}{5} \text{ i.e., } \frac{1}{20}.$$

Therefore in order to verify Euler's theorem we are to show that

$$x\,(\partial u / \partial x) + y\,(\partial u / \partial y) = \frac{1}{20} u.$$

We have $\quad \log u = \log (x^{1/4} + y^{1/4}) - \log (x^{1/5} + y^{1/5})$. ...(1)

Differentiating (1) partially with respect to x, we have

$$\frac{1}{u} \frac{\partial u}{\partial x} = \frac{1}{x^{1/4} + y^{1/4}} \cdot \left(\frac{1}{4} x^{-3/4} \right) - \frac{1}{x^{1/5} + y^{1/5}} \cdot \left(\frac{1}{5} x^{-4/5} \right)$$

$$\therefore \quad \frac{\partial u}{\partial x} = u \left[\frac{1}{4} \frac{x^{-3/4}}{x^{1/4} + y^{1/4}} - \frac{1}{5} \frac{x^{-4/5}}{x^{1/5} + y^{1/5}} \right].$$

$$\therefore \quad x \frac{\partial u}{\partial x} = u \left[\frac{1}{4} \frac{x^{1/4}}{x^{1/4} + y^{1/4}} - \frac{1}{5} \frac{x^{1/5}}{x^{1/5} + y^{1/5}} \right]. \quad \text{...(2)}$$

Again differentiating (1) partially with respect to y, we get

$$\frac{1}{u} \frac{\partial u}{\partial y} = \left[\frac{1}{4} \frac{y^{-3/4}}{x^{1/4} + y^{1/4}} - \frac{1}{5} \frac{y^{-4/5}}{x^{1/5} + y^{1/5}} \right].$$

$$\therefore \quad y \frac{\partial u}{\partial y} = u \left[\frac{1}{4} \frac{y^{1/4}}{x^{1/4} + y^{1/4}} - \frac{1}{5} \frac{y^{1/5}}{x^{1/5} + y^{1/5}} \right]. \quad \text{...(3)}$$

Adding (2) and (3), we get

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u \left[\frac{1}{4} \frac{x^{1/4} + y^{1/4}}{x^{1/4} + y^{1/4}} - \frac{1}{5} \frac{x^{1/5} + y^{1/5}}{x^{1/5} + y^{1/5}} \right]$$

$$= u \left[\frac{1}{4} - \frac{1}{5} \right] = \frac{1}{20} u.$$

This verifies Euler's theorem.

Example 8: *If* $u = \sin^{-1}\{(x^2 + y^2) / (x + y)\}$, *show that*

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \tan u.$$

(Garhwal 2002; Kumaun 08; Avadh 12)

Solution: We have,

$$\sin u = (x^2 + y^2) / (x + y).$$

Let $\quad v = (x^2 + y^2) / (x + y).$

Then v is a homogeneous function of x and y of degree $2 - 1$ *i.e.*, 1. Therefore by Euler's theorem, we have

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = 1 . v = v. \qquad \text{...(1)}$$

Now $\quad v = \sin u.$

$\therefore \quad \dfrac{\partial v}{\partial x} = \cos u \dfrac{\partial u}{\partial x} \quad$ and $\quad \dfrac{\partial v}{\partial y} = \cos u \dfrac{\partial u}{\partial y} \cdot$

Putting these values in (1), we get

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = v$$

or $\quad \left(x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} \right) = \dfrac{v}{\cos u} = \dfrac{\sin u}{\cos u} = \tan u. \quad \left[\because \; v = \sin u = \dfrac{x^2 + y^2}{x + y} \right]$

This proves the result.

Comprehensive Exercise 2

1. State Euler's theorem on homogeneous functions. **(Bundelkhand 2001)**
2. Verify Euler's theorem in the following cases :

   (i) $u = ax^2 + 2hxy + by^2$ $\qquad$ (ii) $u = \dfrac{x(x^3 - y^3)}{x^3 + y^3}$,

   (iii) $u = axy + byz + czx$, $\qquad$ (iv) $u = x^n \sin(y/x)$

   (v) $u = x^n \log(y/x)$, $\qquad$ (vi) $u = 1/\sqrt{(x^2 + y^2)}$.

3. (i) If $u = \tan^{-1}\left(\dfrac{x^3 + y^3}{x - y}\right)$, show that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = \sin 2u.$

   (Rohilkhand 2012B)

   (ii) If $u = \sin^{-1}\left\{\dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right\}$, show that $\dfrac{\partial u}{\partial x} = -\dfrac{y}{x}\dfrac{\partial u}{\partial y} \cdot$

   (Garhwal 2002; Gorakhpur 05)

4. (i) If $u = \sin^{-1}\left(\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right)$, show that $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} = \dfrac{1}{2} \tan u.$

(ii) If $u = \cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \dfrac{1}{2}\cot u = 0.$

(Rohilkhand 2013B)

(iii) If $u = \sin^{-1}\left(\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)$, show that $x\left(\dfrac{\partial u}{\partial x}\right) + y\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{1}{12}\tan u.$

(Garhwal 2014)

(iv) If $u = \cot^{-1}\dfrac{x+y}{x^{1/2}+y^{1/2}}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{4}\sin 2u.$

(Kumaun 2015)

5. (i) If $u = \log\dfrac{x^3+y^3}{x+y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2.$

(ii) If $u = \log\dfrac{x^4+y^4}{x+y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3.$

(iii) If $u = \log\dfrac{x^2+y^2}{x+y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1.$

(Kanpur 2006; Avadh 13)

6. Use Euler's theorem on homogeneous functions to show that if $u = \tan^{-1}\dfrac{y}{x}$, then $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0.$

7. If u be a homogeneous function of x and y of degree n, show that

(i) $x^2\dfrac{\partial^2 u}{\partial x^2} + y\dfrac{\partial^2 u}{\partial x\,\partial y} = (n-1)\dfrac{\partial u}{\partial x}$

(ii) $x\dfrac{\partial^2 u}{\partial x\,\partial y} + y\dfrac{\partial^2 u}{\partial y^2} = (n-1)\dfrac{\partial u}{\partial y}\cdot$

(iii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = n(n-1)u.$

8. If $u = \dfrac{xy}{x+y}$, show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0.$

9. If $u = x\,\phi(y/x) + \psi(y/x)$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\,\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 0.$

(Kanpur 2008)

7 Composite Functions and Total Differential Coefficient

(i) If $u = f(x, y)$ where $x = \phi_1(t)$ and $y = \phi_2(t)$, then x and y are not independent variables. Substituting the values of x and y in u, we can express u as a function of the single variable t and we can find the ordinary differential coefficient du/dt. Here u is called a composite function of the single variable t.

(ii) If $z = f(x, y)$ where $x = \phi(u, v)$ and $y = \psi(u, v)$, then z is called a composite function of two variables u and v and we can find the partial differential coefficients

$$\frac{\partial z}{\partial u} \text{ and } \frac{\partial z}{\partial v}.$$

To distinguish du / dt from the partial differential coefficients $\partial u / \partial x$ and $\partial u / \partial y$, we call du / dt as the *total differential coefficient*.

8 Chain Rule for Differential of Composite Functions

We shall now obtain a formula which will enable us to find du / dt without first expressing u in terms of t only.

Let $u = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$. Suppose $\delta x, \delta y$ and δu are the increments in x, y and u respectively corresponding to an increment δt in t.

Then $\quad u + \delta u = f(x + \delta x, y + \delta y).$

$\therefore \quad \delta u = f(x + \delta x, y + \delta y) - f(x, y).$

$$\therefore \quad \frac{\delta u}{\delta t} = \frac{f(x + \delta x, y + \delta y) - f(x, y)}{\delta t}$$

$$= \frac{\{f(x + \delta x, y + \delta y) - f(x, y + \delta y)\} + \{f(x, y + \delta y) - f(x, y)\}}{\delta t}$$

[adding and subtracting the term $f(x, y + \delta y)$ in the numerator]

$$= \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta t}$$

$$= \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \cdot \frac{\delta x}{\delta t} + \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \cdot \frac{\delta y}{\delta t}$$

Now
$$\frac{du}{dt} = \lim_{\delta t \to 0} \frac{\delta u}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} \cdot \frac{\partial x}{\partial t}$$

$$+ \lim_{\delta t \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y} \cdot \frac{\partial y}{\partial t}. \qquad \ldots(1)$$

Now δx and δy also tend to zero when δt tends to zero.

So we have

$$\lim_{\delta x \to 0} \frac{f(x + \delta x, y + \delta y) - f(x, y + \delta y)}{\delta x} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x},$$

because while x becomes $x + \delta x$, $y + \delta y$ remains unchanged.

Similarly,
$$\lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y} = \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y}.$$

Also
$$\lim_{\delta t \to 0} \frac{\delta x}{\delta t} = \frac{dx}{dt} \quad \text{and} \quad \lim_{\delta t \to 0} \frac{\delta y}{\delta t} = \frac{dy}{dt}.$$

Therefore (1) gives

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}.$$

Similarly, if $u = f(x_1, x_2, \dots, x_m)$ and $x_1, x_2, \dots, x_m$ are all functions of t, we can prove that

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial u}{\partial x_2} \cdot \frac{dx_2}{dt} + \dots + \frac{\partial u}{\partial x_m} \cdot \frac{dx_m}{dt}.$$

Corollary: If $z = f(x, y)$, where x, y are functions of u and v, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}; \text{ and } \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

9 Differential Coefficient of Implicit Functions

Suppose $u = f(x, y)$, where $y = \phi(x)$. Then supposing t to be the same as x in the formula of article 7, we get

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} \quad \text{or} \quad \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}.$$

Now suppose we are given an implicit relation between x and y of the form $u \equiv f(x, y) = c$, where c is a constant and y is a function of x.

Then, we have $du/dx = 0$

or $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} \quad \text{or} \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{f_x}{f_y}.$$

Differentiating $\frac{dy}{dx} = -\frac{f_x}{f_y}$ with respect to x, we get

$$\frac{d^2y}{dx^2} = -\left[\frac{f_y \frac{d}{dx}(f_x) - f_x \frac{d}{dx}(f_y)}{f_y^2}\right]$$

$$= -\frac{f_y\left[\frac{\partial f_x}{\partial x} + \frac{\partial f_x}{\partial y} \cdot \frac{dy}{dx}\right] - f_x\left[\frac{\partial f_y}{\partial x} + \frac{\partial f_y}{\partial y} \cdot \frac{dy}{dx}\right]}{f_y^2}$$

$$= -\frac{f_y\left[f_{xx} - f_{yx}\frac{f_x}{f_y}\right] - f_x\left[f_{xy} - f_{yy}\frac{f_x}{f_y}\right]}{f_y^2}$$

$$= -\left[\frac{f_{xx} f_y^2 - f_x f_y f_{xy} - f_x f_y f_{xy} + f_{yy} f_x^2}{f_y^3}\right]$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\left[\frac{f_{xx} f_y^2 - 2 f_x f_y f_{xy} + f_{yy} f_x^2}{f_y^3}\right].$$

10 Geometrical Interpretation of Partial Derivatives

The graph of a function defines a surface in Euclidian space. To every point on this surface, there are an infinite number of tangent lines. Geometrically partial differentiation is the act of choosing one of these lines and finding its slope. Usually, the lines of most interest are those that are parallel to the xz-plane, and those that are parallel to the yz-plane (which result from holding either y or x constant, respectively.)

Illustrative Examples

Example 9: *If $(\tan x)^y + y^{\cot x} = a$, find the value of dy / dx.* **(Rohilkhand 2014)**

Solution: Let $f(x, y) \equiv (\tan x)^y + y^{\cot x} = a$.

Then, we have $\dfrac{dy}{dx} = -\dfrac{\partial f / \partial x}{\partial f / \partial y}$. ...(1)

Now $\dfrac{\partial f}{\partial x} = y(\tan x)^{y-1} . \sec^2 x + y^{\cot x} . \log y . (-\operatorname{cosec}^2 x)$,

and $\dfrac{\partial f}{\partial y} = (\tan x)^y \log \tan x + (\cot x) . y^{\cot x - 1}$.

Therefore (1) gives

$$\frac{dy}{dx} = -\frac{y(\tan x)^{y-1} \sec^2 x - y^{\cot x} . \log y . \operatorname{cosec}^2 x}{(\tan x)^y \log \tan x + \cot x . y^{\cot x - 1}}.$$

Example 10: *If $u = x^2 y$, where $x^2 + xy + y^2 = 1$, find du / dx.*

Solution: We have, $\dfrac{du}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} \cdot \dfrac{dy}{dx}$. ...(1)

Now , $\dfrac{\partial u}{\partial x} = 2xy$ and $\dfrac{\partial u}{\partial y} = x^2$.

Let $f(x, y) \equiv x^2 + xy + y^2 = 1$.

Then $\dfrac{dy}{dx} = -\dfrac{\partial f / \partial x}{\partial f / \partial y} = -\dfrac{2x + y}{x + 2y}$.

So putting the values in (1), we get

$$\frac{du}{dx} = 2xy + x^2 . \left(-\frac{2x + y}{x + 2y}\right) = 2xy - \frac{x^2 (2x + y)}{x + 2y}.$$

Example 11: *If $u = f(y - z, z - x, x - y)$, prove that* $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

(Kanpur 2009; Avadh 10; Garhwal 10, 11; Kumaun 15)

Solution: We have $u = f(y - z, z - x, x - y)$.

Let $y - z = A, z - x = B$, and $x - y = C$.

Then $u = f(A, B, C)$ where A, B and C are functions of x, y and z.

Now
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial A} \cdot \frac{\partial A}{\partial x} + \frac{\partial u}{\partial B} \cdot \frac{\partial B}{\partial x} + \frac{\partial u}{\partial C} \cdot \frac{\partial C}{\partial x}$$
$$= \frac{\partial u}{\partial A} \cdot (0) + \frac{\partial u}{\partial B} \cdot (-1) + \frac{\partial u}{\partial C} \cdot (1) = -\frac{\partial u}{\partial B} + \frac{\partial u}{\partial C} \cdot \quad \text{...(1)}$$

Similarly
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial A} \cdot \frac{\partial A}{\partial y} + \frac{\partial u}{\partial B} \cdot \frac{\partial B}{\partial y} + \frac{\partial u}{\partial C} \cdot \frac{\partial C}{\partial y}$$
$$= \frac{\partial u}{\partial A} \cdot (1) + \frac{\partial u}{\partial B} \cdot (0) + \frac{\partial u}{\partial C} \cdot (-1)$$
$$= \frac{\partial u}{\partial A} - \frac{\partial u}{\partial C} \quad \text{...(2)}$$

and
$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial A} \cdot \frac{\partial A}{\partial z} + \frac{\partial u}{\partial B} \cdot \frac{\partial B}{\partial z} + \frac{\partial u}{\partial C} \cdot \frac{\partial C}{\partial z}$$
$$= \frac{\partial u}{\partial A} \cdot (-1) + \frac{\partial u}{\partial B} \cdot (1) + \frac{\partial u}{\partial C} \cdot (0)$$
$$= -\frac{\partial u}{\partial A} + \frac{\partial u}{\partial B} \cdot \quad \text{...(3)}$$

Adding (1), (2) and (3), we have
$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Comprehensive Exercise 3

1. Find dy / dx in the following :
   (i) $x^y + y^x = a^b$. (ii) $ax^2 + 2hxy + by^2 = 1$. **(Kashi 2013)**
2. If $\sqrt{(1 - x^2)} + \sqrt{(1 - y^2)} = a(x - y)$, prove that
$$\frac{dy}{dx} = \frac{\sqrt{(1 - y^2)}}{\sqrt{(1 - x^2)}} \cdot$$
3. Find du / dx if $u = \sin(x^2 + y^2)$, where $a^2 x^2 + b^2 y^2 = c^2$.
4. If $u = x^4 y^5$, where $x = t^2$ and $y = t^3$, find du / dt.
5. If $f(x, y) = 0$, $\phi(y, z) = 0$, show that
$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y} \cdot$$
(Lucknow 2009)
6. If $u = \sqrt{(x^2 + y^2)}$ and $x^3 + y^3 + 3axy = 5a^2$, find the value of du / dx when $x = a$, $y = a$.

Answers 3

1. (i) $-\dfrac{\{y\,x^{\,y-1}+y^{\,x}\log y\}}{\{x^{\,y}\log x+xy^{\,x-1}\}}$, (ii) $-\dfrac{(ax+hy)}{(hx+by)}$.

3. $2\,x\,\{\cos(x^2+y^2)\}\left(1-\dfrac{a^2}{b^2}\right)$. **4.** $23t^{\,22}$. **6.** 0.

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If $z=\tan(y+a\,x)+(y-a\,x)^{3/2}$, then $\left(\dfrac{\partial^2 z}{\partial x^2}\right)-a^2\left(\dfrac{\partial^2 z}{\partial y^2}\right)$ is equal to

(a) 0 (b) $(y-a\,x)$

(c) 1 (d) $\sec(y+a\,x)$

2. If u is a homogeneous function of x and y of degree n, then $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}$ is equal to

(a) n (b) $n\,u$

(c) $\dfrac{n}{u}$ (d) u

(Kanpur 2016)

3. If $\dfrac{1}{u}=\sqrt{(x^2+y^2+z^2)}$, then $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}$ is equal to

(a) u (b) $-u$

(c) u^2 (d) 0

4. If x and y are connected by an equation of the form $u\equiv f(x,y)=c$, then $\dfrac{dy}{dx}$ is

(a) $\dfrac{\partial f/\partial x}{\partial f/\partial y}$ (b) $(-1)^n\dfrac{\partial f/\partial x}{\partial f/\partial y}$

(c) $\dfrac{\partial f/\partial y}{\partial f/\partial x}$ (d) $-\dfrac{\partial f/\partial x}{\partial f/\partial y}$

(Garhwal 2007)

5. If $x = r\cos\theta$, $y = r\sin\theta$, then the value of $\frac{\partial\theta}{\partial x}$ is

(a) $-\frac{\sin\theta}{r}$ (b) $\frac{r}{\sin\theta}$

(c) $\frac{\sin\theta}{r}$ (d) $-\frac{1}{r\sin\theta}$ **(Garhwal 2002)**

6. If $f(x, y)$ be a homogeneous function of x and y of degree n, then

(a) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f$ (b) $y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = f$

(c) $y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = nf$ (d) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$ **(Garhwal 2003)**

7. If $\theta = t^n e^{-r^2/4t}$, then what value of n will make : $\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\theta}{\partial t}\right] = \frac{\partial\theta}{\partial t}$

(a) -1 (b) $-\frac{3}{2}$

(c) $-\frac{1}{2}$ (d) $-\frac{5}{2}$ **(Kanpur 2016)**

8. If $z = f(y/x^2)$, then $x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y} =$

(a) $\frac{4y}{x^2}f'(y/x^2)$ (b) $\frac{4y}{x^3}f'(y/x^2)$

(c) 0 (d) none of these **(Kumaun 2015)**

Fill in the Blanks

Fill in the blanks "……", so that the following statements are complete and correct.

1. If $u = e^{my}\cos mx$, then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \ldots\ldots$

2. If $u = \tan^{-1}\frac{x^2 + y^2}{x + y}$, then $\frac{\partial u}{\partial x} = \ldots\ldots$

3. An expression in which every term is of the same degree is called a …… function.

4. If $u = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$, then $\frac{du}{dt} = \ldots\ldots$

5. If $u(x, y)$ is a homogeneous function of x and y of degree n, then
$$x\frac{\partial}{\partial x}(u_x) + y\frac{\partial}{\partial y}(u_x) = \ldots\ldots, \quad \text{where} \quad u_x = \frac{\partial u}{\partial x}.$$

6. If $u = f(y + ax) + \phi(y - ax)$, then $\frac{\partial^2 u}{\partial x^2} - \frac{a^2\partial^2 u}{\partial y^2} = \ldots\ldots$ **(Agra 2002)**

7. If $u = f(y/x)$, then, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \ldots\ldots$ **(Agra 2003)**

True or False

Write 'T' for true and 'F' for false statement.

1. If $u = f(x, y)$ and its partial derivatives are continuous, then the order of differentiation is immaterial.
2. If u is a homogeneous function of x and y of degree n, then $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are also homogeneous functions of x and y each being of degree n.
3. If $f(x, y) = 0$ be an implicit function of x and y and $p = \frac{\partial f}{\partial x}, q = \frac{\partial f}{\partial y}, r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x\,\partial y}$ and $t = \frac{\partial^2 f}{\partial y^2}$, then

$$\frac{d^2 y}{dx^2} = -(q^2 r - 2pqs + p^2 t)/q^3.$$

Answers

Multiple Choice Questions

1. (a) 2. (b) 3. (b) 4. (d) 5. (a)
6. (d) 7. (b)

Fill in the Blank(s)

1. 0 2. $\frac{(x^2 + 2xy - y^2)}{\{(x+y)^2 + (x^2 + y^2)^2\}}$ 3. homogeneous
4. $\frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$ 5. $(n-1)\, u_x$ 6. 0 7. 0

True or False

1. *T* 2. *F* 3. *T*

Chapter

2

Jacobians

1 Jacobian. Definition

(Kanpur 2014)

If $u_1, u_2, \dots, u_n$ are functions of n independent variables $x_1, x_2, \dots, x_n$ then the determinant

$$\begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dots\dots\dots & \dfrac{\partial u_1}{\partial x_n} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dots\dots\dots & \dfrac{\partial u_2}{\partial x_n} \\ \dots\ \dots & \dots & \dots\ \dots\ \dots & \dots \\ \dfrac{\partial u_n}{\partial x_1} & \dfrac{\partial u_n}{\partial x_2} & \dots\dots\dots & \dfrac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, \dots, u_n$ with respect to $x_1, x_2, \dots, x_n$ and is denoted either by $\dfrac{\partial (u_1, u_2, \dots, u_n)}{\partial (x_1, x_2, \dots, x_n)}$ or by $J(u_1, u_2, \dots, u_n)$. The second notation is used when there is no doubt as regards the independent variables.

Thus if u and v are functions of two independent variables x and y, we have

$$\frac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = J (u, v) .$$

Similarly if u, v and w are functions of three independent variables x, y and z, we have

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} = J (u, v, w).$$

Note: If the functions $u_1, u_2, \ldots, u_n$ of n independent variables $x_1, x_2, \ldots, x_n$ are of the following forms,

$$u_1 = f_1 (x_1), u_2 = f_2 (x_1, x_2), \ldots, u_n = f_n (x_1, x_2, \ldots, x_n),$$

then

$$\frac{\partial (u_1, u_2, \ldots, u_n)}{\partial (x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & 0 & 0 & \ldots & 0 \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \dfrac{\partial u_n}{\partial x_1} & \dfrac{\partial u_n}{\partial x_2} & \dfrac{\partial u_n}{\partial x_3} & \ldots & \dfrac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$= \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} \cdots\cdots \frac{\partial u_n}{\partial x_n},$$

i.e., in such cases the Jacobian reduces to the principal diagonal term of the determinant.

(Gorakhpur 2013)

Illustrative Examples

Example 1: *If* $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$, *show that*

$$\frac{\partial (x, y, z)}{\partial (r, \theta, \phi)} = r^2 \sin \theta.$$

(Meerut 2003; Garhwal 02; Lucknow 07; Rohilkhand 12; Avadh 07, 12)

Solution: We have

$$\frac{\partial (x, y, z)}{\partial (r, \theta, \phi)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \phi} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta\,(r^2 \sin\theta\cos\theta\cos^2\phi + r^2 \sin\theta\cos\theta\sin^2\phi)$$
$$+ r\sin\theta\,(r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi),$$

expanding the determinant along the third row

$$= r^2 \sin\theta\cos^2\theta + r^2\sin^3\theta = r^2\sin\theta\,(\cos^2\theta + \sin^2\theta)$$
$$= r^2\sin\theta.$$

Example 2: *Find the Jacobian* $\dfrac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$ *being given*

$$x = r\cos\theta\cos\phi,\ y = r\sin\theta\sqrt{(1 - m^2\sin^2\phi)},$$
$$z = r\sin\phi\sqrt{(1 - n^2\sin^2\theta)},\ \textit{where } m^2 + n^2 = 1.$$

Solution: Here

$$x^2 + y^2 + z^2 = r^2\cos^2\theta\cos^2\phi + r^2\sin^2\theta - r^2 m^2\sin^2\theta\sin^2\phi$$
$$+ r^2\sin^2\phi - r^2 n^2\sin^2\phi\sin^2\theta$$
$$= r^2(\cos^2\theta\cos^2\phi + \sin^2\theta + \sin^2\phi - \sin^2\theta\sin^2\phi)$$
$$[\because\ m^2 + n^2 = 1]$$
$$= r^2(\cos^2\theta\cos^2\phi + \sin^2\theta + \sin^2\phi\cos^2\theta)$$
$$= r^2(\sin^2\theta + \cos^2\theta)$$
$$= r^2.$$

$\therefore$
$$\left.\begin{aligned} x\frac{\partial x}{\partial r} + y\frac{\partial y}{\partial r} + z\frac{\partial z}{\partial r} &= r; \\ x\frac{\partial x}{\partial \theta} + y\frac{\partial y}{\partial \theta} + z\frac{\partial z}{\partial \theta} &= 0; \\ x\frac{\partial x}{\partial \phi} + y\frac{\partial y}{\partial \phi} + z\frac{\partial z}{\partial \phi} &= 0. \end{aligned}\right\} \quad \ldots(1)$$
and

Now
$$J(x, y, z) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \frac{1}{x}\begin{vmatrix} x\frac{\partial x}{\partial r} & x\frac{\partial x}{\partial \theta} & x\frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \frac{1}{x}\begin{vmatrix} r & 0 & 0 \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}, \text{ by adding } y\,R_2 + z\,R_3 \text{ to } R_1 \text{ and using the relations (1)}$$

$$= \frac{r}{x}\begin{vmatrix} \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \frac{r}{x}\left\{\frac{\partial y}{\partial \theta}\frac{\partial z}{\partial \phi} - \frac{\partial z}{\partial \theta}\frac{\partial y}{\partial \phi}\right\}$$

$$= \frac{r}{x}\left\{r\cos\theta\sqrt{(1-m^2\sin^2\phi)}\,.\,r\cos\phi\sqrt{(1-n^2\sin^2\theta)} - \frac{r\sin\phi.\,n^2\sin\theta\cos\theta}{\sqrt{(1-n^2\sin^2\theta)}}\cdot\frac{r\sin\theta\,.m^2\sin\phi\cos\phi}{\sqrt{(1-m^2\sin^2\phi)}}\right\}$$

$$= \frac{r^3\cos\theta\cos\phi}{x}\left[\frac{(1-m^2\sin^2\phi)(1-n^2\sin^2\theta) - n^2m^2\sin^2\theta\sin^2\phi}{\sqrt{[(1-n^2\sin^2\theta)(1-m^2\sin^2\phi)]}}\right]$$

$$= \frac{r^3\cos\theta\cos\phi}{r\cos\theta\cos\phi}\cdot\left[\frac{1-m^2\sin^2\phi - n^2\sin^2\theta + m^2n^2\sin^2\phi\sin^2\theta - m^2n^2\sin^2\theta\sin^2\phi}{\sqrt{[(1-n^2\sin^2\theta)(1-m^2\sin^2\phi)]}}\right]$$

$$= \frac{r^2(m^2\cos^2\phi + n^2\cos^2\theta)}{\sqrt{[(1-n^2\sin^2\theta)(1-m^2\sin^2\phi)]}}. \qquad [\because\ m^2+n^2=1]$$

Example 3: *If* $y_1 = r\sin\theta_1\sin\theta_2$, $y_2 = r\sin\theta_1\cos\theta_2$, $y_3 = r\cos\theta_1\sin\theta_3$, $y_4 = r\cos\theta_1\cos\theta_3$, *find the value of the Jacobian* $\dfrac{\partial(y_1, y_2, y_3, y_4)}{\partial(r, \theta_1, \theta_2, \theta_3)}$.

Solution: Squaring and adding the given relations, we have

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = r^2.$$

$$\left.\begin{aligned} \therefore\quad & y_1\frac{\partial y_1}{\partial r} + y_2\frac{\partial y_2}{\partial r} + y_3\frac{\partial y_3}{\partial r} + y_4\frac{\partial y_4}{\partial r} = r \\ \text{and}\quad & y_1\frac{\partial y_1}{\partial \theta_r} + y_2\frac{\partial y_2}{\partial \theta_r} + y_3\frac{\partial y_3}{\partial \theta_r} + y_4\frac{\partial y_4}{\partial \theta_r} = 0, \quad r = 1, 2, 3. \end{aligned}\right\} \quad ...(1)$$

Also $y_3^2 + y_4^2 = r^2\cos^2\theta_1$, so that

$$\left.\begin{aligned} & y_3\frac{\partial y_3}{\partial \theta_1} + y_4\frac{\partial y_4}{\partial \theta_1} = -r^2\cos\theta_1\sin\theta_1\,; \\ & y_3\frac{\partial y_3}{\partial \theta_r} + y_4\frac{\partial y_4}{\partial \theta_r} = 0, \qquad r = 2, 3. \end{aligned}\right\} \quad ...(2)$$

Now the required Jacobian

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial r} & \frac{\partial y_1}{\partial \theta_1} & \frac{\partial y_1}{\partial \theta_2} & \frac{\partial y_1}{\partial \theta_3} \\ \frac{\partial y_2}{\partial r} & \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & \frac{\partial y_2}{\partial \theta_3} \\ \frac{\partial y_3}{\partial r} & \frac{\partial y_3}{\partial \theta_1} & \frac{\partial y_3}{\partial \theta_2} & \frac{\partial y_3}{\partial \theta_3} \\ \frac{\partial y_4}{\partial r} & \frac{\partial y_4}{\partial \theta_1} & \frac{\partial y_4}{\partial \theta_2} & \frac{\partial y_4}{\partial \theta_3} \end{vmatrix}.$$

Operating $y_1 R_1 + (y_2 R_2 + y_3 R_3 + y_4 R_4)$, and using the results (1), we get

$$J = \frac{1}{y_1} \begin{vmatrix} r & 0 & 0 & 0 \\ \frac{\partial y_2}{\partial r} & \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & \frac{\partial y_2}{\partial \theta_3} \\ \frac{\partial y_3}{\partial r} & \frac{\partial y_3}{\partial \theta_1} & \frac{\partial y_3}{\partial \theta_2} & \frac{\partial y_3}{\partial \theta_3} \\ \frac{\partial y_4}{\partial r} & \frac{\partial y_4}{\partial \theta_1} & \frac{\partial y_4}{\partial \theta_2} & \frac{\partial y_4}{\partial \theta_3} \end{vmatrix} = \frac{r}{y_1} \begin{vmatrix} \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & \frac{\partial y_2}{\partial \theta_3} \\ \frac{\partial y_3}{\partial \theta_1} & \frac{\partial y_3}{\partial \theta_2} & \frac{\partial y_3}{\partial \theta_3} \\ \frac{\partial y_4}{\partial \theta_1} & \frac{\partial y_4}{\partial \theta_2} & \frac{\partial y_4}{\partial \theta_3} \end{vmatrix}$$

$$= \frac{r}{y_1 \, y_3} \begin{vmatrix} \frac{\partial y_2}{\partial \theta_1} & \frac{\partial y_2}{\partial \theta_2} & \frac{\partial y_2}{\partial \theta_3} \\ -r^2 \cos \theta_1 \sin \theta_1 & 0 & 0 \\ \frac{\partial y_4}{\partial \theta_1} & \frac{\partial y_4}{\partial \theta_2} & \frac{\partial y_4}{\partial \theta_3} \end{vmatrix},$$

adding $y_4 R_3$ to $y_3 \, R_2$ and using the results (2)

$$= \frac{r}{y_1 \, y_3} \cdot r^2 \cos \theta_1 \sin \theta_1 \left[\frac{\partial y_2}{\partial \theta_2} \cdot \frac{\partial y_4}{\partial \theta_3} - \frac{\partial y_4}{\partial \theta_2} \cdot \frac{\partial y_2}{\partial \theta_3} \right]$$

$$= \frac{r^3 \cos \theta_1 \sin \theta_1}{y_1 \, y_3} [(-r \sin \theta_1 \sin \theta_2)(-r \cos \theta_1 \sin \theta_3) - 0]$$

$$= \frac{r^5 \sin^2 \theta_1 \cos^2 \theta_1 \sin \theta_2 \sin \theta_3}{r^2 \sin \theta_1 \cos \theta_1 \sin \theta_2 \sin \theta_3} = r^3 \sin \theta_1 \cos \theta_1.$$

Comprehensive Exercise 1

1. If $x = r \cos \theta$, $y = r \sin \theta$, show that

(i) $\dfrac{\partial (x, y)}{\partial (r, \theta)} = r,$

(Kanpur 2005; Meerut 13B; Kashi 13; Gorakhpur 15)

(ii) $\dfrac{\partial (r, \theta)}{\partial (x, y)} = \dfrac{1}{r} \cdot$

(Kanpur 2005; Meerut 13)

2. If $x = u (1 + v), y = v (1 + u)$, find the Jacobian of x, y with respect to u, v.

(Lucknow 2011; Meerut 13)

3. If $x = c \cos u \cosh v, y = c \sin u \sinh v$, prove that

$$\frac{\partial (x, y)}{\partial (u, v)} = \frac{1}{2} c^2 (\cos 2u - \cosh 2v) .$$

(Rohilkhand 2013)

4. If $u = \dfrac{y^2}{2x}, v = \dfrac{x^2 + y^2}{2x}$, find $\dfrac{\partial (u, v)}{\partial (x, y)} \cdot$

(Meerut 2012)

5. If $u_1 = x_2 x_3 / x_1, u_2 = x_3 x_1 / x_2, u_3 = x_1 x_2 / x_3$, prove that

$J (u_1, u_2, u_3) = 4.$ **(Kumaun 2011; Bundelkhand 14; Purvanchal 14; Gorakhpur 14, 15)**

6. If $x = \sin \theta \sqrt{(1 - c^2 \sin^2 \phi)}, y = \cos \theta \cos \phi$, then show that

$$\frac{\partial (x, y)}{\partial (\theta, \phi)} = - \sin \phi \frac{[(1 - c^2) \cos^2 \theta + c^2 \cos^2 \phi]}{\sqrt{(1 - c^2 \sin^2 \phi)}} \cdot$$

7. If $u = xyz, v = xy + yz + zx, w = x + y + z$, compute $\dfrac{\partial (u, v, w)}{\partial (x, y, z)} \cdot$

(Rohilkhand 2013; Kumaun 14)

8. If $y_1 = 1 - x_1, y_2 = x_1 (1 - x_2), y_3 = x_1 x_2 (1 - x_3), \ldots,$
$y_n = x_1 x_2 \ldots x_{n-1} (1 - x_n)$, prove that
$J (y_1, y_2, \ldots, y_n) = (-1)^n x_1^{n-1} x_2^{n-2} \ldots x_{n-1}.$

(Kumaun 2007, 11; Gorakhpur 12, 14)

9. If $y_1 = \cos x_1, y_2 = \sin x_1 \cos x_2, y_3 = \sin x_1 \sin x_2 \cos x_3, \ldots,$
$y_n = \sin x_1 \sin x_2 \sin x_3 \ldots \sin x_{n-1} \cos x_n$, find the Jacobian of $y_1, y_2,$
$\ldots, y_n$ with respect to $x_1, x_2, \ldots, x_n$.

Answers 1

2. $1 + u + v$

4. $-\dfrac{y}{2x}$

7. $(x - y)(y - z)(z - x)$

9. $(-1)^n \sin^n x_1 \sin^{n-1} x_2 \ldots \sin x_n$

2 Case of Functions of Functions (Chain Rule)

We shall establish the formula for two variables and the result can be easily extended to any number of variables.

Theorem: *If* u_1, u_2 *are functions of* y_1, y_2 *and* y_1, y_2 *are functions of* x_1, x_2, *then*

$$\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \frac{\partial(u_1, u_2)}{\partial(y_1, y_2)} \cdot \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}.$$

(Kumaun 2003)

Proof: We have

$$\left.\begin{aligned} \frac{\partial u_1}{\partial x_1} &= \frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial u_1}{\partial y_2}\frac{\partial y_2}{\partial x_1}, \quad \frac{\partial u_1}{\partial x_2} = \frac{\partial u_1}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial u_1}{\partial y_2}\frac{\partial y_2}{\partial x_2}, \\ \frac{\partial u_2}{\partial x_1} &= \frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial u_2}{\partial y_2}\frac{\partial y_2}{\partial x_1}, \quad \frac{\partial u_2}{\partial x_2} = \frac{\partial u_2}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial u_2}{\partial y_2}\frac{\partial y_2}{\partial x_2}. \end{aligned}\right\} \quad \ldots(1)$$

Now
$$\frac{\partial(u_1, u_2)}{\partial(y_1, y_2)} \cdot \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \dfrac{\partial u_1}{\partial y_1} & \dfrac{\partial u_1}{\partial y_2} \\ \dfrac{\partial u_2}{\partial y_1} & \dfrac{\partial u_2}{\partial y_2} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial u_1}{\partial y_1}\dfrac{\partial y_1}{\partial x_1} + \dfrac{\partial u_1}{\partial y_2}\dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial u_1}{\partial y_1}\dfrac{\partial y_1}{\partial x_2} + \dfrac{\partial u_1}{\partial y_2}\dfrac{\partial y_2}{\partial x_2} \\ \dfrac{\partial u_2}{\partial y_1}\dfrac{\partial y_1}{\partial x_1} + \dfrac{\partial u_2}{\partial y_2}\dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial u_2}{\partial y_1}\dfrac{\partial y_1}{\partial x_2} + \dfrac{\partial u_2}{\partial y_2}\dfrac{\partial y_2}{\partial x_2} \end{vmatrix},$$

applying row-by-column multiplication rule

$$= \begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} \end{vmatrix}, \text{ using the relations (1)}$$

$$= \frac{\partial(u_1, u_2)}{\partial(x_1, x_2)}.$$

Note: The above formula resembles very much with the formula $\dfrac{df}{dx} = \dfrac{df}{dt} \cdot \dfrac{dt}{dx}$, for the derivative of the function of a function.

Generalization of the above formula: *If* $u_1, u_2, \ldots, u_n$ *are functions of* $y_1, y_2, \ldots, y_n$ *and* $y_1, y_2, \ldots, y_n$ *are functions of* $x_1, x_2, \ldots, x_n$, *then*

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = \frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(y_1, y_2, \ldots, y_n)} \cdot \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)}.$$

The proof may be easily extended as in the case of two variables and has been left as an exercise for the students.

3 Jacobian of Implicit Functions

Theorem 1: *Suppose $u_1, u_2, \ldots, u_n$ instead of being given explicitly in terms of $x_1, x_2, \ldots, x_n$ are connected with them by equations such as*

$$F_1\,(u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n) = 0,$$
$$F_2\,(u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n) = 0,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$F_n\,(u_1, u_2, \ldots, u_n, x_1, x_2, \ldots, x_n) = 0.$$

Then, we have $$\frac{\partial\,(u_1, u_2, \ldots, u_n)}{\partial\,(x_1, x_2, \ldots, x_n)} = (-1)^n \frac{\dfrac{\partial\,(F_1, F_2, \ldots, F_n)}{\partial\,(x_1, x_2, \ldots, x_n)}}{\dfrac{\partial\,(F_1, F_2, \ldots, F_n)}{\partial\,(u_1, u_2, \ldots, u_n)}}.$$

Proof: Here also we shall establish the result for two variables and the proof can be extended easily for n variables. The students should themselves write the proof for n variables on the basis of the proof given below for two variables.

In the case of two variables, the connecting relations are

$$\left.\begin{aligned} F_1\,(u_1, u_2, x_1, x_2) &= 0, \\ F_2\,(u_1, u_2, x_1, x_2) &= 0. \end{aligned}\right\} \qquad \ldots(1)$$

From relations (1), we have by differentiation

$$\left.\begin{aligned} \frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial F_1}{\partial u_2}\frac{\partial u_2}{\partial x_1} &= 0, \\ \frac{\partial F_1}{\partial x_2} + \frac{\partial F_1}{\partial u_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial F_1}{\partial u_2}\frac{\partial u_2}{\partial x_2} &= 0, \\ \frac{\partial F_2}{\partial x_1} + \frac{\partial F_2}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial F_2}{\partial u_2}\frac{\partial u_2}{\partial x_1} &= 0, \\ \frac{\partial F_2}{\partial x_2} + \frac{\partial F_2}{\partial u_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial F_2}{\partial u_2}\frac{\partial u_2}{\partial x_2} &= 0. \end{aligned}\right\} \qquad \ldots(2)$$

Now $$\frac{\partial\,(F_1, F_2)}{\partial\,(u_1, u_2)} \cdot \frac{\partial\,(u_1, u_2)}{\partial\,(x_1, x_2)} = \begin{vmatrix} \dfrac{\partial F_1}{\partial u_1} & \dfrac{\partial F_1}{\partial u_2} \\ \dfrac{\partial F_2}{\partial u_1} & \dfrac{\partial F_2}{\partial u_2} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial F_1}{\partial u_1}\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial F_1}{\partial u_2}\dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial F_1}{\partial u_1}\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial F_1}{\partial u_2}\dfrac{\partial u_2}{\partial x_2} \\ \dfrac{\partial F_2}{\partial u_1}\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial F_2}{\partial u_2}\dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial F_2}{\partial u_1}\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial F_2}{\partial u_2}\dfrac{\partial u_2}{\partial x_2} \end{vmatrix},$$

applying row-by-column multiplication rule

$$= \begin{vmatrix} \dfrac{-\partial F_1}{\partial x_1} & \dfrac{-\partial F_1}{\partial x_2} \\ \dfrac{-\partial F_2}{\partial x_1} & \dfrac{-\partial F_2}{\partial x_2} \end{vmatrix}, \text{ using the relations (2)}$$

$$= (-1)^2 \frac{\partial (F_1, F_2)}{\partial (x_1, x_2)}.$$

Accordingly, we have

$$\frac{\partial (u_1, u_2)}{\partial (x_1, x_2)} = (-1)^2 \frac{\dfrac{\partial (F_1, F_2)}{\partial (x_1, x_2)}}{\dfrac{\partial (F_1, F_2)}{\partial (u_1, u_2)}}.$$

Illustrative Examples

Example 4: *If J is the Jacobian of u, v with respect to x, y and J' is the Jacobian of x, y with respect to u, v then prove that $JJ' = 1$ or* $\dfrac{\partial (u, v)}{\partial (x, y)} \times \dfrac{\partial (x, y)}{\partial (u, v)} = 1.$

(Kanpur 2008; Bundelkhand 11; Meerut 12; Avadh 13)

Solution: Let $u = f_1 (x, y), v = f_2 (x, y).$ ...(1)

Obviously x and y can also be expressed as functions of u and v. Differentiating relations (1) partially with respect to u and v, we get

$$\left.\begin{aligned} 1 &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u}, \quad 0 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v}, \\ 0 &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u}, \quad 1 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v}. \end{aligned}\right\} \quad ...(2)$$

Now $$\frac{\partial (u, v)}{\partial (x, y)} \times \frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial u} & \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial v} \\ \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial u} & \dfrac{\partial v}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial v}{\partial y}\dfrac{\partial y}{\partial v} \end{vmatrix},$$

applying row-by-column multiplication

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \text{ using the relations (2)}$$

$$= 1.$$

Example 5: *If* $u^3 + v + w = x + y^2 + z^2$, $u + v^3 + w = x^2 + y + z^2$, $u + v + w^3 = x^2 + y^2 + z$, *prove that* $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = \dfrac{1 - 4(xy + yz + zx) + 16xyz}{2 - 3(u^2 + v^2 + w^2) + 27u^2v^2w^2}$.

Solution: The given relations can be written as

$$F_1 \equiv u^3 + v + w - x - y^2 - z^2 = 0,$$
$$F_2 \equiv u + v^3 + w - x^2 - y - z^2 = 0,$$
$$F_3 \equiv u + v + w^3 - x^2 - y^2 - z = 0.$$

Now
$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} \Big/ \frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} \qquad \ldots(1)$$

Here
$$\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} = \begin{vmatrix} -1 & -2y & -2z \\ -2x & -1 & -2z \\ -2x & -2y & -1 \end{vmatrix}$$
$$= -1(1 - 4yz) + 2x(2y - 4yz) - 2x(4yz - 2z)$$
$$= -1 + 4(yz + zx + xy) - 16xyz.$$

And
$$\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} = \begin{vmatrix} 3u^2 & 1 & 1 \\ 1 & 3v^2 & 1 \\ 1 & 1 & 3w^2 \end{vmatrix}$$
$$= 3u^2(9v^2w^2 - 1) - 1(3w^2 - 1) + 1.(1 - 3v^2)$$
$$= 2 - 3(u^2 + v^2 + w^2) + 27u^2v^2w^2.$$

$\therefore$ From (1),
$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{1 - 4(yz + zx + xy) + 16xyz}{2 - 3(u^2 + v^2 + w^2) + 27u^2v^2w^2}.$$

Comprehensive Exercise 2

1. If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$, show that $\dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{1}{2}\dfrac{y^2 - x^2}{uv(u - v)}$.

(Kumaun 2002; Garhwal 03)

2. If $x + y + z = u$, $y + z = uv$, $z = uvw$, show that $\dfrac{\partial(x, y, z)}{\partial(u, v, w)} = u^2 v$.

(Rohilkhand 2005; Gorakhpur 11; Kashi 14)

3. If $u^3 = xyz$, $\dfrac{1}{v} = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$, $w^2 = x^2 + y^2 + z^2$, prove that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = -\frac{v(y - z)(z - x)(x - y)(x + y + z)}{3u^2 w(yz + zx + xy)}.$$

(Kumaun 2009, 13; Rohilkhand 12B)

4. If $u^3 + v^3 + w^3 = x + y + z$, $u^2 + v^2 + w^2 = x^3 + y^3 + z^3$,

$u + v + w = x^2 + y^2 + z^2$, then prove that

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = \frac{(y - z)(z - x)(x - y)}{(u - v)(v - w)(w - u)}.$$

(Purvanchal 2007; Kanpur 12)

5. Compute the Jacobian $\dfrac{\partial (u, v)}{\partial (r, \theta)}$ where

$u = 2xy, v = x^2 - y^2, x = r \cos \theta, y = r \sin \theta.$

6. If $u_1 = x_1 + x_2 + x_3 + x_4, u_1 u_2 = x_2 + x_3 + x_4, u_1 u_2 u_3 = x_3 + x_4,$

$u_1 u_2 u_3\, u_4 = x_4$, show that $\dfrac{\partial (x_1, x_2, x_3, x_4)}{\partial (u_1, u_2, u_3, u_4)} = u_1^3 u_2^2 u_3.$

(Kumaun 2012)

7. Given $y_1 (x_1 - x_2) = 0, y_2 (x_1^2 + x_1 x_2 + x_2^2) = 0$, show that

$$\frac{\partial (y_1, y_2)}{\partial (x_1, x_2)} = 3 y_1\, y_2 \frac{x_1 + x_2}{x_1^3 - x_2^3}.$$

8. If $u = x (1 - r^2)^{-1/2}, v = y (1 - r^2)^{-1/2}, w = z (1 - r^2)^{-1/2}$,

where $r^2 = x^2 + y^2 + z^2$, show that $\dfrac{\partial (u, v, w)}{\partial (x, y, z)} = (1 - r^2)^{-5/2}$.

9. (a) Find the Jacobian of $y_1, y_2, y_3, \ldots, y_n$, being given

$y_1 = x_1 (1 - x_2), y_2 = x_1 x_2 (1 - x_3), \ldots, y_{n-1} = x_1 x_2 \ldots x_{n-1} (1 - x_n),$

$y_n = x_1\, x_2 x_3 \ldots x_n.$

(b) If $y_1 = r \cos \theta_1, y_2 = r \sin \theta_1 \cos \theta_2, y_3 = r \sin \theta_1 \sin \theta_2 \sin \theta_3, \ldots,$

$y_{n-1} = r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-2} \cos \theta_{n-1}$

and $y_n = r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-1}$, prove that

$$\frac{\partial (y_1, y_2, \ldots, y_n)}{\partial (r, \theta_1, \ldots, \theta_{n-1})} = r^{n-1} \sin^{n-2} \theta_1 \sin^{n-3} \theta_2 \ldots \sin \theta_{n-2}.$$

(Kumaun 2010)

10. If λ, μ, ν are the roots of the equation in k,

$$\frac{x}{a + k} + \frac{y}{b + k} + \frac{z}{c + k} = 1,$$

prove that $\dfrac{\partial (x, y, z)}{\partial (\lambda, \mu, \nu)} = -\dfrac{(\mu - \nu)(\nu - \lambda)(\lambda - \mu)}{(b - c)(c - a)(a - b)}.$

11. The roots of the equation in λ,

$$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$$

are u, v, w. Prove that

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = -2 \frac{(y - z)(z - x)(x - y)}{(v - w)(w - u)(u - v)}.$$

(Lucknow 2007; Kumaun 08; Kanpur 10; Gorakhpur 11)

12. If x, y, z are connected by a functional relation $f(x, y, z) = 0$, show that

$$\frac{\partial (y, z)}{\partial (x, z)} = \left(\frac{\partial y}{\partial x}\right)_{z = \text{const.}}$$

13. (i) Prove that $\dfrac{\partial (u, v, w)}{\partial (x, y, z)} \times \dfrac{\partial (x, y, z)}{\partial (u, v, w)} = 1.$

(Kanpur 2008, 09; 11)

(ii) Prove that $\dfrac{\partial (y_1, y_2, \dots, y_n)}{\partial (x_1, x_2, \dots x_n)} \cdot \dfrac{\partial (x_1, x_2, \dots x_n)}{\partial (y_1, y_2, \dots, y_n)} = 1.$

14. If $x^2 + y^2 + u^2 - v^2 = 0$ and $uv + xy = 0$, prove that $\dfrac{\partial (u, v)}{\partial (x, y)} = \dfrac{x^2 - y^2}{u^2 + v^2}$.

(Kumaun 2015)

Answers 2

5. $-4r^3$

9. (a) $x_1^{\,n-1} x_2^{\,n-2} \dots x_{n-1}$

4 Necessary and Sufficient Condition for a Jacobian to Vanish

Theorem 1: *Let $u_1, u_2, \dots, u_n$ be functions of n independent variables $x_1, x_2, \dots, x_n$. In order that these n functions may not be independent, i.e., there may exist between these n functions a relation*

$$F(u_1, u_2, \dots, u_n) = 0, \qquad \dots(1)$$

it is necessary and sufficient that the Jacobian $\dfrac{\partial (u_1, u_2, \dots, u_n)}{\partial (x_1, x_2, .., x_n)}$ should vanish identically.

Proof: The condition is necessary *i.e.*, if there exists between $u_1, u_2, \dots, u_n$ a relation

$$F(u_1, u_2, \dots, u_n) = 0 \qquad \dots(1)$$

their Jacobian is necessarily zero.

Differentiating (1) partially with respect to $x_1, x_2, \dots, x_n$, we get

$$\frac{\partial F}{\partial u_1}\frac{\partial u_1}{\partial x_1} + \frac{\partial F}{\partial u_2}\frac{\partial u_2}{\partial x_1} + \dots + \frac{\partial F}{\partial u_n}\frac{\partial u_n}{\partial x_1} = 0,$$

$$\frac{\partial F}{\partial u_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial F}{\partial u_2}\frac{\partial u_2}{\partial x_2} + \dots + \frac{\partial F}{\partial u_n}\frac{\partial u_n}{\partial x_2} = 0,$$

$$\frac{\partial F}{\partial u_1}\frac{\partial u_1}{\partial x_n} + \frac{\partial F}{\partial u_2}\frac{\partial u_2}{\partial x_n} + \dots + \frac{\partial F}{\partial u_n}\frac{\partial u_n}{\partial x_n} = 0.$$

Eliminating $\frac{\partial F}{\partial u_1}, \frac{\partial F}{\partial u_2}, \ldots, \frac{\partial F}{\partial u_n}$ from these equations, we get

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_1}{\partial x_n} & \frac{\partial u_2}{\partial x_n} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix} = 0 \quad \text{or} \quad \frac{\partial (u_1, u_2, \ldots, u_n)}{\partial (x_1, x_2, \ldots, x_n)} = 0.$$

The condition is sufficient, *i.e.*, if the Jacobian $J\,(u_1, u_2, \ldots, u_n)$ is zero, then there must exist a relation between $u_1, u_2, \ldots, u_n$.

The equations connecting the functions $u_1, u_2, \ldots, u_n$ and the variables $x_1, x_2, \ldots, x_n$ are always capable of being put into the following form :

$$\phi_1\,(x_1, x_2, \ldots, x_n, u_1) = 0$$
$$\phi_2\,(x_2, x_3, \ldots, x_n, u_1, u_2) = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\phi_r\,(x_r, x_{r+1}, \ldots, x_n, u_1, u_2, \ldots, u_r) = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\phi_n\,(x_n, u_1, u_2, \ldots, u_n) = 0.$$

Then, we have $J = \dfrac{\partial (u_1, u_2, \ldots, u_n)}{\partial (x_1, x_2, \ldots, x_n)} = (-1)^n \dfrac{\dfrac{\partial (\phi_1, \phi_2, \ldots, \phi_n)}{\partial (x_1, x_2, \ldots, x_n)}}{\dfrac{\partial (\phi_1, \phi_2, \ldots, \phi_n)}{\partial (u_1, u_2, \ldots, u_n)}}$

$$= (-1)^n \frac{\frac{\partial \phi_1}{\partial x_1} \frac{\partial \phi_2}{\partial x_2} \cdots \frac{\partial \phi_n}{\partial x_n}}{\frac{\partial \phi_1}{\partial u_1} \frac{\partial \phi_2}{\partial u_2} \cdots \frac{\partial \phi_n}{\partial u_n}}. \qquad \text{[See note after article 1]}$$

Now, if $J = 0$, we have

$$\frac{\partial \phi_1}{\partial x_1} \cdot \frac{\partial \phi_2}{\partial x_2} \cdots \frac{\partial \phi_r}{\partial x_r} \cdots \frac{\partial \phi_n}{\partial x_n} = 0$$

i.e., $\dfrac{\partial \phi_r}{\partial x_r} = 0$ for *some value* of r between 1 and n.

Hence, for that particular value of r the function ϕ_r must not contain x_r; and accordingly the corresponding equation is of the form

$$\phi_r\,(x_{r+1}, \ldots, x_n, u_1, u_2, \ldots, u_r) = 0.$$

Consequently between this and the remaining equations $\phi_{r+1} = 0, \phi_{r+2} = 0, \ldots, \phi_n = 0$, the variables $x_{r+1}, x_{r+2}, \ldots, x_n$ can be eliminated so as to give a final equation between $u_1, u_2, \ldots, u_n$ alone.

Hence the theorem is established.

Illustrative Examples

Example 6: *Show that the functions*

$$u = x + y - z, v = x - y + z, w = x^2 + y^2 + z^2 - 2yz$$

are not independent of one another. Also find the relation between them.

(Garhwal 2000; Lucknow 10)

Solution: Here

$$\frac{\partial (u, v, w)}{\partial (x, y, z)} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2(y-z) & 2(z-y) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2x & 2(y-z) & 0 \end{vmatrix},$$

adding C_2 to C_3

$$= 0.$$

Since the Jacobian is zero, the functions are not independent.

Now $u + v = 2x$ and $u - v = 2(y - z)$.

Therefore $(u + v)^2 + (u - v)^2 = 4(x^2 + y^2 + z^2 - 2yz) = 4w$.

This is the required relation between u, v, w.

Example 7: *Show that* $ax^2 + 2hxy + by^2$ *and* $Ax^2 + 2Hxy + By^2$ *are independent unless*

$$\frac{a}{A} = \frac{h}{H} = \frac{b}{B}.$$

Solution: Let $u = ax^2 + 2hxy + by^2, v = Ax^2 + 2Hxy + By^2$. If the functions u, v are not independent, then

$$\frac{\partial (u, v)}{\partial (x, y)} = 0 \quad \text{or} \quad \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = 0$$

or $$\begin{vmatrix} 2(ax + hy) & 2(hx + by) \\ 2(Ax + Hy) & 2(Hx + By) \end{vmatrix} = 0$$

or $$(ax + hy)(Hx + By) - (hx + by)(Ax + Hy) = 0$$

or $$(aH - Ah)x^2 + (aB - Ab)xy + (Bh - bH)y^2 = 0.$$

Since the variables x, y are independent, the coefficients of x^2 and y^2 in the above equation must be separately zero. Hence, we have

$$aH - Ah = 0 \quad \text{and} \quad Bh - bH = 0$$

whence $$\frac{a}{A} = \frac{h}{H} = \frac{b}{B}.$$

Comprehensive Exercise 3

1. If $u = x^2 + y^2 + z^2, v = x + y + z, w = xy + yz + zx$, show that the Jacobian $\dfrac{\partial (u, v, w)}{\partial (x, y, z)}$ vanishes identically. Also find the relation between u, v and w. **(Avadh 2014)**

2. If $u = (x + y) / (1 - xy)$ and $v = \tan^{-1} x + \tan^{-1} y$, find $\dfrac{\partial (u, v)}{\partial (x, y)}$. Are u and v functionally related ? If so, find the relationship.

3. If the functions u, v, w of three independent variables x, y, z are not independent, prove that the Jacobian of u, v, w with respect to x, y, z vanishes.

4. Show that the functions $u = 3x + 2y - z, v = x - 2y + z$ and $w = x(x + 2y - z)$ are not independent and find the relation between them.

5. Show that the functions
$$u = x + y + z, v = xy + yz + zx, w = x^3 + y^3 + z^3 - 3xyz$$
are not independent. Find the relation between them. **(Meerut 2013B)**

6. If $u = x + 2y + z, v = x - 2y + 3z$ and $w = 2xy - xz + 4yz - 2z^2$, show that they are not independent. Find the relation between u, v and w. **(Lucknow 2009, 11)**

7. If $u = \dfrac{x + y}{z}, v = \dfrac{y + z}{x}, w = \dfrac{y(x + y + z)}{xz}$, show that u, v, w are not independent and find the relation between them.

8. If $u = x + y + z + t, v = x + y - z - t, w = xy - zt, r = x^2 + y^2 - z^2 - t^2$, show that $\dfrac{\partial (u, v, w, r)}{\partial (x, y, z, t)} = 0$ and hence find a relation between u, v, w and r.

9. If $f(0) = 0$ and $f'(x) = \dfrac{1}{1 + x^2}$, prove without using the method of integration, that $f(x) + f(y) = f\left(\dfrac{x + y}{1 - xy}\right)$. **(Meerut 2012B, 13; Gorakhpur 13, 14)**

Answers 3

1. $v^2 = u + 2w$	2. $u = \tan v$	4. $u^2 - v^2 = 8w$
5. $u^3 = 3uv + w$	6. $u^2 - v^2 = 4w$	7. $uv = w + 1$
8. $uv = r + 2w$.		

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If $x = r\cos\theta,\ y = r\sin\theta,$ then

(a) $\dfrac{\partial(x, y)}{\partial(r,\theta)} = r$ (b) $\dfrac{\partial(x, y)}{\partial(r,\theta)} = \dfrac{1}{r}$

(c) $\dfrac{\partial(x, y)}{\partial(r,\theta)} = r^2$ (d) $\dfrac{\partial(x, y)}{\partial(r,\theta)} = \dfrac{1}{r^2}$ **(Kumaun 2007)**

2. If $u = \dfrac{y^2}{2x},\ v = \dfrac{x^2 + y^2}{2x}$, then

(a) $\dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{y}{2x}$ (b) $\dfrac{\partial(u, v)}{\partial(x, y)} = -\dfrac{y}{2x}$

(c) $\dfrac{\partial(u, v)}{\partial(x, y)} = \dfrac{2x}{y}$ (d) $\dfrac{\partial(u, v)}{\partial(x, y)} = -\dfrac{2x}{y}$ **(Kumaun 2011)**

3. The Jacobian $\dfrac{\partial(u, v)}{\partial(x, y)}$ for the functions $u = e^x \sin y,\ v = x + \log\sin y$ is

(a) 0 (b) 1

(c) -1 (d) e^x

4. If $u = x\sin y$ and $v = y\sin x$, then $\dfrac{\partial(u, v)}{\partial(x, y)}$ is

(a) 1 (b) -1

(c) 0 (d) $\sin x\sin y - xy\cos x\cos y$

5. If $x = r\cos\theta,\ y = r\sin\theta,\ z = z$, then $\dfrac{\partial(x, y, z)}{\partial(r,\theta, z)}$ is

(a) $-r$ (b) r

(c) $-\dfrac{1}{r}$ (d) $\dfrac{1}{r}$

6. The two functions $u(x, y)$ and $v(x, y)$ are functionally dependent if their

(a) Jacobian is zero (b) Jacobian is not zero

(c) Product is real (d) none of these

7. If $J = \dfrac{\partial(u, v)}{\partial(x, y)}$ and $J' = \dfrac{\partial(x, y)}{\partial(u, v)}$ then

(a) $JJ' = 0$ (b) $JJ' = 1$

(c) $JJ' = -1$ (d) none of these

8. If $x = r\cos\theta,\ y = r\sin\theta$, then $\dfrac{\partial(r,\theta)}{\partial(x,y)} \times \dfrac{\partial(x,y)}{\partial(r,\theta)}$ is equal to

(a) 0 (b) 1

(c) 2 (d) ∞ **(Kumaun 2015)**

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. If u and v are functions of two independent variables x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \cdots \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}.$$

2. If $x = r\cos\theta,\ y = r\sin\theta$, then $\dfrac{\partial(r,\theta)}{\partial(x,y)} = \ldots.$

3. $\dfrac{\partial(u,v)}{\partial(x,y)} \times \dfrac{\partial(x,y)}{\partial(u,v)} = \ldots.$

4. If $x = u(1+v),\ y = v(1+u)$, then $\dfrac{\partial(x,y)}{\partial(u,v)} = \ldots.$

5. If $u = xyz,\ v = xy + yz + xz,\ w = x + y + z$, then $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \ldots.$ **(Lucknow 2010)**

6. If $u = \dfrac{2yz}{x}, v = \dfrac{3zx}{y}, w = \dfrac{4xy}{z}$, then $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \ldots.$

7. If $u = \dfrac{yz}{x}, v = \dfrac{zx}{y}, w = \dfrac{xy}{z}$, then $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \ldots.$

True or False

Write 'T' for true and 'F' for false statement.

1. If $u_1 = \dfrac{x_2 x_3}{x_1}, u_2 = \dfrac{x_3 x_1}{x_2}, u_3 = \dfrac{x_1 x_2}{x_3}$, then $\dfrac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = -4.$

2. If u_1, u_2 are functions of y_1, y_2 and y_1, y_2 are functions of x_1, x_2, then

$$\frac{\partial(u_1,u_2)}{\partial(x_1,x_2)} = \frac{\partial(u_1,u_2)}{\partial(y_1,y_2)} \cdot \frac{\partial(y_1,y_2)}{\partial(x_1,x_2)}.$$

3. If $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$, then $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta.$

4. If u, v and w are functions of three independent variables x, y and z, then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} \cdot \frac{\partial(x,y,z)}{\partial(u,v,w)} = 0.$$

5. If u_1, u_2 and u_3 are functions of three independent variables x_1, x_2 and x_3 connected by the equations

$$F_1(u_1, u_2, u_3, x_1, x_2, x_3) = 0,$$
$$F_2(u_1, u_2, u_3, x_1, x_2, x_3) = 0,$$
and $$F_3(u_1, u_2, u_3, x_1, x_2, x_3) = 0,$$

then $$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = \frac{\dfrac{\partial(F_1, F_2, F_3)}{\partial(x_1, x_2, x_3)}}{\dfrac{\partial(F_1, F_2, F_3)}{\partial(u_1, u_2, u_3)}}.$$

6. The functions $u = x^2 + y^2 + z^2$, $v = x + y + z$, $w = xy + yz + zx$ are not independent of each other.

7. If $x + y + z = u$, $y + z = uv$, $z = uvw$, then $\dfrac{\partial(x, y, z)}{\partial(u, v, w)} = u^2 v$. **(Lucknow 2006, 08)**

8. If u, v, w are functions of three independent variables x, y, z, then u, v, w are not independent of each other if $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} \neq 0$.

9. The functional relation between the functions $u = \dfrac{3x^2}{2(y+z)}$, $v = \dfrac{2(y+z)}{3(x-y)^2}$, $w = \dfrac{x-y}{x}$ is $uvw^2 = 1$.

Answers

Multiple Choice Questions

1.	(a)	2.	(b)	3.	(a)	4.	(d)	5.	(b)
6.	(a)	7.	(b)	8.	(b)				

Fill in the Blank(s)

1.	$\dfrac{\partial u}{\partial y}$.	2.	$\dfrac{1}{r}$.	3.	1	4.	$1 + u + v$.
5.	$(x-y)(y-z)(z-x)$			6.	96	7.	4

True or False

1.	*F*	2.	*T*	3.	*T*	4.	*F*	5.	*F*
6.	*T*	7.	*T*	8.	*F*	9.	*T*		

□

Chapter

3

Maxima and Minima

(Of Functions of Several Independent Variables)

1 Definition

Let $f(x, y, z, \ldots)$ be any function of several independent variables $x, y, z, \ldots$ supposed to be continuous for all values of these variables in the neighbourhood of their values $a, b, c, \ldots$ respectively. Then $f(a, b, c, \ldots)$ is said to be a **maximum** or a **minimum** value of $f(x, y, z, \ldots)$ according as $f(a+h, b+k, c+l, \ldots)$ is **less** or **greater** than $f(a, b, c, \ldots)$ for all sufficiently small independent values of $h, k, l, \ldots$, positive or negative, provided they are not all zero.

2 Necessary Conditions for the Existence of Maxima or Minima

From the definition it is obvious that we shall have a maximum or a minimum of $f(x, y, z, \ldots)$ for those values of $x, y, z, \ldots$ for which the expression $f(x+h, y+k, z+l, \ldots) - f(x, y, z, \ldots)$ is of **invariable sign** for all sufficiently small independent values of $h, k, l, \ldots$ provided they are not all equal to zero. There will be a maximum or a minimum according as this sign is negative or positive.

Expanding by Taylor's theorem for several variables, we have

$$f\,(x+h,\,y+k,z+l,\ldots)$$

$$=\left[1+\frac{1}{1!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}+l\frac{\partial}{\partial z}+\ldots\right)\right.$$

$$\left.+\frac{1}{2\,!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}+l\frac{\partial}{\partial z}+\ldots\right)^2+\ldots\right]\,f\,(x,y,z,\ldots).$$

$\therefore$

$$f\,(x+h,\,y+k,z+l,\ldots)-f\,(x,y,z\,,\ldots)$$

$$=\left(h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}+l\frac{\partial f}{\partial z}+\ldots\right)+\text{terms of the second}$$

and higher orders in $h, k, l, \ldots$. ...(1)

Now by taking $h, k, l, \ldots$ sufficiently small, the first degree terms in $h, k, l, \ldots$ can be made to govern the sign of the right hand side and therefore of the left hand side of (1). But if $h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}+l\frac{\partial f}{\partial z}+\ldots$, is not equal to zero, the sign of this expression will change by changing the sign of each of $h, k, l, \ldots$,. Hence as a necessary condition for the occurrence of a maximum or a minimum of $f\,(x,y,z,\ldots)$, we must have

$$h\frac{\partial f}{\partial x}+k\frac{\partial f}{\partial y}+l\frac{\partial f}{\partial z}+\ldots=0\,. \qquad \ldots(2)$$

Since (2) is true whatever be the values of $h, k, l, \ldots$ independent of each other, we must have as a necessary consequence

$$\boldsymbol{\frac{\partial f}{\partial x}=0,\ \frac{\partial f}{\partial y}=0,\ \frac{\partial f}{\partial z}=0,\ \ldots}$$

If there are n independent variables, we have then obtained n simultaneous equations to give us the values $a, b, c, \ldots$ of the n variables $x, y, z, \ldots$ for which $f\,(x,y\,,z,\ldots)$ may have a maximum or a minimum value.

The conditions $\frac{\partial f}{\partial x}=0$, $\frac{\partial f}{\partial y}=0$, $\frac{\partial f}{\partial z}=0, \ldots$ *are necessary but not sufficient for the existence of maxima and minima.*

3 Stationary and Extreme Points

A point $(a_1, a_2\,,\ldots, a_n)$ is called a **stationary point**, if all the first order partial derivatives of the function $f\,(x_1, x_2\,,\ldots,\ x_n)$ vanish at that point. Also then the value of the function $f\,(x_1, x_2\,,\ldots, x_n)$ is said to be stationary at that point. A stationary point which is either a maximum or a minimum is called an **extreme point** and the value of the function at that point is called an **extreme value.** A stationary point is not necessarily an extreme point. Thus a stationary value may be a maximum or a minimum or neither of these two. To decide whether a stationary point is really an extreme point, a further investigation is required.

4 Lagrange's Necessary and Sufficient Conditions for the Maxima or Minima of a Function of three Independent Variables

Necessary Conditions: Let $f\ (x, y, z)$ be a function of three independent variables x, y and z . Then as derived in article 4.2, for $f\ (x, y, z)$ to be a maximum or a minimum at any point (a, b, c), it is **necessary** that $\frac{\partial f}{\partial x} = \mathbf{0}, \frac{\partial f}{\partial y} = \mathbf{0}$ **and** $\frac{\partial f}{\partial z} = 0$ at that point.

Hence the points where the value of the function $f\ (x, y, z)$ is stationary (*i.e.*, may be a maximum or a minimum) are obtained by solving the simultaneous equations

$$\frac{\partial f}{\partial x} = 0\ , \frac{\partial f}{\partial y} = 0\ , \frac{\partial f}{\partial z} = 0\ .$$

Sufficient Conditions: Before deriving the sufficient conditions for the existence of a maximum or a minimum of a function of three independent variables, we obtain the following two algebraic lemmas regarding the signs of quadratic expressions.

Lemma 1: Let $I_2 = ax^2 + 2hxy + by^2$ be a quadratic expression in two variables x and y . We can write

$$I_2 = \frac{1}{a}[a^2x^2 + 2ahxy + aby^2]\ , \text{if } a \neq 0$$

$$= \frac{1}{a}[(ax + hy)^2 + (ab - h^2)\ y^2]\ .$$

The expression within the square brackets will be positive if $ab - h^2$ is positive and in that case the sign of the expression I_2 will be the same as that of a .

In case $ab - h^2$ is not positive, we can say nothing about the sign of the expression within the square brackets and hence nothing about the sign of the given quadratic expression I_2 .

Lemma 2: In three variables x, y and z ,

$$I_3 \equiv ax^2 + by^2 + cz^2 + 2\,fyz + 2\,gzx + 2hxy$$

$$= \frac{1}{a}[a^2x^2 + aby^2 + acz^2 + 2\,fayz + 2\,gazx + 2haxy]\ , \text{if } a \neq 0$$

$$= \frac{1}{a}[a^2x^2 + 2ax\,(gz + hy) + aby^2 + acz^2 + 2\,fayz]$$

$$= \frac{1}{a}[(ax + hy + gz)^2 + aby^2 + acz^2 + 2\,fayz - (gz + hy)^2]$$

$$= \frac{1}{a}[(ax + hy + gz)^2 + (ab - h^2)\ y^2 + 2\,yz\ (\ fa - gh) + (ac - g^2)\ z^2].$$

Now I_3 will be of the same sign as a provided the expression within the square brackets is positive which will of course be so if

$$ab - h^2 \text{ and } \{(ab - h^2)\,(ac - g^2) - (\ fa - gh)^2\} \text{ are both positive}$$

i.e., if $ab - h^2$ and $a\,(abc + 2fgh - af^2 - bg^2 - ch^2)$ are both positive.

Thus I_3 *will be positive if* $a, \begin{vmatrix} a & h \\ h & b \end{vmatrix}, \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

be all positive and will be negative if these three expressions are alternately negative and positive.

Now we are in a position to derive Lagrange's sufficient conditions for the existence of a maximum or a minimum of a function of three independent variables at a stationary point.

Let a set of the values of x, y, z obtained by solving the equations

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0 \text{ be } a, b, c.$$

Let the values of the six second order partial derivatives

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial z^2}, \frac{\partial^2 f}{\partial y\, \partial z}, \frac{\partial^2 f}{\partial z\, \partial x} \text{ and } \frac{\partial^2 f}{\partial x\, \partial y}$$

at the point (a, b, c) be denoted by A, B, C, F, G and H respectively.

Then, we have

$$f\,(a + h, b + k, c + l) - f\,(a, b, c)$$
$$= \frac{1}{2!}(Ah^2 + Bk^2 + Cl^2 + 2Fkl + 2Glh + 2Hhk) + R_3, \qquad \ldots(1)$$

where R_3 consists of terms of third and higher orders of small quantities h, k and l. By taking h, k and l sufficiently small, the second degree terms in h, k and l can be made to govern the sign of the right hand side and therefore of the left hand side of (1). If this group of terms forms an expression of invariable sign for all such values of h, k and l, we shall have a maximum or a minimum value of $f\,(x, y, z)$ at (a, b, c) according as that sign is negative or positive.

Hence by our lemma 2, if the expressions

$$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

be all positive, we shall have a minimum of $f\,(x, y, z)$ at (a, b, c) and if these expressions be alternately negative and positive, we shall have a maximum of $f\,(x, y, z)$ at (a, b, c), whilst if these conditions are not satisfied, we shall in general have neither a maximum nor a minimum of $f\,(x, y, z)$ at (a, b, c).

5 Working Rule for Finding the Maxima and Minima of a Function of Three Independent Variables

Suppose $f\,(x, y, z)$ is a given function of three independent variables x, y and z. Find $\partial f / \partial x$, $\partial f / \partial y$ and $\partial f / \partial z$ and solve the simultaneous equations $\partial f / \partial x = 0, \partial f / \partial y = 0$

and $\partial f / \partial z = 0$ .All the triads (a, b, c) of the values of x, y and z obtained on solving these equations will give the stationary values of $f(x, y, z)$ *i.e.*, will give the points at which the function $f(x, y, z)$ may be a maximum or a minimum.

To discuss the maximum or minimum of $f(x, y, z)$ at any point (a, b, c) obtained on solving the equations $\partial f / \partial x = 0$, $\partial f / \partial y = 0$ and $\partial f / \partial z = 0$, we find the values at this point of the six partial derivatives of second order of $f(x, y, z)$ symbolically denoted as follows :

$$A = \frac{\partial^2 f}{\partial x^2}, B = \frac{\partial^2 f}{\partial y^2}, C = \frac{\partial^2 f}{\partial z^2}, F = \frac{\partial^2 f}{\partial y\, \partial z}, G = \frac{\partial^2 f}{\partial z\, \partial x} \text{ and } H = \frac{\partial^2 f}{\partial x\, \partial y}.$$

If the expressions

$$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

be all positive, we shall have a minimum of $f(x, y, z)$ at (a, b, c) and if these expressions be alternately negative and positive, we shall have a maximum of $f(x, y, z)$ at (a, b, c) , whilst if these conditions are not satisfied, we shall in general have neither a maximum nor a minimum of $f(x, y, z)$ at (a, b, c) .

Illustrative Examples

Example 1: *Discuss the maximum or minimum values of u where*

$$u = x^2 + y^2 + z^2 + x - 2z - xy .$$

Solution: For a maximum or a minimum of u , we must have

$$\frac{\partial u}{\partial x} = 2x - y + 1 = 0 ,$$

$$\frac{\partial u}{\partial y} = - x + 2y = 0 ,$$

and $$\frac{\partial u}{\partial z} = 2z - 2 = 0 .$$

These equations give $x = -2/3, y = -1/3, z = 1$.

$\therefore$ $(-2/3, -1/3, 1)$ is the only point at which u is stationary *i.e.*, at which u may have a maximum or a minimum.

Now $$\frac{\partial^2 u}{\partial x^2} = 2, \frac{\partial^2 u}{\partial y^2} = 2, \frac{\partial^2 u}{\partial z^2} = 2, \frac{\partial^2 u}{\partial y\, \partial z} = 0, \frac{\partial^2 u}{\partial z\, \partial x} = 0$$

and $$\frac{\partial^2 u}{\partial x\, \partial y} = -1.$$

If A, B, C, F, G and H denote the respective values of these six partial derivatives of second order at the point $(-2/3, -1/3, 1)$, then

$$A = 2, B = 2, C = 2, F = 0, G = 0, H = -1.$$

Now we have $A = 2$, $\begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$

and $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6$.

Since these three expressions are all positive, we have a minimum of u when $x = -2/3$, $y = -1/3$, $z = 1$.

Example 2: *Show that the point such that the sum of the squares of its distances from n given points shall be minimum, is the centre of the mean position of the given points.*

Solution: Let the n given points be (a_1, b_1, c_1), (a_2, b_2, c_2), ..., (a_n, b_n, c_n) and let (x, y, z) be the coordinates of the required point.

If u denotes the sum of the squares of the distances of (x, y, z) from the n given points, then

$$u = \Sigma\,[(x - a_1)^2 + (y - b_1)^2 + (z - c_1)^2]$$
$$= \Sigma\,(x - a_1)^2 + \Sigma\,(y - b_1)^2 + \Sigma\,(z - c_1)^2.$$

For a maximum or a minimum of u, we must have

$$\frac{\partial u}{\partial x} = 2\Sigma\,(x - a_1) = 2nx - 2\Sigma a_1 = 0,$$

$$\frac{\partial u}{\partial y} = 2\Sigma\,(y - b_1) = 2ny - 2\Sigma b_1 = 0,$$

and
$$\frac{\partial u}{\partial z} = 2\Sigma\,(z - c_1) = 2nz - 2\Sigma c_1 = 0.$$

Solving these equations, we get

$$x = \frac{\Sigma a_1}{n},\ y = \frac{\Sigma b_1}{n},\ z = \frac{\Sigma c_1}{n}.$$

Now
$$A = \frac{\partial^2 u}{\partial x^2} = 2n,\ B = \frac{\partial^2 u}{\partial y^2} = 2n,\ C = \frac{\partial^2 u}{\partial z^2} = 2n,$$

$$F = \frac{\partial^2 u}{\partial y\,\partial z} = 0,\ G = \frac{\partial^2 u}{\partial z\,\partial x} = 0,\ H = \frac{\partial^2 u}{\partial x\,\partial y} = 0.$$

We have
$$A = 2n,\ \begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 2n & 0 \\ 0 & 2n \end{vmatrix} = 4n^2,$$

and
$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2n & 0 & 0 \\ 0 & 2n & 0 \\ 0 & 0 & 2n \end{vmatrix} = 8n^3.$$

Since these three expressions are all positive, u is minimum when

$$x = \frac{\Sigma\,a_1}{n},\ y = \frac{\Sigma\,b_1}{n} \text{ and } z = \frac{\Sigma\,c_1}{n}$$

i.e., u is minimum when the point (x, y, z) is the centre of the mean position of the n given points.

Example 3: *Find the maximum value of u where*

$$u = \frac{xyz}{(a+x)(x+y)(y+z)(z+b)}.$$

Solution: We have

$$\log u = \log x + \log y + \log z - \log(a+x) - \log(x+y) - \log(y+z) - \log(z+b).$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{x} - \frac{1}{a+x} - \frac{1}{x+y} = \frac{ay - x^2}{x(a+x)(x+y)}$$

or
$$\frac{\partial u}{\partial x} = \frac{(ay - x^2)\,u}{x(a+x)(x+y)}.$$

Similarly
$$\frac{\partial u}{\partial y} = \frac{(xz - y^2)\,u}{y(x+y)(y+z)}$$

and
$$\frac{\partial u}{\partial z} = \frac{(by - z^2)\,u}{z(y+z)(z+b)}.$$

Now for a maximum or a minimum of u, we must have

$$\frac{\partial u}{\partial x} = 0 \quad i.e., \quad ay - x^2 = 0$$

$$\frac{\partial u}{\partial y} = 0 \quad i.e., \quad xz - y^2 = 0$$

and
$$\frac{\partial u}{\partial z} = 0 \quad i.e., \quad by - z^2 = 0.$$

From the above equations, it follows that a, x, y, z and b are in geometrical progression. Let r be the common ratio of this geometrical progression. Then

$$ar^4 = b \quad \text{or} \quad r = (b/a)^{1/4}.$$

Also $x = ar$, $y = ar^2$, $z = ar^3$.

Substituting these values, we get

$$u = \frac{ar \,.\, ar^2 \,.\, ar^3}{a(1+r)\,.\,ar(1+r)\,.\,ar^2(1+r)\,.\,ar^3(1+r)}$$

$$= \frac{1}{a(1+r)^4} = \frac{1}{a[1+(b/a)^{1/4}]^4} = \frac{1}{(a^{1/4}+b^{1/4})^4}.$$

This gives a stationary value of u. To decide whether this value of u is a maximum or a minimum we proceed to find the second order partial derivatives of u.

We have
$$\frac{\partial^2 u}{\partial x^2} = \frac{-2xu}{x(a+x)(x+y)} + (ay - x^2)\frac{\partial}{\partial x}\left[\frac{u}{x(a+x)(x+y)}\right].$$

$\therefore$ When $x = ar$, $y = ar^2$, $z = ar^3$, we have

$$A = \frac{\partial^2 u}{\partial x^2} = \frac{-2\,.\,ar\,.\,u}{ar\,.\,a(1+r)\,.\,ar(1+r)}$$

$$= \frac{-2u}{a^2 \, r \, (1+r)^2}, \text{ which is negative.}$$

Hence the above stationary value of u is a maximum.

Ans. Maximum value of $u = \dfrac{1}{(a^{1/4} + b^{1/4})^4}$.

Note: In the complicated problems in order to find whether a stationary value of u is a maximum or a minimum, it is sufficient to find the value of a second partial differential coefficient of u with respect to any of the independent variables. The value of u will be maximum or minimum according as the value of this second partial derivative at the stationary point under consideration is –ive or +ive.

Comprehensive Exercise 1

1. Show that $u = (x + y + z)^3 - 3(x + y + z) - 24xyz + a^3$ has minimum at $(1, 1, 1)$ and maximum at $(-1, -1, -1)$.
2. Find the maximum or minimum values of u where
$$u = axy^2z^3 - x^2y^2z^3 - xy^3z^3 - xy^2z^4.$$
3. Find the maximum value of
$$(ax + by + cz)\, e^{-(\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2)}.$$

Answers 1

2. u is maximum when $x = a/7, y = 2a/7, z = 3a/7$ and the maximum value is $108a^7/7$.
3. $\sqrt{\left\{\dfrac{1}{2e}\left(\dfrac{a^2}{\alpha^2} + \dfrac{b^2}{\beta^2} + \dfrac{c^2}{\gamma^2}\right)\right\}}$

6 Lagrange's Method of Undetermined Multipliers

Let $\quad u = f(x_1, x_2, \ldots, x_n)$

be a function of n variables $x_1, x_2, \ldots, x_n$. Let these variables be connected by m equations

$$\phi_1(x_1, x_2, \ldots, x_n) = 0, \phi_2(x_1, x_2, \ldots, x_n) = 0, \ldots, \phi_m(x_1, x_2, \ldots, x_n) = 0$$

so that only $n - m$ of the n variables are independent.

For a maximum or a minimum of u, we have

$$du=\frac{\partial u}{\partial x_1}dx_1+\frac{\partial u}{\partial x_2}dx_2+\frac{\partial u}{\partial x_3}dx_3+\ldots+\frac{\partial u}{\partial x_n}dx_n=0\,.$$

Also differentiating the m given equations connecting the variables, we get

$$\begin{aligned}
d\phi_1&=\frac{\partial \phi_1}{\partial x_1}dx_1+\frac{\partial \phi_1}{\partial x_2}dx_2+\frac{\partial \phi_1}{\partial x_3}dx_3+\ldots+\frac{\partial \phi_1}{\partial x_n}dx_n=0\,,\\
d\phi_2&=\frac{\partial \phi_2}{\partial x_1}dx_1+\frac{\partial \phi_2}{\partial x_2}dx_2+\frac{\partial \phi_2}{\partial x_3}dx_3+\ldots+\frac{\partial \phi_2}{\partial x_n}dx_n=0\,,\\
&\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\
&\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\
d\phi_m&=\frac{\partial \phi_m}{\partial x_1}dx_1+\frac{\partial \phi_m}{\partial x_2}dx_2+\frac{\partial \phi_m}{\partial x_3}dx_3+\ldots+\frac{\partial \phi_m}{\partial x_n}dx_n=0\,.
\end{aligned}$$

Multiplying the above $m+1$ equations obtained on differentiation by $1, \lambda_1, \lambda_2, \ldots, \lambda_m$ respectively and adding, we get an equation which may be written as

$$P_1\,dx_1+P_2\,dx_2+P_3\,dx_3+\ldots+P_n\,dx_n=0\,, \qquad \ldots(1)$$

where $\qquad P_r=\dfrac{\partial u}{\partial x_r}+\lambda_1\dfrac{\partial \phi_1}{\partial x_r}+\lambda_2\dfrac{\partial \phi_2}{\partial x_r}+\ldots+\lambda_m\dfrac{\partial \phi_m}{\partial x_r}\,.$

Now the m multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$ are at our choice. We choose them such that they satisfy the m linear equations

$$P_1=0\,,\, P_2=0\,,\ldots,\, P_m=0\,.$$

Then the equation (1) reduces to

$$P_{m+1}\,dx_{m+1}+P_{m+2}\,dx_{m+2}+\ldots+P_n\,dx_n=0\,. \qquad \ldots(2)$$

It is immaterial which of the $n-m$ of the n variables $x_1, x_2, \ldots, x_n$ are regarded as independent. Let us regard the $n-m$ variables $x_{m+1}, x_{m+2}, \ldots, x_n$ as independent. Then since the $n-m$ quantities $dx_{m+1}, dx_{m+2}, \ldots, dx_n$ are all independent of one another, their coefficients must be separately zero in the relation (2). Hence we must have

$$P_{m+1}=0\,,\, P_{m+2}=0\,,\ldots,\, P_n=0\,.$$

Thus we get $m+n$ equations

$$P_1=0\,,\, P_2=0\,,\ldots,\, P_n=0$$

and $\qquad \phi_1=0\,,\, \phi_2=0\,,\ldots,\, \phi_m=0\,,$

which together with the relation $u=f(x_1, x_2, \ldots, x_n)$ determine the m multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$, the values of $x_1, x_2, \ldots, x_n$ and u at the stationary point. This method is known as **Lagrange's method of undetermined multipliers.** It is very convenient to apply and it often gives us the maximum or minimum values of u without actually determining the values of the multipliers $\lambda_1, \ldots, \lambda_m$. The only drawback of this method is that it does not determine the nature of the stationary point.

Illustrative Examples

Example 4: *If* $u = x^2 + y^2 + z^2$, *where* $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$, *find the maximum or minimum values of* u.

Solution: We have $u = x^2 + y^2 + z^2$, ...(1)

where the variables x, y, z are connected by the relation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1. \quad ...(2)$$

For a maximum or a minimum of u, we have $du = 0$

$\Rightarrow$ $2x dx + 2y dy + 2z dz = 0$

$\Rightarrow$ $x\,dx + y\,dy + z\,dz = 0$. ...(3)

Also differentiating the given relation (2), we get

$$2ax\,dx + 2by\,dy + 2cz\,dz + 2fy\,dz + 2fz\,dy + 2gz\,dx + 2gx\,dz + 2hx\,dy + 2hy\,dx = 0$$

or $(ax + hy + gz)\,dx + (hx + by + fz)\,dy + (gx + fy + cz)\,dz = 0$. ...(4)

Multiplying (3) by 1, (4) by λ and adding, and then equating the coefficients of dx, dy, dz to zero, we have

$x + \lambda (ax + hy + gz) = 0$, ...(5)

$y + \lambda (hx + by + fz) = 0$, ...(6)

and $z + \lambda (gx + fy + cz) = 0$. ...(7)

Multiplying (5) by x, (6) by y, (7) by z and adding, we get

$$x^2 + y^2 + z^2 + \lambda (ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = 0$$

or $u + \lambda .1 = 0$, using (1) and (2).

$\therefore$ $\lambda = -u$.

Hence from (5), we have

$$x - u(ax + hy + gz) = 0$$

or $x(1 - au) - huy - guz = 0$

or $\left(a - \dfrac{1}{u}\right) x + hy + gz = 0$. ...(8)

Similarly from (6) and (7), we have

$hx + (b - 1/u)\, y + fz = 0$, ...(9)

and $gx + fy + (c - 1/u)\, z = 0$. ...(10)

Eliminating x, y, z from (8), (9), (10), we get

$$\begin{vmatrix} a - (1/u) & h & g \\ h & b - (1/u) & f \\ g & f & c - (1/u) \end{vmatrix} = 0. \quad ...(11)$$

Hence the required maximum or minimum values of u are the roots of the equation (11).

Example 5: *Find the stationary values of* $x^2 + y^2 + z^2$ *subject to the conditions*

$$ax^2 + by^2 + cz^2 = 1 \text{ and } lx + my + nz = 0.$$

Interpret the result geometrically. **(Lucknow 2009)**

Solution: Let $u = x^2 + y^2 + z^2$, ...(1)

where the variables x, y and z are connected by the relations

$$ax^2 + by^2 + cz^2 = 1, \qquad ...(2)$$

and $$lx + my + nz = 0. \qquad ...(3)$$

For a stationary value of u, we have

$$du = 0$$

$\Rightarrow$ $$2x\,dx + 2y\,dy + 2z\,dz = 0$$

$\Rightarrow$ $$x dx + y dy + z dz = 0. \qquad ...(4)$$

Also differentiating the given relations (2) and (3), we get

$$2ax\,dx + 2by\,dy + 2cz\,dz = 0$$

i.e., $$ax\,dx + by\,dy + cz\,dz = 0 \qquad ...(5)$$

and $$l\,dx + m\,dy + n\,dz = 0. \qquad ...(6)$$

Multiplying (4) by 1, (5) by λ and (6) by μ and adding, and then equating the coefficients of dx, dy, dz to zero, we get

$$x + \lambda ax + \mu l = 0, \qquad ...(7)$$

$$y + \lambda by + \mu m = 0, \qquad ...(8)$$

and $$z + \lambda cz + \mu n = 0. \qquad ...(9)$$

Multiplying the equations (7), (8) and (9) by x, y and z respectively and adding, we get

$$x^2 + y^2 + z^2 + \lambda (ax^2 + by^2 + cz^2) + \mu (lx + my + nz) = 0,$$

or $$u + \lambda .1 + \mu .0 = 0, \text{ using (1), (2) and (3)}$$

or $$\lambda = -u.$$

Substituting for λ in the equations (7), (8) and (9), we get

$$x = \frac{\mu l}{au - 1}, y = \frac{\mu m}{bu - 1}, z = \frac{\mu n}{cu - 1}.$$

Substituting these values of x, y, z in (3), we get

$$\frac{\mu l^2}{au - 1} + \frac{\mu m^2}{bu - 1} + \frac{\mu n^2}{cu - 1} = 0$$

or $$\frac{l^2}{au - 1} + \frac{m^2}{bu - 1} + \frac{n^2}{cu - 1} = 0. \qquad ...(10)$$

Hence the stationary (*i.e.*, maximum or minimum) values of u are given by the equation (10). The equation (10) is a quadratic in u and so it gives two stationary values of u.

Geometrical Interpretation: The surface $ax^2 + by^2 + cz^2 = 1$ represents an ellipsoid (or a hyperboloid) whose centre is origin, and $lx + my + nz = 0$ is a plane passing through the origin. Therefore the point (x, y, z) satisfying both the conditions

(2) and (3) lies on the conic in which (2) and (3) intersect. Also $x^2 + y^2 + z^2$ gives the square of the distance of (x, y, z) from the origin which is also the centre of the conic of intersection. The maximum and minimum values of this distance are the major and minor semi-axes of the conic. So the equation (10) gives the squares of the lengths of the semi-axes of the conic of intersection.

Example 6: *Find the maximum and minimum values of*

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4},$$

when $lx + my + nz = 0$ *and* $x^2 / a^2 + y^2 / b^2 + z^2 / c^2 = 1$.

Interpret the result geometrically. **(Lucknow 2011)**

Solution: Let $u = x^2 / a^4 + y^2 / b^4 + z^2 / c^4$. Then for a maximum or a minimum of u, we have

$$du = 0 \quad \Rightarrow \quad \frac{x}{a^4}\, dx + \frac{y}{b^4}\, dy + \frac{z}{c^4}\, dz = 0 \qquad \text{...(1)}$$

Also differentiating the two given equations connecting the variables x, y and z, we get

$$l\, dx + m\, dy + n\, dz = 0, \qquad \text{...(2)}$$

and
$$\frac{x}{a^2}\, dx + \frac{y}{b^2}\, dy + \frac{z}{c^2}\, dz = 0. \qquad \text{...(3)}$$

Multiplying (1), (2) and (3) by 1, λ and μ respectively and adding, and then equating to zero the coefficients of dx, dy and dz, we get

$$\frac{x}{a^4} + \lambda l + \mu \frac{x}{a^2} = 0, \qquad \text{...(4)}$$

$$\frac{y}{b^4} + \lambda m + \mu \frac{y}{b^2} = 0, \qquad \text{...(5)}$$

and
$$\frac{z}{c^4} + \lambda n + \mu \frac{z}{c^2} = 0. \qquad \text{...(6)}$$

Multiplying the equations (4), (5) and (6) by x, y and z respectively and adding, we get

$$u + \lambda .0 + \mu .1 = 0 \quad \text{or} \quad \mu = -u.$$

Putting $\mu = -u$ in (4), we get

$$\frac{x}{a^4} + \lambda l - \frac{xu}{a^2} = 0, \quad \text{or} \quad \frac{x}{a^2}\left\{u - \frac{1}{a^2}\right\} = \lambda l, \quad \text{or} \quad x = \frac{\lambda l a^4}{a^2 u - 1}.$$

Similarly from (5) and (6), we get

$$y = \frac{\lambda m b^4}{b^2 u - 1} \quad \text{and} \quad z = \frac{\lambda n c^4}{c^2 u - 1}.$$

Substituting these values of x, y, z in $lx + my + nz = 0$, we get

$$\frac{l^2 a^4}{a^2 u - 1} + \frac{m^2 b^4}{b^2 u - 1} + \frac{n^2 c^4}{c^2 u - 1} = 0. \qquad \text{...(7)}$$

The equation (7) gives the required maximum or minimum values of u.

Geometrical Interpretation: The equation of the tangent plane to the ellipsoid $x^2 / a^2 + y^2 / b^2 + z^2 / c^2 = 1$ at any point (x, y, z) on it is

$$\frac{Xx}{a^2} + \frac{Yy}{b^2} + \frac{Zz}{c^2} = 1. \qquad ...(8)$$

If p be the length of the perpendicular from origin which is also the centre of the ellipsoid to the tangent plane (1), then

$$p^2 = \frac{1}{x^2 / a^4 + y^2 / b^4 + z^2 / c^4}.$$

If the point (x, y, z) on the ellipsoid also lies on the given plane $lx + my + nz = 0$, the problem consists of finding out the maximum or minimum values of the perpendicular distance from the origin to the tangent planes to the ellipsoid at the points common to the plane $lx + my + nz = 0$ and the ellipsoid.

Example 7: *Prove that of all rectangular parallelopipeds of the same volume, the cube has the least surface.*

Solution: Let x, y, z be the dimensions of the rectangular parallelopiped, V be its volume and S be its surface. Then

$$S = 2xy + 2yz + 2zx \qquad ...(1)$$

and $$xyz = V = \text{some constant}. \qquad ...(2)$$

For a maximum or a minimum of S, we have

$$dS = 2(y + z)\,dx + 2(z + x)\,dy + 2(x + y)\,dz = 0$$

i.e., $$(y + z)\,dx + (z + x)\,dy + (x + y)\,dz = 0. \qquad ...(3)$$

Also differentiating (2) and observing that V is constant, we have

$$yz\,dx + zx\,dy + xy\,dz = 0. \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx, dy and dz, we get

$$(y + z) + \lambda yz = 0, \qquad ...(5)$$

$$(z + x) + \lambda zx = 0, \qquad ...(6)$$

and $$(x + y) + \lambda xy = 0. \qquad ...(7)$$

These give $-\lambda = \frac{1}{y} + \frac{1}{z} = \frac{1}{z} + \frac{1}{x} = \frac{1}{x} + \frac{1}{y}$.

$\therefore$ $$\frac{1}{y} - \frac{1}{x} = 0 \quad \text{or} \quad x = y.$$

Similarly $y = z$.

Hence for a stationary value of S, we have

$$x = y = z = V^{1/3}, \text{ from (2)}.$$

Thus S is stationary when the rectangular parallelopiped is a cube.

Let us now find the nature of this stationary value of S.

Here S is a function of three variables x, y, z which are connected by the relation (2) so that only two variables are independent. Let us regard x and y as independent variables and z to be dependent on them.

Then from (1), $\dfrac{\partial S}{\partial x} = 2y + 2y\dfrac{\partial z}{\partial x} + 2z + 2x\dfrac{\partial z}{\partial x}\cdot$

Also from (2), $yz + xy\dfrac{\partial z}{\partial x} = 0$ *i.e.*, $\dfrac{\partial z}{\partial x} = -\dfrac{z}{x}\cdot$

$$\therefore \qquad \frac{\partial S}{\partial x} = 2y - \frac{2yz}{x} + 2z - 2z = 2y - \frac{2yz}{x}$$

and
$$\frac{\partial^2 S}{\partial x^2} = \frac{2yz}{x^2} - \frac{2y}{x}\cdot\frac{\partial z}{\partial x} = \frac{2yz}{x^2} + \frac{2yz}{x^2} = \frac{4yz}{x^2} = 4 \text{ at } x = y = z\,.$$

Similarly by symmetry $\dfrac{\partial^2 S}{\partial y^2} = 4$ at $x = y = z$.

Also
$$\frac{\partial^2 S}{\partial x\,\partial y} = 2 - \frac{2z}{x} - \frac{2y}{x}\frac{\partial z}{\partial y}\cdot$$

But differentiating (2) partially w.r.t. y taking x as constant, we get

$$xz + xy\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad \frac{\partial z}{\partial y} = -\frac{z}{y}\cdot$$

$$\therefore \qquad \frac{\partial^2 S}{\partial x\,\partial y} = 2 - \frac{2z}{x} - \frac{2y}{x}\left(-\frac{z}{y}\right) = 2 - \frac{2z}{x} + \frac{2z}{x} = 2\,.$$

Thus at the stationary point $x = y = z = V^{1/3}$, we have

$$r = \frac{\partial^2 S}{\partial x^2} = 4\,, s = \frac{\partial^2 S}{\partial x\,\partial y} = 2 \text{ and } t = \frac{\partial^2 S}{\partial y^2} = 4\,.$$

$$\therefore \qquad rt - s^2 = 4\times 4 - 2^2 = 12 \text{ which is} > 0\,.$$

Also $r = 4$ which is > 0.

Hence the stationary value of S given by $x = y = z = V^{1/3}$ is a minimum.

Thus of all rectangular parallelopipeds of the same volume, the cube has the least surface.

Example 8: *Discuss the maxima and minima of the function*
$u = \sin x \sin y \sin z$, where x, y, z are the angles of a triangle.

Solution: We have $u = \sin x \sin y \sin z$, ...(1)

where $\qquad x + y + z = \pi$. ...(2)

For a maximum or a minimum of u, we must have

$$du = \cos x \sin y \sin z\, dx + \sin x \cos y \sin z\, dy + \sin x \sin y \cos z\, dz$$
$$= 0\,. \qquad ...(3)$$

Also from (2), we have

$$dx + dy + dz = 0\,. \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ and adding and then equating to zero the coefficients of dx, dy, dz, we get

$$\cos x \sin y \sin z + \lambda = 0,$$

$$\sin x \cos y \sin z + \lambda = 0,$$

and $$\sin x \sin y \cos z + \lambda = 0.$$

From these, we get

$$-\lambda = \cos x \sin y \sin z = \sin x \cos y \sin z = \sin x \sin y \cos z$$

or $$\cot x = \cot y = \cot z \qquad \text{[Dividing by } \sin x \sin y \sin z]$$

i.e., $$x = y = z = \pi/3. \qquad \text{[From (2)]}$$

Thus u is stationary when $x = y = z = \pi/3$.

Let us now find the nature of this stationary value of u.

Since variables x, y and z are connected by the relation (2), only two of them may be regarded as independent.

Let us regard x and y as independent and z to be dependent on them by the relation (2).

Then from (1),

$$\frac{\partial u}{\partial x} = \sin y \sin z \cos x + \sin x \sin y \cos z \, \frac{\partial z}{\partial x}.$$

Also form (2),

$$1 + \frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial z}{\partial x} = -1.$$

$\therefore$ $$\frac{\partial u}{\partial x} = \sin y \sin z \cos x - \sin x \sin y \cos z$$

and $$\frac{\partial^2 u}{\partial x^2} = -\sin y \sin z \sin x + \sin y \cos x \cos z \, \frac{\partial z}{\partial x}$$
$$- \cos x \sin y \cos z + \sin x \sin y \sin z \, \frac{\partial z}{\partial x}$$
$$= -2 \sin x \sin y \sin z - 2 \sin y \cos x \cos z.$$

Also $$\frac{\partial^2 u}{\partial x \, \partial y} = \cos y \sin z \cos x + \sin y \cos x \cos z \, \frac{\partial z}{\partial y}$$
$$- \sin x \cos y \cos z + \sin x \sin y \sin z \, \frac{\partial z}{\partial y}$$
$$= \cos y \sin z \cos x - \sin y \cos x \cos z - \sin x \cos y \cos z.$$
$$- \sin x \sin y \sin z$$

Hence putting $x = y = z = \pi/3$, we get

$$r = \frac{\partial^2 u}{\partial x^2} = -2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = -\frac{3\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = -\sqrt{3},$$

$$s = \frac{\partial^2 u}{\partial x \, \partial y} = \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8} - \frac{3\sqrt{3}}{8} = -\frac{\sqrt{3}}{2}.$$

Also by symmetry, $t = \dfrac{\partial^2 u}{\partial y^2} = -\sqrt{3}$.

$\therefore$ At the stationary point $x = y = z = \pi / 3$, we have

$$rt - s^2 = 3 - (3/4) = 9/4 \text{ which is} > 0$$

and $\quad r = -\sqrt{3}$ which is < 0.

$\therefore$ At the stationary point $x = y = z = \pi / 3$, u is maximum.

Hence u is maximum when $x = y = z = \pi / 3$ and the maximum value of

$$u = \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}.$$

Example 9: *Show that the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \textit{is} \quad \frac{8abc}{3\sqrt{3}}.$$

(Gorakhpur 2010)

Solution: Let (x, y, z) denote the coordinates of the vertex of the rectangular parallelopiped which lies in the positive octant and let V denote its volume. Then, we have to find the maximum value of

$$V = 8xyz \qquad \text{...(1)}$$

subject to the condition

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \qquad \text{...(2)}$$

For a maximum or a minimum of V, we have

$$dV = 8\,yzdx + 8zxdy + 8xydz = 0$$

i.e., $$yz\,dx + zx\,dy + xy\,dz = 0. \qquad \text{...(3)}$$

Also differentiating (2), we get

$$\frac{2x}{a^2}\,dx + \frac{2y}{b^2}\,dy + \frac{2z}{c^2}\,dz = 0$$

i.e., $$\frac{x}{a^2}\,dx + \frac{y}{b^2}\,dy + \frac{z}{c^2}\,dz = 0. \qquad \text{...(4)}$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero, we get

$$yz + \frac{\lambda x}{a^2} = 0,\ zx + \frac{\lambda y}{b^2} = 0 \text{ and } xy + \frac{\lambda z}{c^2} = 0.$$

From these, we get

$$\frac{x}{a^2} = -\frac{yz}{\lambda},\ \frac{y}{b^2} = -\frac{zx}{\lambda},\ \frac{z}{c^2} = -\frac{xy}{\lambda}$$

or $$\frac{x^2}{a^2} = -\frac{xyz}{\lambda},\ \frac{y^2}{b^2} = -\frac{xyz}{\lambda},\ \frac{z^2}{c^2} = -\frac{xyz}{\lambda}$$

or $$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{x^2/a^2 + y^2/b^2 + z^2/c^2}{3} = \frac{1}{3}, \text{ using (2)}$$

or $$x = a/\sqrt{3},\ y = b/\sqrt{3},\ z = c/\sqrt{3}.$$

Thus V is stationary when $x = a/\sqrt{3},\ y = b/\sqrt{3},\ z = c/\sqrt{3}$.

Now regard x and y as independent variables and z as a function of x and y given by (2).

Then from (1), $\dfrac{\partial V}{\partial x} = 8yz + 8xy\dfrac{\partial z}{\partial x}$.

Differentiating (2) partially w.r.t. x taking y as constant, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial z}{\partial x} = -\frac{c^2 x}{a^2 z}.$$

$\therefore$ $$\frac{\partial V}{\partial x} = 8yz + 8xy \cdot \left(-\frac{c^2 x}{a^2 z}\right) = 8yz - \frac{8c^2 x^2 y}{a^2 z}$$

and so $$\frac{\partial^2 V}{\partial x^2} = 8y\frac{\partial z}{\partial x} - \frac{16c^2 x y}{a^2 z} + \frac{8c^2 x^2 y}{a^2 z^2} \cdot \frac{\partial z}{\partial x}$$

$$= 8y \cdot \left(\frac{-c^2 x}{a^2 z}\right) - \frac{16c^2 x y}{a^2 z} - \frac{8c^2 x^2 y}{a^2 z} \cdot \frac{c^2 x}{a^2 z},$$

which is –ive when $x = a/\sqrt{3},\ y = b/\sqrt{3},\ z = c/\sqrt{3}$.

Hence V is maximum when $x = a/\sqrt{3},\ y = b/\sqrt{3},\ z = c/\sqrt{3}$ and the maximum value of $V = \dfrac{8abc}{3\sqrt{3}}$.

Note: In complicated problems to show that whether the stationary value of a function is maximum or minimum, it will be sufficient to see whether the second partial differential coefficient of the function w.r.t. any of the independent variables is negative or positive.

Example 10: *A rectangular box open at the top is to have a given capacity. Find the dimensions of the box requiring least material for its construction.*

Solution: Let the given capacity of the box be V, its three edges be x, y, z and its surface be S. Then

$$S = xy + 2yz + 2zx \qquad \text{...(1)}$$

and $$xyz = V. \qquad \text{...(2)}$$

For a maximum or a minimum of S, we have

$$dS = (y + 2z)\,dx + (x + 2z)\,dy + 2(x + y)\,dz = 0. \qquad \text{...(3)}$$

Also from (2), since V is constant, we have $yz\,dx + zx\,dy + xy\,dz = 0$. ...(4)

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx, dy and dz, we get

$$(y + 2z) + \lambda yz = 0, \qquad \text{...(5)}$$

$$(x + 2z) + \lambda zx = 0, \qquad \text{...(6)}$$

and $$2x + 2y + \lambda xy = 0. \qquad \text{...(7)}$$

Multiplying (5) by x, (6) by y and subtracting, we get

$$2zx - 2zy = 0 \text{ or } 2\,z\,(x - y) = 0\text{, or } x = y.$$

[The root $z = 0$ is inadmissible because the depth of the box cannot be zero.]

Similarly, from the equations (6) and (7), we get $y = 2z$.

Hence the dimensions of the box for a stationary value of S are

$$x = y = 2\,z = (2V)^{1/3}, \text{ from (2).}$$

Let us now find the nature of this stationary value of S.

Regard x and y as independent variables and z as a function of x and y given by (2).

Then from (1), $\dfrac{\partial S}{\partial x} = y + 2\,y\,\dfrac{\partial z}{\partial x} + 2\,z + 2x\,\dfrac{\partial z}{\partial x}\cdot$

Differentiating (2) partially w.r.t. x taking y as constant, we get

$$yz + xy\,\frac{\partial z}{\partial x} = 0 \text{ i.e., } \frac{\partial z}{\partial x} = -\frac{z}{x}\cdot$$

$$\therefore \qquad \frac{\partial S}{\partial x} = y - \frac{2\,yz}{x} + 2z - 2z = y - \frac{2\,yz}{x}$$

and so $$\frac{\partial^2 S}{\partial x^2} = \frac{2\,yz}{x^2} - \frac{2\,y}{x}\cdot\frac{\partial z}{\partial x} = \frac{2\,yz}{x^2} + \frac{2\,yz}{x^2} = \frac{4\,yz}{x^2} = 2 \text{ at } x = y = 2z.$$

Thus at the stationary point $x = y = 2\,z = (2V)^{1/3}$, we have

$$r = \frac{\partial^2 S}{\partial x^2} = 2\text{, which is positive.}$$

Similarly we can find $s = \dfrac{\partial^2 S}{\partial x\,\partial y}$ and $t = \dfrac{\partial^2 S}{\partial y^2}$

at the stationary point $x = y = 2\,z = (2V)^{1/3}$ and can show that $r\,t - s^2$ is positive.

Since at the stationary point $x = y = 2\,z = (2V)^{1/3}$, $r\,t - s^2 > 0$ and $r > 0$, therefore the stationary value of S at this point is a minimum.

Hence the dimensions of the box requiring least material for its construction are given by $x = y = 2\,z = (2V)^{1/3}$.

Comprehensive Exercise 2

1. Find the maximum and minimum values of $u = a^2x^2 + b^2y^2 + c^2z^2$, where $x^2 + y^2 + z^2 = 1$ and $lx + my + nz = 0$.

2. Show that the maximum and minimum values of $u = x^2 + y^2 + z^2$ subject to the conditions $px + qy + rz = 0$ and $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ are given by the equation $\dfrac{a^2p^2}{u - a^2} + \dfrac{b^2q^2}{u - b^2} + \dfrac{c^2r^2}{u - c^2} = 0$.

3. Find the maximum and minimum values of u^2 when

$$u^2 = a^2x^2 + b^2y^2 + c^2z^2$$

while $x^2 + y^2 + z^2 = 1$ and $lx + my + nz = 0$. **(Lucknow 2008)**

4. Find the maximum value of $x^m y^n z^p$ subject to the condition

$$x + y + z = a.$$

5. Find the minimum value of $x + y + z$, subject to the condition

$$(a / x) + (b / y) + (c / z) = 1.$$

6. Find the minimum value of $x^2 + y^2 + z^2$, given that

$$ax + by + cz = p.$$ **(Lucknow 2006)**

7. Find the maximum or minimum value of $x^p y^q z^r$ subject to the condition

$$ax + by + cz = p + q + r.$$

8. Find the maximum or minimum value of $x^p y^q z^r$ subject to the condition

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1.$$

9. Find the minimum value of $x^4 + y^4 + z^4$, where $xyz = c^3$.

10. Divide a number a into three parts such that their product will be maximum.

11. In a plane triangle ABC, find the maximum value of

$$u = \cos A \cos B \cos C.$$ **(Lucknow 2007)**

12. Find a plane triangle ABC such that $u = \sin^m A \sin^n B \sin^p C$

has maximum value.

13. Show that if the perimeter of a triangle is constant, its area is a maximum when it is equilateral.

14. Find the triangle of maximum area inscribed in a circle.

15. Prove that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube.

16. Given $u = 5xyz / (x + 2y + 4z)$. Find the values of x, y, z for which u is maximum subject to the condition $xyz = 8$.

17. Show that the maximum and minimum of the radii vectors of the sections of the surface $(x^2 + y^2 + z^2)^2 = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}$

by the plane $\lambda x + \mu y + \nu z = 0$ are given by the equation

$$\frac{a^2\lambda^2}{1 - a^2 r^2} + \frac{b^2\mu^2}{1 - b^2 r^2} + \frac{c^2\nu^2}{1 - c^2 r^2} = 0.$$

18. If two variables x and y are connected by the relation $ax^2 + by^2 = ab$, show that the maximum and minimum values of the function $u = x^2 + y^2 + xy$ will be the roots of the equation $4(u - a)(u - b) = ab$.

19. Find the maximum or minimum value of $x^2 + y^2 + z^2$, subject to the conditions

$$lx + my + nz = 1, l'x + m'y + n'z = 1.$$

20. Show that the maximum and minimum values of

$$u = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

subject to the conditions $lx + my + nz = 0$ and $x^2 + y^2 + z^2 = 1$ are given by the equation $\begin{vmatrix} a-u & h & g & l \\ h & b-u & f & m \\ g & f & c-u & n \\ l & m & n & 0 \end{vmatrix} = 0$.

21. Find the maximum value of u, when $u = x^2 y^3 z^4$ and $2x + 3y + 4z = a$.

22. Find the points where $u = ax^p + by^q + cz^r$ has extreme values subject to the condition $x^l + y^m + z^n = k$.

23. Determine the maximum value of OP, O being the origin of coordinates where P describes the curve $x^2 + y^2 + 2z^2 = 5, x + 2y + z = 5$.

24. Prove that if $x + y + z = 1$, $ayz + bzx + cxy$ has an extreme value equal to

$$\frac{abc}{2bc + 2ca + 2ab - a^2 - b^2 - c^2}.$$

Prove also that if a, b, c are all positive and c lies between $a + b - 2\sqrt{(ab)}$ and $a + b + 2\sqrt{(ab)}$ this value is true maximum and that if a, b, c are all negative and c lies between $a + b \pm 2\sqrt{(ab)}$, it is true minimum.

25. Find the maxima and minima of $x^2 + y^2$ subject to the condition

$$ax^2 + 2hxy + by^2 = 1.$$

Answers 2

1. The maximum or minimum values of u are the roots of the equation

$$\frac{l^2}{u - a^2} + \frac{m^2}{u - b^2} + \frac{n^2}{u - c^2} = 0.$$

3. The required values are the roots of the equation

$$\frac{l^2}{u^2 - a^2} + \frac{m^2}{u^2 - b^2} + \frac{n^2}{u^2 - c^2} = 0.$$

4. $m^m n^n p^p a^{m+n+p} / (m + n + p)^{m+n+p}$.

5. A minimum when $\dfrac{x}{\sqrt{a}} = \dfrac{y}{\sqrt{b}} = \dfrac{z}{\sqrt{c}} = \sqrt{a} + \sqrt{b} + \sqrt{c}$.

and the minimum value $= (\sqrt{a} + \sqrt{b} + \sqrt{c})^2$.

6. $p^2 / (a^2 + b^2 + c^2)$.

7. A maximum value is $(p/a)^p (q/b)^q (r/c)^r$.

8. A minimum when $\dfrac{px}{a} = \dfrac{qy}{b} = \dfrac{rz}{c} = p + q + r$.

9. A minimum when $x = y = z = c$ and the minimum value $= 3c^4$.

10. $a/3, a/3, a/3$.

11. u is maximum when $A = B = C = \dfrac{\pi}{3}$ and the maximum value of $u = \dfrac{1}{8}$.

12. A maximum when $\dfrac{\tan A}{m} = \dfrac{\tan B}{n} = \dfrac{\tan C}{p}$.

14. An equilateral triangle.

16. $x = 4, y = 2, z = 1$.

19. The minimum value $\Sigma (l - l')^2 / \Sigma(mn' - m'n)^2$.

21. $(a/9)^9$.

22. The values of x, y, z are given by $\dfrac{x^{p-l}}{l/pa} = \dfrac{y^{q-m}}{m/qb} = \dfrac{z^{r-n}}{n/rc}$.

23. $\sqrt{5}$.

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The function $u = \sin x \sin y \sin z$, where x, y, z are the angles of a triangle, is stationary at the point

(a) $x = \dfrac{\pi}{2}, y = \dfrac{\pi}{4}, z = \dfrac{\pi}{4}$

(b) $x = y = z = \dfrac{\pi}{3}$

(c) $x = \dfrac{\pi}{4}, y = \dfrac{\pi}{2}, z = \dfrac{\pi}{4}$

(d) $x = 0, y = \dfrac{\pi}{2}, z = \dfrac{\pi}{2}$

2. The maximum and minimum values of $u = a^2x^2 + b^2y^2 + c^2z^2$ where $x^2 + y^2 + z^2 = 1$ and $lx + my + nz = 0$ are the roots of the equation

(a) $\dfrac{l^2}{u - a^2} + \dfrac{m^2}{u - b^2} + \dfrac{n^2}{u - c^2} = 0$

(b) $\dfrac{l}{u - a^2} + \dfrac{m}{u - b^2} + \dfrac{n}{u - c^2} = 0$

(c) $\dfrac{l^2}{u - a} + \dfrac{m^2}{u - b} + \dfrac{n^2}{u - c} = 0$

(d) $\dfrac{l}{u - a} + \dfrac{m}{u - b} + \dfrac{n}{u - c} = 0$

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. Let $f(x, y, z)$ be a function of three independent variables x, y and z. The necessary conditions for the existence of a maximum or a minimum of $f(x, y, z)$ at $x = a$, $y = b$ and $z = c$ are
$$\frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = 0 \text{ and } \frac{\partial f}{\partial z} = \ldots \text{ at } x = a, y = b, z = c.$$

2. For a maximum or a minimum of $u = x^2 + y^2 + z^2 + x - 2z - xy$, we must have
$$2x - y + 1 = 0, \ldots, \text{ and } 2z - 2 = 0.$$

True or False

Write 'T' for true and 'F' for false statement.

1. The function $u = x^2 + y^2 + z^2 + x - 2z - xy$ has a maximum at the point
$$x = -\frac{2}{3}, y = -\frac{1}{3}, z = 1.$$

2. The volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ is } \frac{8abc}{3\sqrt{3}}.$$

3. The maximum or minimum values of $u = x^2 + y^2 + z^2$ subject to the conditions
$$ax^2 + by^2 + cz^2 = 1 \text{ and } lx + my + nz = 0$$
are the roots of the equation $\dfrac{l^2}{au - 1} + \dfrac{m^2}{bu - 1} + \dfrac{n^2}{cu - 1} = 0.$

4. The minimum value of $x + y + z$, subject to the condition $\dfrac{a}{x} + \dfrac{b}{y} + \dfrac{c}{z} = 1$, is $(a + b + c)^2$.

Answers

Multiple Choice Questions

1. (b)   2. (a)

Fill in the Blank(s)

1. 0   2. $-x + 2y = 0$.

True or False

1. *F*   2. *T*   3. *T*   4. *F*

□

Chapter

4

Limit and Continuity of Functions of Several Variables

1 Functions of Two Variables

Let $\mathbf{R}$ denote the set of real numbers.

We have $\quad \mathbf{R}^2 = \mathbf{R} \times \mathbf{R} = \{(x, y) : x \in \mathbf{R}, y \in \mathbf{R}\}$.

Let $\quad A \subseteq \mathbf{R}^2$.

If $f : A \to \mathbf{R}$, then f is called a real valued function of two real variables.
The set A is called the domain of this function and the range of this function is some subset of $\mathbf{R}$.

Definition: *We say that $z = f(x, y)$ is a real valued function of two independent real variables x and y, if for each pair of values of x and y of a certain set $\{(x, y)\}$ over which the point (x, y) ranges, there exists a unique real value of z.*

For example, $z = x^2 + y^2 - xy$, $z = x \sin y + y \cos x$ etc., are real valued functions of two real independent variables x and y.

2 Neighbourhood of a Point

Rectangular neighbourhood of a point:

Let (a, b) be any point of $\mathbf{R}^2$.

The set $N(a, b) = \{(x, y) : x \in \mathbf{R}, y \in \mathbf{R}, a - h < x < a + h, b - k < y < b + k\}$ is called a rectangular neighbourhood of the point (a, b), where h and k are arbitrarily small positive real numbers.

In particular,

$$N(a, b) = \{(x, y) : x \in \mathbf{R}, y \in \mathbf{R}, a - \delta < x < a + \delta, b - \delta < y < b + \delta\}$$

is a square neighbourhood of the point (a, b), where δ is an arbitrarily small positive real number.

A neighbourhood of a point (a, b) is denoted by $N(a, b)$.

We shall often write the word neighbourhood in the abbreviated form as nhd or nbd.

Circular neighbourhood of a point:

The set $N(a, b) = \{(x, y) : x \in \mathbf{R}, y \in \mathbf{R}$ and $\sqrt{(x - a)^2 + (y - b)^2} < \delta\}$ is called a circular neighbourhood of the point (a, b), where δ is an arbitrarily small positive real number.

It can be seen that every circular neighbourhood of a point contains a rectangular neighbourhood and vice versa.

Deleted neighbourhood of a point: If from the neighbourhood $N(a, b)$ of the point (a, b), we delete the point (a, b), then the remaining set is called a deleted nbd of (a, b).

3 Simultaneous and Iterated Limits

For a function $f(x, y)$ of two variables x and y, we can define several kinds of limits. If (a, b) is the limiting point of a set of values in two dimensional space, then we have the limits

$$\lim_{(x, y) \to (a, b)} f(x, y), \quad \lim_{x \to a} \lim_{y \to b} f(x, y), \quad \lim_{y \to b} \lim_{x \to a} f(x, y).$$

The first type of limit is known as '**simultaneous limit**' or '**double limit**' and the last two types are known as '**iterated limits**' or '**repeated limits**'.

The simultaneous limit, $\lim_{(x, y) \to (a, b)} f(x, y)$ is also written as

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y).$$

The notion of an iterated limit is nothing but a limit of a limit and it can be obtained as in the case of a function of a single variable. However, the notion of simultaneous limit is quite different from that of the limit of a function of single variable.

Simultaneous Limit:

Definition: *We say that the simultaneous limit of* $f(x, y)$ *exists and is equal to* A *as* $(x, y) \to (a, b)$, *if for every* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that*

$$|f(x, y) - A| < \varepsilon$$

for all values of x *and* y *in the nbd of* (a, b) *defined by* $|x - a| < \delta, |y - b| < \delta.$

Non-existence of simultaneous limit:

For the existence of simultaneous limit, not only must we have same limiting value if the variable point (x, y) approaches the limiting point (a, b) through any set of values dense at the point, but we must also have the same limiting value as the variable point approaches its limiting position along any curve whatsoever.

Thus if we can find two methods of approach to the limiting point which give different limiting values then we can conclude that the simultaneous limit does not exist.

Illustrative Examples

Example 1: *Show that the simultaneous limit* $\displaystyle\lim_{(x, y) \to (0,0)} \frac{xy^3}{x^2 + y^6}$ *does not exist.*

(Gorakhpur 2010)

Solution: First let (x, y) approach $(0, 0)$ along the line $y = x$ which is a line through the origin. For this, we first put $y = x$ in the function and then allow x to approach 0. We thus get

$$\lim_{x \to 0} \frac{x^4}{x^2 + x^6} = \lim_{x \to 0} \frac{x^2}{1 + x^4} = 0.$$

Again, let (x, y) approach $(0, 0)$ along the curve $x = y^3$. For this, we put $x = y^3$ in the function and then allow y to approach zero. In this case, we get

$$\lim_{y \to 0} \frac{y^6}{y^6 + y^6} = \frac{1}{2}.$$

Since two methods of approach to the limiting point give different limiting values, the simultaneous limit does not exist.

Note 1: It can be seen that if we approach the origin along any line $y = mx$, the limit comes out to be zero. But we cannot conclude that the simultaneous limit exists and is 0 as we have seen that the limit along $x = y^3$ comes out to be $\frac{1}{2}$.

Note 2: The existence of the simultaneous limit, $\displaystyle\lim_{(x, y) \to (a, b)} f(x, y)$ implies that the single limits $\displaystyle\lim_{x \to a} f(x, b)$, $\displaystyle\lim_{y \to b} f(a, y)$ also exist. However, it does not follow that the single limits $\displaystyle\lim_{x \to a} f(x, y)$, $\displaystyle\lim_{y \to b} f(x, y)$ exist for $y \neq b, x \neq a$ respectively.

It is shown by the following example :

Example 2: *Show that the simultaneous limit* $\lim\limits_{(x, y) \to (0,0)} y \sin \frac{1}{x}$ *exists and is equal to* 0 *but the single limit* $\lim\limits_{x \to 0} y_1 \sin \frac{1}{x} (y_1 \neq 0)$ *does not exist.*

Solution: First we shall show that the simultaneous limit $\lim\limits_{(x, y) \to (0,0)} y \sin \frac{1}{x}$ exists.

Let $\varepsilon > 0$ be given. Taking $\delta = \varepsilon$, we see that for all x, y satisfying the inequalities $0 < |x| < \delta, 0 < |y| < \delta$, we have

$$\left| y \sin \frac{1}{x} - 0 \right| = \left| y \sin \frac{1}{x} \right| = |y| \left| \sin \frac{1}{x} \right|$$

$$\leq |y| \qquad \left[\because \left| \sin \frac{1}{x} \right| \leq 1 \right]$$

$$< \delta = \varepsilon.$$

$\therefore \lim\limits_{(x, y) \to (0,0)} y \sin \frac{1}{x} = 0$ *i.e.*, the simultaneous limit exists.

But, for any constant value of $y = y_1 \neq 0$, we get

$$\lim_{x \to 0} y_1 \sin \frac{1}{x} = y_1 \lim_{x \to 0} \sin \frac{1}{x}, \text{ which does not exist.}$$

Repeated limits or Iterated limits: If a function f is defined in some deleted neighbourhood of (a, b), then the limit $\lim\limits_{y \to b} f(x, y)$, if it exists, is a function of x, say $\phi(x)$. If then the limit, $\lim\limits_{x \to a} \phi(x)$ exists and is equal to λ, we write

$$\lim_{x \to a} \lim_{y \to b} f(x, y) = \lambda$$

and say that λ is a *repeated limit* of f as $y \to b, x \to a$.

If we change the order of taking the limits, we get the other repeated limit

$$\lim_{y \to b} \lim_{x \to a} f(x, y) = \mu \text{ (say)}$$

when first $x \to a$ and then $y \to b$.

These two limits may or may not be equal.

Note: In case the simultaneous limit exists, these two repeated limits if they exist are necessarily equal but the converse is not true. However, *if the repeated limits are not equal, the simultaneous limit cannot exist.*

Example 3: *Let* $f(x, y) = \frac{y - x}{y + x} \cdot \frac{1 + x}{1 + y}, (x, y) \neq (0,0)$. *Show that the two repeated limits exist at the origin but are unequal.*

Solution: We have $\lim\limits_{x \to 0} \lim\limits_{y \to 0} f(x, y) = \lim\limits_{x \to 0} -\left(\frac{1 + x}{1} \right) = -1.$

Again, $\lim_{y \to 0} \lim_{x \to 0} f(x, y) = \lim_{y \to 0} \left(\frac{1}{1+y} \right) = 1.$

Thus the two repeated limits exist at the origin but are not equal.

Now we give a theorem which is only a **sufficient** but not a **necessary** condition for the interchange of the order of iterated limits.

Theorem 1: *Let the simultaneous limit* $\lim_{(x, y) \to (a, b)} f(x, y)$ *exist and be equal to A and let the limit* $\lim_{x \to a} f(x, y)$ *exist for each constant value of y in the nhd of b and likewise let the limit* $\lim_{y \to b} f(x, y)$ *exist for each constant value of x in the nhd of a. Then*

$$\lim_{x \to a} \lim_{y \to b} f(x, y) = \lim_{y \to b} \lim_{x \to a} f(x, y) = A.$$

Proof: Since the limit $\lim_{x \to a} f(x, y)$ exists for each value of y in the nhd of b, we get a set of these limiting values which defines a function of y, say $F(a, y)$. Thus, we can write

$$\lim_{x \to a} f(x, y) = F(a, y), \qquad \ldots(1)$$

where $F(a, y)$ may or may not be identical with $f(a, y)$.

Let $\varepsilon > 0$ be given. By virtue of (1), there exists $\delta_1 > 0$ such that for each value of y in a nhd of b, say in the nhd defined by $|y - b| < \delta_0$, we get

$$|F(a, y) - f(x, y)| < \varepsilon / 2 \qquad \ldots(2)$$

for all x satisfying the inequality $|x - a| < \delta_1$.

Again, since the simultaneous limit of $f(x, y)$ at (a, b) is A, there exists $\delta_2 > 0$ such that

$$|f(x, y) - A| < \varepsilon / 2 \qquad \ldots(3)$$

for all x, y satisfying $|x - a| < \delta_2$, $|y - b| < \delta_2$.

Taking $\delta = \min(\delta_0, \delta_2)$, we get for all y satisfying $|y - b| < \delta$,

$$\begin{aligned} |F(a, y) - A| &= |F(a, y) - f(x, y) + f(x, y) - A| \\ &\le |F(a, y) - f(x, y)| + |f(x, y) - A| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ by (2) and (3).} \end{aligned}$$

It gives that $\lim_{y \to b} F(a, y) = A$

or $\lim_{y \to b} \lim_{x \to a} f(x, y) = A$, using (1).

Similarly, we can prove that

$$\lim_{x \to a} \lim_{y \to b} f(x, y) = A.$$

Thus $\lim_{(x, y) \to (a, b)} f(x, y) = \lim_{x \to a} \lim_{y \to b} f(x, y) = \lim_{y \to b} \lim_{x \to a} f(x, y).$

Example 4: *Give an example to show that the order of iterated limits can be interchanged although the simultaneous limit does not exist.* **(Gorakhpur 2012)**

Solution: We consider the function $f(x, y) = \dfrac{xy}{x^2 + y^2}$ for simultaneous and iterated limits at $(0, 0)$. First let (x, y) approach $(0, 0)$ along the line $y = x$. Putting $y = x$ and then making x approach 0, we get

$$\lim_{x \to 0} \frac{x^2}{x^2 + x^2} = \frac{1}{2}.$$

Again, let (x, y) approach $(0, 0)$ along y-axis. Putting $x = 0$, we get

$$\lim_{y \to 0} \frac{0}{0 + y^2} = 0.$$

Since two methods of approach give different results, the simultaneous limit does not exist.

For iterated limits, we get

$$\lim_{x \to 0} \lim_{y \to 0} \frac{xy}{x^2 + y^2} = \lim_{x \to 0} \frac{x \,.\, 0}{x^2 + 0} = 0$$

and
$$\lim_{y \to 0} \lim_{x \to 0} \frac{xy}{x^2 + y^2} = \lim_{y \to 0} \frac{0 \,.\, y}{0 + y^2} = 0.$$

Thus the order of iterated limits can be interchanged.

Example 5: *Show that* $\displaystyle \lim_{(x, y) \to (0,0)} xy \frac{x^2 - y^2}{x^2 + y^2} = 0.$

Solution: Let $\varepsilon > 0$ be given.

For all $(x, y) \neq (0, 0)$, we have

$$\left| xy \frac{(x^2 - y^2)}{x^2 + y^2} - 0 \right| = \left| xy \frac{(x^2 - y^2)}{x^2 + y^2} \right| = | x^2 - y^2 | \,.\, \frac{|xy|}{x^2 + y^2}. \qquad \ldots(1)$$

Now for all x and y, we have $(|x| - |y|)^2 \geq 0$

$\Rightarrow \qquad |x|^2 + |y|^2 - 2|x|.|y| \geq 0$

$\Rightarrow \qquad x^2 + y^2 \geq 2|xy|$

$\Rightarrow \qquad \dfrac{|xy|}{x^2 + y^2} \leq \dfrac{1}{2}$, if $(x, y) \neq (0, 0)$. $\qquad \ldots(2)$

$\therefore$ From (1) and (2), for all $(x, y) \neq (0, 0)$, we have

$$\left| xy \frac{(x^2 - y^2)}{x^2 + y^2} - 0 \right| \leq \frac{1}{2} | x^2 - y^2 | \leq \frac{1}{2} | x^2 + y^2 |$$

$$= \frac{1}{2} (x^2 + y^2) < \varepsilon,$$

if $x^2 < \varepsilon$ and $y^2 < \varepsilon$ *i.e.*, if $|x| < \sqrt{\varepsilon}$ and $|y| < \sqrt{\varepsilon}$.

Now if we take $\delta = \sqrt{\varepsilon}$, then we see that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{xy\,(x^2 - y^2)}{x^2 + y^2} - 0 \right| < \varepsilon, \text{ whenever } |x| < \delta \text{ and } |y| < \delta.$$

$$\therefore \quad \lim_{(x,y)\to(0,0)} xy\,\frac{x^2 - y^2}{x^2 + y^2} = 0.$$

Alternative Solution: Take any given $\varepsilon > 0$. Putting $x = r\cos\theta$, $y = r\sin\theta$, we have

$$\left| xy\,\frac{x^2 - y^2}{x^2 + y^2} - 0 \right| = \left| xy\,\frac{x^2 - y^2}{x^2 + y^2} \right| = |r^2 \sin\theta\cos\theta\cos 2\theta|$$

$$= \left| \frac{r^2}{4} \sin 4\theta \right| = \frac{r^2}{4} |\sin 4\theta|$$

$$\leq \frac{r^2}{4} \qquad [\because |\sin 4\theta| \leq 1]$$

$$= \frac{x^2 + y^2}{4}$$

$$< \varepsilon, \text{ if } \frac{x^2}{4} < \frac{\varepsilon}{2} \text{ and } \frac{y^2}{4} < \frac{\varepsilon}{2} \text{ i.e.,}$$

$$\text{if } |x| < \sqrt{2\varepsilon} \text{ and } |y| < \sqrt{2\varepsilon}.$$

Now if we take $\delta = \sqrt{2\varepsilon}$, then we see that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| xy\,\frac{x^2 - y^2}{x^2 + y^2} - 0 \right| < \varepsilon, \text{ whenever } |x| < \delta \text{ and } |y| < \delta.$$

Hence $$\lim_{(x,y)\to(0,0)} xy\,\frac{x^2 - y^2}{x^2 + y^2} = 0.$$

Example 6: *Show that* $\lim_{(x,y)\to(0,0)} \dfrac{3x - 2y}{2x - 3y}$ *does not exist.*

Solution: When $(x, y) \to (0,0)$ along the st. line $y = x$, we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{3x - 2x}{2x - 3x} = \lim_{x\to 0} \frac{x}{-x} = \lim_{x\to 0} -1 = -1.$$

Again when $(x, y) \to (0,0)$ along the st. line $y = 0$, we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{3x - 0}{2x - 0} = \lim_{x\to 0} \frac{3x}{2x} = \lim_{x\to 0} \frac{3}{2} = \frac{3}{2}.$$

Since the two methods of approach to the limiting point give different limiting values, the simultaneous limit does not exist.

Example 7: *Show that* $\lim_{(x,y)\to(0,0)} \dfrac{2xy^2}{x^2 + y^4}$ *does not exist.*

Solution: Let (x, y) approach $(0,0)$ along the curve $x = my^2$. For this, we put $x = my^2$ in the function and then allow y to approach zero. Thus in this case we have

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{2my^4}{(m^2+1)\,y^4}$$

$$= \lim_{y\to 0} \frac{2\,m}{1+m^2}$$

$$= \frac{2\,m}{1+m^2}$$

which is different for different values of m.

For example, if $m = 1$, then this limit $= \dfrac{2\,.\,1}{1+1^2} = 1$

and if $m = 2$, then this limit $= \dfrac{2\,.\,2}{1+2^2} = \dfrac{4}{5}\cdot$

Since two methods of approach to the limiting point give different limiting values, therefore the simultaneous limit

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4} \text{ does not exist.}$$

Example 8: *Show that the simultaneous limit exists at the origin but the repeated limits do not, where*

$$f(x,y) = \begin{cases} x \sin \dfrac{1}{y} + y \sin \dfrac{1}{x}, & xy \neq 0 \\ 0, & xy = 0. \end{cases}$$

(Lucknow 2009)

Solution: Here $\lim\limits_{y\to 0} f(x,y)$, $\lim\limits_{x\to 0} f(x,y)$ do not exist and therefore both the repeated limits $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$ and $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$ do not exist.

However, the simultaneous limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists and is equal to 0 as shown below. Take any given $\varepsilon > 0$.

For all $(x,y) \neq (0,0)$ such that $xy = 0$, we have

$$|f(x,y) - 0| = |0 - 0| = 0 < \varepsilon.$$

Again for all $(x,y) \neq (0,0)$ such that $xy \neq 0$, we have

$$|f(x,y) - 0| = |f(x,y)| = \left| x \sin \frac{1}{y} + y \sin \frac{1}{x} \right|$$

$$\leq \left| x \sin \frac{1}{y} \right| + \left| y \sin \frac{1}{x} \right| \qquad [\because |a+b| \leq |a| + |b|]$$

$$= |x| \left| \sin \frac{1}{y} \right| + |y| \left| \sin \frac{1}{x} \right| \qquad [\because |ab| = |a|\,|b|]$$

$$\leq |x| + |y| \qquad \left[\because \left|\sin\frac{1}{y}\right| \leq 1 \text{ and } \left|\sin\frac{1}{x}\right| \leq 1\right]$$

$$\leq 2\sqrt{x^2 + y^2}$$

$$[\because |x| \leq \sqrt{x^2 + y^2} \text{ and } |y| \leq \sqrt{x^2 + y^2}]$$

$$< \varepsilon, \text{ if } x^2 < \frac{\varepsilon^2}{8} \text{ and } y^2 < \frac{\varepsilon^2}{8} \text{ i.e.,}$$

$$\text{if } |x| < \frac{\varepsilon}{2\sqrt{2}} \text{ and } |y| < \frac{\varepsilon}{2\sqrt{2}}.$$

Now if we take $\delta = \dfrac{\varepsilon}{2\sqrt{2}}$, then we see that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x, y) - 0| < \varepsilon$, whenever $|x| < \delta$ and $|y| < \delta$.

Hence, $\lim\limits_{(x, y) \to (0,0)} f(x, y) = 0.$

Comprehensive Exercise 1

1. Show that $\lim\limits_{(x, y) \to (0,0)} \dfrac{2xy^2}{x^2 + y^4}$ does not exist.

2. Show that $\lim\limits_{(x, y) \to (0,0)} (x + y) = 0.$

3. Show that $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{2xy}{x^2 + y^2}$ does not exist.

   (Gorakhpur 2013)

4. Show that $\lim\limits_{(x, y) \to (0,0)} \dfrac{\sqrt{x^2 y^2 + 1} - 1}{x^2 + y^2} = 0.$

5. Show that $\lim\limits_{(x, y) \to (0,0)} \dfrac{x^3 + y^3}{x - y}$ does not exist.

6. Let $f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & \text{if } x^4 + y^2 \neq 0 \\ 0, & \text{if } x = y = 0. \end{cases}$

   Show that the straight line approach gives the limit zero at the origin but the function possesses no limit at the origin.

7. Show that $\lim\limits_{(x, y) \to (0,0)} \dfrac{xy^2}{x^2 + y^2} = 0.$

8. Show that the limit, when $(x, y) \to (0,0)$, exists in each case :

(i) $\lim \dfrac{xy}{\sqrt{x^2 + y^2}}$ (ii) $\lim \dfrac{x^3 y^3}{x^2 + y^2}$

(iii) $\lim \dfrac{x^4 + y^4}{x^2 + y^2}$. (iv) $\lim \dfrac{x^3 - y^3}{x^2 + y^2}$

9. Show that the repeated limits exist at the origin and are equal but the simultaneous limit does not exist, where $f(x, y) = \begin{cases} 1, & \text{if } xy \neq 0 \\ 0, & \text{if } xy = 0. \end{cases}$

10. Show that the limit and the repeated limits exist when $(x, y) \to (0,0)$:

$$f(x, y) = \begin{cases} x\,y\,\dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0). \end{cases}$$

11. Show that $\displaystyle\lim_{(x, y) \to (0,0)} f(x, y)$ and $\displaystyle\lim_{y \to 0} \lim_{x \to 0} f(x, y)$ exist, but $\displaystyle\lim_{x \to 0} \lim_{y \to 0} f(x, y)$ does not exist, where

$$f(x, y) = \begin{cases} y + x \sin\left(\dfrac{1}{y}\right), & \text{if } y \neq 0 \\ 0, & \text{if } y = 0. \end{cases}$$

12. Show that if $f(x, y) = \dfrac{x - y}{x + y}$, then the repeated limits exist but the double limit does not exist when $(x, y) \to (0,0)$. **(Lucknow 2010)**

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4 Continuity of Functions of two Variables

Definition: *The function $f(x, y)$ is said to be continuous at (a, b) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|x - a| < \delta, |y - b| < \delta \Rightarrow |f(x, y) - f(a, b)| < \varepsilon.$$

Equivalently, $f(x, y)$ is said to be continuous at (a, b) if the simultaneous limit

$$\lim_{(x, y) \to (a, b)} f(x, y)$$

exists and is equal to its functional value $f(a, b)$ at (a, b).

If f is not continuous at $(a, b) \in D \subseteq \mathbf{R}^2$, then f is said to be discontinuous at (a, b).

f is said to be continuous on the domain D if f is continuous at each point of D.

Note 1: Let $D \subseteq \mathbf{R}^2$ and $f : D \to \mathbf{R}$ be a function continuous at $(a, b) \in D$.

Let $f_1(x) = f(x, b)$. Then f_1 is a function of a single variable x.

f is continuous at $(a, b) \Rightarrow$ for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x, y) - f(a, b)| < \varepsilon \text{ for } (x, y) \in D \text{ and } |x - a| < \delta, |y - b| < \delta$$

$$\Rightarrow \quad |f(x, b) - f(a, b)| < \varepsilon \text{ for } (x, b) \in D \text{ and } |x - a| < \delta$$

$$\Rightarrow \quad |f_1(x) - f_1(a)| < \varepsilon \text{ for } (x, b) \in D \text{ and } |x - a| < \delta$$

$$\Rightarrow \quad f_1 \text{ is continuous at } a.$$

Similarly, we can show that $f_2(y) = f(a, y)$ is continuous at b.

Hence, if $f(x, y)$ is continuous at (a, b), then

(i) $f(x, b)$ is continuous at $x = a$ and (ii) $f(a, y)$ is continuous at $y = b$.

The converse of the above result is not true. Thus, if $f(x, b)$ is continuous at $x = a$ and $f(a, y)$ is continuous at $y = b$ then $f(x, y)$ need not be continuous at (a, b).

Note 2: For $f(x, y)$ to be continuous in both the variables together, it must have the same limiting value by all possible approaches to the critical point. Thus the necessary and sufficient condition is that the function is not only continuous in each direction but the continuity is uniform for all directions.

In the definition of continuity, if we put $x = a + r\cos\theta$, $y = b + r\sin\theta$, we get

$$|f(a + r\cos\theta, b + r\sin\theta) - f(a, b)| < \varepsilon,$$

which must hold for all values of r less than some number r_0 which is independent of θ. In other words we say that the transformed function must be uniformly continuous in r for all values of $|\theta| \le 2\pi$.

Illustrative Examples

Example 9: *Examine the continuity at* $(1, 2)$ *of the function*

$$f(x, y) = \begin{cases} x^2 + 4y & \text{when } (x, y) \ne (1, 2) \\ 0 & \text{when } (x, y) = (1, 2). \end{cases}$$

Solution: We have $\lim_{(x, y) \to (1, 2)} x^2 + 4y = 1^2 + 4 \times 2 = 9$, so that the limit exists and is equal to 9.

Since $f(1, 2) = 0$ and $\lim_{(x, y) \to (1, 2)} f(x, y) = 9$, we have

$$\lim_{(x, y) \to (1, 2)} f(x, y) \ne f(1, 2).$$

Hence, the function is not continuous at $(1, 2)$.

Example 10: *Show that the function* $f : \mathbf{R}^2 \to \mathbf{R}$ *defined by* $f(x, y) = \dfrac{(y^2 - x^2)\,yx}{x^2 + y^2}$ *for* $(x, y) \ne (0, 0)$ *and* $f(0, 0) = 0$ *is continuous at the point* $(0, 0)$.

Solution: First we shall show that $\lim_{(x, y) \to (0, 0)} f(x, y) = 0$.

Let $\varepsilon > 0$ be given.

For all $(x, y) \neq (0,0)$, we have

$$|f(x, y) - 0| = \left|\frac{(y^2 - x^2)\, yx}{x^2 + y^2}\right| = |y^2 - x^2| \cdot \frac{|xy|}{x^2 + y^2}. \qquad \ldots(1)$$

Now for all x and y, we have $(|x| - |y|)^2 \geq 0$

$$\Rightarrow \quad |x|^2 + |y|^2 - 2|x|.|y| \geq 0 \quad \Rightarrow \quad x^2 + y^2 \geq 2|xy|$$

$$\Rightarrow \quad \frac{|xy|}{x^2 + y^2} \leq \frac{1}{2}, \text{ if } (x, y) \neq (0,0). \qquad \ldots(2)$$

$\therefore$ From (1) and (2), for all $(x, y) \neq (0,0)$, we have

$$|f(x, y) - 0| \leq \frac{1}{2}|y^2 - x^2| \leq \frac{1}{2}|y^2 + x^2| = \frac{1}{2}(x^2 + y^2)$$

$$< \varepsilon, \text{ if } x^2 < \varepsilon \text{ and } y^2 < \varepsilon \text{ } i.e.,$$

$$\text{if } |x| < \sqrt{\varepsilon} \text{ and } |y| < \sqrt{\varepsilon}.$$

Now if we take $\delta = \sqrt{\varepsilon}$, then we see that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x, y) - 0| < \varepsilon, \text{ whenever } |x| < \delta \text{ and } |y| < \delta$$

$$\therefore \quad \lim_{(x, y) \to (0,0)} f(x, y) = 0.$$

But $\quad f(0,0) = 0.$ [Given]

$$\therefore \quad \lim_{(x, y) \to (0,0)} f(x, y) = f(0,0).$$

Hence $f(x, y)$ is continuous at the origin.

Example 11: *Discuss the continuity of the function*

$$f(x, y) = \frac{2xy^2}{x^3 + 3y^3}, (x, y) \neq (0,0) \text{ and } f(0,0) = 0,$$

with respect to both variables.

Solution: First we observe that $\lim_{(x, y) \to (0,0)} f(x, y)$ does not exist.

When $(x, y) \to (0,0)$ along the st. line $y = x$, we have

$$\lim_{(x, y) \to (0,0)} f(x, y) = \lim_{x \to 0} \frac{2x \cdot x^2}{x^3 + 3x^3} = \lim_{x \to 0} \frac{2x^3}{4x^3} = \lim_{x \to 0} \frac{1}{2} = \frac{1}{2}.$$

Again when $(x, y) \to (0,0)$ along the st. line $y = 0$, we have

$$\lim_{(x, y) \to (0,0)} f(x, y) = \lim_{x \to 0} \frac{2x \cdot 0^2}{x^3 + 3 \cdot 0^3} = \lim_{x \to 0} 0 = 0.$$

Since the two methods of approach to the limiting point give different limiting values, the simultaneous limit does not exist. Consequently the function is not continuous at $(0,0)$ in (x, y) together.

However, the function is continuous in x alone and in y alone at the origin. Because if we put either variable zero and then let the other variable approach zero, we find the limiting value zero, which is the value of the function at $(0,0)$.

Example 12: *Show that the function* $f(x, y) = \dfrac{xy^3}{x^2 + y^6}$, $(x, y) \neq (0,0)$ *and* $f(0,0) = 0$ *is not continuous at* $(0,0)$ *in* x, y *together but that the function is continuous in* x *alone and in* y *alone at the origin.* **(Purvanchal 2009)**

Solution: Simultaneous limit of $f(x, y)$ at $(0,0)$ does not exist as shown below :

First let (x, y) approach $(0,0)$ through any line $y = mx$. We have

$$\lim_{x \to 0} \frac{x \cdot m^3 x^3}{x^2 + m^6 x^6} = \lim_{x \to 0} \frac{m^3 x^2}{1 + m^6 x^4} = \frac{0}{1+0} = 0.$$

Now let (x, y) approach $(0,0)$ through the curve $x = y^3$.

Then $$\lim_{y \to 0} \frac{y^3 \cdot y^3}{y^6 + y^6} = \frac{1}{2}.$$

Since the limits obtained by two different approaches are different, the simultaneous limit does not exist. Consequently the function is not continuous at $(0,0)$ in (x, y) together.

The function is however continuous in x alone and in y alone at the origin. For putting either variable zero and then letting the other variable approach zero, we get the limiting value zero, which is the value of $f(0,0)$.

Example 13: *Show that the function* $f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{(x^2 + y^2)}}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$

is continuous at the origin in x-y together.

(Rohilkhand 2008; Garhwal 10; Lucknow 09, 10; Gorakhpur 10)

Solution: First we shall show that $\lim_{(x, y) \to (0,0)} f(x, y) = 0$.

Let $\varepsilon > 0$ be given. For all $(x, y) \neq (0,0)$, we have

$$|f(x, y) - 0| = \left| \frac{xy}{\sqrt{(x^2 + y^2)}} - 0 \right| = \left| \frac{xy}{\sqrt{(x^2 + y^2)}} \right| = \left| \frac{r\cos\theta \, r\sin\theta}{r} \right|$$

[Putting $x = r\cos\theta$, $y = r\sin\theta$]

$$= r|\cos\theta||\sin\theta|$$

$$\leq r \qquad [\because |\cos\theta| \leq 1, |\sin\theta| \leq 1]$$

$$= \sqrt{(x^2 + y^2)}$$

$$< \varepsilon, \text{ if } x^2 < \frac{\varepsilon^2}{2} \text{ and } y^2 < \frac{\varepsilon^2}{2} \text{ } i.e., \text{ if } |x| < \frac{\varepsilon}{\sqrt{2}} \text{ and } |y| < \frac{\varepsilon}{\sqrt{2}}.$$

Now if we take $\delta = \dfrac{\varepsilon}{\sqrt{2}}$, then we see that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x, y) - 0| < \varepsilon$, whenever $|x| < \delta$ and $|y| < \delta$.

Hence, $$\lim_{(x, y) \to (0,0)} f(x, y) = 0.$$

Since $f(0,0)=0$, therefore $\lim_{(x,y)\to(0,0)} f(x,y)=f(0,0)$.

Hence, the given function f is continuous at $(0,0)$ in x-y together.

Note: If $\lim_{r\to 0} f(a+r\cos\theta, b+r\sin\theta)=f(a,b)$ for every value of θ, then it is not necessary that the function is continuous at (a,b).

For example, we have seen in Example 12 of article 4 that the simultaneous limit of $f(x,y)=\dfrac{xy^3}{x^2+y^6}$, $x\neq 0, y\neq 0$ and $f(0,0)=0$ does not exist as $(x,y)\to(0,0)$.

Therefore, this function is discontinuous at the origin. However, if we put $x=r\cos\theta$, $y=r\sin\theta$, we get

$$\lim_{r\to 0}\frac{r^4\cos\theta\sin^3\theta}{r^2\cos^2\theta+r^6\sin^6\theta}=\lim_{r\to 0} r^2\frac{\cos\theta\sin^3\theta}{\cos^2\theta+r^4\sin^6\theta}$$

$$=0=f(0,0)$$

for each constant value of θ.

Comprehensive Exercise 2

1. Investigate for continuity at $(1,2)$, the function

$$f(x,y)=\begin{cases} x^2+2y\ , & (x,y)\neq(1,2) \\ 0\ , & (x,y)=(1,2).\end{cases}$$

2. Show that the function f defined as follows has a removable discontinuity at $(2,3)$.

$$f(x,y)=\begin{cases} 3xy\ , & (x,y)\neq(2,3) \\ 6\ , & (x,y)=(2,3).\end{cases}$$

Suitably redefine the function to make it continuous at $(2,3)$.

3. Show that the function f defined by

$$f(x,y)=\begin{cases} \dfrac{x^2-y^2}{x^2+y^2}\ , & (x,y)\neq(0,0) \\ 0\ , & (x,y)=(0,0)\end{cases}$$

is not continuous at the origin.

4. Prove that $f(x,y)=xy\dfrac{x^2-y^2}{x^2+y^2}$ when $(x,y)\neq(0,0)$ and $f(0,0)=0$ is continuous at the origin. **(Gorakhpur 2015)**

5. A function f is defined as follows :

$$f(x, y)=\begin{cases}\dfrac{2xy}{x^2+y^2}, & \text{when } (x, y)\neq(0,0)\\ 0, & \text{when } (x, y)=(0,0).\end{cases}$$

Show that this function is continuous at the origin in x alone and y alone but is discontinuous at the origin with regard to both the variables x and y.

6. Show that the function f defined by $f(x, y)=\dfrac{xy^2}{x^2+y^4}, (x, y)\neq(0,0)$ and $f(0,0)=0$ is discontinuous in $x-y$ at the origin.

Show also that this function is continuous along the radius vector $\theta=\pi/2$.

7. Show that the function $f(x, y)=\dfrac{xy}{x^3+y^2}$, when $x\neq 0, y\neq 0$ and $f(0,0)=0$ is discontinuous in (x, y) at the origin.

8. Investigate the continuity at $(0,0)$ of the function

$$f(x, y)=\begin{cases}\dfrac{x^2 y}{x^3+y^3}, & x^3+y^3\neq 0\\ 0, & x^3+y^3=0.\end{cases}$$

9. Show that the function f is continuous at the origin, where

$$f(x, y)=\begin{cases}\dfrac{x^3-y^3}{x^2+y^2}, & (x, y)\neq(0,0)\\ 0, & (x, y)=(0,0)\end{cases}$$

(Gorakhpur 2011)

10. Show that the following functions are discontinuous at the origin :

(i) $f(x, y)=\begin{cases}\dfrac{1}{x^2+y^2}, & (x, y)\neq(0,0)\\ 0, & (x, y)=(0,0).\end{cases}$

(ii) $f(x, y)=\begin{cases}\dfrac{x^2 y^2}{x^4+y^4}, & (x, y)\neq(0,0)\\ 0, & (x, y)=(0,0).\end{cases}$

(iii) $f(x, y)=\begin{cases}\dfrac{x^4-y^4}{x^4+y^4}, & (x, y)\neq(0,0)\\ 0, & (x, y)=(0,0).\end{cases}$

(iv) $f(x, y)=\begin{cases}\dfrac{x^2}{x^2+y^2}, & (x, y)\neq(0,0)\\ 0, & (x, y)=(0,0).\end{cases}$

11. Show that the following functions are continuous at the origin :

(i) $f(x, y) = \begin{cases} \dfrac{x^3 y^3}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0). \end{cases}$

(ii) $f(x, y) = \begin{cases} \dfrac{x^2 y^2}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0). \end{cases}$

12. Show that the following function is discontinuous at $(0,0)$:

$$f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y. \end{cases}$$

(Lucknow 2011; Meerut 12)

Answers 2

1. Discontinuous

2. $f(x, y) = \begin{cases} 3xy, & (x, y) \neq (2,3) \\ 18, & (x, y) = (2,3) \end{cases}$

8. Discontinuous

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. $\lim_{(x, y) \to (0,0)} \dfrac{3x - 2y}{2x - 3y} =$

(a) $\dfrac{3}{2}$ (b) $\dfrac{2}{3}$

(c) 1 (d) does not exist

2. $\lim_{(x, y) \to (0,0)} \dfrac{2xy^2}{x^2 + y^4} =$

(a) 0 (b) does not exist

(c) 1 (d) 2

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. We say that the simultaneous limit of $f(x, y)$ exists and is equal to A as $(x, y) \to (a, b)$, if for every given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x, y) - A| < \varepsilon$ for all values of x and y in the nhd of (a, b) defined by $|x - a| < \delta, \ldots\ldots.$

2. $\lim\limits_{(x, y) \to (0,0)} xy \dfrac{y^2 - x^2}{x^2 + y^2} = \ldots\ldots.$

3. The function $f(x, y)$ is said to be continuous at (a, b) if
$$\lim_{(x, y) \to (a, b)} f(x, y) = \ldots.$$

True or False

Write 'T' for true and 'F' for false statement.

1. $\lim\limits_{(x, y) \to (0,0)} \dfrac{xy}{x^2 + y^2} = 0.$

2. The function
$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$$
is discontinuous at the origin.

3. The function
$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{(x^2 + y^2)}}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$$
is continuous at the origin.

4. The function $f(x, y) = \begin{cases} xy \dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$

is discontinuous at the origin.

Answers

Multiple Choice Questions

1. (d) 2. (b)

Fill in the Blank(s)

1. $|y-b|<\delta$ 2. 0 3. $f(a,b)$

True or False

1. F 2. T 3. T 4. F

□

Chapter

5

Partial Derivation and Differentiability of Functions of Several Variables

1 Partial Derivatives

Definition: Let $z = f(x, y)$ be a function of two independent variables x and y. Then the partial derivative of z with respect to x is the ordinary derivative of z with respect to x when y is regarded as a constant and is denoted by

$$f_x \quad or \quad \frac{\partial z}{\partial x} \quad or \quad \frac{\partial f}{\partial x}.$$

Thus if $\lim\limits_{h \to 0} \dfrac{f(a+h, b) - f(a, b)}{h}$ *exists, then this limit is called the partial derivative of* $f(x, y)$ *with respect to* x *at the point* (a, b) *and is denoted by*

$$f_x(a, b) \quad or \quad \left(\frac{\partial z}{\partial x}\right)_{(a, b)} \quad or \quad \left(\frac{\partial f}{\partial x}\right)_{(a, b)}.$$

Similarly, if $\lim\limits_{k \to 0} \dfrac{f(a, b+k) - f(a, b)}{k}$ exists, then this limit is called the partial derivative of $f(x, y)$ with respect to y at (a, b) and is denoted by

$$f_y(a,b) \quad \text{or} \quad \left(\frac{\partial z}{\partial y}\right)_{(a,b)} \quad \text{or} \quad \left(\frac{\partial f}{\partial y}\right)_{(a,b)}$$

Let $f : X \to \mathbf{R}$ and $X \subseteq \mathbf{R}^2$. If the function f has partial derivatives at each point of X then f is partially differentiable on X.

Note 1: We have, by definition

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h} = \lim_{x \to a} \frac{f(x,b) - f(a,b)}{x-a}$$

$$f_y(a,b) = \lim_{k \to 0} \frac{f(a,b+k) - f(a,b)}{k} = \lim_{y \to b} \frac{f(a,y) - f(a,b)}{y-b}.$$

Note 2: In the case of functions of two variables the existence of partial derivatives at a point need not imply continuity at that point.

Illustrative Examples

Example 1: *Find* $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ *at* $(1,2)$ *if* $f(x,y) = 2x^2 - xy + 2y^2$.

Solution: We have $\left(\dfrac{\partial f}{\partial x}\right)_{(1,2)} = \lim\limits_{h \to 0} \dfrac{f(1+h,2) - f(1,2)}{h}$

$$= \lim_{h \to 0} \frac{\{2(1+h)^2 - (1+h).2 + 2.2^2\} - \{2.1^2 - 1.2 + 2.2^2\}}{h}$$

$$= \lim_{h \to 0} \frac{2h^2 + 2h}{h} = \lim_{h \to 0} (2h+2) = 2$$

and

$$\left(\frac{\partial f}{\partial y}\right)_{(1,2)} = \lim_{k \to 0} \frac{f(1,2+k) - f(1,2)}{k}$$

$$= \lim_{k \to 0} \frac{\{2 - (2+k) + 2(2+k)^2\} - \{2 - 2 + 8\}}{k}$$

$$= \lim_{k \to 0} \frac{2k^2 + 7k}{k} = \lim_{k \to 0} (2k+7) = 7.$$

Example 2: *Find* $f_x(0,0)$ *and* $f_y(0,0)$ *if*

$$f(x,y) = \begin{cases} \dfrac{x^2 - xy}{x+y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Solution: We have

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{\dfrac{h^2-h\,.0}{h+0}-0}{h}=\lim_{h\to 0}1=1$$

and $$f_y(0,0)=\lim_{k\to 0}\frac{f(0,0+k)-f(0,0)}{k}=\lim_{k\to 0}\frac{0-0}{k}=\lim_{k\to 0}0=0.$$

Example 3: *If $f(x,y)=\dfrac{x^3+y^3}{x-y}$, $x\neq y$ and 0 if $x=y$ then show that f is discontinuous at the origin but the partial derivatives exist at the origin.* **(Lucknow 2007)**

Solution: If we let (x,y) approach $(0,0)$ through the curve $y=x-mx^3$, we have

$$\lim_{(x,y)\to(0,0)}f(x,y)=\lim_{(x,y)\to(0,0)}f(x,x-mx^3)$$

$$=\lim_{x\to 0}\frac{x^3+(x-mx^3)^3}{x-(x-mx^3)}=\lim_{x\to 0}\frac{x^3\{1+(1-mx^2)^3\}}{mx^3}$$

$$=\lim_{x\to 0}\frac{1}{m}\{2-m^3x^6-3mx^2+3m^2x^4\}$$

$$=\lim_{x\to 0}\left\{\frac{2}{m}-m^2x^6-3x^2+3mx^4\right\}=\frac{2}{m}.$$

Hence the simultaneous limit does not exist since it depends upon m. So the function $f(x,y)$ is not continuous in x-y at the origin.

But $$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{\dfrac{h^3}{h}-0}{h}=\lim_{h\to 0}h=0$$

and $$f_y(0,0)=\lim_{k\to 0}\frac{f(0,0+k)-f(0,0)}{k}=\lim_{k\to 0}\frac{\dfrac{k^3}{-k}-0}{k}=\lim_{k\to 0}(-k)=0.$$

Thus the partial derivatives exist at the origin.

Example 4: *Let $f(x,y)=\dfrac{x^2y}{x^4+y^2}$ for $(x,y)\neq(0,0)$ and $f(0,0)=0$. Show that the partial derivatives f_x, f_y exist everywhere in the region $-1\le x\le 1, -1\le y\le 1$, although $f(x,y)$ is discontinuous in x-y at the origin.*

Solution: We have for $x\neq 0, y\neq 0$

$$f_x=2xy\,\frac{y^2-x^4}{(x^4+y^2)^2},\ f_y=x^2\,\frac{x^4-y^2}{(x^4+y^2)^2}.$$

For $x=0, y=0$, we get

$$f_x = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0-0}{h} = \lim_{h \to 0} 0 = 0$$

$$\left[\because f(h,0) = \frac{h^2 \,.0}{h^4 + 0^2} = 0 \text{ and } f(0,0) = 0\right]$$

and
$$f_y = \lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{0-0}{k} = \lim_{k \to 0} 0 = 0.$$

Also, we can show that

$$f_x(x, y) = 0 \quad \text{for} \quad x = 0, y \neq 0\,;$$
$$f_x(x, y) = 0 \quad \text{for} \quad x \neq 0, y = 0\,;$$
$$f_y(x, y) = 0 \quad \text{for} \quad x = 0, y \neq 0\,;$$
and
$$f_y(x, y) = \frac{1}{x^2} \quad \text{for} \quad x \neq 0, y = 0.$$

Hence the partial derivatives f_x, f_y exist at all points of the given region. However, the function $f(x, y)$ is not continuous in x-y at the origin as the simultaneous limit does not exist. We can see that the limit along the line $y = 0$ is 0 whereas along the curve $y = x^2$ it is $\frac{1}{2}$.

Example 5: *Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a function defined by*

$$f(x, y) = \frac{x y^2}{x^2 + y^4} \textit{ for } (x, y) \neq (0,0) \textit{ and } f(0,0) = 0.$$

Using definition calculate $f_x(0,0)$ and $f_y(0,0)$.

Solution: By definition,

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = \lim_{h \to 0} 0 = 0.$$

$$\left[\because f(h,0) = \frac{h\,.0^2}{h^2 + 0^4} = 0 \text{ and } f(0,0) = 0\right]$$

Again
$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} \frac{0-0}{k} = \lim_{k \to 0} 0 = 0.$$

$$\left[\because f(0,k) = \frac{0\,.k^2}{0^2 + k^4} = 0, f(0,0) = 0\right]$$

2 Differentiability

Definition: *Let $z = f(x, y)$ be a function of two independent variables x and y and suppose $f(x, y)$ possesses a determinate value at a point (a, b) and at any point $(a+h, b+k)$ in the neighbourhood of (a, b). The function $f(x, y)$ is said to be totally differentiable (or simply differentiable) at (a, b) if there exist two numbers A and B independent of h, k such that*

$$f(a+h, b+k) - f(a, b) = Ah + Bk + \sqrt{(h^2 + k^2)}\,\phi(h, k)$$

where $$\lim_{(h,k)\to(0,0)} \phi(h, k) = 0.$$

The part $Ah + Bk$ is called the derivative of $f(x, y)$ at (a, b).

The distance between the fixed point (a, b) and the variable point $(a+h, b+k)$ is $\sqrt{(h^2 + k^2)}$, which may be regarded as a simultaneous increment of the two variables.

Remark:Another definition of differentiability of the function $f(x, y)$ at a point of its domain. The function $f(x, y)$ is said to be differentiable at (a, b) if

$$f(a+h, b+k) - f(a, b) = Ah + Bk + h\,\phi(h, k) + k\,\psi(h, k),$$

where A and B are constants independent of h, k and ϕ, ψ are functions of h, k tending to zero as h, k tend to zero simultaneously.

According to this definition the concept of differentiability of the function $f(x, y)$ at any point (x, y) of its domain can be discussed as below.

Let $(x, y), (x+\delta x, y+\delta y)$ be two neighbouring points in the domain of the function $f(x, y)$. The change δf in the value of the function f as the point changes from (x, y) to $(x+\delta x, y+\delta y)$ is given by

$$\delta f = f(x+\delta x, y+\delta y) - f(x, y).$$

The function f is said to be *differentiable* at (x, y) if the change δf can be expressed in the form

$$\delta f = A\,\delta x + B\,\delta y + \delta x\,\phi(\delta x, \delta y) + \delta y\,\psi(\delta x, \delta y) \qquad \ldots(1)$$

where A and B are expressions independent of $\delta x, \delta y$ and ϕ, ψ are functions of $\delta x, \delta y$ each tending to zero as $\delta x, \delta y$ tend to zero simultaneously.

Also, $A\,\delta x + B\,\delta y$ is then called the *differential* of f at (x, y) and is denoted by df. Thus, we have $df = A\,\delta x + B\,\delta y$.

From (1), when $(\delta x, \delta y) \to (0, 0)$, we get

$$\delta f \to 0 \quad \Rightarrow \quad f(x+\delta x, y+\delta y) - f(x, y) \Rightarrow 0$$

$$\Rightarrow \quad f(x+\delta x, y+\delta y) \to f(x, y) \;\Rightarrow\; f(x, y) \text{ is continuous at } (x, y).$$

Hence, *every differentiable function is continuous.*

The converse is not true *i.e.*, a function $f(x, y)$ may be continuous at (a, b) but may not be differentiable at (a, b).

Again, from (1), when $\delta y = 0$ *i.e.*, when y remains constant, we have

$$\delta f = A\,\delta x + \delta x\,\phi(\delta x, 0).$$

Dividing by δx and taking limits of both sides as $\delta x \to 0$, we get

$$\frac{\partial f}{\partial x} = A.$$

Similarly, $\dfrac{\partial f}{\partial y} = B.$

Thus the expressions A and B are respectively the partial derivatives of f with respect to x and y.

Hence, *a function which is differentiable at a point possesses the first order partial derivatives at that point.*

The converse of this statement is not true *i.e.*, there may exist functions which are continuous and may even possess partial derivatives at a point but are not differentiable at that point. We shall later on give some examples of such functions.

Theorem 1: *If the function $f(x, y)$ is differentiable at the point (a, b), then it is continuous in x-y together at $(a\ b)$.* **(Lucknow 2014)**

Proof: Since $f(x, y)$ is differentiable at the point (a, b) so we have a relation

$$f(a+h, b+k) - f(a, b) = A\,h + B\,k + \sqrt{(h^2 + k^2)}\,\phi(h, k), \quad \ldots(1)$$

where (i) $(a+h, b+k)$ is a point in the nhd of (a, b),

(ii) A, B are independent of h, k and

(iii) $\lim\limits_{(h,k)\to(0,0)} \phi(h, k) = 0.$

Taking limit of both sides of (1) as $(h, k) \to (0, 0)$, we get

$$\lim_{(h,k)\to(0,0)} f(a+h, b+k) = f(a, b),$$

which shows that $f(x, y)$ is continuous in x-y together at $(a\ b)$.

Note: The above theorem shows that continuity in the two variables is a necessary condition for differentiability. However, it is not a sufficient condition as shown in example 8.

Theorem 2: *If the function $z = f(x, y)$ is differentiable at (a, b), then the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ both exist and are finite.* **(Lucknow 2014)**

Proof: Since $f(x, y)$ is differentiable at the point (a, b) so we have a relation

$$f(a+h, b+k) - f(a, b) = A\,h + Bk + \sqrt{(h^2 + k^2)}\,\phi(h, k), \quad \ldots(1)$$

where (i) $(a+h, b+k)$ is a point in the nhd of (a, b)

(ii) A and B are independent of h, k and

(iii) $\lim\limits_{(h,k)\to(0,0)} \phi(h, k) = 0.$

Putting $k = 0$ and proceeding to the limits as $h \to 0$, we see from (1) that

$$\lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h} = A \quad \text{or} \quad f_x(a, b) = A.$$

$\therefore$ $f_x(a, b)$ exists and is equal to A.

Similarly we can show that $f_y(a, b)$ exists and is equal to B.

Note: The converse of the above theorem need not be true *i.e.*, if f is continuous and possesses partial derivatives at a point then f need not be differentiable at that point. This is illustrated in example 10.

Illustrative Examples

Example 6: *Show that*

$$f(x, y) = \begin{cases} xy\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right); & (x, y) \neq (0,0) \\ 0 \quad ; & (x, y) = (0,0) \end{cases}$$

is differentiable at the origin. **(Lucknow 2002)**

Solution: We have $f(0 + h, 0 + k) = f(h, k) = hk\left(\dfrac{h^2 - k^2}{h^2 + k^2}\right)$.

$\therefore$
$$f(0 + h, 0 + k) - f(0,0) = 0 \,.\, h + 0 \,.\, k + \sqrt{(h^2 + k^2)}\, \frac{hk\,(h^2 - k^2)}{(h^2 + k^2)^{3/2}}$$

so that
$$A = 0, B = 0 \text{ and } \phi(h, k) = \frac{hk\,(h^2 - k^2)}{(h^2 + k^2)^{3/2}}.$$

Thus A and B are independent of h, k. Now we shall show that $\phi(h, k) \to 0$ as $(h, k) \to (0,0)$.

Let $\varepsilon > 0$ be given.

For all $(h, k) \neq (0,0)$, we have

$$|\phi(h, k) - 0| = \left| \frac{hk}{\sqrt{(h^2 + k^2)}} \cdot \frac{h^2 - k^2}{h^2 + k^2} \right| = \left| \frac{hk}{\sqrt{(h^2 + k^2)}} \right| \cdot \frac{|h^2 - k^2|}{h^2 + k^2}$$

$$\leq \left| \frac{hk}{\sqrt{(h^2 + k^2)}} \right| \qquad [\because |h^2 - k^2| \leq h^2 + k^2]$$

$$= \frac{|hk|}{h^2 + k^2} \cdot \sqrt{(h^2 + k^2)} \qquad \ldots(1)$$

Now for all h and k, we have $(|h| - |k|)^2 \geq 0$

$$\Rightarrow \quad |h|^2 + |k|^2 - 2|h|.|k| \geq 0$$

$$\Rightarrow \quad h^2 + k^2 \geq 2|hk|$$

$$\Rightarrow \quad \frac{|hk|}{h^2 + k^2} \leq \frac{1}{2}, \text{ if } (h, k) \neq (0,0). \qquad \ldots(2)$$

$\therefore$ From (1) and (2), for all $(h, k) \neq (0,0)$, we have

$$|\phi(h, k) - 0| \leq \frac{1}{2} \sqrt{(h^2 + k^2)}$$
$$< \sqrt{(h^2 + k^2)}$$

$$< \varepsilon, \text{ if } h^2 < \frac{\varepsilon^2}{2} \text{ and } k^2 < \frac{\varepsilon^2}{2} \text{ i.e.,}$$

$$\text{if } |h| < \frac{\varepsilon}{\sqrt{2}} \text{ and } |k| < \frac{\varepsilon}{\sqrt{2}}.$$

Now if we take $\delta = \dfrac{\varepsilon}{\sqrt{2}}$, then we see that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\phi(h,k) - 0| < \varepsilon, \text{whenever } |h| < \delta \text{ and } |k| < \delta.$$

$$\therefore \quad \lim_{(h,k)\to(0,0)} \phi(h,k) = 0.$$

$\therefore$ f is differentiable at $(0,0)$ and the derivative of f at $(0,0) = Ah + Bk = 0$.

Example 7: *Show that $f(x,y) = |x| + |y|$ is not differentiable at the origin.*

Solution: We have $f(0+h, 0+k) = |h| + |k|$.

$$\therefore \quad f(0+h,0+k) - f(0,0) = 0 \,.\, h + 0 \,.\, k + \sqrt{(h^2+k^2)} \,.\, \left\{\frac{|h|+|k|}{\sqrt{(h^2+k^2)}}\right\}$$

so that $\quad A = 0, B = 0$ and $\phi(h,k) = \dfrac{|h|+|k|}{\sqrt{(h^2+k^2)}}$.

Thus A and B are independent of h, k. To find the limit of $\phi(h,k)$ let (h,k) approach $(0,0)$ along the line $k = h$.

Putting $k = h$ and then making h to approach 0, we get

$$\lim_{h\to 0} \frac{2|h|}{\sqrt{(2h^2)}} = \lim_{h\to 0} \frac{2|h|}{\sqrt{2}\,.|h|} = \lim_{h\to 0} \sqrt{2} = \sqrt{2}.$$

Since $\lim_{(h,k)\to(0,0)} \phi(h,k) \neq 0$, f is not differentiable at the origin.

Example 8: *Show that the function defined by*

$$f(x,y) = \frac{xy}{\sqrt{(x^2+y^2)}} \text{ for } (x,y) \neq (0,0) \text{ and } f(0,0) = 0$$

is continuous in x-y at the origin but is not differentiable in x-y at the origin.

(Lucknow 2013)

Solution: **Continuity of $f(x,y)$ in x-y at the origin.**

First we shall show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Let $\varepsilon > 0$ be given. For all $(x,y) \neq (0,0)$, we have

$$|f(x,y) - 0| = \left|\frac{xy}{\sqrt{(x^2+y^2)}} - 0\right| = \left|\frac{xy}{\sqrt{(x^2+y^2)}}\right| = \left|\frac{r\cos\theta\, r\sin\theta}{r}\right|$$

[Putting $x = r\cos\theta$, $y = r\sin\theta$]

$$= r|\cos\theta||\sin\theta|$$

$$\leq r \qquad [\because |\cos\theta| \leq 1, |\sin\theta| \leq 1]$$

$$= \sqrt{(x^2 + y^2)}$$

$$< \varepsilon, \text{ if } x^2 < \frac{\varepsilon^2}{2} \text{ and } y^2 < \frac{\varepsilon^2}{2} \text{ i.e.,}$$

$$\text{if } |x| < \frac{\varepsilon}{\sqrt{2}} \text{ and } |y| < \frac{\varepsilon}{\sqrt{2}}.$$

Now if we take $\delta = \frac{\varepsilon}{\sqrt{2}}$, then we see that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x, y) - 0| < \varepsilon, \text{ whenever } |x| < \delta \text{ and } |y| < \delta.$$

Hence, $$\lim_{(x, y) \to (0,0)} f(x, y) = 0.$$

Since $f(0,0) = 0$, therefore

$$\lim_{(x, y) \to (0,0)} f(x, y) = f(0,0).$$

Hence, the given function f is continuous at $(0,0)$ in x-y together.

Now we shall show that $f(x, y)$ is not differentiable at the origin.

We have $$f(0 + h, 0 + k) = \frac{hk}{\sqrt{(h^2 + k^2)}}.$$

$$\therefore \quad f(0 + h, 0 + k) - f(0,0) = 0 \, . \, h + 0 \, . \, k + \sqrt{(h^2 + k^2)} \cdot \frac{hk}{(h^2 + k^2)}$$

so that $$A = 0, B = 0 \text{ and } \phi(h, k) = \frac{hk}{h^2 + k^2}.$$

Thus A and B are independent of h, k.

Here $$\lim_{(h, k) \to (0,0)} \phi(h, k) = \lim_{(h, k) \to (0,0)} \frac{hk}{h^2 + k^2}.$$

This limit does not exist as shown below.

Taking $k = mh$, we have

$$\lim_{(h, k) \to (0,0)} \frac{hk}{h^2 + k^2} = \lim_{h \to 0} \frac{mh^2}{h^2 + m^2 h^2}$$

$$= \lim_{h \to 0} \frac{mh^2}{h^2 (1 + m^2)}$$

$$= \lim_{h \to 0} \frac{m}{1 + m^2}$$

$$= \frac{m}{1 + m^2}, \text{ which depends upon } m.$$

If $m = 0$, then $\displaystyle\lim_{(h, k) \to (0,0)} = \frac{0}{1 + 0} = 0$

and if $m = 1$, then $\displaystyle\lim_{(h, k) \to (0,0)} = \frac{1}{1 + 1} = \frac{1}{2}.$

Since these two methods of approach to the limiting point give different limiting values, therefore the simultaneous limit, $\lim\limits_{(h,k)\to(0,0)} \dfrac{hk}{h^2+k^2}$ does not exist.

Since $\lim\limits_{(h,k)\to(0,0)} \phi(h,k) \neq 0$, therefore f is not differentiable at the origin.

Example 9: *Let* $f(x,y) = \dfrac{xy}{\sqrt{(x^2+y^2)}}$, $(x,y)\neq(0,0)$ *and* $f(0,0)=0$. *Prove that f has partial derivatives at* $(0,0)$ *but is not differentiable at the origin.* **(Lucknow 2013)**

Solution: We have

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = \lim_{h\to 0} 0 = 0.$$

Again $$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k)-f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = \lim_{k\to 0} 0 = 0.$$

Thus f has partial derivatives at $(0,0)$.

To show that $f(x,y)$ is not differentiable at the origin, see example 8 above.

Example 10: *Let* $f(x,y) = \dfrac{x^3-y^3}{x^2+y^2}$ *when* $(x,y)\neq(0,0)$ *and* $f(0,0)=0$. *Show that the function f is continuous and possesses partial derivatives but not differentiable at the origin.* **(Lucknow 2010)**

Solution: **Continuity of f at (0, 0):**

The function $f(x,y)$ will be continuous at $(0,0)$ if

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0.$$

Now we shall show that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

Take any given $\varepsilon > 0$.

For all $(x,y)\neq(0,0)$, we have

$$|f(x,y)-0| = \left|\frac{x^3-y^3}{x^2+y^2}\right| = \left|\frac{r^3(\cos^3\theta-\sin^3\theta)}{r^2}\right|$$

[Putting $x = r\cos\theta$ and $y = r\sin\theta$]

$$= r\,|\cos^3\theta - \sin^3\theta|$$

$$\leq r\,[\,|\cos^3\theta| + |\sin^3\theta|\,] \qquad [\because |a-b| \leq |a|+|b|]$$

$$\leq 2r \qquad [\because |\cos^3\theta|\leq 1 \text{ and } |\sin^3\theta|\leq 1]$$

$$= 2\sqrt{(x^2 + y^2)}$$

$$< \varepsilon, \text{ if } x^2 < \frac{\varepsilon^2}{8} \text{ and } y^2 < \frac{\varepsilon^2}{8} \quad i.e.,$$

$$\text{if } |x| < \frac{\varepsilon}{2\sqrt{2}} \text{ and } |y| < \frac{\varepsilon}{2\sqrt{2}}.$$

Now if we take $\delta = \dfrac{\varepsilon}{2\sqrt{2}}$, then we see that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that $| f(x, y) - 0 | < \varepsilon$, whenever $|x| < \delta$ and $|y| < \delta$.

$\therefore \quad \lim\limits_{(x, y) \to (0,0)} f(x, y) = 0 = f(0,0).$

Hence f is continuous in x-y together at the origin.

Partial derivatives at (0, 0) :

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{1}{h}\left[\frac{h^3 - 0}{h^2 + 0} - 0\right] = 1;$$

$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k \to 0} \frac{1}{k}\left[\frac{0 - k^3}{0 + k^2} - 0\right] = -1.$$

Thus $f_x(0,0)$ and $f_y(0,0)$ exist.

Differentiability at (0, 0) :

We have $\quad f(0+h, 0+k) = \dfrac{h^3 - k^3}{h^2 + k^2} = h - k + \dfrac{h^2 k - h k^2}{h^2 + k^2}.$

$\therefore \quad f(0+h, 0+k) - f(0,0) = 1 \,.\, h + (-1) \,.\, k + \sqrt{(h^2 + k^2)}\left\{\dfrac{hk(h-k)}{(h^2 + k^2)^{3/2}}\right\}$

so that $\quad A = 1, B = -1$ and $\phi(h, k) = \dfrac{hk(h-k)}{(h^2 + k^2)^{3/2}}.$

Thus A and B are numbers independent of h, k. If we put $k = mh$, then we have

$$\lim_{(h,k) \to (0,0)} \phi(h, k) = \lim_{h \to 0} \frac{mh^2(h - mh)}{(h^2 + m^2 h^2)^{3/2}}$$

$$= \lim_{h \to 0} \frac{h^3 m(1 - m)}{h^3 (1 + m^2)^{3/2}}$$

$$= \frac{m(1 - m)}{(1 + m^2)^{3/2}}.$$

This limit does not exist since it depends upon m and so is not unique.

$\therefore \quad \lim \phi(h, k) \neq 0$ as $(h, k) \to (0,0)$.

It follows that the given function is not differentiable at $(0,0)$.

3 Bounded Function

Let $f : X \to \mathbf{R}$, $X \subseteq \mathbf{R}^2$ *be a function of two variables. If the set*

$$f(X) = \{ f(x, y) \in \mathbf{R} : (x, y) \in X \}$$

is bounded , then we say that f *is bounded on* X.

Theorem 1: (Mean-value theorem):

If $f : X \to \mathbf{R}$, $X \subseteq \mathbf{R}^2$ *and* $(a, b) \in X$ *be such that (i)* $f_y(a, b)$ *exists, (ii)* f_x *exists throughout a nhd* $N(a, b)$ *of* (a, b), *then for every point* $(a + h, b + k) \in N(a, b)$

$$f(a + h, b + k) - f(a, b) = h\, f_x(a + \theta h, b + k) + k\, [\, f_y(a, b) + \phi(k)]$$

where $0 < \theta < 1$ *and* $\lim_{k \to 0} \phi(k) = 0$.

Proof: We have $f(a + h, b + k) - f(a, b)$

$$= f(a + h, b + k) - f(a, b + k) + f(a, b + k) - f(a, b). \quad \ldots(1)$$

Since $(a + h, b + k) \in N(a, b)$ and f_x exists in $N(a, b)$, by Lagrange's mean value theorem we have

$$f(a + h, b + k) - f(a, b + k) = h\, f_x(a + \theta h, b + k), \quad \ldots(2)$$

where $0 < \theta < 1$.

Also, since $f_y(a, b)$ exists, we have

$$\lim_{k \to 0} \frac{f(a, b + k) - f(a, b)}{k} = f_y(a, b).$$

$$\therefore \quad f(a, b + k) - f(a, b) = k\, f_y(a, b) + k\, \phi(k) \quad \ldots(3)$$

where $\phi(k) \to 0$ as $k \to 0$.

Using (2), (3) in (1), we get the required result.

Now we shall prove a **sufficient condition** for continuity.

Theorem 2: *If* $f : X \to \mathbf{R}$, $X \subseteq \mathbf{R}^2$, $(a, b) \in X$ *is such that (i)* f_x *exists and is bounded on a neighbourhood* $N(a, b)$ *of* (a, b) *and (ii)* $f_y(a, b)$ *exists then f is continuous at* (a, b).

Proof: Since the conditions of mean-value theorem are satisfied by f so for every $(a + h, b + k) \in N(a, b)$ we have

$$f(a + h, b + k) - f(a, b) = h\, f_x(a + \theta h, b + k) + k\, [\, f_y(a, b) + \phi(k)] \quad \ldots(1)$$

where $0 < \theta < 1$ and $\phi(k) \to 0$ as $k \to 0$.

Since f_x is bounded on $N(a, b)$ and $(a + h, b + k) \in N(a, b)$, $f_x(a + \theta h, b + k)$ is bounded.

$$\therefore \quad \lim_{h \to 0} h\, f_x(a + \theta h, b + k) = 0.$$

Also $\lim k\, f_y(a, b) = 0$, $\lim \phi(k) = 0$ as $k \to 0$.

Taking limit of both sides of (1) as $(h, k) \to (0, 0)$, we get

$$\lim_{(h,k)\to(0,0)} \{f(a+h,b+k) - f(a,b)\} = 0.$$

$$\therefore \quad \lim_{(h,k)\to(0,0)} f(a+h,b+k) = f(a,b).$$

Hence f is continuous at (a,b).

Theorem 3: (A sufficient condition for differentiability).

If $f: X \to \mathbf{R}$, $X \subseteq \mathbf{R}^2$, $(a,b) \in X$ is such that (i) f_x is continuous at (a,b) and (ii) $f_y(a,b)$ exists then f is differentiable at (a,b).

Proof: Since f_x is continuous at (a,b), f_x exists in a nhd $N(a,b)$ of (a,b). Also $f_y(a,b)$ exists. Thus the function f satisfies the conditions of mean value theorem and so for each $(a+h,b+k) \in N(a,b)$, we have

$$f(a+h,b+k) - f(a,b) = h\, f_x(a+\theta h, b+k) + k\,[\,f_y(a,b) + \phi_1(k)] \qquad \ldots(1)$$

where $0 < \theta < 1$ and $\phi_1(k) \to 0$ as $k \to 0$.

Again f_x is continuous at (a,b) gives

$$\lim_{(h,k)\to(0,0)} f_x(a+\theta h, b+k) = f_x(a,b).$$

$$\therefore \quad f_x(a+\theta h, b+k) = f_x(a,b) + \phi_2(h,k) \qquad \ldots(2)$$

where $\quad \phi_2(h,k) \to 0$ as $(h,k) \to (0,0)$.

Using (2) in (1), we get

$$\begin{aligned} f(a+h,b+k) - f(a,b) &= h\,\{f_x(a,b) + \phi_2(h,k)\} + k\,\{f_y(a,b) + \phi_1(k)\} \\ &= h\, f_x(a,b) + k\, f_y(a,b) + \{h\,\phi_2(h,k) + k\,\phi_1(k)\} \\ &= h\, f_x(a,b) + k\, f_y(a,b) \\ &\quad + \sqrt{(h^2+k^2)} \left\{ \frac{h}{\sqrt{(h^2+k^2)}}\, \phi_2(h,k) + \frac{k}{\sqrt{(h^2+k^2)}}\, \phi_1(k) \right\} \\ &= h\, f_x(a,b) + k\, f_y(a,b) + \sqrt{(h^2+k^2)}\, \phi(h,k) \qquad \ldots(3) \end{aligned}$$

where $\quad \phi(h,k) = \dfrac{h}{\sqrt{(h^2+k^2)}}\, \phi_2(h,k) + \dfrac{k}{\sqrt{(h^2+k^2)}}\, \phi_1(k).$

Since $\dfrac{h}{\sqrt{(h^2+k^2)}}, \dfrac{k}{\sqrt{(h^2+k^2)}}$ are bounded and

$$\lim_{(h,k)\to(0,0)} \phi_2(h,k) = 0, \quad \lim_{k\to 0} \phi_1(k) = 0$$

so we get $\lim \phi(h,k) = 0$ as $(h,k) \to (0,0)$.

Also, the numbers $f_x(a,b)$, $f_y(a,b)$ are independent of h, k.

Hence from (3), f is differentiable at (a,b).

Note 1: Similarly we can prove that if (i) $f_x(a,b)$ exists and (ii) f_y is continuous at (a,b) then f is differentiable at (a,b).

Note 2: The conditions of the theorem are not necessary for differentiability as the following example illustrates :

Illustration: *Let f be the function defined by*

$$f\,(x, y) = x^2 \ sin \left(\frac{1}{x}\right) + y^2 \ sin \left(\frac{1}{y}\right); (x, y) \neq (0,0)$$

$$f\,(x,0) = x^2 \ sin \left(\frac{1}{x}\right); x \neq 0$$

$$f\,(0, y) = y^2 \ sin \left(\frac{1}{y}\right); y \neq 0 \qquad and \qquad f\,(0,0) = 0.$$

(Lucknow 2011)

For $x \neq 0$, $\quad f_x\,(x, y) = 2x \sin\left(\frac{1}{x}\right) + x^2 \cos\left(\frac{1}{x}\right).\left(-\frac{1}{x^2}\right) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$

Also $f_y\,(x, y) = 2y \sin\left(\frac{1}{y}\right) - \cos\left(\frac{1}{y}\right)$ for $y \neq 0$ and $f_x\,(0, y) = 0,\ f_y\,(x,0) = 0.$

Thus
$$f_x\,(x, y) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right); & x \neq 0 \\ 0\,; & x = 0 \end{cases}$$

$$\therefore \qquad \lim_{(x, y) \to (0,0)} f_x\,(x, y) = \lim_{x \to 0} 2x \sin\frac{1}{x} - \lim_{x \to 0} \cos\frac{1}{x},$$

which does not exist since $\lim\limits_{x \to 0} \cos\frac{1}{x}$ does not exist. Hence $f_x\,(x, y)$ is not continuous at $(0,0)$.

Similarly, we can show that $f_y\,(x, y)$ is not continuous at $(0,0)$.

Now
$$f\,(0 + h, 0 + k) - f\,(0,0) = h^2 \sin\frac{1}{h} + k^2 \sin\frac{1}{k}$$

$$= 0\,.\,h + 0\,.\,k + \sqrt{(h^2 + k^2)} \left\{\frac{h^2}{\sqrt{(h^2 + k^2}} \sin\frac{1}{h} + \frac{k^2}{\sqrt{(h^2 + k^2)}} \sin\frac{1}{k}\right\}$$

so that $A = 0, B = 0$ are independent of h, k.

Also
$$\phi\,(h, k) = \frac{h^2}{\sqrt{(h^2 + k^2)}} \sin\frac{1}{h} + \frac{k^2}{\sqrt{(h^2 + k^2)}} \sin\frac{1}{k}.$$

Since $\lim\limits_{h \to 0} h \sin\frac{1}{h} = 0,\ \lim\limits_{k \to 0} k \sin\frac{1}{k} = 0$ and $\frac{h}{\sqrt{(h^2 + k^2)}}, \frac{k}{\sqrt{(h^2 + k^2)}}$

are bounded so $\lim \phi\,(h, k) = 0$ as $(h, k) \to (0,0)$.

$\therefore \quad f$ is differentiable at $(0,0)$.

4 Partial Derivatives of Higher Order

The derivative functions f_x, f_y are in general functions of x and y and hence may themselves have partial derivatives with respect to x or y. Thus we are led to higher order partial derivatives. Second order partial derivatives of f are denoted by

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}=f_{xx}\ ;\qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}=f_{yy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\,\partial y}=f_{xy}\ ;\qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\,\partial x}=f_{yx}$$

The second order partial derivatives at a particular point (a, b) are often denoted by

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,\,b)},\ \frac{\partial^2 f\ (a, b)}{\partial x^2}\ \text{ or }\ f_{xx}(a, b)$$

$$\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_{(a,\,b)},\ \frac{\partial^2 f\ (a, b)}{\partial x\,\partial y}\ \text{ or }\ f_{xy}\,(a, b)\ \text{and so on.}$$

Thus,

$$f_{xx}\,(a, b)=\lim_{h\to 0}\frac{f_x\,(a+h, b)-f_x\,(a, b)}{h}$$

$$f_{xy}\,(a, b)=\lim_{h\to 0}\frac{f_y\,(a+h, b)-f_y\,(a, b)}{h}$$

$$f_{yx}\,(a, b)=\lim_{k\to 0}\frac{f_x\,(a, b+k)-f_x\,(a, b)}{k}$$

$$f_{yy}\,(a, b)=\lim_{k\to 0}\frac{f_y\,(a, b+k)-f_y\,(a, b)}{k}$$

provided the limits exist.

We have

$$f_{xx}\,(a, b)=\lim_{h\to 0}\frac{f_x\,(a+h, b)-f_x\,(a, b)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\lim_{h\to 0}\frac{f\,(a+2h, b)-f\,(a+h, b)}{h}-\lim_{h\to 0}\frac{f\,(a+h, b)-f\,(a, b)}{h}\right]$$

$$=\lim_{h\to 0}\frac{f\,(a+2\,h, b)-2f\,(a+h, b)+f\,(a, b)}{h^2}.$$

Similarly,

$$f_{yy}\,(a, b)=\lim_{k\to 0}\frac{f\,(a, b+2\,k)-2f\,(a, b+k)+f\,(a, b)}{k^2}$$

$$f_{xy}(a, b)=\lim_{h\to 0}\left[\lim_{k\to 0}\frac{f\,(a+h, b+k)-f\,(a+h, b)-f\,(a, b+k)+f\,(a, b)}{hk}\right]$$

$$f_{yx}\,(a, b)=\lim_{k\to 0}\left[\lim_{h\to 0}\frac{f\,(a+h, b+k)-f\,(a+h, b)-f\,(a, b+k)+f\,(a, b)}{hk}\right]$$

Note 1: The existence of higher derivatives requires the existence of the corresponding derivatives of lower order. Thus in order that f_{yx} should exist at a point, it is necessary that the partial derivative f_x should exist in the neighbourhood of that point. But, it is possible for the limit defining f_{yx} to exist without the existence of the partial derivative f_x. In such cases higher derivatives cannot be said to exist.

Note 2: The existence (and even the continuity) of f_{xy} need not imply the existence of f_x .

Note 3: f_{yx} and f_{xy} do not always give the same value. However, under certain conditions the equality $f_{yx} = f_{xy}$ can hold. We shall examine these conditions later on.

Illustrative Examples

Example 11: *Let* $f(x, y) = x \sin \frac{1}{x} + y \sin \frac{1}{y}, (x, y) \neq (0,0)$;

$$f(0, y) = y \sin \frac{1}{y}, y \neq 0 \,;\, f(x,0) = x \sin \frac{1}{x}, x \neq 0, \text{ and } f(0,0) = 0.$$

Examine the existence of f_x *and* f_{yx} *at* $(0,0)$.

Solution: We have

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{h \sin \frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} \sin \frac{1}{h},$$

which does not exist. So $f_x(0,0)$ does not exist.

Now
$$\lim_{k \to 0} \left[\lim_{h \to 0} \left\{ \frac{f(h,k) - f(h,0) - f(0,k) + f(0,0)}{hk} \right\} \right]$$

$$= \lim_{k \to 0} \lim_{h \to 0} \frac{h \sin \frac{1}{h} + k \sin \frac{1}{k} - h \sin \frac{1}{h} - k \sin \frac{1}{k} + 0}{hk}$$

$$= \lim_{k \to 0} \lim_{h \to 0} \frac{0}{hk} = 0.$$

Thus, the limit used for defining the second derivative f_{yx} does exist. In spite of this fact, the derivative $f_{yx}(0,0)$ cannot be said to exist, since $f_x(0,0)$ does not exist.

Example 12: *Let* $f(x, y) = g(x)$, *where g is nowhere differentiable. Show that* f_{xy} *exists and is continuous and yet* f_x *does not exist.*

Solution: We have $f(x, y) = g(x)$.

$\therefore$
$$f_x = \frac{\partial f}{\partial x} = g'(x),$$

which does not exist since g is nowhere differentiable.

Again $g(x)$ is a function of x only so its partial derivative with respect to y is zero.

$\therefore$
$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} g(x) = 0.$$

Now
$$f_{xy} = \frac{\partial}{\partial x} f_y = 0.$$

Thus f_{xy} exists and has the value 0 at every point. Being a constant function, f_{xy} is continuous.

5 Interchange of the Order of Differentiation

If $z = f(x, y)$ is a function of two independent variables x and y, then $\dfrac{\partial^2 z}{\partial y\,\partial x}$ and $\dfrac{\partial^2 z}{\partial x\,\partial y}$ do not always give the same value.

However, under certain conditions the equality $\dfrac{\partial^2 z}{\partial y\,\partial x} = \dfrac{\partial^2 z}{\partial x\,\partial y}$ can hold. These conditions we shall examine later. First we give some examples in which f_{xy} may be different from f_{yx}.

Illustrative Examples

Example 13: *If* $f(x, y) = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$; $(x, y) \neq (0,0)$ *and* $f(0,0) = 0$ *show that* $f_{xy}(0,0) \neq f_{yx}(0,0)$. **(Lucknow 2012)**

Solution: Let us define $f_{xy} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$ and $f_{yx} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$.

Thus
$$f_{xy}(0,0) = \lim_{h \to 0} \frac{f_y(0 + h, 0) - f_y(0,0)}{h}.$$

Now
$$f_y(0 + h, 0) = f_y(h, 0) \lim_{k \to 0} \frac{f(h, 0 + k) - f(h, 0)}{k}$$
$$= \lim_{k \to 0} \frac{\dfrac{hk(h^2 - k^2)}{h^2 + k^2} - 0}{k} = \lim_{k \to 0} \frac{h(h^2 - k^2)}{h^2 + k^2} = h,$$

and
$$f_y(0,0) = \lim_{k \to 0} \frac{f(0, 0 + k) - f(0,0)}{k} = \lim_{k \to 0} \frac{0}{k} = \lim_{k \to 0} 0 = 0.$$

$\therefore$
$$f_{xy}(0,0) = \lim_{h \to 0} \frac{h - 0}{h} = \lim_{h \to 0} 1 = 1.$$

Again
$$f_{yx}(0,0) = \lim_{k \to 0} \frac{f_x(0, 0 + k) - f_x(0,0)}{k}.$$

But
$$f_x(0, 0 + k) = f_x(0, k) = \lim_{h \to 0} \frac{f(0 + h, k) - f(0, k)}{h}$$
$$= \lim_{h \to 0} \frac{\dfrac{hk(h^2 - k^2)}{h^2 + k^2} - 0}{h} = \lim_{h \to 0} \frac{k(h^2 - k^2)}{h^2 + k^2} = -k$$

and
$$f_x(0,0) = \lim_{h \to 0} \frac{f(h, 0) - f(0,0)}{h} = \lim_{h \to 0} 0 = 0.$$

$\therefore \qquad f_{yx}(0,0) = \lim_{k \to 0} \dfrac{-k-0}{k} = \lim_{k \to 0} (-1) = -1.$

Hence $f_{xy}(0,0) \neq f_{yx}(0,0)$ in this case.

Example 14: *If* $f(x, y) = \dfrac{xy(y^2 - x^2)}{y^2 + x^2}$ *when* $(x, y) \neq (0,0)$ *and* $f(0,0) = 0$, *calculate* $f_{xy}(0,0)$ *and* $f_{yx}(0,0)$. *Are they equal ?*

Solution: Let us define $f_{xy} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$ and $f_{yx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$

Then $\qquad f_{xy}(0,0) = \lim_{k \to 0} \dfrac{f_x(0, 0+k) - f_x(0,0)}{k}.$

Now $\qquad f_x(0, 0+k) = f_x(0,k) = \lim_{h \to 0} \dfrac{f(0+h, k) - f(0,k)}{h}$

$$= \lim_{h \to 0} \frac{f(h,k) - f(0,k)}{h} = \lim_{h \to 0} \frac{1}{h}\left[\frac{hk(k^2 - h^2)}{k^2 + h^2} - 0\right]$$

$$= \lim_{h \to 0} \frac{k(k^2 - h^2)}{k^2 + h^2} = k,$$

and $\qquad f_x(0,0) = \lim_{h \to 0} \dfrac{f(0+h, 0) - f(0,0)}{h} = \lim_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h}$

$$= \lim_{h \to 0} \frac{0-0}{h} = \lim_{h \to 0} 0 = 0.$$

$\therefore \qquad f_{xy}(0,0) = \lim_{k \to 0} \dfrac{k-0}{k} = \lim_{k \to 0} 1 = 1.$

Again $\qquad f_{yx}(0,0) = \lim_{h \to 0} \dfrac{f_y(0+h, 0) - f_y(0,0)}{h}.$

But $\qquad f_y(0+h, 0) = f_y(h, 0) = \lim_{k \to 0} \dfrac{f(h, 0+k) - f(h,0)}{k}$

$$= \lim_{k \to 0} \frac{f(h,k) - f(h,0)}{k} = \lim_{k \to 0} \frac{1}{k}\left[\frac{hk(k^2 - h^2)}{k^2 + h^2} - 0\right]$$

$$= \lim_{k \to 0} \frac{h(k^2 - h^2)}{k^2 + h^2} = -h$$

and $\qquad f_y(0,0) = \lim_{k \to 0} \dfrac{f(0, 0+k) - f(0,0)}{k} = \lim_{k \to 0} \dfrac{f(0,k) - f(0,0)}{k}$

$$= \lim_{k \to 0} \frac{0-0}{k} = \lim_{k \to 0} 0 = 0.$$

$\therefore \qquad f_{yx}(0,0) = \lim_{h \to 0} \dfrac{-h-0}{h} = \lim_{h \to 0} -1 = -1.$

We observe that here $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Example 15: *Examine the equality of $f_{xy}(0,0)$ and $f_{yx}(0,0)$ for the function*

$$f(x, y) = \begin{cases} (x^2 + y^2) \tan^{-1}(y/x), & x \neq 0 \\ \pi(y^2/2), & x = 0. \end{cases}$$

(Lucknow 2013)

Solution: Let us define $f_{xy} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$ and $f_{yx} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$

Then $$f_{xy}(0,0) = \lim_{k \to 0} \frac{f_x(0,0+k) - f_x(0,0)}{k} = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k}.$$

Now $$f_x(0,k) = \lim_{h \to 0} \frac{f(0+h,k) - f(0,k)}{h} = \lim_{h \to 0} \frac{f(h,k) - f(0,k)}{h}$$

$$= \lim_{h \to 0} \frac{(h^2 + k^2)\tan^{-1}(k/h) - \pi(k^2/2)}{h} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{h \to 0} \frac{(h^2 + k^2) \cdot \dfrac{1}{1 + (k^2/h^2)} \cdot \left(-\dfrac{k}{h^2}\right) + 2h \tan^{-1}\left(\dfrac{h}{k}\right) - 0}{1}$$

[By L' Hospital's rule]

$$= \lim_{h \to 0} \left[-k + 2h \tan^{-1}\left(\frac{k}{h}\right)\right] = -k$$

and $$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0 - 0}{h} = \lim_{h \to 0} 0 = 0.$$

$\therefore$ $$f_{xy}(0,0) = \lim_{k \to 0} \frac{-k - 0}{k} = \lim_{k \to 0} -1 = -1. \qquad \ldots(1)$$

Again

$$f_{yx}(0,0) = \lim_{h \to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h} = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h}.$$

But $$f_y(h,0) = \lim_{k \to 0} \frac{f(h,0+k) - f(h,0)}{k} = \lim_{k \to 0} \frac{f(h,k) - f(h,0)}{k}$$

$$= \lim_{k \to 0} \frac{(h^2 + k^2)\tan^{-1}(k/h) - 0}{k} = \lim_{k \to 0} \left(h + \frac{k^2}{h}\right) \cdot \frac{\tan^{-1}(k/h)}{k/h}$$

$$= h$$

$$\left[\because \lim_{k \to 0} \frac{\tan^{-1}(k/h)}{k/h} = \lim_{t \to 0} \frac{\tan^{-1} t}{t} = \lim_{t \to 0} \frac{1/(1+t^2)}{1} = 1\right]$$

and $$f_y(0,0) = \lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{\pi(k^2/2) - 0}{k} = \lim_{k \to 0} \frac{\pi}{2} k = 0.$$

$$\therefore \qquad f_{yx}(0,0) = \lim_{h \to 0} \frac{h-0}{h} = \lim_{h \to 0} 1 = 1. \qquad \ldots(2)$$

From (1) and (2), we observe that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

6 Sufficient Conditions for the Equality of f_{xy} and f_{yx}

We have seen that the equality $f_{xy} = f_{yx}$ does not always hold. We now give two theorems the object of which is to set out precisely under what conditions it is safe to assume that $f_{xy} = f_{yx}$ at a point *i.e.*, **sufficient conditions** for the equality of f_{xy} and f_{yx}.

Theorem 1: (Schwarz's Theorem):

If $f : X \to \mathbf{R}$, $X \subseteq \mathbf{R}^2$ and $(a,b) \in X$ is such that (i) f_x exists in a neighbourhood $N(a,b)$ of (a,b) and (ii) f_{xy} is continuous at (a,b), then $f_{yx}(a,b)$ exists and is equal to $f_{xy}(a,b)$.

(Lucknow 2013)

Proof: Since f_{xy} is continuous at (a,b), f_y and f_{xy} exist at every point in $N(a,b)$.

Let $\quad \phi(h,k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b) \qquad \ldots(1)$

where $\quad (a+h, b+k) \in N(a,b)$.

If we take $\quad g(y) = f(a+h, y) - f(a, y) \qquad \ldots(2)$

then $\quad g(b+k) = f(a+h, b+k) - f(a, b+k)$

and $\quad g(b) = f(a+h, b) - f(a,b)$.

$\therefore \quad \phi(h,k) = g(b+k) - g(b). \qquad \ldots(3)$

Since f_y exists at every point in $N(a,b)$, $f(a+h, y) - f(a, y)$ is differentiable at each point of $N(a,b)$ *i.e.*, $g(y)$ is differentiable in $[b, b+k] \subseteq N(a,b)$.

$\therefore$ Using mean value theorem for $g(y)$ in $[b, b+k]$, we get

$$g(b+k) - g(b) = kg'(b+\theta k) \text{ where } 0 < \theta < 1.$$

$$\therefore \quad \phi(h,k) = k[f_y(a+h, b+\theta k) - f_y(a, b+\theta k)] \qquad \ldots(4)$$

[From (2), $g'(y) = f_y(a+h, y) - f_y(a, y)$]

Now f_{xy} exists at each point in $N(a,b)$ so $f_y(x, b+\theta k)$ is differentiable at each point in $N(a,b)$ *i.e.*, $f_y(x, b+\theta k)$ is differentiable in $[a, a+h] \subseteq N(a,b)$.

$\therefore$ Using mean-value theorem for $f_y(x, b+\theta k)$ in $[a, a+h]$, we get

$$f_y(a+h, b+\theta k) - f_y(a, b+\theta k) = h f_{xy}(a+\theta_1 h, b+\theta k)$$

where $\quad 0 < \theta_1 < 1$.

$\therefore \quad \phi(h,k) = k[h f_{xy}(a+\theta_1 h, b+\theta k]; 0 < \theta, \theta_1 < 1$

or $$\frac{\phi(h,k)}{hk} = f_{xy}(a+\theta_1 h, b+\theta k)$$

or $$\frac{1}{k}\left[\frac{f(a+h, b+k) - f(a, b+k)}{h} = \frac{f(a+h, b) - f(a,b)}{h}\right] = f_{xy}(a+\theta_1 h, b+\theta k).$$

Taking the limits as $h \to 0$, we get

$$\frac{1}{k}[f_x(a,b+k)-f_x(a,b)] = \lim_{h\to 0} f_{xy}(a+\theta_1 h, b+\theta k)$$

$$\Rightarrow \quad \lim_{k\to 0}\frac{f_x(a,b+k)-f_x(a,b)}{k} = \lim_{k\to 0}\left[\lim_{h\to 0} f_{xy}(a+\theta_1 h, b+\theta k\right]$$

$$= f_{xy}(a,b).$$

Thus f_{yx} exists and $f_{yx}(a,b) = f_{xy}(a,b)$.

Similarly, we can prove that if f_y exists in $N(a,b)$ and f_{yx} is continuous at (a,b), then $f_{xy}(a,b)$ exists and is equal to $f_{yx}(a,b)$.

Note: If f_{xy} and f_{yx} are both continuous at (a,b), then $f_{xy}(a,b) = f_{yx}(a,b)$.

Theorem 2: (Young's Theorem):

If $f : X \to \mathbf{R}$, $X \subseteq \mathbf{R}^2$ and $(a,b) \in X$ is such that f_x, f_y are differentiable at (a,b), then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Proof: Since f_x, f_y are differentiable at (a,b) so f_x, f_y exist in a neighbourhood $N(a,b)$ of (a,b) and $f_{xx}, f_{yx}, f_{xy}, f_{yy}$ exist at (a,b).

Let $\quad \phi(h,h) = f(a+h, b+h) - f(a+h, b) - f(a, b+h) + f(a,b)$

and $\quad g(y) = f(a+h, y) - f(a, y)$.

Then $\quad \phi(h,h) = g(b+h) - g(b)$.

Now, using mean value theorem to $g(y)$ in $[b, b+h]$,

$$g(b+h) - g(b) = hg'(b+\theta h), 0 < \theta < 1.$$

$$\therefore \quad \phi(h,h) = h[f_y(a+h, b+\theta h) - f_y(a, b+\theta h)]. \qquad \ldots(1)$$

Since f_y is differentiable at (a,b), by definition, we have

$$f_y(a+h, b+\theta h) - f_y(a,b)$$
$$= h f_{xy}(a,b) + \theta h f_{yy}(a,b) + \sqrt{(h^2+\theta^2 h^2)}\,\phi_1(h,h) \qquad \ldots(2)$$

and $\quad f_y(a, b+\theta h) - f_y(a,b) = \theta h\, f_{yy}(a,b) + \theta h\, \phi_2(h,h) \qquad \ldots(3)$

where $\quad \phi_1 \to 0$ and $\phi_2 \to 0$ as $h \to 0$.

From (1), (2) and (3) we conclude that

$$\frac{\phi(h,h)}{h^2} = f_{xy}(a,b) + \sqrt{(1+\theta^2)}\,\phi_1(h,h) - \theta\,\phi_2(h,h)$$

$$\lim_{h\to 0}\frac{\phi(h,h)}{h^2} = f_{xy}(a,b).$$

Similarly, if in place of $g(y)$ we take $\quad F(x) = f(x, b+h) - f(x,b)$

we can prove that

$$\lim_{h\to 0}\frac{\phi(h,h)}{h^2} = f_{yx}(a,b).$$

Hence $\quad f_{xy}(a,b) = f_{yx}(a,b)$.

Note: The conditions of above two theorems are only sufficient but not necessary.

Illustrative Examples

Example 16: *Let* $f(x, y) = \frac{1}{4}(x^2 + y^2) \log(x^2 + y^2)$ *for* $(x, y) \neq (0,0)$ *and* $f(0,0) = 0$. *Show that* $f_{xy} = f_{yx}$ *at all points* (x, y). *Also show that neither of the derivatives is continuous in* x-y *at the origin.*

Solution: We have for $(x, y) \neq (0,0)$

$$f_x = \frac{1}{2} x \{1 + \log(x^2 + y^2)\}, f_y = \frac{1}{2} y \{1 + \log(x^2 + y^2)\},$$

and hence $f_{xy} = f_{yx} = \dfrac{xy}{x^2 + y^2}$.

Using the definition of partial derivatives, it can be easily shown that for $x = 0, y = 0$

$$f_x = f_y = f_{xy} = f_{yx} = 0.$$

Thus $f_{xy} = f_{yx}$ at all points (x, y).

However, $f_{xy} = f_{yx}$ is not continuous at $(0,0)$.

We find that the simultaneous limit $\lim\limits_{(x, y) \to (0,0)} \dfrac{xy}{x^2 + y^2}$ does not exist. For if we let (x, y) approach $(0,0)$ through the line $y = mx$, we get

$$\lim_{(x, y) \to (0,0)} \frac{xy}{x^2 + y^2} = \lim_{x \to 0} \frac{x \cdot mx}{x^2 + m^2 x^2} = \frac{m}{1 + m^2}.$$

This limit depends upon m so it does not exist. It implies that $f_{xy} = f_{yx}$ is not continuous at the origin.

Comprehensive Exercise 1

1. If $f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0), \end{cases}$

show that f is discontinuous at the origin but both the partial derivatives exist at the origin.

2. If $f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0), \end{cases}$

show that $f_x(0,0) = 0$ and $f_y(0,0) = 0$.

3. If $f(x, y) = xy\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right)$ when $x^2 + y^2 \neq 0$ and $f(0,0) = 0$, show that

$f_x(x,0) = 0 = f_y(0, y)$ and $f_x(0, y) = -y$.

4. Show that the function $f(x, y) = \dfrac{x^2 y}{x^4 + y^2}$ for $(x, y) \neq (0,0)$ and $f(0,0) = 0$ possesses partial derivatives at the origin but the function is discontinuous at the origin.
5. If $f(x, y) = \sqrt{(|xy|)}$, find $f_x(0,0)$ and $f_y(0,0)$.
6. Show that the function $f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$

   is continuous in x-y together at the origin but not differentiable at $(0,0)$.
7. Prove that the function $f(x, y) = \sqrt{(|xy|)}$ is not (totally) differentiable at $(0,0)$ but that f_x and f_y both exist at the origin and have the value 0.
8. Show that for the function $f(x, y) = |x^2 - y^2|$, we have $f_{xy}(0,0) = f_{yx}(0,0)$.
9. Show that for the function $f(x, y) = \begin{cases} \dfrac{x^2 y^2}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$

   $f_{xy}(0,0) = f_{yx}(0,0)$, even though the conditions of Schwarz's theorem and also of Young's theorem are not satisfied.
10. Show that the function $f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$

    does not satisfy the conditions of Schwarz's theorem and $f_{xy}(0,0) \neq f_{yx}(0,0)$.
11. State a set of conditions under which $f_{xy} = f_{yx}$ where f is a function of two variables. If $f(x, y) = \dfrac{xy(y^2 - x^2)}{y^2 + x^2}$ where $(x, y) \neq (0,0)$ and $f(0,0) = 0$, calculate $f_{xy}(0,0)$ and $f_{yx}(0,0)$. Are they equal ?
12. Show that the function

$$f(x, y) = \begin{cases} x^2 \sin(1/x) + y^2 \sin(1/y), & xy \neq 0 \\ x^2 \sin(1/x), & x \neq 0 \text{ and } y = 0 \\ y^2 \sin(1/y), & x = 0 \text{ and } y \neq 0 \\ 0, & x = y = 0 \end{cases}$$

    is differentiable at the origin.
13. Show that if f_y exists in a certain neighbourhood of a point (a, b) of the domain of definition of a function f, and f_{yx} is continuous at (a, b), then $f_{xy}(a, b)$ exists, and is equal to $f_{yx}(a, b)$.

Answers 1

5. $f_x(0,0) = 0,\ f_y(0,0) = 0$

11. No

7 Implicit Functions

Whenever we come across a functional equation $f(x, y) = 0$, we generally assume that it determines y as a function of x. But in some cases such an equation may not define any such function or it may define one or more than one such functions.

For example, the equation $x^2 + y^2 - 16 = 0$ determines two functions

$$y = \sqrt{16 - x^2} \quad \text{and} \quad y = -\sqrt{16 - x^2}, \text{ for } x^2 \le 16$$

whereas the equations $x^2 + y^2 + 7 = 0$ and $x^4 + y^4 + 9 = 0$ determine no such function.

Definition: *Let $f(x, y)$ be a function of two variables x and y and $y = \phi(x)$ be a function of x such that, for every value of x, for which $\phi(x)$ is defined, $f(x, \phi(x))$ vanishes identically, i.e., $y = \phi(x)$ is a solution of the functional equation $f(x, y) = 0$. Then we say that $y = \phi(x)$ is an* **implicit function** *defined by the functional equation $f(x, y) = 0$.*

It is only in elementary cases, such as those given above, that an implicit function determined by a functional equation $f(x, y) = 0$ may be expressed in the explicit form $y = \phi(x)$. In the case of complicated functional equations the determination of the implicit function in an explicit form may be too laborious or in some cases it may not even be possible. However, the difficulty of actual determination of an analytical expression does not rule out the possibility of the existence of the implicit function or functions, defined by a functional equation. We shall now consider an existence theorem, known as **Implicit function theorem**, that mentions conditions which guarantee that a functional equation does define an implicit function even though its actual determination in explicit form may not be possible.

For many mathematical purposes the real importance lies in the fact that a given functional equation defines an implicit function rather than in finding an expression for the implicit function defined by it.

Hence, the implicit function theorem has its own utility in mathematics.

8 Implicit Function Determined by a Single Functional Equation

Implicit Function Theorem: *Let $f(x, y)$ be a function of two variables x and y and let (a, b) be a point of its domain such that*

(i) $f(a, b) = 0$

(ii) *the function $f(x, y)$ possesses continuous partial derivatives f_x and f_y in a certain neighbourhood of (a, b) and*

(iii) $f_y(a, b) \neq 0$;

then there exists a rectangle $[a-h, a+h; b-k, b+k]$ about (a, b) such that for every value of x in the interval $[a-h, a+h]$, the equation $f(x, y) = 0$ determines one and only one value $y = \phi(x)$, lying in the interval $[b-k, b+k]$, with the following properties :

(1) $b = \phi(a)$,

(2) $f[x, \phi(x)] = 0$, *for every* x *in* $[a-h, a+h]$, *and*

(3) $\phi(x)$ *is derivable and both* $\phi(x)$ *and* $\phi'(x)$ *are continuous in* $[a-h, a+h]$.

Proof: It is given that $f_y(a, b) \neq 0$.

So, either $f_y(a, b) > 0$ or $f_y(a, b) < 0$.

Without any loss of generality we may assume that $f_y(a, b) > 0$, for if $f_y(a, b) < 0$, we should only have to replace $f(x, y)$ by $-f(x, y)$ and this change would leave the equation $f(x, y) = 0$ unaltered.

Uniqueness and the existence of the implicit function:

Let f_x, f_y be continuous in a rectangular neighbourhood

$$R_1 = [a-h_1, a+h_1 ; b-k_1, b+k_1] \text{ of } (a, b).$$

Since f_x, f_y are continuous in R_1, therefore, f is differentiable and hence continuous in R_1.

Again, since f_y is continuous at (a, b) and $f_y(a, b) > 0$, there exists a rectangle

$$R_2 = [a-h_2, a+h_2 ; b-k, b+k], h_2 < h_1, k < k_1$$

(*i.e.*, $R_2 \subseteq R_1$) such that for every point (x, y) of this rectangle R_2, $f_y(x, y) > 0$.

Now, since $f_y(x, y) > 0$ for all $(x, y) \in R_2$, therefore, for all $x \in [a-h_2, a+h_2]$, the function f of y strictly increases as y increases from $b-k$ to $b+k$.

In particular, since $f(a, b) = 0$, we have $f(a, b-k) < 0$, $f(a, b+k) > 0$.

In view of this and the fact that f is continuous, there exists a positive real number $h < h_2$ such that for every x in $[a-h, a+h]$, $f(x, b-k)$ is as close to $f(a, b-k)$ and $f(x, b+k)$ is as close to $f(a, b+k)$ as we please and, therefore, we have

$$f(x, b-k) < 0, f(x, b+k) > 0 \text{ for all } x \in [a-h, a+h].$$

Now for every fixed value of x in $[a-h, a+h]$, the continuous function f of y *strictly* increases from a negative to a positive value as y increases from $b-k$ to $b+k$ and therefore there exists *one and only one* value of y for which the function f vanishes.

Thus, for each x in $[a-h, a+h]$ there is a uniquely determined value of y in $[b-k, b+k]$ for which $f(x, y) = 0$; this y is a function of x, say $\phi(x)$ such that the properties (1) and (2) are true.

Hence, there exists a rectangular neighbourhood

$$R_3 = [a-h, a+h ; b-k, b+k] \text{ of } (a, b)$$

such that for each x belonging to $[a-h, a+h]$ there exists a unique value of $y = \phi(x)$ belonging to $[b-k, b+k]$ such that $b = \phi(a)$ and $f(x, \phi(x)) = 0$ for all x in $[a-h, a+h]$.

This completes the proof of the *existence* and the *uniqueness* of the implicit function ϕ.

Continuity and derivability of the implicit function:

Let $(x, y), (x+\delta x, y+\delta y)$ be two points in $R_3 = [a-h, a+h ; b-k, b+k]$ such that

$$y = \phi(x), y + \delta y = \phi(x + \delta x)$$

and $$f(x, y) = 0, f(x+\delta x, y+\delta y) = 0.$$

Since f is differentiable in R_1 and $R_3 \subseteq R_1$, therefore f is also differentiable in R_3.

$\therefore \qquad f(x+\delta x, y+\delta y) - f(x, y) = \delta x\, f_x + \delta y\, f_y + \delta x\, \psi_1 + \delta y\, \psi_2$,

where ψ_1, ψ_2 are functions of δx and δy, each tending to zero as $(\delta x, \delta y) \to 0$.

But $\qquad f(x+\delta x, y+\delta y) - f(x, y) = 0 - 0 = 0$.

$\therefore$ In R_3, $\qquad \delta x\, f_x + \delta y\, f_y + \delta x\, \psi_1 + \delta y\, \psi_2 = 0$

or $\qquad \delta y\, f_y = -\delta x\, f_x - \delta x\, \psi_1 - \delta y\, \psi_2$. ...(1)

Since $f_y > 0$ in R_2 and $R_3 \subseteq R_2$, therefore $f_y \neq 0$ in R_3. So, dividing both sides of (1) by $\delta x\, f_y$, we have

$$\frac{\delta y}{\delta x} = -\frac{f_x}{f_y} - \frac{\psi_1}{f_y} - \frac{\delta y}{\delta x}\frac{\psi_2}{f_y}.$$

Proceeding to limits as $(\delta x, \delta y) \to (0, 0)$, we get

$$\phi'(x) = \frac{dy}{dx} = -\frac{f_x}{f_y}.$$

Thus $\phi(x)$ is derivable and continuous in R_3. Also $\phi'(x)$, being a quotient of two continuous functions, is itself continuous in R_3.

Note 1: The function $y = \phi(x)$ is said to be the unique solution of $f(x, y) = 0$ *near* (a, b) or the unique implicit function determined by $f(x, y) = 0$ *near* (a, b).

Note 2: It should be clearly understood that the theorem just proved is essentially of a *local character*. It states that if f is a function of two variables satisfying certain assumptions in respect of continuity and derivability in a neighbourhood of a point (a, b) and if $f(a, b) = 0$, then there exists a rectangular neighbourhood $[a-h, a+h; b-k, b+k]$ of (a, b) such that for each x belonging to $[a-h, a+h]$ there exists a unique y belonging to $[b-k, b+k]$ such that $f(x, y) = 0$. Thus, $f(x, y) = 0$ determines a unique implicit function $y = \phi(x)$ in $[a-h, a+h]$, y lying in $[b-k, b+k]$ and this function ϕ of x is derivable. The implicit function $y = \phi(x)$ is a unique Solution of $f(x, y) = 0$ in a certain neighbourhood $[a-h, a+h; b-k, b+k]$ of (a, b). The functional equation $f(x, y) = 0$ may have a different Solution if a different neighbourhood of (a, b) is considered.

Illustrative Examples

Example 17: *Examine the following equation for the existence of a unique implicit function near the point indicated and verify your assertion by direct calculation. Find also the first derivative of the solution :* $x^2 + y^2 - 1 = 0$, *in the nbd of the point* $(0, 1)$.

Solution: Let $f(x, y) = x^2 + y^2 - 1$.

Then $\qquad f_x = 2x,\ f_y = 2y$.

We have $\qquad f(0, 1) = 0^2 + 1^2 - 1 = 0$

$\qquad f_y(0, 1) = 2 \cdot 1 = 2 \neq 0$.

Obviously f_x and f_y are continuous functions in the whole x-y plane and so also in some neighbourhood of $(0, 1)$.

Thus $f(x, y)$ satisfies all the conditions of implicit function theorem in some neighbourhood of $(0,1)$. Hence, by implicit function theorem, there exists a rectangular neighbourhood $[0-h, 0+h; 1-k, 1+k]$ of $(0,1)$ such that for each x belonging to $[0-h, 0+h]$ there exists a unique y belonging to $[1-k, 1+k]$ such that $f(x, y) = 0$. Thus, $f(x, y) = 0$ determines a unique implicit function $y = \phi(x)$ in $[0-h, 0+h]$, y lying in $[1-k, 1+k]$ and this function ϕ of x is derivable.

Solving the equation $x^2 + y^2 - 1 = 0$ as a quadratic in y, we get

$$y^2 = 1 - x^2$$

or $$y = \pm\sqrt{1-x^2}.$$

Obviously $(0,1)$ satisfies the equation

$$y = \sqrt{1-x^2}.$$

Hence, $y = \sqrt{1-x^2}$ is the unique implicit function $y = \phi(x)$ determined by the equation $f(x, y) = 0$ in some neighbourhood of $(0,1)$, where $|x| \le 1$, $y > 0$.

We have $$\left(\frac{dy}{dx}\right)_{(0,1)} = -\left(\frac{f_x}{f_y}\right)_{(0,1)} = -\left(\frac{2x}{2y}\right)_{(0,1)} = 0$$

Remark: Here, $y = -\sqrt{x^2 - 1}$ is the unique implicit function determined by the equation $f(x, y) = 0$ in some neighbourhood of $(0, -1)$, where $|x| \le 1$, $y < 0$.

Example 18: *Show that the following equation determines a unique Solution near the point indicated ; find also the first derivative of the solution :*

$$xy \sin x + \cos y = 0, \left(0, \frac{1}{2}\pi\right).$$

Solution: Let $f(x, y) = xy \sin x + \cos y$.

Then $$f_x = y \sin x + xy \cos x, \quad f_y = x \sin x - \sin y.$$

We have $$f\left(0, \frac{1}{2}\pi\right) = 0 + \cos\frac{1}{2}\pi = 0,$$

$$f_y\left(0, \frac{1}{2}\pi\right) = 0 - \sin\frac{1}{2}\pi = -1 \ne 0$$

Obviously, the partial derivatives f_x and f_y are continuous functions in a neighbourhood of $\left(0, \frac{1}{2}\pi\right)$.

Thus $f(x, y)$ satisfies all the conditions of implicit function theorem in a neighbourhood of $\left(0, \frac{1}{2}\pi\right)$. Hence, by implicit function theorem, the equation $f(x, y) = 0$ determines a unique implicit function, say $y = \phi(x)$, in a neighbourhood of $\left(0, \frac{1}{2}\pi\right)$.

We have $\dfrac{dy}{dx} = \phi'(x) = -\dfrac{f_x}{f_y} = -\dfrac{y \sin x + xy \cos x}{x \sin x - \sin y}$.

$\therefore$ $\left(\dfrac{dy}{dx}\right)_{(0, \frac{1}{2}\pi)} = -\dfrac{0+0}{0 - \sin \frac{1}{2}\pi} = -\dfrac{0}{-1} = 0.$

Example 19: *Examine the following equation for the existence of a unique implicit function near the point indicated and verify your assertion by direct calculation :*

$$y^2 - yx^2 - 2x^5 = 0, \text{ point } (1, -1).$$

Solution: Let $f(x, y) = y^2 - yx^2 - 2x^5$.

Then $f_x = -2xy - 10x^4$, $f_y = 2y - x^2$.

We have $f(1, -1) = (-1)^2 - (-1) \cdot 1^2 - 2 \cdot 1^5 = 0$

$f_y(1, -1) = 2 \cdot (-1) - 1^2 = -3 \neq 0.$

Obviously, the partial derivatives $f_x(x, y) = -2xy - 10x^4$ and $f_y(x, y) = 2y - x^2$ are continuous in a neighbourhood of $(1, -1)$.

Hence, by implicit function theorem, the equation $f(x, y) = 0$ determines a unique implicit function $y = \phi(x)$ in a neighbourhood of $(1, -1)$.

Solving the equation $y^2 - yx^2 - 2x^5 = 0$ as a quadratic in y, we get

$$y = \frac{x^2 \pm \sqrt{x^4 + 8x^5}}{2} = \frac{x^2}{2}[1 \pm \sqrt{1 + 8x}], x \geq -\frac{1}{8}.$$

Of the above two possible solution, $y = \dfrac{x^2}{2}[1 - \sqrt{1+8x}], x \geq -\dfrac{1}{8}$ is the unique solution of $f(x, y) = 0$ in a neighbourhood of $(1, -1)$, since, $-1 = y(1)$.

Comprehensive Exercise 2

Show that the following equations determine unique solution near the points indicated; find also the first derivatives of the solution :

1. $x^3 + y^3 - 3xy + y = 0$, point $(0, 0)$.
2. $y^3 \cos x + y^2 \sin^2 x = 7$, point $\left(\frac{1}{3}\pi, 2\right)$
3. $2xy - \log xy = 2$, point $(1, 1)$.

Examine the following equations for the existence of a unique implicit function near the point indicated and verify your assertion by direct calculations :

4. $y^2 - yx^2 - 2x^5 = 0$, near $(0, 0)$.

5. $y^2 + 2x^2 y + x^5 = 0$, near $(1, -1)$.

6. $y^2 + yx^3 + x^2 = 0$, near $(0, 0)$.

7. $y^4 + x^2 y^2 - 2x^5 = 0$, near $(1, 1)$.

8. $x^2 + xy + y^2 - 1 = 0$, near $(1, 0)$.

Answers 2

1. 0
2. $\dfrac{2\sqrt{3}}{9}$
3. -1
4. $\phi(x)$ not unique near $(0, 0)$
5. $\phi(x)$ not unique near $(1, -1)$
6. $\phi(x)$ does not exist near $(0, 0)$
7. $\phi(x)$ unique near $(1, 1)$
8. $\phi(x)$ unique near $(1, 0)$

9 Taylor's Theorem for Functions of Several Variables

(Lucknow 2007, 10; Gorakhpur 10, 11, 13, 14, 15)

To expand $f(x+h, y+k)$ in powers of h and k in case $f(x, y)$ and all its partial derivatives are continuous in a certain domain of the point (x, y).

Regarding $f(x+h, y+k)$ as a function of one variable only, say that of x (that is, supposing that only x varies while y remains constant), and expanding by Taylor's theorem for one variable only, we have

$$f(x+h, y+k) = f(x, y+k) + h\frac{\partial f(x, y+k)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f(x, y+k)}{\partial x^2} + \frac{h^3}{3!}\frac{\partial^3 f(x, y+k)}{\partial x^3} + \dots \qquad \dots(1)$$

Now expanding each term on the right hand side of (1) by Taylor's theorem regarding y as variable and x as constant, we have

$$\begin{aligned} f(x+h, y+k) &= f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \frac{k^3}{3!}\frac{\partial^3 f(x, y)}{\partial y^3} + \dots \\ &\quad + h\frac{\partial}{\partial x}\left\{f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \dots\right\} \\ &\quad + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left\{f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \dots\right\} \\ &\quad + \frac{h^3}{3!}\frac{\partial^3}{\partial x^3}\{f(x, y) + \dots\} + \dots \end{aligned}$$

Hence,
$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk \cdot \frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)$$
$$+\frac{1}{3!}\left(h^3\frac{\partial^3 f}{\partial x^3} + 3h^2 k\frac{\partial^3 f}{\partial x^2\,\partial y} + 3hk^2\frac{\partial^3 f}{\partial x\,\partial y^2} + k^3\frac{\partial^3 f}{\partial y^3}\right) + \dots$$

Symbolically we may put this result as
$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \dots,$$
$$f(x+h, y+k) = f(x, y) + (hf_x + kf_y) + \frac{1}{2!}(h^2 f_{xx} + 2hk\, f_{xy} + k^2 f_{yy})$$
$$+\frac{1}{3!}(h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}) + \dots\dots \quad \dots(2)$$

Corollary 1: Putting $x = a$ and $y = b$ in equation (2), we get
$$f(a+h, b+k) = f(a, b) + \{hf_x(a, b) + kf_y(a, b)\} + \frac{1}{2!}\{h^2 f_{xx}(a, b) + 2hk\, f_{xy}(a, b) + k^2 f_{yy}(a, b)\} + \frac{1}{3!}\{h^3 f_{xxx}(a, b)$$
$$+3h^2 k\, f_{xxy}(a, b) + 3hk^2 f_{xyy}(a, b) + k^3 f_{yyy}(a, b)\} + \dots$$

Corollary 2: Putting $a = 0, b = 0, h = x, k = y$, in equation (2), we get
$$f(x, y) = f(0,0) + x\, f_x(0,0) + y\, f_y(0,0) + \frac{x^2}{2!} f_{xx}(0,0) + xy\, f_{xy}(0,0) + \frac{y^2}{2!} f_{yy}(0,0) + \dots,$$

where
$$f_x(0,0) = \left(\frac{\partial f}{\partial x}\right)_{(0,0)}, \; f_y(0,0) = \left(\frac{\partial f}{\partial y}\right)_{(0,0)}, \; f_{xx}(0,0) = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)},$$
$$f_{xy}(0,0) = \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_{(0,0)}, \; f_{yy}(0,0) = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)} = \dots$$

which is **Maclaurin's theorem** for two variables.

Illustrative Examples

Example 20: Find the first six terms of the expansion of the function $e^x \log(1+y)$ in a Taylor series in the neighbourhood of the point $(0,0)$.

Solution: Let $F(x, y) = e^{x} \log(1+y)$. Then

$$F(x, y) = e^x \log(1+y) \quad \Rightarrow \quad F(0,0) = 0$$

$$F_x(x, y) = e^x \log(1+y) \quad \Rightarrow \quad F_x(0,0) = 0$$

$$F_{xx}(x, y) = e^x \log(1+y) \quad \Rightarrow \quad F_{xx}(0,0) = 0$$

$$F_y(x, y) = \frac{e^x}{(1+y)} \quad \Rightarrow \quad F_y(0,0) = 1$$

$$F_{yy}(x, y) = -\frac{e^x}{(1+y)^2} \quad \Rightarrow \quad F_{yy}(0,0) = -1$$

$$F_{xy}(x, y) = \frac{e^x}{(1+y)} \quad \Rightarrow \quad F_{xy}(0,0) = 1.$$

Using Taylor's theorem, we get

$$F(x, y) = F(0,0) + xF_x(0,0) + yF_y(0,0)$$

$$+\frac{x^2}{2!} F_{xx}(0,0) + x\,y\, F_{xy}(0,0) + \frac{y^2}{2!} F_{yy}(0,0) + \ldots$$

i.e., $$e^{x} \log(1+y) = 0 + x(0) + y(1) + \frac{x^2}{2!}(0) + x\,y(1) + \frac{y^2}{2!}(-1) + \ldots$$

i.e., $$e^x \log(x+y) = y + xy - \frac{y^2}{2} + \ldots$$

Example 21: *Find the expansion for cos x cos y in powers of x , y upto fourth order terms.*

Solution: Let $f(x, y) = \cos x \cos y$. Then

$$f(x, y) = \cos x \cos y \quad \Rightarrow \quad f(0,0) = 1$$

$$f_x(x, y) = -\sin x \cos y \quad \Rightarrow \quad f_x(0,0) = 0$$

$$f_y(x, y) = -\cos x \sin y \quad \Rightarrow \quad f_y(0,0) = 0$$

$$f_{x^2}(x, y) = -\cos x \cos y \quad \Rightarrow \quad f_{x^2}(0,0) = -1$$

$$f_{xy}(x, y) = \sin x \sin y \quad \Rightarrow \quad f_{xy}(0,0) = 0$$

$$f_{y^2}(x, y) = -\cos x \cos y \quad \Rightarrow \quad f_{y^2}(0,0) = -1$$

$$f_{x^3}(x, y) = \sin x \cos y \quad \Rightarrow \quad f_{x^3}(0,0) = 0$$

$$f_{x^2 y}(x, y) = \cos x \sin y \quad \Rightarrow \quad f_{x^2 y}(0,0) = 0$$

$$f_{xy^2}(x, y) = \sin x \cos y \quad \Rightarrow \quad f_{xy^2}(0,0) = 0$$

$$f_{y^3}(x, y) = \cos x \sin y \quad \Rightarrow \quad f_{y^3}(0,0) = 0$$

$$f_{x^4}(x, y) = \cos x \cos y \quad \Rightarrow \quad f_{x^4}(0,0) = 1$$

$$f_{x^3 y}(x, y) = -\sin x \sin y \quad \Rightarrow \quad f_{x^3 y}(0,0) = 0$$

$$f_{x^2 y^2}(x, y) = \cos x \cos y \quad \Rightarrow \quad f_{x^2 y^2}(0,0) = 1$$

$$f_{xy^3}(x, y) = -\sin x \sin y \quad \Rightarrow \quad f_{xy^3}(0,0) = 0$$

$$f_{y^4}(x, y) = \cos x \cos y \quad \Rightarrow \quad f_{y^4}(0,0) = 1.$$

Using Taylor's theorem, we get

$$f(x, y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0)$$

$$+ \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0, 0) + \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(0,0)$$

$$+ \frac{1}{4!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^4 f(0,0) + \ldots$$

$$= f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{x^2}{2} f_{xx}(0,0)$$

$$+ xyf_{xy}(0,0) + \frac{y^2}{2} f_{yy}(0,0) + \frac{1}{6} x^3 f_{xxx}(0,0)$$

$$+ \frac{1}{2} x^2 y f_{xxy}(0,0) + \frac{1}{2} xy^2 f_{xyy}(0,0)$$

$$+ \frac{1}{6} y^3 f_{yyy}(0,0) + \frac{1}{24} x^4 f_{x^4}(0,0)$$

$$+ \frac{1}{6} x^3 y f_{x^3 y}(0,0) + \frac{1}{4} x^2 y^2 f_{x^2 y^2}(0,0)$$

$$+ \frac{1}{6} xy^3 f_{xy^3}(0,0) + \frac{1}{24} y^4 f_{y^4}(0,0) + \ldots\ldots$$

i.e.,

$$\cos x \cos y = 1 + 0 + 0 + \frac{x^2}{2}(-1) + x y(0) + \frac{y^2}{2}(-1) + \frac{x^3}{6}(0)$$

$$+ \frac{x^2 y}{2}(0) + \frac{xy^2}{2}(0) + \frac{y^3}{6}(0) + \frac{x^4}{24}(1) + \frac{x^3 y}{6}(0)$$

$$+ \frac{x^2 y^2}{4}(1) + \frac{xy^3}{6}(0) + \frac{y^4}{24}(1) + \ldots\ldots$$

i.e.,

$$\cos x \cos y = 1 - \frac{x^2}{2} - \frac{y^2}{2} + \frac{x^4}{24} + \frac{x^2 y^2}{4} + \frac{y^4}{24} + \ldots\ldots.$$

Example 22: *If $F(x, y) = \tan^{-1}(xy)$, compute an approximate value of $F(0 \cdot 9, -1 \cdot 2)$.*

(Gorakhpur 2014)

Solution: Let us expand $F(x, y)$ near the point $(1, -1)$.

By Taylor's theorem, we get

$$F(0 \cdot 9, -1 \cdot 2) = F(1 - 0 \cdot 1, -1 - 0 \cdot 2)$$

$$= F(1,1) + \{(1 - 0 \cdot 9) F_x(1,1) + (1 - 0 \cdot 2) F_y(1,1)\}$$

$$+ \frac{1}{2!}\{(1 - 0 \cdot 9)^2 F_{xx}(1, 1) + 2(1 - 0 \cdot 9)(1 - 0 \cdot 2) F_{xy}(1,1)$$

$$+ (1 - 0 \cdot 2)^2 F_{yy}(1,1)\} + \ldots \qquad \ldots(1)$$

Here $a = 1, b = -1$.

$$F(x, y) = \tan^{-1} xy \quad \Rightarrow \quad F(1, -1) = \tan^{-1}(-1) = -\frac{\pi}{4}\cdot$$

$$F_x = \frac{y}{1 + x^2 y^2} \quad \Rightarrow \quad F_x(1, -1) = \frac{-1}{1+1} = -\frac{1}{2}\cdot$$

$$F_y = \frac{x}{1 + x^2 y^2} \quad \Rightarrow \quad F_y(1, -1) = \frac{1}{1+1} = \frac{1}{2}\cdot$$

$$F_{x^2} = -\frac{(2x)\, y}{(1 + x^2 y^2)^2} \quad \Rightarrow \quad F_{x^2}(1, -1) = \frac{-(2)(-1)}{(1+1)^2} = \frac{1}{2}\cdot$$

$$F_{y^2} = \frac{-x\,(2x^2 y)}{(1 + x^2 y^2)^2} \quad \Rightarrow \quad F_{y^2}(1, -1) = \frac{2}{(1+1)^2} = \frac{1}{2}\cdot$$

$$F_{xy} = \frac{1 + x^2 y^2 - x\,(2xy^2)}{(1 + x^2 y^2)^2}$$

$$= \frac{1 - x^2 y^2}{(1 + x^2 y^2)^2} \quad \Rightarrow \quad F_{xy}(1, -1) = \frac{1-1}{(1+1)^2} = 0.$$

Putting all these values in (1), we get

$$F(0\cdot 9, -1\cdot 2) = -\frac{\pi}{4} + (-0\cdot 1)\left(-\frac{1}{2}\right) + (-0\cdot 2)\,\frac{1}{2} + \frac{1}{2}\left[(-0\cdot 1)^2 \left(\frac{1}{2}\right)\right.$$

$$\left. + 2\,(-0\cdot 1)(-0\cdot 2)\, 0 + (-0\cdot 2)^2 \left(\frac{1}{2}\right)\right] + \ldots.$$

$$= -\frac{\pi}{4} + 0\cdot 05 - 0\cdot 1 + 0\cdot 0125 = -0\cdot 823.$$

Example 23: *Expand e^x. cos y near the point $\left(1, \frac{\pi}{4}\right)$ by Taylor's Theorem.*

Solution: We have

$$F(x + h, y + k) = F(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) F$$

$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 F + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 F + \ldots \quad \ldots(1)$$

Again $\quad e^x \cos y = F(x, y) = F\left[1 + (x-1), \frac{\pi}{4} + \left(y - \frac{\pi}{4}\right)\right] = F\left(1 + h, \frac{\pi}{4} + k\right)$

where $\quad h = x - 1, k = y - \frac{\pi}{4}\cdot$

$$F(x, y) = e^x \cos y \quad \Rightarrow \quad F\left(1, \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$$

$$\frac{\partial F}{\partial x} = e^x \cos y \quad \Rightarrow \quad \frac{\partial F}{\partial x}\left(1, \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$$

$$\frac{\partial F}{\partial y} = -e^x \sin y \quad \Rightarrow \quad \frac{\partial F}{\partial y}\left(1, \frac{\pi}{4}\right) = \frac{-e}{\sqrt{2}}$$

$$\frac{\partial^2 F}{\partial x^2} = e^x \cos y \qquad \Rightarrow \qquad \frac{\partial^2 F}{\partial x^2}\left(1, \frac{\pi}{4}\right) = \frac{e}{\sqrt{2}}$$

$$\frac{\partial F}{\partial y^2} = -e^x \cos y \qquad \Rightarrow \qquad \frac{\partial^2 F}{\partial y^2}\left(1, \frac{\pi}{4}\right) = \frac{-e}{\sqrt{2}}$$

$$\frac{\partial^2 F}{\partial x\, dy} = -e^x \sin y \qquad \Rightarrow \qquad \frac{\partial^2 F}{\partial y\, \partial y}\left(1, \frac{\pi}{4}\right) = \frac{-e}{\sqrt{2}}.$$

Substituting these values in Taylor's theorem, we get

$$e^x \cos y = \frac{e}{\sqrt{2}} + \left[(x-1)\frac{e}{\sqrt{2}} + \left(y - \frac{\pi}{4}\right)\frac{-e}{\sqrt{2}}\right]$$
$$+ \frac{1}{2!}\left[(x-1)^2 \frac{e}{\sqrt{2}} + 2(x-1)\left(y-\frac{\pi}{4}\right)\left(\frac{-e}{\sqrt{2}}\right) + \left(y-\frac{\pi}{4}\right)^2\left(\frac{-e}{\sqrt{2}}\right)\right] + \ldots$$

Comprehensive Exercise 3

1. Expand $e^x . \sin y$ in powers of x and y as far as terms of third degree.
2. Expand : $F(x, y) = x^2 y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$ by Taylor's theorem. **(Gorakhpur 2015)**
3. Obtain Taylor's expansion of $\tan^{-1} \frac{y}{x}$ about $(1, 1)$ upto and including the second degree terms. Hence compute $F(1 \cdot 1, 0 \cdot 9)$. **(Gorakhpur 2012, 13)**
4. Expand x^y in powers of $(x-1)$ and $(y-1)$ upto the third degree terms.

Answers 3

1. $e^x \sin y = y + xy + \frac{x^2 y}{2} - \frac{y^3}{6} + \ldots$
2. $x^2 y + 3y - 2 = -10 - 4(x-1) + 4(y+2) - 2(x-1)^2$
   $+ 2(x-1)(y+2) + (x-1)^2 (y+2)$
3. $\tan^{-1} \frac{y}{x} = \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2 + \ldots;\ 0 \cdot 7862$
4. $x^y = 1 + (x-1) + (x-1)(y-1) + \frac{1}{2}(x-1)^2 (y-1)$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Let $f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{(x^2 + y^2)}}, & (x, y) \neq (0,0) \\ 0, & (x, y) = 0. \end{cases}$

Then

(a) $f(x, y)$ is discontinuous at $(0,0)$ (b) $f(x, y)$ is continuous at $(0,0)$
(c) $f_x(0,0) = 1$ (d) $f_y(0,0) = 1$.

2. Let $f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0). \end{cases}$

Then

(a) $f(x, y)$ is not defined at the origin (b) $f_x(0,0) = 0$
(c) $f_x(0,0) = 1$ (d) $f_y(0,0)$ does not exist.

Fill in the Blank(s)

Fill in the blanks "……", so that the following statements are complete and correct.

1. The partial derivative $f_x(a, b)$ of a function $f(x, y)$ at the point (a, b) is given by

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - \ldots\ldots}{h},$$

provided the limit exists.

2. If a function $f(x, y)$ is differentiable at (a, b), then it is …… in x-y together at (a, b).

3. If $f : X \to \mathbf{R}, X \subseteq \mathbf{R}^2$ and $(a, b) \in X$ is such that f_x, f_y are differentiable at (a, b), then $f_{xy}(a, b) = \ldots\ldots$.

True or False

Write 'T' for true and 'F' for false statement.

1. If a function $f(x, y)$ is continuous at a point (a, b), it must also be differentiable at (a, b).

2. If a function $f(x, y)$ possesses both the partial derivatives $f_x(a, b)$ and $f_y(a, b)$, it must be differentiable at (a, b).

3. If a function $f(x, y)$ is differentiable at (a, b), it must be continuous at (a, b).

Answers

Multiple Choice Questions

1. (b).
2. (b).

Fill in the Blanks

1. $f\,(a, b)$.
2. continuous.
3. $f_{yx}\,(a, b)$.

True or False

1. F.
2. F.
3. T.

□

Chapter

6

Partial Differential Equations of the First Order

1 Partial Differential Equations

Equations which contain one or more partial derivatives are called **Partial Differential Equations**. Such equations arise in geometry and physics when the number of independent variables in the problem under consideration is two or more. Whenever we consider the case of two independent variables, x and y will usually be taken as the independent variables and z as the dependent variable. The partial differential coefficients $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ will be denoted by p and q respectively. The second order partial derivatives are denoted by $r, s, t,$ so that

$$\frac{\partial^2 z}{\partial x^2} = r, \ \frac{\partial^2 z}{\partial x \, \partial y} = s, \ \frac{\partial^2 z}{\partial y^2} = t.$$

Some examples of partial differential equations are :

$$pz - qz = z^2 + (x + y)^2, \qquad \ldots(1)$$

$$p \tan x + q \tan y = \tan z, \qquad \ldots(2)$$

$$p^2 + q^2 - 2px - 2qy + 2xy = 0, \qquad \ldots(3)$$

$$p^2 - q y^2 = y^2 - x^2, \qquad \ldots(4)$$

$$r - 2s + t = \sin(2x + 3y), \qquad \ldots(5)$$

$$y^2 \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \, \partial y} + x^2 \frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x} \frac{\partial z}{\partial x} + \frac{x^2}{y} \frac{\partial z}{\partial y}. \qquad \ldots(6)$$

Order and Degree of a Partial Differential equation.

As in the case of ordinary differential equations, *we define the* **order** *of a partial differential equation to be the order of the highest order derivative occurring in the equation.*

The **degree** *of a partial differential equation is the degree of the highest order derivative which occurs in it after the differential equation has been rationalised (i.e., made free from radicals and fractions so far as derivatives are concerned).*

In the above mentioned examples equations (1) and (2) are of first order and first degree.

Equations (3) and (4) are of first order and second degree. Equations (5) and (6) are of second order and first degree.

In the present chapter we shall discuss the partial differential equations of the first order.

2 Formation of a Partial Differential Equation

Partial differential equations can be derived in two ways :

(a) By the elimination of arbitrary constants from a relation between x, y and z

(b) By the elimination of arbitrary functions of these variables.

Now we illustrate these methods.

(a) By the elimination of arbitrary constants:

Let z be a function of x and y such that

$$f(x, y, z, a, b) = 0, \qquad \ldots(1)$$

where a and b denote arbitrary constants.

Differentiating (1) partially with respect to x and y, we get the relations

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \;\; i.e., \;\; \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p = 0 \qquad \ldots(2)$$

and

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0 \;\; i.e., \;\; \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q = 0. \qquad \ldots(3)$$

By means of the three equations (1), (2) and (3) two constants a and b can be eliminated and we obtain a relation of the form

$$F(x, y, z, p, q) = 0. \qquad \ldots(4)$$

This shows that the system of surfaces (1) gives rise to a partial differential equation of the first order given by (4).

Note: We observe that if the number of constants to be eliminated is equal to the number of independent variables then the derived partial differential equation is of the first order. But if the number of constants to be eliminated is greater than the number of independent variables then, in general, the derived partial differential equations will be of the second or higher order.

(b) By the elimination of arbitrary functions:

Suppose we have a relation between x, y and z of the type

$$f\ (u, v) = 0, \qquad \ldots(1)$$

where u and v are known functions of x, y and z and f is an arbitrary function of u and v.

This relation can also be expressed in the form $u = \phi\ (v)$, where ϕ is arbitrary.

Differentiating (1) partially with respect to each of the independent variables x and y regarding z as dependent variable, we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\,p\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\,p\right)=0 \qquad \ldots(2)$$

and
$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\,q\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\,q\right)=0. \qquad \ldots(3)$$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ between (2) and (3), we get

$$\left(\frac{\partial u}{\partial x}+p\,\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y}+q\,\frac{\partial v}{\partial z}\right)=\left(\frac{\partial u}{\partial y}+q\,\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x}+p\,\frac{\partial v}{\partial z}\right)$$

or
$$\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial z}\right)p+\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

or
$$Pp + Qq = R \qquad \ldots(4)$$

where
$$P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial z}=\frac{\partial\,(u, v)}{\partial\,(y, z)},$$

$$Q=\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}=\frac{\partial\,(u, v)}{\partial\,(z, x)},$$

and
$$R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=\frac{\partial\,(u, v)}{\partial\,(x, y)}.$$

The equation (4) is a partial differential equation of the first order.

Note 1: If the given relation between x, y, z contains two arbitrary functions then the derived partial differential equation will contain partial derivatives of an order higher than two, except in particular cases.

Note 2: It should be observed that the partial differential equation (4) derived in (**b**) is a linear equation *i.e.*, powers of p and q are both unity while the partial differential equation (4) derived in (**a**) need not be linear.

Illustrative Examples

Example 1: *Form a partial differential equation by the elimination of the constants h and k from*

$$(x-h)^2+(y-k)^2+z^2=c^2. \qquad ...(1)$$

Solution: Differentiating (1) partially w.r.t. x and y, we get

$$x-h+zp=0 \quad \text{and} \quad y-k+zq=0.$$

Putting the values of $x-h$ and $y-k$ from the last two equations in the given equation (1), we get

$$z^2(p^2+q^2+1)=c^2,$$

which is the required partial differential equation.

Example 2: *Form a partial differential equation by eliminating a, b, c from*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \qquad ...(1)$$

Solution: Differentiating (1) partially w.r.t. x and y, we get

$$\frac{x}{a^2}+\frac{z}{c^2}p=0 \qquad ...(2)$$

and

$$\frac{y}{b^2}+\frac{z}{c^2}q=0. \qquad ...(3)$$

Since the relations (1), (2) and (3) are not sufficient to eliminate the constants a, b and c so we require one more relation.

Differentiating (2) partially w.r.t. x, we get

$$\frac{1}{a^2}+\frac{p^2}{c^2}+\frac{z}{c^2}r=0. \qquad ...(4)$$

Multiplying (4) by x and then subtracting (2) from it, we get

$$\frac{1}{c^2}\{xzr+xp^2-pz\}=0$$

or

$$pz=xp^2+xzr.$$

Thus after the elimination of a, b and c we obtain a partial differential equation of order 2.

Note: In this case one more partial differential equation can also be obtained.

Differentiating (3) partially w.r.t. y, we get

$$\frac{1}{b^2}+\frac{q^2}{c^2}+\frac{z}{c^2}t=0.$$

Multiplying it by y and then subtracting (3) from it, we get

$$qz=yq^2+yzt,$$

which is also a partial differential equation of order 2.

Example 3: *Form a partial differential equation by eliminating the arbitrary function ϕ from*

$$z = e^{ny} \phi (x - y). \qquad \text{...(1)}$$

Solution: Differentiating (1) partially w.r.t. x and y, we get

$$p = e^{ny} \phi' (x - y), \qquad \text{...(2)}$$

and $$q = n e^{ny} \phi (x - y) - e^{ny} \phi' (x - y). \qquad \text{...(3)}$$

From (1), (2) and (3), we get

$$q = nz - p \quad \text{or} \quad p + q = nz,$$

which is the required partial differential equation of order one.

Example 4: *Form a partial differential equation by eliminating the functions f and F from*

$$z = f (x + iy) + F (x - iy). \qquad \text{...(1)}$$

Solution: Differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = f' (x + iy) + F' (x - iy) \qquad \text{...(2)}$$

and $$\frac{\partial z}{\partial y} = if' (x + iy) - iF' (x - iy) \qquad \text{...(3)}$$

Differentiating (2) and (3) partially w.r.t. x and y respectively, we get

$$\frac{\partial^2 z}{\partial x^2} = f'' (x + iy) + F'' (x - iy),$$

and $$\frac{\partial^2 z}{\partial y^2} = - f'' (x + iy) - F'' (x - iy).$$

Hence $$\frac{\partial^2 z}{\partial y^2} = - \frac{\partial^2 z}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0,$$

which is the required partial differential equation of the second order.

Comprehensive Exercise 1

Form partial differential equations by eliminating arbitrary constants from the following relations :

1. $z = (x + a)(y + b)$.
2. $z = ax + a^2 y^2 + b$.
3. $z = ax e^{y} + \frac{1}{2} a^2 e^{2y} + b$.
4. $z = A e^{pt} \sin px$.

Form partial differential equations by eliminating the arbitrary functions from the following equations :

5. $z = f(y/x)$.

6. $z = f(x + ay) + F(x - ay)$.

7. $z = y^2 + 2f\left(\dfrac{1}{x} + \log y\right)$.

8. $f(x + y + z, x^2 + y^2 - z^2) = 0$.

Answers 1

1. $z = pq$
2. $q = 2yp^2$
3. $q = px + p^2$
4. $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial t^2} = 0$
5. $px + qy = 0$
6. $\dfrac{\partial^2 z}{\partial y^2} = a^2 \dfrac{\partial^2 z}{\partial x^2}$
7. $x^2 \dfrac{\partial z}{\partial x} + y \dfrac{\partial z}{\partial y} = 2y^2$
8. $(y + z)p - (z + x)q = x - y$

3 Linear Partial Differential Equation

Definition: *A partial differential equation is said to be* **linear** *if the dependent variable z and all its partial differential coefficients occur in it in first degree.*

A partial differential equation is said to be **non-linear** if some or all the partial differential coefficients appearing in it do not occur in first degree.

Every linear partial differential equation is necessarily of first degree but a partial differential equation of first degree may or may not be linear. For example, the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} + 4\left(\frac{\partial z}{\partial x}\right)^2 + 5\frac{\partial z}{\partial y} + 6z = 9$$

is of first degree but it is not linear.

Below we give some examples of linear partial differential equations.

(i) The partial differential equation

$$(x^2 + y^2)\frac{\partial z}{\partial x} + 3xy\frac{\partial z}{\partial y} + (x + y)z = e^{x + y}$$

is a linear partial differential equation of first order.

(ii) The partial differential equation

$$x\frac{\partial^2 z}{\partial x^2} + (3x + 4y)\frac{\partial^2 z}{\partial x\,\partial y} + e^x\frac{\partial^2 z}{\partial y^2} + 5x\frac{\partial z}{\partial x} + 6y\frac{\partial z}{\partial y} + 9z = x^2 + y^2$$

is a linear partial differential equation of second order.

(iii) The partial differential equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

is a linear partial differential equation of second order.

(iv) The partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

is a linear partial differential equation of second order.

(v) The partial differential equation

$$x^4 \frac{\partial^3 z}{\partial x^3} + y^2 \frac{\partial^2 z}{\partial x\, \partial y} + x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} + 6z = 9\,(x^2 + y^2)$$

is a linear partial differential equation of third order.

(vi) The partial differential equation

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 3x + 4y$$

is a linear partial differential equation of first order.

(vii) The partial differential equation

$$\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0$$

is a linear partial differential equation of order 4.

(viii) The partial differential equation

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$$

is a first-order linear partial differential equation in three variables.

Here are some examples of partial differential equations which are not linear.

(i) The partial differential equation $p^2 x + q^2 y = z$ is not linear.

(ii) The partial differential equation $(x^2 - y^2)\, pq - x\, y\, (p + q) - 1 = 0$ is not linear.

(iii) The partial differential equation $\frac{\partial^2 z}{\partial x^2} + 4 \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} + 5 \frac{\partial^2 z}{\partial y^2} = 0$ is not linear.

However the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} + 4 \frac{\partial^2 z}{\partial x\, \partial y} + 5 \frac{\partial^2 z}{\partial y^2} = 0 \text{ is linear.}$$

4 Classification of Partial Differential Equations of First Order into Linear, Semi-Linear, Quasi-Linear and Non-Linear

A first order partial differential equation in two variables in its most general form is given by

$$F\,(x, y, z, p, q) = 0, \qquad \ldots(1)$$

where $$p = \frac{\partial z}{\partial x}$$

and $$q = \frac{\partial z}{\partial y}.$$

In this differential equation z is dependent variable and x, y are independent variables. When the function F is not a linear expression in p and q, the equation (1) is said to be *non-linear.*

(a) Non-linear partial differential equation of first order:

A first order partial differential equation in which dependent variable is z and is a function of two independent variables x and y is called a *non-linear equation* if the partial derivatives $\frac{\partial z}{\partial x}$ *i.e.*, p and $\frac{\partial z}{\partial y}$ *i.e.*, q do not occur in it in first degree.

For example, the partial differential equations of the first order such as

$$(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1, \; x^2 y^3 p^2 q = z^3,$$

$$(x^2+y^2)(p^2+q^2) = 1, \; z(p^2-q^2) = x-y,$$

$z^2(p^2+q^2) = x^2+y^2$ etc., are all non-linear equations.

(b) Quasi-linear partial differential equation of first order:

A first order partial differential equation $F(x, y, z, p, q) = 0$ in which dependent variable is z and is a function of two independent variables x and y is called a *quasi-linear equation* if the function F is a linear expression in p and q but not necessarily linear in z.

A quasi-linear partial differential equation of first order is of the form

$$f(x, y, z)\frac{\partial z}{\partial x} + g(x, y, z)\frac{\partial z}{\partial y} = h(x, y, z)$$

where the functions f and g depend on z also.

Thus the first order partial differential equations such as

$$(x+y+z)\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} + xz = 3x^2 + 5y^2 + 6z^2,$$

$$(x^2+y^2)\frac{\partial z}{\partial x} + 4xyz\frac{\partial z}{\partial y} = 3z + e^{x+y},$$

$$2z\frac{\partial z}{\partial x} + 5y\frac{\partial z}{\partial y} = 6z^2 + \log x + e^{y} \text{ etc.,}$$

are all quasi-linear equations.

(c) Semi-linear partial differential equation of first order:

A first order partial differential equation $F(x, y, z, p, q) = 0$ in which dependent variable is z and is a function of two independent variables x and y is called a *semi-linear equation* if it is of the form

$$f(x, y)\frac{\partial z}{\partial x} + g(x, y)\frac{\partial z}{\partial y} = h(x, y, z).$$

In a semi-linear partial differential equation of first order the coefficients of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are functions of x and y only and they do not depend on z. The terms that do not involve $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ contain some terms that are not of first degree in z.

Thus the first order partial differential equations such as

$$(x + y^2) \frac{\partial z}{\partial x} + x \log y \frac{\partial z}{\partial y} = 2z^2 x + xy + e^x,$$

$$p \cos (x + y) + q \sin (x + y) = z^3 + \sin x + e^y,$$

$$(y^3 x - 2x^4) \frac{\partial z}{\partial x} + (2y^4 - x^3 y) \frac{\partial z}{\partial y} = 9 (x^2 - y^2) \log z, \text{etc.},$$

are all semi-linear equations.

(d) Linear partial differential equation of first order:

A first order partial differential equation $F(x, y, z, p, q) = 0$ in which dependent variable is z and is a function of two independent variables x and y is called a *linear equation* if in this differential equation the dependent variable z and its partial differential coefficients $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ all occur in first degree.

Thus a linear partial differential equation of first order is of the form

$$f(x, y) \frac{\partial z}{\partial x} + g(x, y) \frac{\partial z}{\partial y} + h(x, y) z = c(x, y)$$

where $f(x, y)$, $g(x, y)$, $h(x, y)$ and $c(x, y)$ are functions of x and y only and they do not contain any term of z.

For example, the first order partial differential equations such as

$$(x^2 + y^3) \frac{\partial z}{\partial x} + (3x + 5y^2) \frac{\partial z}{\partial y} + (x - y) z = \sin (x + y),$$

$$p \cos (x + y) + q \sin (x + y) = z + e^y \sin x,$$

$$\log x \frac{\partial z}{\partial x} + e^y \frac{\partial z}{\partial y} = x(y - z),$$

$$p + 3q = 5z + \tan (y - 3x),$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 5u + \frac{xy}{z}, \quad x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^3 \frac{\partial u}{\partial z} = u + xyz,$$

$$(y^3 x - 2x^4) p + (2y^4 - x^3 y) q = 9z (x^3 - y^3), \text{etc.},$$

are all linear equations.

Remark: In many standard text books on partial differential equations the concepts of quasi-linear and semi-linear differential equations of first order are not introduced. There we find only two categories of first order partial differential equations — one linear and the other non-linear. According to this classification a partial differential equation is said to be linear if all the partial derivatives occurring in it appear only in first degree and there is no restriction on the dependent variable, it may or may not occur in first degree. Also, a differential equation is said to be non-linear if it is not linear. Thus a first order partial differential equation $F(x, y, z, p, q) = 0$ is linear if in this differential equation the partial derivatives p and q occur only in first degree and there is no restriction on the dependent variable z, it may occur in any form. Accordingly every first order partial differential equation of the form

$$f(x, y, z)\frac{\partial z}{\partial x} + g(x, y, z)\frac{\partial z}{\partial y} = h(x, y, z)$$

is linear where f, g, h are any functions of x, y, z. Thus the partial differential equations such as

$$(x^2 - yz)\, p + (y^2 - zx)\, q = z^2 - xy,$$

$$(mz - ny)\, p + (nx - lz)\, q = ly - mx,$$

$$(y^2 + z^2 - x^2)\, p - 2\, xyq + 2zx = 0,$$

$$x(y - z)\, p + y(z - x)\, q = z(x - y), \text{ etc., are all linear.}$$

On the other hand the partial differential equations such as

$$x^2p^2 + y^2q^2 = z^2,\ p^2 + q^2 = 3pq,$$

$$pq = 5,\ (y - x)(qy - px) = (p - q)^2,$$

$$p^2 = z^2(1 - pq),\ p^3 + q^3 = 27z \text{ etc., are all non-linear.}$$

5 Lagrange's Linear Partial Differential Equation

The partial differential equation $Pp + Qq = R$, where P, Q and R are any functions of x, y, and z is called Lagrange's linear partial differential equation of first order.

Thus Lagrange's linear partial differential equation is of the form

$$f(x, y, z)\frac{\partial z}{\partial x} + g(x, y, z)\frac{\partial z}{\partial y} = h(x, y, z),$$

where f, g, h are any functions of x, y, z.

The partial differential equations such as

$$(y^2 + z^2 - x^2)\, p - 2\, xyq + 2zx = 0,$$

$$p\cos(x + y) + q\sin(x + y) = z,$$

$$x(y - z)\frac{\partial z}{\partial x} + y(z - x)\frac{\partial z}{\partial y} = z(x - y), \text{ etc.,}$$

are all Lagrange's linear partial differential equations of first order.

Comprehensive Exercise 2

Tell whether the following partial differential equations are linear or non-linear:

1. $x^2 \dfrac{\partial^2 u}{\partial x^2} + y^2 \dfrac{\partial^2 u}{\partial y^2} + z^2 \dfrac{\partial^2 u}{\partial z^2} = 3u.$

2. $\dfrac{\partial^3 z}{\partial x^3} + 3 \dfrac{\partial^2 z}{\partial x^2} + 4 \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial z}{\partial y} + 5z = 9e^x.$

3. $4 \dfrac{\partial^2 z}{\partial x^2} + x \dfrac{\partial^2 z}{\partial y^2} + 9 \dfrac{\partial^2 z}{\partial x \, \partial y} + 5 \dfrac{\partial z}{\partial x} + 6 \dfrac{\partial z}{\partial y} = \log x.$

4. $\dfrac{\partial^2 z}{\partial x^2} - 4 \dfrac{\partial^2 z}{\partial x \, \partial y} = \dfrac{4x}{y^2} - \dfrac{y}{x^2} \cdot$

5. $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \, \partial y} - 2 \left(\dfrac{\partial z}{\partial y} \right)^2 = (y - 1) e^x.$

Which of the following partial differential equations of first order are non-linear, quasi-linear, semi-linear or linear ?

6. $(x^2 + y^2) \dfrac{\partial z}{\partial x} + (x - y) \dfrac{\partial z}{\partial y} + xz = 3x + 4y.$

7. $zy^2 p - xyq = x(z - 2y).$

8. $(y^2 - x^2) \dfrac{\partial z}{\partial x} - xy \dfrac{\partial z}{\partial y} = x^2 + y^2 - xz^2.$

9. $(y^3 + x^3) \dfrac{\partial z}{\partial x} + 4x^2 y \dfrac{\partial z}{\partial y} = x^3 z - 4y^2 + yz.$

10. $4 \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial z}{\partial y} + 5 \dfrac{\partial z}{\partial x} + 6 \dfrac{\partial z}{\partial y} + 3z = 9(x^2 + y^2).$

11. $(x^2 - yz) p + (y^2 - zx) q = z^2 - xy.$

12. $(mx - ny) p + (nx + my) q = z + ly - mx^2.$

13. $x^2 \dfrac{\partial z}{\partial x} + (y^3 + x^3) \dfrac{\partial z}{\partial y} = x^2 + y^2 + z^2.$

14. $x \dfrac{\partial u}{\partial x} + y \left(\dfrac{\partial u}{\partial y} \right)^2 + 3 \dfrac{\partial u}{\partial z} = x^2 + y^2 + z^2.$

15. $(z^2 - 2yz - y^2) p + x(y + z) q = x(y - z).$

Answers 2

1. Linear	2. Non-linear	3. Linear
4. Linear	5. Non-linear	6. Linear
7. Quasi-linear	8. Semi-linear	9. Linear
10. Non-linear	11. Quasi-linear	12. Linear
13. Semi-linear	14. Non-linear	15. Quasi-linear

6 Equation Equivalent to the Linear Equation

A partial differential equation which is linear in p and q is of the type

$$Pp + Qq = R, \qquad \ldots(1)$$

where P, Q, R are functions of x, y, z.

Let any relation $u = a$ be an integral of (1). Differentiating it partially with respect to x and y, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q = 0.$$

These give $\quad p = -\dfrac{\partial u}{\partial x} \Big/ \dfrac{\partial u}{\partial z},\ q = -\dfrac{\partial u}{\partial y} \Big/ \dfrac{\partial u}{\partial z}.$

Putting these values of p and q in (1) it changes to

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0. \qquad \ldots(2)$$

Hence, if $u = a$ satisfies (1), $u = a$ also satisfies (2).

Conversely, dividing (2) by $\dfrac{\partial u}{\partial z}$ and substituting p and q for their values given above we see that if $u = a$ is an integral of (2), it is also an integral of (1).

Thus equation (2) *can be taken as equivalent to equation* (1).

7 Lagrange's Solution of the Linear Partial Differential Equation of First Order

The first systematic theory of the linear partial differential equations was given by **Lagrange.** For that reason the equation $Pp + Qq = R$ is referred to as **Lagrange's equation.** The method of solving a linear equation of this form is contained in the theorem given below :

Theorem: *The general solution of the linear partial differential equation*

$$Pp + Qq = R \qquad \ldots(1)$$

is

$$f(u, v) = 0, \qquad \ldots(2)$$

where f is an arbitrary function and

$$u(x, y, z) = c_1 \quad \text{and} \quad v(x, y, z) = c_2 \qquad \ldots(3)$$

form a solution of the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \qquad \ldots(4)$$

Proof: We shall give a purely analytic proof of this theorem. If the relation $u(x, y, z) = c_1$ satisfies the equations (4) then the equations

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \quad \text{and} \quad \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

must be compatible *i.e.*, we should have

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0. \qquad \ldots(5)$$

Similarly we should have

$$P\frac{\partial v}{\partial x} + Q\frac{\partial v}{\partial y} + R\frac{\partial v}{\partial z} = 0. \qquad \ldots(6)$$

Solving the equations (5) and (6), we get

$$\frac{P}{\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z} - \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}} = \frac{Q}{\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}} = \frac{R}{\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}}. \qquad \ldots(7)$$

Earlier we have seen that the relation $f(u, v) = 0$ gives the partial differential equation

$$\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p + \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}. \qquad \ldots(8)$$

Substituting from the equations (7) into the equation (8), we find that (2) is a solution of the equation (1) if u and v are given by the equations (3).

Note: In place of $f(u, v) = 0$ the functional relation can be written as $u = \phi(v)$, where ϕ denotes an arbitrary function.

It is called **Lagrange's solution** of the linear equation (1) and the equations (4) are called **Lagrange's auxiliary equations** or **Lagrange's subsidiary equations.**

8 The Linear Equation Containing more than two Independent Variables

The generalisation of Lagrange's method is as follows :

Let the linear equation with n independent variables $x_1, x_2, \ldots, x_n$ be

$$P_1 p_1 + P_2 p_2 + P_3 p_3 + \ldots + P_n p_n = R. \qquad \ldots(1)$$

where $P_1, P_2, \ldots, P_n$ and R are functions of $x_1, x_2, \ldots, x_n$ and z. Here p_i denotes $\dfrac{\partial z}{\partial x_i}$ for $i = 1, 2, \ldots, n$.

Then the general solution of (1) is given by

$$f\ (u_1, u_2, \dots, u_n) = 0$$

where $\quad u_i\ (x_1, x_2, \dots, x_n, z) = c_i,\ i = 1, 2, \dots, n$

are any independent integrals of the auxiliary equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \dots = \frac{dx_n}{P_n} = \frac{dz}{R}.$$

9 Geometrical Interpretation of Lagrange's Linear Equation

Lagrange's linear equation is

$$Pp + Qq = R \text{ or } Pp + Qq + (-1)\ R = 0. \qquad \dots(1)$$

Since d.r.'s of the normal at the point (x, y, z) on the surface $f\ (x, y, z) = c$ are given by

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \quad \text{or} \quad -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial z}, -\frac{\partial f}{\partial y} \Big/ \frac{\partial f}{\partial z}, -1$$

i.e.,
$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1 \text{ or } p, q, -1,$$

hence the equation (1) shows that the normal to a certain surface is perpendicular to the line whose d.r.'s are P, Q, R.

But we know that the simultaneous equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \dots(2)$$

represent a family of curves in space such that the direction ratios of the tangent to any member of this family at any point (x, y, z) are P, Q, R. If $u = a$ and $v = b$ are two independent integrals of (2) then $f\ (u, v) = 0$ represents a surface through such curves. Through every point of such a surface passes a curve of the family, lying wholly on the surface.

Thus the general solution of (1) is the family of surfaces such that the normal to a surface at any point must be perpendicular to the tangent to a curve of the family represented by (2).

10 The Equation $Pp + Qq = R$ Represents a Family of Surfaces Orthogonal to the Family of Surfaces Represented by $P\,dx + Q\,dy + R\,dz = 0$ (If Integrable)

We know that d.r.'s of the normal at the point (x, y, z) to a surface of the family represented by $Pp + Qq = R$ are $p, q, -1$.

Also d.r.'s of the normal at the point (x, y, z) to a surface of the family represented by $P\,dx + Q\,dy + R\,dz = 0$ are P, Q, R.

Since the equation $Pp + Qq + (-1)\,R = 0$ shows that the two lines whose d.r.'s are $p, q, -1$ and P, Q, R are perpendicular, hence the surfaces represented by $Pp + Qq = R$ are orthogonal to the surfaces represented by $P\,dx + Q\,dy + R\,dz = 0$.

11 Integral Surfaces Passing Through a Given Curve

Now we shall indicate how the general solution of a linear partial differential equation may be used to find the integral surface passing through a given curve. Suppose that we have obtained two solutions

$$u = a \text{ and } v = b \quad \ldots(1)$$

of the auxiliary equations (4) of article 7. Then we know that the general solution of the corresponding linear equation is of the form

$$f\,(u, v) = 0 \quad \ldots(2)$$

arising from a relation

$$f\,(a, b) = 0 \quad \ldots(3)$$

between the constants a and b. We have to consider the problem of determining the function f in special cases.

If we want to find the integral surface passing through the curve whose parametric equations are $x = x\,(t),\ y = y\,(t),\ z = z\,(t)$,

t being a parameter, then the solutions in (1) must be such that

$$u\,(x\,(t), y\,(t), z\,(t)) = a,\ v\,(x\,(t), y\,(t), z\,(t)) = b.$$

Eliminating the single variable t from these two equations we find a relation of the type (3). The required solution is then given by the equation (2).

12 Working Rule to Find the Solution of Lagrange's Equation $Pp + Qq = R$

Form the auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

Find two independent integrals of these auxiliary equations, say $u = a$ and $v = b$. Then the general solution of the partial differential equation $Pp + Qq = R$ is given by $f\,(u, v) = 0$, where f is an arbitrary function or it can also be written as $u = \phi\,(v)$, where ϕ is an arbitrary function.

Illustrative Examples

Example 5: *Solve* $(z^2 - 2yz - y^2)\, p + (xy + zx)\, q = xy - zx.$ ...(1)

Solution: The Lagrange's auxiliary equations of the given differential equation (1) are

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{xy + zx} = \frac{dz}{xy - zx}.$$

Taking the last two members, we get

$$(y - z)\, dy = (y + z)\, dz \quad \text{or} \quad y\, dy - (z\, dy + y\, dz) - z\, dz = 0.$$

Integrating, $y^2 - 2yz - z^2 = c_1$.

Again choosing x, y, z as multipliers, we get

$$\text{each fraction} = \frac{x\, dx + y\, dy + z\, dz}{0} \quad \Rightarrow \quad x\, dx + y\, dy + z\, dz = 0.$$

Integrating, $x^2 + y^2 + z^2 = c_2$.

Hence the general solution of (1) is given by

$$f\,(y^2 - 2yz - z^2, x^2 + y^2 + z^2) = 0$$

or

$$y^2 - 2yz - z^2 = \phi\,(x^2 + y^2 + z^2).$$

Example 6: *Solve* $(mz - ny)\, p + (nx - lz)\, q = ly - mx.$ ...(1)

(Garhwal 2001, 14)

Solution: The Lagrange's auxiliary equations of the given differential equation (1) are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}.$$

Choosing x, y, z as multipliers, we get

$$\text{each fraction} = \frac{x\, dx + y\, dy + z\, dz}{0} \quad \Rightarrow xdx + ydy + zdz = 0.$$

Integrating, $x^2 + y^2 + z^2 = c_1$.

Again choosing l, m, n as multipliers, we get

$$\text{each fraction} = \frac{l\, dx + m\, dy + n\, dz}{0} \quad \Rightarrow \quad l\, dx + m\, dy + n\, dz = 0.$$

Integrating, $lx + my + nz = c_2$.

Hence the general solution of (1) is given by

$$f\,(x^2 + y^2 + z^2, lx + my + nz) = 0,$$

where f is an arbitrary function.

Example 7: *Solve* $(y^2 + z^2 - x^2)\, p - 2xyq + 2zx = 0.$ ...(1)

(Garhwal 2004; Kumaun 11)

Solution: The Lagrange's auxiliary equations of (1) are

$$\frac{dx}{y^2 + z^2 - x^2} = \frac{dy}{-2xy} = \frac{dz}{-2zx}.$$

Taking the last two members, we get

$$\frac{dy}{y} = \frac{dz}{z}.$$

Integrating, $\log \dfrac{y}{z} = \log c_1$ or $\dfrac{y}{z} = c_1$.

Again choosing x, y, z as multipliers, we get

$$\text{each fraction} = \frac{x\,dx + y\,dy + z\,dz}{-x\,(x^2 + y^2 + z^2)}.$$

$$\therefore \quad \frac{x\,dx + y\,dy + z\,dz}{-x\,(x^2 + y^2 + z^2)} = \frac{dz}{-2zx} \quad \Rightarrow \quad 2\,\frac{x\,dx + y\,dy + z\,dz}{x^2 + y^2 + z^2} = \frac{dz}{z}.$$

Integrating, $\log (x^2 + y^2 + z^2) = \log z + \log c_2$ or $x^2 + y^2 + z^2 = zc_2$.

Hence the general solution of (1) is

$$f\left(\frac{y}{z}, \frac{x^2 + y^2 + z^2}{z}\right) = 0, \quad \text{or} \quad x^2 + y^2 + z^2 = z\phi\left(\frac{y}{z}\right),$$

where f and ϕ are arbitrary functions.

Example 8: *Solve* $(y + z)\,p + (z + x)\,q = x + y.$ ...(1)

(Garhwal 2001, 12)

Solution: The Lagrange's auxiliary equations of (1) are

$$\frac{dx}{y + z} = \frac{dy}{z + x} = \frac{dz}{x + y}.$$

$$\therefore \quad \frac{dx - dy}{y - x} = \frac{dy - dz}{z - y} = \frac{dx + dy + dz}{2\,(x + y + z)}.$$

Taking the first two members, we get

$$\log (y - x) = \log (z - y) + \log c_1$$

or $\quad y - x = c_1 (z - y).$

Again taking the first and the last members, we get

$$-2 \log (x - y) = \log (x + y + z) - \log c_2$$

or $\quad (x - y)^2 (x + y + z) = c_2.$

Hence the general solution of (1) is given by

$$f\left[\frac{y - x}{z - y}, (x - y)^2 (x + y + z)\right] = 0.$$

Example 9: *Solve* $\dfrac{(y - z)}{yz}\,p + \dfrac{(z - x)}{zx}\,q = \dfrac{x - y}{xy}.$ ...(1)

(Kumaun 2008)

Solution: The auxiliary equations of (1) are

$$\frac{dx}{x\,(y - z)} = \frac{dy}{y\,(z - x)} = \frac{dz}{z\,(x - y)}.$$

Each fraction $= \dfrac{dx + dy + dz}{0}$ and also $= \dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$.

$\therefore \qquad dx + dy + dz = 0$ and $\dfrac{1}{x}dx + \dfrac{1}{y}dy + \dfrac{1}{z}dz = 0.$

Integrating, $x + y + z = c_1$ and $xyz = c_2$.

Hence the general solution of (1) is given by

$$f(x + y + z,\ xyz) = 0.$$

Example 10: *Solve* $p + 3q = 5z + \tan(y - 3x).$ ...(1)

(Garhwal 2012)

Solution: The auxiliary equations of (1) are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)}.$$

Taking the first two members, we get

$$dy - 3dx = 0.$$

$\therefore \qquad y - 3x = c_1.$

Again taking the first and the last members, we get

$$dx = \frac{dz}{5z + \tan c_1}.$$

$\therefore \qquad \log(5z + \tan c_1) = 5x + \log c_2$

or $\qquad e^{-5x}\{5z + \tan(y - 3x)\} = c_2.$

Hence the general solution of (1) is given by

$$e^{-5x}\{5z + \tan(y - 3x)\} = f(y - 3x).$$

Example 11: *Solve* $x^2(y - z)p + y^2(z - x)q = z^2(x - y).$ ...(1)

Solution: The auxiliary equations of (1) are

$$\frac{dx}{x^2(y - x)} = \frac{dy}{y^2(z - x)} = \frac{dz}{z^2(x - y)}.$$

Each fraction $= \dfrac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{0}$ and also $= \dfrac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$.

$\therefore \qquad \dfrac{dx}{x^2} + \dfrac{dy}{y^2} + \dfrac{dz}{z^2} = 0$ and $\dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{dz}{z} = 0.$

Integrating, $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = c_1$ and $x\,y\,z = c_2$.

Hence the general solution of (1) is given by

$$f\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z},\ x\,y\,z\right) = 0.$$

Example 12: *Solve* $x(y^2+z)\,p - y(x^2+z)\,q = z(x^2-y^2)$. ...(1)

Solution: The auxiliary equations of (1) are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}.$$

Choosing $1/x, 1/y, 1/z$ as multipliers, we get

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0.$$

Integrating, $\log x + \log y + \log z = \log c_1$

or $\quad xyz = c_1$.

Again choosing $x, y, -1$ as multipliers, we get

$$x\,dx + y\,dy - dz = 0.$$

Integrating, $x^2 + y^2 - 2z = c_2$.

Hence the general solution of (1) is given by $\quad f(xyz, x^2 + y^2 - 2z) = 0$.

Example 13: *Find the equation of the integral surface of the differential equation*

$$2y(z-3)\,p + (2x-z)\,q = y(2x-3), \quad ...(1)$$

which passes through the circle $z = 0, x^2 + y^2 = 2x$.

Solution: The auxiliary equations of (1) are

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)}.$$

Taking the first and the third members, we get

$$(2x-3)\,dx = 2(z-3)\,dz.$$

Integrating, $x^2 - 3x - z^2 + 6z = a$. ...(2)

Again, using $\frac{1}{2}, y, -1$ as multipliers, we get

$$\frac{1}{2}\,dx + y\,dy - dz = 0.$$

Integrating, $\frac{1}{2}x + \frac{1}{2}y^2 - z = \frac{1}{2}b$ or $x + y^2 - 2z = b$. ...(3)

The parametric equations of the circle are

$$x = t,\ y = \sqrt{(2t - t^2)},\ z = 0.$$

Putting these values in the equations (2) and (3), we get

$$t^2 - 3t = a \quad \text{and} \quad t + (2t - t^2) = b.$$

Eliminating t from these, we find the relation

$$a + b = 0,$$

showing that the required integral surface is

$$(x^2 - 3x - z^2 + 6z) + (x + y^2 - 2z) = 0$$

or

$$x^2 + y^2 - z^2 - 2x + 4z = 0.$$

Comprehensive Exercise 3

Solve the following equations :

1. $xzp + yzq = xy$. **(Kumaun 2012; Garhwal 13)**
2. (i) $p \tan x + q \tan y = \tan z$. **(Garhwal 2000)**

   (ii) $p \cos (x + y) + q \sin (x + y) = z$ **(Kumaun 2008; Garhwal 10)**
3. $x^2 p + y^2 q = z^2$.
4. $yzp + zxq = xy$.
5. $\dfrac{y^2 z}{x} p + zxq = y^2$.
6. $(x^2 - yz) p + (y^2 - zx) q = z^2 - xy$.
7. $z - px - qy = a \sqrt{(x^2 + y^2 + z^2)}$.
8. $z (xp - yq) = y^2 - x^2$
9. $(3x + y - z) p + (x + y - z) q = 2 (z - y)$.

   $\left[\textbf{Hint.} \quad \dfrac{dx}{3x + y - z} = \dfrac{dy}{x + y - z} = \dfrac{dz}{2 (z - y)} \right.$.

   (i) Choose $-1, 3, 1$ as multipliers.

   (ii) $\left. \dfrac{dx - dy + dz}{\ldots} = \dfrac{dx + dy - dz}{\ldots} \right]$
10. $p_2 + p_3 = 1 + p_1$.
11. $x \dfrac{\partial u}{\partial x} + y \dfrac{\partial u}{\partial y} + z \dfrac{\partial u}{\partial z} = xyz$.
12. Find the integral surface of the linear partial differential equation
$$x (y^2 + z) p - y (x^2 + z) q = (x^2 - y^2) z$$
which contains the line $x + y = 0, z = 1$.
13. Find the general integral of the partial differential equation
$$(2xy - 1) p + (z - 2x^2) q = 2 (x - yz)$$
and also the particular integral which passes through the line $x = 1, y = 0$.
14. Find the family of surfaces orthogonal to the family of surfaces given by the differential equation
$$(y + z) p + (z + x) q = x + y.$$
15. Find the integral surface of the partial differential equation
$$x^2 p + y^2 q + z^2 = 0$$
which passes through the hyperbola $xy = x + y,\ z = 1$.

Answers 3

1. $f\left(z^2 - xy, \dfrac{y}{x}\right) = 0$

2. (i) $\dfrac{\sin y}{\sin z} = f\left(\dfrac{\sin x}{\sin y}\right)$

(ii) $[\cos(x+y) + \sin(x+y)]\, e^{y-x} = f\left\{z^{\sqrt{2}} \cot\left(\dfrac{x}{2} + \dfrac{y}{2} + \dfrac{\pi}{8}\right)\right\}.$

3. $\dfrac{1}{x} - \dfrac{1}{y} = f\left(\dfrac{1}{x} - \dfrac{1}{z}\right)$

4. $f(x^2 - y^2, x^2 - z^2) = 0$

5. $f(x^3 - y^3, x^2 - z^2) = 0$

6. $f\left(\dfrac{x-y}{y-z}, \dfrac{y-z}{z-x}\right) = 0$

7. $y^{1-a} = \{z + \sqrt{(x^2 + y^2 + z^2)}\}\, f\left(\dfrac{x}{y}\right)$

8. $f(xy, x^2 + y^2 + z^2) = 0$

9. $(x - y + z)^2 = (x + y - z)\, f(x - 3y - z)$

10. $f(x_1 + z, x_1 + x_2, x_1 + x_3) = 0$

11. $f\left(\dfrac{x}{y}, \dfrac{y}{z}, xyz - 3u\right) = 0$

12. $x^2 + y^2 + 2xyz - 2z + 2 = 0$

13. $x^2 + y^2 - xz - y + z - 1 = 0$

14. $xy + yz + zx = c$

15. $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{2}{z} = 3$

13 The Integrals of the Non-Linear Equation

The complete and particular integrals: We have seen that the relation of the type

$$f(x, y, z, a, b) = 0 \qquad \ldots(1)$$

gives rise to a partial differential equation of the first order of the form

$$F(x, y, z, p, q) = 0 \qquad \ldots(2)$$

on the elimination of arbitrary constants a and b. Here x, y are independent variables and z is dependent variable. The relation (1) is a solution of (2). *Any such relation which contains as many arbitrary constants as there are independent variables and is a solution of a partial differential equation of the first order is called* **a complete solution or a complete integral** *of that equation.*

A particular integral *of* (2) *can be obtained by giving particular values to a and b in* (1).

The singular integral: If the envelope of the doubly infinite system of surfaces represented by (1) exists, it is also a solution of the equation (2). The reason is that the envelope of all the surfaces represented by (1) is touched at each of its points by some one of these surfaces. Hence the coordinates of any point on the envelope of surfaces (1) with the corresponding values of p and q, being identical with the x, y, z, p, q of some point on one of these surfaces, must satisfy (2). The equation of the envelope of the surfaces represented by (1) can be obtained by eliminating a and b between the three equations

$$f = 0, \frac{\partial f}{\partial a} = 0 \text{ and } \frac{\partial f}{\partial b} = 0.$$

This equation of the envelope is called the **singular integral** *of the differential equation* (2).

It differs from a particular integral in the sense that it is not contained in the complete integral, *i.e.*, it cannot be obtained from the complete integral by giving particular values to the constants.

The general integral: If in the equation (1), one of the constants is a function of the other, say $b = \phi(a)$, then this equation becomes

$$f(x, y, z, a, \phi(a)) = 0. \qquad \ldots(3)$$

It is a one-parameter subfamily of the family (1). The equation of the envelope of the family of surfaces represented by (3) is also a solution of the equation (2).

It is called the general integral of (2) *corresponding to the complete integral* (1).

The equation of the envelope of the surfaces represented by (3) is obtained by eliminating a between

$$f(x, y, z, a, \phi(a)) = 0 \text{ and } \frac{\partial f}{\partial a} = 0.$$

14 General Method of Solution of a Non-Linear Partial Differential Equation of Order One with two Independent Variables. (Charpit's Method)

Let the given partial differential equation be

$$f(x, y, z, p, q) = 0. \qquad \ldots(1)$$

Since z depends upon x and y, it gives that

$$dz = p\,dx + q\,dy. \qquad \ldots(2)$$

The fundamental idea in Charpit's method is to introduce a second partial differential equation of the first order $F(x, y, z, p, q, a) = 0$, ...(3)

containing an arbitrary constant a and which is such that :

(i) The equations (1) and (3) can be solved to find

$$p = p(x, y, z, a), q = q(x, y, z, a).$$

(ii) Substituting these values of p and q in (2), the equation

$$dz = p(x, y, z, a)\,dx + q(x, y, z, a)\,dy \qquad \ldots(4)$$

is integrable.

If such a relation (3) has been found, the solution of equation (4),

$$\phi(x, y, z, a, b) = 0 \qquad \ldots(5)$$

containing two arbitrary constants a and b will be a solution of the equation (1). Also it is a complete integral of the equation (1).

Now the main problem is to devise a method of finding the relation (3). Let us assume that (3) is the relation which when taken along with (1) gives those values of p and q which make (2) integrable. Differentiating (1) and (3) with respect to x, we get

$$\left.\begin{aligned} &\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}=0 \\ \text{and}\quad &\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}p+\frac{\partial F}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}=0 \end{aligned}\right\} \qquad \ldots(6)$$

Again, differentiating (1) and (3) w.r.t. y, we get

$$\left.\begin{aligned} &\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q+\frac{\partial f}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial y}=0 \\ \text{and}\quad &\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}q+\frac{\partial F}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}=0 \end{aligned}\right\} \qquad \ldots(7)$$

Eliminating $\dfrac{\partial p}{\partial x}$ from the equations in (6) and $\dfrac{\partial q}{\partial y}$ from the equations in (7),

we get

$$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial F}{\partial x}\right)+p\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial F}{\partial z}\right)+\frac{\partial q}{\partial x}\left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial F}{\partial q}\right)=0, \qquad \ldots(8)$$

and $$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q}-\frac{\partial f}{\partial q}\frac{\partial F}{\partial y}\right)+q\left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q}-\frac{\partial f}{\partial q}\frac{\partial F}{\partial z}\right)+\frac{\partial p}{\partial y}\left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q}-\frac{\partial f}{\partial q}\frac{\partial F}{\partial p}\right)=0. \qquad \ldots(9)$$

Since $\dfrac{\partial q}{\partial x}=\dfrac{\partial^2 z}{\partial x\,\partial y}=\dfrac{\partial p}{\partial y}$, hence adding (8) and (9) and re-arranging, we get

$$\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial q}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z}$$
$$+\left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y}=0. \qquad \ldots(10)$$

This is a linear partial differential equation of the first order with x, y, z, p, q as independent variables and F as dependent variable. Lagrange's auxiliary equations of (10) are

$$\frac{dp}{\dfrac{\partial f}{\partial x}+p\dfrac{\partial f}{\partial z}}=\frac{dq}{\dfrac{\partial f}{\partial y}+q\dfrac{\partial f}{\partial z}}=\frac{dz}{-p\dfrac{\partial f}{\partial p}-q\dfrac{\partial f}{\partial q}}=\frac{dx}{-\dfrac{\partial f}{\partial p}}=\frac{dy}{-\dfrac{\partial f}{\partial q}}=\frac{dF}{0}\cdot \qquad \ldots(11)$$

These equations are known as **Charpit's auxiliary equations.**

Any of the integrals of (11) will satisfy (10). If such an integral contains p or q, it can be taken as the required second relation. It should be noted that not all of Charpit's equations (11) need be used, but that p or q must occur in the solution obtained. Of course, the simpler the integral containing p or q, or both p and q that is derived from (11), the easier will be the subsequent labour in finding the solution of (1).

Illustrative Examples

Example 14: *Find a complete integral of the equation*

$$2zx - px^2 - 2qxy + pq = 0.$$ **(Garhwal 2000, 14; Kumaun 10)**

Solution: The given differential equation is

$$f(x, y, z, p, q) \equiv 2zx - px^2 - 2qxy + pq = 0. \quad \ldots(1)$$

Charpit's auxiliary equations are

$$\frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dz}{px^2 - pq + 2qxy - pq} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p}$$

from which it follows that $dq = 0$ or $q = a$ (a constant). ...(2)

Solving the equations (1) and (2) for p and q, we get

$$p = \frac{2x(z - ay)}{x^2 - a}, q = a.$$

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{2x(z - ay)}{x^2 - a}\,dx + a\,dy$$

or $$\frac{dz - a\,dy}{z - ay} = \frac{2x}{x^2 - a}\,dx.$$

Integrating, $$\log(z - ay) = \log(x^2 - a) + \log b$$

or $$z - ay = b(x^2 - a) \quad \text{or} \quad z = ay + b(x^2 - a),$$

which is a complete integral of (1).

Example 15: *Find a complete integral of $px + qy = pq$.* **(Kumaun 2008; 10)**

Solution: Here $f(x, y, z, p, q) \equiv px + qy - pq = 0.$ ...(1)

Charpit's auxiliary equations are

$$\frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-p(x - q) - q(y - p)} = \frac{dx}{-(x - q)} = \frac{dy}{-(y - p)}$$

from which it follows that

$$\frac{dp}{p} = \frac{dq}{q}, \text{ or } p = aq. \quad \ldots(2)$$

Solving the equations (1) and (2) for p and q, we get

$$q = \frac{y + ax}{a}, p = y + ax.$$

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = (y + ax)\,dx + \frac{(y + ax)}{a}\,dy$$

or $$a\,dz = (y + ax)(dy + a\,dx).$$

Integrating, $az = \frac{1}{2}(y + ax)^2 + b$, which is a complete integral of (1).

Example 16: *Find a complete integral of* $(p^2+q^2)\,y=qz$.

(Garhwal 2000, 08; Kumaun 11, 13)

Solution: The given differential equation is

$$f \equiv (p^2+q^2)\,y - qz = 0. \qquad \ldots(1)$$

Charpit's auxiliary equations are

$$\frac{dp}{-pq}=\frac{dq}{(p^2+q^2)-q^2}=\frac{dz}{-2p^2y-2q^2y+qz}=\frac{dx}{-2py}=\frac{dy}{-2qy+z}$$

from which it follows that

$$\frac{dp}{-pq}=\frac{dq}{p^2} \quad \text{or} \quad p\,dp+q\,dq=0$$

or $$p^2+q^2=a^2 \text{ (say)} \qquad \ldots(2)$$

Solving the equations (1) and (2) for p and q, we get

$$q=\frac{a^2y}{z},\ p=\frac{q}{z}\sqrt{(z^2-a^2y^2)}.$$

Putting these values of p and q in $dz=p\,dx+q\,dy$, we get

$$dz=\frac{a}{z}\sqrt{(z^2-a^2y^2)}\,dx+\frac{a^2y}{z}\,dy$$

or $$\frac{z\,dz-a^2y\,dy}{\sqrt{(z^2-a^2y^2)}}=a\,dx.$$

Integrating, $\sqrt{(z^2-a^2y^2)}=ax+b$

or $z^2=a^2y^2+(ax+b)^2$, which is a complete integral of (1).

Example 17: *Find a complete integral of* $p=(qy+z)^2$.

Solution: The given differential equation is

$$f \equiv p-(qy+z)^2=0. \qquad \ldots(1)$$

Charpit's auxiliary equations are

$$\frac{dp}{2p(qy+z)}=\frac{dq}{4q(qy+z)}=\frac{dy}{-2y(qy+z)}=\ldots\ldots$$

from which it follows that $\dfrac{dp}{p}=-\dfrac{dy}{y}\cdot$

Integrating, $py=a$. $\qquad \ldots(2)$

Putting $p=a/y$ in (1), we get

$$(qy+z)^2=\frac{a}{y}\cdot$$

$\therefore$ $$q=\frac{1}{y}\left\{\sqrt{\left(\frac{a}{y}\right)}-z\right\}.$$

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{a}{y}\,dx + \frac{1}{y}\left\{\sqrt{\left(\frac{a}{y}\right)} - z\right\} dy$$

or $$y\,dz + z\,dy = a\,dx + \sqrt{\left(\frac{a}{y}\right)}\,dy.$$

Integrating, $yz = ax + 2\sqrt{(ay)} + b$,

which is a complete integral of (1).

Example 18: *Find a complete integral of* $px + qy = z\,(1 + pq)^{1/2}$. **(Garhwal 2011)**

Solution: Here $f \equiv px + qy - z\,(1 + pq)^{1/2} = 0$. ...(1)

Charpit's auxiliary equations are

$$\frac{dp}{p - p\,(1 + pq)^{1/2}} = \frac{dq}{q - q\,(1 + pq)^{1/2}} = \ldots\ldots$$

from which it follows that

$$\frac{dp}{p} = \frac{dq}{q} \;\Rightarrow\; p = aq. \qquad \ldots(2)$$

Putting $p = aq$ in (1), we get

$$q\,(ax + y) = z\,(1 + aq^2)^{1/2}$$

or $$q^2\,[(ax + y)^2 - az^2\,] = z^2.$$

$\therefore$ $$q = \frac{z}{[(ax + y)^2 - az^2\,]^{1/2}} \quad \text{and} \quad p = aq = \frac{az}{[(ax + y)^2 - az^2\,]^{1/2}}.$$

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{z\,(a\,dx + dy)}{\sqrt{\{(ax + y)^2 - az^2\}}}$$

or $$\frac{dz}{z} = \frac{a\,dx + dy}{\sqrt{\{(ax + y)^2 - az^2\}}}$$

Let $ax + y = \sqrt{a}\,u$ so that $a\,dx + dy = \sqrt{a}\,du$.

$\therefore$ $$\frac{dz}{z} = \frac{\sqrt{a}\,du}{\sqrt{(au^2 - az^2)}}$$

or $$\frac{du}{dz} = \frac{\sqrt{(u^2 - z^2)}}{z}$$

To solve this, putting $u = vz$, we get

$$v + z\,\frac{dv}{dz} = \frac{1}{z}\sqrt{(v^2 z^2 - z^2)}$$

or $$z\,\frac{dv}{dz} = \sqrt{(v^2 - 1)} - v$$

or $$\frac{dz}{z} = \frac{dv}{\sqrt{(v^2 - 1)} - v}$$

or $$\frac{dz}{z} = -\,\{\sqrt{(v^2 - 1)} + v\}\,dv.$$

Integrating,

$$\log z = -\left[\frac{v}{2}\sqrt{(v^2-1)} - \frac{1}{2}\log\{v+\sqrt{(v^2-1)}\}\right] - \frac{v^2}{2} + b$$

or
$$\log z + \frac{v^2}{2} + \frac{v}{2}\sqrt{(v^2-1)} - \frac{1}{2}\log\{v+\sqrt{(v^2-1)}\} = b,$$

which is a complete integral of (1), where $v = \dfrac{u}{z} = \dfrac{ax+y}{z\sqrt{a}}$.

Example 19: *Find a complete integral of* $p^2 x + q^2 y = z$.

Solution: The given differential equation is

$$f \equiv p^2 x + q^2 y - z = 0. \qquad \text{...(1)}$$

Charpit's auxiliary equations are

$$\frac{dp}{-p+p^2} = \frac{dq}{-q+q^2} = \frac{dx}{-2px} = \frac{dy}{-2qy} = \ldots$$

from which it follows that

$$\frac{p^2\,dx + 2px\,dp}{p^2 x} = \frac{q^2\,dy + 2qy\,dq}{q^2 y}.$$

Integrating, $\log p^2 x = \log q^2 y + \log a$.

$\therefore$
$$p^2 x = aq^2 y. \qquad \text{...(2)}$$

Solving (1) and (2) simultaneously for p and q, we get

$$q = \left\{\frac{z}{(1+a)\,y}\right\}^{1/2} \text{ and } p = \left\{\frac{az}{(1+a)\,x}\right\}^{1/2}.$$

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \left\{\frac{az}{(1+a)\,x}\right\}^{1/2} dx + \left\{\frac{z}{(1+a)\,y}\right\}^{1/2} dy$$

or
$$\sqrt{(1+a)}\,\frac{dz}{\sqrt{z}} = \sqrt{a}\,\frac{dx}{\sqrt{x}} + \frac{dy}{\sqrt{y}}.$$

Integrating, $\sqrt{\{(1+a)\,z\}} = \sqrt{(ax)} + \sqrt{y} + b$,
which is a complete integral of (1).

Comprehensive Exercise 4

Apply Charpit's method to find the complete integrals of the following equations:

1. $z^2(p^2 z^2 + q^2) = 1$.
2. $(p+q)(px+qy) = 1$. **(Garhwal 2009)**
3. $p\,xy + pq + qy = yz$.

4. $z = px + qy + p^2 + q^2$. **(Garhwal 2010)**

5. $z = pq$. **(Garhwal 2009; Kumaun 15)**

6. $\sqrt{p} + \sqrt{q} = 2x$.

7. $q = -px + p^2$.

8. $q = 3p^2$. **(Garhwal 2006)**

9. $yz\,p^2 - q = 0$.

10. $zpq = p + q$.

11. $2(z + px + qy) = yp^2$.

12. $z^2 = pqxy$. **(Kumaun 2014)**

13. $p^2 + q^2 - 2px - 2qy + 2xy = 0$. **(Garhwal 2010B,15)**

14. $p^2 + q^2 - 2px - 2qy + 1 = 0$. **(Garhwal 2012)**

Answers 4

1. $(a^2z^2 + 1)^3 = 9a^4 (ax + y + b)^2$

2. $\sqrt{(1+a)}\, z = 2\sqrt{(ax + y)} + b$

3. $(z - ax)(y + a)^a = be^y$

4. $z = ax + by + a^2 + b^2$

5. $2\sqrt{z} = \sqrt{a}\,x + (1/\sqrt{a})\, y + b$

6. $z = \dfrac{1}{6}(2x - a)^3 + a^2 y + b$

7. $z = \dfrac{x^2}{4} \pm \dfrac{1}{2}\left[\dfrac{x}{2}\sqrt{(x^2 + 4a)} + 2a \log\{x + \sqrt{(x^2 + 4a)}\}\right] + ay + b$

8. $z = ax + 3a^2 y + b$

9. $z^2 (a - y^2) = (x + b)^2$

10. $z^2 = 2(a + 1)\{x + (y/a)\} + b$

11. $z = \dfrac{ax}{y^2} - \dfrac{a^2}{4y^3} + \dfrac{b}{y}$

12. $z = bx^a y^{1/a}$

13. $2z = x^2 + ax + y^2 + ay + \dfrac{1}{\sqrt{2}}\left[\dfrac{u}{2}\sqrt{(u^2 - a^2)} - \dfrac{a^2}{2}\log\{u + \sqrt{(u^2 - a^2)}\}\right] + b,$

where $u = \sqrt{2}\,(x - y)$.

14. $(a^2 + 1)z = \dfrac{t^2}{2} \pm \left\{\dfrac{t}{2}\sqrt{\{t^2 - (a^2 + 1)\}} - \dfrac{a^2 + 1}{2}\log\left[t + \sqrt{\{t^2 - (a^2 + 1)\}}\right]\right\} + b,$

where $t = ax + y$.

15 Application of Charpit's Method to Standard Forms

There are a few standard forms to which many first order differential equations are reducible and which can be integrated by methods which are sometimes shorter than the general method.

Standard I: Equations involving only p and q and no x, y, z

Let the equation be written as

$$f\,(p, q) = 0. \qquad \ldots(1)$$

The complete integral is given by

$$z = ax + by + c, \qquad \ldots(2)$$

where a and b are connected by

$$f\,(a, b) = 0. \qquad \ldots(3)$$

Obviously, from (2) we have

$$p = \frac{\partial z}{\partial x} = a$$

and $$q = \frac{\partial z}{\partial y} = b.$$

Substituting $p = a$ and $q = b$ in (3), we get (1).

From (3) we may find b in terms of a

i.e., $b = \phi\,(a)$ say.

The complete integral of (1) is

$$z = ax + \phi\,(a).\,y + c. \qquad \ldots(4)$$

General Integral: Putting $c = \psi(a)$ in (4), where ψ denotes an arbitrary function, we get $$z = ax + \phi\,(a)\,y + \psi(a). \qquad \ldots(5)$$

Differentiating (5) partially w.r.t. a, we get

$$0 = x + \phi'\,(a)\,y + \psi\,'\,(a). \qquad \ldots(6)$$

The *general integral* is obtained by eliminating a between (5) and (6).

Singular Integral: The singular integral, if it exists, is obtained by eliminating a and c between the complete integral (4) and the equations formed by differentiating (4) partially w.r.t. a and c *i.e.*, between the equations

$$z = ax + \phi\,(a)\,y + c,$$

$$0 = x + \phi'\,(a)\,.\,y$$

and $$0 = 1.$$

Since $1 = 0$ is inconsistent, therefore, in this case there is no singular integral.

Note: In many cases, using some transformations, equations can be reduced to the form of the standard I.

Illustrative Examples

Example 20: *Solve* $p^2 + q^2 = m^2$. ...(1)

Solution: The given equation is of the form $f(p, q) = 0$.

Therefore a complete integral is given by

$$z = ax + by + c \quad \text{where} \quad a^2 + b^2 = m^2$$

i.e.,
$$z = ax + \sqrt{(m^2 - a^2)}\, y + c. \qquad ...(2)$$

To find the general integral put $c = \phi(a)$ in (2).

Then
$$z = ax + \sqrt{(m^2 - a^2)}\, y + \phi(a). \qquad ...(3)$$

Differentiating (3) partially w.r.t. a, we get

$$0 = x - \frac{a}{\sqrt{(m^2 - a^2)}}\, y + \phi'(a). \qquad ...(4)$$

Eliminating a from (3) and (4) the general integral is obtained.

Example 21: *Find a complete integral of*

$$(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1. \qquad ...(1)$$

Solution: Put $x + y = X^2$, $x - y = Y^2$

so that
$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial x} = \frac{1}{2X} \cdot \frac{\partial z}{\partial X} + \frac{1}{2Y} \cdot \frac{\partial z}{\partial Y}$$

and
$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y} = \frac{1}{2X} \frac{\partial z}{\partial X} - \frac{1}{2Y} \frac{\partial z}{\partial Y}.$$

These give $p + q = \dfrac{1}{X} \dfrac{\partial z}{\partial X}$, $p - q = \dfrac{1}{Y} \dfrac{\partial z}{\partial Y}$.

Substituting these values in the equation (1), we get

$$\left(\frac{\partial z}{\partial X}\right)^2 + \left(\frac{\partial z}{\partial Y}\right)^2 = 1, \qquad ...(2)$$

which is of the form of standard I.

Hence, a complete integral of (2) is given by

$$z = aX + bY + c, \quad \text{where} \quad a^2 + b^2 = 1.$$

$\therefore$ a complete integral of (1) is given by

$$z = a\sqrt{(x + y)} + \sqrt{(1 - a^2)}\, \sqrt{(x - y)} + c,$$

where a and c are arbitrary constants.

Example 22: *Solve* $x^2 p^2 + y^2 q^2 = z^2$.

Solution: The given equation can be written as

$$\left(\frac{x}{z} \frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z} \frac{\partial z}{\partial y}\right)^2 = 1. \qquad ...(1)$$

Put $\dfrac{dx}{x} = dX,\ \dfrac{dy}{y} = dY,\ \dfrac{dz}{z} = dZ$

i.e., $X = \log x,\ Y = \log y,\ Z = \log z$

so that $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial X} \cdot \dfrac{\partial X}{\partial x} + \dfrac{\partial z}{\partial Y} \cdot \dfrac{\partial Y}{\partial x} = \dfrac{\partial z}{\partial X} \cdot \dfrac{1}{x} \cdot$

$\therefore$ $\dfrac{\partial Z}{\partial X} = \dfrac{\partial Z}{\partial z} \cdot \dfrac{\partial z}{\partial X} = \dfrac{1}{z} \cdot p \,.\, x = \dfrac{x}{z} \cdot \dfrac{\partial z}{\partial x} \cdot$

Similarly $\dfrac{\partial Z}{\partial Y} = \dfrac{y}{z} \dfrac{\partial z}{\partial y} \cdot$

Substituting these values in (1), we get

$$\left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1, \qquad \ldots(2)$$

which is of the form of standard I *i.e.*, of the form

$$f\left(\frac{\partial Z}{\partial X}, \frac{\partial Z}{\partial Y}\right) = 0.$$

Hence, a complete integral of (2) is given by

$$Z = aX + bY + c_1, \text{ where } a^2 + b^2 = 1.$$

$\therefore$ A complete integral of (1) is given by

$$\log z = a \log x + \sqrt{(1 - a^2)} \log y + c_1.$$

If we take $a = \cos\alpha, c_1 = \log c$ then complete integral can be written as

$$\log z = \cos\alpha \log x + \sin\alpha \log y + \log c$$

or $z = c\, x^{\cos\alpha} .\, y^{\sin\alpha}$, where α and c are arbitrary constants.

General integral is obtained by eliminating α from

$$z = \phi(\alpha)\, x^{\cos\alpha} .\, y^{\sin\alpha}, \text{ where } c = \phi(\alpha)$$

and $0 = \phi'(\alpha)\, x^{\cos\alpha} .\, y^{\sin\alpha} + \phi(\alpha)\, \{x^{\cos\alpha} .\log x .(-\sin\alpha) .\, y^{\sin\alpha}$
$+ x^{\cos\alpha} .\, y^{\sin\alpha} .\log y .\cos\alpha\}$

Singular integral is obtained by eliminating α and c between the equations

$$z = c\, x^{\cos\alpha} .\, y^{\sin\alpha},$$

$$0 = c\, \{x^{\cos\alpha} .\log x .(-\sin\alpha) .\, y^{\sin\alpha} + x^{\cos\alpha} .\, y^{\sin\alpha} .\log y .\cos\alpha\}$$

and $0 = x^{\cos\alpha} .\, y^{\sin\alpha}$.

Hence the singular integral is $z = 0$.

Example 23: *Find a complete integral of* $(y - x)(qy - px) = (p - q)^2$.

Solution: Put $x + y = X,\ xy = Y$, so that

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial x} = \frac{\partial z}{\partial X} \cdot 1 + \frac{\partial z}{\partial Y} \cdot y$$

and $q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial X} \cdot \dfrac{\partial X}{\partial y} + \dfrac{\partial z}{\partial Y} \cdot \dfrac{\partial Y}{\partial y} = \dfrac{\partial z}{\partial X} \cdot 1 + \dfrac{\partial z}{\partial Y} \cdot x.$

Substituting these values of p and q in the given equation, we get

$$(y-x)\left[\left(\frac{\partial z}{\partial X}+x\frac{\partial z}{\partial Y}\right)y-\left(\frac{\partial z}{\partial X}+y\frac{\partial z}{\partial Y}\right)x\right]=(y-x)^2\left(\frac{\partial z}{\partial Y}\right)^2$$

or $$(y-x)^2\frac{\partial z}{\partial X}=(y-x)^2\left(\frac{\partial z}{\partial Y}\right)^2$$

or $$\frac{\partial z}{\partial X}=\left(\frac{\partial z}{\partial Y}\right)^2,$$ which is of the form of standard I.

Hence, a complete integral is given by

$$z=aX+bY+c, \text{ where } a=b^2.$$

$\therefore$ A complete integral of the given equation is

$$z=a(x+y)+\sqrt{a}\,(xy)+c,$$

where a and c are arbitrary constants.

Example 24: *Find a complete integral of $pq=x^m y^n z^{2l}$.*

Solution: *The given equation can be written as*

$$\frac{pz^{-l}}{x^m}\cdot\frac{qz^{-l}}{y^n}=1.$$

Put $$X=\frac{x^{m+1}}{m+1},\ Y=\frac{y^{n+1}}{n+1},\ Z=\frac{z^{1-l}}{1-l},$$

so that $$\frac{\partial Z}{\partial X}=\frac{\partial Z}{\partial x}\cdot\frac{dx}{dX}=z^{-l}\cdot p\frac{1}{x^m},\ \frac{\partial Z}{\partial Y}=\frac{\partial Z}{\partial y}\cdot\frac{dy}{dY}=z^{-l}q\frac{1}{y^n}.$$

Putting these values in (1), we get

$$\frac{\partial Z}{\partial X}\cdot\frac{\partial Z}{\partial Y}=1,$$ which is of the form of standard I.

Hence a complete integral is given by

$$Z=aX+bY+c, \text{ where } ab=1.$$

$\therefore$ a complete integral of the given equation is

$$\frac{z^{1-l}}{1-l}=a\frac{x^{m+1}}{m+1}+\frac{y^{n+1}}{a(n+1)}+c,$$

where a and c are arbitrary constants.

Standard II: Equations involving only p, q and z *i.e.*, equations of the form

$$f(z,p,q)=0. \quad \ldots(1)$$

Let us assume $z=f(x+ay)$ as a trial solution of given equation (1), where a is an arbitrary constant.

$\therefore$ $$z=f(X) \text{ where } X=x+ay.$$

$\therefore$ $$p=\frac{\partial z}{\partial x}=\frac{dz}{dX}\frac{\partial X}{\partial x}=\frac{dz}{dX}$$

and $$q=\frac{\partial z}{\partial y}=\frac{dz}{dX}\frac{\partial X}{\partial y}=a\frac{dz}{dX}.$$

$\therefore$ Equation (1) reduces to the form $f\left(z, \frac{dz}{dX}, a\frac{dz}{dX}\right) = 0$

which is an ordinary differential equation of order one. Integrating it we may get the complete integral.

The general and the singular integrals are to be found in the usual way.

Rule : The method of solving the equations of the Standard II can be formulated in the following rule :

Put $\frac{dz}{dX}$ for $p, a\frac{dz}{dX}$ for q, where $X = x + ay$.

Now solve the resulting ordinary differential equation in the variables z and X. Then substitute x + ay for X.

This gives a complete solution.

Note: Sometimes using transformations equations reduce to the form of standard II.

Illustrative Examples

Example 25: *Find a complete integrals of the following*

(i) $9(p^2z + q^2) = 4$. *(ii)* $9(p^2z + q^2) = 1$. **(Kumaun 2013)**

Solution: **(i)** The given equation is of the form $f(z, p, q) = 0$.

Putting $\frac{dz}{dX}$ for $p, a\frac{dz}{dX}$ for q, the given equation becomes

$$9\left[z\left(\frac{dz}{dx}\right)^2 + a^2\left(\frac{dz}{dX}\right)^2\right] = 4, \text{where } X = x + ay$$

or $$9(z + a^2)\left(\frac{dz}{dX}\right)^2 = 4, \quad \text{or} \quad \frac{dz}{dX} = \frac{2}{3\sqrt{(z + a^2)}}$$

or $$\frac{3}{2}\sqrt{(z + a^2)}\, dz = dX.$$

Integrating, $$(z + a^2)^{3/2} = X + b$$

or $$(z + a^2)^{3/2} = x + ay + b$$

or $$(z + a^2)^3 = (x + ay + b)^2,$$

which is a complete integral of the given equation.

(ii) Proceed as in part (i). Ans. $4(z + a^2)^3 = (x + ay + b)^2$.

Example 26: *Find a complete integral of* $z^2(p^2z^2 + q^2) = 1$.

Solution: Putting $\frac{dz}{dX}$ for $p, a\frac{dz}{dX}$ for q, the given equation becomes

$$z^2\left[\left(\frac{dz}{dX}\right)^2 . z^2 + a^2\left(\frac{dz}{dX}\right)^2\right] = 1, \text{where } X = x + ay$$

or $$z^2(z^2 + a^2)\left(\frac{dz}{dX}\right)^2 = 1$$

or $$z\sqrt{(z^2+a^2)}\,dz = dX.$$

Integrating, $$\frac{1}{3}(z^2+a^2)^{3/2} = X + b$$

or $$9\,(x+ay+b)^2 = (z^2+a^2)^3,$$

which is a complete integral of the given equation.

Example 27: *Find a complete integral of* $pz = 1 + q^2$.

Solution: The given equation is of the form $f\,(p, q, z) = 0$.

Putting $\dfrac{dz}{dX}$ for p, $a\,\dfrac{dz}{dX}$ for q, the given equation becomes

$$z\,\frac{dz}{dX} = 1 + a^2\left(\frac{dz}{dX}\right)^2, \text{ where } X = x + ay$$

or $$a^2\left(\frac{dz}{dX}\right)^2 - z\,\frac{dz}{dX} + 1 = 0.$$

$\therefore$ $$\frac{dz}{dX} = \frac{z \pm \sqrt{(z^2-4a^2)}}{2a^2}$$

or $$\frac{dz}{z \pm \sqrt{(z^2-4a^2)}} = \frac{dX}{2a^2} \quad \text{or} \quad \frac{z \mp \sqrt{(z^2-4a^2)}}{4a^2}\,dz = \frac{dX}{2a^2}$$

or $$\{z \mp \sqrt{(z^2-4a^2)}\}\,dz = 2\,dX.$$

Integrating, we get

$$\frac{z^2}{2} \mp \left[\frac{z}{2}\sqrt{(z^2-4a^2)} - \frac{1}{2}\cdot 4a^2 \log\{z + \sqrt{(z^2-4a^2)}\}\right] = 2X + b$$

or $$z^2 \mp [z\sqrt{(z^2-4a^2)} - 4a^2 \log\{z+\sqrt{(z^2-4a^2)}\}] = 4\,(x+ay) + b,$$

which is a complete integral of the given equation.

Example 28: *Find a complete integral of* $p^2 = z^2\,(1 - pq)$.

Solution: The given equation is of the form $f\,(p, q, z) = 0$.

Putting $\dfrac{dz}{dX}$ for p, $a\,\dfrac{dz}{dX}$ for q, the given equation becomes

$$\left(\frac{dz}{dX}\right)^2 = z^2\left\{1 - \frac{dz}{dX}\,a\,\frac{dz}{dX}\right\}, \text{ where } X = x + ay$$

or $$\left(\frac{dz}{dX}\right)^2 (1 + az^2) = z^2$$

or $$\frac{\sqrt{(1+az^2)}}{z}\,dz = dX$$

or $$\frac{1+az^2}{z\sqrt{(1+az^2)}}\,dz = dX$$

or $$\left\{\frac{1}{z\sqrt{(1+az^2)}} + \frac{az}{\sqrt{(1+az^2)}}\right\}dz = dX.$$

Integrating, $$\frac{1}{\sqrt{a}} \log [z \sqrt{a} + \sqrt{(1 + az^2)}] + \sqrt{(1 + az^2)} = X + b$$

or $$\frac{1}{\sqrt{a}} \log [z \sqrt{a} + \sqrt{(1 + az^2)}] + \sqrt{(1 + az^2)} = x + ay + b,$$

which is a complete integral of the given equation.

Standard III: Equation of the form $f(x, p) = F(y, q)$.

As a trial solution let us put each side equal to an arbitrary constant

i.e., $$f(x, p) = F(y, q) = a \qquad \ldots(1)$$

from which we obtain

$$p = f_1(x, a) \quad \text{and} \quad q = f_2(y, a).$$

Now from $dz = p\,dx + q\,dy$, we have $dz = f_1(x, a)\,dx + f_2(y, a)\,dy$.

Integrating , we get $z = \int f_1(x, a)\,dx + \int f_2(y, a)\,dy + b$

which is the complete integral.

The general integral can be obtained in the usual way. As in the case of standard I, there is no singular integral.

Rule: Write the differential equation in the form (1). Put both the sides of the equation equal to an arbitrary constant. Solving them find the values of p and q. Substitute the values of p and q in $dz = p\,dx + q\,dy$ and integrate to find a complete integral.

Note: Sometimes using transformations equations reduce to the form of standard III.

Illustrative Examples

Example 29: *Find a complete integral of* $p^2 + q^2 = x + y$. **(Kumaun 2012)**

Solution: Separating q and y from p and x, the given equation can be written as

$$p^2 - x = y - q^2 = a, \text{ (say).}$$

$\therefore$ $p = \sqrt{(x + a)}$ and $q = \sqrt{(y - a)}$.

Putting the values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \sqrt{(x + a)}\,dx + \sqrt{(y - a)}\,dy.$$

Integrating, $z = \frac{2}{3}(x + a)^{3/2} + \frac{2}{3}(y - a)^{3/2} + b,$

which is a complete integral of the given equation.

Example 30: *Find a complete integral of* $yp = 2yx + \log q$.

Solution: The given equation can be written as

$$p = 2x + \frac{1}{y} \log q$$

or $$p - 2x = \frac{1}{y} \log q = a, \text{ (say).}$$

$\therefore \qquad p = 2x + a, \log q = ay \;\; i.e., \;\; q = e^{ay}.$

Putting the values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = (2x + a)\,dx + e^{ay}\,dy.$$

Integrating, $\qquad z = x^2 + ax + \frac{1}{a} e^{ay} + b,$

which is a complete integral of the given equation.

Example 31: *Find a complete integral of*

$$z^2 (p^2 + q^2) = x^2 + y^2.$$

Solution: Put $z\,dz = dZ$; *i.e.*, $Z = \frac{1}{2} z^2$, so that

$$z \frac{\partial z}{\partial x} = \frac{\partial Z}{\partial x} = P \text{ (say)}, \; z \frac{\partial z}{\partial y} = \frac{\partial Z}{\partial y} = Q \text{ (say)}.$$

Putting these values the given equation becomes

$$P^2 + Q^2 = x^2 + y^2 \text{ or } P^2 - x^2 = y^2 - Q^2 = a^2 \text{ (say)}.$$

$\therefore \qquad P = \sqrt{(a^2 + x^2)}, Q = \sqrt{(y^2 - a^2)}.$

Now $\qquad dZ = P\,dx + Q\,dy = \sqrt{(a^2 + x^2)}\,dx + \sqrt{(y^2 - a^2)}\,dy.$

Integrating ,

$$Z = \frac{1}{2} x \sqrt{(a^2 + x^2)} + \frac{a^2}{2} \log \{x + \sqrt{(a^2 + x^2)}\}$$
$$+ \frac{1}{2} y \sqrt{(y^2 - a^2)} - \frac{a^2}{2} \log \{y + \sqrt{(y^2 - a^2)}\} + b$$

or

$$z^2 = x \sqrt{(a^2 + x^2)} + a^2 \log \{x + \sqrt{(a^2 + x^2)}\} + y \sqrt{(y^2 - a^2)}$$
$$- a^2 \log \{y + \sqrt{(y^2 - a^2)}\} + b,$$

which is a complete integral of the given equation.

Example 32: *Find a complete integral of* $x^2 y^3 p^2 q = z^3$.

Solution: The given equation can be written as

$$x^2 y^3 \left(\frac{1}{z} \frac{\partial z}{\partial x}\right)^2 \left(\frac{1}{z} \frac{\partial z}{\partial y}\right) = 1. \qquad \ldots(1)$$

Put $\frac{1}{z} dz = dZ$ *i.e.*, $Z = \log z$. Then

$$\frac{1}{z} \frac{\partial z}{\partial x} = \frac{\partial Z}{\partial x} = P \text{ (say)}, \; \frac{1}{z} \frac{\partial z}{\partial y} = \frac{\partial Z}{\partial y} = Q \text{ (say)}.$$

The equation (1) reduces to $x^2 y^3 P^2 Q = 1$

or $\qquad x^2 P^2 = \frac{1}{Q y^3} = a^2$, (say).

$\therefore \qquad P = \frac{a}{x}, \; Q = \frac{1}{a^2 y^3}.$

Now $dZ = P\,dx + Q\,dy = \frac{a}{x}\,dx + \frac{1}{a^2 y^3}\,dy.$

Integrating, $\qquad Z = a \log x - \frac{1}{2a^2 y^2} + b$

or $\qquad \log z = a \log x - \frac{1}{2a^2 y^2} + b,$

which is a complete integral of the given equation.

Example 33: *Find a complete integral of* $z\,(p^2 - q^2) = x - y.$

Solution: The given equation can be written as

$$\left[\left(\sqrt{z}\,\frac{\partial z}{\partial x}\right)^2 - \left(\sqrt{z}\,\frac{\partial z}{\partial y}\right)^2\right] = x - y. \qquad \ldots(1)$$

Putting $\sqrt{z}\,dz = dZ$ *i.e.,* $Z = \frac{2}{3} z^{3/2}$,

so that $\qquad \frac{\partial Z}{\partial x} = \sqrt{z}\,\frac{\partial z}{\partial x} = P$ (say), $\frac{\partial Z}{\partial y} = \sqrt{z}\,\frac{\partial z}{\partial y} = Q$ (say),

the equation (1) reduces to

$$P^2 - Q^2 = x - y \text{ or } P^2 - x = Q^2 - y = a \text{ (say)}.$$

$\therefore \qquad P = \sqrt{(a + x)}, Q = \sqrt{(a + y)}.$

Now $\qquad dZ = P\,dx + Q\,dy = \sqrt{(a+x)}\,dx + \sqrt{(a+y)}\,dy.$

Integrating, $\qquad Z = \frac{2}{3}(a + x)^{3/2} + \frac{2}{3}(a + y)^{3/2} + b$

or $\qquad z^{3/2} = (a + x)^{3/2} + (a + y)^{3/2} + b,$

which is a complete integral of the given equation.

Standard IV: Equation of the form

$$\boldsymbol{z = px + qy + f\,(p, q)} \qquad \ldots(1)$$

(analogous to Clairaut's form)

We know that the solution of Clairaut's equation

$$y = px + f\,(p) \text{ where } p = \frac{dy}{dx} \text{ is } y = cx + f\,(c).$$

Similarly the complete integral of Clairaut's equation (1) is

$$z = ax + by + f\,(a, b).$$

Rule: To get the complete integral of the equation of this type replace p and q by a, b (two arbitrary constants) respectively.

General Integral is obtained as in other cases.

Singular Integral: The complete integral is

$$F = z - ax - by - f(a,b) = 0 \qquad \ldots(1)$$

$$\frac{\partial F}{\partial a} = 0, \text{ gives } x + \frac{\partial f}{\partial a} = 0, \qquad \ldots(2)$$

and $$\frac{\partial F}{\partial b} = 0, \text{ gives } y + \frac{\partial f}{\partial b} = 0. \qquad \ldots(3)$$

Singular integral is obtained by eliminating a, b from (1), (2) and (3).

Illustrative Examples

Example 34: *Find a complete integral of*

$$z = px + qy + p^2 + q^2.$$

Solution: The given equation is of the form of standard IV *i.e.*, of the form

$$z = px + qy + f(p, q).$$

Hence a complete integral is given by

$$z = ax + by + a^2 + b^2.$$

Example 35: *Find the singular integral of*

$$z = px + qy + \log pq.$$

Solution: The complete integral of the given equation is

$$z = ax + by + \log ab. \qquad \ldots(1)$$

Differentiating (1) partially w.r.t. a and b, we get

$$0 = x + \frac{1}{a}$$

and $$0 = y + \frac{1}{b}\cdot$$

$\therefore$ $$a = -1/x$$

and $$b = -1/y. \qquad \ldots(2)$$

Eliminating a and b between (1) and the equations (2), we get

$$z = x\left(-\frac{1}{x}\right) + y\left(-\frac{1}{y}\right) + \log\left(\frac{1}{xy}\right)$$

or $z = -2 - \log xy$, which is the required singular integral.

Example 36: *Find a complete integral and the singular integral of*

$$4xyz = pq + 2px^2y + 2qxy^2.$$

Solution: Put $x^2 = X,\ y^2 = Y$, so that

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{dX}{dx} = 2X^{1/2}\frac{\partial z}{\partial X},\ q = 2Y^{1/2}\frac{\partial z}{\partial Y}\cdot$$

The given equation then reduces to

$$z = X\frac{\partial z}{\partial X} + Y\frac{\partial z}{\partial Y} + \frac{\partial z}{\partial X}\cdot\frac{\partial z}{\partial Y}\cdot$$

$\therefore$ A complete integral is $z = aX + bY + ab$

or $$z = ax^2 + by^2 + ab. \quad \ldots(1)$$

Differentiating (1) partially w.r.t. a and b, we get

$$0 = x^2 + b \text{ and } 0 = y^2 + a. \quad \ldots(2)$$

Eliminating a and b between (1) and the equations (2), the singular integral is

$$z = -x^2 y^2 - x^2 y^2 + x^2 y^2 \text{ i.e., } z + x^2 y^2 = 0.$$

Example 37: *Find the singular integral of*

$$z = px + qy + c\sqrt{(1 + p^2 + q^2)}.$$

Solution: The complete integral of the given equation is

$$z = ax + by + c\sqrt{(1 + a^2 + b^2)}. \quad \ldots(1)$$

Singular integral. Differentiating (1) partially w.r.t. a and b, we get

$$0 = x + \frac{ac}{\sqrt{(1 + a^2 + b^2)}}, \quad 0 = y + \frac{bc}{\sqrt{(1 + a^2 + b^2)}}, \quad \ldots(2)$$

so that $$x^2 + y^2 = \frac{a^2 c^2 + b^2 c^2}{1 + a^2 + b^2}$$

i.e., $$c^2 - x^2 - y^2 = \frac{c^2}{1 + a^2 + b^2}$$

i.e., $$1 + a^2 + b^2 = \frac{c^2}{c^2 - x^2 - y^2}. \quad \ldots(3)$$

Also, from (2)

$$a = -\frac{x\sqrt{(1 + a^2 + b^2)}}{c} = \frac{-x}{\sqrt{(c^2 - x^2 - y^2)}} \quad \ldots(4)$$

and $$b = -\frac{y\sqrt{(1 + a^2 + b^2)}}{c} = \frac{-y}{\sqrt{(c^2 - x^2 - y^2)}} \quad \ldots(5)$$

Putting the values from (3), (4) and (5) in (1), we get the singular solution as

$$z = -\frac{x^2}{\sqrt{(c^2 - x^2 - y^2)}} - \frac{y^2}{\sqrt{(c^2 - x^2 - y^2)}} + \frac{c^2}{\sqrt{(c^2 - x^2 - y^2)}}$$

or $$z = \frac{c^2 - x^2 - y^2}{\sqrt{(c^2 - x^2 - y^2)}} \text{ i.e., } z^2 = c^2 - x^2 - y^2 \text{ i.e., } x^2 + y^2 + z^2 = c^2.$$

Comprehensive Exercise 5

Find complete integrals of the following equations :

1. $q = e^{-p/\alpha}$.
2. $p^2 - q^2 = \lambda$.

3. $\sqrt{p} + \sqrt{q} = 1.$

4. $p^2 = zq.$

5. $p^3 + q^3 - 3pqz = 0.$

6. $p(1+q) = qz.$

7. $\sqrt{p} + \sqrt{q} = 2x.$

8. $pq = xy.$

9. $p - 3x^2 = q^2 - y.$

10. $z = px + qy + pq.$

11. $z = px + qy - 2\sqrt{(pq)}.$

12. $z = px + qy - p^2 q.$

13. $p^2 + q^2 = z^2 (x + y).$

14. $z = px + qy + \dfrac{q}{p} - p.$

15. $z = px + qy + 3p^{1/3} q^{1/3}.$

Answers 5

1. $z = ax + ye^{-a/\alpha} + c$

2. $z = ax + (a^2 - \lambda)^{1/2} y + c$

3. $z = ax + (1 - \sqrt{a})^2 y + c$

4. $z = be^{a(x+ay)}$

5. $3a(x + ay) + b = (1 + a^3) \log z$

6. $az - 1 = be^{x+ay}$

7. $z = \dfrac{1}{6}(a + 2x)^3 + a^2 y + b$

8. $2az = a^2 x^2 + y^2 + 2ab$

9. $z = x^3 + ax + \dfrac{2}{3}(y + a)^{3/2} + b$

10. $z = ax + by + ab$

11. $z = ax + by - 2\sqrt{(ab)}$

12. $z = ax + by - a^2 b$

13. $\dfrac{3}{2} \log z = (a + x)^{3/2} + (y - a)^{3/2} + b$

14. $z = ax + by + \dfrac{b}{a} - a$

15. $z = ax + by + 3(ab)^{1/3}$

16 Compatible Systems of First Order Partial Differential Equations

Definition: *If every solution of the first order partial differential equation*

$$f(x, y, z, p, q) = 0 \qquad \ldots(1)$$

is also a solution of the first order partial differential equation

$$g(x, y, z, p, q) = 0, \qquad \ldots(2)$$

then the differential equations (1) *and* (2) *are said to be* **compatible.**

To find the condition for the pair of equations (1) and (2) to be compatible.

If $J = \dfrac{\partial(f, g)}{\partial(p, q)} \neq 0$, then the equations (1) and (2) can be solved to obtain the explicit expressions $p = \phi(x, y, z), q = \psi(x, y, z)$ ...(3)

for p and q.

The condition that the pair of equations (1) and (2) should be compatible reduces then to the condition that the system of equations (3) should be completely integrable.

If z is a function of x and y, then we know that

$$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy = p\, dx + q\, dy.$$

Therefore, the system of equations (3) will be completely integrable if and only if the single differential equation

$$dz = \phi\, dx + \psi\, dy \quad i.e., \quad \phi\, dx + \psi\, dy - dz = 0 \qquad \ldots(4)$$

is integrable.

We know that the necessary and sufficient condition for the single differential equation $P\, dx + Q\, dy + R\, dz = 0$ to be integrable is that

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0.$$

Therefore, the single differential equation (4) is integrable if and only if

$$\phi\,(\psi_z - 0) + \psi\,(0 - \phi_z) - 1\,(\phi_y - \psi_x) = 0 \qquad [\because \; P = \phi, Q = \psi, R = -1]$$

i.e., if and only if $\psi_x + \phi\psi_z = \phi_y + \psi\,\phi_z$. ...(5)

Substituting from equations (3) into equation (1) and differentiating with respect to x and z, respectively, we obtain the equations

$$f_x + f_p\,\phi_x + f_q\,\psi_x = 0 \qquad \ldots(6)$$

and

$$f_z + f_p\,\phi_z + f_q\,\psi_z = 0. \qquad \ldots(7)$$

Multiplying (7) by ϕ and adding to (6), we get

$$f_x + \phi\, f_z + f_p\,(\phi_x + \phi\,\phi_z) + f_q\,(\psi_x + \phi\,\psi_z) = 0. \qquad \ldots(8)$$

In a similar manner, equation (2) will give

$$g_x + \phi\, g_z + g_p\,(\phi_x + \phi\,\phi_z) + g_q\,(\psi_x + \phi\,\phi_z) = 0. \qquad \ldots(9)$$

Now solving the equations (8) and (9), we obtain

$$\psi_x + \phi\,\psi_z = \frac{1}{J}\left[\frac{\partial(f, g)}{\partial(x, p)} + \phi\,\frac{\partial(f, g)}{\partial(z, p)}\right] \qquad \ldots(10)$$

where

$$J = \frac{\partial(f, g)}{\partial(p, q)}.$$

If we had differentiated the given pair of equations (1) and (2) with respect to y and z after substituting in them from equations (3), then we would have obtained

$$\phi_y + \psi\,\phi_z = -\frac{1}{J}\left[\frac{\partial(f, g)}{\partial(y, q)} + \psi\,\frac{\partial(f, g)}{\partial(z, q)}\right]. \qquad \ldots(11)$$

Substituting from equations (10) and (11) in (5), we get the desired condition that the two equations (1) and (2) should be compatible as

$$[f, g] = 0, \qquad \ldots(12)$$

where

$$[f, g] = \frac{\partial(f, g)}{\partial(x, p)} + p\,\frac{\partial(f, g)}{\partial(z, p)} + \frac{\partial(f, g)}{\partial(y, q)} + q\,\frac{\partial(f, g)}{\partial(z, q)}. \qquad \ldots(13)$$

$$[\because \; p = \phi \text{ and } q = \psi]$$

Particular Case:

Theorem: *The first order partial differential equations*

$$p = P(x, y), \quad q = Q(x, y)$$

are compatible if and only if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

Proof: We know that if z is a function of two independent variables x and y, then

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = p\, dx + q\, dy.$$

So, the partial differential equations $p = P(x, y), q = Q(x, y)$ are compatible if and only if the single differential equation

$$dz = P\, dx + Q\, dy$$

is integrable.

Now we know that if P and Q are functions of two variables x and y, then $P\, dx + Q\, dy$ is an exact differential $d\,\phi(x, y)$ if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

$\therefore$ The single differential equation $dz = P\, dx + Q\, dy$ is integrable if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Hence, the differential equations $p = P(x, y), q = Q(x, y)$ are compatible if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

17 The Method of Characteristics

The method of characteristics is a technique for solving partial differential equations. It is a powerful method that allows one to reduce any first order linear partial differential equation to an ordinary differential equation, which can be subsequently solved using ODE techniques. Typically it applies to first-order equations, more generally this method is valid for any hyperbolic partial differential equation.

18 Characteristics of First Order Partial Differential Equations

The method of characteristics discovers curves for a first order partial differential equation, along which the PDE becomes an ODE. These curves are called the characteristic curves or simply characteristics. Once the ODE is found, it can be solved along the characteristic curves and transformed into a solution for the original PDE.

Here we confine our study to the case of a function of two independent variables x and y. Consider a quasi-linear PDE of the form

$$a(x, y\, z)\frac{\partial z}{\partial x} + b(x, y, z)\frac{\partial z}{\partial y} = c(x, y, z). \qquad \ldots(1)$$

Suppose that a solution z is known. Consider the surface graph $z = z(x, y)$ in $\mathbf{R}^3$. A normal vector to this surface is given by

$$\left(\frac{\partial z}{\partial x}(x, y),\ \frac{\partial z}{\partial y}(x, y),\ -1\right).$$

The equation (1) is equivalent to the geometrical statement that the vector field

$$(a(x, y\, z),\ b(x, y, z),\ c(x, y, z))$$

is tangent to the surface $z = z(x, y)$ at every point, for the dot product of this vector field with the above normal vector is zero.

In other words, the graph of the solution must be a union of integral curves of this vector field. These integral curves are called the characteristic curves of the original partial differential equation.

The equations of the characteristic curve may be expressed invariantly by the Lagrange-Charpit equations

$$\frac{dx}{a(x, y\, z)} = \frac{dy}{b(x, y\, z)} = \frac{dz}{c(x, y\, z)}.$$

If a particular parametrization t of the curves is fixed, then these equations may be written as a system of ordinary differential equations for $x(t)$, $y(t)$, $z(t)$ as

$$\frac{dx}{dt} = a(x, y\, z);\ \frac{dy}{dt} = b(x, y\, z);\ \frac{dz}{dt} = c(x, y\, z). \qquad \ldots(2)$$

These are the characteristic equations for the original system.

19 Characteristics of Linear and Quasi-linear Partial Differential Equations

Consider a partial differential equation of the form

$$\sum_{i=1}^{n} a_i(x_1, \ldots, x_n, u)\frac{\partial u}{\partial x_i} = c(x_1, \ldots, x_n, u). \qquad \ldots(1)$$

For this partial differential equation to be linear, the coefficients a_i may be functions of the spatial variables only, and independent of u. And for it to be quasi-linear a_i may also depend on the value of the function, but not on any derivatives.

For a linear or quasi-linear partial differential equation, the characteristic curves are given parametrically by

$$(x_1, \ldots, x_n, u) = (x_1(s), \ldots, x_n(s), u(s)) \;\; u(X(s)) = U(s)$$

such that the following system of ordinary differential equations is satisfied

$$\frac{dx_i}{ds} = a_i(x_1, \ldots, x_n, u) \qquad \ldots(2)$$

$$\frac{du}{ds} = c(x_1, \ldots, x_n, u). \qquad \ldots(3)$$

Equations (2) and (3) give the characteristics of the partial differential equation.

Illustrative Examples

Example 38: *Show that the equations*

$$xp - yq = x, \;\; x^2 p + q = xz$$

are compatible and find their solution.

Solution: We know that the first order partial differential equations

$$f(x, y, z, p, q) = 0$$

and
$$g(x, y, z, p, q) = 0$$

are compatible if and only if $[f, g] = 0$,

where
$$[f, g] = \frac{\partial(f, g)}{\partial(x, p)} + p\frac{\partial(f, g)}{\partial(z, p)} + \frac{\partial(f, g)}{\partial(y, q)} + q\frac{\partial(f, g)}{\partial(z, q)}.$$

Here, the given equations are $f(x, y, z, p, q) \equiv xp - yq - x = 0$

and
$$g(x, y, z, p, q) \equiv x^2 p + q - xz = 0.$$

$$\therefore \quad \frac{\partial(f, g)}{\partial(x, p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} p-1 & x \\ 2xp - z & x^2 \end{vmatrix} = x^2(p-1) - x(2xp - z),$$

$$\frac{\partial(f, g)}{\partial(z, p)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & x \\ -x & x^2 \end{vmatrix} = 0 - (-x^2) = x^2,$$

$$\frac{\partial(f, g)}{\partial(y, q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} -q & -y \\ 0 & 1 \end{vmatrix} = -q,$$

and
$$\frac{\partial(f, g)}{\partial(z, q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -y \\ -x & 1 \end{vmatrix} = -xy.$$

Therefore, $[f, g] = x^2(p-1) - 2x^2 p + xz + px^2 - q - qxy$

$$= x^2 p - x^2 - 2x^2 p + xz + px^2 - q - qxy$$

$$= -x^2 + xz - q - qxy$$

$$= -x^2 + x^2 p + q - q - qxy$$

[$\because$ from the given differential equations, $xz = x^2 p + q$]

$$= -x^2 + x(xp - yq)$$

$$= -x^2 + x \,.\, x$$

[$\because$ from the given differential equations, $xp - yq = x$]

$$= -x^2 + x^2 = 0.$$

Hence, the given equations are compatible.

Now, let us find p and q by solving the equations

$$xp - yq = x \quad \text{...(1)}$$

and
$$x^2 p + q = xz \quad \text{...(2)}$$

Multiplying (2) by y and adding to (1), we get

$$yx^2 p + xp = xyz + x$$

or
$$xp(1 + xy) = x(1 + yz)$$

or
$$p = \frac{1 + yz}{1 + xy}.$$

$\therefore$ From (2), $q = xz - x^2 p = xz - \dfrac{x^2 + x^2 yz}{1 + xy}$

$$= \frac{xz(1 + xy) - x^2 - x^2 yz}{1 + xy}$$

$$= \frac{x(z - x)}{1 + xy}.$$

Thus,
$$p = \frac{1 + yz}{1 + xy}, q = \frac{x(z - x)}{1 + xy}.$$

Now
$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = p\,dx + q\,dy$$

$$= \left(\frac{1 + yz}{1 + xy}\right) dx + \left[\frac{x(z - x)}{1 + xy}\right] dy.$$

So we have to solve the single differential equation

$$(1 + xy)\,dz - (1 + yz)\,dx - x(z - x)\,dy = 0$$

$\Rightarrow$
$$(1 + xy)\,dz - (1 + xy)\,dx + (1 + xy)\,dx - (1 + yz)\,dx - x(z - x)\,dy = 0$$

$\Rightarrow$
$$(1 + xy)(dz - dx) - y(z - x)\,dx - x(z - x)\,dy = 0$$

$$\Rightarrow \qquad \frac{(1+xy)(dz-dx)-(z-x)(y\,dx+x\,dy)}{(1+xy)^2}=0$$

$$\Rightarrow \qquad \frac{(1+xy)(dz-dx)-(z-x)\,d(1+xy)}{(1+xy)^2}=0$$

$$\Rightarrow \qquad d\left(\frac{z-x}{1+xy}\right)=0$$

$$\Rightarrow \qquad \frac{z-x}{1+xy}=c, \text{ where } c \text{ is an arbitrary constant.}$$

Hence, the required solution of the given equations is

$$z-x=c\,(1+xy)$$

or $$z=x+c\,(1+xy), \text{ where } c \text{ is a constant.}$$

Example 39: *Show that the differential equations* $p=5x-4y+3, q=4x+5y+2$ *do not possess any common solution.*

Solution: We know that the first order partial differential equations

$$p=P(x,y),\ q=Q(x,y)$$

are compatible if and only if

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}.$$

Here, $P=5x-4y+3,\ Q=4x+5y+2.$

We have $\dfrac{\partial P}{\partial y}=-4$ and $\dfrac{\partial Q}{\partial x}=4.$

Since $\dfrac{\partial P}{\partial y}\neq\dfrac{\partial Q}{\partial x}$, therefore the given differential equations are not compatible.

Hence, the given differential equations do not possess any common solution.

Example 40: *Show that the differential equations*

$$\frac{\partial z}{\partial x}=6x+3y,\ \frac{\partial z}{\partial y}=3x-4y$$

are compatible and find their solution.

Solution: We know that the first order partial differential equations

$$p=P(x,y),\ q=Q(x,y)$$

are compatible if and only if

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}.$$

Here, $P(x,y)=6x+3y,\ Q(x,y)=3x-4y.$

We have $\dfrac{\partial P}{\partial y}=3$ and $\dfrac{\partial Q}{\partial x}=3.$

Since $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, therefore the given differential equations are compatible.

Now $$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = p\,dx + q\,dy$$

$$= (6x + 3y)\,dx + (3x - 4y)\,dy$$
$$= 6x\,dx + 3\,(y\,dx + x\,dy) - 4y\,dy$$
$$= 6x\,dx + 3d\,(xy) - 4y\,dy.$$

Integrating both sides, we get

$$z = 6 \cdot \frac{x^2}{2} + 3xy - 4 \cdot \frac{y^2}{2} + c$$

or $$z = 3x^2 + 3xy - 2y^2 + c$$

as the required solution of the given differential equations.

Example 41: *Show that the differential equations*

$$p = x^2 - ay, q = y^2 - ax$$

are compatible and find their common solution.

Solution: We know that the first order partial differential equations

$$p = P\,(x, y), q = Q\,(x, y)$$

are compatible if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Here, $$P\,(x, y) = x^2 - ay, Q\,(x, y) = y^2 - ax.$$

We have $$\frac{\partial P}{\partial y} = -a \quad \text{and} \quad \frac{\partial Q}{\partial x} = -a.$$

Since $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, therefore the given differential equations are compatible.

Now $$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = p\,dx + q\,dy$$

$$= (x^2 - ay)\,dx + (y^2 - ax)\,dy$$
$$= x^2\,dx + y^2\,dy - a\,(x\,dy + y\,dx)$$
$$= x^2\,dx + y^2\,dy - ad\,(xy).$$

Integrating both sides, we get

$$z = \frac{x^3}{3} + \frac{y^3}{3} - axy + c$$

as the required common solution of the given differential equations.

Example 42: *Solve the simultaneous differential equations*

$$\frac{\partial z}{\partial x} = x^4 - 2xy^2 + y^4, \; \frac{\partial z}{\partial y} = -(2x^2 y - 4xy^3 + \sin y).$$

Solution: We know that the first order partial differential equations

$$p = P(x, y), \; q = Q(x, y)$$

are compatible if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Here, $\quad P = x^4 - 2xy^2 + y^4, Q = -2x^2 y + 4xy^3 - \sin y.$

We have $\quad \dfrac{\partial P}{\partial y} = -4xy + 4y^3, \; \dfrac{\partial Q}{\partial x} = -4xy + 4y^3.$

Since $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$, therefore the given differential equations are compatible.

Now
$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = p\, dx + q\, dy$$
$$= P\, dx + Q\, dy$$
$$= (x^4 - 2xy^2 + y^4)\, dx + (-2x^2 y + 4xy^3 - \sin y)\, dy.$$

Integrating both sides, we get

$$z = \underset{\text{(treating } y \text{ as a constant)}}{\int P\, dx} + \underset{\substack{\text{(taking in } Q \text{ only those} \\ \text{terms which do not contain } x)}}{\int Q\, dy} + c$$

$$= \underset{\text{(treating } y \text{ as a constant)}}{\int (x^4 - 2xy^2 + y^4)\, dx} + \int (-\sin y)\, dy + c$$

$$= \frac{1}{5} x^5 - x^2 y^2 + x y^4 + \cos y + c.$$

Hence, the required common solution of the given simultaneous equations is

$$z = \frac{1}{5} x^5 - x^2 y^2 + xy^4 + \cos y + c,$$

where c is an arbitrary constant.

Comprehensive Exercise 6

1. Show that the differential equations

$$\frac{\partial z}{\partial x} = 7x + 8y - 5, \; \frac{\partial z}{\partial y} = 9x + 11y + 3$$

are not compatible.

2. Show that the differential equations $p = 4x + 3y + 1,\ q = 3x + 2y + 1$ are compatible and find their solution.
3. Show that the differential equations $p = 1 + 4xy + 2y^2,\ q = 1 + 4xy + 2x^2$ are compatible and find their common solution.
4. Show that the differential equations in each of the following systems are compatible and find their solution :

(i) $p = ax + hy + g,\ q = hx + by + f$

(ii) $p = 2x - y,\ q = 5 - x - 2y$

(iii) $p = (2ax + by)\, y,\ q = (ax + 2\,by)\, x$

(iv) $p = x - \dfrac{y}{x^2 + y^2},\ q = y + \dfrac{x}{x^2 + y^2}$

(v) $p = 1 + e^{x/y},\quad q = e^{x/y}\left(1 - \dfrac{x}{y}\right)$

(vi) $p = y^2 e^{xy^2} + 4x^3,\ q = 2xy\, e^{xy^2} - 3y^2$

(vii) $p = (e^y + 1)\cos x,\ q = e^y \sin x$

(viii) $p = \sin x \cos y + e^{2x},\ q = \cos x \sin y + \tan y.$

5. Show that the equations $f(x, y, p, q) = 0,\ g(x, y, p, q) = 0$ are compatible if $\dfrac{\partial(f, g)}{\partial(x, p)} + \dfrac{\partial(f, g)}{\partial(y, q)} = 0.$

Answers 6

2. $z = 2x^2 + 3xy + x + y^2 + y + c$
3. $z = x + 2x^2 y + 2xy^2 + y + c$
4. **(i)** $z = \dfrac{1}{2}ax^2 + hxy + gx + \dfrac{1}{2}by^2 + fy + c$

(ii) $z = x^2 - xy - y^2 + 5y + c$

(iii) $z = ayx^2 + bxy^2 + c$

(iv) $z = \dfrac{1}{2}x^2 - \tan^{-1}\left(\dfrac{x}{y}\right) + \dfrac{1}{2}y^2 + c$

(v) $z = x + ye^{x/y} + c$

(vi) $z = e^{xy^2} + x^4 - y^3 + c$

(vii) $z = (e^y + 1)\sin x + c.$

(viii) $z = -\cos x \cos y + \dfrac{1}{2}e^{2x} + \log \sec y + c.$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Equation $p \tan y + q \tan x = \sec^2 z$ is of order
   (a) one (b) two
   (c) zero (d) none of these

2. Equation $r^2 + 2s - t^2 = 0$ is of order
   (a) one (b) two
   (c) three (d) none of these

3. Equation $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \, \partial y} + \left(\dfrac{\partial z}{\partial y}\right)^3 = 0$ is of degree
   (a) one (b) two
   (c) three (d) none of these

4. Equation $p^3 + qx^2 + z^4 = 0$ is of degree
   (a) two (b) three
   (c) four (d) one
   (Garhwal 2011; Kumaun 11)

5. The equation $Pp + Qq = R$ is known as
   (a) Charpit's equation (b) Lagrange's equation
   (c) Bernoulli's equation (d) Clairaut's equation
   (Kumaun 2007)

6. Out of the following four partial differential equations, the differential equation which is linear is
   (a) $\dfrac{\partial^3 z}{\partial x^3} - 3\dfrac{\partial^2 z}{\partial x^2} \cdot \dfrac{\partial z}{\partial y} + 8\dfrac{\partial^2 z}{\partial y^2} = \sin x$
   (b) $\dfrac{\partial^2 z}{\partial x^2} + \left(\dfrac{\partial z}{\partial x}\right)^2 + \dfrac{\partial z}{\partial y} + 9z = 0$
   (c) $4\dfrac{\partial^2 z}{\partial x^2} + 5\dfrac{\partial^2 z}{\partial x \, \partial y} + 6\dfrac{\partial^2 z}{\partial y^2} + 7\dfrac{\partial z}{\partial x} + 8\dfrac{\partial z}{\partial y} + 3z = (x^3 + y^3)\sin x$
   (d) $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 = x^2 + y^2$

7. The differential equation $(2x + 3y)\,p + 4xq - 8pq = x + y$ is
   (a) linear (b) non-linear
   (c) quasi-linear (d) semi-linear **(Kumaun 2012)**

8. The differential equation $(x^2 + y^2)\dfrac{\partial z}{\partial x} + (x + y)\dfrac{\partial z}{\partial y} + (3x - 4y)\, z = x^2 + y^2$ is

(a) linear (b) quasi-linear
(c) semi-linear (d) non-linear

9. The differential equation $(x + y - 3z)\dfrac{\partial z}{\partial x} + (3x + 4y)\dfrac{\partial z}{\partial y} + 2\, z = x + y$ is

(a) linear (b) quasi-linear
(c) semi-linear (d) non-linear

10. The differential equation $x^2 \dfrac{\partial z}{\partial x} + y^2 \dfrac{\partial z}{\partial y} = (x + y)\, z^2 + 4x$ is

(a) linear (b) quasi-linear
(c) semi-linear (d) non-linear

11. Out of the following four pairs of first order partial differential equations, mention the pair in which the differential equations are compatible :

(a) $p = 4y - 7x + 3, q = 7x + 4y + 2$ (b) $p = x + y, q = y - x$
(c) $p = 3x + 8y, q = 8x - 3y$ (d) $p = x + y + 1, q = 2x - y + 1$

12. Lagrange's auxiliary equations of $Pp + Qq = R$ are given by

(a) $\dfrac{dx}{R} = \dfrac{dy}{Q} = \dfrac{dz}{P}$ (b) $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

(c) $\dfrac{dx}{Q} = \dfrac{dy}{R} = \dfrac{dz}{P}$ (d) none of these

(Garhwal 2005, 10, 15; Kumaun 09, 11, 13)

13. For the equation $z = pq$, Charpit's auxiliary equations are :

(a) $\dfrac{dp}{p} = \dfrac{dq}{q} = \dfrac{dz}{2\,pq} = \dfrac{dx}{q} = \dfrac{dy}{p}$ (b) $\dfrac{dp}{q} = \dfrac{dq}{p} = \dfrac{dz}{pq} = \dfrac{dx}{p} = \dfrac{dy}{q}$

(c) $\dfrac{dp}{p} = \dfrac{dq}{q} = \dfrac{dz}{pq} = \dfrac{dx}{p} = \dfrac{dy}{q}$ (d) $\dfrac{dp}{p} = \dfrac{dq}{q} = \dfrac{dz}{2\,pq} = \dfrac{dx}{p} = \dfrac{dy}{q}$

(Garhwal 2014)

14. For the equation $(p^2 + q^2)\, y = qz$, Charpit's auxiliary equation is

(a) $\dfrac{dp}{-q} = \dfrac{dq}{p}$ (b) $\dfrac{dp}{q} = \dfrac{dq}{p}$

(c) $\dfrac{dp}{q} = \dfrac{dq}{p + 1}$ (d) $\dfrac{dp}{q + 1} = \dfrac{dq}{p}$

(Garhwal 2015)

15. A partial differential equation has

(a) One independent variables
(b) Two or more independent variables
(c) More than one dependent variables
(d) Equal number of independent and dependent variables

(Garhwal 2013, 15)

16. Solution of the equation $z = px + qy + f(p, q)$ is

(a) $z = f(ax, by)$ (b) $z = ax + by + f(a, b)$

(c) $z = f(ax/by)$ (d) none of these

(Kumaun 2006)

17. The complete integral of equations of the type $f(p, q) = 0$ is $z = ax + by + c$, where a and b are connected by the relation

(a) $f(a, b) = 0$ (b) $f(ab) = 0$

(c) $f(a + b) = 0$ (d) $f(a/b) = 0$

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. The order of the differential equation

$$4\frac{\partial^3 z}{\partial x^2\, \partial y} + 9\frac{\partial^2 z}{\partial x^2} + 8\frac{\partial^2 z}{\partial y^2} + 6\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} + 8 = 0 \text{ is } \dots .$$

2. In a linear partial differential equation all the partial derivatives occurring in it are in ... degree.
3. Lagrange's linear equation is of the form ... .
4. Lagrange's auxiliary equations or Lagrange's subsidiary equations are ... .
5. Lagrange's auxiliary equations of $(y^2 + z^2 - x^2)\, p - 2xyq + 2zx = 0$ are ... .
6. Lagrange's subsidiary equations of $z\,(xp - yq) = y^2 - x^2$ are ... .
7. Equations $\dfrac{dp}{\dfrac{\partial f}{\partial x} + p\dfrac{\partial f}{\partial z}} = \dfrac{dq}{\dfrac{\partial f}{\partial y} + q\dfrac{\partial f}{\partial z}} = \dfrac{dz}{-p\dfrac{\partial f}{\partial p} - q\dfrac{\partial f}{\partial q}} = \dfrac{dx}{-\dfrac{\partial f}{\partial p}} = \dfrac{dy}{-\dfrac{\partial f}{\partial q}} = \dfrac{dF}{0}$

   are known as ... .
8. For the differential equation $p^2 x + q^2 y = z$, Charpit's auxiliary equations are ... .
9. The first order partial differential equations

$$\frac{\partial z}{\partial x} = P(x, y), \frac{\partial z}{\partial y} = Q(x, y) \text{ are compatible if } \frac{\partial P}{\partial y} = \dots$$

10. If every solution of the differential equation $f(x, y, z, p, q) = 0$ is also a solution of the differential equation $g(x, y, z, p, q) = 0$, then the two differential equations are said to be ...

True or False

Write 'T' for true and 'F' for false statement.

1. The differential equation $\dfrac{\partial z}{\partial x} \cdot \dfrac{\partial z}{\partial y} = 3xy$ is non-linear.
2. The differential equation $\dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x\, \partial y} + 5\dfrac{\partial u}{\partial x} + 6\dfrac{\partial u}{\partial y} + 7u = x^2$ is linear.

3. The differential equation $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ is non-linear.
4. The differential equation $z = px + qy$ is linear.
5. $pz - qy = z^2 + (x + y)^2$ is a linear differential equation.
6. The differential equations $\frac{\partial z}{\partial x} = 5x - 7y, \frac{\partial z}{\partial y} = 6x + 8y$ possess a common solution.
7. The differential equations $p = 12x + 7y + 1, q = 7x + 4y + 1$ are compatible.
8. The differential equations $p = \frac{y}{x^2 + y^2} - 1, q = -\frac{x}{x^2 + y^2}$ are compatible.
9. The differential equations $\frac{\partial z}{\partial x} = y \sin 2x, \frac{\partial z}{\partial y} = -(1 + y^2 + \cos^2 x)$ are not compatible.
10. The differential equations $p = \cos x (\cos x - \sin\alpha \sin y), q = \cos y (\cos y - \sin\alpha \sin x)$ are compatible.
11. Singular integral of a differential equation is obtained by giving particular values to arbitrary constants in its general solution.
12. The equation of envelope to a surface is the singular integral of its differential equation.

Answers

Multiple Choice Questions

1.	(a)	2.	(b)	3.	(a)	4.	(b)	5.	(b)
6.	(c)	7.	(b)	8.	(a)	9.	(b)	10.	(c)
11.	(c)	12.	(b)	13.	(a)	14.	(a)	15.	(b)
16.	(b)	17.	(a)						

Fill in the Blank(s)

1. 3
2. first
3. $Pp + Qq = R$
4. $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$
5. $\frac{dx}{y^2 + z^2 - x^2} = \frac{dy}{-2xy} = \frac{dz}{-2zx}$
6. $\frac{dx}{zx} = \frac{dy}{-yz} = \frac{dz}{y^2 - x^2}$
7. Charpit's auxiliary equations
8. $\frac{dp}{-p + p^2} = \frac{dq}{-q + q^2} = \frac{dz}{-2p^2 x - 2q^2 y} = \frac{dx}{-2px} = \frac{dy}{-2qy}$
9. $\frac{\partial Q}{\partial x}$
10. compatible

True or False

1.	*T*	2.	*T*	3.	*F*	4.	*T*	5.	*T*
6.	*F*	7.	*T*	8.	*T*	9.	*F*	10.	*T*
11.	*F*	12.	*T*						

□

Chapter

7

Linear Partial Differential Equations with Constant Coefficients

1 The General Linear Partial Differential Equation of an Order Higher than the First

A partial differential equation in which the dependent variable and its derivatives appear only in the first degree and are not multiplied together, their coefficients all being constants or functions of x and y, is called a **linear partial differential equation.** The general form of such an equation is

$$\frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\,\partial y} + \ldots + A_n \frac{\partial^n z}{\partial y^n} + B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + \ldots$$

$$+ M \frac{\partial z}{\partial x} + N \frac{\partial z}{\partial y} + P z = f(x, y), \quad \ldots(1)$$

where the coefficients $A_1, \ldots, A_n, B_0, \ldots, M, N, P$ are constants or functions of x and y. If the coefficients of various terms are constants then it is called a **linear partial differential equation with constant coefficients.**

2 The Homogeneous Linear Partial Differential Equation with Constant Coefficients

In this equation all the partial derivatives appearing in the equation are of the same order. *A linear homogeneous partial differential equation of order n with constant coefficients is of the form*

$$\frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \ldots + A_n \frac{\partial^n z}{\partial y^n} = f(x, y), \qquad \ldots(1)$$

where $A_1, \ldots, A_n$ are constants.

Denoting the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ by D, D' respectively, the equation (1) can also be

written as $\qquad (D^n + A_1 D^{n-1} D' + \ldots + A_n D'^n) z = f(x, y)$

or $\qquad F(D, D') z = f(x, y), \qquad \ldots(2)$

where $\qquad F(D, D') \equiv D^n + A_1 D^{n-1} D' + \ldots + A_n D'^n.$

Note that $F(D, D')$ is a homogeneous function in D, D' of degree n.

3 Solution of a Linear Homogeneous Partial Differential Equation with Constant Coefficients.

As in the case of ordinary linear differential equations the basic theorem is :

Theorem 1: *If u is the complementary function and z_1 a particular integral of a linear partial differential equation $F(D, D') z = f(x, y)$ then $u + z_1$ is a general solution of the equation.*

Proof: The complementary function of the equation

$$F(D, D') z = f(x, y) \qquad \ldots(1)$$

is the most general solution of the equation $F(D, D') z = 0. \qquad \ldots(2)$

It must contain as many arbitrary constants as is the order of the differential equation (2).

Any solution of (1) is called a particular integral of (1). It does not contain any arbitrary constant.

Since the equations (1) and (2) are of the same order, the solution $u + z_1$ will contain as many arbitrary constants as the general solution of (1) requires. Also

$$F(D, D') u = 0,$$

$$F(D, D') z_1 = f(x, y)$$

so that $\qquad F(D, D')(u + z_1) = f(x, y).$

This shows that, in fact, $u + z_1$ is a solution of (1). This completes the proof.

The next result is of extensive use in the solution of the partial differential equations.

Theorem 2: *If* $u_1, u_2, \ldots, u_n$, *are solutions of the homogeneous linear partial differential equation* $F(D, D')z = 0$ *then* $\sum_{r=1}^{n} c_r u_r$ *is also a solution, where* c_r*'s are arbitrary constants.*

Proof: The proof of this theorem is obvious.

We have $F(D, D')(c_r u_r) = c_r F(D, D') u_r$.

Also $\quad F(D, D') \sum_{r=1}^{n} v_r = \sum_{r=1}^{n} F(D, D') v_r$, for any set of functions v_r.

Hence
$$F(D, D') \sum_{r=1}^{n} c_r u_r = \sum_{r=1}^{n} F(D, D')(c_r u_r)$$
$$= \sum_{r=1}^{n} c_r F(D, D') u_r = \sum_{r=1}^{n} c_r .0 = 0.$$

4 Determination of the Complementary Function (C.F.) of the Linear Homogeneous Partial Differential Equation with Constant Coefficients

Consider a linear homogeneous nth order partial differential equation with constant coefficients of the form $F(D, D')z = f(x, y)$. ...(1)

The complementary function of (1) is the general solution of
$$F(D, D')z = 0$$
i.e.,
$$(D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \ldots + A_n D'^n)z = 0 \qquad \ldots(2)$$
This is equivalent to
$$[(D - m_1 D')(D - m_2 D')\ldots(D - m_n D')]z = 0, \qquad \ldots(3)$$
where $m_1, m_2, \ldots, m_n$ are some constants.

The solution of any one of the equations
$$(D - m_1 D')z = 0, (D - m_2 D')z = 0, \ldots, (D - m_n D')z = 0 \qquad \ldots(4)$$
is also a solution of (3) and we know that the general solution of $(D - mD')z = 0$ is $z = \phi(y + mx)$, where ϕ is an arbitrary function. Hence we can assume that a solution of the equation (3) is of the form $z = \phi(y + mx)$.

Differentiation will give
$$Dz = m\phi'(y + mx), D^n z = m^n \phi^{(n)}(y + mx), D'^n z = \phi^{(n)}(y + mx)$$
and, in general,
$$D^r D'^s z = m^r \phi^{(r+s)}(y + mx).$$
Therefore, the substitution of $\phi(y + mx)$ for z in (2) gives
$$(m^n + A_1 m^{n-1} + \ldots + A_n)\phi^{(n)}(y + mx) = 0.$$

This is true if m is a root of the equation

$$m^n + A_1 m^{n-1} + \ldots + A_n = 0. \qquad \ldots(5)$$

The equation (5) *is called the* **auxiliary equation (A.E.)** *and is obtained by putting* $D = m, D' = 1$ *in* $F(D, D') = 0$.

It will give in general n roots, say, $m_1, m_2, \ldots m_n$. Each value of m will give a solution of (2). Hence if all the roots of the Auxiliary equation are distinct, the general solution of (2) *i.e.*, the complementary function of (1) is

$$z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \ldots + \phi_n(y + m_n x) \qquad \ldots(6)$$

where $\phi_1, \phi_2, \ldots, \phi_n$ are arbitrary functions.

Solution when the auxiliary equation has equal roots

i.e., the roots of the A.E. are repeated.

The equation corresponding to two repeated roots each equal to m is

$$(D - mD')(D - mD')\, z = 0.$$

Putting $(D - mD')\, z = u$, this becomes $(D - mD')\, u = 0$, the solution of which is $u = \phi(y + mx)$.

Hence $(D - mD')\, z = \phi(y + mx)$

or $$p - mq = \phi(y + mx).$$

Lagrange's auxiliary equations of this linear equation are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi(y + mx)}.$$

Taking the first two members, we get

$$dy + m\, dx = 0.$$

$\therefore$ $$y + mx = a.$$

Taking the first and the third members, we get

$$dz = \phi(y + mx)\, dx = \phi(a)\, dx.$$

$\therefore$ $$z = \phi(a).x + b.$$

Hence $$z = x\phi(y + mx) + \psi(y + mx).$$

Proceeding in the same way it can be shown that when a root m is repeated r times, the corresponding part of the complementary function is

$$\phi_1(y + mx) + x\,\phi_2(y + mx) + x^2\,\phi_3(y + mx) + \ldots + x^{r-1}\,\phi_r(y + mx).$$

Illustrative Examples

Example 1: *Solve* $2r + 5s + 2t = 0$. $\ldots(1)$

Solution: We know that

$$r = \frac{\partial^2 z}{\partial x^2} = D^2 z,\ s = \frac{\partial^2 z}{\partial x\, \partial y} = DD'z,\ t = \frac{\partial^2 z}{\partial y^2} = D'^2 z.$$

Hence the given equation can be written as

$$(2D^2 + 5DD' + 2D'^2)\, z = 0.$$

A.E. is $2m^2 + 5m + 2 = 0$ or $(2m+1)(m+2) = 0.$

$\therefore \quad m = -\frac{1}{2}, -2.$

Therefore the general solution of (1) is

$$z = f\left(y - \frac{1}{2}x\right) + \psi(y - 2x) \quad \text{or} \quad z = \phi(2y - x) + \psi(y - 2x).$$

Example 2: *Solve* $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)\, z = 0.$ ...(1)

Solution: The auxiliary equation is

$$m^3 - 6m^2 + 11m - 6 = 0 \quad \text{or} \quad (m-1)(m-2)(m-3) = 0.$$

$\therefore \quad m = 1, 2, 3.$

Therefore the general solution of (1) is

$$z = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x).$$

Example 3: *Solve* $(D^3 - 3D^2D' + 2DD'^2)\, z = 0.$ **(Lucknow 2010)**

Solution: The auxiliary equation is

$$m^3 - 3m^2 + 2m = 0 \quad \text{or} \quad m(m-1)(m-2) = 0.$$

$\therefore \quad m = 0, 1, 2.$

Therefore the general solution of (1) is

$$z = \phi_1(y) + \phi_2(y + x) + \phi_3(y + 2x).$$

Example 4: *Solve* $\dfrac{\partial^4 z}{\partial x^4} - \dfrac{\partial^4 z}{\partial y^4} = 0.$ **(Avadh 2012)** ...(1)

Solution: The equation (1) can be written as $(D^4 - D'^4)\, z = 0.$

A.E. is $m^4 - 1 = 0$ or $(m-1)(m+1)(m^2+1) = 0.$

$\therefore \quad m = 1, -1, \pm i.$

Therefore the general solution of (1) is

$$z = \phi_1(y + x) + \phi_2(y - x) + \phi_3(y + ix) + \phi_4(y - ix).$$

Example 5: *Solve* $(D^4 - 2D^3D' + 2DD'^3 - D'^4)\, z = 0.$ **(Lucknow 2006, 09)** ...(1)

Solution: The auxiliary equation is

$$m^4 - 2m^3 + 2m - 1 = 0 \quad \text{or} \quad (m+1)(m-1)^3 = 0.$$

$\therefore \quad m = -1, 1, 1, 1.$

Therefore the general solution of (1) is

$$z = \phi_1(y - x) + \phi_2(y + x) + x\phi_3(y + x) + x^2\phi_4(y + x).$$

Example 6: *Solve* $25r - 40s + 16t = 0.$ ...(1)

Solution: The equation (1) can be written as

$$(25D^2 - 40\, DD' + 16\, D'^2)\, z = 0.$$

A.E. is $\quad 25m^2 - 40m + 16 = 0 \quad$ or $\quad (5m - 4)^2 = 0.$

$\therefore \quad m = 4/5, 4/5.$

Therefore the general solution of (1) is

$$z = \phi_1 \left(y + \frac{4}{5} x \right) + x\, \phi_2 \left(y + \frac{4}{5} x \right)$$

or $\quad z = f_1\, (5y + 4x) + x\, f_2\, (5y + 4x).$

Comprehensive Exercise 1

1. Solve $r = a^2 t$. **(Rohilkhand 2010, Avadh 10)**
2. Solve $(D^3 - 4D^2 D' + 4\, DD'^2)\, z = 0.$
3. Solve $\dfrac{\partial^3 z}{\partial x^3} - 7 \dfrac{\partial^3 z}{\partial x\, \partial y^2} + 6 \dfrac{\partial^3 z}{\partial y^3} = 0.$
4. Solve $(D^2 - 3aDD' + 2a^2 D'^2)\, z = 0.$
5. Solve $2 \dfrac{\partial^2 z}{\partial x^2} - 3 \dfrac{\partial^2 z}{\partial x\, \partial y} - 2 \dfrac{\partial^2 z}{\partial y^2} = 0.$
6. Solve $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = 0.$
7. Solve $r + t + 2s = 0.$ **(Kanpur 2009)**
8. Solve $(4D^2 + 12DD' + 9D'^2)\, z = 0.$
9. Solve $(D^4 + D'^4 - 2\, D^2 D'^2)\, z = 0.$
10. Solve $(D^4 + D'^4)\, z = 0.$

Answers 1

1. $z = \phi_1\, (y + ax) + \phi_2\, (y - ax)$
2. $z = \phi_1\, (y) + \phi_2\, (y + 2x) + x\, \phi_3\, (y + 2x)$
3. $z = \phi_1\, (y + x) + \phi_2\, (y + 2x) + \phi_3\, (y - 3x)$
4. $z = \phi_1\, (y + ax) + \phi_2\, (y + 2ax)$
5. $z = \phi_1\, (2y - x) + \phi_2\, (y + 2x)$
6. $z = \phi_1\, (y + x) + \phi_2\, (y - x)$
7. $z = \phi_1\, (y - x) + x\, \phi_2\, (y - x)$
8. $z = \phi_1\, (2y - 3x) + x\, \phi_2\, (2y - 3x)$

9. $z = \phi_1 (y + x) + x \phi_2 (y + x) + \phi_3 (y - x) + x \phi_4 (y - x)$

10. $z = \phi_1 (y + \alpha x) + \phi_2 (y + \bar{\alpha} x) + \phi_3 (y + \beta x) + \phi_4 (y + \bar{\beta} x)$

where $\alpha = \frac{1}{\sqrt 2} + i \frac{1}{\sqrt 2}$ and $\beta = - \frac{1}{\sqrt 2} + i \frac{1}{\sqrt 2}$.

5 Determination of the Particular Integral (P.I.)

The particular integral of the equation (1) of 9.4 will be denoted by $\frac{1}{F (D, D')} f (x, y)$.

$\frac{1}{F (D, D')} V$ is defined as the function which gives V when it is operated upon by $F (D, D')$.

The symbolic function $F (D, D')$ can be treated as an algebraic function of D and D'. It can be factorised, resolved into partial fractions or can be expanded in ascending powers of D or D'.

Note: $\frac{1}{D}$ means integration w.r.t. x, $\frac{1}{D'}$ means integration w.r.t y, and so on and particular integral would be different according as $F (D, D')$ is expanded in ascending powers of D or D'.

Illustrative Examples

Example 7: *Solve* $(D^2 + 3DD'^2 + 2D'^2) z = x + y$. **(Avadh 2012; Kanpur 14)**

Solution: The auxiliary equation is

$$m^2 + 3m + 2 = 0 \quad \text{or} \quad (m + 2)(m + 1) = 0.$$

$$\therefore \quad m = -1, -2.$$

$$\therefore \quad \text{C.F.} = \phi_1 (y - x) + \phi_2 (y - 2x).$$

$$\text{Now P.I.} = \frac{1}{D^2 + 3DD' + 2D'^2} (x + y) = \frac{1}{D^2} \left(1 + \frac{3D'}{D} + \frac{2 D'^2}{D^2} \right)^{-1} (x + y)$$

$$= \frac{1}{D^2} \left(1 - \frac{3D'}{D} + \ldots \right) (x + y) = \frac{1}{D^2} (x + y) - \frac{3}{D^3} D' (x + y)$$

$$= \frac{x^3}{6} + y \cdot \frac{x^2}{2} - \frac{3}{D^3} 1 = \frac{x^3}{6} + \frac{1}{2} x^2 y - 3 \cdot \frac{x^3}{6} = - \frac{1}{3} x^3 + \frac{1}{2} x^2 y.$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = \phi_1 (y - x) + \phi_2 (y - 2x) - \frac{1}{3} x^3 + \frac{1}{2} x^2 y.$$

Example 8: *Solve* $(D^2 - 6DD' + 9D'^2) z = 12x^2 + 36xy$.

Solution: A.E. is $m^2 - 6m + 9 = 0$ or $(m-3)^2 = 0$.

$\therefore$ $m = 3, 3.$

$\therefore$ $\text{C.F.} = \phi_1(y+3x) + x\,\phi_2(y+3x).$

Now
$$\text{P.I.} = \frac{1}{D^2 - 6DD' + 9D'^2}(12x^2 + 36xy)$$
$$= \frac{1}{(D-3D')^2}(12x^2 + 36xy)$$
$$= \frac{1}{D^2}\left(1 - \frac{3D'}{D}\right)^{-2}(12x^2 + 36xy)$$
$$= \frac{1}{D^2}\left\{1 + \frac{6D'}{D} + 27\frac{D'^2}{D^2} + \dots\right\}(12x^2 + 36xy)$$
$$= \frac{1}{D^2}(12x^2 + 36xy) + \frac{6}{D^3}D'(12x^2 + 36xy)$$
$$= x^4 + 6x^3y + \frac{6}{D^3}(36x)$$
$$= x^4 + 6x^3y + 9x^4 = 10x^4 + 6x^3y.$$

Hence the general solution of the given equation is
$$z = \text{C.F.} + \text{P.I.} = \phi_1(y+3x) + x\,\phi_2(y+3x) + 10x^4 + 6x^3y.$$

Example 9: *Solve* $r + (a+b)s + abt = xy$. **(Purvanchal 2014)**

Solution: The given equation can be written as
$$\{D^2 + (a+b)DD' + ab\,D'^2\}z = xy.$$

A.E. is $m^2 + (a+b)m + ab = 0$ or $(m+a)(m+b) = 0$.

$\therefore$ $m = -a, -b.$

$\therefore$ $\text{C.F.} = \phi_1(y-ax) + \phi_2(y-bx).$

Now
$$\text{P.I.} = \frac{1}{D^2 + (a+b)DD' + abD'^2}xy$$
$$= \frac{1}{D^2}\left\{1 + (a+b)\frac{D'}{D} + ab\frac{D'^2}{D^2}\right\}^{-1}xy$$
$$= \frac{1}{D^2}\left\{1 - (a+b)\frac{D'}{D}\dots\right\}(xy) = \frac{1}{D^2}(xy) - (a+b)\frac{1}{D^3}\{D'(xy)\}$$
$$= \frac{x^3y}{6} - (a+b)\frac{x^4}{24}.$$

Hence the general solution of the given equation is
$$z = \text{C.F.} + \text{P.I.} = \phi_1(y-ax) + \phi_2(y-bx) + \frac{x^3y}{6} - (a+b)\frac{x^4}{24}.$$

Example 10: *Solve*
$$(2D^2 - 5DD' + 2D'^2)z = 24(y-x).$$

Solution: A.E. is $2m^2 - 5m + 2 = 0$

or $(2m-1)(m-2) = 0.$

$\therefore \quad m = \frac{1}{2}, 2.$

$\therefore \quad \text{C.F.} = \phi_1(2y + x) + \phi_2(y + 2x).$

Now $\text{P.I.} = \frac{1}{2D^2 - 5DD' + 2D'^2} 24(y - x)$

$$= \frac{1}{2D^2}\left(1 - \frac{5D'}{2D} + \frac{D'^2}{D^2}\right)^{-1} 24(y - x)$$

$$= \frac{1}{2D^2}\left(1 + \frac{5D'}{2D} \dots\right) 24(y - x) = \frac{1}{2D^2} 24(y - x) + \frac{5}{4D^3} 24$$

$$= 12\left(y\frac{x^2}{2} - \frac{x^3}{6}\right) + \frac{5}{4} \cdot 24 \cdot \frac{x^3}{6} = 6x^2 y + 3x^3.$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = \phi_1(2y + x) + \phi_2(y + 2x) + 6x^2 y + 3x^3.$$

Example 11: *Solve* $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y.$

(Lucknow 2008)

Solution: The given equation can be written as

$$(D^2 - D'^2) z = x - y.$$

A.E. is $m^2 - 1 = 0.$

$\therefore \quad m = 1, -1.$

$\therefore \quad \text{C.F.} = \phi_1(y + x) + \phi_2(y - x).$

Now $\text{P.I.} = \frac{1}{D^2 - D'^2}(x - y) = \frac{1}{D^2}\left(1 - \frac{D'^2}{D^2}\right)^{-1}(x - y)$

$$= \frac{1}{D^2}\left(1 + \frac{D'^2}{D^2} + \dots\right)(x - y)$$

$$= \frac{1}{D^2}(x - y) = \frac{1}{6}x^3 - y \cdot \frac{1}{2}x^2.$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = \phi_1(y + x) + \phi_2(y - x) + \frac{1}{6}x^3 - \frac{1}{2}yx^2.$$

Example 12: *Solve*

$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial y^3} + \frac{\partial^3 u}{\partial z^3} - 3\frac{\partial^3 u}{\partial x\, \partial y\, \partial z} = x^3 + y^3 + z^3 - 3xyz. \quad \dots(1)$$

(Avadh 2012)

Solution: The given equation can be written as

$$(D_1^3 + D_2^3 + D_3^3 - 3D_1 D_2 D_3) u = x^3 + y^3 + z^3 - 3xyz$$

or $$(D_1 + D_2 + D_3)(D_1^2 + D_2^2 + D_3^2 - D_1 D_2 - D_2 D_3 - D_3 D_1) u = x^3 + y^3 + z^3 - 3xyz$$

or $$(D_1 + D_2 + D_3)(D_1 + \omega D_2 + \omega^2 D_3)(D_1 + \omega^2 D_2 + \omega D_3) u = x^3 + y^3 + z^3 - 3\,xyz$$

where ω is an imaginary cube root of unity.

Now consider $(D_1 + \omega D_2 + \omega^2 D_3) u = 0$. ...(2)

Auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{\omega} = \frac{dz}{\omega^2}.$$

These give $y - \omega x = a,\ z - \omega^2 x = b$.

Hence solution of (2) is $\phi_1(y - \omega x,\ z - \omega^2 x) = 0$.

Similarly $(D_1 + D_2 + D_3) u = 0$ and $(D_1 + \omega^2 D_2 + \omega D_3) u = 0$ give

$\phi_2(y - x, z - x) = 0$ and $\phi_3(y - \omega^2 x, z - \omega x) = 0$ respectively.

$\therefore$ C.F. of (1) $= \phi_1(y - \omega x, z - \omega^2 x) + \phi_2(y - x, z - x) + \phi_3(y - \omega^2 x, z - \omega x)$.

Now P.I. corresponding to x^3 is

$$= \frac{1}{D_1^3 + D_2^3 + D_3^3 - 3\,D_1 D_2 D_3} x^3$$

$$= \frac{1}{D_1^3}\left(1 + \frac{D_2^3}{D_1^3} + \dots\right)^{-1} x^3 = \frac{1}{D_1^3} x^3 = \frac{x^6}{4 \cdot 5 \cdot 6} = \frac{x^6}{120}.$$

Similarly particular integrals corresponding to y^3 and z^3 are $\dfrac{y^6}{120}$ and $\dfrac{z^6}{120}$ respectively and P.I. corresponding to $(-3xyz)$

$$= \frac{1}{D_1^3 + D_2^3 + D_3^3 - 3\,D_1 D_2 D_3} (-3xyz)$$

$$= \frac{1}{-3D_1 D_2 D_3}\left(1 - \frac{D_1^2}{D_2 D_3} \dots\right)^{-1} (-3xyz)$$

$$= -\frac{1}{3D_1 D_2 D_3} (-3\,xyz) = \frac{x^2 y^2 z^2}{8}.$$

Hence the general solution of the equation (1) is

$$z = \text{C.F.} + \text{P.I.}$$

Comprehensive Exercise 2

1. Solve $(D^2 - 2DD' + D'^2) z = 12xy$. **(Lucknow 2008; Avadh 10, 11; Kanpur 12)**

2. Solve $\dfrac{\partial^3 z}{\partial x^3} - \dfrac{\partial^3 z}{\partial y^3} = x^3 y^3$.

3. Solve $(D^2 - a^2 D'^2) z = x^2$.

4. Solve $\dfrac{\partial^2 z}{\partial x^2} + 3 \dfrac{\partial^2 z}{\partial x \, \partial y} + 2 \dfrac{\partial^2 z}{\partial y^2} = 2x + 3y$.

5. Solve $\dfrac{\partial^2 z}{\partial x^2} + 3 \dfrac{\partial^2 z}{\partial x \, \partial y} + 2 \dfrac{\partial^2 z}{\partial y^2} = 6(x + y)$.

6. Solve $(D^2 - DD' - 6D'^2) z = xy$.

Answers 2

1. $z = \phi_1(y + x) + x\,\phi_2(y + x) + 2x^3 y + x^4$

2. $z = \phi_1(y + x) + \phi_2(y + \omega x) + \phi_3(y + \omega^2 x) + \dfrac{1}{120} x^6 y^3 + \dfrac{x^9}{10080}$

3. $z = \phi_1(y + ax) + \phi_2(y - ax) + \dfrac{1}{12} x^4$

4. $z = \phi_1(y - x) + \phi_2(y - 2x) - \dfrac{7}{6} x^3 + \dfrac{3}{2} x^2 y$

5. $z = \phi_1(y - x) + \phi_2(y - 2x) + 3x^2 y - 2x^3$

6. $z = \phi_1(y - 2x) + \phi_2(y + 3x) + \dfrac{1}{6} x^3 y + \dfrac{1}{24} x^4$

6 Short Method for Finding the Particular Integral

When $f(x, y)$ is a function of $ax + by$, we have a shorter method for determining the particular integral.

Let $\quad f(x, y) = \phi(ax + by)$.

Then $\quad D^r \phi(ax + by) = a^r \phi^{(r)}(ax + by), \; D'^r \phi(ax + by) = b^r \phi^{(r)}(ax + by)$,

and $\quad D^r D'^s \phi(ax + by) = a^r b^s \phi^{(r+s)}(ax + by)$,

where $\phi^{(r)}$ is the rth derivative of ϕ w.r.t. $ax + by$ as a whole.

Since $F(D, D')$ is homogeneous in D and D' of degree n,

hence $\quad F(D, D')\,\phi(ax + by) = F(a, b)\,\phi^{(n)}(ax + by)$

or
$$\frac{1}{F(D, D')} \phi^{(n)}(ax + by) = \frac{1}{F(a, b)} \phi(ax + by),$$

provided that $F(a, b) \neq 0$.

Putting $ax + by = t$, we get $\dfrac{1}{F(D, D')} \phi^{(n)}(t) = \dfrac{1}{F(a, b)} \phi(t)$.

Now integrating both the sides n times w.r.t. 't', we get

$$\frac{1}{F(D, D')}\phi(t) = \frac{1}{F(a,b)}\iint \ldots \int \phi(t)\,dt \ldots dt, \text{where } t = ax + by .$$

Working Rule: To find the particular integral of an equation $F(D, D')z = \phi(ax + by)$, where $F(D, D')$ is a homogeneous function of D, D' of degree n, proceed as follows :

(i) Put $ax + by = t$ and integrate $\phi(t), n$ times with respect to t.

(ii) Find $F(a, b)$, replacing D, D' by a, b respectively in $F(D, D')$.

(iii) Now P.I. $= \dfrac{1}{F(a,b)} \times n$th integral of $\phi(t)$ w.r.t. 't', where $t = ax + by$.

Exceptional case when $F(a, b) = 0$.

If $F(a,b) = 0$ then the above method fails.

Now $F(a,b) = 0$ if and only if $(bD - aD')$ is a factor of $F(D, D')$.

Let us consider the equation $(bD - aD')z = x^r \phi(ax + by)$ ...(1)

or $\quad bp - aq = x^r \phi(ax + by)$.

Lagrange's auxiliary equations are

$$\frac{dx}{b} = \frac{dy}{-a} = \frac{dz}{x^r \phi(ax + by)}.$$

Taking the first two members, we get

$$a\,dx + b\,dy = 0.$$

$$\therefore \quad ax + by = c \quad \text{(constant)}.$$

Again taking the first and the last members, we get

$$dz = \frac{x^r}{b}\phi(ax + by)\,dx = \frac{x^r}{b}\phi(c)\,dx.$$

$$\therefore \quad z = \frac{x^{r+1}}{b(r+1)}\phi(c) = \frac{x^{r+1}}{b(r+1)}\phi(ax + by). \qquad \text{...(2)}$$

It gives the solution of (1).

From (1), $$z = \frac{1}{bD - aD'}x^r \phi(ax + by). \qquad \text{...(3)}$$

Thus $$\boldsymbol{\frac{1}{(bD - aD')}x^r \phi(ax + by) = \frac{x^{r+1}}{b(r+1)}\phi(ax + by)}. \qquad \text{...(4)}$$

Hence if $$z = \frac{1}{(bD - aD')^n}\phi(ax + by),$$

then $$z = \frac{1}{(bD - aD')^{n-1}}\left[\frac{1}{(bD - aD')}\phi(ax + by)\right]$$

$$= \frac{1}{(bD - aD')^{n-1}}\frac{x}{b}\phi(ax + by), \text{ using (4)}$$

$$= \frac{1}{(bD - aD')^{n-2}} \left[\frac{1}{(bD - aD')} \frac{x}{b} \phi(ax + by) \right]$$

$$= \frac{1}{(bD - aD')^{n-2}} \frac{1}{b} \cdot \frac{x^2}{2b} \phi(ax + by), \text{ using (4)}$$

… … … … … … … … … … … …

… … … … … … … … … … … …

$$= \frac{x^n}{b^n \, n!} \phi(ax + by), \text{ by repeated application of (4).}$$

Thus $$\boldsymbol{\frac{1}{(bD - aD')^n} \phi(ax + by) = \frac{x^n}{b^n \, n!} \phi(ax + by)}.$$

Illustrative Examples

Example 13: *Solve* $4r - 4s + t = 16 \log(x + 2y)$. **(Lucknow 2007; Purvanchal 09; Rohilkhand 14)**

Solution: The given equation can be written as

$$(4D^2 - 4DD' + D'^2) z = 16 \log(x + 2y).$$

A.E. is $4m^2 - 4m + 1 = 0$ or $(2m - 1)^2 = 0$.

$\therefore$ C.F. $= \phi_1(2y + x) + x \phi_2(2y + x)$.

Now $$\text{P.I.} = \frac{1}{(2D - D')^2} 16 \log(x + 2y)$$

$$= \frac{x^2}{2^2 \cdot 2!} 16 \log(x + 2y) = 2x^2 \log(x + 2y). \qquad [\because F(a, b) = 0]$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = \phi_1(2y + x) + x \phi_2(2y + x) + 2x^2 \log(x + 2y).$$

Example 14: *Solve* $(D^3 - 4D^2 D' + 4DD'^2) z = 4 \sin(2x + y)$. **(Rohilkhand 2011)**

Solution: A.E. is $m^3 - 4m^2 + 4m = 0$ or $m(m - 2)^2 = 0$.

$\therefore$ $m = 0, 2, 2$.

$\therefore$ C.F. $= \phi_1(y) + \phi_2(y + 2x) + x \phi_3(y + 2x)$.

Now $$\text{P.I.} = \frac{1}{D^3 - 4D^2 D' + 4DD'^2} 4 \sin(2x + y)$$

$$= \frac{1}{(D - 2D')^2} \cdot \frac{1}{D} 4 \sin(2x + y)$$

$$= \frac{1}{(D - 2D')^2} \{-2 \cos(2x + y)\}$$

$$= -2 \frac{1}{(D - 2D')^2} \cos(2x + y)$$

$$= -2 \cdot \frac{x^2}{2!} \cos(2x + y) = -x^2 \cos(2x + y). \qquad [\because\ F(a, b) = 0]$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.}$$
$$= \phi_1(y) + \phi_2(y + 2x) + x\,\phi_3(y + 2x) - x^2 \cos(2x + y).$$

Example 15: *Solve* $(D^2 - 2DD' + D'^2)z = e^{x+2y} + x^3.$

(Lucknow 2010; Rohilkhand 14)

Solution: A.E. is $m^2 - 2m + 1 = 0$ or $(m-1)^2 = 0.$

$\therefore$ $m = 1, 1.$

$\therefore$ $\text{C.F.} = \phi_1(y + x) + x\,\phi_2(y + x).$

Now $$\text{P.I.} = \frac{1}{(D - D')^2} e^{x+2y} + \frac{1}{(D - D')^2} x^3$$

$$= \frac{1}{(1-2)^2} \cdot e^{x+2y} + \frac{1}{D^2}\left(1 - \frac{D'}{D}\right)^{-2} x^3$$

$$= e^{x+2y} + \frac{1}{D^2}\left(1 + \frac{2D'}{D} + \ldots\right) x^3$$

$$= e^{x+2y} + \frac{1}{D^2} x^3 = e^{x+2y} + \frac{1}{20} x^5.$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = \phi_1(y + x) + x\,\phi_2(y + x) + e^{x+2y} + \frac{1}{20} x^5.$$

Example 16: *Solve* $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \cos mx \cos ny.$

(Lucknow 2006; Kanpur 07; Avadh 14)

Solution: The given equation can be written as

$$(D^2 + D'^2)z = \frac{1}{2}[\cos(mx + ny) + \cos(mx - ny)].$$

A.E. is $m^2 + 1 = 0.$ $\therefore$ $m = \pm i.$

$\therefore$ $\text{C.F.} = \phi_1(y + ix) + \phi_2(y - ix).$

Now $$\text{P.I.} = \frac{1}{2} \cdot \frac{1}{D^2 + D'^2} \cos(mx + ny) + \frac{1}{2} \frac{1}{D^2 + D'^2} \cos(mx - ny)$$

$$= \frac{1}{2} \frac{1}{m^2 + n^2} \{-\cos(mx + ny)\} + \frac{1}{2} \cdot \frac{1}{m^2 + n^2} \{-\cos(mx - ny)\}$$

$$= -\frac{1}{2(m^2 + n^2)} \{\cos(mx + ny) + \cos(mx - ny)\}.$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = \phi_1(y + ix) + \phi_2(y - ix)$$
$$- \frac{1}{2(m^2 + n^2)} [\cos(mx + ny) + \cos(mx - ny)].$$

Example 17: *Solve* $2r - s - 3t = 5e^x / e^y$. **(Rohilkhand 2009)**

Solution: The given equation can be written as

$$(2D^2 - DD' - 3D'^2)\, z = 5e^{x-y}.$$

A.E. is $\quad 2m^2 - m - 3 = 0$ or $(2m-3)(m+1) = 0$.

$\therefore \quad$ C.F. $= \phi_1(2y + 3x) + \phi_2(y - x)$.

Now $\quad$ P.I. $= \dfrac{1}{2D^2 - DD' - 3D'^2} 5e^{x-y} = \dfrac{1}{(D + D')(2D - 3D')} 5e^{x-y}$

$$= \frac{1}{D + D'} \cdot \frac{1}{2+3} \cdot 5e^{x-y} = \frac{1}{D + D'} e^{x-y}$$

$$= \frac{x}{1!} e^{x-y} = x e^{x-y}. \qquad [\because F(a, b) = 0]$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = \phi_1(2y + 3x) + \phi_2(y - x) + x e^{x-y}.$$

Example 18: *Solve* $(D^2 + 3DD' + 2D'^2)\, z = x + y$.

(Rohilkhand 2010; Agra 02; Avadh 14; Purvanchal 14)

Solution: A.E. is $m^2 + 3m + 2 = 0$ or $(m + 2)(m + 1) = 0$.

$\therefore \quad m = -1, -2.$

$\therefore \quad$ C.F. $= \phi_1(y - x) + \phi_2(y - 2x)$.

Now $\quad$ P.I. $= \dfrac{1}{D^2 + 3DD' + 2D'^2}(x + y)$

$$= \frac{1}{(1 + 3 \cdot 1 \cdot 1 + 2 \cdot 1)} \cdot \frac{(x + y)^3}{6} = \frac{(x + y)^3}{36}.$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = \phi_1(y - x) + \phi_2(y - 2x) + \frac{(x + y)^3}{36}.$$

Example 19: *Solve* $(D^2 - 5DD' + 4D'^2)\, z = \sin(4x + y)$. **(Lucknow 2006)**

Solution: A.E. is $m^2 - 5m + 4 = 0$

or $\quad (m - 1)(m - 4) = 0 \qquad \therefore \quad m = 1, 4.$

$\therefore \quad$ C.F. $= \phi_1(y + x) + \phi_2(y + 4x)$.

Now $\quad$ P.I. $= \dfrac{1}{D^2 - 5DD' + 4D'^2} \sin(4x + y)$

$$= \frac{1}{(D - 4D')} \cdot \frac{1}{(D - D')} \sin(4x + y)$$

$$= \frac{1}{(D - 4D')} \cdot \frac{1}{(4 - 1)} \{-\cos(4x + y)\}$$

$$= \frac{1}{D - 4D'} \{-\frac{1}{3} \cos(4x + y)\}$$

$$= -\frac{1}{3} \cdot x \cos (4x + y). \qquad [\because F(a, b) = 0]$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = \phi_1 (y + x) + \phi_2 (y + 4x) - \frac{1}{3} x \cos (4x + y).$$

Comprehensive Exercise 3

1. Solve $(D^2 - 6DD' + 9D'^2) z = 6x + 2y$.
2. Solve $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 12 (x + y)$.
3. Solve $(D^3 - 4D^2 D' + 4DD'^2) z = \cos (y + 2x)$. **(Purvanchal 2011)**
4. Solve $r - 2s + t = \sin (2x + 3y)$. **(Kanpur 2010; Meerut 13)**
5. Solve $(2D^2 - 5DD' + 2D'^2) z = 5 \sin (2x + y)$.
6. Solve $(D^2 + 2DD' + D'^2) z = e^{2x + 3y}$. **(Purvanchal 2007)**
7. Solve $(D^2 + D'^2) z = 30 (2x + y)$.
8. Solve $(D^3 - 2D^2 D' - DD'^2 + 2D'^3) z = e^{x + y}$.
9. Solve $(D^3 - 7DD'^2 - 6D'^3) z = x^2 + xy^2 + y^3 + \cos (x - y)$.
10. Solve $r + s - 2t = \sqrt{(2x + y)}$. **(Lucknow 2010)**
11. Solve $(D^3 - 4D^2 D' + 5DD'^2 - 2D'^3) z = e^{y + 2x} + (y + x)^{1/2}$.
12. Solve $(D^2 - 3DD' + 2D'^2) z = e^{2x - y} + e^{x + y} + \cos (x + 2y)$.
13. Solve $\log s = x + y$.
14. Solve $(D_x^3 - 7D_x D_y^2 - 6D_y^3) z = \sin (x + 2y) + e^{3x + y}$. **(Meerut 2013)**

Answers 3

1. $z = \phi_1 (y + 3x) + x \phi_2 (y + 3x) + \frac{1}{4} x^2 (3x + y)$
2. $z = \phi_1 (y + ix) + \phi_2 (y - ix) + (x + y)^3$
3. $z = \phi_1 (y) + \phi_2 (y + 2x) + x \phi_3 (y + 2x) + \frac{1}{4} x^2 \sin (y + 2x)$
4. $z = \phi_1 (y + x) + x \phi_2 (y + x) - \sin (2x + 3y)$
5. $z = \phi_1 (2y + x) + \phi_2 (y + 2x) - \frac{1}{3} \cdot 5x \cos (2x + y)$
6. $z = \phi_1 (y - x) + x \phi_2 (y - x) + \frac{1}{25} e^{2x + 3y}$

7. $z = \phi_1(y + ix) + \phi_2(y - ix) + (2x + y)^3$

8. $z = \phi_1(y - x) + \phi_2(y + x) + \phi_3(y + 2x) - \frac{1}{2} xe^{x+y}$

9. $z = \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y + 3x) + \frac{5}{72} x^6 + \frac{1}{60} x^5 + \frac{7}{20} x^5 y$
$+ \frac{1}{24} x^4 y^2 + \frac{1}{6} x^3 y^3 - \frac{1}{4} x \cos(x - y).$

10. $z = \phi_1(y + x) + \phi_2(y - 2x) + \frac{1}{15}(2x + y)^{5/2}$

11. $z = \phi_1(y + x) + x\,\phi_2(y + x) + \phi_3(y + 2x) + xe^{y+2x} - \frac{1}{3} x^2 (y + x)^{3/2}$

12. $z = \phi_1(y + x) + \phi_2(y + 2x) + \frac{1}{12} e^{2x-y} - xe^{x+y} - \frac{1}{3}\cos(x + 2y)$

13. $z = \phi_1(x) + \phi_2(y) + e^{x+y}$

14. $z = \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y + 3x) + \frac{1}{75}\cos(x + 2y) + \frac{x}{20} e^{3x+y}$

7 A General Method of Finding the Particular Integral

Consider the equation $(D - mD')z = \phi(x, y)$ or $p - mq = \phi(x, y)$.

Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi(x, y)}.$$

Taking the first two members, we get $dy + m\,dx = 0$.

$\therefore$ $y + mx = a$ (a constant).

Again taking the first and the last members, we get

$$dz = \phi(x, y)\,dx = \phi(x, a - mx)\,dx.$$

$$z = \int \phi(x, a - mx)\,dx.$$

Thus $\boldsymbol{z = \frac{1}{D - mD'}\phi(x, y) = \int \phi(x, a - mx)\,dx}$,

where after integration the constant a is to be replaced by $y + mx$, as the particular integral does not contain any arbitrary constant.

Now if the equation is $F(D, D')z = \phi(x, y)$

where $F(D, D') = (D - m_1 D')(D - m_2 D')\ldots(D - m_n D')$,

then $$\text{P.I.} = \frac{1}{F(D, D')}\phi(x, y)$$

$$= \frac{1}{D - m_1 D'} \cdot \frac{1}{D - m_2 D'} \cdots \frac{1}{D - m_n D'}\phi(x, y),$$

which can be evaluated by the repeated application of the above method.

Illustrative Examples

Example 20: *Solve $r + s - 6t = y \cos x$.* **(Avadh 2009, 10, 11; Purvanchal 07)**

Solution: The given equation can be written as

$$(D^2 + DD' - 6D'^2) z = y \cos x.$$

A.E. is $m^2 + m - 6 = 0$ or $(m+3)(m-2) = 0$.

$\therefore$ $m = 2, -3$.

$\therefore$ C.F. $= \phi_1 (y + 2x) + \phi_2 (y - 3x)$.

Now

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2} y \cos x = \frac{1}{(D - 2D')(D + 3D')} y \cos x$$

$$= \frac{1}{D - 2D'} \int (a + 3x) \cos x \, dx, \text{ where } y - 3x = a$$

$$= \frac{1}{D - 2D'} [a \sin x + 3x \sin x + 3 \cos x]$$

$$= \frac{1}{D - 2D'} [(y - 3x) \sin x + 3x \sin x + 3 \cos x]$$

$$= \frac{1}{(D - 2D')} [y \sin x + 3 \cos x]$$

$$= \int [(b - 2x) \sin x + 3 \cos x] \, dx, \text{ where } y + 2x = b$$

$$= - b \cos x - 2(- x \cos x + \sin x) + 3 \sin x$$

$$= - (y + 2x) \cos x + 2x \cos x + \sin x = - y \cos x + \sin x.$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = \phi_1 (y + 2x) + \phi_2 (y - 3x) - y \cos x + \sin x.$$

Example 21: *Solve $(D^2 + 2DD' + D'^2) z = 2 \cos y - x \sin y$.* **(Lucknow 2011)**

Solution: A.E. is $m^2 + 2m + 1 = 0$ or $(m + 1)^2 = 0$.

$\therefore$ $m = -1, -1$.

$\therefore$ C.F. $= \phi_1 (y - x) + x \phi_2 (y - x)$.

Now

$$\text{P.I.} = \frac{1}{D^2 + 2DD' + D'^2} (2 \cos y - x \sin y)$$

$$= \frac{1}{(D + D')(D + D')} (2 \cos y - x \sin y)$$

$$= \frac{1}{D + D'} \int [2 \cos (x + a) - x \sin (x + a)] \, dx, \text{ where } y - x = a$$

$$= \frac{1}{D + D'} [2 \sin (x + a) - \{- x \cos (x + a) + \sin (x + a)\}]$$

$$= \frac{1}{D + D'} [\sin (x + a) + x \cos (x + a)]$$

$$= \frac{1}{D + D'} [\sin y + x \cos y]$$

$$= \int [\sin (x + b) + x \cos (x + b)]\, dx, \text{ where } y - x = b$$

$$= -\cos (x + b) + x \sin (x + b) + \cos (x + b) = x \sin (x + b) = x \sin y.$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = \phi_1 (y - x) + x\, \phi_2 (y - x) + x \sin y.$$

Example 22: *Solve* $(D^2 - 2DD' - 15D'^2)\, z = 12\, x\, y.$ **(Purvanchal 2010)**

Solution: A.E. is $m^2 - 2m - 15 = 0$ or $(m - 5)(m + 3) = 0.$

$\therefore$ $m = 5, -3.$

$\therefore$ $\text{C.F.} = \phi_1 (y + 5x) + \phi_2 (y - 3x).$

Now $\text{P.I.} = \dfrac{1}{D^2 - 2DD' - 15D'^2} 12xy = \dfrac{1}{(D + 3D')(D - 5D')} 12xy$

$$= \frac{12}{D + 3D'} \int x (a - 5x)\, dx, \text{where } y + 5x = a$$

$$= \frac{12}{D + 3D'} \left\{ \frac{1}{2} ax^2 - \frac{5}{3} x^3 \right\}$$

$$= \frac{12}{D + 3D'} \left\{ \frac{1}{2} (y + 5x)\, x^2 - \frac{5}{3} x^3 \right\}$$

$$= \frac{2}{D + 3D'} \{3\, yx^2 + 5x^3\}$$

$$= 2 \int [3x^2 (3x + b) + 5x^3]\, dx \text{, where } y - 3x = b$$

$$= 2 \int (14x^3 + 3x^2 b)\, dx$$

$$= 2 \left(14 \cdot \frac{x^4}{4} + 3 \cdot \frac{x^3}{3} b \right) = 7x^4 + 2x^3 b$$

$$= 7x^4 + 2x^3 (y - 3x) = x^4 + 2x^3 y.$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = \phi_1 (y + 5x) + \phi_2 (y - 3x) + x^4 + 2x^3 y.$$

Example 23: *Solve* $r - t = \tan^3 x \tan y - \tan x \tan^3 y.$ **(Lucknow 2006; Avadh 10)**

Solution: The given equation can be written as

$$(D^2 - D'^2)\, z = \tan^3 x \tan y - \tan x \tan^3 y.$$

A.E. is $m^2 - 1 = 0.$ $\therefore$ $m = 1, -1.$

$\therefore$ $\text{C.F.} = \phi_1 (y + x) + \phi_2 (y - x).$

Now $\text{P.I.} = \dfrac{1}{D^2 - D'^2} (\tan^3 x \tan y - \tan x \tan^3 y)$

$$= \frac{1}{(D + D')(D - D')} (\tan^3 x \tan y - \tan x \tan^3 y)$$

$$= \frac{1}{D + D'} \int [\tan^3 x \tan (a - x) - \tan x \tan^3 (a - x)] \, dx,$$

where $y + x = a$

$$= \frac{1}{D + D'} \int [\{-1 + \sec^2 x\} \tan x \tan (a - x)]$$

$$- \tan x \tan (a - x) \{-1 + \sec^2 (a - x)\}] \, dx$$

$$= \frac{1}{D + D'} \left[\int \tan (a - x) \tan x \sec^2 x \, dx \right.$$

$$\left. - \int \tan x \tan (a - x) \sec^2 (a - x) \, dx \right]$$

$$= \frac{1}{D + D'} \left[\tan (a - x) \cdot \frac{\tan^2 x}{2} + \frac{1}{2} \int \sec^2 (a - x) \cdot \tan^2 x \, dx \right.$$

$$\left. + \tan x \cdot \frac{\tan^2 (a - x)}{2} - \frac{1}{2} \int \sec^2 x \tan^2 (a - x) \, dx \right]$$

$$= \frac{1}{2 (D + D')} [\tan^2 x \tan (a - x) + \tan x \tan^2 (a - x)$$

$$- \int \sec^2 x \{\sec^2 (a - x) - 1 - \sec^2 (a - x) \cdot (\sec^2 x - 1)\} \, dx]$$

$$= \frac{1}{2 (D + D')} [\tan^2 x \tan (a - x) + \tan x \tan^2 (a - x)$$

$$+ \int \{\sec^2 x - \sec^2 (a - x)\} \, dx]$$

$$= \frac{1}{2 (D + D')} [\tan^2 x \tan y + \tan x \tan^2 y + (\tan x + \tan y)]$$

$$= \frac{1}{2 (D + D')} [\tan x . \sec^2 y + \tan y . \sec^2 x]$$

$$= \frac{1}{2} \int [\tan x \sec^2 (b + x) + \tan (b + x) \sec^2 x] \, dx, \text{where } y - x = b$$

$$= \frac{1}{2} \tan x \tan (b + x) - \frac{1}{2} \int \sec^2 x \tan (b + x) \, dx$$

$$+ \frac{1}{2} \int \tan (b + x) \sec^2 x \, dx$$

$$= \frac{1}{2} \tan x \tan (b + x) = \frac{1}{2} \tan x \tan y.$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = \phi_1 (y + x) + \phi_2 (y - x) + \frac{1}{2} \tan x \tan y.$$

Comprehensive Exercise 4

1. Solve $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = \sin x.$

2. Solve $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \, \partial y} - 6 \dfrac{\partial^2 z}{\partial y^2} = y \sin x.$ **(Kanpur 2011)**

3. Solve $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial y^2} = \dfrac{4x}{y^2} - \dfrac{y}{x^2}$. **(Rohilkhand 2000; Agra 03)**

4. Solve $r - s - 2t = (2x^2 + xy - y^2) \sin xy - \cos xy$.

5. Solve $(D^2 - DD' - 2D'^2)\, z = (y - 1)\, e^x$.

Answers 4

1. $z = \phi_1 (y - x) - \cos x$
2. $z = \phi_1 (y + 2x) + \phi_2 (y - 3x) - (y \sin x + \cos x)$
3. $z = \phi_1(y + 2x) + \phi_2(y - 2x) + x \log y + y \log x + 3x$.
4. $z = \phi_1 (y + 2x) + \phi_2 (y - x) + \sin xy$.
5. $z = \phi_1 (y + 2x) + \phi_2 (y - x) + ye^x$.

8 Non-Homogeneous Linear Equations with Constant Coefficients

A linear partial differential equation which is not homogeneous is called a non-homogeneous linear equation.

Consider the differential equation $F(D, D')\, z = f(x, y)$. ...(1)

When $F(D, D')$ is a homogeneous function in D, D' it can always be resolved into linear factors. But the result is not always true when $F(D, D')$ is non-homogeneous. We classify linear differential operators $F(D, D')$ into two main types. These are :

(*i*) $F(D, D')$ is **reducible** if it can be written as the product of linear factors of the form $D + aD' + b$, with a, b constants.

(*ii*) $F(D, D')$ is **irreducible** if it cannot be so written.

Complementary function of non-homogeneous linear equation:

When $F(D, D')$ can be resolved into linear factors.

The complementary function of non-homogeneous linear equation (1) is the general solution of the equation

$$F(D, D')\, z = 0. \qquad \text{...(2)}$$

Let us consider a simple non-homogeneous equation

$$(D - mD' - k)\, z = 0 \qquad \text{...(3)}$$

or $$p - mq = kz.$$

Lagrange's auxiliary equations for it are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{kz}.$$

The first two members give

$$dy + m\,dx = 0.$$

$\therefore$ $$y + mx = a \text{ (a constant).}$$

Again taking the first and the third members, we get

$$\frac{dz}{z} = k\,dx.$$

$\therefore$ $$\log z = k\,x + \log b$$

or $$z = be^{kx}.$$

$\therefore$ $z = e^{kx}\,\phi(y + mx)$, which is the solution of (3).

If $F(D, D')$ can be factorised into **non-repeated** linear factors

$$(D - m_1 D' - k_1), (D - m_2 D' - k_2), \ldots, (D - m_n D' - k_n),$$

then the equation (2) is equivalent to

$$[(D - m_1 D' - k_1)(D - m_2 D' - k_2) \ldots (D - m_n D' - k_n)]\, z = 0 \qquad \ldots(4)$$

The complete solution of (2) or (4) is made up of the solutions of

$$(D - m_1 D' - k_1)\, z = 0, (D - m_2 D' - k_2)\, z = 0, \ldots, (D - m_n D' - k_n)\, z = 0.$$

Hence the general solution (complete solution) of (2) is

$$z = e^{k_1 x}\,\phi_1(y + m_1 x) + e^{k_2 x}\,\phi_2(y + m_2 x) + \ldots + e^{k_n x}\,\phi_n(y + m_n x). \qquad \ldots(5)$$

Note 1: If the operator $F(D, D')$ is reducible, the order in which the linear factors occur is immaterial.

Note 2: It can be shown that if the equation is

$$(\alpha D + \beta D' + \gamma)\, z = 0, \text{ then } z = e^{(-\gamma x/\alpha)}\,\phi(\beta x - \alpha y).$$

***F* (*D*, *D′*) has repeated factors.**

Let a factor $(D - mD' - k)$ occur twice in $F(D, D')$.

Consider the equation $(D - mD' - k)^2\, z = 0$. $\qquad \ldots(6)$

Let $(D - mD' - k)\, z = u$. Then (6) reduces to

$$(D - mD' - k)\, u = 0.$$

This gives $u = e^{kx}\,\phi_1(y + mx)$.

Hence $$(D - mD' - k)\, z = e^{kx}\,\phi_1(y + mx)$$

or $$p - mq = kz + e^{kx}\,\phi_1(y + mx).$$

Lagrange's auxiliary equations for this are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{kz + e^{kx}\,\phi_1(y + mx)}.$$

The first two members give, $y + mx = a$ (a constant).

From the first and the last members, we get

$$\frac{dz}{dx} - kz = e^{kx}\,\phi_1(y + mx) = e^{kx}\,\phi_1(a),$$

which is an ordinary linear equation.

$$\text{I. F.} = e^{\int -k\,dx} = e^{-kx}.$$

$$\therefore \quad ze^{-kx} = \int e^{-kx}\, e^{kx}\, \phi_1(a)\, dx + b = x\,\phi_1(a) + b\,.$$

$$\therefore \quad z = e^{kx}\,[\,x\,\phi_1(\,y + mx) + \phi_2(\,y + mx)], \qquad \ldots(7)$$

which is the general solution of (6).

Hence if $(D - mD' - k)$ occurs twice in $F(D, D')$ then the C.F. corresponding to this factor is

$$e^{kx}\,[x\,\phi_1(y + mx) + \phi_2\,(y + mx)] \qquad \ldots(8)$$

It can be shown that the general solution of

$$(D - mD' - k)^r\, z = 0$$

is $$z = e^{kx}\,[\,\phi_1(\,y + mx) + x\,\phi_2(\,y + mx) + \ldots + x^{\,r-1}\,\phi_r(\,y + mx)].$$

Hence if $(D - mD' - k)$ occurs r times in $F(D, D')$ then the C.F. corresponding to this factor is

$$e^{kx}\,[\phi_1(y + mx) + x\,\phi_2(y + mx) + \ldots + x^{r-1}\,\phi_r(y + mx)]\,. \qquad \ldots(9)$$

Case when linear factors of $F(D, D')$ are not possible:

In case $F(D, D')$ is irreducible, *i.e.*, it cannot be resolved into linear factors in D and D', the above methods of finding the complementary function fail. In such cases a trial method is used to find solutions.

9 Particular Integral

The complete solution of $F(D, D')\,z = f(x, y)$

is $$z = \text{C.F.} + \text{P.I.}$$

where $$\text{P.I.} = \frac{1}{F(D, D')}\, f(x, y).$$

The methods of obtaining particular integrals of non-homogeneous partial differential equations are very similar to those of ordinary linear equations with constant coefficients. We give some cases of finding the particular integrals.

Case I: $\dfrac{1}{F(D, D')}\, e^{\,ax+by} = \dfrac{1}{F(a, b)}\, e^{\,ax+by}$, if $F(a, b) \neq 0$.

We have $$D^r\,(e^{\,ax+by}) = a^r\, e^{\,ax+by},$$

$$D'^s\,(e^{\,ax+by}) = b^s\, e^{\,ax+by}$$

and $$(D^r\, D'^s)\,(e^{\,ax+by}) = a^r\, b^s\, e^{\,ax+by}.$$

$$\therefore \quad F(D, D')\,(e^{\,ax+by}) = F(a, b)\, e^{\,ax+by}.$$

Operating both the sides by $\dfrac{1}{F(D, D')}$, we get

$$e^{\,ax+by} = F(a, b)\,\frac{1}{F(D, D')}\, e^{\,ax+by}$$

$$\Rightarrow \qquad \frac{1}{F(D, D')} e^{ax+by} = \frac{1}{F(a, b)} e^{ax+by}, \text{ provided } F(a, b) \neq 0.$$

Case II: The value of $\frac{1}{F(D, D')} \sin(ax + by)$ is obtained by putting

$$D^2 = -a^2, DD' = -ab \text{ and } D'^2 = -b^2,$$

provided the denominator is not zero. Similar is the rule for cos $(ax + by)$.

Case III: $\frac{1}{F(D, D')} x^m y^n = [F(D, D')]^{-1} x^m y^n,$

which can be evaluated after expanding $[F(D, D')]^{-1}$ in ascending powers of D or D'.

Case IV: $\frac{1}{F(D, D')} (e^{ax+by} . V) = e^{ax+by} \frac{1}{F(D + a, D' + b)} V.$

It follows from the fact that

$$F(D, D') \{e^{ax+by} . V\} = e^{ax+by} F(D + a, D' + b) V.$$

Illustrative Examples

Example 24: *Solve* $r + 2s + t + 2p + 2q + z = 0.$

Solution: The given equation can be written as

$$(D^2 + 2DD' + D'^2 + 2D + 2D' + 1) z = 0$$

or $\qquad (D + D' + 1)^2 z = 0 \quad \text{or} \quad \{D - (-1) D' - (-1)\}^2 z = 0.$

There are repeated linear factors.

Hence the solution is

$$z = e^{-x} \phi_1(y - x) + xe^{-x} \phi_2(y - x).$$

Example 25: *Solve* $(D^2 - a^2 D'^2 + 2ab\, D + 2abD') z = 0.$

Solution: The given equation can be written as

$$(D + aD')(D - aD' + 2ab) z = 0.$$

There are distinct linear factors. Hence the solution is

$$z = \phi_1(y - ax) + e^{-2abx} \phi_2(y + ax).$$

Example 26: *Solve* $(2D^4 - 3D^2 D' + D'^2) z = 0.$

Solution: The given equation can be written as

$$(2D^2 - D')(D^2 - D') z = 0.$$

Let $z = Ae^{hx+ky}$ be the solution corresponding to

$$(D^2 - D') z = 0.$$

$$\therefore \qquad (D^2 - D') z = A h^2 e^{hx+ky} - Ake^{hx+ky}$$

$$= A(h^2 - k) e^{hx+ky} = 0$$

$$\Rightarrow \qquad h^2 - k = 0 \quad \Rightarrow \quad k = h^2.$$

Hence the general solution of $(D^2 - D')\, z = 0$ is

$$z = \Sigma A e^{hx + h^2 y}.$$

Similarly the general solution of $(2D^2 - D')\, z = 0$ is

$$z = \Sigma\, B e^{h'x + 2h'^2 y}.$$

Hence the most general solution of the given equation is

$$z = \Sigma\, A e^{hx + h^2 y} + \Sigma\, B e^{h'x + 2h'^2 y}.$$

Example 27: *Solve* $(D - D' - 1)(D - D' - 2)\, z = e^{2x - y} + x.$

Solution: $\text{C.F.} = e^x\, \phi_1(y + x) + e^{2x}\, \phi_2(y + x).$

Now P.I. corresponding to $e^{2x - y}$

$$= \frac{1}{(D - D' - 1)(D - D' - 2)}\, e^{2x - y}$$

$$= \frac{1}{(2 + 1 - 1)(2 + 1 - 2)}\, e^{2x - y} = \frac{1}{2}\, e^{2x - y}$$

and P.I. corresponding to x

$$= \frac{1}{(D - D' - 1)(D - D' - 2)}\, x$$

$$= \frac{1}{2}(1 - D + D')^{-1}\left(1 - \frac{1}{2} D + \frac{1}{2} D'\right)^{-1} x$$

$$= \frac{1}{2}(1 + D - D' \ldots)\left(1 + \frac{1}{2} D - \frac{1}{2} D' \ldots\right) x$$

$$= \frac{1}{2}\left(1 + D + \frac{1}{2} D + \ldots\right) x$$

$$= \frac{1}{2}\left(1 + \frac{3}{2} D\right) x = \frac{1}{2}\left(x + \frac{3}{2}\right).$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = e^x\, \phi_1(y + x) + e^{2x}\, \phi_2(y + x) + \frac{1}{2}\, e^{2x - y} + \frac{x}{2} + \frac{3}{4}.$$

Example 28: *Solve* $(D^2 - DD' + D' - 1)\, z = \cos(x + 2y) + e^y.$ **(Avadh 2010, 11)**

Solution: The given equation can be written as

$$(D - 1)(D - D' + 1)\, z = \cos(x + 2y) + e^y.$$

$$\therefore \qquad \text{C.F.} = e^x\, \phi_1(y) + e^{-x}\, \phi_2(y + x).$$

Now P.I. corresponding to $\cos(x + 2y)$

$$= \frac{1}{D^2 - DD' + D' - 1} \cos(x + 2y)$$

$$= \frac{1}{-1^2 - (-1 \cdot 2) + D' - 1} \cos(x + 2y)$$

$$= \frac{1}{D'} \cos(x + 2y) = \frac{D'}{D'^2} \cos(x + 2y)$$

$$= \frac{D'}{-2^2} \cos(x + 2y) = -\frac{1}{4} \{-2 \sin(x + 2y)\}$$

$$= \frac{1}{2} \sin(x + 2y)$$

and P.I. corresponding to e^y

$$= \frac{1}{D^2 - DD' + D' - 1} e^y = \frac{1}{(D-1)(D - D' + 1)} e^y$$

$$= \frac{1}{(0-1)(D - D' + 1)} e^y = -e^y \frac{1}{D - (D' + 1) + 1} 1$$

$$= -e^y \frac{1}{D - D'} 1 = -e^y \frac{1}{D} \left(1 - \frac{D'}{D}\right)^{-1} 1 = -e^y \frac{1}{D} 1 = -xe^y.$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = e^x \phi_1(y) + e^{-x} \phi_2(y + x) + \frac{1}{2} \sin(x + 2y) - xe^y.$$

Example 29: *Solve*

$$(D^2 - D'^2 - 3D + 3D') z = xy + e^{x+2y}.$$

(Rohilkhand 2007; Lucknow 08; Purvanchal 2007)

Solution: The given equation can be written as

$$(D - D')(D + D' - 3) z = xy + e^{x+2y}.$$

$\therefore$ $\quad$ $\text{C.F.} = \phi_1(y + x) + e^{3x} \phi_2(y - x).$

Now P.I. corresponding to xy

$$= \frac{1}{(D - D')(D + D' - 3)} xy$$

$$= -\frac{1}{3D} \left(1 - \frac{D'}{D}\right)^{-1} \left(1 - \frac{D}{3} - \frac{D'}{3}\right)^{-1} xy$$

$$= -\frac{1}{3D} \left(1 + \frac{D'}{D} + \dots\right) \left(1 + \frac{D}{3} + \frac{D'}{3} + \frac{2DD'}{9} + \dots\right) xy$$

$$= -\frac{1}{3D} \left(1 + \frac{D}{3} + \frac{D'}{3} + \frac{D'}{D} + \frac{D'}{3} + \frac{2DD'}{9} + \dots\right) xy$$

$$= -\frac{1}{3D} \left(xy + \frac{y}{3} + \frac{2}{3} x + \frac{1}{D} x + \frac{2}{9} \right)$$

$$= -\frac{1}{3} \left(\frac{1}{2} x^2 y + \frac{1}{3} xy + \frac{1}{3} x^2 + \frac{1}{6} x^3 + \frac{2}{9} x \right)$$

and P.I. corresponding to e^{x+2y}

$$= \frac{1}{(D - D')(D + D' - 3)} e^{x+2y}$$

$$= \frac{1}{(1-2)(D+D'-3)} e^{x+2y} = -\frac{1}{(D+D'-3)} e^{x+2y}$$

$$= -e^x \frac{1}{D'-2} e^{2y}, \quad \text{putting } D=1$$

$$= -e^x . e^{2y} \frac{1}{(D'+2)-2} 1 = -e^{x+2y} \frac{1}{D'} 1$$

$$= -ye^{x+2y}.$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = \phi_1(y+x) + e^{3x}\phi_2(y-x)$$

$$-\frac{1}{3}\left(\frac{1}{2}x^2y + \frac{1}{3}xy + \frac{1}{3}x^2 + \frac{1}{6}x^3 + \frac{2}{9}x\right) - ye^{x+2y}.$$

Example 30: *Solve* $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y} + xy + \sin(2x+y)$.

(Avadh 2013)

Solution: The given equation can be written as

$$(D+D')(D-2D'+2)z = e^{2x+3y} + xy + \sin(2x+y).$$

$\therefore$
$$\text{C.F.} = \phi_1(y-x) + e^{-2x}\phi_2(y+2x).$$

Now P.I. corresponding to e^{2x+3y}

$$= \frac{1}{(D+D')(D-2D'+2)} e^{2x+3y}$$

$$= \frac{1}{(2+3)(2-6+2)} e^{2x+3y} = -\frac{1}{10} e^{2x+3y},$$

P.I. corresponding to xy

$$= \frac{1}{(D+D')(D-2D'+2)} xy$$

$$= \frac{1}{2D}\left(1+\frac{D'}{D}\right)^{-1} \{1+\frac{1}{2}(D-2D')\}^{-1} xy$$

$$= \frac{1}{2D}\left(1-\frac{D'}{D}+\ldots\right)\{1-\frac{1}{2}(D-2D')+\frac{1}{4}(D-2D')^2+\ldots\} xy$$

$$= \frac{1}{2D}\left(1-\frac{D'}{D}+\ldots\right)(xy - \frac{1}{2}y + x - 1)$$

$$= \frac{1}{2D}(xy - \frac{1}{2}y + x - 1 - \frac{1}{2}x^2 + \frac{1}{2}x)$$

$$= \frac{1}{2D}\left(xy - \frac{1}{2}y - \frac{1}{2}x^2 + \frac{3}{2}x - 1\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}x^2y - \frac{1}{2}xy - \frac{1}{6}x^3 + \frac{3}{4}x^2 - x\right)$$

and P.I. corresponding to $\sin(2x+y)$

$$= \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'} \sin(2x+y)$$

$$= \frac{1}{-2^2 - (-2\cdot 1) - 2\cdot(-1^2) + 2D + 2D'} \sin(2x+y)$$

$$= \frac{1}{2(D+D')} \sin(2x+y) = \frac{D-D'}{2(D^2-D'^2)} \sin(2x+y)$$

$$= \frac{1}{2} \frac{D-D'}{-2^2-(-1^2)} \sin(2x+y) = -\frac{1}{6}(D-D') \sin(2x+y)$$

$$= -\frac{1}{6}\{2\cos(2x+y) - \cos(2x+y)\} = -\frac{1}{6}\cos(2x+y).$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.}$$

Example 31: *Solve* $(D-3D'-2)^2 z = 2e^{2x} \tan(y+3x)$.

Solution: $\text{C.F.} = e^{2x} \phi_1(y+3x) + xe^{2x} \phi_2(y+3x)$.

Now $\text{P.I.} = \dfrac{1}{(D-3D'-2)^2} 2e^{2x} \tan(y+3x)$

$$= 2e^{2x} \frac{1}{\{(D+2)-3D'-2\}^2} \tan(y+3x)$$

$$= 2e^{2x} \frac{1}{(D-3D')^2} \tan(y+3x)$$

$$= 2e^{2x} \cdot \frac{x^2}{2!} \tan(y+3x) = x^2 e^{2x} \tan(y+3x).$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.}$$

Example 32: *Solve* $(D^3 - 3DD' + D + 1) z = e^{2x+3y}$. **(Kanpur 2008)**

Solution: Since $D^3 - 3DD' + D + 1$ cannot be resolved into linear factors in D and D' hence $\text{C.F.} = \Sigma\, Ae^{hx+ky}$

where $h^3 - 3hk + h + 1 = 0$ *i.e.*, $k = \dfrac{h^3+h+1}{3h}$.

Now $\text{P.I.} = \dfrac{1}{D^3 - 3DD' + D + 1} e^{2x+3y}$

$$= \frac{1}{2^3 - 3\cdot 2\cdot 3 + 2 + 1} e^{2x+3y} = -\frac{1}{7} e^{2x+3y}.$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = \Sigma\, Ae^{hx+ky} - \frac{1}{7} e^{2x+3y},$$

where $h^3 - 3hk + h + 1 = 0$.

Example 33: *Solve* $(D^2 - DD' - 2D) z = \sin(3x+4y)$.

Solution: Since $D^2 - DD' - 2D$ cannot be resolved into linear factors in D and D', hence

$$\text{C.F.} = \Sigma\, Ae^{hx+ky}, \text{ where } h^2 - hk - 2h = 0 \text{ i.e., } k = h - 2.$$

Now $$\text{P.I.} = \frac{1}{D^2 - DD' - 2D} \sin(3x+4y)$$

$$= \frac{1}{-9-(-3.4)-2D} \sin(3x+4y)$$

$$= \frac{1}{3-2D} \sin(3x+4y) = \frac{3+2D}{9-4D^2} \sin(3x+4y)$$

$$= \frac{3+2D}{9-4(-9)} \sin(3x+4y)$$

$$= \frac{1}{45} \{3 \sin(3x+4y) + 2 \cdot 3 \cos(3x+4y)\}$$

$$= \frac{1}{15} \{\sin(3x+4y) + 2\cos(3x+4y)\}.$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.}$$

Example 34: *Solve* $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x \, \partial y} - 6 \dfrac{\partial^2 z}{\partial y^2} = x^2 \sin(x+y).$

Solution: The given equation can be written as

$$(D^2 + DD' - 6D'^2)\, z = x^2 \sin(x+y)$$

or $$(D - 2D')(D + 3D')\, z = x^2 \sin(x+y).$$

$\therefore$ $$\text{C.F.} = \phi_1(y+2x) + \phi_2(y-3x).$$

Now $$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2} x^2 \sin(x+y)$$

$$= \text{imaginary part of } \frac{1}{D^2 + DD' - 6D'^2} x^2 e^{i(x+y)}$$

$$= \text{I.P. of } e^{i(x+y)} \frac{1}{(D+i)^2 + (D+i)(D'+i) - 6(D'+i)^2} x^2$$

$$= \text{I.P. of } e^{i(x+y)} \frac{1}{D^2 + 3iD + DD' - 11D'i - 6D'^2 + 4} x^2$$

$$= \text{I.P. of } e^{i(x+y)} \cdot \frac{1}{4}\left[1 + \left(\frac{D^2}{4} + \frac{3iD}{4} + \frac{DD'}{4} - \frac{11D'i}{4} - \frac{6D'^2}{4}\right)\right]^{-1} x^2$$

$$= \text{I.P. of } e^{i(x+y)} \cdot \frac{1}{4}\left[1 - \frac{D^2}{4} - \frac{3iD}{4} - \frac{DD'}{4} + \frac{11D'i}{4} + \frac{6}{4}D'^2 + \frac{9i^2 D^2}{16} + \ldots\right] x^2$$

$$= \text{I.P. of } e^{i(x+y)} \cdot \frac{1}{4}\left[x^2 - \frac{1}{2} - \frac{3i}{2}x - \frac{9}{8}\right]$$

$$= \text{I.P. of } \frac{1}{4}\{\cos(x+y) + i\sin(x+y)\}\left\{x^2 - \frac{13}{8} - \frac{3ix}{2}\right\}$$

$$= \frac{1}{4}\left(x^2 - \frac{13}{8}\right)\sin(x+y) - \frac{3}{8}x\cos(x+y).$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = \phi_1(y+2x) + \phi_2(y-3x) + \frac{1}{4}\left(x^2 - \frac{13}{8}\right)\sin(x+y) - \frac{3}{8}x\cos(x+y).$$

Comprehensive Exercise 5

1. Solve $(D^2 - D'^2 + D - D')z = 0$.
2. Solve $DD'(D - 2D' - 3)z = 0$.
3. Solve $(D - 2D' - 1)(D - 2D'^2 - 1)z = 0$.
4. Solve $t + s + q = 0$. **(Avadh 2007)**
5. Solve $r - s + p = 1$. **(Avadh 2010)**
6. Solve $(D^2 - D')z = 2y - x^2$.
7. Solve $(D^2 + DD' + D' - 1)z = \sin(x + 2y)$. **(Rohilkhand 2008)**
8. Solve $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x\,\partial y} + 4\dfrac{\partial^2 z}{\partial y^2} + \dfrac{\partial z}{\partial x} - 2\dfrac{\partial z}{\partial y} = e^{x+y}$.
9. Solve $(D^2 - DD' - 2D)z = \sin(3x + 4y) - e^{2x+y}$. **(Avadh 2010)**
10. Solve $(D^2 - D')z = xe^{ax + a^2 y}$.
11. Solve $(D^2 - D'^2 + D - D')z = e^{2x+3y}$.
12. Solve $(D^2 - 4DD' + D - 1)z = e^{3x-2y}$.
13. Solve $(D^2 - D' - 1)z = x^2 y$.
14. Solve $(D^2 - DD' + D' - 1)z = 2\cos(x + 2y) - e^y$. **(Avadh 2009)**
15. Solve $\dfrac{\partial^2 z}{\partial x^2} - 3\dfrac{\partial^2 z}{\partial x\,\partial y} + 2\dfrac{\partial^2 z}{\partial y^2} - \dfrac{\partial z}{\partial x} + 2\dfrac{\partial z}{\partial y} = (2 + 4x)e^{-y}$.

Answers 5

1. $z = \phi_1(y + x) + e^{-x}\phi_2(y - x)$
2. $z = \phi_1(x) + \phi_2(y) + e^{3x}\phi_3(y + 2x)$

3. $z = e^x \phi_1(y + 2x) + \Sigma\, Ae^{(2k^2 + 1)x + ky}$

4. $z = \phi_1(x) + e^{-x} \phi_2(y - x)$

5. $z = \phi_1(y) + e^{-x} \phi_2(y + x) + x$

6. $z = \Sigma\, Ae^{hx + h^2 y} + yx^2$

7. $z = e^{-x} \phi_1(y) + e^x \phi_2(y - x) - \frac{1}{10}\{\cos(x + 2y) + 2\sin(x + 2y)\}$

8. $z = \phi_1(y + 2x) + e^{-x} \phi_2(y + 2x) + \frac{1}{2} ye^{x + y}$

9. $z = \phi_1(y) + e^{2x} \phi_2(y + x) + \frac{1}{15}\{\sin(3x + 4y) + 2\cos(3x + 4y)\} + \frac{1}{2} e^{2x + y}$.

10. $z = \Sigma\, Ae^{hx + h^2 y} + \left(\frac{x^2}{4a} - \frac{x}{4a^2}\right) e^{ax + a^2 y}$

11. $z = \phi_1(y + x) + e^{-x} \phi_2(y - x) - \frac{1}{6} e^{2x + 3y}$

12. $z = \Sigma\, Ae^{hx + ky} + \frac{1}{35} e^{3x - 2y}$, where $k = \frac{1}{4h}(h^2 + h - 1)$

13. $z = \Sigma\, Ae^{hx + (h^2 - 1)y} + x^2 - x^2 y - 2y + 4$

14. $z = e^x \phi_1(y) + e^{-x} \phi_2(y + x) + \sin(x + 2y) + x\, e^y$

15. $z = \phi_1(y + 2x) + e^x \phi_2(y + x) + e^{-y}(x + x^2)$

10 Equations Reducible to Linear Form with Constant Coefficients

A partial differential equation having variable coefficients can sometimes be reduced to an equation with constant coefficients by suitable substitutions. We shall discuss the reduction of an equation of the form

$$\left[A_0\, x^n \frac{\partial^n z}{\partial x^n} + A_1\, x^{n-1}\, y \frac{\partial^n z}{\partial x^{n-1}\, \partial y} + \ldots + A_n\, y^n \frac{\partial^n z}{\partial y^n} + \ldots\right] = f(x, y) \qquad \ldots(1)$$

into a linear equation with constant coefficients.

It should be noted that in the equation (1) the term $\frac{\partial^n z}{\partial x^r\, \partial y^{n-r}}$ is multiplied by the variable expression $x^r\, y^{n-r}$.

To transform the equation (1), we put $x = e^X, y = e^Y$ so that, $X = \log x$ and $Y = \log y$.

Then $$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x}=\frac{1}{x}\frac{\partial z}{\partial X}$$

or $$x\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}$$

$\therefore$ $$x\frac{\partial}{\partial x}\equiv\frac{\partial}{\partial X}\equiv D \text{ (say).} \qquad \ldots(2)$$

Now $$x\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial x}\right)=x^2\frac{\partial^2 z}{\partial x^2}+x\frac{\partial z}{\partial x}$$

$\Rightarrow$ $$x^2\frac{\partial^2 z}{\partial x^2}=\left(x\frac{\partial}{\partial x}-1\right)x\frac{\partial z}{\partial x}$$
$$=(D-1)\,Dz$$
$$=D\,(D-1)\,z. \qquad \ldots(3)$$

In general $$x^n\frac{\partial^n z}{\partial x^n}=D\,(D-1)\,(D-2)\ldots(D-n+1)\,z. \qquad \ldots(4)$$

Similarly differentiating w.r.t. y, we get

$$y\frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}=D'z \quad i.e., \quad y\frac{\partial}{\partial y}\equiv\frac{\partial}{\partial Y}\equiv D',$$

$$y^2\frac{\partial^2 z}{\partial y^2}=D'\,(D'-1)\,z,$$

$$y^n\frac{\partial^n z}{\partial y^n}=D'\,(D'-1)\ldots(D'-n+1)\,z.$$

Also $$xy\frac{\partial^2 z}{\partial x\,\partial y}=DD'z$$

and $$x^m y^n\frac{\partial^{m+n} z}{\partial x^m\,\partial y^n}=D\,(D-1)\ldots(D-m+1)\,D'\,(D'-1)\ldots(D'-n+1)\,z.$$

These substitutions reduce the equation (1) to an equation having constant coefficients and now it can easily be solved by the methods discussed for homogeneous and non-homogeneous linear equations with constant coefficients.

Illustrative Examples

Example 35: *Solve* $x^2\dfrac{\partial^2 z}{\partial x^2}+2xy\dfrac{\partial^2 z}{\partial x\,\partial y}+y^2\dfrac{\partial^2 z}{\partial y^2}=0.$

Solution: Putting $x=e^X$, $y=e^Y$

and denoting $\dfrac{\partial}{\partial X}$ and $\dfrac{\partial}{\partial Y}$ by D and D' respectively, the given equation transforms to

$$[\,D\,(D-1)+2DD'+D'\,(D'-1)]\,z=0$$

or $\qquad [D^2 + 2DD' + D'^2 - D - D']\, z = 0$

or $\qquad (D + D')(D + D' - 1)\, z = 0.$

Hence the general solution of the given equation is

$$z = \phi_1 (Y - X) + e^X \phi_2 (Y - X)$$

$$= \phi_1 (\log y - \log x) + x\, \phi_2 (\log y - \log x)$$

$$= \phi_1 \left(\log \frac{y}{x} \right) + x\, \phi_2 \left(\log \frac{y}{x} \right) = f_1 \left(\frac{y}{x} \right) + x\, f_2 \left(\frac{y}{x} \right).$$

Example 36: *Solve* $x^2 \dfrac{\partial^2 z}{\partial x^2} - 4xy \dfrac{\partial^2 z}{\partial x\, \partial y} + 4y^2 \dfrac{\partial^2 z}{\partial y^2} + 6y \dfrac{\partial z}{\partial y} = x^3 y^4.$

Solution: Putting $x = e^X$, $y = e^Y$

and denoting $\dfrac{\partial}{\partial X}$ and $\dfrac{\partial}{\partial Y}$ by D and D' respectively, the given equation transforms to

$$[D(D - 1) - 4DD' + 4D'(D' - 1) + 6D']\, z = e^{3X + 4Y}$$

or $\qquad (D^2 - D - 4DD' + 4D'^2 + 2D')\, z = e^{3X + 4Y}.$

or $\qquad (D - 2D')(D - 2D' - 1)\, z = e^{3X + 4Y}.$

$\therefore \qquad$ C.F. $= \phi_1 (Y + 2X) + e^X \phi_2 (Y + 2X)$

$$= \phi_1 (\log y + 2 \log x) + x\, \phi_2 (\log y + 2 \log x)$$

$$= \phi_1 (\log (yx^2)) + x\, \phi_2 (\log (yx^2))$$

$$= f_1 (yx^2) + x\, f_2 (yx^2).$$

Now $\qquad$ P.I. $= \dfrac{1}{(D - 2D')(D - 2D' - 1)} e^{3X + 4Y}$

$$= \frac{1}{(3 - 2.4)(3 - 2.4 - 1)} e^{3X + 4Y} = \frac{1}{30} x^3 y^4.$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = f_1 (yx^2) + x\, f_2 (yx^2) + \frac{1}{30} x^3 y^4.$$

Example 37: *Solve* $x^2 \dfrac{\partial^2 z}{\partial x^2} + 2xy \dfrac{\partial^2 z}{\partial x\, \partial y} + y^2 \dfrac{\partial^2 z}{\partial y^2} = (x^2 + y^2)^{n/2}.$

Solution: Putting $x = e^X$, $y = e^Y$ and denoting $\dfrac{\partial}{\partial X}$ and $\dfrac{\partial}{\partial Y}$ by D and D' respectively,

the given equation transforms to

$$\{D(D - 1) + 2DD' + D'(D' - 1)\}\, z = (e^{2X} + e^{2Y})^{n/2}$$

or $\qquad \{D^2 + 2DD' + D'^2 - D - D'\}\, z = (e^{2X} + e^{2Y})^{n/2}$

or $\qquad (D + D')(D + D' - 1)\, z = (e^{2X} + e^{2Y})^{n/2}.$

$\therefore$ $$\text{C.F.} = \phi_1(Y - X) + e^X \phi_2(Y - X)$$

$$= \phi_1(\log y - \log x) + x\,\phi_2(\log y - \log x)$$

$$= \phi_1\left(\log \frac{y}{x}\right) + x\,\phi_2\left(\log \frac{y}{x}\right) = f_1\left(\frac{y}{x}\right) + x\,f_2\left(\frac{y}{x}\right).$$

Now $$\text{P.I.} = \frac{1}{(D + D')(D + D' - 1)}(e^{2X} + e^{2Y})^{n/2}$$

$$= \frac{1}{(D + D')(D + D' - 1)} e^{nX}\{1 + e^{2(Y - X)}\}^{n/2}$$

$$= \frac{1}{(D + D')(D + D' - 1)}\left[e^{nX} + \frac{1}{2} n e^{(n-2)X + 2Y} + \frac{\frac{1}{2} n\left(\frac{1}{2} n - 1\right)}{2\,!} e^{(n-4)X + 4Y} + \ldots\right]$$

$$= \frac{e^{nX}}{n(n-1)}\left[1 + \frac{1}{2} n e^{2(Y - X)} + \ldots\right]$$

$$= \frac{e^{nX}[1 + e^{2(Y - X)}]^{n/2}}{n(n-1)} = \frac{(e^{2X} + e^{2Y})^{n/2}}{n(n-1)} = \frac{(x^2 + y^2)^{n/2}}{n(n-1)}.$$

Hence the general solution of the given equation is

$$z = \text{C.F.} + \text{P.I.} = f_1\left(\frac{y}{x}\right) + x\,f_2\left(\frac{y}{x}\right) + \frac{(x^2 + y^2)^{n/2}}{n(n-1)}.$$

Example 38: *Solve* $yt - q = xy$. **(Avadh 2009)**

Solution: The given equation can be written as

$$y\frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial y} = xy$$

or $$y^2\frac{\partial^2 z}{\partial y^2} - y\frac{\partial z}{\partial y} = xy^2.$$

Putting $x = e^X$, $y = e^Y$ and denoting $\dfrac{\partial}{\partial X}$ and $\dfrac{\partial}{\partial Y}$ by D and D' respectively the given equation transforms to

$$\{D'(D' - 1) - D'\} z = e^{X + 2Y}$$

or $$D'(D' - 2) z = e^{X + 2Y}.$$

$\therefore$ $$\text{C.F.} = \phi_1(X) + e^{2Y}\phi_2(X)$$

$$= \phi_1(\log x) + y^2\,\phi_2(\log x)$$

$$= f_1(x) + y^2 f_2(x).$$

Now
$$\text{P.I.} = \frac{1}{D'(D'-2)} e^{X+2Y}$$
$$= \frac{1}{2(D'-2)} e^{X+2Y}$$
$$= \frac{1}{2} e^{X+2Y} \frac{1}{D'+2-2} 1$$
$$= \frac{1}{2} e^{X+2Y} \frac{1}{D'} 1 = \frac{1}{2} Ye^{X+2Y}$$
$$= \frac{1}{2} xy^2 \log y.$$

Hence the general solution of the given equation is
$$z = \text{C.F.} + \text{P.I.} = f_1(x) + y^2 f_2(x) + \frac{1}{2} xy^2 \log y.$$

Example 39: *Solve* $\dfrac{1}{x^2}\dfrac{\partial^2 z}{\partial x^2} - \dfrac{1}{x^3}\dfrac{\partial z}{\partial x} = \dfrac{1}{y^2}\dfrac{\partial^2 z}{\partial y^2} - \dfrac{1}{y^3}\dfrac{\partial z}{\partial y}$.

Solution: Put $\frac{1}{2} x^2 = X$ and $\frac{1}{2} y^2 = Y$,

so that $x\,dx = dX$ and $y\,dy = dY$.

Hence
$$\frac{\partial z}{\partial X} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial X} = \frac{1}{x}\frac{\partial z}{\partial x}$$

and
$$\frac{\partial^2 z}{\partial X^2} = \frac{\partial}{\partial X}\left(\frac{\partial z}{\partial X}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z}{\partial x}\right)\cdot\frac{\partial x}{\partial X}$$
$$= \frac{1}{x}\cdot\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z}{\partial x}\right)$$
$$= -\frac{1}{x^3}\frac{\partial z}{\partial x} + \frac{1}{x^2}\frac{\partial^2 z}{\partial x^2}$$

i.e.,
$$\frac{1}{x^2}\frac{\partial^2 z}{\partial x^2} - \frac{1}{x^3}\frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial X^2}.$$

Similarly
$$\frac{1}{y^2}\frac{\partial^2 z}{\partial y^2} - \frac{1}{y^3}\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial Y^2}.$$

Thus the given equation transforms to
$$\frac{\partial^2 z}{\partial X^2} = \frac{\partial^2 z}{\partial Y^2} \quad \text{or} \quad \frac{\partial^2 z}{\partial X^2} - \frac{\partial^2 z}{\partial Y^2} = 0$$

or
$$(D^2 - D'^2)\,z = 0$$

where
$$D \equiv \frac{\partial}{\partial X}, \; D' \equiv \frac{\partial}{\partial Y}$$

or $$(D+D')(D-D')z=0.$$

Hence the solution is

$$z=\phi_1(Y-X)+\phi_2(Y+X)$$
$$=f_1(y^2-x^2)+f_2(y^2+x^2).$$

Comprehensive Exercise 6

1. Solve $x^2\dfrac{\partial^2 z}{\partial x^2}-y^2\dfrac{\partial^2 z}{\partial y^2}-y\dfrac{\partial z}{\partial y}+x\dfrac{\partial z}{\partial x}=0.$

2. Solve $x^2\dfrac{\partial^2 z}{\partial x^2}+2xy\dfrac{\partial^2 z}{\partial x\,\partial y}+y^2\dfrac{\partial^2 z}{\partial y^2}+x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}-z=0.$

3. Solve $x^2r-3xys+2y^2t+px+2qy=x+2y.$

4. Solve $x^2\dfrac{\partial^2 z}{\partial x^2}-y^2\dfrac{\partial^2 z}{\partial y^2}=x^2y.$

5. Solve $(x^2D^2+2xyDD'+y^2D'^2)z=x^my^n.$

6. Solve $x^2\dfrac{\partial^2 z}{\partial x^2}+2xy\dfrac{\partial^2 z}{\partial x\,\partial y}-x\dfrac{\partial z}{\partial x}=\dfrac{x^3}{y^2}\cdot$

7. Solve $x^2r-y^2t+px-qy=\log x.$ **(Lucknow 2006)**

Answers 6

1. $z=f_1(xy)+f_2\left(\dfrac{y}{x}\right)$

2. $z=x\,f_1\left(\dfrac{y}{x}\right)+\dfrac{1}{x}f_2\left(\dfrac{y}{x}\right)$

3. $z=f_1(xy)+f_2(x^2y)+x+y$

4. $z=f_1(xy)+x\,f_2(y/x)+\dfrac{1}{2}x^2y$

5. $z=f_1\left(\dfrac{y}{x}\right)+x\,f_2\left(\dfrac{y}{x}\right)+\dfrac{1}{(m+n)(m+n-1)}x^my^n$

6. $z=f_1(y)+x^2f_2(y/x^2)-\dfrac{1}{9}(x^3/y^2)$

7. $z=f_1(y/x)+f_2(xy)+\dfrac{1}{6}(\log x)^3$

11 Classification of Linear Partial Differential Equation of Second Order

The linear partial differential equation of the second order in n independent variables $x_1, x_2, \ldots, x_n$ can be written as

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + c\, u = 0 \qquad \ldots(1)$$

where a_{ij}, b_i and c are constants or functions of $x_1, x_2, \ldots, x_n$.

Let δ_i represent $\frac{\partial}{\partial x_i}$, for $i = 1, 2, \ldots, n$

and $\quad \delta_i \delta j$ represent $\frac{\partial^2}{\partial x_i \partial x_j}$, for $i = 1, 2, 3, \ldots, n$ and $j = 1, 2, 3, \ldots, n$.

Consider the operator $\phi = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \delta_i \delta_j \qquad \ldots(2)$

for all real values of δ_i positive or negative.

Now at a point $x_1, x_2, \ldots, x_n$ we call the linear partial differential equation given by (1) as :

(i) **elliptic** if ϕ is positive for all real values of δ_i and it reduces to zero only when all δ_i's are zero.

(ii) **hyperbolic** if ϕ can be both positive or negative.

(iii) **parabolic** if the determinant Δ vanishes, where

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0 .$$

If a_{ij} are functions of $x_1, x_2, \ldots, x_n$ the same differential equation can be elliptic, hyperbolic and parabolic at different points.

If a_{ij} are constants, the equation will have the same nature throughout.

12 Classification of Linear Partial Differential Equation of Second Order in Two Independent Variables

Let us consider the equation of second order in two independent variables x and y

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 , \qquad \ldots(1)$$

where A is positive.

Here $\phi = A\delta_1^2 + B\,\delta_1\,\delta_2 + C\,\delta_2^2$.

The equation (1) is

(i) elliptic if $B^2 - 4AC < 0$,

(ii) hyperbolic if $B^2 - 4AC > 0$,

(iii) parabolic if $B^2 - 4AC = 0$.

Note 1: If A, B, C are constants then the nature of the equation (1) will be the same in the whole region *i.e.*, for all values of x and y. The nature will depend on $B^2 - 4\,AC$.

The equation (1) will be elliptic if $B^2 - 4AC < 0$.

The equation (1) will be hyperbolic if $B^2 - 4AC > 0$.

The equation (1) will be parabolic if $B^2 - 4AC = 0$.

Note 2: If A, B, C are functions of x and y then the nature of equation (1) will not be same in the whole region *i.e.*, for all values of x and y.

The equation (1) will be elliptic in the region where $B^2 - 4AC < 0$.

The equation (1) will be hyperbolic in the region where $B^2 - 4AC > 0$.

The equation (1) will be parabolic in the region where $B^2 - 4AC = 0$.

Illustrative Examples

Example 40: *Classify the operator*

(i) $$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x\,\partial t} + \frac{\partial^2 u}{\partial x^2}$$ **(Meerut 2011)**

(ii) $$\frac{\partial^2 u}{\partial t^2} - 4\,\frac{\partial^2 u}{\partial x\,\partial t} + \frac{\partial^2 u}{\partial x^2}$$

(iii) $$\frac{\partial^2 u}{\partial t^2} + 4\,\frac{\partial^2 u}{\partial x\,.\,\partial t} + 4\,\frac{\partial^2 u}{\partial x^2}.$$

Solution: (i) Here $A = 1, B = 1, C = 1$ and so $B^2 - 4\,AC = 1 - 4 = -3 < 0$.

Therefore, the given operator is elliptic.

(ii) Here $A = 1,\ B = -4,\ C = 1$ and so $B^2 - 4\,AC = 16 - 4 = 12 > 0$.

Therefore the given operator is hyperbolic.

(iii) Here $A = 1,\ B = 4,\ C = 4$ and so $B^2 - 4\,AC = 16 - 16 = 0$.

Therefore the given operator is parabolic.

Example 41: *Classify the equation*

$$(1-x^2)\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x\,\partial y}+(1-y^2)\frac{\partial^2 z}{\partial y^2}+x\frac{\partial z}{\partial x}+3x^2y\frac{\partial z}{\partial y}-2z=0.$$

Solution: Consider the operator

$$\phi=A\delta_1^2+B\delta_1\delta_2+C\delta_2^2,\quad \text{where } \delta_1\equiv\frac{\partial}{\partial x},\delta_2\equiv\frac{\partial}{\partial y}.$$

Here $\qquad A=1-x^2,\ B=-2xy,\ C=1-y^2,$

and so $\qquad B^2-4AC=4x^2y^2-4(1-x^2)(1-y^2)=4(-1+x^2+y^2).$

Since A, B, C are functions of x and y, the given differential equation is hyperbolic in the region where

$$B^2-4AC>0 \ \ i.e.,\ x^2+y^2>1,$$

parabolic in the region where

$$B^2-4AC=0 \ \ i.e., \text{ at points on the circle } x^2+y^2=1,$$

and elliptic in the region where

$$B^2-4AC<0 \ \ i.e.,\ x^2+y^2<1.$$

Comprehensive Exercise 7

1. Classify the following equations :

(i) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0$ (Laplace's equation) **(Meerut 2007, 08, 11)**

(ii) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=\dfrac{1}{C^2}\dfrac{\partial^2 u}{\partial t^2}$ (Wave equation) **(Meerut 2007, 10)**

(iii) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=\dfrac{1}{C^2}\dfrac{\partial u}{\partial t}.$ (Heat equation) **(Meerut 2007)**

2. Find where the following operator is hyperbolic, parabolic and elliptic

(i) $\dfrac{\partial^2 u}{\partial t^2}+t\dfrac{\partial^2 u}{\partial x\,\partial t}+x\dfrac{\partial^2 u}{\partial x^2}$

(ii) $x^2\dfrac{\partial^2 u}{\partial t^2}-\dfrac{\partial^2 u}{\partial x^2}+u$

(iii) $t\dfrac{\partial^2 u}{\partial t^2}+2\dfrac{\partial^2 u}{\partial x\,\partial t}+x\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial u}{\partial x}.$

3. Show that the equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ is hyperbolic. **(Rohilkhand 2008, 10)**

4. Classify the following as elliptic, parabolic or hyperbolic :

(i) $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial z}{\partial y}$ **(ii)** $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial^2 z}{\partial y^2}$

(iii) $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0.$

5. Classify the partial differential equation

$$x^2 \frac{\partial^2 u}{\partial t^2} + 3 \frac{\partial^2 u}{\partial x \partial t} + x \frac{\partial^2 u}{\partial x^2} + 17 \frac{\partial u}{\partial t} = 100 u.$$

6. Classify the following differential equation as to type in the second quadrant of xy-plane

$$\sqrt{(y^2 + x^2)}\, u_{xx} + 2(x - y)\, u_{xy} + \sqrt{(y^2 + x^2)}\, u_{yy} = 0.$$

Answers 7

1. **(i)** elliptic **(ii)** hyperbolic **(iii)** parabolic
2. hyperbolic if $t^2 > 4x$, parabolic if $t^2 = 4x$ and elliptic if $t^2 < 4x$
4. **(i)** parabolic **(ii)** hyperbolic **(iii)** elliptic
5. hyperbolic if $9 > 4x^3$, parabolic if $9 = 4x^3$ and elliptic if $9 < 4x^3$
6. hyperbolic

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Out of the following four P.D.E., the equation which is linear:

(a) $\dfrac{\partial^3 z}{\partial x^3} - 3 \dfrac{\partial^2 z}{\partial x^2} \dfrac{\partial z}{\partial y} + 8 \dfrac{\partial^2 z}{\partial y^2} = \sin x$ (b) $\dfrac{\partial^2 z}{\partial x^2} + \left(\dfrac{\partial z}{\partial y}\right)^2 + 9z = 0$

(c) $\dfrac{\partial^2 z}{\partial x^2} + 5 \dfrac{\partial^2 z}{\partial y^2} = 0$ (d) none of these

(Rohilkhand 2003)

2. The A.E. of the equation $2r + 5s + 2t = 0$ is
   (a) $2m^2 + 5m + 2 = 0$ (b) $2m^2 - 5m + 2 = 0$
   (c) $2m^2 + 5m - 2 = 0$ (d) $2m^2 - 5m - 2 = 0$
3. The general solution of the differential equation $(D^2 - 2\,DD' + D'^2)\,z = 0$ is
   (a) $z = c_1\,e^{x} + c_2\,e^{y}$
   (b) $z = c\,e^{(x + c_2\,y)}$
   (c) $z = \phi_1(x + y) + \phi_2(y + x)$
   (d) $z = \phi_1(y + x) + x\,\phi_2(y + x)$ **(Rohilkhand 2002)**
4. The C.F. of the equation $(D^3 - 3\,DD'^2 + 2\,D'^3)\,z = (x + 2y)^{1/2}$ is
   (a) $\phi_1(y + x) + \phi_2(y - 2x)$
   (b) $\phi_1(y + x) + x\,\phi_2(y + x) + \phi_3(y + 2x)$
   (c) $\phi_1(y + x) + x\,\phi_2(y + x) + \phi_3(y - 2x)$
   (d) none of these.
5. The P.I. of the differential equation $(D^2 + 3\,DD' + 2\,D'^2)\,z = x + y$ is
   (a) $\dfrac{(x + y)^3}{6}$ (b) $\dfrac{(x + y)^3}{12}$
   (c) $\dfrac{(x + y)^3}{36}$ (d) none of these
6. The C.F. of the equation $\log s = x + y$ is
   (a) $\phi_1(x) + \phi_2(y)$ (b) $\phi_1(y) + x\,\phi_2(y)$
   (c) $\phi_1(x) + x\,\phi_2(y)$ (d) none of these
7. The C.F. of the equation $(D^2 + 2\,DD' + D'^2)\,z = 2\cos y - x\sin y$ is
   (a) $\phi_1(y + x) + \phi_2(y - x)$ (b) $\phi_1(y + x) + x\,\phi_2(y + x)$
   (c) $\phi_1(y - x) + x\,\phi_2(y - x)$ (d) none of these
8. The solution of non-homogeneous equation $(D - mD' - k)\,z = 0$ is
   (a) $z = e^{kx}\,\phi(y + mx)$ (b) $z = e^{kx}\,\phi(y - mx)$
   (c) $z = e^{x}\,\phi(y + mx)$ (d) $z = e^{x}\,\phi(y - mx)$
9. If $z = Ae^{hx + ky}$ be the solution of $(D - D'^2)\,z = 0$ then
   (a) $h = k$ (b) $h = k^2$
   (c) $h = k^3$ (d) none of these
10. The C.F. of $(D^2 - DD' - 2\,D)\,z = \sin(3x + 4y)$ is $\Sigma\,Ae^{hx + ky}$ where
   (a) $h^2 - hk - 2h = 0$ (b) $h^2 + hk - 2h = 0$
   (c) $h^2 - hk + 2h = 0$ (d) none of these

11. P.I. of the equation $(D^3 - 3\,DD' + D' + 1)\,z = e^{4x+5y}$ is

(a) $\frac{1}{4}e^{4x+5y}$ (b) $\frac{1}{5}e^{4x+5y}$

(c) $\frac{1}{10}e^{4x+5y}$ (d) none of these

12. The equation

$$A_0\,x^n\frac{\partial^n z}{\partial x^n} + A_1\,x^{n-1}y\frac{\partial^n z}{\partial x^{n-1}\partial y} + \ldots + A_n\,y^n\frac{\partial^n z}{\partial y^n} + \ldots = f(x, y)$$

can be reduced into a linear equation with constant coefficients in the variables X and Y by substitutions

(a) $x = \log X,\ y = \log Y$ (b) $x = e^X,\ y = e^Y$

(c) $x = e^{2X},\ y = e^{2Y}$ (d) none of these

13. The partial differential equation

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + f(x, y, u_x, u_y) = 0$$

is hyperbolic if

(a) $B^2 - 4AC = 0$ (b) $B^2 - 4AC < 0$

(c) $B^2 - 4AC > 0$ (d) $A = B = C = 0$

14. The partial differential equation

$$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + f(x, y, u_x, u_y) = 0$$

is parabolic if

(a) $B^2 - 4AC = 0$ (b) $B^2 - 4AC < 0$

(c) $B^2 - 4AC > 0$ (d) $A = B = C = 0$

15. The partial differential equation $2\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x\partial y} + 3\frac{\partial^2 u}{\partial y^2} = 0$

is classified as

(a) parabolic (b) elliptic

(c) hyperbolic (d) none of these

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. In the homogeneous linear partial differential equation with constant coefficients all the partial derivatives appearing in the equation are of the ...... order.
2. If u is the complementary function and z_1 a particular integral of a linear partial differential equation $F(D, D')\,z = f(x, y)$ then ...... is a general solution of the equation.
3. A linear partial differential equation which is not homogeneous is called a ...... linear equation.

4. The A.E. of a linear homogeneous nth order partial differential equation with constant coefficients will give in general ...... roots.
5. The A.E. of $(D^2 + D'^2)\, z = 30\,(2x + y)$ is ....
6. The partial differential equation
$$A u_{xx} + B u_{xy} + C u_{yy} + f(x, y, u_x, u_y) = 0$$
is classified as elliptic if ....
7. The partial differential equation $2\dfrac{\partial^2 u}{\partial x^2} + 6\dfrac{\partial^2 u}{\partial x \partial y} + 3\dfrac{\partial^2 u}{\partial y^2} = 0$
is classified as ....
8. The equation $A\dfrac{\partial^2 u}{\partial x^2} + B\dfrac{\partial^2 u}{\partial x\, \partial y} + C\dfrac{\partial^2 u}{\partial y^2} + f\left(x, y, u, \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right) = 0$
is elliptic, hyperbolic or parabolic according as $B^2 - 4AC$.......

True or False

Write 'T' for true and 'F' for false statement.

1. The A.E. of a linear homogeneous nth order partial differential equation with constant coefficients of the form $F(D, D')\, z = f(x, y)$ is obtained by putting $D = 1, D' = m$ in $F(D, D') = 0$.
2. The value of $\dfrac{1}{(bD - aD')} x^r\, \phi(ax + by) = \dfrac{x^{r+1}}{b\,(r+1)}\, \phi(ax + by)$.
3. The solution of $(D^2 - 2aDD' + a^2 D'^2)\, z = 0$ is
$$z = \phi_1(y + ax) + x\, \phi_2(y + ax).$$
4. In case of non-homogeneous linear partial differential equation with constant coefficients the value of $\dfrac{1}{F(D, D')} e^{(ax+by)} . V = e^{ax+by} \dfrac{1}{(D - a, D' - b)} V$.
5. The C.F. of $(D^3 - 3DD' + D + 1)\, z = e^{2x+3y}$ is $\Sigma\, Ae^{hx+ky}$ where $h^3 - 3hk + h + 1 = 0$.
6. In the equation $(D - mD')\, z = \phi(x, y)$ the value of
$$z = \frac{1}{D - mD'} \phi(x, y) = \int \phi(x, a - mx)\, dx, \text{ where } a = y + mx.$$

Answers

Multiple Choice Questions

1.	(c)	2.	(a)	3.	(d)
4.	(c)	5.	(c)	6.	(a)
7.	(c)	8.	(a)	9.	(b)

10. (a)	**11.** (c)	**12.** (b)
13. (c)	**14.** (a)	**15.** (b)

Fill in the Blank(s)

1. same

2. $u + z_1$

3. non-homogeneous.

4. n

5. $m^2 + 1 = 0$

6. $B^2 - 4AC < 0$

7. hyperbolic

8. < 0 , > 0, $= 0$

True or False

1. F	**2.** T	**3.** T
4. F	**5.** T	**6.** T

□