UNIT (V)

Homogeneous Linear D.E with ~~const~~ Variable Cofficients.

General form. $\left(x^n D^n + a_1 x^{n-1} D^{n-1} + a_2 x^{n-2} D^{n-2} + \text{———} - + a_n\right) y = Q.$

where

$a_1, a_2, \ldots a_n$ = Constants

Q = Either const. or function of x.

Solving Method: Variable interchanging Method.

Put $x = e^z$ $\Rightarrow z = \log x$, we get

$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$

or $\frac{dy}{dx} = \frac{1}{e^z}\frac{dy}{dz}$

~~write~~ $\boxed{x\frac{dy}{dx} = \frac{dy}{dz}} \Rightarrow \boxed{x\frac{d}{dx} = \Delta}$

i.e. $xDy = \Delta y \Rightarrow xD = \Delta$

and $\boxed{x^2\frac{d^2y}{dx^2} = \Delta(\Delta-1)y}$

Similarly $x^3\frac{d^3y}{dx^3} = \Delta(\Delta-1)(\Delta-2)y$.

$x^n\frac{d^ny}{dx^n} = \Delta(\Delta-1)(\Delta-2) - \cdots (\Delta-n+1)y.$

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Ex: solve $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 3y = x^2 + x.$

~~it can also be written as~~

put $x = e^z$ then.

$$\left[D(D-1)(D-2) + 2D(D-1) + 3D - 3\right]y = e^{2z} + e^{z}$$

$$\Rightarrow \quad (D-1)(D^2+3)y = e^{2z} + e^{z}.$$

its Auxillary eqn is $(m-1)(m^2+3) = 0$

$$\Rightarrow m = 1, \pm\sqrt{3}\,i$$

C.F. $(y) = c_1 e^z + [c_2 \cos\sqrt{3}z + c_3 \sin\sqrt{3}z]$

$$P.I = \frac{1}{f(D)}(e^{2z} + e^{z})$$

Notes

$$= \frac{1}{(D-1)(D^2+3)}(e^{2z} + e^{z})$$

Success is simply a matter of luck Ask any failure

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$$= \frac{1}{(D-1)(D^2+3)} e^{2z} + \frac{1}{(D-1)(D^2+3)} e^{z}$$

$$= \frac{e^{2z}}{(2-1)(2^2+3)} + \frac{e^{z}}{(1-1)(1^2+3)}$$

$$= \frac{1}{7} e^{2z} + \frac{1}{4} \cdot \frac{1}{(D-1)} e^{z}$$

$$= \frac{1}{7} e^{z} + \frac{1}{4} e^{z} \left(\frac{1}{D+1-1}\right)(1)$$

$$= \frac{1}{7} e^{z} + \frac{z}{4} e^{z}$$

General sol. $= c_1 e^{z} + c_2 \cos\sqrt{3}z + c_3 \sin\sqrt{3}z + \frac{1}{7} e^{2z} + \frac{z}{4} e^{z}$

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Put $x = e^z$

$\Rightarrow z = \log x.$

$$= c_1 x + c_2 \cos\sqrt{3}(\log x) + c_3 \sin\sqrt{3}(\log x) + \frac{1}{7}x^2 + \frac{x\log x}{4}.$$

Notes

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May 2014

15 wk 20 • day 135
Thursday

MAY						2014
M	T	W	T	F	S	S
			1	2	3	4
5	6	7	8	9	10	11
12	13	14	15	16	17	18
19	20	21	22	23	24	25
26	27	28	29	30	31	

(2). $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^4$.

(3) $x^3\left(\frac{d^3y}{dx^3}\right) + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left[x + 1/x\right]$.

(4) Solve $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$.

(5). $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2$.

Solution

Sol. Put $x = e^{z} \Rightarrow z = \log x$

let us denoted $\frac{d}{dz} = D$. then we can write the given equation. is as

$(D(D-1) - 4D + 6)y = x^4$.

$\Rightarrow (D^2 - 5D + 6)y = x^4$

its Auxillary equation is

$m^2 - 5m + 6 = 0$

$(m-3)(m-2) = 0 \Rightarrow m = 3, 2.$

Hence

C.F. $(y) = c_1 e^{3z} + c_2 e^{2z}$.

$= c_1 x^3 + c_2 x^2$

Now $P.I. = \frac{1}{f(D)} x^4 = \frac{1}{f(D)} e^{4z}$

$= \frac{1}{(D-3)(D-2)} e^{4z}$

$= \frac{e^{4z}}{2}$. if $D = 4$ then $f(D) = f(4) = 2 \neq 0$.

Notes

$= \frac{x}{2}$.

$\boxed{\text{Hence Gen. Sol.} = C.F + P.I.}$

(3). $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left[x + \frac{1}{x}\right]$

Sol. Put. $x = e^z \Rightarrow z = \log x$.

and write $\frac{d}{dz} = D$. then we get.

$D(D-1)(D-2) + 2D(D-1) + 2 = 10\left[e^z + e^{-z}\right]$

$\Rightarrow$ its Auxillary eq. is

$m(m-1)(m-2) + 2m(m-1) + 2 = 0.$

$m(m-1)(m\cancel{-2}+\cancel{2}) + 2 = 0$

15 $m^3 - m^2 + 2 = 0$

$\Rightarrow (m+1)(m^2 - 2m + 2) = 0 \Rightarrow m = -1, (1 \pm i)$

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Hence $C.F. (y) = c_1 e^{-z} + e^{z} (c_2 \cos z + c_3 \sin z)$

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$= c_1 x^{-1} + x\, c_2 \cos(\log x) + c_3 x \sin(\log x)$

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$P.I. = \frac{1}{f(D)} 10 [e^2 + e^{-2}]$

18 Sunday

$= \frac{1}{(D^3 - D^2 + 2)} 10 [e^2 + e^{-2}]$

Notes

$= 10 \left[\frac{e^2}{(D^3 - D^2 + 2)} + \frac{e^{-2}}{(D^3 - D^2 + 2)} \right]$

$y_p = \frac{1}{f(D)} 10e^2$ taking first part

$$= 10\,\frac{e^2}{D^3 - D^2 + 2}$$

$$= \frac{10e^2}{1-1+2} = \frac{10e^2}{2}.$$

$f(D) = D^3 - D^2 + 2$

$f(1) = \cancel{1} - \cancel{1} + 2$
$= 2.$

$f(D) =$

$f(-1)$
$= -1-1+2$
$= 0.$

II part $= \frac{10e^{-2}}{(D^3 - D^2 + 2)}$

$$= 10e^{-2}\,\frac{1}{((D-1)^3 - (D-1)^2 + 2)} \quad 1$$

$$= 10e^{-2}\,\frac{1}{(D-1)^2((D-1)^2 - 1) + 2} \quad (1)$$

$$= 10e^{-2}\,\frac{1}{(D-1)(D^2 + \cancel{1} - 2D - \cancel{1}) + 2}$$

$$= 10e^{-2}\,\frac{1}{(D-1)(D^2 - 2D) + 2.}$$

$$= 10e^{-2}\,\frac{1}{D(D-1)(D-2) + 2}$$

$$= 10e^{-2}\,\frac{1}{D(D^2 - 3D + 2) + 2} = 10e^{-2}\,\frac{1}{(D^3 - 3D^2 + 2D + 2)}$$

Notes

$$= 10e^{-2}\,\frac{1}{2\left(1 + \frac{D^3 - 3D^2 + 2D}{2}\right)}. \quad (1)$$

Solve it